Stack and Queue Layouts of Posets

Lenwood S. Heath and Sriram V. Pemmaraju

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Department of Computer Science
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

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Virginia Polytechnic Institute & State University
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Abstract

The stacknumber (queuenumber) of a poset is defined as the stacknumber (queuenumber) of its Hasse diagram viewed as a directed acyclic graph. Upper bounds on the queuenumber of a poset are derived in terms of its jumpnumber, its length, its width, and the queuenumber of its covering graph. A lower bound of $\Omega(\sqrt{n})$ is shown for the queuenumber of the class of planar posets. The queuenumber of a planar poset is shown to be within a small constant factor of its width. The stacknumber of posets with planar covering graphs is shown to be $\Theta(n)$. These results exhibit sharp differences between the stacknumber and queuenumber of posets as well as between the stacknumber (queuenumber) of a poset and the stacknumber (queuenumber) of its covering graph.

Keywords: book embedding, Hasse diagram, jumpnumber, poset, queuelayout, stacklayout

1 Introduction

Stack layouts and queue layouts of undirected graphs appear in a variety of contexts such as VLSI, fault-tolerant processing, parallel processing, and sorting networks. In a new context, Heath and Pemmaraju [10] use queue layouts as the basis of a flexible scheme to efficiently perform matrix computations on a data driven network. Bernhart and Kainen [2] introduce the concept of a stack layout, which they call *book embedding.* Chung, Leighton, and Rosenberg [4] study stack layouts of undirected graphs and provide optimal stack layouts for a variety of classes of graphs. Heath, and Rosenberg [11] develop the notion of queue layouts and provide optimal queue layouts for many classes of undirected graphs. In some applications of stack and queue layouts, it is more realistic to model the application domain with directed acyclic graphs (dags) or with posets, rather than with undirected graphs. Various questions that have been asked about stack and queue layouts of undirected graphs acquire a new flavor when there are directed edges (arcs). This is because the direction of the arcs imposes restrictions on the node orders that can be considered. Barrett, Heath, and Pemmaraju [1] initiate the study of stack and queue layouts of dags and provide optimal stack and queue layouts for several classes of dags.

In this paper, we focus on stack and queue layouts of posets. Posets are ubiquitous mathematical objects and various measures of their structure have been defined. Some of these measures are bump-number, jumpnumber, length, width, dimension, and thickness [3,8]. Nowakowski and Parker [13] define the stacknumber of a poset as the stacknumber of its Hasse diagram viewed as a dag. They derive a general lower bound on the stacknumber of a planar poset and an upper bound on the stacknumber of a lattice. Nowakowski and Parker conclude by asking whether the stacknumber of the class of planar posets is unbounded. Hung [12] shows that there exists a planar poset with stacknumber 4; moreover, no planar poset with stacknumber 5 is known. Syslo [15] provides a lower bound on the stacknumber of a poset in terms of its bumpnumber. He also shows that while posets with jumpnumber 1 have stacknumber at most 2, posets with jumpnumber 2 can have an arbitrarily large stacknumber.

The organization of this paper is as follows. Section 2 contains definitions. In Section 3, we derive upper bounds on the queuenumber of a poset in terms of its jumpnumber, its length, its width, and the queuenumber of its covering graph. In Section 4, we show that the queuenumber of the class of planar posets is unbounded. A complementary upper bound result shows that the queuenumber of a planar poset is within a small constant factor of the width of the poset. In Section 5, we show that the stacknumber of the class of posets with planar covering graphs is $\Theta(n)$. In Section 6, the decision problems of recognizing a 4-queue poset and a 5-stack poset are defined; Barrett, Heath, and Pemmaraju [1] show that both problems are NP-complete. In Section 7 we present several open questions and conjectures concerning stack and queue layouts of posets.

2 Definitions

This section contains the definitions of stack and queue layouts of undirected graphs, dags, and posets. Other measures of the structure of posets are also defined.

Let $G = (V, E)$ be an undirected graph without multiple edges or loops. A ***k*-stack layout** of G consists of a total order σ on V along with an assignment of each edge in E to one of k stacks, $s_1, s_2, \ldots, s_k$. Each stack s_j operates as follows. The vertices of V are scanned in left-to-right (ascending) order according to σ. When a vertex v is encountered, any edges assigned to s_j that have v as their right endpoint must be at the top of the stack and are popped. Any edges that are assigned to s_j and have left endpoint v are pushed onto s_j in descending order (according to σ) of their right endpoints. The *stacknumber* $SN(G)$ of G is the smallest k such that G has a *k-stack layout.* G is said to be a *k-stack* graph if $SN(G) = k$. The *stacknumber* of a class of graphs $\mathcal{C}$, denoted by $SN(\mathcal{C})$, is the largest k such that there exists a graph $G \in \mathcal{C}$ with $SN(G) = k$.

A *k-queue layout* of G consists of a total order σ on V along with an assignment of each edge in E to one of k queues, $q_1, q_2, \ldots, q_k$. Each queue q_j operates as follows. The vertices of V are scanned in left-to-right (ascending) order according to σ. When a vertex v is encountered, any edges assigned to q_j that have v as their right endpoint must be at the front of the queue and are dequeued. Any edges that are assigned to q_j and have left endpoint v are enqueued into q_j in ascending order (according to σ) of their right endpoints. The *queuenumber* $QN(G)$ of G is the smallest k such that G has a *k-queue layout.* The *queuenumber* of a class of graphs $\mathcal{C}$, denoted by $QN(\mathcal{C})$, is the largest k such that there exists a graph $G \in \mathcal{C}$ with $QN(G) = k$.

