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# Charge assignments in multiple-$U(1)$ gauge theories

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We discuss the choice of gauge field basis in multiple-$U(1)$ gauge theories. We find that there is a preferred basis, specified by the *charge orthogonality condition*, in which the $U(1)$ gauge fields do not mix under one-loop renormalization group running. [S0556-2821(99)04219-8]

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## I. INTRODUCTION

Many models of new physics extend the standard model gauge group with one or more $U(1)$ factors. For instance, the gauge group of the topcolor–assisted technicolor [1] models introduced in Refs. [2,3] is given by

$$\mathrm{SU}(3)_s \times \mathrm{SU}(3)_w \times U(1)_s \times U(1)_w \times \mathrm{SU}(2)_L .$$

At a scale of about one TeV, the two SU(3)'s and the two $U(1)$'s are assumed to break to their diagonal subgroups which are identified with the standard model color and hypercharge groups. Multiple $U(1)$'s have also figured prominently in supersymmetric model building (see, e.g. Ref. [4]).

In most models, the charges of the matter fields under the multiple $U(1)$'s are restricted by anomaly cancellation conditions and phenomenological requirements, but beyond that they are often unconstrained within a relatively large parameter space. However, in any model with multiple $U(1)$'s, radiative corrections will in general mix the $U(1)$'s at different scales, even if the gauge fields were orthogonal at the initial scale. As a result, the meaning of the $U(1)$ charges assigned to the matter fields becomes ambiguous and the phenomenology is difficult to decipher or control.

In this paper, we argue that for any multiple-$U(1)$ gauge theory, one can always find a basis for the gauge fields in which this scale-dependent mixing does not occur (at one-loop order). In this basis, the $U(1)$ charges of the matter fields satisfy a constraint which we call the charge orthogonality condition (COC), and both the $U(1)$ fields and charges become scale-independent concepts.

Since any multiple-$U(1)$ gauge theory is equivalent to another in which the COC is satisfied, one can impose the COC on the matter–field charges without loss of generality. Not only does this simplify the study of the model at different scales, but also places a welcome additional restriction on the allowed charge parameter space.

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## II. $U(1)$ GAUGE BOSON MIXING

Consider a theory consisting of the gauge group $U_1(1) \times U_2(1) \times \cdots \times U_N(1)$ coupled to fermions $\psi_a$.[1] The Lagrangian one might naively write down for such a theory is

$$\mathcal{L} = -\sum_{i=1}^{N} \frac{1}{4g_i^2} F^i_{\mu\nu} F_i^{\mu\nu} + \sum_a \overline{\psi}_a \left( \partial_\mu - \sum_{i=1}^{N} Y_i^a A_\mu^i \right) \gamma^\mu \psi_a .$$

Here, $g_i$ is the gauge coupling of the $i$th $U(1)$ gauge group and $Y_i^a$ is the $i$th $U(1)$ charge of the $a$th fermion. However, for $U(1)$ gauge fields the field strength tensors

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu$$

are by themselves gauge invariant, so terms of the form

$$F^i_{\mu\nu} F_j^{\mu\nu} \quad (i \neq j)$$

are also permitted. So, the most general Lagrangian allowed by the symmetry is actually

$$\mathcal{L} = -\frac{1}{4} \sum_{i,j} F^i_{\mu\nu} K_i^j F_j^{\mu\nu} + \sum_a \overline{\psi}_a \left( \partial_\mu - \sum_{i=1}^{N} Y_i^a A_\mu^i \right) \gamma^\mu \psi_a ,$$

where the $N \times N$ matrix $K_i^j$ can be assumed to be real, symmetric, and positive definite. Therefore, in general, the $U(1)$ gauge fields will mix, and the diagonal elements of $K_i^j$ do not have a simple interpretation as coupling constants.

Of course, it is always possible to choose a basis which diagonalizes $K_i^j$ and to identify its diagonal elements as the inverse coupling constants squared in that particular basis. However, even if $K_i^j$ is diagonalized at one scale, renormalization reintroduces the off-diagonal mixing terms at other scales, and the mutually orthogonal $U(1)$ fields (and the charges to which they couple) must be redefined at every new scale. For a general $U(1)_1 \times U(1)_2 \times \cdots \times U(1)_N$ theory (as might be embedded in models such as topcolor–assisted technicolor) this scale-dependent mixing of the gauge fields, the gauge couplings, and the charges of the matter fields will likely make the analysis of its phenomenology confusing.

