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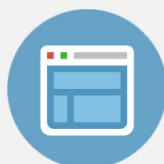
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## Stability of relative equilibria of three vortices

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Three point vortices on the unbounded plane have relative equilibria wherein the vortices either form an equilateral triangle or are collinear. While the stability analysis of the equilateral triangle configurations is straightforward, that of the collinear relative equilibria is considerably more involved. The only comprehensive analysis available in the literature, by Tavantzis and Ting [Phys. Fluids **31**, 1392 (1988)], is not easy to follow nor is it very physically intuitive. The symmetry between the three vortices is lost in this analysis. A different analysis is given based on explicit formulas for the three eigenvalues determining the stability, including a new formula for the angular velocity of rotation of a collinear relative equilibrium. A graphical representation of the space of vortex circulations is introduced, and the resultants between various polynomials that enter the problem are used. This approach adds considerable transparency to the solution of the stability problem and provides more physical understanding. The main results are summarized in a diagram that gives both the stability and instability of the various collinear relative equilibria and their sense of rotation. © 2009 American Institute of Physics. [doi:10.1063/1.3216063]

### I. INTRODUCTION

A relative equilibrium of point vortices is a configuration that moves without change in shape or size, as if the vortices formed a rigid body. All configurations of two point vortices on the unbounded plane are relative equilibria. Three point vortices have relative equilibria of two kinds. First, three vortices of any circulations placed at the vertices of an equilateral triangle are a relative equilibrium. Second, for almost all vortex triples it is possible to find one or more collinear configurations that are relative equilibria. In this paper we determine all relative equilibria of three vortices as a function of the vortex circulations and analyze their linear stability.

As would be expected for this classical problem, numerous partial results are scattered about in the literature going back, at least, to Thomson.<sup>1</sup> For a given triple of vortex circulations whether a relative equilibrium is stable or unstable is usually easy to discover from the geometrical analyses of three-vortex motion pioneered in 1877 by Gröbli<sup>2</sup> and refined over the course of a century by Synge<sup>3</sup> and, independently, by Aref.<sup>4</sup> See also Ref. 5 for a recent account of Synge's approach. However, the only systematic analysis of the linear stability problem for the collinear relative equilibria, of which the author is aware, is the 1988 paper by Tavantzis and Ting.<sup>6</sup> The present study arose from an attempt to make this analysis more transparent in terms of physics. In particular, the author felt that the symmetry between the three vortices inherent in the problem, which is so useful in much of the theory of three-vortex motion, was lost in the analysis in Ref. 6.

So far as the author can tell, all the results obtained here are consistent with those of Ref. 6. Additional details and insights have emerged in the present analysis, in particular as regards the sense of rotation of the collinear equilibrium configurations. The analysis given here also produces explicit

formulas for the angular frequency of rotation and for the oscillation frequency or growth rate of infinitesimal perturbations in terms of the physical variables characterizing the relative equilibrium. This information should also be obtainable from the analysis in Ref. 6, but it seems quite laborious to do so in general. The "phase diagram" in vortex circulation space, which we introduce in Fig. 2, is very helpful in getting one's bearings in this problem. It was, to the best of the author's knowledge, first used in a 1978 *Saclay* report by Conte and de Seze.<sup>7</sup> These authors also used the notion of resultants between two polynomials to track how various relationships depend on the circulations that enter the coefficients of those polynomials. They had access to and used a symbolic mathematics package called AMP,<sup>8</sup> something that is routine today but was quite novel 30 years ago. The stability analysis in the report<sup>7</sup> is incomplete (it takes up only about a page). Unfortunately, a full account was never published in the archival journal literature. According to Conte<sup>9</sup> (private communication, 2004) a version of the report was included in his *Thèse d'État* defended on June 14, 1979. The ideas introduced in this work prove very useful, as we shall see. The stability problem for the equilateral triangle configurations is straightforward by comparison with the case of collinear relative equilibria and is included mostly for completeness.

### II. GENERAL ANALYSIS OF RELATIVE EQUILIBRIA

We start with some general considerations regarding relative equilibria of an arbitrary number of point vortices  $N$ . This section initially follows the analysis in the review paper by Aref *et al.*<sup>10</sup> and establishes our notation. The point vortex equations on the unbounded plane, taken as the complex plane, without background flow are

$$\overline{\frac{dz_\alpha}{dt}} = \frac{1}{2\pi i} \sum_{\beta=1}^N \prime \frac{\Gamma_\beta}{z_\alpha - z_\beta}, \quad (1)$$

where the line over a symbol means complex conjugation, and the prime on the summation means  $\beta \neq \alpha$ . The time-dependent vortex positions are denoted as  $z_\alpha(t)$ ,  $\alpha=1, \dots, N$ . The constant circulations of the vortices are  $\Gamma_\alpha$ . These equations may be found in many places in the literature, e.g., Ref. 11 or Ref. 12.

We now assume that the configuration of vortices is instantaneously moving as a rigid body, i.e., that the velocity of every vortex is made up of a translation and a rotation,

$$\frac{dz_\alpha}{dt} = V + i\Omega z_\alpha, \quad (2)$$

where  $V$  is complex and  $\Omega$  is real. Both are the same for all vortices. Substituting the *Ansatz* (2) into Eq. (1), we obtain in place of the ordinary differential equations (ODEs) a set of algebraic equations

$$\bar{V} - i\Omega \bar{z}_\alpha = \frac{1}{2\pi i} \sum_{\beta=1}^N \prime \frac{\Gamma_\beta}{z_\alpha - z_\beta}. \quad (3)$$

To establish necessary conditions for a solution to exist, we multiply Eq. (3) by  $\Gamma_\alpha$  and  $\Gamma_\alpha z_\alpha$ , in turn, and sum over  $\alpha$ . Thus, multiplying by  $\Gamma_\alpha$  we obtain

$$\gamma_1 V + i\Omega(X + iY) = 0, \quad (4)$$

where  $\gamma_1$  is the first symmetric function of the vortex circulations,

$$\gamma_1 = \sum_{\alpha=1}^N \Gamma_\alpha, \quad (5)$$

and the real quantities  $X$  and  $Y$  are the components of linear impulse,

$$X + iY = \sum_{\alpha=1}^N \Gamma_\alpha z_\alpha. \quad (6)$$

Next, multiplying Eq. (3) by  $\Gamma_\alpha z_\alpha$ , and summing yields, after a brief calculation,

$$V(X - iY) + i\Omega I = -\frac{\gamma_2}{2\pi i}, \quad (7)$$

where  $\gamma_2$  is the second symmetric function of the vortex circulations,

$$\gamma_2 = \sum_{1 \leq \alpha < \beta \leq N} \Gamma_\alpha \Gamma_\beta, \quad (8)$$

and  $I$  is the angular impulse,

$$I = \sum_{\alpha=1}^N \Gamma_\alpha |z_\alpha|^2. \quad (9)$$

We note for future reference that

$$\gamma_1^2 = \sum_{\alpha=1}^N \Gamma_\alpha^2 + 2\gamma_2. \quad (10)$$

Equations (4) and (7) allow us to classify relative equilibria of point vortices. The form of these equations is simple: two linear equations in two unknowns,  $V$  and  $\Omega$ . The condition for a unique solution is that the determinant of the coefficient matrix on the left hand side be nonzero, i.e., that

$$L = \gamma_1 I - (X^2 + Y^2) \neq 0. \quad (11)$$

For  $L \neq 0$ , then, we find unique solutions for  $V$  and  $\Omega$ ,

$$V = \frac{1}{iL} \begin{vmatrix} 0 & i(X + iY) \\ -\frac{\gamma_2}{2\pi i} & iI \end{vmatrix} = \frac{\gamma_2}{2\pi i} \frac{X + iY}{L}, \quad (12a)$$

$$\Omega = \frac{1}{iL} \begin{vmatrix} \gamma_1 & 0 \\ X - iY & -\frac{\gamma_2}{2\pi i} \end{vmatrix} = \frac{\gamma_1 \gamma_2}{2\pi L}. \quad (12b)$$

The right hand sides of these equations depend only on combinations of the first and second symmetric functions of the vortex circulations,  $\gamma_1$  and  $\gamma_2$ , and on the integrals of motion  $X$ ,  $Y$ , and  $I$  (or  $L$ ). Thus, if the *Ansatz* (2) is valid at some instant, it will be valid for all time.

We may now reason as follows (maintaining the assumption  $L \neq 0$ ). For  $\gamma_1 \neq 0$ , we may assume  $X=Y=0$ , since an inconsequential shift in the origin of coordinates will otherwise assure this. This amounts to choosing the *center of vorticity*

$$z_{cv} = \frac{\sum_{\alpha=1}^N \Gamma_\alpha z_\alpha}{\sum_{\alpha=1}^N \Gamma_\alpha} = \frac{X + iY}{\gamma_1} \quad (13)$$

as the origin of coordinates. Then, using the new coordinates,  $V=0$  from Eq. (4), and  $L = \gamma_1 I_c$ , where the subscript reminds us that  $I$  is being calculated about the center of vorticity. The vortices rotate as a rigid body about the center of vorticity with an angular velocity given by Eq. (7) or

$$I_c \Omega = \frac{\gamma_2}{2\pi}. \quad (14)$$

Equation (14) includes the possibility for  $\gamma_2=0$  that the vortex configuration is stationary.

For  $\gamma_1=0$  and  $X^2 + Y^2 = -L \neq 0$ , we have  $\Omega=0$  by Eq. (12b) and the motion consists of pure translation with a velocity given by Eq. (12a),

$$V = \frac{\sum_{\alpha=1}^N \Gamma_\alpha^2}{4\pi i} \frac{X + iY}{X^2 + Y^2}, \quad (15)$$

where we have used Eq. (10) with  $\gamma_1=0$ .

The case  $L=0$  remains to be considered. Equations (4) and (7) are no longer independent. For  $\gamma_1 \neq 0$  we may again arrange  $X=Y=0$  by a shift in the origin of coordinates. When the origin is so chosen,  $L=0$  implies  $I_c=0$ . It also follows from Eq. (4) that  $V=0$  and from Eq. (7) that we must have  $\gamma_2=0$ . The configuration can either rotate or be stationary. The angular velocity must be found by returning to the basic equations in their algebraic form, Eq. (3). It is unknown at

present whether stationary configurations with  $L=0$  exist. (We shall, however, see momentarily that a stationary configuration with  $L=0$  requires at least four vortices.)

For  $\gamma_1=0$ ,  $L=0$  implies  $X=Y=0$ . Equation (4) is then satisfied identically and Eq. (7) becomes

$$I\Omega = \frac{\gamma_2}{2\pi}.$$

Since  $\gamma_2 \neq 0$  for  $\gamma_1=0$ , we must have  $I \neq 0$ , and the configuration rotates with an angular frequency given by this equation. However, the center of rotation needs to be determined as part of the analysis (since the center of vorticity is “at infinity”). For this determination we must return to the point vortex equations in their algebraic form, Eq. (3).

We now specialize the analysis to three-vortex motion. Among three circulations two must be of the same sign, and we can assume these two circulations to be positive without loss of generality. The case of two negative vortices then follows by reflection in the plane of flow or by time reversal. Since the numbering of the vortices is also at our disposal, we may adopt the convention that the three circulations satisfy  $\Gamma_3 \leq \Gamma_2 \leq \Gamma_1$  in every case.

For three vortices every relative equilibrium is known to be either an equilateral triangle or a collinear configuration.<sup>2-4</sup> The equilateral triangle configuration is a relative equilibrium for every choice of vortex circulations. Given a triple of vortex circulations  $(\Gamma_1, \Gamma_2, \Gamma_3)$ , there are, in principle, two equilateral triangle relative equilibria, one for clockwise orientation of vortices 1, 2, 3, and one for counterclockwise orientation. The motion and stability of these two configurations parallel one another modulo the obvious adjustments, and so we shall treat only one orientation. Since the geometry of these relative equilibria is independent of the vortex circulations, the stability analysis for the equilateral triangle case is rather straightforward. On the other hand, for the collinear relative equilibria, of which there can be three, two, one or even none depending on the circulations, the analysis is considerably more complicated. For these the configuration geometry—albeit just three points along a line—depends on the vortex circulations, and this coupling between configuration and vortex circulations is, as we shall see, the main source of complexity both in determining the configurations and in the stability calculations.

For an equilateral triangle of side  $s$  we have  $L = \gamma_2 s^2$ . Hence,  $L \neq 0$  is equivalent to  $\gamma_2 \neq 0$ . For  $\gamma_1 \neq 0$  our present analysis tells us that  $L = \gamma_1 I_c = \gamma_2 s^2$  and the vortices must rotate about the center of vorticity with an angular frequency given by Eq. (14) or  $\Omega = \gamma_1 / 2\pi s^2$ . As we shall see from the explicit solution for the equilateral triangle relative equilibria, this formula also gives the angular velocity of rotation in the case  $\gamma_2=0$  when Eq. (14) becomes indeterminate. For  $\gamma_1=0$  the triangle translates with the velocity of translation found previously. See also the considerations leading to Eq. (25) below.

The special case of the three-vortex problem with  $\gamma_1=0$  can be solved in full detail as shown by Rott<sup>13</sup> and Aref,<sup>14</sup> including for the case  $L=0$ . One finds that the vortices form

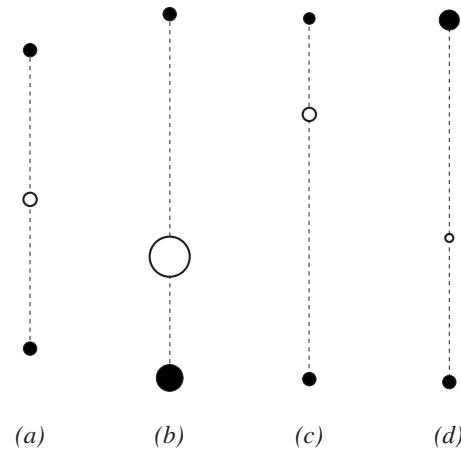


FIG. 1. Four collinear relative equilibria singled out by the analysis. (a)  $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 1, -1)$ , unstable; (b)  $(2, 1, -3)$ , sum of circulations is zero, stable; (c)  $(1, \sqrt{3}/2, -1)$ , marginally stable; (d)  $(\frac{3}{2}, 1, -\frac{3}{2})$ , unstable. Configurations (a)–(c) rotate clockwise; (d) is stationary. Linear scale differs from configuration to configuration. Solid dots correspond to positive, open dots to negative circulation. The radii of the dots scale with the magnitude of the circulation.

a collinear relative equilibrium in this case. The positions are given by

$$\begin{aligned} z_1(t) &= \left( \frac{\Gamma_3}{\Gamma_2} - \frac{\Gamma_2}{\Gamma_3} \right) \xi e^{i\Omega t}, \\ z_2(t) &= \left( \frac{\Gamma_1}{\Gamma_3} - \frac{\Gamma_3}{\Gamma_1} \right) \xi e^{i\Omega t}, \\ z_3(t) &= \left( \frac{\Gamma_2}{\Gamma_1} - \frac{\Gamma_1}{\Gamma_2} \right) \xi e^{i\Omega t}, \end{aligned} \tag{16a}$$

where  $\xi$  is a scale factor, and

$$\Omega = - \frac{\gamma_3}{2\pi\gamma_2\xi^2}, \tag{16b}$$

with

$$\gamma_3 = \Gamma_1\Gamma_2\Gamma_3, \tag{17}$$

the third symmetric function of the three vortex circulations. We note that  $\Omega < 0$  (since both  $\gamma_2$  and  $\gamma_3$  are negative when  $\gamma_1=0$ ), i.e., the configuration rotates clockwise. An example of such a configuration is shown in Fig. 1. The relative equilibrium Eq. (16a) appears already in Gröbli’s thesis,<sup>2</sup> although the general analysis of the three-vortex problem with total circulation zero was not given in detail in that work.

Another important special case arises by considering a configuration of three vortices that is stationary:  $V=0$ ,  $\Omega=0$ . Then Eq. (4) is identically satisfied and, according to Eq. (7),  $\gamma_2=0$ . Since  $\gamma_2=0$  implies  $\gamma_1 \neq 0$ , we may assume (or arrange by a shift in coordinates) that

$$\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3 = 0. \tag{18a}$$

For a stationary configuration Eqs. (3), for  $N=3$ , take the form

$$\frac{\Gamma_2}{z_1 - z_2} + \frac{\Gamma_3}{z_1 - z_3} = 0, \quad \frac{\Gamma_1}{z_2 - z_1} + \frac{\Gamma_3}{z_2 - z_3} = 0,$$

$$\frac{\Gamma_1}{z_3 - z_1} + \frac{\Gamma_2}{z_3 - z_2} = 0.$$

These are clearly equivalent to

$$\frac{z_1 - z_2}{\Gamma_2} = \frac{z_3 - z_1}{\Gamma_3}, \quad \frac{z_2 - z_1}{\Gamma_1} = \frac{z_3 - z_2}{\Gamma_3}, \quad \frac{z_3 - z_1}{\Gamma_1} = \frac{z_2 - z_3}{\Gamma_2},$$

and then, recalling  $\gamma_2=0$ , to

$$\frac{z_1}{\Gamma_1} + \frac{z_2}{\Gamma_2} + \frac{z_3}{\Gamma_3} = 0. \quad (18b)$$

Equations (18a) and (18b) are two linear relations between the coordinates  $z_1$ ,  $z_2$ , and  $z_3$  and thus show that the vortices must be collinear. We take this common line to be the  $x$ -axis. The solution has the form

$$x_1 = \frac{\Gamma_2 - \Gamma_3}{\Gamma_1} \xi, \quad x_2 = \frac{\Gamma_3 - \Gamma_1}{\Gamma_2} \xi, \quad x_3 = \frac{\Gamma_1 - \Gamma_2}{\Gamma_3} \xi, \quad (19)$$

where  $\xi$  is a real scale factor. An example of such a configuration is shown in Fig. 1. This solution also appears in Ref. 2. In fact, the solutions (16a), (16b), and (19) are closely related, as we shall see.

Recalling that  $\gamma_2=0$ , one finds easily for this solution that  $I_c = 3\gamma_1\xi^2$ . For three vortices, then, all stationary configurations have  $L = \gamma_1 I_c = 3(\gamma_1\xi)^2 > 0$ . Hence, stationary configurations with  $L=0$  (or, equivalently,  $I_c=0$ ) must involve at least four vortices. In fact, Hampton *et al.*<sup>15</sup> showed that all stationary four-vortex configurations also have  $L \neq 0$ . If stationary  $N$ -vortex configurations with  $L=0$  exist, they must involve at least five vortices.