For a fixed order σ on V, we identify sets of edges that are obstacles to minimizing the number of stacks or queues. A *k-rainbow* is a set of k edges $\left\{(a_i, b_i) \mid 1 \leq i \leq k\right\}$ such that

$$a_1 <_\sigma a_2 <_\sigma \cdots <_\sigma a_{k-1} <_\sigma a_k <_\sigma b_k <_\sigma b_{k-1} <_\sigma \cdots <_\sigma b_2 <_\sigma b_1,$$

i.e., a rainbow is a *nested* matching. Any two edges in a rainbow are said to *nest.*

A *k-twist* is a set of k edges $\left\{(a_i, b_i) \mid 1 \leq i \leq k\right\}$ such that

$$a_1 <_\sigma a_2 <_\sigma \cdots <_\sigma a_{k-1} <_\sigma a_k <_\sigma b_1 <_\sigma b_2 <_\sigma \cdots <_\sigma b_{k-1} <_\sigma b_k,$$

i.e., a twist is a *fully crossing* matching. Any two edges in a twist are said to *cross.*

A rainbow is an obstacle for a queue layout because no two edges that nest can be assigned to the same queue, just as a twist is an obstacle for a stack layout because no two edges that cross can be assigned to the same stack. Intuitively, we can think of a stack layout or a queue layout of a graph as a drawing of the graph in which the vertices are laid out on a horizontal line and the edges appear as arcs above the line. In a stack layout no two edges that intersect can be assigned to the same stack, while in a queue layout no two edges that nest can be assigned to the same queue. Clearly, the size of the largest twist (rainbow) in a layout is a lower bound on the number of stacks (queues) required for that layout.

The size of the largest rainbow in a layout equals the minimum queue requirement of the layout [11], while the size of the largest twist in a layout may be strictly less than the minimum stack requirement of the layout [7].

The definitions of stack and queue layouts are now extended to dags by requiring that the layout order be a topological order. Following a common distinction, we use *vertices* and *edges* for undirected graphs, but *nodes* and *arcs* for directed graphs. If $G = (V, E)$ is an undirected graph then, $\vec{G} = (V, \vec{E})$ is a dag whose arc set $\vec{E}$ is obtained by directing the edges in E. A *topological order* of $\vec{G}$ is a total order σ on V such that $(u, v) \in \vec{E}$ implies $u <_\sigma v$. A k-stack (k-queue) layout of the dag $\vec{G} = (V, \vec{E})$ is a k-stack (k-queue) layout of the graph G such that the total order is a *topological order* of $\vec{G}$. As before, $SN(\vec{G})$ is the smallest k such that $\vec{G}$ has a k-stack layout and $QN(\vec{G})$ is the smallest k such that $\vec{G}$ has a k-queue layout.

A *partial order* is a reflexive, transitive, anti-symmetric binary relation. A *poset* $P = (S, \leq)$ is a set S with a partial order $\leq$ (see Birkhoff [3] or Stanton and White [14]). The cardinality $|P|$ of a poset P equals $|S|$. We only consider posets with finite cardinality in this paper. We write $u < v$ if $u \leq v$ and $u \neq v$. The *Hasse diagram* $\vec{H}(P) = (S, \vec{E})$ of a poset $P = (S, \leq)$ is a dag with arc set

$$\vec{E} = \{(u, v) \mid u < v \text{ and there is no } w \text{ such that } u < w < v\}$$

(see Stanton and White [14]). A Hasse diagram is a minimal representation of a poset because it contains none of the arcs implied by transitivity of $\leq$. The stacknumber $SN(P)$ of a poset P is $SN(\vec{H}(P))$; P is a *k-stack poset* if $SN(P) = k$. Similarly, the queuenumber $QN(P)$ of a poset P is $QN(\vec{H}(P))$; P is a *k-queue poset* if $QN(P) = k$. The underlying undirected graph, $H(P)$, of $\vec{H}(P)$ is called the *covering graph* of P. Clearly, for any poset P

$$SN(H(P)) \leq SN(P) \quad \text{and} \quad QN(H(P)) \leq QN(P).$$

An example of a 2-stack poset is given in Figure 1. and example of a 2-queue poset is given in Figure 2. The stacknumber and the queuenumber of the covering graphs of the posets shown in Figure 1 and Figure 2 is 1.

A *diagram invariant* is a property of posets that is shared by all posets with the same covering graph. It is easy to verify that neither stacknumber nor queuenumber is a *diagram invariant*. In fact, in the subsequent sections we show that stacknumber and queuenumber are not even *approximate diagram*

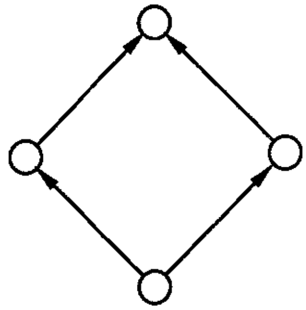

Figure 1: A 2-stack poset.

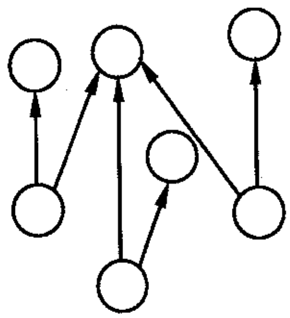

Figure 2: A 2-queue poset.

invariants in the sense that we are able to exhibit pairs of posets that share a covering graph, but one poset has a constant stacknumber (queuenumber), while the other has a stacknumber (queuenumber) that is arbitrarily large.

A poset P is *planar* if its Hasse diagram $\vec{H}(P)$ has a planar embedding in which all arcs are drawn as straight line segments with the tail of each arc strictly below its head with respect to a Cartesian coordinate system. Note that $H(P)$ may be planar even though the poset P is not. The posets shown in Figure 1 and in Figure 2 are both planar.

