

Subitizing Activity: Item Orientation with Regard to Number Abstraction

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ABSTRACT

Subitizing, a quick apprehension of the numerosity of a small set of items, is inconsistently utilized by preschool educators to support early number understandings (Sarama & Clements, 2009). The purpose of this qualitative study is to investigate the relationship between children's number understanding and subitizing activity. Sarama and Clements (2009) consider students' subitizing activity as shifting from reliance upon perceptual processes to conceptual processes. Hypothesized mental actions carried into subitizing activity by children have not yet been empirically investigated (Sarama & Clements, 2009). Drawing upon Piaget's (1968/1970) three mother structures of mathematical thinking, the theoretical implications of this study consider expanding the scope of Piaget's (1968/1970) definition of topological thinking structures to include patterned orientations. Increasing the scope of this definition would allow for the investigation of the development of topological thinking structures and subitizing activity.

An 11-week teaching experiment was conducted with six preschool aged children in order to analyze student engagement with subitizing tasks (Steffe & Ulrich, in press). To infer what perceptual and conceptual processes students relied upon when subitizing, tasks were designed to either assess or provoke cognitive changes. Analysis of interactions between students and the teacher-researcher informed this teacher-researcher of cognitive changes relative to each student's thinking structure.

Results indicated that students rely upon the space between items, symmetrical aspects of items, and color of items when perceptually subitizing. Seven different types of subitizing activity were documented and used to more explicitly describe student reliance upon perceptual

or conceptual processes. Conceptual subitizing activity was redefined in this study, as depending upon mental reversibility and sophisticated number schemes. Students capable of conceptual subitizing were also able to conserve number. Students capable of conserving number were not always capable of conceptual subitizing. The symmetrical aspects of an item's arrangement elicited students' attention towards subgroups and transitioning students' perceptual subitizing to conceptual subitizing. Combinations of counting and subitizing activity explained students' reliance upon serial and classification thinking structures when transitioning from perceptual subitizing to conceptual subitizing. Implications of this study suggest effectively designed subitizing activity can both assess students' number understandings, and appropriately differentiate preschool curriculum.

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Dedication

This dissertation is dedicated to my best friend and loving husband, Mark Frederick MacDonald. I could not have done this alone. My husband's willingness to step in as a single father, at times, and support a stressed out partner by listening to me discuss work each day for eight and a half years shows true love and compassion. For this I am forever thankful.

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Chapter One: Introduction

For more than 100 years, researchers have investigated how number is understood by children (Sarama & Clements, 2009). However, questions regarding the relationship between subitizing activity and number development subsist. Subitizing, as defined by Sarama and Clements (2009), is derived from the Latin term *subitus*, which means “to arrive at suddenly” (Sarama & Clements, 2009, p.29), and describes a process that results in the apprehension of the numerosity of a small set of objects without counting (Dehaene, 2011). Most research investigates subitizing to better understand how individuals process perceptual information, as most subitizing research has been carved out in the psychology research field. The general purpose of this study is to consider the relationship between conceptual understandings regarding number and subitizing activity.

Mathematics education researchers have a limited set of investigations through which the evolutions of subitizing and number development and possible connections between them have been studied. For instance, Sarama and Clements (2009) suggest that subitizing relates to early number development, explaining why certain item orientations can be effectively subitized as children grow and develop a more sophisticated understanding of number. Simply, Sarama and Clements (2009) suggest that the greater the number of orientations that children are able to subitize, the more sophisticated those children’s number understandings. The current investigation extends these and similar findings, through qualitative analysis of the relationship between particular item orientations and children’s subitizing activity.

The Importance of Understanding Preschool Experiential Mathematics Development

Early mathematical activities and experiences have become more of a focus in the elementary classroom, as the demands of a global economy now require present day educators to meet all

students' mathematical needs (English, 2008). Meeting all students' academic needs can be difficult at times for many teachers, as students have a wide variety of academic strengths and weaknesses prior to entering kindergarten (Bowman, Donovan & Burns, 2001). Specifically, achievement gaps between children from low socioeconomic status (SES) households and children from middle or high SES households can be explained by differences in early childhood development and prekindergarten (preK) programs (Bowman et al., 2001).

This achievement gap in early mathematics learning has been found to predict later achievement in both reading and mathematics, demonstrating a need for early mathematics instructional interventions (Duncan et al., 2007). Thus, as a wide variety of children's academic needs is supported by educators, it is necessary to begin by designing effective preK and kindergarten mathematics curriculum (Clements & Sarama, 2007).

Strategies upon which preK students rely when counting, adding, and subtracting vary in students from different SES households (Ginsburg & Pappas, 2004). For instance, all preK students benefit from adult assistance when counting, adding, and subtracting, but children from high SES households are often able to work independently, not relying solely on this assistance (Ginsburg & Pappas, 2004). This ability to work independently is further explained by Ginsburg and Pappas (2004), as evidence of students' reliance upon more abstract strategies. Thus, it can be inferred that early informal mathematical experiences influence preK students' flexible use of counting, adding, and subtracting strategies (Ginsburg & Pappas, 2004).

Instructional activities which motivate students' intellectual engagement in preK through grade 2 classrooms typically take the form of a game (Clements & Sarama, 2007; Ramani & Seigler, 2011). Instructional activities in which children are required to build upon prior mathematical knowledge have been found to raise preK children's mathematical achievement, as

children connect informal mathematical knowledge to more formal mathematical concepts (Clements & Sarama, 2007; Ramani & Seigler, 2011). Building on prior mathematical knowledge has also been shown to increase children's motivation to learn mathematics, as the activities are more meaningful for them (Clements & Sarama, 2007). More specifically, the use of instructional activities embedded in linear number board games with preK children has been found to have a dramatic effect on the mathematical knowledge of preK children from low SES households (Ramani & Seigler, 2011). Thus, instructional activities embedded in game play are foundational aspects of children's early informal mathematical experiences; these games both motivate children to learn more formal mathematical concepts and also improve children's number understanding (Clements & Sarama, 2007; Ramani & Seigler, 2011).