### III. EQUILATERAL TRIANGLE RELATIVE EQUILIBRIA

Let us now give the complete solution for the equilateral triangle relative equilibria. If vortices 1, 2, and 3 appear counterclockwise, we have

$$z_3 - z_2 = \epsilon(z_2 - z_1), \quad (20)$$

$$z_1 - z_3 = \epsilon(z_3 - z_2) = \epsilon^2(z_2 - z_1),$$

where  $\epsilon = e^{i2\pi/3}$ . From the basic equation (1) we then have

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{i}{2\pi s^2} [\Gamma_2(z_1 - z_2) + \Gamma_3(z_1 - z_3)] \\ &= \frac{i}{2\pi s^2} [\gamma_1 z_1 - (X + iY)], \\ \frac{dz_2}{dt} &= \frac{i}{2\pi s^2} [\Gamma_1(z_2 - z_1) + \Gamma_3(z_2 - z_3)] \\ &= \frac{i}{2\pi s^2} [\gamma_1 z_2 - (X + iY)], \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{dz_3}{dt} &= \frac{i}{2\pi s^2} [\Gamma_1(z_3 - z_1) + \Gamma_2(z_3 - z_2)] \\ &= \frac{i}{2\pi s^2} [\gamma_1 z_3 - (X + iY)], \end{aligned}$$

where  $s$  is the common length of the three sides of the triangle.

For  $\gamma_1 \neq 0$  we introduce the center of vorticity  $z_{cv}$  from Eq. (13),

$$z_{cv} = \frac{X + iY}{\gamma_1} = \frac{\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3}{\Gamma_1 + \Gamma_2 + \Gamma_3}. \quad (22)$$

Equation (21) may then be written as

$$\frac{dz_\alpha}{dt} = i\Omega(z_\alpha - z_{cv}), \quad \alpha = 1, 2, 3, \quad (23a)$$

where

$$\Omega = \frac{\gamma_1}{2\pi s^2}. \quad (23b)$$

Since  $z_{cv}$  is a fixed point in the plane, these equations show that the vortices rotate about the center of vorticity with angular frequency  $\Omega$ . This derivation includes the case  $\gamma_2=0$ .

Solving Eq. (20) along with Eq. (22) we obtain the solution

$$\begin{aligned} z_1(t) &= z_{cv} + \frac{\epsilon^2 \Gamma_3 - \Gamma_2}{\gamma_1} s e^{i\Omega t}, \\ z_2(t) &= z_{cv} + \frac{\Gamma_1 - \epsilon \Gamma_3}{\gamma_1} s e^{i\Omega t}, \\ z_3(t) &= z_{cv} + \frac{\epsilon \Gamma_2 - \epsilon^2 \Gamma_1}{\gamma_1} s e^{i\Omega t}, \end{aligned} \quad (24)$$

with  $\Omega$  given by Eq. (23b). The solution in Eq. (24) is the counterpart for three vortices of the well-known solution to the two-vortex problem. There is a free scale parameter,  $s$ , the side length of the equilateral triangle. The form of Eq. (24) makes the side 12 parallel to the  $x$ -axis of coordinates at  $t=0$ :  $z_2(0) - z_1(0) = s$ . It is not difficult to check that the solution (24) satisfies Eq. (1) with  $N=3$ . The reader may find it instructive to write the formulas corresponding to Eq. (24) when vortices 1, 2, and 3 appear clockwise.

For  $\gamma_1=0$  the right hand sides of Eq. (21) may all be written as the same velocity

$$V = \frac{X + iY}{2\pi i s^2},$$

which is constant throughout the motion. Hence, the vortex triangle translates like a rigid body. Now, when  $\gamma_1=0$ ,  $L = -(X^2 + Y^2) = \gamma_2 s^2$  for an equilateral triangle. Using Eq. (10) we get

$$X^2 + Y^2 = \frac{1}{2}(\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2) s^2.$$

The speed of translation  $|V|$  is then

$$|V| = \frac{\sqrt{\frac{1}{2}(\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2)}}{2\pi s}. \quad (25)$$

The formula for the speed of translation of a vortex pair with constituent vortices of circulation  $\Gamma_1 = -\Gamma_2$  can be written in the somewhat contrived form

$$|V| = \frac{\sqrt{\frac{1}{2}(\Gamma_1^2 + \Gamma_2^2)}}{2\pi s}$$

to bring out the analogy with Eq. (25). This is also the limiting form of Eq. (25) when one of the three circulations tends to 0.

#### IV. COLLINEAR EQUILIBRIA OF THREE VORTICES

For a collinear relative equilibrium we take the instantaneous line through the vortices to be the  $x$ -axis, and we then have the conditions for a relative equilibrium in the form

$$\begin{aligned} a + bx_1 &= \frac{\Gamma_2}{x_1 - x_2} + \frac{\Gamma_3}{x_1 - x_3}, \\ a + bx_2 &= \frac{\Gamma_1}{x_2 - x_1} + \frac{\Gamma_3}{x_2 - x_3}, \\ a + bx_3 &= \frac{\Gamma_1}{x_3 - x_1} + \frac{\Gamma_2}{x_3 - x_2}, \end{aligned} \quad (26)$$

where  $a = 2\pi i \bar{V}$  and  $b = 2\pi \Omega$  are related to the velocity of translation and the angular frequency of rotation, respectively. Since  $b$  is real, and the right hand sides in Eq. (26) are real, it follows that  $a$  must also be real, i.e.,  $V$  is pure imaginary. In other words, if the line of vortices is to translate, it must do so in a direction perpendicular to the line.

We have already dealt with the special case  $a = b = 0$ . Solutions for this case exist only if  $\gamma_2 = 0$  and are given by Eq. (19). The reader may wish to verify that, indeed, Eq. (19) satisfies Eq. (26) with 0 on the left hand side.

If  $b = 0$  but  $a \neq 0$ , solutions will only exist for  $\gamma_1 = 0$ . This follows from the general theory, but is not hard to verify independently by multiplying the first of Eq. (26) by  $\Gamma_1$ , the second by  $\Gamma_2$ , the third by  $\Gamma_3$ , and adding. Now, if we subtract the second of Eq. (26) from the first, we find

$$\frac{\Gamma_2}{x_1 - x_2} + \frac{\Gamma_3}{x_1 - x_3} - \frac{\Gamma_1}{x_2 - x_1} - \frac{\Gamma_3}{x_2 - x_3} = 0,$$

or, using  $\gamma_1 = 0$ ,

$$\frac{1}{x_1 - x_2} + \frac{1}{x_2 - x_3} + \frac{1}{x_3 - x_1} = 0. \quad (27)$$

This equation, which does not contain the circulations, has no solutions, however. To see this note that it may be rewritten as

$$\frac{x_1 - x_3}{(x_1 - x_2)(x_2 - x_3)} = \frac{1}{x_1 - x_3}, \quad (x_1 - x_2)(x_2 - x_3) = (x_1 - x_3)^2.$$

This shows that  $0 < (x_1 - x_2)(x_2 - x_3)$ . But

$$(x_1 - x_3)^2 = (x_1 - x_2)^2 + (x_2 - x_3)^2 + 2(x_1 - x_2)(x_2 - x_3),$$

so we also have

$$(x_1 - x_2)^2 + (x_2 - x_3)^2 = -(x_1 - x_2)(x_2 - x_3),$$

which shows that  $(x_1 - x_2)(x_2 - x_3) < 0$ . Hence, there can be no solution of Eq. (26) for  $b = 0$ ,  $a \neq 0$ . For a neutral vortex triple there are no translating collinear relative equilibria. This result also follows, of course, from the full analysis of the case  $\gamma_1 = 0$ .<sup>13,14</sup>

If  $b \neq 0$ , we may assume  $a = 0$ , since we can always shift the origin of coordinates by  $-a/b$  and this will eliminate any nonzero  $a$  from Eq. (26). We now have

$$\Gamma_1 x_1 + \Gamma_2 x_2 + \Gamma_3 x_3 = 0. \quad (28a)$$

Multiplying the first of Eq. (26) by  $(x_2 - x_3)^{-1}$ , the second by  $(x_3 - x_1)^{-1}$ , the third by  $(x_1 - x_2)^{-1}$ , and adding, we obtain an equation that does not contain the circulations,

$$\frac{x_1}{x_2 - x_3} + \frac{x_2}{x_3 - x_1} + \frac{x_3}{x_1 - x_2} = 0. \quad (28b)$$

This elegant result was given by Gröbli.<sup>2</sup>

For vortex triples with  $\Gamma_1 = \Gamma_2 = \Gamma$ , and any value of  $\Gamma_3$ , we have the solution  $x_3 = 0$ ,  $x_1 = -x_2$  of Eqs. (28a) and (28b). These include the stationary configuration for  $\Gamma_3 = -\Gamma/2$ , a special case of Eq. (19). In the restricted space of vortex triples with two identical vortices these configurations have “invariant” geometry as the strength of the third vortex is varied. This suggests that the stability analysis will simplify for such configurations since geometry and circulations are “decoupled.” This family of relative equilibria may not be the most mathematically “exciting” solutions of Eq. (26), but we shall see that they play an important role in the theory. They are also important physically, and they are certainly among the most celebrated relative equilibria of three vortices. Three vortices on a line became a much studied topic in geophysical fluid dynamics with the discovery of the *vortex tripole* by van Heijst and Kloosterziel,<sup>16</sup> Kloosterziel and van Heijst,<sup>17</sup> and van Heijst *et al.*<sup>18</sup> The vortex triple (1,1,-1) with the same geometry is shown in Fig. 1. An asymmetrical rotating configuration with the sum of the circulations equal to 0 is also shown.

For a general discussion of Eqs. (28a) and (28b) consider the variables

$$\xi_1 = x_2 - x_3, \quad \xi_2 = x_3 - x_1, \quad \xi_3 = x_1 - x_2, \quad (29)$$

which are not independent since

$$\xi_1 + \xi_2 + \xi_3 = 0. \quad (30)$$

Equation (28b) may then be written as

$$\frac{x_1}{\xi_1} + \frac{x_2}{\xi_2} + \frac{x_3}{\xi_3} = 0. \quad (31)$$

From Eq. (28a) and any two of Eq. (29) we get the transformation formulas inverse to Eq. (29) when  $\gamma_1 \neq 0$ ,

$$x_1 = \frac{\Gamma_2 \xi_3 - \Gamma_3 \xi_2}{\gamma_1}, \quad x_2 = \frac{\Gamma_3 \xi_1 - \Gamma_1 \xi_3}{\gamma_1}, \quad x_3 = \frac{\Gamma_1 \xi_2 - \Gamma_2 \xi_1}{\gamma_1}. \tag{32}$$

For  $\gamma_1=0$  the condition (28a) tells us that

$$\Gamma_1 \xi_2 = \Gamma_2 \xi_1,$$

which with Eq. (30) gives

$$\frac{\xi_1}{\Gamma_1} = \frac{\xi_2}{\Gamma_2} = \frac{\xi_3}{\Gamma_3}. \tag{33}$$

It follows that  $L$  is proportional to  $\gamma_1 \gamma_3$  and so vanishes. Hence, we return to the solution (16a) and (16b).

We note in passing that the general solution of the system

$$\Gamma_1 x_1 + \Gamma_2 x_2 + \Gamma_3 x_3 = 0, \quad \frac{x_1}{\Gamma_1} + \frac{x_2}{\Gamma_2} + \frac{x_3}{\Gamma_3} = 0, \tag{34a}$$

is

$$x_1 = \left( \frac{\Gamma_3}{\Gamma_2} - \frac{\Gamma_2}{\Gamma_3} \right) \xi, \quad x_2 = \left( \frac{\Gamma_1}{\Gamma_3} - \frac{\Gamma_3}{\Gamma_1} \right) \xi, \quad x_3 = \left( \frac{\Gamma_2}{\Gamma_1} - \frac{\Gamma_1}{\Gamma_2} \right) \xi, \tag{34b}$$

where  $\xi$  is an arbitrary parameter. For  $\gamma_1=0$ , Eq. (33) combined with Eq. (31) yields the second equation (34a). Equation (34b) then gives the solution (16a). For  $\gamma_2=0$ , Eqs. (18a) and (18b) are equivalent to Eq. (34a). We get the solution (19) by rewriting Eq. (34b) using  $\gamma_2=0$ . This common algebraic origin of the two special solutions (16a) and (19) seems to have gone unnoticed.

Continuing with the general case  $\gamma_1 \neq 0$ , Eq. (28a) is now identically satisfied, and ‘‘Gröbli’s relation,’’ Eq. (31), becomes

$$\Gamma_1 \left( \frac{\xi_2}{\xi_3} - \frac{\xi_3}{\xi_2} \right) + \Gamma_2 \left( \frac{\xi_3}{\xi_1} - \frac{\xi_1}{\xi_3} \right) + \Gamma_3 \left( \frac{\xi_1}{\xi_2} - \frac{\xi_2}{\xi_1} \right) = 0. \tag{35}$$

Choose one of the ratios between the  $\xi$ ’s as a new independent variable, e.g.,

$$z = \frac{\xi_1}{\xi_3} = \frac{x_2 - x_3}{x_1 - x_2}, \tag{36a}$$

and note that we then have

$$\frac{\xi_2}{\xi_3} = -\frac{\xi_1 + \xi_3}{\xi_3} = -1 - z; \quad \frac{\xi_1}{\xi_2} = \frac{\xi_1 \xi_3}{\xi_3 \xi_2} = -\frac{z}{1+z}. \tag{36b}$$

Thus Eq. (35) becomes

$$\Gamma_1 \left( -1 - z + \frac{1}{1+z} \right) + \Gamma_2 \left( \frac{1}{z} - z \right) + \Gamma_3 \left( -\frac{z}{1+z} + 1 + \frac{1}{z} \right) = 0,$$

which is a cubic equation for determining  $z$ ,

$$p(z) = (\Gamma_1 + \Gamma_2)z^3 + (2\Gamma_1 + \Gamma_2)z^2 - (\Gamma_2 + 2\Gamma_3)z - (\Gamma_2 + \Gamma_3) = 0. \tag{37}$$

This equation (with a relabeling of vortices 1 and 2) appears as Eq. (51) of Ref. 6. A related equation appears in Ref. 19.

The cubic equation (37) will always have at least one real root. According to Eq. (36a), in order for a root of  $p(z)$  to be physically acceptable, it must be  $\neq 0, -1$ . Since  $p(0) = -(\Gamma_2 + \Gamma_3)$  and  $p(-1) = \Gamma_1 + \Gamma_3$ , the cases  $\Gamma_2 + \Gamma_3 = 0$  and  $\Gamma_1 + \Gamma_3 = 0$  require special consideration. If we imagine vortices 1 and 2 placed on the real axis, with vortex 1 to the left of vortex 2,  $x_1 < x_2$ , we see that vortex 3 to the right of vortex 2,  $x_2 < x_3$ , implies  $0 < z$ . When vortex 3 approaches vortex 2 from the right,  $z$  tends to the disallowed value  $z=0$ . For vortex 3 between vortices 1 and 2,  $x_1 < x_3 < x_2$ , we have  $-1 < z < 0$ . The disallowed value  $z=-1$  corresponds to vortex 3 coinciding with vortex 1. Finally, for vortex 3 to the left of vortex 1,  $x_3 < x_1$ , we have  $z < -1$ . Three special values of  $z$  deserve highlighting. For  $z=1$  vortex 2 is at the midpoint of 13. For  $p(z)$  to have the root  $z=1$  we must have  $\Gamma_1 = \Gamma_3$ , and so the three vortex circulations are identical. For  $z=-\frac{1}{2}$  vortex 3 is at the midpoint of 12;  $p(-\frac{1}{2})=0$  implies  $\Gamma_1 = \Gamma_2$ . For  $z=-2$  vortex 1 is at the midpoint of 32;  $p(-2)=0$  implies  $\Gamma_2 = \Gamma_3$ . If vortex 1 is to the right of vortex 2, we will return to the situation explored above after half a turn of the configuration. Hence, this analysis covers all possible cases.

The real roots of the cubic polynomial  $p(z)$ , Eq. (37), except for  $z=0, -1$ , give the possible collinear relative equilibria. The explicit formulas for the roots of  $p(z)$  as functions of  $\Gamma_1, \Gamma_2$ , and  $\Gamma_3$  are known from what is usually called *Cardano’s formula*; see Ref. 20 for an elegant exposition of the theory of the algebraic solution of the cubic. If we set

$$Z = (\Gamma_1 + \Gamma_2)z + \frac{1}{3}(2\Gamma_1 + \Gamma_2), \quad z = \frac{Z - \frac{1}{3}(2\Gamma_1 + \Gamma_2)}{\Gamma_1 + \Gamma_2}, \tag{38a}$$

Eq. (37) is transformed to

$$Z^3 + 3HZ + G = 0, \tag{38b}$$

where

$$H = -\left[ \frac{2}{3}(\Gamma_1 + \Gamma_2)\Gamma_3 + \frac{4}{9}(\Gamma_1^2 + \Gamma_2^2) + \frac{7}{9}\Gamma_1\Gamma_2 \right], \tag{38c}$$

$$G = (\Gamma_1 - \Gamma_2) \left[ \frac{1}{3}(\Gamma_1 + \Gamma_2)\Gamma_3 + \frac{16}{27}(\Gamma_1^2 + \Gamma_2^2) + \frac{31}{27}\Gamma_1\Gamma_2 \right].$$

Remarkably, one finds that the *discriminant*  $D$  of this cubic is quite simple (this result was found using the symbolic algebra program MATHEMATICA),

$$D = \frac{1}{4}G^2 + H^3 = \frac{1}{108}(\Gamma_1 + \Gamma_2)^2(3\gamma_2^2 + 36\gamma_1\gamma_3 - 32\gamma_1^2\gamma_2), \tag{38d}$$

where the symmetric functions of the vortex circulations,  $\gamma_1, \gamma_2$ , and  $\gamma_3$ , have been introduced previously.

### A. Special cases

When  $\gamma_1=0$ , we see that  $D > 0$ . Hence, there is only one real root of  $p(z)$ . For the solution (16a) we have  $\xi_1 : \xi_2 : \xi_3 = \Gamma_1 : \Gamma_2 : \Gamma_3$ , cf. Eq. (33). Thus, from Eq. (36a), we find the root  $\Gamma_1/\Gamma_3$  and, indeed, direct substitution shows that  $p(\Gamma_1/\Gamma_3)=0$  when  $\gamma_1=0$ .

When  $\gamma_2=0$ , we have  $\gamma_3 < 0$  but  $\gamma_1 > 0$ . Thus  $D < 0$  and there are three real roots of  $p(z)$ . For the stationary solution (19) we have  $\xi_1 : \xi_2 : \xi_3 = \Gamma_1^{-1} : \Gamma_2^{-1} : \Gamma_3^{-1}$ , see the equations lead-

ing up to Eq. (18b) and Eq. (65) later. Thus, from Eq. (36a), we find the root  $\Gamma_3/\Gamma_1$  and, indeed, direct substitution shows that  $p(\Gamma_3/\Gamma_1)=0$  when  $\gamma_2=0$ .

Let us also consider the special cases  $\Gamma_2+\Gamma_3=0$  and  $\Gamma_1+\Gamma_3=0$  where a “disallowed” root of  $p(z)$ , i.e.,  $z=0$  or  $z=-1$ , needs to be factored out since it leads to two of the vortices coinciding, a physically unacceptable situation.