Let γ be a fixed topological order on $\vec{H}(P)$. Two elements u and v are *adjacent* in γ if there is no w such that $u <_\gamma w <_\gamma v$ or $v <_\gamma w <_\gamma u$. A *spine arc* in $\vec{H}(P)$ with respect to γ is an arc (u, v) in $\vec{H}(P)$ such that u and v are adjacent in γ. A *break* in $\vec{H}(P)$ with respect to γ is a pair (u_1, u_2) of adjacent elements such that $u_1 <_\gamma u_2$ and (u_1, u_2) is not an arc in $\vec{H}(P)$. A *connection* C in $\vec{H}(P)$ with respect to γ is a maximal sequence of elements $u_1 <_\gamma u_2 <_\gamma \cdots <_\gamma u_k$ such that (u_i, u_{i+1}) is a spine arc for all $i, 1 \leq i < k$; in other words a connection is a maximal path of spine arcs without a break. The *jumpnumber of* P, denoted by $JN(P)$, is the minimum number of breaks in any topological order on $\vec{H}(P)$.

A *chain* in a poset P is a sequence of elements $u_1, u_2, \ldots, u_n$ such that $u_1 < u_2 < \cdots < u_n$. The *length* $L(P)$ of a poset P is the maximum cardinality of a any chain in P. An *antichain* in a poset P is a subset of elements of S that does not contain a chain of size 2. The *width* $W(P)$ of a poset P is the maximum cardinality of any anti-chain in P.

3 Upper Bounds on Queuenumber

In this section we derive upper bounds on the queuenumber of a poset in terms of its jumpnumber, its length, its width, and the queuenumber of its covering graph.

3.1 Jumpnumber and Queuenumber

We show that, for any poset P, $JN(P) + 1$ is an upper bound on $QN(P)$. This bound is tight within a constant factor. This result is in contrast with the following result of Syslo.

Proposition 1 Syslo [15] *For any poset P, with $JN(P) = 1$, $SN(P) \leq 2$. For every n, there exists a poset P such that $|P| = n$, $JN(P) = 2$, but $SN(P) = \Omega(n)$.*

Theorem 2 *For any poset P, $QN(P) \leq JN(P) + 1$. There exists a poset P such that $|P| = 2n$ and $JN(P)/2 \leq QN(P)$.*

Proof: For the upper bound, let P be any poset and let $JN(P) = k$. Let γ be a topological order on $\vec{H}(P)$ that has exactly k breaks and $k+1$ connections. Lay out $\vec{H}(P)$ according to γ and label these connections $C_0, C_1, \ldots, C_k$ from left to right. Let (u_1, v_1) and (u_2, v_2) be two nonspine arcs such that u_1 and u_2 are in C_i and v_1 and v_2 are in C_j, where $1 \leq i < j \leq k$. If (u_1, v_1) and (u_2, v_2) nest, then one of (u_1, v_1) and (u_2, v_2) is a transitive arc. The fact that no two nonspine arcs between a pair of connections nest suggests the following assignment of arcs to queues. Assign all non-spine arcs between pairs of connections C_i and C_j, where $|i - j| = \ell, 1 \leq \ell \leq k$ to queue q_ℓ. Assign all the spine arcs to a queue q_0. Hence, we use k queues for non-spine arcs and one queue for spine arcs, for a total of $k+1$ queues.

To show a lower bound, construct the Hasse diagram of a poset P from the complete bipartite graph $K_{n,n} = (V_1, V_2, E)$ by directing all the edges from vertices in V_1 to vertices in V_2. Hence, $JN(P) = 2(n-1)$, $QN(P) = n$, and

$$QN(P) = \frac{n}{2(n-1)} JN(P).$$

□

Proposition 1 and Theorem 2 have the following corollary.

Corollary 3 *There exists a class of posets $\mathcal{P}$ for which the ratio $\frac{SN(\mathcal{P})}{QN(\mathcal{P})}$ is unbounded.*

Theorem 10, in contrast, shows a class of posets $\mathcal{Q}$ for which $SN(\mathcal{Q})/QN(\mathcal{Q})$ is unbounded.

3.2 Length and Queuenumber

In this section, we show an upper bound on the queuenumber of a poset in terms of its length and the queuenumber of its covering graph. In order to prove the theorem we need the following lemma that gives a bound on the queuenumber of a layout of a graph whose vertices have been rearranged in a limited fashion.

Lemma 4 (Heath and Pemmaraju [9]) *Suppose that σ is an order on the vertices of of an m-partite graph $G = (V_1, V_2, \ldots, V_m, E)$ that yields a k-queue layout of G. Let σ' be an order on the vertices of G in which the vertices in each set V_i, $1 \leq i \leq m$ appear consecutively and in the same order as in σ. Then σ' yields a $2(m-1)k$-queue layout of G.*

Theorem 5 *For any poset P,*

$$QN(P) \leq 2 \cdot (L(P) - 1) \cdot QN(H(P)).$$

There exists a poset P for which $QN(P) = 2 \cdot (L(P) - 1) \cdot QN(H(P))$.

Proof: Let $QN(H(P)) = k$ and let $\vec{H}(P) = (V, \vec{E})$. Let σ be a total order on V that yields a k-queue layout of $H(P)$. The nodes of $\vec{H}(P)$ can be labeled by a function $l : V \rightarrow \{1, \ldots, L(P)\}$ such that $l(u) < l(v)$ if $u < v$ in P, as follows. Let $\vec{H}_0 = \vec{H}(P)$. Label all the nodes with indegree 0 in $\vec{H}_0$ with the label 1. Delete all the labeled nodes in $\vec{H}_0$ to obtain $\vec{H}_1$. In general, label the nodes with indegree 0 in $\vec{H}_i$ with the label $i+1$. Delete the labeled nodes in $\vec{H}_i$ to obtain $\vec{H}_{i+1}$. By an inductive proof, it can be checked that the labeling so obtained satisfies the required conditions. Let $V_i = \{u \in V \mid l(u) = i\}$ for each i, $1 \leq i \leq L(P)$. For any arc $(u, v) \in \vec{E}$, if $u \in V_i$, then $v \in V_j$ for some j, $1 \leq i < j \leq L(P)$. Therefore $\vec{H}(P) = (V_1, V_2, \ldots, V_{L(P)}, \vec{E})$ is an $L(P)$-partite dag. Construct a total order γ on the nodes of $\vec{H}(P)$ such that

1. The elements in each set V_i, $1 \leq i \leq L(P)$ occur contiguously and in the order prescribed by σ.

2. The elements in V_i occur before the elements in V_{i+1} for all i, $1 \leq i < L(P)$.

Since every arc in $\vec{H}(P)$ is from a node in V_i to a node in V_j, $1 \leq i < j \leq L(P) - 1$, γ is a topological order on $\vec{H}(P)$. By Lemma 4, γ yields a layout that requires no more than $2 \cdot (L(P) - 1) \cdot k$ queues.

