# Gauge Fields, Membranes, and Subdeterminant Vector Models

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We present a class of classically marginal $N$-vector models in $d = 4$ and $d = 3$ whose scalar potentials can be written as subdeterminants of symmetric matrices. The $d = 3$ case can be thought of as a generalization of the scalar sector of the Bagger-Lambert-Gustavsson model. Using the Hubbard-Stratonovich transformation we calculate their effective potentials which exhibit intriguing large-$N$ scaling behaviors. We comment on the possible relevance of our models to strings, membranes, and also to a class of novel spin systems that are based on ternary commutation relations.

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The relationship between $D3$ branes and 4D Yang-Mills theories is a fundamental ingredient in our current understanding of string theory. One of the simplest manifestations of such a relationship arises in the study of the scalar part of the action of $\mathcal{N} = 4$ $U(N)$ super-Yang-Mills (SYM) theory

$$I = \int d^4x\, \mathrm{Tr}\left(\frac{1}{2}\partial_\mu \Phi^I \partial_\mu \Phi^I - \frac{1}{4}[\Phi^I, \Phi^J]^2\right), \tag{1}$$

where $\Phi^I$ are matrix fields $\Phi^I = \Phi^I_\alpha T^\alpha$, $I = 1, 2, \ldots, 6$, and $T^\alpha$, $\alpha = 1, 2, \ldots, N^2$ are the adjoint generators of $U(N)$. Expanding around the Coulomb vacuum, i.e., the $N$ Cartan generators $\bar{\Phi}^I_a$, $a = 1, 2, \ldots, N$, of $U(N)$ one obtains an effective potential of the form [1]

$$V_{\mathrm{eff}} \sim \sum_{a<b} m^4_{ab} \ln\frac{m^2_{ab}}{\Lambda^2}, \qquad a, b = 1, 2, \ldots, N, \tag{2}$$

where $m^2_{ab} = |\bar{\Phi}^I_a - \bar{\Phi}^I_b|^2$ and $\Lambda$ is a cutoff. The Cartan elements $\bar{\Phi}^I_a$ are interpreted as the positions of the $N$ $D3$ branes and the potential is minimized when the branes form a spherical shell of radius $|\bar{\Phi}^I_a| = R \propto \Lambda$ whose energy density is $\mathcal{E} \propto N^2\Lambda^4$. This simplified analysis reveals the crucial physical property of the system: the potential energy of $N$ $D3$ branes is due to the two-body interactions among them. The latter are naturally interpreted as strings stretched between pairs of branes, having masses given by $m^2_{ab}$. Then, for large $N$ the usual $N^2$ YM scaling arises from the combinatoric factor counting two-body interactions.

The effective potential (2) depends essentially on the $N$-vector fields $\bar{\Phi}^I_a$, through the symmetric composite quantities $m^2_{ab}$; however, only the $N(N-1)/2$ off-diagonal elements of $m_{ab}$ enter the result (2). This effect is a consequence of the underlying $U(N)$ algebraic structure of the system. Such a point of view motivates us to ask whether we could capture the essential physics of the $D3$-brane system using a simpler $N$-vector model. Indeed, using scalars $\bar{\Phi}^I_a$ we could construct various composite quantities that are symmetric in the Cartan indices and then we could conceive an alternative mechanism leading to an effective potential similar to (2) without having to assume detailed knowledge of the underlying algebraic structure. We will present such a model below. Of course, the knowledge of the $U(N)$ structure provides us with a wealth of additional information regarding the $D3$-brane system and its relationship with string theory. A corresponding model exists in $d = 3$, and as we will see below, its features might be taken to suggest that it is in the universality class of the conformal theory describing $N$ $M2$-branes [2], in which case the underlying algebraic structure is still not understood.