For  $\Gamma_3=-\Gamma_2$  the polynomial  $p(z)$  in Eq. (37) has the factor  $z$ , and

$$p(z) = z[(\Gamma_1 + \Gamma_2)z^2 + (2\Gamma_1 + \Gamma_2)z + \Gamma_2]. \quad (39a)$$

If  $(2\Gamma_1 + \Gamma_2)^2 - 4(\Gamma_1 + \Gamma_2)\Gamma_2 = 4\Gamma_1^2 - 3\Gamma_2^2 > 0$ , i.e., if  $(\sqrt{3}/2)\Gamma_2 < \Gamma_1$ , this equation has three real solutions: the physically unacceptable  $z=0$  and the two roots of the quadratic in square brackets,

$$z_{\pm} = \frac{-(2\Gamma_1 + \Gamma_2) \pm \sqrt{4\Gamma_1^2 - 3\Gamma_2^2}}{2(\Gamma_1 + \Gamma_2)}.$$

Our convention on relative magnitudes is  $\Gamma_2 \leq \Gamma_1$  so the inequality  $(\sqrt{3}/2)\Gamma_2 < \Gamma_1$  is always satisfied. Of these two solutions the other unacceptable value,  $z=-1$ , arises if, and only if,  $\Gamma_2 = \Gamma_1$ . In this case there is only the one collinear relative equilibrium for the vortex circulation triple  $(1, 1, -1)\Gamma$ :  $z = -\frac{1}{2}$  or  $x_3 = \frac{1}{2}(x_1 + x_2) = 0$ . This configuration is shown in Fig. 1.

For  $\Gamma_3 = -\Gamma_1$  the polynomial  $p(z)$  in Eq. (37) has the factor  $z+1$ ,

$$p(z) = (z+1)[(\Gamma_1 + \Gamma_2)z^2 + \Gamma_1 z + \Gamma_1 - \Gamma_2]. \quad (39b)$$

If  $\Gamma_1^2 - 4(\Gamma_1 + \Gamma_2)(\Gamma_1 - \Gamma_2) = 4\Gamma_2^2 - 3\Gamma_1^2 > 0$ , i.e., if  $(\sqrt{3}/2)\Gamma_1 < \Gamma_2$ , this equation has three real solutions: the physically unacceptable  $z=-1$  and the two roots of the quadratic in square brackets,

$$z_{\pm} = \frac{-\Gamma_1 \pm \sqrt{4\Gamma_2^2 - 3\Gamma_1^2}}{2(\Gamma_1 + \Gamma_2)}.$$

Our convention on relative magnitudes restricts us to consider the interval  $(\sqrt{3}/2)\Gamma_1 < \Gamma_2 \leq \Gamma_1$ . Of the two solutions the other unacceptable value  $z=0$  arises if, and only if,  $\Gamma_2 = \Gamma_1$ . For this case there is, again, only the collinear equilibrium  $z = -\frac{1}{2}$  or  $x_3 = \frac{1}{2}(x_1 + x_2) = 0$ . For  $\Gamma_2 = \Gamma_1(\sqrt{3}/2)$ , i.e., for the vortex circulations  $(\Gamma_1, \Gamma_2, \Gamma_3) = (2, \sqrt{3}, -2)\Gamma$ , there is also only one collinear relative equilibrium corresponding to  $z = \sqrt{3} - 2$ . Up to a scale factor, the vortex positions in this configuration may be given as  $x_1 = \sqrt{3} - 2$ ,  $x_2 = 2\sqrt{3} - 2$ , and  $x_3 = 1$ . This configuration is shown in Fig. 1. For  $0 < \Gamma_2 < \Gamma_1(\sqrt{3}/2)$ , i.e., for vortex triples  $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, r, -1)\Gamma$  with  $0 < r < \sqrt{3}/2$  there are no collinear relative equilibria. We now return to the main analysis.

## B. The general case

In the general case, the number of real roots of the cubic equation (37) will depend on the sign of the discriminant [Eq. (38d)]. There will be three real solutions to Eq. (37) when the discriminant, Eq. (38d), is negative, i.e., when

$$3\gamma_2^2 + 36\gamma_1\gamma_3 < 32\gamma_1^2\gamma_2.$$

In this case the roots of Eq. (38b) are given as<sup>20</sup>

$$Z_n = 2\sqrt{-H} \cos \frac{2n\pi + \theta}{3}, \quad n = 0, 1, 2, \quad \cos \theta = \frac{-G}{2\sqrt{-H^3}}, \quad (40)$$

where  $H$  and  $G$  are as in Eq. (38c), and  $H$  is negative. The corresponding values of  $z$ , obtained through Eq. (38a), will be designated  $z_0^{(p)}$ ,  $z_1^{(p)}$ , and  $z_2^{(p)}$ . (The superscripts are to avoid confusion with the vortex positions  $z_1$  and  $z_2$ .) We have  $z_1^{(p)} \leq z_2^{(p)} \leq z_0^{(p)}$  with equality at points or curves in vortex circulation space where two or more of the roots merge.

When the discriminant vanishes, two of the roots  $Z_0$ ,  $Z_1$ , and  $Z_2$  coincide. The cubic has three coincident roots when  $G=0$  and  $H=0$ . This can only occur for the circulation triple  $(1, 1, -\frac{5}{4})\Gamma$ .

When the discriminant is positive, the cubic has just one real root. This root can be written as<sup>20</sup>

$$Z = \left(-\frac{G}{2} + \sqrt{D}\right)^{1/3} - H \left(-\frac{G}{2} + \sqrt{D}\right)^{-1/3}, \quad (41)$$

where the quantities  $H$ ,  $G$ , and  $D$  were given in Eqs. (38c) and (38d). This  $Z$ -value must be converted to a value of  $z$  through Eq. (38a).

The formulas developed here clearly imply a rather complicated dependence of the positions of the vortices in a collinear relative equilibrium on the circulations.

Given a root of the cubic equation (37),  $z$ , and imposing the condition that  $\Gamma_1 x_2 + \Gamma_2 x_2 + \Gamma_3 x_3 = 0$ , we find for  $\gamma_1 \neq 0$  that the positions of the three vortices along the line are, cf. Eq. (32) and Eqs. (36a) and (36b),

$$x_1 = (\Gamma_3 z + \Gamma_2 + \Gamma_3) \frac{\xi}{\gamma_1}, \quad x_2 = (\Gamma_3 z - \Gamma_1) \frac{\xi}{\gamma_1}, \quad (42)$$

$$x_3 = -[(\Gamma_1 + \Gamma_2)z + \Gamma_1] \frac{\xi}{\gamma_1},$$

where  $\xi$  is a scale factor,  $\xi = x_1 - x_2 = \xi_3$ , that may be chosen freely. For  $\gamma_1 = 0$  the positions were given by Eq. (16a).

The angular frequency of rotation  $\Omega$  of a collinear relative equilibrium is given by the general relation (14). However, for collinear relative equilibria we shall find an expression for  $\Omega$  derived directly from the determining equations to be more readily applicable. We have

$$2\pi\Omega x_1 = \frac{\Gamma_2}{\xi_3} - \frac{\Gamma_3}{\xi_2}, \quad 2\pi\Omega x_2 = \frac{\Gamma_3}{\xi_1} - \frac{\Gamma_1}{\xi_3}, \quad (43)$$

$$2\pi\Omega x_3 = \frac{\Gamma_1}{\xi_2} - \frac{\Gamma_2}{\xi_1}.$$

From these by subtraction, and using Eq. (30),

$$2\pi\Omega \xi_3 = \frac{\Gamma_1 + \Gamma_2}{\xi_3} - \Gamma_3 \left[ \frac{1}{\xi_1} + \frac{1}{\xi_2} \right] = \frac{\Gamma_1 + \Gamma_2}{\xi_3} + \Gamma_3 \frac{\xi_3}{\xi_1 \xi_2}.$$

By permutation of indices we obtain the following set of equations:



$$2\pi\Omega = \frac{\Gamma_1 + \Gamma_2}{\xi_3^2} + \frac{\Gamma_3}{\xi_1\xi_2} = \frac{\Gamma_2 + \Gamma_3}{\xi_1^2} + \frac{\Gamma_1}{\xi_2\xi_3}$$

$$= \frac{\Gamma_3 + \Gamma_1}{\xi_2^2} + \frac{\Gamma_2}{\xi_3\xi_1}.$$

From these one obtains, again using Eq. (30) and assuming  $\gamma_2 \neq 0$ ,

$$\Omega = \frac{1}{2\pi\gamma_2} \left[ \frac{\Gamma_1\Gamma_2(\Gamma_1 + \Gamma_2)}{\xi_3^2} + \frac{\Gamma_2\Gamma_3(\Gamma_2 + \Gamma_3)}{\xi_1^2} + \frac{\Gamma_3\Gamma_1(\Gamma_3 + \Gamma_1)}{\xi_2^2} \right]. \tag{44}$$

For the special case  $\gamma_2=0, L=\gamma_1I_c=0$ , the frequency of rotation requires a different manipulation of Eqs. (43), as we shall see in Sec. VII C.

We pause to consider the special case of two identical vortices, which is of interest in itself but also will play an important role in the general development of the theory.

### C. The case of two identical vortices

For  $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 1, r)\Gamma$ , we have  $G=0, H=-\frac{1}{3}\Gamma^2 \times (5+4r)$ . There are three solutions of the cubic when  $-\frac{5}{4} < r$ . For these we have  $\theta = \pi/2$  and

$$Z_0 = \Gamma\sqrt{5+4r}, \quad Z_1 = -\Gamma\sqrt{5+4r}, \quad Z_2 = 0.$$

The corresponding values of  $z$  are, from Eq. (38a),

$$z_0^{(p)} = \frac{1}{2}(-1 + \sqrt{5+4r}), \quad z_1^{(p)} = \frac{1}{2}(-1 - \sqrt{5+4r}), \quad z_2^{(p)} = -\frac{1}{2}.$$

These could have been obtained directly from Eq. (37), which in this case takes the form  $2z^3 + 3z^2 - (1+2r)z - (1+r) = 0$ . The cubic is seen to have the root  $z = -\frac{1}{2}$ . Factoring out a linear term corresponding to this root gives a quadratic equation from which the remaining two solutions are obtained.

Equation (42) with the roots  $z_0^{(p)}, z_1^{(p)}$ , and  $z_2^{(p)}$  leads to the following configurations (up to a scale factor  $\xi_3$ ):

---


$$\begin{aligned} \text{(i):} \quad & x_1 = \frac{1}{2} \left[ 1 + \frac{r\sqrt{5+4r}}{2+r} \right], \quad x_2 = \frac{1}{2} \left[ -1 + \frac{r\sqrt{5+4r}}{2+r} \right], \quad x_3 = -\frac{\sqrt{5+4r}}{2+r}, \\ \text{(ii):} \quad & x_1 = \frac{1}{2} \left[ 1 - \frac{r\sqrt{5+4r}}{2+r} \right], \quad x_2 = \frac{1}{2} \left[ -1 - \frac{r\sqrt{5+4r}}{2+r} \right], \quad x_3 = \frac{\sqrt{5+4r}}{2+r}, \\ \text{(iii):} \quad & x_1 = -x_2 = \frac{1}{2}, \quad x_3 = 0, \end{aligned} \tag{45}$$

respectively. Configuration (i) is identical to configuration (ii) if we rotate it half a turn and relabel the identical vortices 1 and 2. For  $r=1$  we get the three collinear arrangements of three identical vortices, labeled 1, 2, 3, on a line, from left to right: (i) 3-2-1; (ii) 2-1-3; (iii) 2-3-1.

We reiterate that the solution (iii) exists for all  $r$ , including the important case  $r=-2$  when the total circulation is zero. Solutions (i) and (ii) exist only for  $-\frac{5}{4} < r \leq 1$ . At  $r=-1$  the roots  $z_0^{(p)}$  and  $z_1^{(p)}$  attain the unacceptable values 0 and  $-1$ , respectively. In (i) we see that  $x_2$  and  $x_3$  coincide for  $r=-1$ . In (ii)  $x_1$  and  $x_3$  coincide. For this value of  $r$ , then, there is again just the one collinear relative equilibrium (iii). In the ranges  $-\frac{5}{4} < r < -1$  and  $-1 < r \leq 1$  there are three collinear relative equilibria.

Turning to the angular frequency of rotation, we see from Eq. (44) that for solution (iii),

$$\Omega_2 = \frac{\Gamma}{2\pi(1+2r)\xi_3^2} [2 + 8r(1+r)] = 2(1+2r) \frac{\Gamma}{2\pi\xi_3^2}.$$

For  $r=-\frac{1}{2}$  this configuration is stationary. Indeed, except for a scale factor, it is consistent with the general form (19). For  $r=-\frac{5}{4}, Z=0$  is a triple root of the cubic. This configuration rotates clockwise with angular frequency  $\Omega_2 = -3\Gamma/2\pi\xi_3^2$ . For even smaller  $r$  we have only the one real root  $Z=0$ . It is

this solution that persists into the range  $r < -\frac{5}{4}$ . For  $r=-2$  we have  $\gamma_1=0$  and  $\Omega = -3\Gamma/\pi\xi_3^2$ , twice the clockwise angular frequency of the  $r=-\frac{5}{4}$  configuration. This is a special case of the solution given in Eqs. (16a) and (16b).

Since solutions (i) and (ii) for  $r=-\frac{1}{2}$  are not of the form (19), in particular,  $x_3 \neq 0$ , they must correspond to rotating configurations and must, therefore, have  $I_c=0$ . In general, we get the rotation frequency from Eq. (44) with  $z=z_{0,1}^{(p)}$ . Note that

$$(z_{0,1}^{(p)})^2 (z_{0,1}^{(p)} + 1)^2 = (1+r)^2, \quad (z_{0,1}^{(p)})^2 + (z_{0,1}^{(p)} + 1)^2 = 3 + 2r.$$

Thus,

$$\begin{aligned} \Omega_{0,1} &= \frac{\Gamma}{2\pi(1+2r)(1+r)^2\xi_3^2} [2(1+r)^2 + r(1+r)(3+2r)] \\ &= \frac{2+r}{1+r} \frac{\Gamma}{2\pi\xi_3^2}. \end{aligned}$$

This has the finite, positive value  $\Omega = 3\Gamma/2\pi\xi_3^2$  for  $r=-\frac{1}{2}$ . For  $r=-1$  these solutions disappear. They do so by vortex 3 coinciding either with vortex 2 [solution (i)] or with vortex 1 [solution (ii)]. As this occurs parametrically, through a sequence of collinear relative equilibria, the expression for the

TABLE I. Different regimes of three-vortex motion in terms of  $\gamma_1$  and  $\gamma_2$  (from Ref. 4).

Case	$\Gamma_3$	$\gamma_1$	$\gamma_2$	Example
1	$>0$	$>0$	$>0$	(1,1,1)
2	$<0$	$>0$	$>0$	(3,2,-1)
3	$<0$	$>0$	$=0$	$(\frac{3}{2}, 1, -\frac{3}{2})$
4	$<0$	$>0$	$<0$	(1,1,-1)
5	$<0$	$=0$	$<0$	(2,1,-3)
6	$<0$	$<0$	$<0$	(2,1,-4)

angular frequency of rotation of these solutions must diverge.

#### D. Main analysis continued

With these results in hand we return, once again, to the main analysis. In the general case—and we exclude the “singular cases”  $\Gamma_1 = -\Gamma_3$  and  $\Gamma_2 = -\Gamma_3$ , which we dealt with separately—the number of real solutions (one or three) is determined by the sign of the discriminant [Eq. (38d)]. There will be three real solutions to Eq. (37) when

$$3\gamma_2^2 + 36\gamma_1\gamma_3 < 32\gamma_1^2\gamma_2. \quad (46)$$

Let us consider the various cases in Table I, which is the basis for the geometrical classification of three-vortex motion in Refs. 3 and 4, in light of this condition. (The examples are merely intended to provide a concrete set of circulations that satisfy the conditions stated for each particular regime. They do not necessarily capture all possibilities in that regime.) We shall need some well known estimates. First, by *Cauchy's inequality*

$$\gamma_2 = \Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 \leq \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = \gamma_1^2 - 2\gamma_2,$$

i.e.,  $3\gamma_2 \leq \gamma_1^2$ . This inequality is valid regardless of the signs of the vortex circulations. Next, the inequality between the arithmetic mean  $\gamma_1/3$  and the harmonic mean  $3\gamma_3/\gamma_2$  implies  $9\gamma_3 \leq \gamma_1\gamma_2$ . This inequality is valid when all vortex circulations are positive.

Thus, in “case 1” in Table I we have

$$\begin{aligned} 3\gamma_2^2 + 36\gamma_1\gamma_3 &< 3\gamma_2^2 + 31 \times 9\gamma_1\gamma_3 \leq \gamma_1^2\gamma_2 + 31\gamma_1^2\gamma_2 \\ &= 32\gamma_1^2\gamma_2, \end{aligned}$$

so Eq. (46) is satisfied, and there must always be three collinear equilibria.

In “case 2” the negative circulation  $\Gamma_3$  is larger than  $-\Gamma_1\Gamma_2/(\Gamma_1+\Gamma_2)$  and thus, *a fortiori*, larger than  $-\Gamma_2$ . We have  $\gamma_3 < 0$  but  $\gamma_1$  and  $\gamma_2$  are both  $>0$ , so

$$3\gamma_2^2 + 36\gamma_1\gamma_3 < 3\gamma_2^2 \leq \gamma_1^2\gamma_2 < 32\gamma_1^2\gamma_2,$$

and there must again always be three collinear equilibria.

In “case 3” the inequality (46) degenerates to  $\gamma_3 < 0$ , which is, of course, correct for that case. There are, thus, three collinear equilibria in “case 3” as well. One of these is stationary and we found it explicitly in Eq. (19).

“Case 4” is the most complicated: we have the possibility that Eq. (37) reduces to a quadratic polynomial, and there

may then be two collinear equilibria, or one, or none. In general, for given  $\Gamma_1$  and  $\Gamma_2$ , we can consider the two sides of the inequality (46) as functions of  $\Gamma_3$  in the range  $-(\Gamma_1 + \Gamma_2) \leq \Gamma_3 \leq -\Gamma_1\Gamma_2/(\Gamma_1 + \Gamma_2)$ . The right hand side of (46) is a cubic polynomial in  $\Gamma_3$  which vanishes at the two ends of this interval and is negative everywhere in between. At the lower limit of the interval it has a double root. The left hand side is a quadratic polynomial, which is positive (equal to  $3\gamma_2^2$ ) at the lower limit of the  $\Gamma_3$  interval, but negative (equal to  $36\gamma_1\gamma_3$ ) at the upper limit. Closer examination [e.g., calculating the discriminant for the cubic in  $\Gamma_3$  given by Eq. (46) with an equal sign, again most simply done by using a symbolic algebra program] shows that there is a unique value of  $\Gamma_3$  within the interval in question, let us call it  $\Gamma_\times$ , at which the two sides of Eq. (46) are equal. Thus, for  $\Gamma_3$  between the lower limit  $-(\Gamma_1 + \Gamma_2)$  and  $\Gamma_\times$ , the inequality (46) will be violated, and there is just one collinear equilibrium (except for the special case  $\Gamma_1 + \Gamma_3 = 0$  discussed previously). For  $\Gamma_3$  between  $\Gamma_\times$  and  $-\Gamma_1\Gamma_2/(\Gamma_1 + \Gamma_2)$  the inequality (46) is satisfied and there are three collinear equilibria. Unfortunately, the general expression for  $\Gamma_\times$  in terms of  $\Gamma_1$  and  $\Gamma_2$  is rather complicated and does not seem to offer further insight. For  $\Gamma_1 = \Gamma_2 = \Gamma$ , however, we have the simple result  $\Gamma_\times = -5\Gamma/4$ . This algebraic discussion is clarified in Sec. IV E when we visualize the “space” of circulation triples.