Effective instructional activities also incorporate students' reliance upon subitizing activity (Clements & Sarama, 2007). *Building Blocks*, a National Science Foundation (NSF)-funded curriculum for students in preK through grade 2, was designed to use games in order to address standards in early mathematics for all students (Clements & Sarama, 2007). Essentially, students engaged in subitizing activities (a subtopic within the *Building Blocks* program), were found to have significant gains in subitizing, suggesting an underlying dependence upon effective mathematical experiences (Clements & Sarama, 2007). Furthermore, the results from this study informed Sarama and Clements' (2009) hypothesized set of mental actions young students may rely upon in order to explain changes in subitizing activity. It is explained by Sarama and Clements (2009) that these mental actions support a student's ability to subitize more items and a greater variety of orientations as the student relies upon more sophisticated number understandings.

In brief, mathematical achievement gaps between different students in many kindergarten classrooms can be explained by considering kindergarten students' experiential development. These gaps are evidence of differences in mathematical experiential knowledge that students have prior to entering kindergarten. Results also indicate that instructional activities embedded in games often support students' construction of meaningful mathematical knowledge and allow preK and kindergarten students to build more abstract strategies when counting, adding, and subtracting. Since students' subitizing activity changes in coordination with number development, educators can benefit from conceptually designed mathematical activities that utilize subitizing, counting, adding, and subtracting to both instruct and assess children's numerical understandings prior to kindergarten. Results from this current study will contribute further toward development of effective kindergarten instructional activities with subitizing activity relative to counting, adding, and subtracting.

Physiological Development of Mathematics

When considering young students' experiential development in mathematics, important questions arise concerning what necessary experiences should be embedded in preK curriculum to support young students' construction of number. Additionally, a student's physiological development, resulting from organic growth and brain development, influences a student's construction of number (Piaget, 1941/1965). Physiological development regarding the construction of number has been researched primarily by Jean Piaget (1941/1965) and explains how the formation of certain thinking structures supports a student's construction of number.

Number conservation is explained by Piaget (1941/1965) as indicative of when number is truly understood by children. When number is conserved by children, their understanding of number allows them to consistently describe the number of objects within a set, regardless of

how the objects change relative to the space the objects take up (Piaget, 1941/1965).

Cognitively, Piaget (1941/1965) explains that for children to conserve number, two of the three *mother structures*, or larger thinking structures, the *ordering relations thinking structure* (serial thinking structure) and the *algebraic thinking structure* (classification thinking structure), need to be coordinated simultaneously. In other words, children capable of conserving number simultaneously draw upon the notion that numbers follow a serial order (e.g., 1, 2, 3, 4, 5,) and that numbers are mentally composed of smaller subgroups (e.g., 2 and 3 compose 5, 1 and 4 compose 5) (Piaget, 1941/1965). When both of these thinking structures are simultaneously coordinated, a child's expression of number is based on logical reasoning grounded in the understanding that order and class size both describe number (Piaget, 1941/1965).

The third mother structure, *topological thinking structure*, has not been empirically studied as directly influencing the construction of number, but only as directly influencing the early development of the classification thinking structure. Topological thinking structures, as defined by Piaget (1968/1970), are "based on notions such as, neighborhoods, borders, and approaching limits" (p. 25-26) and relate primarily to geometry, but are not limited to geometrical concepts. For instance, when infants interact with their environment visually, the concept of a line is first understood as a visual trajectory (Piaget & Inhelder, 1948/1967). Piaget and Inhelder (1948/1967) describe *dimensionality* of items as representing the flexible thinking necessary for conceptualizing formal Euclidean Geometry. The following five areas of development are included in dimensionality: proximity (nearbyness), separation (betweenness), continuity (connecting objects in spatial fields), and enclosure of shape (surrounding) (Piaget & Inhelder, 1948/1967). As lines intersect, a two dimensional plane is conceptualized, directly

supporting children's understanding of Euclidean Geometry later taught in schools (Piaget & Inhelder, 1948/1967).

Topological thinking structures, as defined by Piaget and Inhelder (1948/1967), may or may not directly relate to number development, but may directly relate to subitizing activity. Early development of the classification thinking structure is supported by topological thinking structures, as children initially rely on the spatial relations of items when deciding if the items belong in a group or not (Piaget & Inhelder, 1948/1967). This overreliance upon items' locations indicates that children are not capable of considering abstract features when grouping items (Piaget & Inhelder, 1948/1967). Until a child attends to abstract features of items and uses these features to group items, the child's classification thinking structure is not considered to be sophisticated (Piaget & Inhelder, 1948/1967).

Piaget (1968/1970) describes the three mother structures as aligning directly with an individual's psychological structures. Considering psychological development and mathematical development as operating in tandem supports the argument that mathematical development is not grounded in language, but directly in an individual's psychological expression of her perceptual environment and logical reality. Thus, it would be important to investigate the direct influences that topological thinking structures have upon a student's construction of number, explaining reliance upon the coordination of all three mother structures when understanding number through subitizing activity.

The perceived regular separation of items influences subitizing, suggesting that topological thinking structures may directly relate to individuals' subitizing activity (Arp & Fagard, 2005). Patterned arrangements of items, not included in Piaget's definition of topological thinking structures, have also been found to decrease time necessary for children to

subitize (Mandler & Shebo, 1982). Thus, it would be important to explore the possibility of modifying Piaget and Inhelder's (1948/1967) definition for topological thinking structures to increase the scope of this definition's application. This modification would include patterned arrangements of items, defined as an individual's perceived amount of regular space between items that construct visual patterns.