To motivate the $d = 4$ model, consider the scalar potential of the action (1) for $SU(2)$ generalized to $I, J = 1, 2, \ldots, N_f$

$$V_{SU(2)}(\Phi^I_a) = \frac{1}{4}\Phi^I_a\Phi^J_b\Phi^I_e\Phi^J_f\epsilon^{abc}\epsilon_c^{\ ef}, \tag{3}$$

where $a$, $b$, $c = 1, 2, 3$. Generalizing $\epsilon_{abc}$ to the structure constants $f_{abc}$ of $U(N)$ with $a$, $b$, $c = 1, 2, \ldots, N^2$ one obtains the scalar potential $U(N)$ YM theory with $N_f$ flavors. There is, however, another generalization of (3) with $a$, $b$, $c = 1, 2, \ldots, N$ which is

$$V^{(4)}_N(\Phi^I_a) = \frac{1}{2(N-2)!}\Phi^I_a\Phi^J_b\Phi^I_e\Phi^J_f\epsilon^{abc\cdots}\epsilon_{c\cdots}^{ef}. \tag{4}$$

 

Here, we use the appropriate $N$-index $\epsilon$ tensors with $N-2$ indices contracted. We refer to the classically marginal $N$-vector potential (4) as the 2-subdeterminant potential, and we will find that it has a natural generalization to other dimensions. A nice property of the $N$-vector model with the potential (4) is that it can be analyzed via a Hubbard-Stratonovich (HS) transformation. To do so, we first note that the potential depends only on the $N \times N$ symmetric matrix field $\rho_{ab} = \Phi_a^I \Phi_b^I$ in terms of which the potential may be written as the sum of all possible $2 \times 2$ subdeterminants of $\rho$ as

$$V_N^{(4)}(\rho) \equiv \underset{2}{\overset{(N)}{\det}}(\rho) = \frac{1}{2}[(\mathrm{tr}\rho)^2 - \mathrm{tr}\rho^2]. \tag{5}$$

Now we introduce the 4D subdeterminant model as

$$L_{4\mathrm{D}} = \frac{1}{2}\Phi_a^I(-\partial^2)\Phi_a^I + \frac{\mu_0^2}{2}\Phi_a^I\Phi_a^I - g_0 V_N^{(4)}(\Phi_a^I), \tag{6}$$

where $\mu_0^2$ is a bare mass term and $g_0$ a dimensionless coupling. Rescaling then $\Phi_a^I \mapsto \Phi_a^I/\sqrt{g_0}$ and introducing the HS fields $\sigma_{ab}$ and $\rho_{ab}$ we write the partition function as

$$Z = \int [\mathcal{D}\Phi_a^I \mathcal{D}\sigma \mathcal{D}\rho] e^{-(1/g_0)\int L_{\mathrm{HS}}}, \tag{7}$$

$$L_{\mathrm{HS}} = \frac{1}{2}\Phi_a^I(-\partial^2)\Phi_a^I + \frac{1}{2}\sigma_{ab}(\Phi_a^I\Phi_b^I - \rho_{ab}) + \frac{\mu_0^2}{2}\mathrm{tr}\rho - \underset{2}{\overset{(N)}{\det}}(\rho). \tag{8}$$

The measure factors are Haar measures for the real symmetric $N \times N$ matrix fields $\sigma$ and $\rho$. We will analyze this theory in the saddle point approximation at large $N_f$, assuming that the saddle points are homogeneous in space-time. Note that the argument of the exponential in the path integral is not invariant under separate transformations of $\rho$ and $\sigma$ (because of the Tr $\sigma\rho$ term). However, the form of the saddle point equation for $\rho$ is $\bar{\sigma} = 2\bar{\rho} + (\mu_0^2 - 2(\mathrm{tr}\bar{\rho}))1$ (where overlines denote saddle point values). This is a local equation which has the following important property (of a saddle point): in a basis where $\bar{\rho}$ is diagonal, $\bar{\sigma}$ is diagonal as well. In this sense, we can simultaneously diagonalize $\sigma_{ab}$ and $\rho_{ab}$, and perform the $\Phi$ path integral. As a result we get

$$Z = \int [\mathcal{D}\sigma \mathcal{D}\rho] e^{-(N_f/g_0)S_{\mathrm{eff}}(\sigma,\rho)}, \tag{9}$$

$$S_{\mathrm{eff}} = \frac{1}{2}\sum_{a=1}^{N}\left(g_0 \,\mathrm{Tr}\ln(-\partial^2 + \sigma_a) - \int \sigma_a\rho_a\right) + \frac{\mu_0^2}{2}\sum_{a=1}^{N}\int \rho_a - \sum_{a<b}^{N}\int \rho_a\rho_b, \tag{10}$$

where $\sigma_a$ and $\rho_a$ are eigenvalues of $\sigma_{ab}$ and $\rho_{ab}$, respectively. To obtain (10), we rescaled $g_0 \mapsto g_0/N_f$. For constant configurations, the effective action yields the effective potential as $V_{\mathrm{eff}} = S_{\mathrm{eff}}/g_0(Vol_4)$. For large $N_f$, the uniform saddle points $\bar{\sigma}_a$, $\bar{\rho}_a$ satisfy

$$\frac{\delta V_{\mathrm{eff}}}{\delta \bar{\sigma}_a} = 0 \Rightarrow \bar{\rho}_a = g_0 \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2 + \bar{\sigma}_a}. \tag{11}$$