In “case 5” we have, trivially, that Eq. (46) is never satisfied and there is just one collinear equilibrium, which we have already determined, Eqs. (16a) and (16b). Similarly, in “case 6,” the right hand side of Eq. (46) is negative, whereas the left hand side is positive. There is again just one collinear equilibrium.

In summary, as we progress through the cases listed in Table I, we go, roughly speaking, from having three collinear equilibria to having just one. The crossover occurs within “case 4,” which precipitates a more detailed analysis. These results are in accord with the geometrically based conclusions from the phase diagrams of Synge<sup>3</sup> and Aref.<sup>4</sup> The more precise quantitative criteria given here and the expression for the discriminant of the cubic in terms of symmetric functions appear to be new.

#### E. Graphical representation

To obtain an overview of solutions it helps to represent things graphically. In the main case where the sum of the three vortex circulations  $\gamma_1$  is nonzero, let  $\kappa_\alpha = \Gamma_\alpha/\gamma_1$ ,  $\alpha = 1, 2, 3$ . Consider a trilinear plot with coordinates  $\kappa_1$ ,  $\kappa_2$ , and  $\kappa_3$ , as shown in Fig. 2. The equilateral triangle has height 1. The central triangle corresponds to  $\kappa_1$ ,  $\kappa_2$ , and  $\kappa_3$  being all positive. This is the region that we called “case 1” in Table I. If we allow  $\Gamma_3$  to become negative, still keeping  $\gamma_1 > 0$ , we move into the wedge-shaped region to the upper left in the diagram in Fig. 2. This region encompasses “case 2,” “case 3,” and “case 4” from Table I (although we restrict ourselves to half the region  $0 < \kappa_2 \leq \kappa_1$ ). Consider the curve that in trilinear coordinates is given by

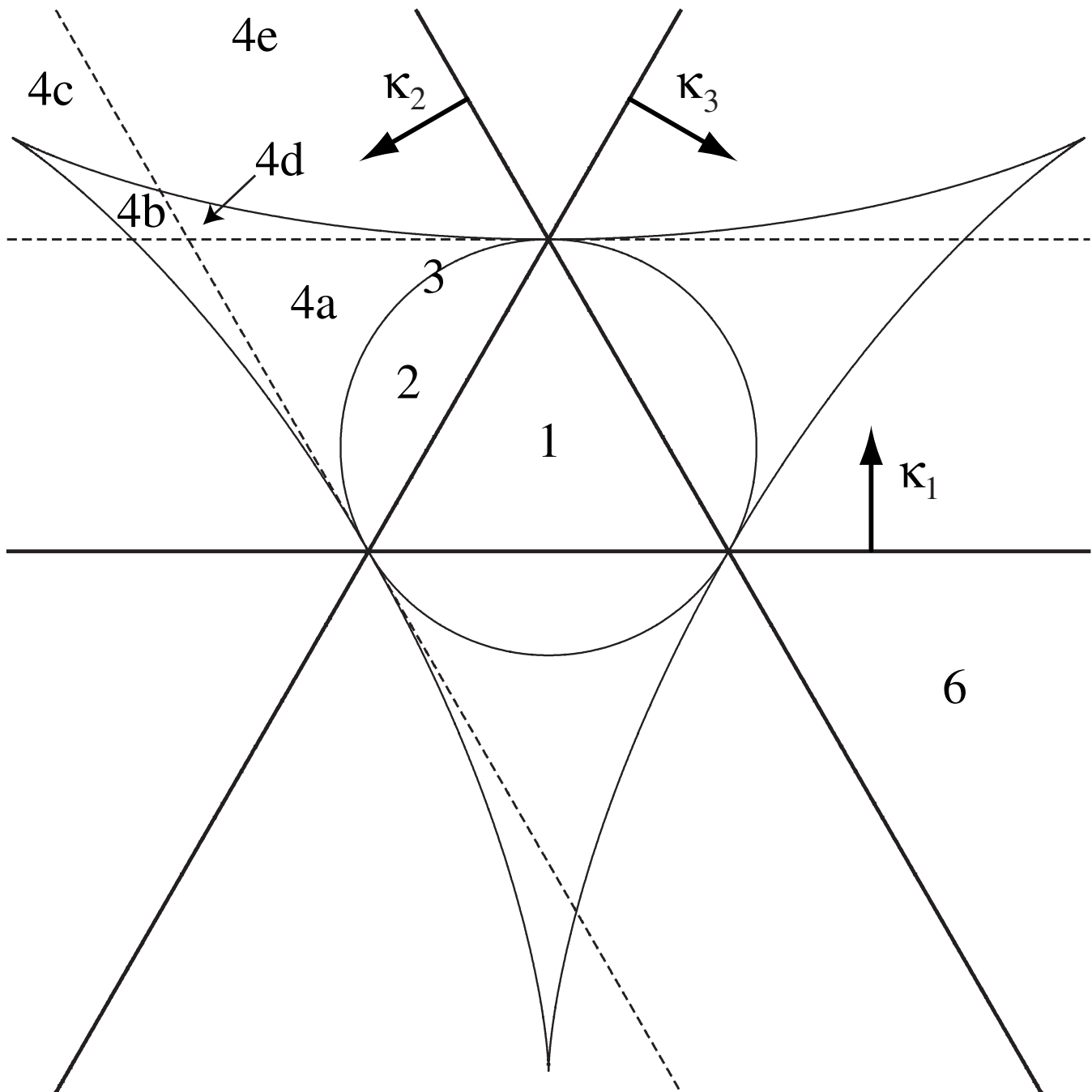


FIG. 2. The trilinear  $(\kappa_1, \kappa_2, \kappa_3)$  diagram used to classify regimes of three-vortex motion. The various curves dividing up “circulation space” are explained in the text. The circle circumscribed about the basic equilateral triangle is given by Eq. (47). The dashed lines are  $\Gamma_1 + \Gamma_3 = 0$  and  $\Gamma_2 + \Gamma_3 = 0$ , respectively. The cusped curve is given by Eq. (46). Labels 1, 2, etc., give the various regimes summarized in Table I. The convention  $\Gamma_3 \leq \Gamma_2 \leq \Gamma_1$  on the three circulations restricts interest to a subset of the diagram.

$$\kappa_1 \kappa_2 + \kappa_2 \kappa_3 + \kappa_3 \kappa_1 = 0. \quad (47)$$

This is the equation for a circle. It passes through the three vertices of the equilateral triangle and, thus, is the circumscribed circle of that triangle. In Fig. 2 we show this circle, and the regimes corresponding to “cases 1–4” and “case 6” from Table I. “Case 3” consists of (half of) the arc of the circle separating the two regimes “2” and “4.” The further partitioning of regime “4” into several pieces is explained below. The regime corresponding to “case 5” is not available in the diagram in Fig. 2 since this representation relies on  $\gamma_1 \neq 0$ . Fortunately, detailed results for the case of vanishing

total circulation are available independently.<sup>13,14</sup> The regime corresponding to “case 6” is (half of) the lower right wedge in the diagram. Although we show the full diagram, and curves such as Eq. (47) extended throughout, our convention  $\Gamma_3 \leq \Gamma_2 \leq \Gamma_1$  restricts us in Fig. 2 to a subset of the full plane. This very useful diagram appears in the work by Conte and de Seze<sup>7</sup> and Conte.<sup>9</sup>

We may now draw the curve corresponding to Eq. (46) in the diagram in Fig. 2. A threefold symmetric curve with three cusps arises. Points inside this curve correspond, in general, to three collinear relative equilibria. Exceptions

arise on the lines  $\kappa_1 + \kappa_3 = 0$  and  $\kappa_2 + \kappa_3 = 0$  and their point of intersection. Points outside the curve correspond, in general, to only one collinear relative equilibrium. (Along the line  $\kappa_1 + \kappa_3 = 0$  outside the curve there are no collinear relative equilibria.) We also need to analyze points on the curve. We clearly see why “case 4” is the most complicated regime. It contains portions corresponding to three, two, one, and zero collinear relative equilibria. The cubic curve and the lines  $\kappa_1 + \kappa_3 = 0$  and  $\kappa_2 + \kappa_3 = 0$  partition regime “4” into five pieces, labeled “4a” through “4e” (and our interest is confined to the half wherein  $0 < \kappa_2 \leq \kappa_1$ ). On the lines  $\kappa_1 + \kappa_3 = 0$  and  $\kappa_2 + \kappa_3 = 0$  there are just two collinear relative equilibria, as we have seen, and at their point of intersection just one. This is true only for the piece of these lines that is inside the cusped cubic curve. As we cross the cubic, the significance of the lines  $\kappa_1 + \kappa_3 = 0$  and  $\kappa_2 + \kappa_3 = 0$  changes from implying two solutions (or exceptionally just one) to implying no solutions at all. On the cusped curve itself the discriminant of the cubic  $p(z)$  vanishes, and we have, in general, just two solutions, and exceptionally just one.

Fixing  $\Gamma_1$  and  $\Gamma_2$  and varying  $\Gamma_3$  in the range  $-(\Gamma_1 + \Gamma_2) < \Gamma_3 < -(\Gamma_1 + \Gamma_2)/\Gamma_1\Gamma_2$  corresponds in Fig. 2 to moving along a line  $\kappa_1 : \kappa_2 = \Gamma_1 : \Gamma_2$  from where it intersects the circle  $\gamma_2 = 0$  to infinity. At some point this line intersects the cusped curve. That point of intersection defines the intermediate value of  $\Gamma_3$  that we designated  $\Gamma_\times$  earlier. For  $\Gamma_1 = \Gamma_2$ , i.e.,  $\kappa_1 = \kappa_2$ , the line goes through the cusp itself.

**V. LINEAR STABILITY OF THE RELATIVE EQUILIBRIA OF THREE VORTICES: GENERAL THEORY**

We are now ready to investigate the linear stability of the equilateral triangle and collinear relative equilibria of three vortices. Since the geometry of the equilateral triangle equilibria is decoupled from the vortex circulations, the analysis for this case will prove much simpler than for the case of the collinear relative equilibria.

The equations of motion are Eqs. (1) specialized to  $N=3$ ,

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{1}{2\pi i} \left[ \frac{\Gamma_2}{z_1 - z_2} + \frac{\Gamma_3}{z_1 - z_3} \right], \\ \frac{dz_2}{dt} &= \frac{1}{2\pi i} \left[ \frac{\Gamma_1}{z_2 - z_1} + \frac{\Gamma_3}{z_2 - z_3} \right], \\ \frac{dz_3}{dt} &= \frac{1}{2\pi i} \left[ \frac{\Gamma_1}{z_3 - z_1} + \frac{\Gamma_2}{z_3 - z_2} \right]. \end{aligned} \tag{48}$$

Most of the relative equilibria correspond to solid body rotation  $z_\alpha(t) = z_\alpha^{(0)} e^{i\Omega t}$ ,  $\alpha = 1, 2, 3$ , where the angular frequency of rotation  $\Omega$  is determined as part of the solution. We perturb such a solution by setting  $z_\alpha(t) = [z_\alpha^{(0)} + \eta_\alpha(t)] e^{i\Omega t}$ , where the  $\eta_\alpha(t)$  are infinitesimal perturbations with respect to which we shall linearize. Inclusion of the factor  $e^{i\Omega t}$  in the perturbation is for convenience in calculation, as we shall see presently.

Expanding to first order in the small perturbations  $\eta_\alpha$ , we obtain the linearized perturbation equations

$$\begin{aligned} \frac{d\bar{\eta}_1}{dt} - i\Omega \bar{\eta}_1 &= -\frac{1}{2\pi i} \left[ \Gamma_2 \frac{\eta_1 - \eta_2}{(z_1^{(0)} - z_2^{(0)})^2} + \Gamma_3 \frac{\eta_1 - \eta_3}{(z_1^{(0)} - z_3^{(0)})^2} \right], \\ \frac{d\bar{\eta}_2}{dt} - i\Omega \bar{\eta}_2 &= -\frac{1}{2\pi i} \left[ \Gamma_1 \frac{\eta_2 - \eta_1}{(z_2^{(0)} - z_1^{(0)})^2} + \Gamma_3 \frac{\eta_2 - \eta_3}{(z_2^{(0)} - z_3^{(0)})^2} \right], \\ \frac{d\bar{\eta}_3}{dt} - i\Omega \bar{\eta}_3 &= -\frac{1}{2\pi i} \left[ \Gamma_1 \frac{\eta_3 - \eta_1}{(z_3^{(0)} - z_1^{(0)})^2} + \Gamma_2 \frac{\eta_3 - \eta_2}{(z_3^{(0)} - z_2^{(0)})^2} \right]. \end{aligned} \tag{49}$$

Factors  $e^{i\Omega t}$  have been canceled in these equations. The coefficient matrix on the right hand side is, thus, independent of time. We may write these linearized perturbation equations in an easily understood vector-matrix notation,

$$\dot{\bar{\eta}} = i(\Omega \bar{\eta} + A \eta), \tag{50a}$$

where

$$A = \frac{1}{2\pi} \begin{bmatrix} \frac{\Gamma_2}{(z_1^{(0)} - z_2^{(0)})^2} + \frac{\Gamma_3}{(z_1^{(0)} - z_3^{(0)})^2} & -\frac{\Gamma_2}{(z_1^{(0)} - z_2^{(0)})^2} & -\frac{\Gamma_3}{(z_1^{(0)} - z_3^{(0)})^2} \\ -\frac{\Gamma_1}{(z_2^{(0)} - z_1^{(0)})^2} & \frac{\Gamma_1}{(z_2^{(0)} - z_1^{(0)})^2} + \frac{\Gamma_3}{(z_2^{(0)} - z_3^{(0)})^2} & -\frac{\Gamma_3}{(z_2^{(0)} - z_3^{(0)})^2} \\ -\frac{\Gamma_1}{(z_3^{(0)} - z_1^{(0)})^2} & -\frac{\Gamma_2}{(z_3^{(0)} - z_2^{(0)})^2} & \frac{\Gamma_1}{(z_3^{(0)} - z_1^{(0)})^2} + \frac{\Gamma_2}{(z_3^{(0)} - z_2^{(0)})^2} \end{bmatrix}. \tag{50b}$$

Differentiating Eq. (50a) once more, we get

$$\ddot{\bar{\eta}} = i(\Omega \dot{\bar{\eta}} + A \dot{\eta}) = -\Omega(\Omega \bar{\eta} + A \eta) + iA[-i(\Omega \eta + \bar{A} \bar{\eta})],$$

or, taking the complex conjugate and effecting the cancellation,

$$\ddot{\eta} = (-\Omega^2 + \bar{A}A) \eta. \tag{51}$$

We may now proceed as follows. First, we find two general eigenvectors of  $\bar{A}A$ . The most obvious of these is (1,1,1), which is an eigenvector of  $A$  with eigenvalue 0 since the rows in  $A$  sum to zero. Hence, (1,1,1) is also an eigenvector

of  $\overline{\mathbf{A}\mathbf{A}}$  with eigenvalue 0. Only slightly less obvious is the eigenvector  $\mathbf{z}^{(0)} = (z_1^{(0)}, z_2^{(0)}, z_3^{(0)})$ . Clearly, the first row of  $\mathbf{A}\mathbf{z}^{(0)}$  is

$$\begin{aligned} & \frac{1}{2\pi} \left[ \Gamma_2 \frac{z_1^{(0)} - z_2^{(0)}}{(z_1^{(0)} - z_2^{(0)})^2} + \Gamma_3 \frac{z_1^{(0)} - z_3^{(0)}}{(z_1^{(0)} - z_3^{(0)})^2} \right] \\ &= \frac{1}{2\pi} \left[ \frac{\Gamma_2}{z_1^{(0)} - z_2^{(0)}} + \frac{\Gamma_3}{z_1^{(0)} - z_3^{(0)}} \right] = \Omega \overline{z_1^{(0)}}. \end{aligned}$$

The two other rows similarly produce  $\Omega \overline{z_2^{(0)}}$  and  $\Omega \overline{z_3^{(0)}}$ , respectively. Thus,  $\mathbf{A}\mathbf{z}^{(0)} = \Omega \overline{\mathbf{z}^{(0)}}$  and  $\overline{\mathbf{A}\mathbf{A}}\mathbf{z}^{(0)} = \Omega^2 \mathbf{z}^{(0)}$ . The vector  $\mathbf{z}^{(0)}$  is, therefore, an eigenvector of  $\overline{\mathbf{A}\mathbf{A}}$  with eigenvalue  $\Omega^2$ .

Next, since  $\text{Tr}(\overline{\mathbf{A}\mathbf{A}}) = \text{Tr}(\mathbf{A}\overline{\mathbf{A}}) = \text{Tr}(\overline{\mathbf{A}\mathbf{A}})$ , the trace of the  $3 \times 3$  matrix  $\overline{\mathbf{A}\mathbf{A}}$  is real, and since we know two of the eigenvalues to be real, the third eigenvalue must also be real. If we expand the perturbation  $\boldsymbol{\eta}$  in eigenmodes of  $\overline{\mathbf{A}\mathbf{A}}$ , we see from Eq. (51) that  $\rho(t)$ , the amplitude of an eigenmode with eigenvalue  $\Lambda$ , will evolve according to the ODE,

$$\frac{d^2 \rho}{dt^2} = (-\Omega^2 + \Lambda)\rho. \quad (52)$$

If  $\Lambda \leq \Omega^2$  the eigenmode is stable. For  $\Lambda < \Omega^2$  the solution for  $\rho(t)$  is oscillatory. For  $\Lambda = \Omega^2$  we would have  $\rho(t) = \dot{\rho}(0)t + \rho(0)$ . For the stability problem under investigation here we do not envision perturbations with a rate of change in the coordinates, only with a displacement at time  $t=0$ , i.e.,  $\dot{\rho}(0)=0$ . The eigenmodes found so far, corresponding to eigenvalues 0 and  $\Omega^2$ , thus, are stable and neutrally stable, respectively, to linear order. The issue is whether the third

eigenvalue, which we now designate as  $\Lambda$ , is larger or smaller than  $\Omega^2$ . We have  $\text{Tr}(\overline{\mathbf{A}\mathbf{A}}) = \Omega^2 + \Lambda$ . Hence, if we calculate this trace, we have determined the last eigenvalue  $\Lambda$ , which, in turn, is decisive insofar as linear stability of the relative equilibrium is concerned.

Stability results aside, the perturbation (1,1,1) changes the linear impulse of the configuration unless  $\gamma_1=0$ . The perturbation  $\mathbf{z}^{(0)}$  changes the angular impulse (unless it happens to vanish for the relative equilibrium under consideration). In a stability calculation we would typically avoid perturbations that change the integrals of motion.