Subitizing in the Field of Psychology

The quick apprehension of the numerosity of a small set of items is described as subitizing in both the fields of psychology and mathematics education. For instance, when an individual is shown a triangular arrangement of three dots, that individual most likely will quickly state that this arrangement is comprised of three dots, without having to count the dots. In the field of psychology, subitizing has been explained, broadly, as a form of perceptual processing with the following four theories: *spatial indexing* theory (e.g., Pylyshyn, 1989), *working memory* theory (e.g., Klahr, 1963a), *patterned orientation* theory (e.g., Mandler & Shebo, 1982), and a *density-based* theory (e.g., Arp & Fagard, 2005). The orientation of items, as influencing subitizing activity, are described in the latter two theories. It has been indicated in recent empirical findings resulting from functional Magnetic Resonance Imaging (fMRI) scans that subitizing activity resides primarily in the occipital area of the brain that is mainly responsible for object detection (Malach et al., 1995). The premise that object orientation largely relates to subitizing activity is further supported by these findings, and suggests that topological thinking structures may also relate to subitizing activity and number development.

Subitizing in the Field of Mathematics Education

Few empirical studies have investigated subitizing in the field of mathematics education, as subitizing investigations are situated primarily in the field of psychology and focused mainly

on the perceptual mechanisms that support subitizing activity. Resulting from these types of studies is an understanding that item orientation (e.g., patterned versus unpatterned items, clustered versus linear orientations) influences activity relying upon perceptual mechanisms (Arp & Fagard, 2005; Mandler & Shebo, 1982; Sophian & Crosby, 2008).

The few discussions framing subitizing in the field of mathematics are theoretical in nature and suggest only hypothetical connections between number development and subitizing activity (Clements & Sarama, 2007; Sarama & Clements, 2009). It was found by Clements and Sarama (2007) that subitizing activity changed as children were able to subitize sets with a larger number of items and a wider variety of orientations. This change in activity was explained as the children's ability to link spatial imagery to number development. The notion that as children's understanding of number increases in sophistication, more item orientations and larger sets of items are able to be subitized are supported by these theories (Clements & Sarama, 2007; Sarama & Clements, 2009). However, mental actions regarding subitizing activity and number understanding have never been explicitly investigated (Sarama & Clements, 2009).

Researchers grounding their theoretical discussions in *radical constructivism* have also offered a unique perspective on subitizing activity. Radical constructivism, as defined by Glasersfeld (1995), is an educational philosophy which describes how knowledge is constructed and changed as an individual actively engages with her own perceived and functional reality. Philosophies grounded in radical constructivism are largely informed by Piaget's (1968/1970) philosophical discussion centered on genetic epistemology, describing the evolution of a child's mathematical thinking as parallel to that of the evolution of a human society's mathematical thinking. This present study's theoretical framework is situated in radical constructivism, as each child's perceived reality indicates conceptual change relative to each child's subitizing acts.

Glaserfeld (1982), and Steffe, Cobb, and Glaserfeld (1988) described subitizing as being sometimes dependent only upon an individual's visual scan of a patterned orientation of items. This implies that children may not need to understand number when subitizing, as they may only be naming a shape as "four" just as a four sided shape might be named as a "square" (e.g., four dots construct a square when placed in two symmetrical rows) (Glaserfeld, 1982; Steffe et al., 1988). In comparison, it is suggested by Sarama and Clements study (2009), that children engage in some conceptual types of subitizing that relates to their conceptual understandings of number, further expanding Glaserfelds' (1982) argument. Once children begin to rely upon more sophisticated notions of number, orientation can be considered more flexibly, which suggests that children do not need to rely primarily on a visual scan when expressing the set size of an orientation of items. Thus, the arrangement of items that children subitize may not only relate to their subitizing activity, but also indicate a child's number understanding.

Rationale for the Study

In mathematics education research, Sarama and Clements (2009) suggest that subitizing activity can change from reliance upon patterned orientations to reliance upon number understandings. Moreover, children's accuracy when subitizing patterned rectangular orientation of items versus circular or linear orientation of items may also indicate what type of number understanding children are relying upon (Sarama & Clements, 2009). In psychology research, Arp and Fagard (2005) and Mandler and Shebo (1982) describe patterned orientations of items as influencing subitizing activity. These findings suggest that purposively designed orientations of small sets of items that engage young students in subitizing activity could be used to both assess and support early number understanding.

Additionally, subitizing activity and estimating have been found to rely upon different cognitive mechanisms, as individuals' accuracy rate does not follow Weber's law when engaging in subitizing activity (Revkin, Piazza, Izard, Cohen, & Dehaene, 2008). An understanding of the different processing mechanisms which children rely upon when subitizing, estimating, and counting would inform mathematics educators of multiple routes to support young children's construction of number. Such understanding would offer educators opportunities to leverage one cognitive mechanism in support of another. For instance, children with visual processing disabilities have difficulty counting, as keeping track of items either prevents items from being counted, or items are counted more than once (Demeyere, Lestou, and Humphreys, 2010). However, when these children numerically expressed items in groups of two, they were more accurate (Demeyere et al., 2010). This change in accuracy was explained by Demeyere et al. (2010) as participants' reliance upon a combination of subitizing activity and counting activity, relieving children's visual processing resources (Demeyere et al., 2010). Thus, it seems important to consider how item orientation may support these early subitizing experiences to better understand how children with a wide variety of experiential and physiological forms of development form a more comprehensive understanding of number.

In brief, the rationale for this study is to understand how three, four, and five year-old students' subitizing activity relates to their number understanding. Regarding this research statement, this study could be considered more specifically to investigate three relationships; between students' subitizing activity and item orientation, between students' subitizing activity and students' empirical activity, and between students' subitizing activity and students' number understanding.

Purpose Statement and Research Questions

The purpose of this study is to investigate the relationship between three, four, and five year-old students' subitizing activity and number understanding by using Sarama and Clements' (2009) hypothetical mental actions. Different orientations of items and students' descriptions of perceived groups of items will be considered to determine relationships between item orientation and subitizing activity, relative to number understanding. To investigate this, three central research questions follow:

1. Does item arrangement relate to students' subitizing activity? If so, how might patterned arrangements, as described by Sarama and Clements (2009), relate to students' number understanding?
2. How might students' counting activity and re-presentations of a variety of item arrangements relate to students' number understanding and subitizing activity?
3. How might students' present thinking models with regard to subitizing change relative to item arrangements? How might different item arrangements elicit student reliance upon different types of number understandings?