Introducing a cutoff $\Lambda$ in Eq. (11) yields

$$r_a \equiv \bar{\rho}_a - \bar{\rho}_{\mathrm{cr}} = \frac{g_0}{16\pi^2}\bar{\sigma}_a \ln(\bar{\sigma}_a/\Lambda^2) + O\left(\frac{\bar{\sigma}_a}{\Lambda^2}\right), \tag{12}$$

$$\bar{\rho}_{\mathrm{cr}} = g_0 \int^{\Lambda} \frac{d^4p}{(2\pi)^4}\frac{1}{p^2} = \frac{g_0}{16\pi^2}\Lambda^2. \tag{13}$$

Substituting $r_a$ back into (10) we first encounter the usual field independent quartic divergence (vacuum energy) which we drop. We also encounter a quadratic divergence which can be cancelled by a fine tuning of the bare mass (in the sense of tuning to a UV fixed point). Namely, we require that

$$\frac{\delta V_{\mathrm{eff}}}{\delta r_a} = -\frac{1}{2}\bar{\sigma}_a + \frac{\mu_0^2}{2} - \sum_{b\neq a}(r_b + \bar{\rho}_{\mathrm{cr}}), \tag{14}$$

is finite and zero for $\bar{\sigma}_a$, $r_a = 0$, which leads to $\mu_0^2 = 2(N-1)\bar{\rho}_{\mathrm{cr}}$. Alternatively, we may renormalize this model by subtracting an infinite contribution to the $\sigma$ tadpole. Integrating then (14) and using (12) we obtain

$$V_{\mathrm{eff}} = \frac{-1}{64\pi^2}\left[\sum_{a=1}^{N}\bar{\sigma}_a^2 \ln\frac{e^{1/2}\bar{\sigma}_a}{\Lambda^2} + \frac{g_0}{4\pi^2}\sum_{a<b}^{N}\bar{\sigma}_a\bar{\sigma}_b \ln\frac{\bar{\sigma}_a}{\Lambda^2}\ln\frac{\bar{\sigma}_b}{\Lambda^2}\right]. \tag{15}$$

To find the saddle point, we should minimize (15) with respect to the $\bar{\sigma}_a$'s. The last term in (15) demonstrates the following property of the 4D subdeterminant model. Although the validity of the saddle point is given by large $N_f$, the effective potential is dominated by the sum over the off-diagonal elements of a symmetric matrix, in close proximity with the $D3$-brane potential (2) if we take $N$ to be large. Moreover, the $\bar{\sigma}_a$ are proportional to the (mass)$^2$ of the $N$ vector fields $\Phi_a^I$. For simplicity we can consider the homogeneous configuration $\bar{\sigma}_a = \sigma$, $a = 1, 2, \ldots, N$ when the potential becomes in the large-$N$ limit

$$V_{\mathrm{eff}} = -g_0\frac{N^2}{512\pi^4}\sigma^2 \ln^2\frac{\sigma}{\Lambda^2} + O(N). \tag{16}$$

In Fig. 1 we sketch the effective potential (16) for $g_0 > 0$, which shows the similarity with the corresponding behavior of the effective potential (2). However, since the simplicity of our model does not allow us to fix either the value or the sign of $g_0$, such a similarity cannot be taken too far. Nevertheless, this is enough motivation to apply the idea to the more intriguing case of $M2$ branes in what follows.