We now show that the upper bound in the above theorem is tight. As in the proof of Theorem theorem:jumpnumber, construct the Hasse diagram of a poset P by directing the edges of the complete bipartite graph $K_{n,n} = (V_1, V_2, E)$ from V_1 to V_2. The length of the poset, $L(P)$, is 2, the queuenumber of the covering graph, $QN(K_{n,n})$, is $\lceil \frac{n}{2} \rceil$ (see Heath and Rosenberg [11]), and the queuenumber of the poset P, $QN(P)$, is n. □

Note that Theorem 5 holds for dags as well as for posets as its proof does not rely on the absence of transitive arcs. An immediate corollary is the following.

Corollary 6

$$QN(H(P)) \leq QN(P) \leq 2 \cdot (L(P) - 1) \cdot QN(H(P)).$$

Consider a class of posets $\mathcal{P}$. Corollary 6 implies that if there exists a constant K such that $L(P) \leq K$, for all $P \in \mathcal{P}$, then $QN(P) = \Theta(QN(H(P))$ for all $P \in \mathcal{P}$.

3.3 Width and Queuenumber

In this section, we establish an upper bound on the queuenumber of a poset in terms of its width. We need the following result of Dilworth.

Lemma 7 (Dilworth [5]) *Let $P = (S, \leq)$ be a poset. Then S can be partitioned into $W(P)$ chains.*

For a poset $P = (S, \leq)$, let $S_1, S_2, \ldots, S_{W(P)}$ be a partition of S into $W(P)$ chains. Define an *i-chain arc* as an arc in $\vec{H}(P)$, both of whose end points belong to chain S_i, $1 \leq i \leq W(P)$. An *(i,j)-cross arc*, $i \neq j$ is an arc whose tail belongs to chain S_i and whose head belongs to chain S_j.

Theorem 8 *The largest rainbow in any layout of a poset P is of size no greater than $W(P)^2$. Hence, the queuenumber of any layout of P is at most $W(P)^2$.*

Proof: Fix an arbitrary topological order of $\vec{H}(P)$. For any i, no two i-chain arcs nest. Therefore, the largest rainbow of chain arcs has size no greater than $W(P)$. If $i \neq j$ then no two (i,j)-cross arcs can nest without one of them being a transitive arc. Therefore, the largest rainbow of cross arcs has size no greater than $W(P)(W(P)-1)$. The size of the largest rainbow is at most $W(P) + W(P)(W(P)-1) = W(P)^2$.
□

The bound established in the above theorem is not known to be tight. In fact, we believe that the queuenumber of a poset is bounded above by its width (see Conjecture 1 in Section 7).

4 The Queuenumber of Planar Posets

In this section, we first show that the queuenumber of the class of planar posets is unbounded. This is in contrast with the fact that the stacknumber of the class of planar posets is unknown. We then establish an upper bound on the queuenumber of a planar poset in terms of its width.

4.1 A Lower Bound on the Queuenumber of Planar Posets

We construct a sequence of planar posets P_n with $|P_n| = 3n+3$ elements and $QN(P_n) = \Theta(\sqrt{n})$. In fact, we determine the queuenumber of P_n almost exactly. To prove the theorem, we need the following result of Erdös and Szekeres.

Lemma 9 (Erdös and Szekeres [6]) *Let $(x_i)_{i=1}^n$ be a sequence of distinct elements from a set X. Let δ be a total order on X. Then $(x_i)_{i=1}^n$ either contains a monotonically increasing subsequence of size $\lceil\sqrt{n}\rceil$ or a monotonically decreasing subsequence of size $\lceil\sqrt{n}\rceil$ with respect to δ.*

Theorem 10 *There exists a planar poset P_n with $3n+3$ elements such that*

$$\lfloor\sqrt{n}\rfloor + 1 \leq QN(P_n) \leq \lfloor\sqrt{n}\rfloor + 2.$$

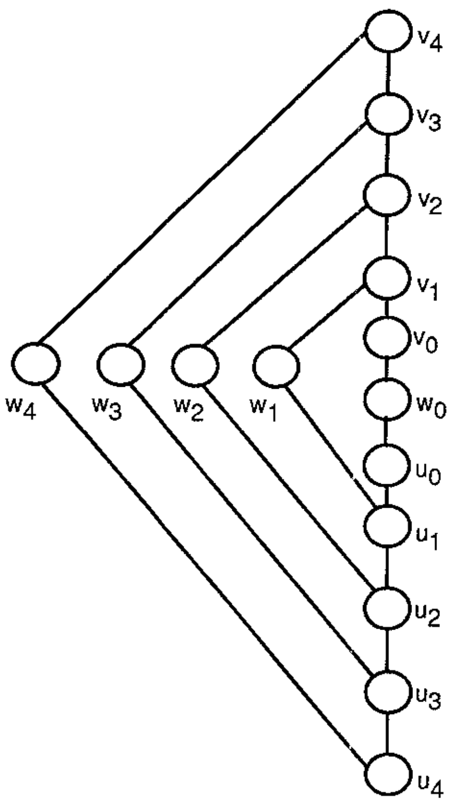

Figure 3: The planar poset P_4.