Outline

In Chapter 2, a review of the literature will provide a theoretical framework for this study. The current and historical literature regarding critical aspects of subitizing from both the psychology field and the mathematics education field will be considered relative to student learning. Results of this discussion will ground this study's purpose, as item orientation will be considered as an influential factor when subitizing. Second, grounded within radical constructivism philosophies, the study's explanations of different stages of number understanding will be discussed relative to subitizing activity through *scheme theory*. Scheme

theory describes the construction and development of particular thinking structures. Finally, student reliance on number understandings when engaging in some forms of subitizing activity will be used to explain connections between the subitizing literature and schemes concerning early number construction.

Methodology for the study is discussed in Chapter 3. A constructivist teaching experimental structure is used in the present study as methodology (Steffe & Ulrich, in press; Steffe & Thompson, 2000). Inductively designed tasks, meant to purposively assess and change cognitive structures upon which the students rely on to subitize, were given to students engaged in this study. Utilizing a teaching experiment structure to answer this study's research questions resulted in the construction of students' thinking models (Steffe & Thompson, 2000). On-going analyses of student responses provide this researcher with glimpses of small cognitive changes between tasks, and retrospective analyses informs this researcher of overall cognitive changes (Steffe & Thompson, 2000). Both forms of analysis assess a static model of thinking and analyze changes made in children's thinking models.

Chapter 4 provides an analysis of the results from the teaching experiment sessions (Steffe & Thompson, 2000). Responses, which reveal interactions between item orientations and subitizing ability, are discussed. Moreover, analyses were considered relative to subitizing activity and each student's number understanding. These analyses further informed this researcher of different types of conceptual processes that students relied upon when engaging in different subitizing activity.

An overview of the entire study will be considered in Chapter 5, as results are connected back to the literature and the theoretical framework discussed in Chapter 2. Implications

resulting from this study will also be addressed in the final chapter, informing future research in connecting subitizing to preK children's mathematical learning and assessment.

Chapter Two: Literature Review and Theoretical Framework

This chapter is comprised of two main sections. In the first section, a review of the literature concerning subitizing in the psychology field will provide a history of *attentional mechanisms* and *preattentional mechanisms* empirically investigated and found effective visual subitizing. A clear purpose for this study's focus on object orientation when children subitize is informed by a critical review of these results. Attentional mechanisms--psychological mechanisms that rely upon working and long-term memory resources-- are considered when an individual subitizes because what individuals visually attend to reveals perceptual and cognitive development attributed toward development of subitizing. Comparatively, preattentional mechanisms - early perceptual processing mechanism that encode perceptual material – are considered when an individual subitizes because what individuals encode when subitizing influences what an individual is capable of perceptually attending towards. An overview of cognitive changes that individuals make when learning, which reveals the construction of thinking models described by Jean Piaget (1968/1970) and Ernst von Glasersfeld (1995), will be provided and defined in this section. Finally in the second section, connections will be made between the subitizing literature and schemes concerning young students' number construction which will explain student reliance upon number understandings when engaging in some forms of subitizing activity.

Historical Aspects of Subitizing

This literature review is composed of two subsections. The first section specifically defines visual subitizing and underlying perceptual mechanisms engaged in subitizing activity by reviewing historical aspects of the literature in the psychology field. These aspects also provide an overview of methods used for investigating subitizing, as task design elicits individuals'

dependency upon different psychological mechanisms. Historical aspects of subitizing are considered in the second section, in relation to mathematics education. To inform mathematics educators about cognitive development with regard to conceptual change in mathematical development, psychological research should inform mathematics education. However, presently a gap exists between the two areas of research indicating a need to consider both sets of findings together.

Subitizing is described as an apprehension of the numerosity of a small set of objects without counting (Kaufman, Lord, Reese, & Volkman, 1949). This definition remains generally the same in both the psychology and mathematics education fields and will be used in this study. Interestingly, infants, just days old, recognize the difference between two and three items, indicating that subitizing is an innate system (Dehaene, 2011). This suggests that human beings are all born with visual processing mechanisms which support subitizing processes. The bulk of the psychology research investigates these processes, but research in the mathematics educational field suggests a connection between subitizing activity and the construction of number. Research findings from the psychology field and the mathematics educational field are considered in order to better understand how number may develop in coordination with subitizing development.

Visual Processing Mechanisms and Subitizing

As early as the late 1800s, experiments designed by Jevons (1871) and Cattell (1886) indicated that as the number of items in a perceptual field increase, so does the time it takes to name the items. Adult participants in Cattell's (1886) research viewed a letter, word, color or picture for a half second, and results indicated that the more ideas the participant were shown (e.g., letter versus word), the longer the response time. Jevons (1871) calculated a formulaic

amount of error as the number of items being subitized or estimated increased. Conclusions from research conducted in the late 1800s have indicated that individuals' subitizing limit is typically between four and six (Jevons, 1871).

In the mid-1900s psychologists investigated subitizing activity in comparison to estimating and counting activities, further refining how these cognitive tools differ when individuals engage with quantification processes. Kaufman, Lord, Reese, and Volkman (1949) investigated influential factors of visual number discrimination, describing subitizing for the first time in the psychology field as an entirely different mechanism than estimation and counting. *Numerousness*, a broader term which includes subitizing and estimating, could be comparative or absolute, but in Kaufman et al.'s (1949) study numerousness was defined as the absolute numerical reporting of a collection of items. Numerousness is defined as an individual's ability to numerically report a collection of items without counting (Kaufman et al., 1949).

Kaufman et al.'s five point scale (1949) quantitatively measured recall time, recall accuracy, and confidence levels as individuals engaged with activity that required numerousness versus counting. Results from this study indicated that aspects of individuals' numerousness when numerically expressing sets of six or more objects differed from an individual's numerousness when numerically expressing sets with five or fewer objects. Estimating, a type of numerousness, had an increase in recall time, and a decrease in recall accuracy and confidence level (Kaufman et al., 1949). As a result of this study, subitizing was initially named, and defined as a different type of numerousness with a quicker, more accurate, and more confident expression of a small set of objects (Kaufman et al., 1949).