In $d = 3$, the corresponding subdeterminant potential is a special $\varphi^6$ model

$$V_N^{(3)}(\Phi_a^I) = \frac{1}{6(N-3)!}\epsilon^{abcd\cdots}\epsilon_{d\cdots}^{efg}\Phi_a^I\Phi_b^J\Phi_c^K\Phi_e^I\Phi_f^J\Phi_g^K, \tag{17}$$

where $a, b, \ldots, f = 1, 2, \ldots, N$ and $I, J, K = 1, 2, \ldots, N_f$. Notice that (17) is proportional to the Bagger-Lambert-

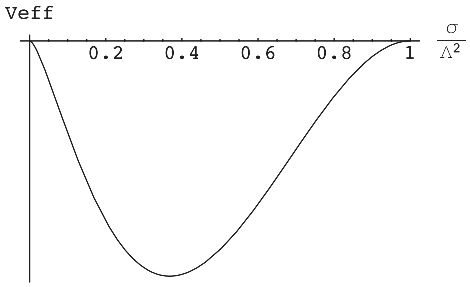


FIG. 1. Plot of $V_{\rm eff}$ for $g_0 > 0$.

Gustavsson (BLG) scalar potential [2] for $N = 4$ and $N_f = 8$. Hence, (17) may be viewed as a generalization of the BLG potential in the same way that (4) is a generalization of the $SU(2)$ YM potential. Introducing as above the symmetric matrices $\rho_{ab} = \Phi^I_a \Phi^I_b$, the potential becomes

$$V_N^{(3)} \equiv \overset{(N)}{\underset{3}{\det}}(\rho) = \frac{1}{6}[(\mathrm{tr}\rho)^3 - 3(\mathrm{tr}\rho)(\mathrm{tr}\rho^2) + 2\mathrm{tr}\rho^3]. \quad (18)$$

Now we introduce the 3D subdeterminant model with Lagrangian

$$L_{3\rm D} = \frac{1}{2}\Phi^I_a(-\partial^2)\Phi^I_a + \frac{\mu_0^2}{2}\Phi^I_a\Phi^I_a + g_0 V_N^{(4)}(\Phi^I_a) - \lambda_0^2 V_N^{(3)}(\Phi^I_a), \quad (19)$$

where now $\mu_0^2$, $g_0$ are dimensionful couplings [3], while $\lambda_0$ is a dimensionless one. As in $d = 4$, the scalar theory (19) at large $N_f$ (but $N$ arbitrary) can be studied using the HS procedure with an auxiliary scalar field $\sigma_{ab}$. After the rescaling $\Phi^I_a \mapsto \Phi^I_a/\sqrt{\lambda_0}$ we can follow the argument above and diagonalize $\rho_{ab}$ and $\sigma_{ab}$ to obtain the effective action

$$Z = \int[\mathcal{D}\sigma\mathcal{D}\rho]e^{-(N_f/\lambda_0)S_{\rm eff}(\sigma,\rho)}, \quad (20)$$

$$S_{\rm eff} = \frac{1}{2}\sum_a^N\left(\lambda_0\,\mathrm{Tr}\ln(-\partial^2 + \sigma_a) - \int\sigma_a\rho_a\right) + \frac{\mu_0^2}{2} \times \sum_{a=1}^N\int\rho_a + \frac{g_0}{\lambda_0}\sum_{a<b}^N\int\rho_a\rho_b - \sum_{a<b<c}^N\int\rho_a\rho_b\rho_c. \quad (21)$$

To achieve this form we rescaled $\lambda_0 \mapsto \lambda/N_f$ and $g_0 \mapsto g_0/N_f$. Following the same procedure as in the 4-dimensional case, we look for uniform saddle points $\bar{\sigma}_a$, $\bar{\rho}_a$ of the large-$N_f$ effective potential $V_{\rm eff} = S_{\rm eff}/\lambda_0(Vol_3)$ which satisfy

$$\frac{\delta V_{\rm eff}}{\delta\bar{\sigma}_a} = 0 \Rightarrow \frac{1}{\lambda_0}\bar{\rho}_a = \int\frac{d^3p}{(2\pi)^3}\frac{1}{p^2+\bar{\sigma}_a} = \frac{\Lambda}{2\pi^2} - \frac{\bar{\sigma}_a^{1/2}}{4\pi} + \cdots. \quad (22)$$