Proof: Define the disjoint sets U, V, and W as follows.

$$\begin{aligned} U &= \{u_i \mid 0 \le i \le n\} \\ V &= \{v_i \mid 0 \le i \le n\} \\ W &= \{w_i \mid 0 \le i \le n\} \end{aligned}$$

Let $S = U \cup V \cup W$. The planar poset $P_n = (S, \le)$ is given by

$$\begin{array}{llllll} u_i & < & u_{i-1}, & & & 1 \le i \le n \\ v_i & < & v_{i+1}, & & & 0 \le i \le n-1 \\ u_i & < & w_i & < & v_i, & 1 \le i \le n \end{array}$$

Figure 3 shows the Hasse diagram of P_4. Let σ be an arbitrary order on the elements of S. The elements of $U \cup V \cup \{w_0\}$ appear in the order $u_n, u_{n-1}, \ldots, u_0, w_0, v_0, v_1, \ldots, v_n$ in σ, and all elements of W appear between u_n and v_n. Define a total order δ on the elements of W by $w_i <_\delta w_j$ if $i < j$. Suppose

$$w_{i_1}, w_{i_2}, \ldots, w_{i_k}$$

is an increasing sequence of nodes in W with respect to δ. Since w_{i_1} appears after u_{i_1} in any topological order of $\vec{H}(P_n)$, the following sequence of nodes is a subsequence of σ:

$$u_{i_k}, u_{i_{k-1}}, \ldots, u_{i_1}, w_{i_1}, w_{i_2}, \ldots, w_{i_k}.$$

Therefore, the set $\{(u_{i_j}, w_{i_j}) \mid 1 \leq j \leq k\}$ is a k-rainbow in σ. Similarly, if

$$w_{i_1}, w_{i_2}, \ldots, w_{i_k}$$

is a decreasing sequence of nodes in W with respect to δ, then the set $\{(w_{i_j}, v_{i_j}) \mid 1 \leq j \leq k\}$ is a k-rainbow in σ. By Lemma 9, in σ, there is an increasing subsequence of size $\lceil \sqrt{n} + 1 \rceil$ or a decreasing subsequence of size $\lceil \sqrt{n} + 1 \rceil$ with respect to δ. Thus there is a rainbow of size $\lceil \sqrt{n} + 1 \rceil \geq \lfloor \sqrt{n} \rfloor + 1$ in any topological order on $\vec{H}(P_n)$ and therefore $QN(P_n) \geq \lfloor \sqrt{n} \rfloor + 1$.

We now lay out P_n in $\lfloor \sqrt{n} \rfloor + 2$ queues. Let $s = \lceil \sqrt{n} \rceil$ and let $t = \lceil \frac{n}{s} \rceil \leq \lceil \sqrt{n} \rceil$. Partition $W - \{w_0\}$ into s nearly equal-sized subsets

$$W_1, W_2, \ldots, W_s$$

such that

$$W_i = \begin{cases} \{w_j \mid 1 \leq j \leq t\} & 1 \leq i \leq s-1 \\ \{w_j \mid (s-1)t \leq j \leq n\} & i = s \end{cases}$$

Construct an order σ on the elements of S by first placing the elements in $U \cup V \cup \{w_0\}$ in the order

$$u_n, u_{n-1}, \ldots, u_0, w_0, v_0, v_1, \ldots, v_n.$$

Now place the elements of $W - \{w_0\}$ between u_0 and v_0 such that the elements belonging to each set W_i appear contiguously and the sets themselves appear in the order

$$W_s, W_{s-1}, \ldots, W_1.$$

Within each set W_i, $1 \leq i \leq s$, place the elements in increasing order. The arcs from U to W form s mutually intersecting rainbows each of size at most t. Therefore t queues suffice for these arcs. The arcs from W to V form s nested twists each of size at most t. Therefore s queues suffice for these arcs. Since no two arcs, one from U to W and the other from W to V nest, they can all be assigned to the same set of s queues. An additional queue is required for the remaining arcs. This is a layout of P_n in $\lceil \sqrt{n} \rceil + 1$ queues (Figure 4). Thus $QN(P_n) \leq \lfloor \sqrt{n} \rfloor + 2$.

□

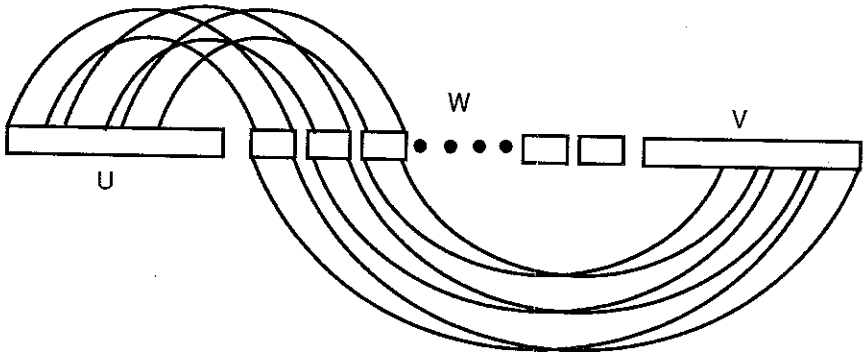

Figure 4: Layout of planar poset P_n.

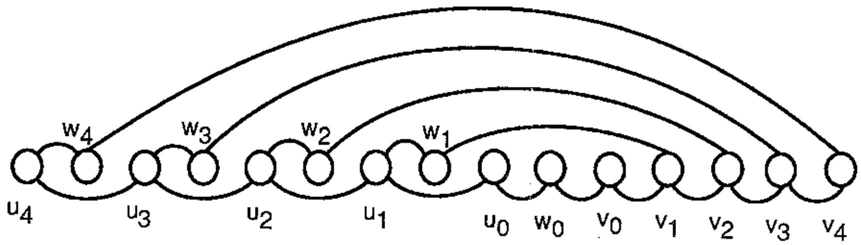

Figure 5: A 2-stack layout of the planar poset P_4.