Subitizing was described as different from estimating in Kaufman et al.'s (1949) research findings because certain characteristics such as recall time, recall accuracy, and confidence levels

differed when individuals numerically expressed larger sets of items versus smaller sets of items. Moreover, this research discovered that those individuals subitizing small sets of items responded within a typical timeframe; it was calculated that each item in a perceptual field could be subitized in about 40 msec (Kaufman et al., 1949).

In the 1970's, subitizing began to be discussed in the psychology field as a form of visual *information processing* by Klahr (1973a). Information processing is based on a philosophy that a cognitive structure exists which explains how memory is stored and acted upon (Feigenbaum, 1967). Information processing adopted a learning model based on *dual-store models* (Shiffrin & Atkinson, 1969). Dual-store models are described as three areas where memory is stored or processed; 1) *sensory register*, 2) *short-term memory*, and 3) *long-term memory*. Perceptual data is stored and processed by the sensory register within a matter of milliseconds (Shiffrin & Atkinson, 1969). Short-term memory, also called working memory, processes and stores a limited amount of perceptual data (Miller, 1956) that individuals utilize with present thinking structures stored in long-term memory (Shiffrin & Atkinson, 1969). *Attention*, a cognitive mechanism which shifts or sustains focus, and filters perceptual data for relevance (Simon, 1986), plays a key role in the working memory, as perceptual data is filtered and bridged with thinking structures stored in long-term memory (Shiffrin & Atkinson, 1969). The *encoding*, of perceptual data in working memory also adds or builds thinking structures (Shiffrin & Atkinson, 1969). Encoding is defined by Atkinson and Shiffrin (1971) as a set of "control processes in which the information to be remembered is put in a context of additional easily retrievable information" (p. 3). Long-term memory has unlimited capacity and thinking structures are stored here (Shiffrin & Atkinson, 1969).

Research from these studies indicated that subitizing did not *rely* on an encoding process, but in fact *was* an encoding process, explaining why items are quantified so quickly when individuals subitize items between one and six (Klahr, 1973a). It is also indicated by these findings that subitizing activity utilized the working memory resources, but did not rely upon these resources for encoding, introducing the idea that subitizing might be supported by preattentive mechanisms (Klahr, 1973a; Trick & Pylyshyn, 1994). Preattentive mechanisms relying on these preattentive resources are defined in the following section.

An investigation on the *quantification operators* was elicited by the counting, subitizing, and estimating processing models hypothesized by Klahr (1973a). Quantification operators are defined as mental operations that individuals rely upon when numerically expressing a set of items, and which are developed by young children and are carried by adults into mathematical tasks. Subitizing is described by Klahr (1973b) as a subsystem of counting because after items were encoded, individuals matched patterned stimuli to numerical thinking structures stored in long-term memory. If no patterned stimuli existed in long-term memory relative to number, then items needed to be coded as a number through the act of counting on or back. This counting on or back activity would need to begin at one of the individual's known numbers to accurately express the quantity of items (Klahr, 1973b). Within this decade, tracking eye movement also began to be utilized and explained response error (Klahr, 1973a). Thus, this research began to define subitizing for the first time as a type of numerical encoding system and utilized eye scans when researching subitizing.

By the late 1900s, four theories existed, explaining why subitizing occurs so quickly and why the number of items subitized remains as four or less (Trick & Pylyshyn, 1994). The four typical theories at this point were a 1) *spatial indexing* explanation, 2) *limited working memory*

explanation, a 3) *patterned orientation* explanation, and a 4) *density-based* explanation (Trick & Pylyshyn, 1994). Two theories, spatial indexing and limited working memory, will be considered in relation to each other in the next section, before considering the last two theories, patterned orientation and density-based, together.

Spatial indexing and limited working memory theories. *Spatial indexing* is a visual *preattentive mechanism* based on the “INSTantiation FINger” theory (Pylyshyn, 1989, p. 4) or *FINST* theory. Pylyshyn (1989) states that the FINST theory explains how individuals perceptually point towards and encode items in their perceptual field, marking it as an item which should be given attention. Preattentive mechanisms are defined by Pylyshyn (1989) as early perceptual processing stages which serve cognitive attentional mechanisms in the working memory. *Individuation*, the actions that the FINST theory relies upon, does not consider feature or location of the item until the perceptual items are encoded and enter the working memory (Pylyshyn, 2001).

For subitizing to result in a numerical expression, relationships between items' location and items' attributes need to be considered. The working memory system resources have limited capacity, so to relieve some of these resources and offer a greater capacity to particular situations, items are typically chunked visually (Miller, 1956). This chunking of items occurs by connecting item locations, relative to each other, and item attributes, like or different groups based on attributes (Miller, 1956). This chunking allows working memory to draw from long-term memory stores to connect thinking structures to perceptual data being considered in working memory (Pylyshyn, 2001).

This attention towards relationships between items' location and items' attributes is described by Pylyshyn (2001), as *situated vision*. For example, individuation is the visual

referencing of items as “that [which] refers to something we have picked out in our field of view without reference to what category it falls under or what properties it may have” (Pylyshyn, 2001, p.129). “That” is *demonstrative*, meaning individuals consider language to be grounded *preconceptually*, or prior to entering working memory, when relating concepts to an experience (Pylyshyn, 2001). To quantify more than one item from our perceptual field, our working memory needs to connect the set of items to a numerical thinking structure from long-term memory (Pylyshyn, 2001). The items’ location and attributes need to now be described in a statement such as: there is ‘that’ green dot to the left of ‘that’ red dot (Pylyshyn, 2001). This is substantiated by Nissen (1985), as location of items was found to largely influence what an individual attended to when processing information. Therefore, it seems that even though a present dichotomy exists between the reliance of subitizing activity on preattentive resources versus attentional resources, many psychological research findings suggest that these two mechanisms actually work in coordination with each other to support both encoding and quantification of sets of items.