[1] Our results apply equally to models with scalars coupled to multiple $U(1)$ fields, but for simplicity we restrict our attention to a model with only fermions.

The natural question arises whether it is possible to find a basis in which $K_i^j$ is diagonal regardless of scale. The answer, at least at one loop, is *yes*.

To construct such a basis, first diagonalize $K_i^j$ at some initial scale $\mu$:

$$K_i^j(\mu)=\begin{bmatrix} \frac{1}{\kappa_1^2} & & & 0 \\ & \frac{1}{\kappa_2^2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\kappa_N^2} \end{bmatrix}.$$

Next, rescale the rotated gauge fields and charges:

$$\frac{1}{\kappa_i}A_\mu^i \rightarrow A_\mu^i,$$

$$\kappa_i Y_i^a \rightarrow Y_i^a.$$

This rescaling leaves the coupling term between the gauge bosons and fermions invariant while reducing $K_i^j(\mu)$ to a unit matrix:

$$K_i^j(\mu)\rightarrow \delta_i^j.$$

In other words, we can set all the gauge coupling constants to one by absorbing them into the definition of the charges.

Under renormalization, fermion loops induce scale–dependent mixing of the gauge fields which were diagonalized at scale $\mu$. That is, generically $K_i^j$ receives both diagonal and off-diagonal contributions. At one loop, one finds at scale $\mu'$:

$$K_i^j(\mu')=\delta_i^j - M_i^j \ln\left(\frac{\mu'}{\mu}\right)^2,$$

where

$$M_i^j=\frac{1}{24\pi^2}\sum_a Y_i^a Y_j^a$$

in which the index $a$ runs over the chiral degrees of freedom.

If the $U(1)$ charges satisfy the condition

$$\sum_a Y_i^a Y_j^a = 0, \quad \text{if} \quad i\neq j, \tag{1}$$

then $M_i^j$ is diagonal. If it is not, one can always diagonalize $M_i^j$ by an orthonormal rotation of the gauge fields and charges which leaves both $\Sigma_{i=1}^N Y_i^a A_\mu^i$ and $K_i^j(\mu)$ (the unit matrix) invariant. In this new basis, both $K_i^j(\mu)$ and $M_i^j$ are diagonal and as a result, $K_i^j(\mu')$ remains diagonal for all scales $\mu'$. Note that the new rotated charges now satisfy Eq. (1).

This discussion illustrates that it is always possible to find a basis in which $K_i^j(\mu)$ and $M_i^j$ are simultaneously diagonalized through *rescaling* and *orthonormal rotations*. [For $K_i^j(\mu)$ and $M_i^j$ to be simultaneously diagonalizable by orthonormal rotations alone, they must commute.] In this basis, which is characterized by the fact that Eq. (1) is satisfied, the $U(1)$ fields and charges do not mix under one loop running. Hence, this should be the basis of choice for analyzing the phenomenology of the theory.

However, starting with a general multiple-$U(1)$ theory with arbitrary (anomaly-free) charge assignments and then finding the preferred basis in which Eq. (1) is satisfied is tantamount to imposing Eq. (1) on the charges to begin with. Therefore, without loss of generality, one can impose Eq. (1) as an additional condition to anomaly cancellations when making the initial charge assignments. We call Eq. (1) the charge orthogonality condition (COC).

Since $M_i^j$ is real, symmetric, and positive semidefinite, all of its eigenvalues are real and non-negative. Let us denote them by

$$0\leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N.$$

After diagonalization

$$K_i^j(\mu')=\delta_i^j\left[1-\lambda_i \ln\left(\frac{\mu'}{\mu}\right)^2\right],$$

and we can make the identification

$$\frac{1}{g_i^2(\mu')}\equiv 1-\lambda_i \ln\left(\frac{\mu'}{\mu}\right)^2.$$

We make the following observations:

(1) The ratios of the $U(1)$ coupling constants $g_i(\mu)$ flow towards infrared fixed points determined by the ratios of the eigenvalues of $M_i^j$:

$$\frac{g_i^2(\mu)}{g_j^2(\mu)} \overset{\mu\to 0}{\Rightarrow} \frac{\lambda_j}{\lambda_i}.$$

(2) If the fermions are massive and decouple in the course of renormalization group running, the effective theory will change as each threshold is crossed. Consequently, $M_i^j$ will differ across each threshold, and it will be necessary to rediagonalize $M_i^j$ in each new effective theory.