At this point the details of the calculation depend on which relative equilibrium we are considering. In fact, for the collinear relative equilibria we shall be able to work directly with matrix  $\mathbf{A}$  rather than  $\overline{\mathbf{A}\mathbf{A}}$ . Thus, we treat the stability of the equilateral triangle relative equilibria and the collinear equilibria in separate sections.

## VI. LINEAR STABILITY OF THE EQUILATERAL TRIANGLE RELATIVE EQUILIBRIA

With vortices 1, 2, and 3 in counterclockwise order, we get  $z_2^{(0)} - z_1^{(0)} = s$ ,  $z_3^{(0)} - z_2^{(0)} = \epsilon s$ , and  $z_1^{(0)} - z_3^{(0)} = \epsilon^2 s$ , where  $\epsilon = e^{i2\pi/3}$ . Thus,

$$\mathbf{A} = \frac{1}{2\pi s^2} \begin{bmatrix} \Gamma_2 + \epsilon^2 \Gamma_3 & -\Gamma_2 & -\epsilon^2 \Gamma_3 \\ -\Gamma_1 & \Gamma_1 + \epsilon \Gamma_3 & -\epsilon \Gamma_3 \\ -\epsilon^2 \Gamma_1 & -\epsilon \Gamma_2 & \epsilon^2 \Gamma_1 + \epsilon \Gamma_2 \end{bmatrix}. \quad (53)$$

A straightforward, if slightly tedious, calculation gives

$$\overline{\mathbf{A}\mathbf{A}} = \left( \frac{1}{2\pi s^2} \right)^2 \begin{bmatrix} (\Gamma_2 - \Gamma_3)^2 + \gamma_2 & -\Gamma_2(\gamma_1 + 3\epsilon \Gamma_3) & -\Gamma_3(\gamma_1 + 3\epsilon^2 \Gamma_2) \\ -\Gamma_1(\gamma_1 + 3\epsilon^2 \Gamma_3) & (\Gamma_3 - \Gamma_1)^2 + \gamma_2 & -\Gamma_3(\gamma_1 + 3\epsilon \Gamma_1) \\ -\Gamma_1(\gamma_1 + 3\epsilon \Gamma_2) & -\Gamma_2(\gamma_1 + 3\epsilon^2 \Gamma_1) & (\Gamma_1 - \Gamma_2)^2 + \gamma_2 \end{bmatrix}, \quad (54)$$

with  $\gamma_1$  and  $\gamma_2$  as the two symmetric functions of the vortex circulations introduced previously.

The trace of this matrix is

$$\begin{aligned} \text{Tr}(\overline{\mathbf{A}\mathbf{A}}) &= \left( \frac{1}{2\pi s^2} \right)^2 [(\Gamma_2 - \Gamma_3)^2 + (\Gamma_3 - \Gamma_1)^2 \\ &+ (\Gamma_1 - \Gamma_2)^2 + 3\gamma_2] = \frac{2\gamma_1^2 - 3\gamma_2}{(2\pi s^2)^2}. \end{aligned} \quad (55)$$

In the last step we have used the result that any polynomial that is symmetric in its arguments, here  $\Gamma_1, \Gamma_2, \Gamma_3$ , can be expressed in terms of the symmetric polynomials  $\gamma_1$ ,  $\gamma_2$ , and  $\gamma_3$ . For this case  $\Omega = \gamma_1 / 2\pi s^2$ , Eq. (23b). Hence, the expression for the trace gives the third eigenvalue  $\Lambda$  of  $\overline{\mathbf{A}\mathbf{A}}$ ,

$$\Omega^2 + \Lambda = 2\Omega^2 - \frac{3\gamma_2}{(2\pi s^2)^2}, \quad \text{or} \quad \Lambda = \Omega^2 - \frac{3\gamma_2}{(2\pi s^2)^2}. \quad (56)$$

We see that  $\Lambda < \Omega^2$  if  $\gamma_2 > 0$ , that  $\Lambda = \Omega^2$  if  $\gamma_2 = 0$ , and that  $\Omega^2 < \Lambda$  if  $\gamma_2 < 0$ . The equilateral triangle relative equilib-

rium, then, is linearly stable if  $\gamma_2 = \Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1 > 0$ , marginally stable if  $\gamma_2 = 0$ , and unstable if  $\gamma_2 < 0$ . The results of the graphical analysis of the three-vortex problem<sup>3,4</sup> show that this conclusion from linearized stability holds for finite amplitude perturbations as well.

For  $\gamma_2 > 0$  the triangle will pulsate around the equilibrium configuration when perturbed with angular frequency

$$\omega = \frac{\sqrt{3\gamma_2}}{2\pi s^2}, \quad (57)$$

since from Eq. (52) the eigenmode evolves as  $e^{t\sqrt{\Lambda - \Omega^2}} = e^{\pm i\omega t}$ . For  $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma$  this frequency is  $\omega = 3\Gamma / 2\pi s^2 = \Omega$ . The observation that the pulsation frequency and the rotation frequency coincide for an equilateral triangle of identical vortices is due to Thomson. In general, in the stable case,

$$\left(\frac{\omega}{\Omega}\right)^2 = \frac{3\gamma_2}{\gamma_1^2} = 1 - \frac{1}{2\gamma_1^2}[(\Gamma_1 - \Gamma_2)^2 + (\Gamma_2 - \Gamma_3)^2 + (\Gamma_3 - \Gamma_1)^2], \tag{58}$$

so the frequency of oscillation is always smaller than the frequency of rotation with equality only in the case of identical vortices. This elegant form of the result was given in Ref. 7.

The eigenvector corresponding to eigenvalue  $\Lambda$  is found to be  $[\epsilon^2/\Gamma_1, \epsilon/\Gamma_2, 1/\Gamma_3]$  (after a short calculation), or in general terms,

$$\left[ \frac{z_3^{(0)} - z_2^{(0)}}{\Gamma_1}, \frac{z_1^{(0)} - z_3^{(0)}}{\Gamma_2}, \frac{z_2^{(0)} - z_1^{(0)}}{\Gamma_3} \right]. \tag{59}$$

If the sum of the circulations vanishes, the equilateral triangle translates  $z_\alpha^{(0)}(t) = z_\alpha^{(0)} + Vt$  with  $V$  given by Eq. (25). We consider perturbations of the form  $z_\alpha(t) = z_\alpha^{(0)} + Vt + \eta_\alpha(t)$ . The analysis follows that already given but with  $\Omega = 0$ . Hence, eigenvalues 0 (now a double root of the characteristic polynomial) and  $-3\gamma_2/(2\pi s^2)^2$  are obtained. From  $\gamma_1 = 0$  and Eq. (10), however, we see that  $\gamma_2$  is negative and the translating equilateral triangle of vortices is always unstable. This is in accord with the detailed solution for this case.<sup>13,14</sup>

We now turn to the calculation that is the main motivation for this paper.

**VII. LINEAR STABILITY OF THE COLLINEAR RELATIVE EQUILIBRIA**

In this case  $A$  is a real matrix,

$$A = \bar{A} = \frac{1}{2\pi} \begin{bmatrix} \frac{\Gamma_2}{\xi_3^2} + \frac{\Gamma_3}{\xi_2^2} & -\frac{\Gamma_2}{\xi_3^2} & -\frac{\Gamma_3}{\xi_2^2} \\ -\frac{\Gamma_1}{\xi_3^2} & \frac{\Gamma_3}{\xi_1^2} + \frac{\Gamma_1}{\xi_3^2} & -\frac{\Gamma_3}{\xi_1^2} \\ -\frac{\Gamma_1}{\xi_2^2} & -\frac{\Gamma_2}{\xi_1^2} & \frac{\Gamma_1}{\xi_2^2} + \frac{\Gamma_2}{\xi_1^2} \end{bmatrix}. \tag{60}$$

The eigenvector  $z^{(0)} = (x_1, x_2, x_3)$  of  $\bar{A}A$ , which we shall call  $x$  in this case, is now real, so the general relation  $Az^{(0)} = \Omega z^{(0)}$  becomes  $Ax = \Omega x$ , and  $x$  is already an eigenvector of  $A$  with eigenvalue  $\Omega$ . The vector  $(1,1,1)$  is, of course, still an eigenvector of  $A$  with eigenvalue 0. If we let  $\Lambda$  denote the third eigenvalue—now of  $A$  not of  $\bar{A}A$ —we have from the coefficient of the quadratic term in the characteristic polynomial of  $A$ , or simply from setting the trace of the matrix to the sum of the eigenvalues, that

$$\text{Tr } A = \frac{1}{2\pi} \left[ \frac{\Gamma_2 + \Gamma_3}{\xi_1^2} + \frac{\Gamma_3 + \Gamma_1}{\xi_2^2} + \frac{\Gamma_1 + \Gamma_2}{\xi_3^2} \right] = \Omega + \Lambda. \tag{61a}$$

We now recall the previously derived expression for  $\Omega$ , Eq. (44),

$$\Omega = \frac{1}{2\pi\gamma_2} \left[ \frac{\Gamma_2\Gamma_3(\Gamma_2 + \Gamma_3)}{\xi_1^2} + \frac{\Gamma_3\Gamma_1(\Gamma_3 + \Gamma_1)}{\xi_2^2} + \frac{\Gamma_1\Gamma_2(\Gamma_1 + \Gamma_2)}{\xi_3^2} \right]. \tag{61b}$$

Using this expression in the result for  $\Omega + \Lambda$ , Eq. (61a), we arrive at a similar expression for  $\Lambda$ ,

$$\Lambda = \frac{1}{2\pi\gamma_2} \left[ \frac{\Gamma_1(\Gamma_2 + \Gamma_3)^2}{\xi_1^2} + \frac{\Gamma_2(\Gamma_3 + \Gamma_1)^2}{\xi_2^2} + \frac{\Gamma_3(\Gamma_1 + \Gamma_2)^2}{\xi_3^2} \right]. \tag{61c}$$

As a complement to Eq. (61a) we have

$$\Lambda - \Omega = \frac{1}{2\pi\gamma_2} \left[ \frac{(\Gamma_1\Gamma_2 - \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1)(\Gamma_2 + \Gamma_3)}{\xi_1^2} + \frac{(\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 - \Gamma_3\Gamma_1)(\Gamma_3 + \Gamma_1)}{\xi_2^2} + \frac{(-\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1)(\Gamma_1 + \Gamma_2)}{\xi_3^2} \right] \tag{61d}$$

a result that will be put to good use shortly.

Although we shall focus exclusively on the eigenvalue  $\Lambda$ , Eq. (61c), in what follows, we note that the eigenvector corresponding to this eigenvalue is in all cases given by a similar expression to Eq. (59) for the equilateral triangle relative equilibrium, viz.,

$$\left[ \frac{\xi_1}{\Gamma_1}, \frac{\xi_2}{\Gamma_2}, \frac{\xi_3}{\Gamma_3} \right]. \tag{62}$$

This vector consists of displacements of the vortices along the line of the relative equilibrium.

Further,  $\bar{A}A = A^2$  and our general criterion for linear instability is that  $A^2$  has an eigenvalue larger than  $\Omega^2$ . The eigenvalues of  $A^2$  are 0,  $\Omega^2$ , and  $\Lambda^2$ , where  $\Lambda$  is given by Eq. (61c). We have linear stability so long as  $|\Lambda| \leq |\Omega|$ . In practice, we shall first determine the sign of  $\Omega$  for a given collinear relative equilibrium. If  $\Omega$  is positive, we then consider the condition for absence of linear instability in the form  $-\Omega \leq \Lambda \leq \Omega$ . Similarly, if  $\Omega$  is negative, the condition to be checked is  $\Omega \leq \Lambda \leq -\Omega$ .

Let us use Eqs. (61b) and (61c) to determine the stability of collinear relative equilibria in the special case treated in Sec. IV C,  $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 1, r)\Gamma$ . (For  $r = 0$  this corresponds to a passive particle in the field of two identical vortices.) With the circulations  $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 1, r)\Gamma$ , the polynomial  $p(z)$  simplifies considerably,

$$p(z) = 2z^3 + 3z^2 - (1 + 2r)z - (1 + r).$$

(In several of the following formulas we have, for simplicity, omitted factors of  $\Gamma$  or, equivalently, used units where  $\Gamma = 1$ . We have restored  $\Gamma$  in formulas where physical units seem essential, such as angular frequency of rotation and the like.) As we saw in Sec. IV C,  $p(z)$  has the roots

$$z_0^{(p)} = \frac{1}{2}(-1 + \sqrt{5 + 4r}), \quad z_1^{(p)} = \frac{1}{2}(-1 - \sqrt{5 + 4r}), \quad z_2^{(p)} = -\frac{1}{2}.$$

Consider first the solution corresponding to the root  $z_2^{(p)}$  of  $p(z)$  which exists for all values of  $r$ . Vortex 3 is at the origin,  $x_3=0$ ; vortices 1 and 2 at equal distances from it,  $x_1=-x_2$ . From Sec. IV C or Eq. (61b) we have the angular frequency of rotation

$$\Omega_2 = 2(1 + 2r) \frac{\Gamma}{2\pi\xi_3^2}.$$

Hence, for  $-\frac{1}{2} < r$  we have counterclockwise rotation. For  $r = -\frac{1}{2}$  we have a stationary equilibrium. For  $r < -\frac{1}{2}$  the configuration rotates clockwise. In particular, for  $r = -2$ , when the sum of the circulations vanishes,  $\Omega_2 < 0$ . From Eq. (61c) we get

$$\Lambda_2 = 4(2 + r) \frac{\Gamma}{2\pi\xi_3^2}.$$

For  $-\frac{1}{2} < r \leq 1$  (because of our convention  $\Gamma_3 \leq \Gamma_2 \leq \Gamma_1$  we are not interested in  $1 < r$ ), we have  $\Omega_2$  and  $\Lambda_2$  both positive, the condition for linear stability is  $-\Omega_2 \leq \Lambda_2 \leq \Omega_2$ , and the configuration is unstable since  $\Omega_2 < \Lambda_2$ . For  $r = 0$  vortex 3 degenerates to a passive particle. The instability is clear in this case from considering the flow induced by the two vortices which has a saddle point at the origin.

For  $r = -\frac{1}{2}$  the equilibrium is stationary,  $\Omega_2 = 0$ . It is unstable since  $0 < \Lambda_2$ , whereas the stability criterion has degenerated to the requirement  $\Lambda_2 = 0$ .

For  $-\frac{5}{4} < r < -\frac{1}{2}$  we have  $\Omega_2 < 0$ , the condition for linear stability is  $\Omega_2 \leq \Lambda_2 \leq -\Omega_2$ , and the configuration is unstable since  $-\Omega_2 < \Lambda_2$ . We have marginal stability for  $r = -\frac{5}{4}$  since  $\Lambda_2 = -\Omega_2$ .

For  $r < -\frac{5}{4}$  we still have  $\Omega_2 < 0$ , the condition for linear stability is still  $\Omega_2 \leq \Lambda_2 \leq -\Omega_2$ , but the configuration is linearly stable since  $\Lambda_2$  now satisfies the criterion. One may think of the strong, negative central vortex as the stabilizing agent. We see that a vortex of absolute strength  $\frac{5}{4}\Gamma$  is sufficient to accomplish the task of stabilizing the configuration. *A fortiori* we have linear stability for  $r = -2$ .

Consider next the solutions corresponding to  $z_{0,1}^{(p)}$ . These are of interest only for  $-\frac{5}{4} < r \leq 1$ . For  $r = -1$ ,  $z_0^{(p)}$  ( $z_1^{(p)}$ ) takes on the disallowed value 0 ( $-1$ ) so vortices 2 (1) and 3 coincide and the angular frequency must diverge. For  $r = -\frac{5}{4}$  these two roots coincide with  $z_2^{(p)}$ , which becomes a triple root. The angular frequency of rotation,  $\Omega_n$ ,  $n=0, 1$ , as determined in Sec. IV C or from Eq. (61b), is the same for both roots,

$$\Omega_{0,1} = \frac{\Gamma}{2\pi\xi_3^2} \frac{2+r}{1+r}.$$

From Eq. (61c) we see that for these configurations  $\Lambda_{0,1}$  is independent of  $r$ ,

$$\Lambda_{0,1} = \frac{3\Gamma}{2\pi\xi_3^2}.$$

The stability results follow. For  $-1 < r \leq 1$ , we have  $0 < \Omega_{0,1}$ , and the condition for stability is  $-\Omega_{0,1} \leq \Lambda_{0,1} \leq \Omega_{0,1}$ . The inequality  $-\Omega_{0,1} \leq \Lambda_{0,1}$  is clearly satisfied. To satisfy

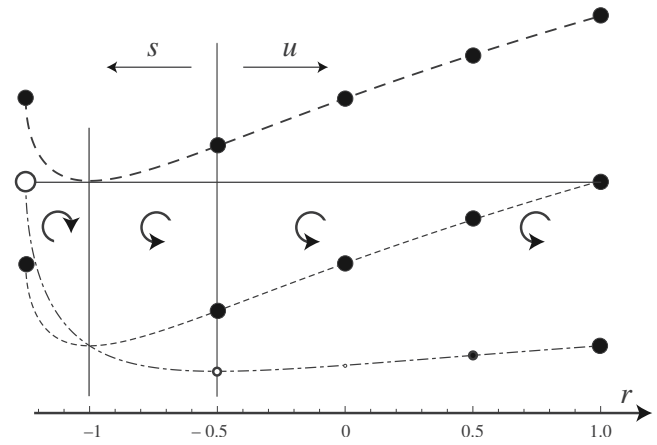


FIG. 3. Schematic of the  $n=0$  collinear relative equilibrium solution for  $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 1, r)\Gamma$ . The positions along the common (vertical) line are shown as a function of  $r$ . Solid dots correspond to positive vortices, open dots to negative vortices. The radii of the dots scale with the magnitude of the circulation. Clockwise and counterclockwise rotation are indicated with the dividing line at  $r = -1$ . Stability and instability are indicated by  $s$  and  $u$ , respectively, with the dividing line at  $r = -\frac{1}{2}$ . Note the singularity at  $r = -1$  (two vortices occupy the same position) and that vortex 3 has become a passive particle at  $r = 0$ . The vortex positions are plotted from Eq. (45).

$\Lambda_{0,1} \leq \Omega_{0,1}$ , we require  $r \leq -\frac{1}{2}$ . Thus, the collinear relative equilibria corresponding to the roots  $z_{0,1}^{(p)}$  are linearly unstable for  $-\frac{1}{2} < r \leq 1$ . This includes the case  $r = 0$  where vortex 3 has degenerated to a passively advected particle. We have marginal stability for  $r = -\frac{1}{2}$ , and these relative equilibria are stable for  $-1 < r < -\frac{1}{2}$ . For  $r = -1$  we have the triple root and a singularity in  $\Omega_{0,1}$ .

For  $-\frac{5}{4} < r < -1$ , on the other side of the singularity,  $\Omega_{0,1} < 0$ , and the condition for stability is  $\Omega_{0,1} \leq \Lambda_{0,1} \leq -\Omega_{0,1}$ . The inequality  $\Omega_{0,1} \leq \Lambda_{0,1}$  is clearly satisfied. The inequality  $\Lambda_{0,1} \leq -\Omega_{0,1}$  again requires  $r \leq -\frac{1}{2}$ . Hence, the collinear relative equilibria corresponding to the roots  $z_{0,1}^{(p)}$  are linearly stable also for  $-\frac{5}{4} < r \leq -1$ , i.e., we have linear stability for the entire interval  $-\frac{5}{4} < r < -\frac{1}{2}$ .

