Chapter 4

The Theoretical Model

Central place theory is the theoretical foundation for most economic models of urban form. A straightforward model based on the theory provides a useful framework for the household travel time problem at issue here. The model also shows how in central place theory housing prices and residential densities are decreasing functions of access to economic centers.[1]

Consider a household choosing a location in a city. In the traditional model the household will choose housing (h) and consumer goods (x) to maximize utility. Traditional urban economic theory provides that the distance to the city center (k) is critical to the household, as economic activity is concentrated at that center. Under the model households receive a benefit by less costly access to the center, as any activity outside the center comes at a greater cost or a lower wage.[2] In a limited polycentric city, k is some weighted measure of the travel distance or time of the household from the various centers of the city. Under either of these models, k may be viewed as a measure of access to economic opportunities. Under this formulation, the household's consumption of each of these goods will vary with the cost of access provided by the chosen housing (k) and the household's utility will be represented by the utility function:

$$U = U(h(k), x(k)) \qquad (1)$$

Each household will be constrained by its budget.[3] Housing and consumer goods are purchased at their respective market prices. The household purchases access to destinations by incurring travel expenses. These expenses will be equal to the amount the household travels (t) times the unit cost of that travel (c). The amount traveled is expected to be an

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[1] The role that joined trips may have in mitigating the apparent travel time disadvantage of a housing location with less access to economic centers is also observable in the model. Discussion of that effect, however, is reserved for the section that examines the use of joined trips.

[2] The model allows for employment outside the center, however, wages are a decreasing function of distance to the center.

[3] The budget in this case is a financial budget. The opportunity cost of time is incorporated into the specification by making the cost of time spent in travel an increasing function of income.

increasing function of both the number of destinations accessed and the measure of the cost of access. The cost of travel will be an increasing function of the household's income (*y),* since the opportunity cost of travel increases with income. Using these assumptions, the budget constraint, normalized by the purchase price of consumer goods, may be written as:

$$y = p(k)h(k) + x(k) + c(y)t(k) \qquad (2)$$

In equilibrium, the household will maximize utility to obtain some given level of utility that may be represented by the function:

$$U^*(y) = U^*(h(k), x(k)) \qquad (3)$$

Solving this expression for *x* we find that:

$$x^* = x(h(k), y) \qquad (4)$$

Substituting this expression into the budget constraint and solving that for *t* yields:

$$t = \frac{y - p(k)h(k) - x(h, y)}{c(y)} \qquad (5)$$

Differentiating this expression allows the examination of changes in the household's travel. For example, differentiating with respect to *k*, relying on the envelope theorem, we find one of the fundamental results in urban economics:

$$c(y)\frac{\partial t}{\partial k} = -p_k h \qquad (6)$$

This expression is typically interpreted to justify rent gradients found in cities. The increase in travel resulting from choosing a residential site with more costly access is equal to the savings on housing costs realized through a drop in the price in housing as the cost of access is sacrificed. The assumption driving the result is that as *k* increases *t* must also increase. For this to be true *k* must provide a meaningful measure of access to economic activity and travel expenditures are strictly increasing in *k*. The objective of this research is to examine the validity of these assumptions. If the purported relationship between access to centers (*k*)

and travel expenditures is not discernable in contemporary cities, the form of cities may differ from that of the monocentric and limited polycentric models.

The model also can be used to generate a density gradient. Although not tested in this research, the density gradient is important to this research since it suggests a particular urban spatial form and its existence in the model is dependent entirely on the purported relationship between household travel and access to economic centers that is tested here. The density gradient can be easily derived by first solving the budget constraint of equation (5) for *h*:

$$h = \frac{y - x - c(y)t}{p} \qquad (7)$$

Differentiating this with respect to *k* we find:

$$\frac{\partial h}{\partial k} = \frac{pc(y)t_k - p_k(y - x - c(y)t)}{p^2} \qquad (8)$$

This expression can be easily signed. We know that *p* and *c* are both positive. If t_k is assumed positive, then p_k is negative by equation (6). The parenthetical term in the numerator is equal to *p* times *h* if the budget constraint is assumed satisfied with equality. As a result, *h* must be an increasing function of *k*. So, under the assumptions of the model, housing densities will be a decreasing function of access to economic centers. This is a theoretical justification of the density gradient observed by economists. If *k* is simply the distance to the city center one can see that housing densities are a declining function of distance to the city center; the density gradient found in the traditional monocentric city. If *k* is instead viewed as a general measure of the cost of access to economic centers, equation (8) justifies the density gradient estimated under a model of a limited polycentric city.

As is apparent, both of the results above depend on the assumption that t_k is positive. The negative rent gradient (p_k) found in equation (6) is derived from the assumption that travel, and therefore travel expenditure, increases in areas with more costly access to economic centers. The positive density gradient that is found in equation (8) is a result of the travel assumption both directly and through the negative rent gradient. This assumption forms the crux of the model on which its predictions and explanations of urban form rely.

The validity of this assumption is tested by this research. In the next chapter the empirical model and data used for this test is described.