Technical Report CS75007-E

A PEDAGOGICAL TOOL: CALCULATING ORDER OF CONVERGENCE

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ABSTRACT

In this report we give a technique for computing the order of convergence of a method which avoids the complexity of theory of difference equations. The technique may be easily perfected by any student in an undergraduate introductory numerical analysis class.

Keywords and Phrases: order of convergence, difference equations, numerical analysis pedagogy.

INTRODUCTION

Given the problem of solving the equation

(1.1) $$f(x) = 0,$$

usual approaches are to construct an equivalent fixed point problem

(1.2) $$x = F(x)$$

which inspires the method

(1.3) $$x_{n+1} = F(x_n) \qquad n \geq 0$$

or; more generally,

(1.4) $$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n+1-k})$$

if indeed F is 2 function of k variables. Assuming the method produces a convergent sequence with limit x^*:

(1.5) $$x^* = F(x^*),$$

the order of convergence is defined as the largest real p for which

(1.6) $$0 < \lim \frac{e_{n+1}}{e_n^p} < \infty$$ [1]

holds, where

(1.7) $$e_n \equiv |x_n - x^*|.$$

As a prelude to finding p, (1.7) is used with (1.3) or (1.4) to produce

[1]We suppress the phrase "$n \to \infty$" from all limit operations.

an error formula of the type

(1.8) $$e_{n+1} = g_n \prod_{j=\phi}^{k} e_{n-j}^{r_{j+1}}$$

where lim g_n exists and is not 0. (See Ralston [4], p. 321). The production of (1.8) usually involves multiple invocations of the mean value theorem and is tedious but within the grasp of a 1st year calculus student. If (1.8) has the special form

(1.9) $$e_{n+1} = g_n e_n^{r_1}$$

then the order of convergence is immediate. (See Ralston [4], p. 329) for a summarizing theorem).

However misunderstood the notion of order of convergence may be (see Greenberg [3], for some enlightenment), the derivation of such for a variety of popular methods not leading to (1.9) is generally beyond the scope of any undergraduate numerical analysis classes and usually is a burden to the beginning graduate unless he is securely versed in difference equations. We present a technique for deriving order of convergence which can be fathomed by any student who can get to (1.8). Most important is that difference equation theory need not be involved.

Background Material

Though the following is either obvious or trivial, for sure it appears elsewhere. (e.g., Conte deBoor [2] pp. 17-21). It is given merely for completeness.

Let a_n, b_n, c_n, and d_n be positive sequences with limit 0.

Definition 2.1 a_n is equivalent to b_n, written

$$a_n \sim b_n$$

if for some constant $k > 0$.

$$\lim \frac{a_n}{b_n} = k$$

Theorem 2.2 $a_n \sim b_n$ if and only if for every integer k, $a_{n+k} \sim b_{n+k}$.

Proof. Obvious.

Theorem 2.3 1. $a_n \sim b_n,\ b_n \sim c_n \Rightarrow a_n \sim c_n$

2. $a_n \sim b_n \Leftrightarrow b_n \sim a_n$

3. $a_n \sim a_n$

Proof. Obvious

Theorem 2.4. $a_n \sim b_n,\ c_n \sim d_n \Rightarrow a_n c_n \sim b_n d_n$

Proof: Trivial

Theorem 2.5 $a_n \sim b_n \Rightarrow a_n^p \sim b_n^p$ for every positive p,

Proof: $h(x) = x^p$ is continuous for $x>0$.

Therefore,

$$h(\lim \frac{a_n}{b_n}) = \lim h(\frac{a_n}{b_n})$$

or

$$(\lim \frac{a_n}{b_n})^p = \lim \frac{a_n^p}{b_n^p} \quad .$$

Theorem 2.6: $a_n^p \sim a_n^q \Rightarrow p=q$.

Proof: $0 < \lim \frac{a_n^p}{a_n^q} < \infty$

since

$$0 < \lim a_n^{(p-q)} < \infty.$$

$$\lim a_n = 0, \; p-q=0$$

All of the above (and more) may easily be covered within a single assignment by any student of calculus.

Direct derivations

We give 2 examples of how order of convergence may be derived directly from an error recursion of the form (1.8). It is of great note that elementary numerical analysis texts either completely ignore this notion (e.g., [1], [2]) or simply state without further hints how they come to be.