Then, writing $r_a \equiv \bar{\rho}_a - \bar{\rho}_{\rm cr}$, $\bar{\rho}_{\rm cr} = \frac{\lambda_0\Lambda}{2\pi^2}$, we express the effective potential in terms of the $r_a$ and fine-tune the bare couplings $\mu_0^2$ and $g_0/\lambda_0$ to renormalize it (after dropping as usual the cubic divergence that corresponds to the 3D vacuum energy). In this case we need to impose two renormalization conditions

$$\frac{\delta V_{\rm eff}}{\delta r_a}\bigg|_{\bar{\sigma}_a, r_a=0} = 0 = \frac{\delta^2 V_{\rm eff}}{\delta r_a \delta r_b}\bigg|_{r_c=0}. \quad (23)$$

We then obtain $\frac{\mu_0^2}{2} + \binom{N-1}{2}\bar{\rho}_{\rm cr}^2 = 0$ as well as $\frac{g_0}{\lambda_0} = (N-2)\bar{\rho}_{\rm cr}$. Finally, integrating the first equation of (23) for generic $\bar{\sigma}_a$ and $\bar{\rho}_a$ we obtain the effective potential as

$$V_{\rm eff} = \frac{1}{24\pi}\left(\sum_{a=1}^N\bar{\sigma}_a^{3/2} + \frac{6\lambda_0^2}{(4\pi)^2}\sum_{a<b<c}(\bar{\sigma}_a\bar{\sigma}_b\bar{\sigma}_c)^{1/2}\right). \quad (24)$$

As before, for homogenous configurations $\bar{\sigma}_a = \sigma$, large-$N$ and assuming that $\lambda_0^2 \sim O(1)$ the effective potential becomes

$$V_{\rm eff} = \lambda_0^2\frac{N^3}{384\pi^3}\sigma^{3/2} + O(N^2). \quad (25)$$

The stable vacuum is at $\sigma = 0$ [4]. Notice the peculiar $N^3$ scaling of the effective potential that arises from the three-body nature of the subdeterminant interaction, i.e., from the ($N$ choose 3) term. We should note that this scaling could have been obtained just from a $(\Phi^I_a\Phi^I_a)^3$ term, which is the large-$N$ limit of the (17) potential. In that sense, although the potential (17) does coincide with the BLG potential for $N = 4$, its algebraic structure does not play a significant role in the large-$N$ result (25). The latter result compares favorably to that of the large-$N$ effective description of a system of $M5$ branes. In such a picture the three-body interactions would correspond to string junctions [5].

We conclude with some observations regarding our subdeterminant potentials and their algebraic properties. Much of the combinatoric structure that we have discussed here arises from properties of the symmetric polynomials involved in the potentials. With this in mind, it is tempting to view our $d = 3$ and $d = 4$ subdeterminant models as arising from a more general scheme. As an example, note that if we were to identify $\rho$ with some sort of curvature 2-form, then the subdeterminant potentials correspond to their Chern characters.

Next, we comment on an intriguing relationship of our models with spin systems. In the 4D case we can define the $N \times N$ matrices $T^A$ as $\epsilon_{ab}{}^{c_1,\ldots,c_{N-2}} \equiv (T^A)_{ab}$, where we have introduced the collective index $\{c_1,\ldots,c_N\} \mapsto A = 1, 2, \ldots, \frac{1}{2}N(N-1)$ and $N \geq 3$. One can show that in fact $(T^A)_{ab}$ is the fundamental $N$-dimensional representation of $O(N)$, namely $[T^A, T^B] = f^{AB}_C T^C$. Hence, the potential (4) can be written as

$$V_N^{(4)} = \frac{1}{2(N-2)!}\delta_{AB}(\Phi^I_a(T^A)_{ab}\Phi^J_b)(\Phi^I_e(T^B)_{ef}\Phi^J_f) = \frac{1}{2(N-2)!}\delta_{AB}(S^A)^{IJ}(S^B)^{IJ}, \quad (26)$$