(3) The position of the Landau pole ($\Lambda$) is determined by the largest eigenvalue $\lambda_N$ of $M_i^j$:

$$\log\left(\frac{\Lambda^2}{\mu^2}\right)=\frac{1}{\lambda_N}.$$

In a general basis (i.e., one in which $M_i^j$ is not diagonal) one might naively calculate the Landau pole from the diagonal elements alone. However, the largest eigenvalue of a positive semidefinite matrix is always at least as large as its largest diagonal element. Thus, the *actual* Landau pole of the model is always lower than one would estimate neglecting gauge field mixing [5].

TABLE I. Quark and technifermion hypercharges and electromagnetic charges in the model of Ref. [2]. Leptons are not shown.

| Particle | $Y_1$ | $Y_2$ | $Q=T_3+Y_1+Y_2$ |
|---|---|---|---|
| $q_L^l$ | 0 | $\frac{1}{6}$ | $\frac{2}{3},-\frac{1}{3}$ |
| $c_R,u_R$ | 0 | $\frac{2}{3}$ | $\frac{2}{3}$ |
| $d_R,s_R$ | 0 | $-\frac{1}{3}$ | $-\frac{1}{3}$ |
| $q_L^h$ | $\frac{1}{6}$ | 0 | $\frac{2}{3},-\frac{1}{3}$ |
| $t_R$ | $\frac{2}{3}$ | 0 | $\frac{2}{3}$ |
| $b_R$ | $-\frac{1}{3}$ | 0 | $-\frac{1}{3}$ |
| $T_L^l$ | $x_1$ | $x_2$ | $\pm\frac{1}{2}+x_1+x_2$ |
| $U_R^l$ | $x_1$ | $x_2+\frac{1}{2}$ | $\frac{1}{2}+x_1+x_2$ |
| $D_R^l$ | $x_1$ | $x_2-\frac{1}{2}$ | $-\frac{1}{2}+x_1+x_2$ |
| $T_L^t$ | $y_1$ | $y_2$ | $\pm\frac{1}{2}+y_1+y_2$ |
| $U_R^t$ | $y_1+\frac{1}{2}$ | $y_2$ | $\frac{1}{2}+y_1+y_2$ |
| $D_R^t$ | $y_1+\frac{1}{2}$ | $y_2-1$ | $-\frac{1}{2}+y_1+y_2$ |
| $T_L^b$ | $z_1$ | $z_2$ | $\pm\frac{1}{2}+z_1+z_2$ |
| $U_R^b$ | $z_1-\frac{1}{2}$ | $z_2+1$ | $\frac{1}{2}+z_1+z_2$ |
| $D_R^b$ | $z_1-\frac{1}{2}$ | $z_2$ | $-\frac{1}{2}+z_1+z_2$ |

## III. CHARGE ORTHOGONALITY IN GUT's

If the multiple $U(1)$'s under consideration are unbroken subgroups of a simple Lie group, as they would be in a grand unified theory (GUT), then it is possible to show that the charge orthogonality condition Eq. (1) is automatically satisfied. If the unbroken $U(1)$'s result from the breaking of a simple Lie group, the charge $Y_i^a$ of the $a$th fermion under the $i$th $U(1)$ is proportional to the $a$th diagonal element of the $i$th unbroken diagonal generator of the simple Lie group in some irreducible representation. The diagonal generators $H_i$ ($i=1,2,\ldots,N$) of a simple Lie group constitute the Cartan subalgebra of the generating Lie algebra and can always be normalized to the form

$$\mathrm{tr}(H_iH_j)=\delta_{ij}k_D$$

where $k_D$ is a representation-dependent constant [6]. Since $\Sigma_a Y_i^a Y_j^a$ is proportional to $\mathrm{tr}(H_iH_j)$, it immediately follows that, for $U(1)$'s arising from the breaking of simple Lie groups, the COC is always satisfied.

Of course, if any of the fermions decouple in the course of renormalization group running before the $U(1)$'s themselves break, then a rediagonalization of the fields and charges would be necessary every time a particle threshold is crossed even for GUT's.