Example 3.1

For the secant method,

$$e_{n+1} = g_n e_n e_{n-1} \tag{3.1}$$

or

$$e_{n+1} \sim e_n e_{n-1}. \tag{3.2}$$

The order of convergence is the largest p for which

$$e_{n+1} \sim e_n^p \tag{3.3}$$

For such p we have from the previous section that

$$e_n \sim e_{n-1}^p$$

(3.4)
$$e_n^{1/p} \sim e_{n-1}$$

Then (3.2) and (3.4) yield

(3.5)
$$\begin{aligned} e_{n+1} &\sim e_n e_n^{1/p} \\ &\sim e_n^{1+1/p} \end{aligned}$$

From (3.3) and (3.5),

$$e_n^p \sim e_n^{\frac{p+1}{p}}$$

or

$$e_n \sim e_n^{\frac{p+1}{p^2}}.$$

By (2.6)

$$\frac{p+1}{p^2} = 1$$

or

$$p^2 - p - 1 = 0.$$

The largest root is $\frac{1+\sqrt{5}}{2}$, which is, indeed, the order of convergence of the second method.

Example 3.2

For Mullers method.

$$e_{n+1} \sim e_n e_{n-1} e_{n-2}$$

and p is largest such that

$$e_{n+1} \sim e_n^p.$$

As before,

$$e_{n-1} \sim e_{n-2}^p$$

and

$$e_n \sim e_{n-1}^p$$

So

$$e_n^{1/p} \sim e_{n-1}$$

and

$$e_n^{1/p^2} \sim e_{n-2}.$$

Thus

$$e_{n+1} \sim e_n e_n^{1/p} e_n^{1/p^2}$$
$$\sim e_n^{\frac{p^2+p+1}{p^2}},$$

and we must have

$$\frac{p^2+p+1}{p^2} = p$$

or

$$p^3 - p^2 - p - 1 = 0$$

The largest root is 1.83$\underline{9}$, which is the order of convergence of this method.

The general approach is clear; if e_{n+1} is given by

(3.1) $\quad e_{n+1} = G(e_n, \ldots, e_{n-k})$ with G continuous at 0, use the definition of order of convergence to get

(3.2) $\quad e_{n-j} \sim e_n^{1/p^j} \quad (j=k,\ldots,0).$

and

(3.3) $$e_n^p \sim G(e_n, e_n^{1/p}, \ldots, e_n^{1/p^k}).$$

Algebraic exercising together with invocations of (2.6) will lead to the result. In fact, the resulting equation for p is generally the indicial equation for the governing difference equation for e_n! (Refer to Examples 3.1 and 3.2).

This simple approach resolves even a problem thought to require very sophisticated understanding. For example, a general multistep method leads to the error recursion (Traub [5], p 60-62)

(3.4) $$e_{n+1} = g_n \prod_{j=0}^{k} e_{n-j}^{r_j}.$$

Immediately we get from (3.3) that

$$e_n^p \sim \prod_{j=0}^{k} e_n^{r_j/p^j} \sim e_n^{\sum_{j=0}^{k} r_j/p^j}$$

from which we deduce that

$$p = \sum_{j=0}^{k} r_j/p^j$$

or

$$p^{k+1} = \sum_{j=0}^{k} r_j p^{k-j}$$

(This example appears in Ralston [4], pp. 336-339). and is taken from the work by Traub [5, pp 62-67]. The analysis is certainly beyond the scope of undergraduates an introductory course in numerical analysis).

SUMMARY

The technique displayed here has been "tested" on students in the introductory numerical methods course (CS 3410) and has been grasped by same with no difficulty whatsoever. An earlier attempt to derive orders of convergence in the "classical" way met with classical disaster -- and should have! The author is convinced that this naive but rigorous approach places the notion and derivation of order of convergence within the domain of introductory numerical analysis courses.

REFERENCES

1. Conte, S. D., Elementary Numerical Analysis, McGraw Hill, New York (1965).

2. ________ and C. deBoor, Elementary Numerical Analysis, 2nd Ed, McGraw-Hill, New York (1972).

3. Greenberg, H. J., Interpreting Rate Analysis, SIGMAP Newsletter.

4. Ralston, A., A First Course in Numerical Analysis, McGraw-Hill, New York (1965).

5. Traub, J. F., Iterative Methods for the Solution of Equations, Prentice-Hall, Englewood Cliffs, N.J. (1964).