We believe that the upper bound in the above proof can be tightened to exactly match the lower bound. In fact, we have been able to show that for $m^2 \leq n \leq m(m+1)$, $QN(P_n) = m + 1 = \lfloor \sqrt{n} \rfloor + 1$.

The above result contrasts with the lack of known bounds on the stacknumber of planar posets. Two observations about P_n are in order. The first observation is that $SN(P_n) = 2$. A 2-stack layout of $\vec{H}(P_4)$ is shown in Figure 5. The second observation is that the stacknumber *and* the queuenumber of $H(P_n)$ is

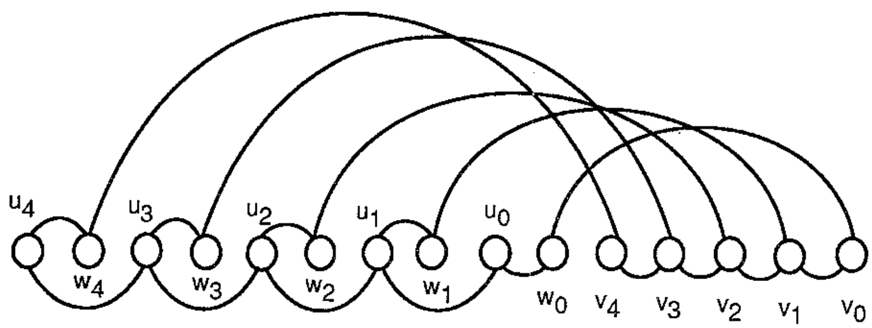

Figure 6: A 2-queue layout of the covering graph of P_4.

2. A 2-queue layout of $H(P_4)$ is shown in Figure 6. Theorem 10 and the above observations imply the following corollaries.

Corollary 11 *There exists a sequence of planar posets P_n, $n \geq 1$, $|P_n| = 3n + 3$ such that*

$$\frac{QN(P)}{SN(P)} = \Omega\left(\sqrt{n}\right).$$

Corollary 12 *There exists a sequence of planar posets P_n, $n \geq 1$, $|P_n| = 3n + 3$, such that*

$$\frac{QN(P)}{QN(H(P))} = \Omega\left(\sqrt{n}\right).$$

While Theorem 10 establishes a lower bound of $\Omega(\sqrt{n})$ on the queuenumber of the class of planar posets with n elements, a matching upper bound is not known (see Conjecture 2 in Section 7).

4.2 An Upper Bound on the Queuenumber of Planar Posets

In this subsection, we show that the queuenumber of a planar poset is within a small constant factor of its width.

Theorem 13 *For any planar poset P, any topological order of $\vec{H}(P)$ has queuenumber no greater than $3 \cdot W(P) - 2$.*

Proof: Without loss of generality, assume that $\vec{H}(P) = (V, \vec{E})$ is embedded in the plane such that every arc is drawn as a straight arrow pointing upwards and no two nodes are on the same horizontal line. (If two nodes are on the same horizontal line, a slight vertical perturbation of either of them yields another planar embedding with the nodes on different horizontal lines). Furthermore, we may assume that $\vec{H}(P)$ has a unique source (the lowest node) and a unique sink (the highest node) by the following argument. If v is a source and not the lowest node, then some node u that is below v is visible from v along a straight line. If the arc (u, v) is added, then we get a Hasse diagram of a new poset with a planar embedding in which every arc is drawn as a straight arrow pointing upwards. The width of the new poset is no greater than $W(P)$, and it has one fewer source than $\vec{H}(P)$. By induction we may assume that $\vec{H}(P)$ has a unique source and a unique sink; call it s. By an analogous argument we may assume that $\vec{H}(P)$ has a unique sink; call it t.

By Lemma 7, V can be partitioned into $W(P)$ chains

$$V_1, V_2, \ldots, V_{W(P)}.$$

For each chain V_i, there is a directed path D_i from s to t that contains every element of V_i. We conclude that the nodes of $\vec{H}(P)$ can be covered by $W(P)$ directed paths, not necessarily disjoint, from s to t.

For an arbitrary directed path D from s to t, there is an obvious notion of a node or an arc being on the left of D, the right of D, or on D. We say that a directed path D' from s to t is *to the left (right) of* D if every arc in D' is either to the left (right) of D or on D. Clearly, D' is to the left of D if and only if D is to the right of D'. Write $D' <_L D$ if D' is to the left of D. It is easy to verify that $<_L$ is a partial order on paths from s to t.

Without loss of generality, we may assume that the paths $D_1, D_2, \ldots, D_{W(P)}$ covering the nodes of $H(P)$ are in the order

$$D_1 <_L D_2 <_L \ldots <_L D_{W(P)}$$

by the following argument. Suppose that D_i and D_j are unrelated in the partial order $<_L$ and that $i < j$. Construct two paths D_i' and D_j' from s to t. Place all arcs that are on both D_i and D_j in both D_i' and D_j'. Place all arcs in D_i to the left of D_j and all arcs in D_j to the left of D_i in D_i'. Place all arcs in D_i to the right of D_j and all arcs in D_j to the right of D_i in D_j'. It is easy to verify that D_i' and D_j' are directed paths from s to t and that $D_i' <_L D_j'$. An inductive proof completes the argument.

We are now prepared to complete the proof of the theorem by an induction on $W(P)$. In the base case, $W(P) = 1$ and the entire Hasse diagram is covered by a single path D_1 from s to t. Assigning all arcs in D_1 to a single queue yields the claimed bound.