In summary, the solution for  $n=2$  has a fixed geometry with vortex 3 at the midpoint of the two positive, identical vortices 1 and 2. This configuration is unstable everywhere within the cusped curve, i.e., for  $-\frac{5}{4} < r \leq 1$ . It reverses sense of rotation, from counterclockwise to clockwise, as it passes through the stationary equilibrium for  $r = -\frac{1}{2}$ . This solution continues as the sole real root of  $p(z)$  outside the cusped curve, i.e., for  $r < -\frac{5}{4}$ . It is marginally stable for  $r = -\frac{5}{4}$  and becomes stable as one crosses the cusped curve. The configuration retains its clockwise sense of rotation for all  $r < -\frac{5}{4}$ . The solutions for  $n=0, 1$ , on the other hand, change from unstable for  $-\frac{1}{2} < r \leq 1$  to stable for  $-1 < r < -\frac{1}{2}$  but their counterclockwise sense of rotation is unchanged. They become marginally stable for  $r = -\frac{1}{2}$ . They become unphysical for  $r = -1$ . When they are “reborn” for  $-\frac{5}{4} < r < -1$ , they have reversed their sense of rotation but remain stable. At  $r = -\frac{5}{4}$  they merge smoothly with the solution for  $n=2$  and disappear.

We summarize this analysis in Fig. 3, which shows a schematic of the configurations for  $n=0$  as a function of  $r$ . (The configurations for  $n=1$  produce a similar diagram.) For

$n=2$  the configuration is fixed and looks like the configuration in Fig. 3 for  $r=-\frac{5}{4}$ . In Fig. 3 positive vortices are shown by solid dots, negative vortices by open dots. The radius of each dot is proportional to the vortex circulation. We see how vortex 3 “migrates” from its central position for  $r=-\frac{5}{4}$  to being beyond the line segment connecting vortices 1 and 2. At  $r=-1$  vortex 3 coincides with vortex 2, and we have a singularity. The diagram shows how the angular velocity changes sign, following the sign of the mutually induced rotation in the unbalanced pair formed by vortex 3 and its closest positive neighbor. We have seen that the configurations change from being stable to being unstable at  $r=-\frac{1}{2}$ , but this seems less intuitive. The orbiting motion of the two identical vortices is certainly quite stable. As  $r$  is increased, vortex 3 continues to weaken relative to these two vortices, at  $r=0$  even becoming a passively advected particle at a corotating point in the flow field induced by them. As vortex 3 strengthens and its circulation becomes positive, we eventually converge on the unstable configuration of three identical vortices on a line.

Ideally, for a given configuration, i.e., a root  $z^{(p)}$  of  $p(z)$ , we would calculate  $\Omega$  as a function of  $\Gamma_1, \Gamma_2$ , and  $\Gamma_3$  from Eq. (61b), as we have just done for the special case  $(\Gamma_1, \Gamma_2, \Gamma_3)=(1, 1, r)\Gamma$ . Once the sign of  $\Omega$  is determined, we need to check the validity of the stability inequalities using Eq. (61c), i.e., whether  $\Lambda + \Omega$  and  $\Lambda - \Omega$  are positive or negative. Unfortunately, except for the simplest choices of the vortex circulations, this straightforward procedure leads to insurmountable algebraic complications (even with the aid of a symbolic mathematics program). One must proceed, therefore, by a circuitous route in which the information about  $\Omega$  and  $\Lambda \pm \Omega$  is “encoded” in various polynomials and the variation in these polynomials, evaluated at the roots of  $p(z)$ , is analyzed as a function of the three vortex circulations. We proceed to explain how this is done.

There are several polynomials to be considered. First, of course, we have the cubic polynomial equation (37), the real roots of which (except for 0 and  $-1$ ) give the possible colinear relative equilibria,

$$p(z) = (\Gamma_1 + \Gamma_2)z^3 + (2\Gamma_1 + \Gamma_2)z^2 - (\Gamma_2 + 2\Gamma_3)z - (\Gamma_2 + \Gamma_3).$$

Second, the sign of  $\Omega$  is related to the polynomial

$$\omega(z) = \Gamma_2\Gamma_3(\Gamma_2 + \Gamma_3)(z + 1)^2 + \Gamma_3\Gamma_1(\Gamma_3 + \Gamma_1)z^2 + \Gamma_1\Gamma_2(\Gamma_1 + \Gamma_2)z^2(z + 1)^2, \tag{63a}$$

which arises from Eq. (61b) through

$$\omega(z) = 2\pi\gamma_2\Omega z^2(z + 1)^2\xi_3^2.$$

Third, we have a quartic polynomial appearing in the expression for  $\Omega + \Lambda$ , Eq. (61a),

$$s(z) = (\Gamma_2 + \Gamma_3)(z + 1)^2 + (\Gamma_3 + \Gamma_1)z^2 + (\Gamma_1 + \Gamma_2)z^2(z + 1)^2 + (\Gamma_1 + \Gamma_2)z^4 + 2(\Gamma_1 + \Gamma_2)z^3 + 2\gamma_1z^2 + 2(\Gamma_2 + \Gamma_3)z + \Gamma_2 + \Gamma_3. \tag{63b}$$

This polynomial enters via

$$s(z) = 2\pi(\Lambda + \Omega)z^2(z + 1)^2\xi_3^2,$$

and its sign thus determines one half of the condition of linear stability. Fourth, there is a second quartic polynomial appearing in the expression for  $\Lambda - \Omega$ , Eq. (61d),

$$d(z) = (\Gamma_2 + \Gamma_3)(\Gamma_1\Gamma_2 - \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1)(z + 1)^2 + (\Gamma_3 + \Gamma_1)(\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 - \Gamma_3\Gamma_1)z^2 + (\Gamma_1 + \Gamma_2)(-\Gamma_1\Gamma_2 + \Gamma_2\Gamma_3 + \Gamma_3\Gamma_1)z^2(z + 1)^2. \tag{63c}$$

This polynomial enters via

$$d(z) = 2\pi\gamma_2(\Lambda - \Omega)z^2(z + 1)^2\xi_3^2,$$

and its sign thus determines the second half of the condition of linear stability.

Next, we introduce a tool from algebraic geometry called the *resultant* (also sometimes *eliminant*). The resultant  $\rho(a, b)$  of two polynomials

$$a(z) = a_0 + a_1z + a_2z^2 + \dots + a_mz^m = a_m \prod_{j=1}^m (z - \alpha_j),$$

$$b(z) = b_0 + b_1z + b_2z^2 + \dots + b_nz^n = b_n \prod_{k=1}^n (z - \beta_k),$$

say, with roots  $\alpha_1, \dots, \alpha_m$  and  $\beta_1, \dots, \beta_n$ , respectively, is

$$\rho(a, b) = a_m^n b_n^m \prod_{j=1}^m \prod_{k=1}^n (\alpha_j - \beta_k).$$

Clearly, the resultant vanishes when, and only when, the polynomials have a common root. The *Sylvester matrix* of the two polynomials is given by the following construction (illustrated for  $m=4, n=3$ ):

$$S_{a,b} = \begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{pmatrix}.$$

The remarkable and powerful result is that  $\rho(a, b) = \det S_{a,b}$ . The theory is set out in many places. For a classic exposition see Chap. VIII of Ref. 20. Thus, the resultant of two polynomials, which tells us whether they have a common root or not, can be calculated without actually solving for the roots themselves! This is extremely useful in situations, such as ours, where the evaluation of the roots, although feasible, is very cumbersome and direct substitution into later formulas makes these all but impenetrable.

We calculate in succession the resultants of the polynomials  $\omega(z), s(z), d(z)$ , and  $p(z)$ . Direct computation, conveniently done with a symbolic manipulation package such as MATHEMATICA, which has a routine for calculating the resultant of two polynomials, shows that



$$\rho(p, \omega) = (\gamma_1 \gamma_2 - \gamma_3) \gamma_1^2 \gamma_2^4, \quad (64a)$$

$$\rho(p, s) = (\gamma_1 \gamma_2 - \gamma_3)(32\gamma_1^2 \gamma_2 - 3\gamma_2^2 - 36\gamma_1 \gamma_3), \quad (64b)$$

and

$$\rho(p, d) = 3(\gamma_1 \gamma_2 - \gamma_3) \gamma_2^5. \quad (64c)$$

In the preceding formulas,

$$\gamma_1 \gamma_2 - \gamma_3 = (\Gamma_1 + \Gamma_2)(\Gamma_2 + \Gamma_3)(\Gamma_3 + \Gamma_1). \quad (64d)$$

The resultants (64a)–(64c) only vanish in the  $\kappa$ -space of Fig. 2 along the curves that we have already identified in discussing the solutions of  $p(z)$ , viz., the circle  $\gamma_2=0$ , the cusped curve  $3\gamma_2^2+36\gamma_1\gamma_3=32\gamma_1^2\gamma_2$ , and the lines  $\Gamma_1+\Gamma_3=0$  and  $\Gamma_2+\Gamma_3=0$ . The case  $\gamma_1=0$ , not included in Fig. 2, is also singled out in the resultant of  $p(z)$  and  $\omega(z)$ .

To see how this information can be used, assume that we know the inequality  $\Lambda < \Omega$  to hold for a certain root  $z^{(p)}$  of  $p(z)$  at a special triple of circulations  $\Gamma^{(0)} = (\Gamma_1^{(0)}, \Gamma_2^{(0)}, \Gamma_3^{(0)})$ . Assume, further, that the resultant of  $p(z)$  and  $d(z)$  is nonzero throughout a region of  $(\Gamma_1, \Gamma_2, \Gamma_3)$ -space that includes the point  $\Gamma^{(0)}$ . Consider a second triple of circulations in this region different from  $\Gamma^{(0)}$ . We claim that also for this second triple must we have  $\Lambda < \Omega$ . To prove this, note that the inequality  $\Lambda < \Omega$  is equivalent to saying that  $d(z^{(p)})$  is either positive or negative (depending on the sign of  $\gamma_2$ ) for the vortex triple  $\Gamma^{(0)}$ . Since the resultant of  $p(z)$  and  $d(z)$  is nonzero everywhere in the particular domain in  $(\Gamma_1, \Gamma_2, \Gamma_3)$ -space,  $d(z)$  and  $p(z)$  cannot have a common root anywhere in that domain. In other words,  $d(z^{(p)})$ , considered as a function of the circulations, must be nonzero everywhere in the domain. Hence,  $\Lambda < \Omega$  must hold everywhere in the domain. In order for this inequality to “flip” for the root  $z^{(p)}$  of  $p(z)$  that we are following,  $d(z^{(p)})$  would have had to become zero, i.e.,  $z^{(p)}$  would have had to become a root of  $d(z)$  for some triple of circulations, which is ruled out by the nonvanishing of the resultant.

We see, then, that linear stability or instability of a family of collinear relative equilibria, corresponding to a certain root of  $p(z)$ , will be invariant throughout each of the regions “1,” “2,” “4a,” “4b,” “4c,” “4d,” “4e,” and “6” in Fig. 2. Hence, if we can establish stability or instability for just one configuration within each of these regions, we will have the same result throughout the region. The special solutions for  $(\Gamma_1, \Gamma_2, \Gamma_3) = (1, 1, r)\Gamma$  that we explored in Sec. IV C will prove very useful in this regard. According to the resultants, the polynomials  $\omega(z)$ ,  $s(z)$ , and  $d(z)$  may all have common roots with  $p(z)$  when  $\Gamma_2 + \Gamma_3 = 0$  or  $\Gamma_1 + \Gamma_3 = 0$ . The common roots in question are the “unphysical” values  $z=0$  and  $z=-1$ . Hence, we can anticipate singularities as we cross these lines in the diagram of Fig. 2, e.g., the angular velocity would be expected to diverge. We shall see that crossing these lines does not lead to a change in stability, but may lead to a reversal of the sense of rotation. Changes in stability do occur when we cross the circle  $\gamma_2=0$ . Along with the regions “1,” “2,” etc., the case  $\gamma_1=0$  and the curves  $\gamma_2=0$ ,  $3\gamma_2^2+36\gamma_1\gamma_3=32\gamma_1^2\gamma_2$ ,  $\Gamma_2+\Gamma_3=0$ , and  $\Gamma_1+\Gamma_3=0$ , and their points of intersection need to be considered separately.

The case  $\gamma_1=0$  is particularly simple. From our earlier analysis we know the collinear relative equilibrium for each triple of vortex circulations in this case rotates clockwise, i.e., that  $\Omega < 0$ . The condition to be checked with respect to linear stability (including marginal stability), then, is  $\Omega \leq \Lambda \leq -\Omega$ . Since the resultants of  $p(z)$  with  $s(z)$  and  $d(z)$  are both nonzero for  $\gamma_1=0$ , the collinear relative equilibrium must either always be stable or always unstable as the vortex strengths are varied subject to the constraint that they sum to zero. For the case  $(1, 1, -2)$ , as we have just seen, the relative equilibrium is linearly stable. Hence, all collinear relative equilibria with  $\gamma_1=0$  must be linearly stable.

We now turn to the various cases from Table I.

### A. All vortex circulations positive

This is “case 1” of Table I or region “1” of Fig. 2. There are three collinear relative equilibria that all rotate counterclockwise. Consider the relative equilibrium corresponding to any one of the roots  $z_n^{(p)}$ ,  $n=0, 1, 2$  of  $p(z)$  as the vortex circulations  $\Gamma_1$ ,  $\Gamma_2$ , and  $\Gamma_3$  are varied. Since  $0 < \Omega$  from Eq. (61b), the condition for linear stability (including marginal stability) is that  $-\Omega \leq \Lambda \leq \Omega$ . From Eq. (61c) we see that  $0 < \Lambda$  when all vortex circulations are positive. For three identical vortices we know from our analysis of the system  $(1, 1, r)$ , or directly from the explicit solution of this system,<sup>21</sup> that the three collinear relative equilibria are unstable, i.e., we have  $\Omega < \Lambda$  for a collinear relative equilibrium of three identical vortices. This inequality translates into  $0 \leq d(z_n^{(p)})$ ,  $n=0, 1, 2$  when  $\Gamma_1 = \Gamma_2 = \Gamma_3$ . Now, according to the resultant (64c),  $d(z_n^{(p)})$  cannot become 0 anywhere in region “1” of Fig. 2. Thus, we must have  $\Omega < \Lambda$  throughout this region of vortex circulations. It follows that all three collinear relative equilibria are unstable for all vortex circulation triples in this region.

We shall use the following short hand notation. A collinear relative equilibrium corresponding to the solution for  $n=0, 1, 2$  is designated as  $nc^\sigma$ . The character  $c$  is  $s$  for a (linearly) *stable*,  $m$  for a *marginally stable*, and  $u$  for an *unstable* configuration. The superscript  $\sigma$  is “+” if the configuration rotates counterclockwise ( $\Omega > 0$ ), “0” for a stationary configuration, and “−” for clockwise rotation. In region “1” of Fig. 2, then, we summarize the stability characteristics of the three collinear relative equilibria as  $0u^+$ ,  $1u^+$ ,  $2u^+$  or  $0$ ,  $1$ ,  $2u^+$ . This has been written in region “1” of Fig. 4.

### B. Two positive vortices and a weak negative vortex

By “weak” we mean “case 2” from Table I, i.e.,  $-\Gamma_1\Gamma_2/(\Gamma_1+\Gamma_2) < \Gamma_3 < 0$ . We can now have values of  $L$  of either sign. However, the collinear relative equilibria occur only for  $L > 0$ , i.e., the rotation is always counterclockwise. This follows analytically from Eq. (61b) since  $\Gamma_1 + \Gamma_3$  and  $\Gamma_2 + \Gamma_3$  are both positive when  $\gamma_2$  is positive. Alternatively, we could use that the resultant (64a) between  $p(z)$  and  $\omega(z)$  remains nonzero so long as  $\gamma_2 > 0$ . Hence,  $\omega(z^{(p)})$  for any

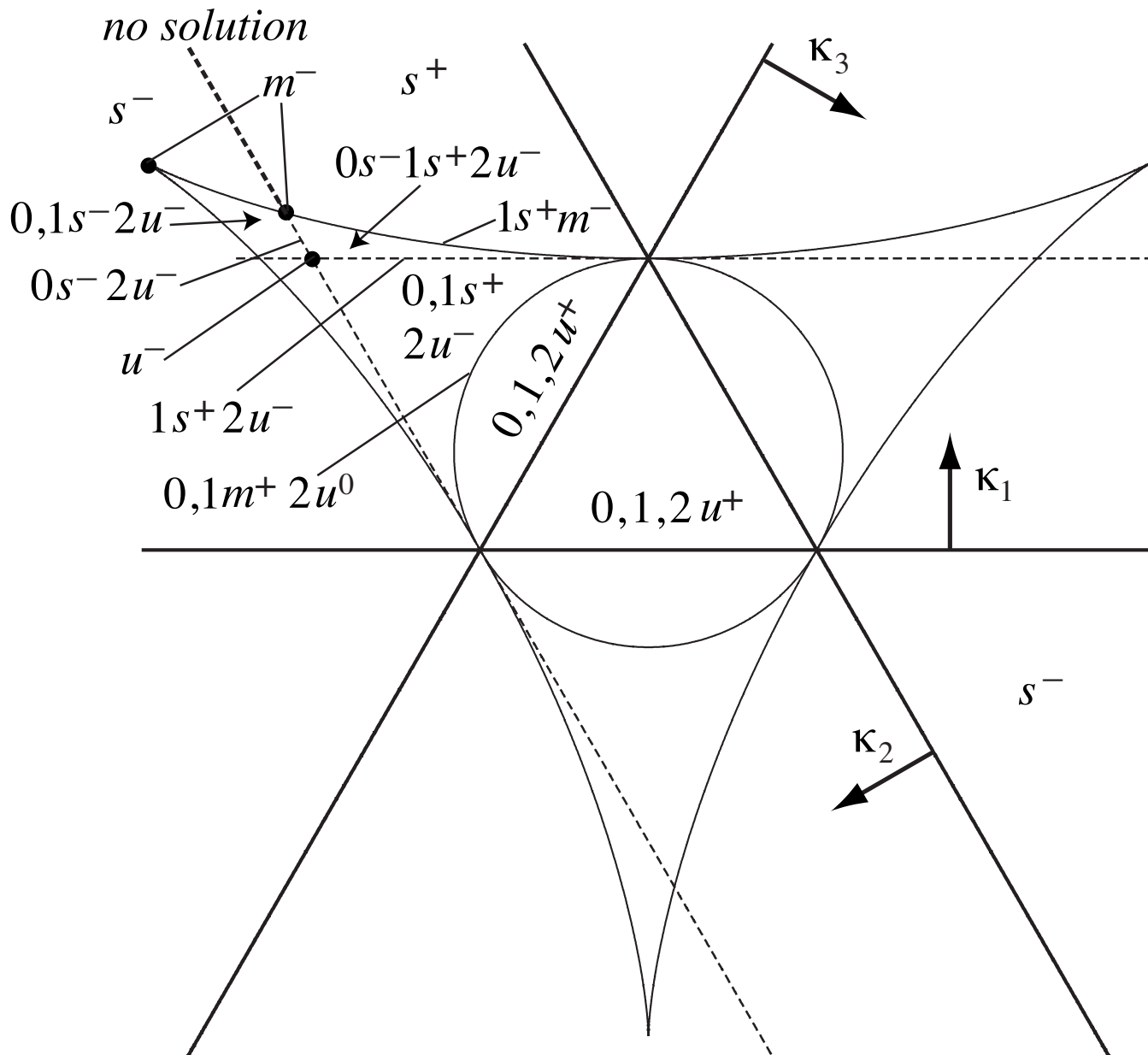


FIG. 4. Stability results for collinear relative equilibria in the trilinear  $(\kappa_1, \kappa_2, \kappa_3)$  diagram of Fig. 2. Symbols are  $s$  for “stable,”  $u$  for “unstable,” and  $m$  for “marginal.” Superscripts signify direction of rotation: “+” for counterclockwise rotation, “-” for clockwise, and “0” for a stationary equilibrium. When there is more than one equilibrium, the numerals 0, 1, and 2 refer to the indexing of the three real roots of the cubic determining the collinear relative equilibria. The portion of the line  $\kappa_2=1$  where there are no solutions is shown as bold and dashed and labeled “no solution.”

root  $z^{(p)}$  of  $p(z)$ , viewed as a function of the circulations, cannot become 0 so long as we stay inside the circle  $\gamma_2=0$  in Fig. 2. Since  $\Omega > 0$  for identical vortices,  $\Omega$  must remain positive as the circulations are varied in the region  $\gamma_2 > 0$ . In particular, the condition for linear stability continues to be  $-\Omega \leq \Lambda \leq \Omega$  for all three roots of  $p(z)$  throughout region “2.”