Regardless of this perspective, the dichotomy present in psychology research suggests that subitizing either relies solely on preattentive mechanisms (e.g., Piazza, Fumarola, Chinello, & Melcher, 2011; Shimomura, & Kumada, 2011; Trick & Pylyshyn, 1993), or upon working memory resources (e.g., Burr, Anobile, & Turi, 2011; Olivers & Watson, 2008; Poiese, Spalek, & Lollo, 2008; Railo, Kovisto, Revonsuo, & Hannula, 2008; Xu, & Liu, 2008). *Dual-task experiments* (Chesney, & Haladjian, 2011) have resulted from this controversy, where the difficulty of one task is raised, and the achievement level of another task is measured. The purpose for these types of research designs is to provide evidence that both tasks rely on the same cognitive resources (Chesney, & Haladjian, 2011).

Psychology researchers investigating cognitive resources, which support subitizing, typically follow the structure of a dual-task experiment to determine whether subitizing relies on attentional resources within the working memory, or relies on preattentive resources. Findings from dual-task experiments have indicated that although preattentive mechanisms, such as individuation, influence perceptual capabilities necessary for subitizing, attentive mechanisms also influence accurate enumeration when subitizing (Anobile, Turi, Cicchini, & Burr, 2012; Burr et al., 2010; Poise et al., 2008; Railo et al., 2008; Xu & Liu, 2008). For instance, the division of participants' attentional focus on tasks in Railo et al.'s (2008) study decreased confidence ratings and subitizing accuracy. Moreover, as the subitized quantity was increased, attentional demands were also increased (Railo et al., 2008).

Olivers and Watson (2008) initially investigated the *attentional blink* with regard to subitizing. The attentional blink, a type of dual-task design, requires participants to remember results from an encoding task, while subitizing or vice versa, remembering a result from subitizing while engaged in an encoding task (Olivers & Watson, 2008). For example, participants were shown one large letter on a piece of paper and then shown dots to subitize (Olivers & Watson, 2008). While participants encoded the letter, the dots were shown, requiring participants to use their working memory storage and processing resources while subitizing (Olivers & Watson, 2008). The basis for these types of experiments considered attentional mechanisms necessary in task control as a requirement for subitizing. The notion that subitizing relies solely on preattentive mechanisms is challenged by this perspective (Olivers & Watson, 2008). Results indicated that short lags in time between letter identification and subitizing tasks resulted in more errors compared to longer lags in time between tasks (Olivers & Watson, 2008). This higher error rate in short lag times indicated that subitizing depends on

working memory, as encoding prevents full use of working memory resources when engaging in other tasks (Olivers & Watson, 2008).

Patterned orientations and density-based theories. Just as the FINST theory and working memory theory relate in the psychology research, the *density-based* theory and the *patterned orientation* theory also relate (Arp & Fagard, 2005; Sophian & Crosby, 2008). Density-based theories are typically referred to as the *gestalt principle of perception* (Arp & Fagard, 2005). The gestalt principle of perception basically considers *point-groupings*, where distance between items suggests that some items are clustered together and some items are in separate groups (Zahn, 1971). *Patterned orientation* theories describe *canonical patterns* as influencing subitizing ability (Logan & Zbrodoff, 2003; Mandler & Shebo, 1982). Canonical patterns are patterns composed of twos (linear) and threes (triangular) which explain that individuals attend to Euclidian type shapes when subitizing (Mandler & Shebo, 1982).

To investigate these two theories, some of these studies have investigated individuals' subitizing ability with fMRIs and other neuroimaging equipment to detect active areas of the brain when subitizing, revealing areas of the brain where object activity resides (Cantlon, Brannon, Carter & Pelphrey, 2006; Demeyere, Rotshtein, & Humphreys, 2012; Heine, Tamm, Wißmann, & Jacobs, 2011). Many studies also considered individuals with visual and developmental disabilities to determine how patterned orientations and grouped items influence individuals not capable of subitizing (Arp, & Fagard, 2005; Schleifer, & Landerl, 2011). Typically, eye scans, or *saccades*, were measured to inform researchers of how visual memory relates to orientation is influenced by eye span lengths (Desoete, Ceulemans, Roeyers, & Huylebroeck, 2009; Schleifer, & Landerl, 2011; Sophian & Crosby, 2008). A critical review of these investigations considers how item orientation may influence subitizing ability.

It was first discovered that canonical patterns influence subitizing ability when results from a series of experiments indicated that children and adults attended to items patterned in groups of twos and threes (Mandler & Shebo, 1982). Arp and Fagard's (2005) research purposely considered canonical patterns as influential when investigating the subitizing impairments exhibited by children with Cerebral Palsy. Findings have indicated that individuals with right hemisphere lesions had significantly deficient subitizing abilities when numerically expressing patterned orientations versus random orientations (Arp & Fagard, 2005). Specific areas of the brain typically utilized in subitizing activity, such as the occipital area, were not found to have a significant effect on subitizing ability compared to other areas within the right hemisphere (Arp & Fagard, 2005). The parietal cortex encompasses the occipital area of the brain, so Arp and Fagard (2005) inferred that the fMRI scans analyzed in this study may not have offered enough detail to consider the occipital area of the brain as significantly influential in an individual's subitizing activity. These findings have suggested that individuals' subitizing activity may rely upon the right parietal cortex, where number processing occurs (Chochon, Cohen, Moortele, & Dehaene, 1999) and at times, more specifically, the occipital cortex, where object activity resides. Resolving this issue in the literature would mean that individuals' subitizing activity may rely on each part of the brain as subitizing activity changes, suggesting that both object activity and number processing might be necessary when an individual's subitizing activity changes.

Subitizing ability has also been investigated by Logan and Zbrodoff (2003), relative to patterned orientations. College-aged students were shown a variety of patterned orientations and asked to rate the orientations on a nine point scale as being very similar (9), or very different (1) (Logan & Zbrodoff, 2003). It is indicated by these findings that students typically found

patterned orientations to be similar when arrangements were limited to three or fewer items (Logan & Zbrodoff, 2003). It is suggested by further findings, that individual's subitizing activity is more effective when a smaller possible set of orientations exist (Logan & Zbrodoff, 2003). Whereas, when a large set of possible orientations exist, with more items in a set, individuals subitizing activity is less effective (Logan & Zbrodoff, 2003). These findings explain that individuals are limited to three or fewer items when subitizing because the variety of orientations that exist with four items compared to three items increases substantially.