To complete the induction, assume that $W(P) > 1$ and that the bound holds for all posets of smaller width. Cover the nodes of $\vec{H}(P)$ with paths $D_1, D_2, \ldots, D_{W(P)}$ from s to t so that

$$D_1 <_L D_2 <_L \ldots <_L D_{W(P)}.$$

Clearly, $D_{W(P)}$ is the directed path from s to t along all the right most arcs in the planar embedding of $\vec{H}(P)$. There must be at least one node in $D_{W(P)}$ that is in no other D_i (otherwise, the width of P is less than $W(P)$, a contradiction). Let P' be the poset derived by removing all such nodes from P. We have that $W(P') = W(P) - 1$ and that the nodes of $\vec{H}(P')$ are covered by $D_1, D_2, \ldots, D_{W(P)-1}$. By the inductive assumption, any topological order of $H(P')$ has queuenumber at most $3 \cdot W(P') - 2 = 3 \cdot W(P) - 5$.

Consider any topological order of $\vec{H}(P)$. Assign the arcs of $\vec{H}(P')$ to $3 \cdot W(P) - 5$ queues. Any arcs not in $\vec{H}(P')$ fall into one of three classes: (i) on $D_{W(P)}$, (ii) *incoming* into $D_{W(P)}$, or (iii) *outgoing* from $D_{W(P)}$. Note that an incoming arc must have its tail on $D_{W(P)-1}$ and an outgoing arc must have its head on $D_{W(P)-1}$. Use one queue for each of these classes of arcs. By the fact that $\vec{H}(P)$ is in topological order, no two arcs *on* $D_{W(P)}$ can nest. Suppose (u, v) and (x, y) are two incoming arcs. Then u and x are on $D_{W(P)-1}$ and v and y are on $D_{W(P)}$. Suppose u precedes x on $D_{W(P)-1}$ and hence in the topological order. By the planarity of the embedding of $\vec{H}(P)$ with arcs drawn as straight arrows pointing upwards, v precedes y in topological order. Similarly, if x precedes u on $D_{W(P)-1}$, then y precedes v on $D_{W(P)}$. In either case (u, v) and (x, y) do not nest. Similarly, it can be shown that no two outgoing arcs nest. Hence, the assignment of arcs to queues described above results in a queue layout of $\vec{H}(P)$ in $3 \cdot W(P) - 2$ queues.

By induction, the theorem follows. □

In the above theorem, we show that any topological order can be used to obtain a $(3W(P)-2)$-queue layout of $\vec{H}(P)$. We believe that using a particular topological order, chosen with care, will be able to provide a $W(P)$-queue layout of $\vec{H}(P)$. (See Conjecture 1).

5 Stacknumber of Posets with Planar Covering Graphs

In this section, we construct $3n$-element posets P_n, $n \geq 1$, such that $H(P_n)$ is planar and hence $SN(H(P_n)) \leq 4$ (see Yannakakis [18]), and yet $SN(P_n) = \Theta(n)$.

Theorem 14 *There exist posets P_n, $n \geq 1$, such that $|P_n| = 3n$, $H(P_n)$ is planar, and*

$$\left\lfloor \frac{n}{2} \right\rfloor \leq SN(P_n) \leq n.$$

Proof: Let U, V, and W be disjoint sets

$$\begin{aligned} U &= \{u_i \mid 1 \leq i \leq n\} \\ V &= \{v_i \mid 1 \leq i \leq n\} \\ W &= \{w_i \mid 1 \leq i \leq n\}. \end{aligned}$$

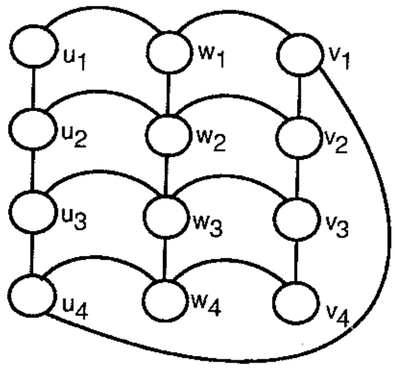

Figure 7: The covering graph of P_4.

The poset $P_n = (U \cup V \cup W, \leq)$ is given by

$$\begin{array}{ll} u_i < u_{i+1}, \quad v_i < v_{i+1}, \quad w_i < w_{i+1}, & 1 \leq i \leq n-1 \\ u_i < w_i < v_i, & 1 \leq i \leq n \\ u_n < v_1. & \end{array}$$

Figure 7 shows $H(P_4)$.

To prove the lower bound on $SN(P_n)$, let σ be any order on the elements of P_n. The order σ contains the elements of $U \cup V$ in the order $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$, and the elements of W in the order $w_1, w_2, \ldots, w_n$. The elements of W are mingled among the elements $U \cup V$. If $w_{\lfloor n/2 \rfloor} <_\sigma u_n$, then

$$(w_1, v_1), (w_2, v_2), \ldots, (w_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor})$$

form an $\lfloor n/2 \rfloor$-twist. If $w_{\lfloor n/2 \rfloor} >_\sigma u_n$, then

$$(u_{\lfloor n/2 \rfloor+1}, w_{\lfloor n/2 \rfloor+1}), (u_{\lfloor n/2 \rfloor+2}, w_{\lfloor n/2 \rfloor+2}), \ldots, (u_n, w_n)$$

form an $\lceil n/2 \rceil$-twist. In either case, the layout contains an $\lfloor \frac{n}{2} \rfloor$ twist. Therefore, $SN(P_n) \geq \lfloor \frac{n}{2} \rfloor$.