For a given root  $z_n^{(p)}$ ,  $n=0, 1, 2$  of  $p(z)$ , we start from any set of vortex circulations in region “1.” The corresponding collinear relative equilibrium is unstable because  $\Omega < \Lambda$ , as we saw in the previous subsection. According to the resultant (64c), however,  $d(z_n^{(p)})=0$  cannot be attained so long as  $\gamma_2 > 0$ . Hence,  $\Omega < \Lambda$  continues to hold throughout region “2.” All three collinear relative equilibria are unstable. In the notation introduced at the end of Sec. VII B, we have, once again,  $0, 1, 2u^+$ .

### C. The case $\gamma_2=0$

This is “case 3” in Table I. There are three collinear relative equilibria. In one the vortices are stationary. In the other two they rotate. The expressions (61b) and (61c) are not valid for this case.

The stationary equilibrium is given by Eq. (19). In the general solution of the cubic  $p(z)$  it corresponds to  $n=2$ :  $z_2^{(p)} = \xi_1 / \xi_3 = \Gamma_3 / \Gamma_1$ , as we saw in Sec. IV A. Since  $\Omega_2=0$ , the necessary condition for stability reduces to  $\Lambda_2=0$ . From the equations leading up to Eqs. (18b) and (19) we have

$$\Gamma_1 \xi_1 = \Gamma_2 \xi_2 = \Gamma_3 \xi_3 = -\gamma_1 \xi. \tag{65}$$

Hence, by direct substitution into Eq. (61a) we find, since  $\Omega_2=0$ , that  $\Lambda_2 = -3\gamma_3 / 2\pi\gamma_1^2 \xi^2$  and, thus, that  $\Lambda$  is positive.

The collinear relative equilibrium in Eq. (19) is linearly unstable.

The resultant of  $p(z)$  and  $d(z)$  varies as  $\gamma_2^5$  in the vicinity of  $\gamma_2=0$ . Unfortunately, the polynomial  $d(z)$  contains a factor  $\gamma_2$  so one cannot conclude from this that  $\Omega=\Lambda$  at a root of  $p(z)$ . (In the case of  $z_2^{(p)}$  one would then have concluded that  $\Lambda_2=\Omega_2=0$  which, as we have just seen, it is not.) We shall, nevertheless, show that  $\Omega_{0,1}=\Lambda_{0,1}$  but we have to proceed a bit differently.

Factoring out a term  $z-z_2^{(p)}=z-\Gamma_3/\Gamma_1$  in  $p(z)$  or, equivalently, setting  $L=0$ , gives the quadratic polynomial

$$(\Gamma_1 + \Gamma_2)z^2 + 2\Gamma_1z - \Gamma_2,$$

which has the roots

$$z_{0,1}^{(p)} = \frac{-\Gamma_1 \pm \Gamma_*}{\Gamma_1 + \Gamma_2}, \quad \Gamma_* = \sqrt{\Gamma_1^2 + \Gamma_1\Gamma_2 + \Gamma_2^2} = \sqrt{(\Gamma_1 + \Gamma_2)\gamma_1}.$$

We see that for the upper sign the root is positive. For the lower sign the root is less than  $-1$ . Hence, in both these relative equilibria vortex 3 is outside the line segment 12. It follows from this that the  $\Omega_{0,1}$  must both be positive, and these relative equilibria rotate counterclockwise.

Writing the polynomial equation for  $z$  in terms of  $\xi_1, \xi_2$ , and  $\xi_3$  gives

$$(\Gamma_1 + \Gamma_2)\xi_1^2 + 2\Gamma_1\xi_1\xi_3 - \Gamma_2\xi_3^2 = 0,$$

or, using Eq. (30),

$$\begin{aligned} \Gamma_1\xi_1(\xi_1 + 2\xi_3) + \Gamma_2(\xi_1^2 - \xi_3^2) \\ = \Gamma_1\xi_1(\xi_3 - \xi_2) - \Gamma_2\xi_2(\xi_1 - \xi_3) = 0. \end{aligned}$$

Thus,

$$(\Gamma_1\xi_1 + \Gamma_2\xi_2)\xi_3 = (\Gamma_1 + \Gamma_2)\xi_1\xi_2. \tag{66}$$

Consider, once again, Eqs. (43) defining the relative equilibrium

$$2\pi\Omega_{0,1x_1} = \frac{\Gamma_2}{\xi_3} - \frac{\Gamma_3}{\xi_2}, \quad 2\pi\Omega_{0,1x_2} = \frac{\Gamma_3}{\xi_1} - \frac{\Gamma_1}{\xi_3}, \tag{43'}$$

$$2\pi\Omega_{0,1x_3} = \frac{\Gamma_1}{\xi_2} - \frac{\Gamma_2}{\xi_1}.$$

Subtracting the second of these from the first gives

$$2\pi\Omega_{0,1} = \frac{\Gamma_1 + \Gamma_2}{\xi_3^2} - \frac{\Gamma_3}{\xi_3} \left[ \frac{1}{\xi_1} + \frac{1}{\xi_2} \right] = \frac{\Gamma_1 + \Gamma_2}{\xi_3^2} + \frac{\Gamma_3}{\xi_1\xi_2}.$$

Thus, using Eq. (66),

$$\begin{aligned} 2\pi\Omega_{0,1}(\Gamma_1\xi_1 + \Gamma_2\xi_2) &= \frac{\Gamma_1 + \Gamma_2}{\xi_3^2}(\Gamma_1\xi_1 + \Gamma_2\xi_2) + \frac{\Gamma_3}{\xi_3}(\Gamma_1 + \Gamma_2) \\ &= \frac{\Gamma_1 + \Gamma_2}{\xi_3^2}(\Gamma_1\xi_1 + \Gamma_2\xi_2 + \Gamma_3\xi_3). \end{aligned}$$

By permutation of indices, summation over the three resulting equations, and using Eq. (61a), we obtain

$$\begin{aligned} 2\Omega_{0,1}(\Gamma_1\xi_1 + \Gamma_2\xi_2 + \Gamma_3\xi_3) &= (\Lambda_{0,1} + \Omega_{0,1}) \\ &\quad \times (\Gamma_1\xi_1 + \Gamma_2\xi_2 + \Gamma_3\xi_3). \end{aligned}$$

Now, Eq. (66) assures us that  $\Gamma_1\xi_1 + \Gamma_2\xi_2 + \Gamma_3\xi_3 \neq 0$ . For if this quantity did vanish, Eq. (66) would become  $-\Gamma_3\xi_3^2 = (\Gamma_1 + \Gamma_2)\xi_1\xi_2$ . Since the left hand side is positive, this would imply that  $\xi_1 = x_2 - x_3$  and  $\xi_2 = x_3 - x_1$  have the same sign. But when vortex 3 is outside the line segment 12, these two position differences always have opposite signs. Hence,  $2\Omega_{0,1} = \Lambda_{0,1} + \Omega_{0,1}$ , i.e.,  $\Lambda_{0,1} = \Omega_{0,1}$ , and the relative equilibria corresponding to the roots  $z_{0,1}^{(p)}$  are both marginally stable.

Again, from Eq. (43') we get

$$\begin{aligned} 2\pi\Omega_{0,1} &= \frac{\Gamma_1 + \Gamma_2}{\xi_3^2} + \frac{\Gamma_3}{\xi_1\xi_2} = \frac{\Gamma_2 + \Gamma_3}{\xi_1^2} + \frac{\Gamma_1}{\xi_2\xi_3} \\ &= \frac{\Gamma_3 + \Gamma_1}{\xi_2^2} + \frac{\Gamma_2}{\xi_3\xi_1}. \end{aligned}$$

Adding these we obtain

$$\begin{aligned} 3\Omega_{0,1} &= \frac{1}{2\pi} \left[ \frac{\Gamma_1 + \Gamma_2}{\xi_3^2} + \frac{\Gamma_2 + \Gamma_3}{\xi_1^2} + \frac{\Gamma_3 + \Gamma_1}{\xi_2^2} \right] \\ &\quad + \frac{\Gamma_1\xi_1 + \Gamma_2\xi_2 + \Gamma_3\xi_3}{2\pi\xi_1\xi_2\xi_3}. \end{aligned}$$

However, according to Eq. (61a) the first term is just  $\Lambda_{0,1} + \Omega_{0,1}$ , i.e.,  $2\Omega_{0,1}$  by the marginal stability result we have just obtained, and we have the formula

$$\Omega_{0,1} = \frac{\Gamma_1\xi_1 + \Gamma_2\xi_2 + \Gamma_3\xi_3}{2\pi\xi_1\xi_2\xi_3} = \frac{1}{2\pi} \left[ \frac{\Gamma_1}{\xi_2\xi_3} + \frac{\Gamma_2}{\xi_3\xi_1} + \frac{\Gamma_3}{\xi_1\xi_2} \right]. \tag{67}$$

We have already seen that these two angular frequencies are positive.

These results could have been anticipated, since the two relative equilibria we are considering arise as part of the analysis of *vortex collapse*.<sup>2,4</sup> We remind the reader of the construction of initial conditions for such motions given in Fig. 8 of Ref. 4. Place vortices 1 and 2 on the  $x$ -axis. Make their center of vorticity the origin of coordinates. Construct an equilateral triangle with base 12. Construct a circle, centered at the origin and passing through the third vertex of the equilateral triangle. This circle intersects the  $x$ -axis in two points, labeled  $P_1$  and  $P_3$  in Fig. 8 of Ref. 4. These two points correspond to  $z_{0,1}$ . This identification again shows that  $\Omega_{0,1}^{(p)}$  must be positive. If we displace vortex 3 infinitesimally upward or downward perpendicular to the line through the vortices, we get self-similar expansion or collapse according to the discussion in Ref. 4, i.e., in both cases a slow (nonexponential) departure from the relative equilibrium configuration. In the linearized stability analysis this shows up as marginal stability. The equilateral triangle relative equilibrium for  $\gamma_2=0$  is part of this same family of initial conditions and, for similar reasons, is also marginally stable in linear theory, as we saw in Sec. VI.

In summary, there are three collinear relative equilibria for  $\gamma_2=0$ . One is stationary, while in the other two the vortices rotate counterclockwise. In the stationary configuration

vortex 3 is situated between vortices 1 and 2. In the two rotating configurations vortex 3 is on the line 12 extended, in one case beyond vortex 1, in the other beyond vortex 2. The stationary configuration is unstable. The two rotating configurations are marginally stable. In our short hand notation we have along the curve  $\gamma_2=0$  in vortex circulation space  $0, 1m^+2u^0$ .

**D. Two positive vortices and a comparable negative vortex**

By “comparable” we mean “case 4” from Table I,  $-(\Gamma_1 + \Gamma_2) < \Gamma_3 < -\Gamma_1\Gamma_2/(\Gamma_1 + \Gamma_2)$ . This is the most complicated parameter regime with several subcases that must be considered.

Let us first explore the regions labeled “4a,” “4b,” and “4c” in Fig. 2. For these regions the solutions for the case  $(1, 1, r)$  are “representative,” in the sense that the values of the resultants of  $p(z)$  with  $\omega(z)$ ,  $s(z)$ , and  $d(z)$  for any point in one of these regions do not become 0 along a curve separating the point in question from the corresponding point  $(1, 1, r)$ . Hence, the sense of rotation of a collinear relative equilibrium and the signs of  $\Lambda \pm \Omega$  are the same as for the  $(1, 1, r)$ -configurations within that region of parameters. In particular, in region “4a” and region “4b” the collinear relative equilibrium corresponding to the root  $z_2^{(p)}$  rotates clockwise and is unstable. In region “4c” this relative equilibrium still rotates clockwise but has become stable. The relative equilibria corresponding to the roots  $z_{0,1}^{(p)}$  rotate counterclockwise and are stable in region “4a.” In region “4b” they rotate clockwise and are again stable. Summarizing using our short hand notation, we have  $0, 1s^+2u^-$  in region “4a,”  $0, 1s^-2u^-$  in “4b,” and  $s^-$  in “4c,” where there is just one collinear relative equilibrium.

Next, let us move to the line  $\Gamma_2 + \Gamma_3 = 0$  or, equivalently, the system  $(1, r, -r)$ ,  $0 < r < 1$ , already considered in Sec. IV A. The polynomial  $p(z)$  has the root  $z_0^{(p)} = 0$  and in the current notation reduces to

$$p(z) = z[(1+r)z^2 + (2+r)z + r].$$

The two other “physical” roots are, as we saw in Sec. IV A,

$$z_1^{(p)} = \frac{-(2+r) - \sqrt{4-3r^2}}{2(1+r)}, \quad z_2^{(p)} = \frac{-(2+r) + \sqrt{4-3r^2}}{2(1+r)}. \tag{68}$$

The identification with the general solutions  $z_n^{(p)}$  is made by considering the limiting values for  $r=1$ . In the general solution for  $Z_n$ , Eq. (40), we have for  $r=1$  that  $H=-\frac{1}{3}$ ,  $G=0$  and thus  $\theta=\frac{\pi}{2}$ . Then,  $Z_0=1$ ,  $Z_1=-1$ , and  $Z_2=0$ . Since, according to Eq. (38a),  $z=\frac{1}{2}(Z-1)$  when  $r=1$ , we have  $z_0^{(p)}=0$ ,  $z_1^{(p)}=-1$ , and  $z_2^{(p)}=-\frac{1}{2}$ . The roots of  $p(z)$  must have these limiting values for  $r \rightarrow 1$ . Hence, the identification of index values  $n$  in Eq. (68). The root  $z_1^{(p)}$  varies from  $-2$  to  $-1$  as  $r$  varies from 0 to 1. Vortex 3 is outside the line segment 12 on the side of vortex 1. The root  $z_2^{(p)}$  varies from 0 to  $-\frac{1}{2}$ . Vortex 3 is between vortices 1 and 2.

From Eqs. (61b) and (61c), noting that  $\gamma_2=-r^2$  in this case, we find

$$\Omega_1 = \frac{\Gamma}{4\pi\xi_2^2} \frac{-r + \sqrt{4-3r^2}}{1+r} = -\frac{\Gamma(z_1^{(p)} + 1)}{2\pi\xi_2^2},$$

$$\Omega_2 = -\frac{\Gamma}{4\pi\xi_2^2} \frac{r + \sqrt{4-3r^2}}{1+r} = -\frac{\Gamma(z_2^{(p)} + 1)}{2\pi\xi_2^2},$$

$$\Lambda_1 = \frac{\Gamma}{4\pi\xi_2^2} (4 - 3r - \sqrt{4 - 3r^2}),$$

$$\Lambda_2 = \frac{\Gamma}{4\pi\xi_2^2} (4 - 3r + \sqrt{4 - 3r^2}).$$

Thus,  $\Omega_1 > 0$ , and it is not difficult to verify that  $0 < \Lambda_1 \leq \Omega_1$ . The corresponding collinear relative equilibrium is linearly stable. On the other hand,  $\Omega_2 < 0$ , and one finds that  $-\Omega_2 < \Lambda_2$ . The corresponding collinear relative equilibrium is linearly unstable. As we step onto the line  $\Gamma_2 + \Gamma_3 = 0$  in Fig. 4 from region “4a” to “4d,” we “lose” the collinear relative equilibrium corresponding to  $n=0$ . The corresponding angular frequency of rotation,  $\Omega_0$ , diverges since  $z_0^{(p)} \rightarrow 0$  implies that vortices 2 and 3 come arbitrarily close. However, the sense of rotation and stability of the two remaining configurations  $n=1, 2$  is unchanged from region “4a.” Thus, on the line segment  $(1, r, -r)$ ,  $0 < r < 1$ , in Fig. 4 we have  $1s^+2u^-$ .

As  $r \rightarrow 0$  and  $z_2^{(p)} \rightarrow 0$ , vortices 2 and 3 approach as they weaken to two passively advected particles. There is no singularity in  $\Omega_2$ ; also  $\Omega_1 \rightarrow \Gamma/2\pi\xi_2^2$ , the orbital frequency of the passive particles about vortex 1. As  $r \rightarrow 1$  and  $z_1^{(p)} \rightarrow -1$ , vortices 1 and 3 approach as the circulations tend to  $(1, 1, -1)\Gamma$ . Since  $z_1^{(p)} + 1 = -\xi_2/\xi_3$ , cf. Eq. (36b), the angular velocity of the configuration,  $\Omega_1$ , diverges. In this limit  $\Omega_2 \rightarrow -\Gamma/4\pi\xi_2^2$ , the angular frequency of the  $(1, 1, -1)\Gamma$  collinear relative equilibrium. Hence, it is the  $n=2$  solution that converges to this configuration, just as it was for the case  $(1, 1, r)\Gamma$  in region “4a.”

Similarly, along the line  $\Gamma_1 + \Gamma_3 = 0$ , or equivalently for the system  $(r, 1, -r)$ , the polynomial  $p(z)$  has the root  $z_1^{(p)} = -1$  and reduces to

$$p(z) = (z+1)[(1+r)z^2 + rz + r-1].$$

The two other physical roots are, as we saw in Sec. IV A,

$$z_0^{(p)} = \frac{-r + \sqrt{4-3r^2}}{2(1+r)}, \quad z_2^{(p)} = \frac{-r - \sqrt{4-3r^2}}{2(1+r)}. \tag{69}$$

We must, thus, limit the consideration to  $1 < r < 2/\sqrt{3}$ , the lower limit due to our general convention  $\Gamma_2 \leq \Gamma_1$  (but we avoid  $r=1$ , which is a degenerate point), the upper to have real solutions in Eq. (69). The identification with the general solutions  $z_n^{(p)}$  is made by considering the limiting values for  $r=1$ . Clearly,  $z_0^{(p)}$  and  $z_2^{(p)}$  have the correct limiting values 0 and  $-\frac{1}{2}$ , respectively. The root  $z_1^{(p)}$  varies from 0 to  $\sqrt{3}-2$  as  $r$  varies from 1 to  $2/\sqrt{3}$ . The root  $z_2^{(p)}$  varies from  $-\frac{1}{2}$  to  $\sqrt{3}-2$ . In both cases vortex 3 is between vortices 1 and 2.