with the obvious identifications. We can think of the $(S^A)^{IJ}$ as classical $O(N)$ spins that carry the antisymmetric indices $IJ$. It is necessary that $N_f \geq 2$ for the potential to be nontrivial. It is furthermore interesting to note that the $(S^A)^{IJ}$ can be elevated to quantum $O(N)$ spins obeying $[S^A, S^B] = f_C^{AB} S^C$ if the $\Phi_a^I$'s are promoted to operators with nontrivial commutation relations. Consider $N_f = 2$, namely $I$, $J = 1, 2$. Then there is only one independent $(S^A)^{IJ}$ in (8), namely, $(S^A)^{IJ} \mapsto S^A = \phi_a^1 \epsilon_{ab}^A \phi_b^2$. Now we can compute the commutator $[S^A, S^B] = [\epsilon_{ab}^A \epsilon_{cd}^B - \epsilon_{ab}^B \epsilon_{cd}^A] \phi_a^1 \phi_b^2 \phi_c^1 \phi_d^2$. By imposing the commutation relations $[\phi_a^1, \phi_b^2] = \delta_{ab}$, $[\phi_a^1, \phi_b^1] = [\phi_a^2, \phi_b^2] = 0$ which is equivalent to a set of $N$ Heisenberg algebras, we obtain $[S^A, S^B] = f_C^{AB} S^C$. Therefore the $S^A$ are a representation of $O(N)$. For $N = 3$, the above is the usual Schwinger boson representation of $O(3)$.

Now, it is tempting to generalize this construction to the 3D case. We define the cubic matrices $T^A$ as $\epsilon_{abc}{}^{c_1,\ldots,c_{N-3}} = (T^A)_{abc}$ where $A \equiv \{c_1, \ldots, c_{N-3}\} = 1, 2, \ldots, \frac{1}{3!} N(N-1)(N-2)$ and $N \geq 4$. Given that, we can express the 3D subdeterminant potential in terms of "generalized spins" $(S^A)^{IJK}$ as

$$\begin{aligned} V_N^{(3)} &= \frac{\delta_{AB}}{6!(N-3)!} [(T^A)_{abc} \Phi_a^I \Phi_b^J \Phi_c^K][(T^B)_{efg} \Phi_e^I \Phi_f^J \Phi_g^K] \\ &= \delta_{AB} (S^A)^{IJK} (S^B)^{IJK}. \end{aligned} \tag{27}$$

For generic $N$ and $N_f$ one should be able to study the algebraic structure of the cubic matrices $T^A$ [6] as well as of the "generalized spins" $(S^A)^{IJK}$. However, for the minimal case $N = 4$ and $N_f = 3$, the cubic matrices $T^A$ become the usual 4-indexed Levi-Civita tensors $(T^A)_{abc} \equiv (\epsilon^A)_{abc}$ and also $(S^A)^{IJK} \mapsto S^A = (\epsilon^A)_{abc} \Phi_a^1 \Phi_b^2 \Phi_c^3$. In this case, one can define the cubic matrices [6] $(T^A)_{abc} = |\epsilon_{abc}^A| e^{(i\pi/8)\epsilon_{Aabc}}$ and the following multiplication rule

$$\sum_{m,n,k} (T^A)_{abm} (T^B)_{anc} (T^C)_{kbc} \Delta_{mnk} = (T^A T^B T^C)_{abc}. \tag{28}$$

We have introduced the generalized Kronecker $\Delta_{abc}$ which is 1 for $a = b = c$ and zero otherwise. All indices run from $1, \ldots, 4$, but no summation over repeated indices is implied. With the above definitions one can show that the standard ternary commutator satisfies

$$[T^A, T^B, T^C]_{abc} = -i \epsilon_D^{ABC} (T^D)_{abc}; \tag{29}$$

i.e., $(T^A)_{abc}$ are a representation of the $\mathcal{A}_4$ 3-algebra [2,6].

In conclusion, we have initiated the study of the 4D and 3D subdeterminant models. These models can be studied using a Hubbard-Stratonovich transformation and exhibit quite interesting large-$N$ behavior which is reminiscent of the behavior of the effective theory describing the stringy interactions among $D3$ branes. Hence, it is conceivable that the 3D subdeterminant model might be related to the effective theory that describes the interactions among $M2$ branes. The $N^3$ scaling of the 3D effective potential might then imply that this effective theory is related to $M5$ branes. Alternative accounts of this scaling have also appeared in Refs. [7]. Finally, we have pointed out the versatility of our subdeterminant potentials, which in 4D provide the $O(N)$ generalization of Schwinger bosons, while in 3D give a new "generalized spin" model. We note that we have also done a preliminary study of the supersymmetric extension of our $d = 3$ model, its relation to the $d = 3$ SYM theory (following [8]) and the appearance of new algebraic structures [9].

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