An n-stack layout of P_n is obtained by laying out the elements of $U \cup V$ in the only possible order, and then placing each element w_i immediately after u_i for all i, $1 \leq i \leq n$. Figure 8 shows a 2-queue layout of P_4, but the total order shown in the figure is what we use to obtain a n-stack layout of P_n. The assignment of arcs to stacks is as follows. Assign each arc in the set $\{(u_i, w_i), (w_i, v_i), (w_i, w_{i+1})\}$ to stack s_i for all i, $1 \leq i \leq n-1$ and assign each arc in the set $\{(u_n, w_n), (w_n, v_n)\}$ to stack s_n. Note that

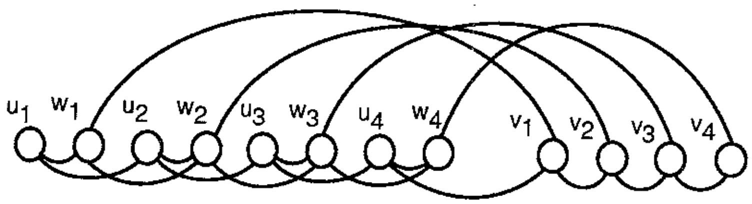

Figure 8: A 2-queue layout of P_4.

no two arcs assigned to the same stack intersect. The only arcs remaining to be assigned are the arcs in the set

$$\{(u_i, u_{i+1}) \mid 1 \le i \le n-1\} \cup \{(v_i, v_{i+1}) \mid 1 \le i \le n-1\} \cup \{(u_n, v_1)\}.$$

The arcs (v_i, v_{i+1}) for i, $1 \le i \le n-1$ do not intersect any other arc and can be assigned to any stack. Each arc (u_i, u_{i+1}) can be assigned to stack s_{i+1} for all i, $1 \le i \le n-1$ and arc (u_n, v_1) can be assigned to stack s_1.

□

Two observations about the poset P_n constructed in the above proof are in order. The first observation is that $QN(P_n) = 2$. A 2-queue layout of P_4 is shown in Figure 8. In general, the n-stack layout of P_n described in the above proof yields a 2-queue layout of P_n. The second observation is that the stacknumber and the queuenumber of the $H(P_n)$ is 2. A 2-stack layout of $H(P_4)$ is shown in Figure 9. In general, a 2-stack layout of $H(P_n)$ can be obtained because $H(P_n)$ is a hamiltonian planar graph [2].

Theorem 14 and the above observations lead to the following corollaries.

Corollary 15 *There exists a sequence of posets P_n, $n \ge 1$, $|P_n| = 3n$, such that $H(P_n)$ is planar and*

$$\frac{SN(P)}{QN(P)} = \Omega(n).$$

Corollary 16 *There exists a sequence of posets P_n, $n \ge 1$, $|P_n| = 3n$, such that $H(P_n)$ is planar and*

$$\frac{SN(P)}{SN(H(P))} = \Omega(n).$$

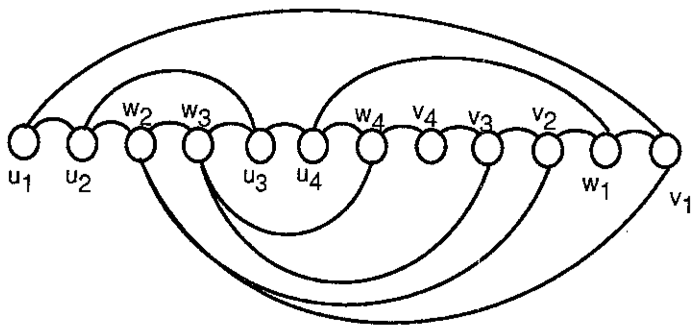

Figure 9: A 2-stack layout of the covering graph of P_4.

6 NP-completeness Results

Heath and Rosenberg [11] show that the problem of recognizing a 1-queue graph is NP-complete. Since a 1-stack graph is an outerplanar graph, a 1-stack graph can be recognized in linear time (Syslo and Iri [16]). But Wigderson [17] shows that the problem of recognizing a 2-stack graph is NP-complete. Barrett, Heath and Pemmaraju [1] show that the problems of recognizing a 4-queue poset and of recognizing a 5-stack poset are both NP-complete.

The decision problem for stack layouts of posets is POSETSN.

POSETSN

INSTANCE: A poset P.

QUESTION: Does P have a 5-stack layout?

The decision problem for queue layouts of posets is POSETQN.

POSETQN

INSTANCE: A poset P.

QUESTION: Does P have a 4-queue layout?

Theorem 17 (Barrett, Heath, Pemmaraju [1]) *The decision problems POSETSN and POSETQN are both NP-complete.*

Since, the Hasse diagram of a poset is a dag, this result hold for dags in general. This result is in the spirit of the result of Yannakakis [19] that recognizing a 3-dimensional poset is NP-complete.

7 Conclusions and Open Questions

In this paper, we have initiated the study of queue layouts of posets and have proved a lower bound result for stack layouts of posets with planar covering graph. The upper bounds on the queuenumber of a poset in terms of its jumpnumber, its length, its width, and the queuenumber of its covering graph, proved in Section 3, may be useful in proving specific upper bounds on the queuenumber of various classes of posets. We believe that the upper bound of $W(P)^2$ on the queuenumber of an arbitrary poset P, proved in Section 3, and the upper bound of $3 \cdot W(P) - 2$ on the queuenumber of any planar poset P, proved in Section 4 are not tight. We conjecture that:

Conjecture 1 *For any poset P, $QN(P) \leq W(P)$.*

We have established a lower bound of $\Omega(\sqrt{n})$ on the queuenumber of the class of planar posets. We believe that this bound is tight and conjecture that:

Conjecture 2 *For any n-element planar poset P, $QN(P) = O(\sqrt{n})$.*

Another upper bound that we believe exists on the queuenumber of a planar poset P is given by the length $L(P)$. We believe that it is possible to embed a planar poset, in a plane in an "almost" leveled-planar fashion with $L(P)$ levels. From such an embedding, a queue layout of P in $L(P)$ queues can be obtained. Therefore we conjecture:

Conjecture 3 *For any planar poset P, $QN(P) \leq L(P)$.*

In Section 5 we show that the stacknumber of posets whose covering graph is planar is $\Theta(n)$. This is in contrast with the fact that the stacknumber of planar posets is still unresolved.

Acknowledgements

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