Note that for vortex circulations  $(r, 1, -r)$ , we have  $\gamma_1 = 1$ ,  $\gamma_2 = \gamma_3 = -r^2$ . Hence  $32\gamma_1^2\gamma_2 - 3\gamma_2^2 - 36\gamma_1\gamma_3 = r^2(4-3r^2)$ .

Thus, when  $r=2/\sqrt{3}$ , the point  $(r, 1, -r)$  is on the cusped boundary in the vortex circulation diagram, Fig. 2.

From Eqs. (61b) and (61c) we find

$$\Omega_0 = \frac{\Gamma}{4\pi\xi_1^2} \frac{-r + \sqrt{4-3r^2}}{1+r} = \frac{\Gamma z_0^{(p)}}{2\pi\xi_1^2},$$

$$\Omega_2 = -\frac{\Gamma}{4\pi\xi_1^2} \frac{r + \sqrt{4-3r^2}}{1+r} = \frac{\Gamma z_2^{(p)}}{2\pi\xi_1^2},$$

$$\Lambda_0 = \frac{\Gamma}{4\pi\xi_1^2} (4-3r - \sqrt{4-3r^2}),$$

$$\Lambda_2 = \frac{\Gamma}{4\pi\xi_1^2} (4-3r + \sqrt{4-3r^2}).$$

Then,  $\Omega_0 < 0$ , and an elementary calculation shows that  $0 < \Lambda_0 \leq -\Omega_0$ . The corresponding collinear relative equilibrium is linearly stable. Again,  $\Omega_2 < 0$ , and  $-\Omega_2 < \Lambda_2$ . The corresponding collinear relative equilibrium is linearly unstable. As we step onto the line  $\Gamma_1 + \Gamma_3 = 0$  in Fig. 2 from region “4b” to “4d,” we “lose” the collinear relative equilibrium corresponding to  $n=1$ . The corresponding angular frequency of rotation  $\Omega_1$  diverges since  $z_1^{(p)} \rightarrow -1$  implies that vortices 1 and 3 come arbitrarily close. The sense of rotation and stability of the two remaining configurations  $n=0, 2$  is unchanged from region “4c.” Thus, on the line segment  $(r, 1, -r)$ ,  $1 < r < 2/\sqrt{3}$  in Fig. 2 we have  $0s^-2u^-$ .

As  $r \rightarrow 1$  and  $z_0^{(p)} \rightarrow 0$ , vortices 2 and 3 approach and the vortex circulations tend to  $(1, 1, -1)\Gamma$ . There is a singularity in  $\Omega_0$ ; it diverges as  $\xi_1^{-1}$  since  $z_0^{(p)} = \xi_1/\xi_3$ . In this same limit  $z_2^{(p)} \rightarrow -\frac{1}{2}$  and  $\Omega_2 \rightarrow \Gamma/4\pi\xi_1^2$ , the angular frequency of the  $(1, 1, -1)\Gamma$  collinear relative equilibrium. Again, it is the  $n=2$  solution that converges to this configuration and the  $n=0$  solution becomes singular.

As  $r \rightarrow 2/\sqrt{3}$ , the two solutions  $z_0^{(p)}$  and  $z_2^{(p)}$  approach one another, and they fuse to one solution,  $z_{0,2}^{(p)} = \sqrt{3}-2$ , when the cusped curve is reached. For this solution we see that  $\Omega_0$  and  $\Omega_2$ , respectively,  $\Lambda_0$  and  $\Lambda_2$ , take on common values  $\Omega < 0$  and  $\Lambda$ , respectively, which satisfy

$$\Omega = -\frac{\Gamma}{4\pi\xi_1^2} \frac{2}{2+\sqrt{3}} = -\frac{\Gamma}{2\pi\xi_1^2} (2-\sqrt{3}) = -\Lambda.$$

The fused solution thus rotates clockwise and is marginally stable. We designate it  $m^-$ .

In region “4d” we again have all three solutions  $n=0, 1, 2$ . Within region “4d” the sense of rotation and stability or instability of the relative equilibria for a given  $n$  is independent of circulations since the resultants of  $\omega(z)$ ,  $s(z)$ , and  $d(z)$  with  $p(z)$  are all nonzero in this region. We do not have a representative configuration of the type  $(1, 1, r)$  here, however, so we reason by continuity from what happens in regions “4a” and “4b” and on the bounding lines separating these regions from “4d.” By continuity across the line  $\Gamma_1 + \Gamma_3 = 0$  for  $n=0$ , and across  $\Gamma_2 + \Gamma_3 = 0$  for  $n=1$ , we have in region “4d” that the collinear relative equilibrium corresponding to  $n=0$  rotates clockwise and is linearly stable, and that for  $n=1$  rotates counterclockwise and is also stable. The

configuration for  $n=2$  rotates clockwise and is unstable, regardless of which region we use to cross into region “4d.” In order for an angular frequency to change sign, it would have had to become 0, or have a singularity, on a bounding line. Thus, we have continuity of sign of angular velocity for the  $n=0$  solution as we go from region “4b” to “4d,” but a change in sign—and, indeed, a singularity—as we go from “4a” to “4d.” Similarly, for the  $n=1$  solution we have the same sense of rotation in regions “4a” and “4d” but different senses—and a singularity along the bounding line—in regions “4b” and “4d.” The stability criteria are continuous in the same way. In order for one or the other of the bounding inequalities to change, we must have marginal stability on the bounding line leading to region “4d.” In conclusion, in region “4d” we have  $0s^-1s^+2u^-$ .

The system  $(1, 1, -1)$  is quite special since for this vortex triple there is just one collinear relative equilibrium. From the explicit solution for this system<sup>2,4</sup> we know that it is unstable. One can, in fact, explore the full nonlinear evolution, including the important problem of scattering of a vortex pair by a single vortex.<sup>4</sup> We already found that this relative equilibrium is unstable in our analysis of the case  $(1, 1, r)$ . This follows immediately from Eqs. (61b) and (61c), with the common circulation of vortices 1 and 2 designated as  $\Gamma$ ,

$$\Omega = -\frac{\Gamma}{\pi\xi_3^2}, \quad \Lambda = \frac{2\Gamma}{\pi\xi_3^2}.$$

The configuration rotates clockwise. The condition for stability is  $\Omega \leq \Lambda \leq -\Omega$ . Since  $\Lambda = -2\Omega$ , this relative equilibrium is unstable. In our short hand notation, at the point  $(1, 1, -1)$  in Fig. 4 we have  $u^-$ .

On the cusped curve  $32\gamma_1^2\gamma_2 = 3\gamma_2^2 + 36\gamma_1\gamma_3$  the three solutions of the cubic  $p(z)$  have reduced to two. Since the discriminant of the cubic is 0, and  $G$ , Eq. (38c), may be shown to be positive in region “4” for  $\Gamma_2 < \Gamma_1$ , we have in Eq. (40) that  $\cos \theta = -1$ , i.e.,  $\theta = \pi$ . Hence,  $z_0^{(p)} = z_2^{(p)}$ . On the piece of the cusped curve for  $0 < \Gamma_2 \leq (\sqrt{3}/2)\Gamma_1$ , i.e., between the vertex  $(1, 0, 0)$  in the circulation diagram of Fig. 2 and the intersection of this curve with the line  $\Gamma_1 + \Gamma_3 = 0$ , the fusion of the solutions for  $n=0$  and  $n=2$  is also clear by considering the sense of rotation of the corresponding relative equilibria.

Indeed, it follows from the vanishing of the resultant of  $s(z)$  and  $p(z)$  on the cusped curve that for at least one of the roots of  $p(z)$  we must have marginal stability,  $\Lambda = -\Omega$ . For  $n=0$  we have in both regions “4b” and “4d” that  $\Omega_0 < 0$ , and since the configurations are stable, the condition for linear stability  $\Omega_0 \leq \Lambda_0 \leq -\Omega_0$  is satisfied. For the solution  $n=2$  we also have  $\Omega_2 < 0$ . However, the relative equilibrium corresponding to this root is unstable. From our earlier analyses, we have in both regions that the condition for stability is violated by  $-\Omega_2 < \Lambda_2$ . As we step onto the cusped curve from region “4b” or region “4d,” and the solutions  $z_0^{(p)}$  and  $z_2^{(p)}$  fuse, we have by continuity that  $\Omega_0 = \Omega_2$  and  $\Lambda_0 = \Lambda_2$ . Hence, for the fused solution we must have  $-\Omega = \Lambda$  along the entire cusped curve. In our short hand notation we have  $1s^+m^-$ .

The point  $(1, 1, -\frac{5}{4})$  in Fig. 2 is again quite special. As we saw in Sec. IV C, at this point the coefficients  $H$  and  $G$  in the cubic equation (38b) vanish. Hence, all three roots of the cubic coalesce. The resulting single collinear relative equilibrium is marginally stable. This follows either by considering that it is the fusion of two stable and one unstable collinear relative equilibria, or by calculating  $\Omega$  and  $\Lambda$  directly and finding  $\Lambda = -\Omega = 3\Gamma/2\pi\xi_3^2$ . At the point  $(1, 1, -\frac{5}{4})$  we have  $m^-$ .

In region “4c” we have just one collinear relative equilibrium. The collinear relative equilibrium that we found for  $(1, 1, r)$  is representative since the resultants of  $\omega(z)$ ,  $s(z)$ , and  $d(z)$  with  $p(z)$  are all nonzero. We have  $s^-$  in this region.

On the line  $\Gamma_1 = -\Gamma_3$ , separating regions “4c” and “4e,” there are no collinear relative equilibria, since the cubic  $p(z)$  only has the real root  $-1$ , and this value would correspond to vortices 1 and 3 having the same position. The parameter range  $(2/\sqrt{3})\Gamma_2 < \Gamma_1 = -\Gamma_3$  (with our convention on relative magnitudes of the three circulations) is the only one for which there is no collinear relative equilibrium.

In region “4e” the single real solution of the cubic  $p(z)$  is the one that connects smoothly to the  $n=1$  solution in region “4d.” The solutions for  $n=0$  and  $n=2$  have fused on the cusped curve and then lifted off the real axis. Since the direction of rotation cannot change, because the resultant of  $\omega(z)$  and  $p(z)$  is nonzero across the cusped curve, the collinear relative equilibrium in region “4e” rotates counterclockwise,  $0 < \Omega$ . The condition for stability is then  $-\Omega \leq \Lambda \leq \Omega$ . Since we had  $\Lambda_1 < \Omega_1$  in region “4d,” and since the resultant of  $d(z)$  and  $p(z)$  is nonzero across the cusped curve, we must still have  $\Lambda < \Omega$  in region “4e.”

We now argue that  $-\Omega < \Lambda$  for the collinear relative equilibrium in region “4e.” Let  $z^{(p)}$  be the root of  $p(z)$  that determines the single, collinear relative equilibrium in regions “4c” and “4e.” We know that  $s(z^{(p)}) < 0$  in region “4c” since  $\Omega < 0$  and the relative equilibrium is stable,  $\Lambda < -\Omega$ . We also know that on the line  $\Gamma_1 + \Gamma_3 = 0$  the root  $z^{(p)} = -1$ . If we expand  $s(z)$ , Eq. (63b), about  $z = -1$ , we find to leading order  $s(z) = \Gamma_1 + \Gamma_3$ . Hence, as we cross the line  $\Gamma_1 + \Gamma_3 = 0$ , the sign of  $s(z^{(p)})$  changes. In region “4e,” then, we have  $s(z^{(p)}) > 0$ , initially in the immediate vicinity of the line  $\Gamma_1 + \Gamma_3 = 0$ , but then everywhere because the resultant of  $s(z)$  with  $p(z)$  is nonzero everywhere in region “4e.” Hence,  $0 < \Omega + \Lambda$  in region “4e,” and we have the condition for stability  $-\Omega < \Lambda < \Omega$ . We conclude that in region “4e” we have  $s^+$ .

Another and operationally simpler approach is to pick a set of circulations in region “4e,” e.g.,  $(3, 1, -2)$ , determine  $z^{(p)}$  (a high accuracy numerical determination suffices, in this case  $z^{(p)} \approx -1.32673$ ), and evaluate  $\omega(z^{(p)})$ ,  $s(z^{(p)})$ , and  $d(z^{(p)})$ . One finds  $\omega(z^{(p)}) \approx -8.09 < 0$ ,  $s(z^{(p)}) \approx 2.41 > 0$ , and  $d(z^{(p)}) \approx 4.16 > 0$ . In view of the negative prefactor  $\gamma_2$  in the expressions for  $\omega(z)$  and  $d(z)$ , Eqs. (63a) and (63c), respectively, but not in  $s(z)$ , these inequalities translate to  $0 < \Omega$ ,  $\Lambda - \Omega < 0$ , and  $0 < \Lambda + \Omega$ . By the resultants being nonzero in region “4e” these results are extended to the entire region.

Either way we conclude that in region “4e” we have  $s^+$ .

TABLE II. Stability of collinear relative equilibria of three vortices as a function of circulations,  $\Gamma_3 \leq \Gamma_2 \leq \Gamma_1$ .

Regime defined by $(\Gamma_1, \Gamma_2, \Gamma_3)$	$\gamma_1$	$\gamma_2$	Stability code
$\Gamma_3 > 0$	$> 0$	$> 0$	$0, 1, 2u^+$
$-\Gamma_1\Gamma_2/(\Gamma_1+\Gamma_2) < \Gamma_3 < 0$	$> 0$	$> 0$	$0, 1, 2u^+$
$\Gamma_3 = -\Gamma_1\Gamma_2/(\Gamma_1+\Gamma_2)$	$> 0$	$= 0$	$0, 1m^+2u^0$
$-\Gamma_2 < \Gamma_3 < -\Gamma_1\Gamma_2/(\Gamma_1+\Gamma_2)$	$> 0$	$< 0$	$0, 1s^+2u^-$
$\Gamma_3 = -\Gamma_2$	$> 0$	$< 0$	$1s^+2u^-$
$\Gamma_1 = \Gamma_2 = -\Gamma_3$	$> 0$	$< 0$	$u^-$
$(\sqrt{3}/2)\Gamma_1 < \Gamma_2, \Gamma_3 = -\Gamma_1$	$> 0$	$< 0$	$0s^-2u^-$
$\Gamma_3 < -\Gamma_1, 3\gamma_2^2 + 36\gamma_1\gamma_3 < 32\gamma_1^2\gamma_2$	$> 0$	$< 0$	$0, 1s^-2u^-$
$-\Gamma_1 < \Gamma_3 < -\Gamma_2, 3\gamma_2^2 + 36\gamma_1\gamma_3 < 32\gamma_1^2\gamma_2$	$> 0$	$< 0$	$0s^-1s^+2u^-$
$3\gamma_2^2 + 36\gamma_1\gamma_3 = 32\gamma_1^2\gamma_2$	$> 0$	$< 0$	$1s^+m^-$
$\Gamma_3 = -\frac{5}{4}\Gamma_1, \Gamma_2 = \Gamma_1$	$> 0$	$< 0$	$m^-$
$\Gamma_3 = -\Gamma_1, \Gamma_2 = (\sqrt{3}/2)\Gamma_1$	$> 0$	$< 0$	$m^-$
$-\Gamma_1 < \Gamma_3, 3\gamma_2^2 + 36\gamma_1\gamma_3 > 32\gamma_1^2\gamma_2$	$> 0$	$< 0$	$s^+$
$\Gamma_3 < -\Gamma_1, 3\gamma_2^2 + 36\gamma_1\gamma_3 > 32\gamma_1^2\gamma_2$	$> 0$	$< 0$	$s^-$
$\Gamma_3 = -\Gamma_1, \Gamma_2 < (\sqrt{3}/2)\Gamma_1$	$> 0$	$< 0$	No solution
$\Gamma_3 = -(\Gamma_1 + \Gamma_2)$	$= 0$	$< 0$	$s^-$
$\Gamma_3 < -(\Gamma_1 + \Gamma_2)$	$< 0$	$< 0$	$s^-$

**E. The case  $\gamma_1 = 0$**

There is just one collinear relative equilibrium for  $\gamma_1 = 0$ , since  $32\gamma_1^2\gamma_2 - 3\gamma_2^2 - 36\gamma_1\gamma_3 = -3\gamma_2^2 < 0$ . We have already argued that this collinear relative equilibrium is always linearly stable. We did so by “continuity” from the case  $(1, 1, r)$  using the resultants. We may give a different argument based directly on the formulas (61b) and (61c). We find that

$$\Omega = -\frac{\gamma_3}{2\pi\gamma_2} \left[ \frac{1}{\xi_1^2} + \frac{1}{\xi_2^2} + \frac{1}{\xi_3^2} \right], \quad \Lambda = \frac{1}{2\pi\gamma_2} \left[ \frac{\Gamma_1^3}{\xi_1^3} + \frac{\Gamma_2^3}{\xi_2^3} + \frac{\Gamma_3^3}{\xi_3^3} \right].$$

We know from the solution (16a), or from Eq. (33), that  $\xi_1 : \xi_2 : \xi_3 = \Gamma_1 : \Gamma_2 : \Gamma_3$ . Hence,  $\Lambda = 0$ . Setting  $\xi_\alpha = \Gamma_\alpha \xi$ , for  $\alpha = 1, 2, 3$ , we again obtain Eq. (16b) for the angular frequency. As we saw then,  $\Omega < 0$  and the condition for linear stability is  $\Omega \leq \Lambda \leq -\Omega$ , which is always satisfied when  $\Lambda = 0$ . In our short hand notation we have  $s^-$ .

**F. Two positive vortices and a strong negative vortex**

By “strong” we mean “case 6” from Table I,  $\Gamma_3 < 0$ , and both  $\gamma_1$  and  $\gamma_2$  are negative. We know that there is just one collinear relative equilibrium, that the special solution for  $(1, 1, r)$  rotates counterclockwise and is stable, and that the resultants of  $s(z)$  and  $d(z)$  with  $p(z)$  are nonzero within the entire parameter region. Hence, we know that the special solution is representative and that we have  $s^-$  throughout this region.

**G. Stability data and diagram**

We summarize this lengthy discussion of stability of collinear relative equilibria by the diagram in Fig. 4 and the results in Table II. The various curves from Fig. 2 have been reproduced in Fig. 4, but in each region, along each special curve, and at each special point the stability results obtained have been written in. Arrows point to regions. Lines ending

on curves show the stability properties on that curve. Similarly, lines ending at special points show the stability properties at that point. The data used to construct Fig. 4 are summarized in Table II.

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