Results from Sophian and Crosby's (2008) research also further supports the notion that subitizing relies upon patterned configurations as eye fixations increased when set size increased (Sophian & Crosby, 2008). Eye tracking began as a methodological approach in Klahr's (1973a) research to determine how visual scans related to individuals subitizing sets of items. The development of this approach continues to inform the psychology field as measured saccades and patterns in eye fixation determine where and how often individuals visually engage with perceptual data when subitizing (Desoete, Ceulemans, Roeyers, & Huylebroeck, 2009; Schleifer, & Landerl, 2011; Sophian & Crosby, 2008). Typically, measuring eye fixations begins by establishing predetermined regions and placing some items within those targeted regions (Klahr, 1973; Sophian & Crosby, 2008). Determining eye fixations on non-targeted regions versus targeted regions offers researchers' use of attentional resources when visually processing targeted items with and without distracters (e.g., Sophian & Crosby, 2008).

Watson, Maylor, and Bruce (2007) investigated eye movements in coordination with enumeration of small and large numerosities. Results indicated that when high density items, items, described as items clustered together more, were controlled for, restricted eye movements did not affect the response time when subitizing (Watson et al., 2007). These results indicate

that while still engaged with cognitive visual search mechanisms, such as *inhibition of return*, individuals that could not visually scan items, to exercise visual memory mechanisms (Watson et al., 2007). Inhibition of return is an innate foraging search mechanism in which individuals engage when searching an area in short spans of time (Posner & Cohen, 1984). Individuals visually scanning items typically do not return to previously searched areas, explaining that a type of visual memory engages with visual scans (Posner & Cohen, 1984). Results stemming from investigations utilizing eye scans were able to better understand how visual processing mechanisms and eye scans were related to each other.

Moeller, Neuburger, Kaufmann, Landerl, and Nuerk (2009) also considered visual processing of small numerosities versus large numerosities by measuring saccade lengths exhibited by individuals diagnosed with *dyscalculia*, a learning disability typically described as a gap between mathematics ability and achievement. Shorter saccades or shorter eye movement lengths typically impair individuals' ability to count and numerically express sets of items and results in lower mathematical achievement scores (Moeller et al., 2009). Dyscalculia is similar to dyslexia, which influences reading achievement scores (Moeller et al., 2009). However, until Moeller et al.'s (2009) study, eye tracking had not been used to investigate the nature of dyscalculia with the enumeration of small quantities of items. Specifically, this study's results describe participants as engaging in elongated fixation time periods which indicate a "noisier representation of numerical magnitude" (Moeller et al., 2009, p. 381), indicating an impaired ability to numerically encode small quantities of items.

Measured eye tracking was not the sole influential factor, as accuracy and *response latencies* also informed Moeller et al.'s (2009) study. Response latencies typically measure the response time it takes for an individual to subitize. Results from this study indicated that

children with dyscalculia did not have perceptual difficulties, but may be developmentally delayed in their ability to encode non-symbolic quantities (Moeller et al., 2009). It is also suggested by these results that subitizing is foundational for subsequent arithmetic skills, explaining the gaps which individuals with dyscalculia have in mathematical achievement (Moeller et al., 2009). Dual-task and eye tracking approaches also successfully countered the argument that subitizing relies solely on preattentive mechanisms, suggesting an additional dependency upon working memory and long-term memory resources (Moeller et al., 2009).

Eye tracking approaches introduced questions about visual processing mechanisms which individuals rely on when subitizing. Therefore, several researchers have investigated the underlying visual processing mechanisms individuals rely upon when subitizing by studying individuals with intellectual and perceptual disabilities (Arp & Fagard, 2005; O'Hearn et al., 2011). Findings from these studies indicate that cognitive development, and perceptual disabilities significantly influence subitizing ability (Arp & Fagard, 2005; O'Hearn et al., 2011).

O'Hearn et al. (2011) investigated subitizing by studying children ranging from five to six and a half years old with *Williams syndrome*, a genetic impairment affecting individuals' visual processing abilities. Results indicated that individuals with impaired visual processing abilities were unable to subitize beyond three items, whereas children the same age could accurately subitize four or five items (O'Hearn et al., 2011). Also, the ability of the Williams syndrome group's to track multiple items on a screen was limited, indicating that visual processing development influences individuation which in turn also limits the number of items that can be subitized (O'Hearn et al., 2011).

Recent psychology research suggests that the occipital area of the brain is utilized when individuals subitize (Dehaene, 2011), but when discriminating or processing number, the right

parietal cortex is engaged (Chochon et al., 1999). The occipital area of the brain is encompassed in the right parietal cortex which suggests subitizing activity changes, as number processing may begin to relate to subitizing. These findings suggest that subitizing is largely influenced by item orientation as object activity resides specifically in the occipital area of the brain (Malach et al., 1995), or item orientation and number processing, as these two types of activity reside in the right parietal cortex (Arp & Fagard, 2005; Chochon et al., 1999).

Importantly, Piazza, Mechelli, Butterworth, and Price (2002) found that subitizing and counting brain activity resided primarily in both the middle occipital and interparietal cortexes, but counting intensified use of both of these areas. However, this claim was countered by Demeyere et al. (2012), who stated that fMRI results from their study indicated that only the occipital cortex is responsible for subitizing activity. Thus, it may be questioned as to how often subitizing and counting are integrated, resulting in a combination of both processing mechanisms. An example of this is when three items are subitized twice, resulting in the numerical expression of six. Does an individual combine acts of subitizing and counting to arrive at the number six? Thus, considering these findings critically, it can be said that individuals subitizing and then drawing on the notion of number may be relying on both the parietal cortex, where mathematical thinking resides (Dehaene, 2011), and the occipital cortex, where object activity resides (Malach et al., 1995), suggesting that subitizing can draw upon both object activity and mathematical thinking processes. This does not suggest that subitizing relies on the same mechanisms as counting, but that at times individuals may unknowingly combine counting and subitizing activity (Beckwith & Restle, 1966; Wang, Resnick, & Boozer, 1971).