## IV. APPLICATION TO TOPCOLOR-ASSISTED TECHNICOLOR MODELS

To see the consequence of the COC in model building, we consider its application in the context of topcolor–assisted technicolor models [1]. As a concrete example, we consider the model proposed by Lane and Eichten in Ref. [2], which has two $U(1)$ factors in its gauge group. The particle content and charge assignments of the model are shown in Table I, where the $U(1)$ charges of technifermions are expressed in terms of six parameters $x_{1,2}$, $y_{1,2}$, and $z_{1,2}$. [Note that the standard model particles are charged under only one of the $U(1)$'s.]

Anomaly cancellation requirements lead to the following five equations for the six unknowns:

$$0=x_1+y_1+z_1,$$

$$0=x_2+y_2+z_2,$$

$$0=x_1\left(y_1-z_1+\frac{1}{2}\right),$$

$$0=x_2\left(y_2-z_2-\frac{1}{2}\right),$$

$$0=x_1\left(y_2-z_2-\frac{1}{2}\right)+x_2\left(y_1-z_1+\frac{1}{2}\right).$$

There are two classes of solutions to these equations: the first with $x_1=x_2=0$, and the other with $(y_1-z_1+\frac{1}{2})=(y_2-z_2-\frac{1}{2})=0$. Each of these classes has two free parameters remaining. The COC provides an additional constraint

$$4(x_1x_2+y_1y_2+z_1z_2)-\left(y_1-z_1+\frac{1}{2}\right)+\left(y_2-z_2-\frac{1}{2}\right)=0,$$

which reduces the number of free parameters to one in each class.

Instead of applying the COC, however, Lane and Eichten require the existence of the four-technifermion operator

$$\overline{T_L^l}\gamma^\mu T_L^t\overline{D_R^t}\gamma_\mu D_R^b$$

to provide mixing between the third and first two fermion generations. This leads to two additional constraints

$$0=x_1-z_1+1,$$

$$0=x_2-z_2-1,$$

which are sufficient to fix one solution in each of the two classes. However, these conditions, and hence the two solutions, are incompatible with the COC.

This means that the charge assignments of Ref. [2] characterize a basis for the $U(1)$ fields which is *not* charge orthogonal, and in this basis the $U(1)$ fields will necessarily mix with a scale-dependent mixing angle under RG running. To avoid this, one must go to the preferred basis in which the COC is satisfied, but in that basis the standard model (SM) particles will carry two $U(1)$ charges instead of just one. In other words, even though one started out with a model restricting the SM particles to carry only one of the $U(1)$ charges, the restriction becomes meaningless if the charges do not satisfy the COC.

We emphasize that the COC does *not* restrict the introduction of the four-technifermion operator shown above, or any other operator one may want to add to the theory. For the operator to be gauge invariant, its $U(1)$ charges must be zero. This condition is obviously basis independent and if it is imposed in one basis, it is automatically satisfied in any other basis. Rather, the original charge assignment of Eichten and Lane does not make scale-independent sense and thus we are forced to consider more general (but physically equivalent) charge assignments, such as those considered by Lane in Ref. [3].

The model constructed by Lane in Ref. [3] is similar to that of Ref. [2], but the $U(1)$ charge assignments of all the particles, including those in the standard model, are kept generic resulting in 26 parameters which must be specified. Anomaly cancellations provide a set of five linear and three cubic equations. Thirteen additional linear equations are imposed from phenomenological considerations. With 21 conditions on 26 parameters, any solution will have at least five free parameters. The COC will provide an additional quadratic equation which will again reduce the number of free parameters by removing the (unphysical) rescaling and rotational degree of freedom in the $U(1)$ field and charge definitions.

## V. CONCLUSION

We have argued that in multiple-$U(1)$ gauge theories, there exists a particular choice of basis for the $U(1)$ fields which is to be preferred over other rescaling and orthonormal rotations of the $U(1)$ fields. In this basis, characterized by the fact that the COC is satisfied, the $U(1)$ gauge bosons do not mix under one-loop renormalization group running. Choosing this basis in model building avoids the ambiguity in charge definitions associated with rescaling and rotations of the gauge fields and expedites study of the models at different scales.

Since a theory with arbitrary $U(1)$ charge assignments can always be mapped onto another in which the COC is satisfied with a simple change of basis, one can impose the COC as an additional constraint on the charges without loss of generality. This will reduce the dimension of the available charge parameter space and simplify the analysis of models considerably.

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