Investigating psychological mechanisms that individuals rely on when subitizing provides the psychology field information regarding different functions of the brain, perceptual

approaches and limitations, and attentional resources. Findings from the psychology field also clearly indicate that subitizing relies on both preattentive and attentive mechanisms, and is distinctly different from counting and estimating. However, mathematical tools and the development of early enumeration skills are rarely considered relative to subitizing in the psychology field. Implications from this research suggest that future investigations consider some attentional mechanisms that coordinate with Pylyshyn's (1989) FINST model and investigate subitizing as a higher order processing thinking model (Simon, Peterson, Patel, & Sathian, 1998).

Considering how individuals engage with particular orientations also suggests that visual preattentive and attentive resources are influenced when individuals subitize patterned orientations of items and clustered items (Anobile et al., 2012; Arp & Fagard, 2005; Railo et al., 2008; Sophian & Crosby, 2008). However, qualitative data has been lacking in this field, illustrating a need to consider transformations in subitizing development, where subitizing activity is leveraged to support counting activity or vice versa (Piazza et al., 2002). Thus, it would be important to connect these findings to mathematics education in which to better understand how number development might relate to subitizing activity.

Mathematical Implications for Subitizing

Subitizing research typically places less emphasis on mathematical implications, as perceptual processing mechanisms remain a priority in most subitizing research. As interesting as the findings from investigating the perceptual and attentional mechanisms which support subitizing are, struggles arise as attempts are made to directly relate these findings to the mathematics classroom. Thus, this section of the literature review considers the mathematical implications resulting from subitizing research. Interestingly, many researchers studying

subitizing and number understanding are affiliated with the psychology field, which tends to limit their perspectives. Regardless, these perspectives consider the differences and similarities between counting and subitizing to infer how children's subitizing can support counting or vice versa.

Subitizing activity and connections to number understanding. Freeman (1912) initially considered early number understanding to be constructed through subitizing type processes. Freeman (1912) designed patterned orientations into groups to elicit these types of observations. For instance, the typical "X" design which describes the quantity "five" and is similar to dot orientations on a die was duplicated to support individuals' numerical expression of the number 10, as individuals might subitize two sets of five to compose ten. The results from this study indicated that there were few differences between the number of items subitized by children as compared to the number of items subitized by adults (Freeman, 1912). However, Freeman (1912) noted that adults could more effectively subitize larger groups by composing subgroups. These results indicated that adults were capable of subitizing more orientations of groups, but children were limited to subitizing a smaller set of orientations because subgroups were not considered when subitizing.

Demeyere et al.'s (2010) findings also indicate that grouping small sets of items to form subgroups, in groups of two, have been considered as a way to connect subitizing and counting activities. Demeyere et al. (2010) argue that both counting and subitizing depend on different psychological mechanisms, as individuals with impaired visual processing mechanisms were more limited in their counting abilities than their subitizing abilities. Interestingly, items grouped into subgroups, composed of two items, significantly improved in their counting ability, as keeping track of items was not a concern (Demeyere et al., 2010). Thus, these findings

indicate that combining subitizing and counting activity within tasks alleviates individual's visual processing working memory resources (Demeyere et al., 2010).

A dissertation study conducted by Whelley (2002) expanded on the notion that subitizing connects to the construction of early numerical concepts. Findings from this study suggest that a *stroop interference effect*, defined as competing attributes, lengthened response time (Liotti, Worldorff, Perez, & Mayber, 1999), and prevented young children before the age of five from connecting numerical concepts to subitizing and counting activity (Whelley, 2002). For instance, when participants were shown three items to subitize, the stroop interference effect tasks required participants to subitize items that looked like the digit 5 (Whelley, 2002). Attempting to visually process three number 5 digits were found to be disruptive to children's subitizing activity, as children did not know if the number of items or the digit 5 should be given attention (Whelley, 2002).

These types of tasks elicited conflict with the participants' attention, preventing the abstract property of number to be attended towards, as items' features (the digit five) contradicted the number of items in the set (three). Findings indicated that children younger than five were incapable of coordinating competing features when subitizing as their level of maturation of *executive function*, a set of "domain-general cognitive abilities" (Kroesbergen, Luit, Lieshout, Loosbroek, & Rijt, 2009, p. 226), prevents the coordination of these competing features (Whelley, 2002). Thus, findings indicated that connections between subitizing and early construction of number is mediated through the maturation of executive function, which eventually leads towards more robust understandings of quantity and number (Whelley, 2002). These findings suggest that subitizing activity changes as organic and experiential development occurs.

Evidence of connections between subitizing and number development is provided as well by Sarama and Clements (2009) (Table 2.1). Effects from a preK through grade 2, NSF-funded program titled *Building Blocks*, were measured by Clements and Sarama (2007) to determine if particular activities within the program would support all students. Embedded in the eight number topics was a subtopic related to subitizing (Clements & Sarama, 2007). Subitizing mean scores were found to increase as much as 2.63 points throughout the course of a school year (Clements & Sarama, 2007). Findings from Clements and Sarama's (2007) study informed the hypothesized mental actions that children rely on when shifting in subitizing ability.

Table 2.1

Learning Trajectory for Recognition of Number and Subitizing

Age	Developmental Progression	Mental Actions on Objects
4	Perceptual Subitizer to Four Recognizes instantly sets of items made of four or fewer and verbally names these sets.	Low level thinking structures act on perceptual data to verbally identify sets of zero to four. Only associates with number name.
5	Perceptual Subitizer to Five Recognizes instantly sets of items made of five or fewer and verbally names these sets.	Low level thinking structures act on perceptual data to verbally identify sets of zero to five. Only associates with number name, but begins to recognize visual patterns.
5	Conceptual Subitizer to Five Verbally labels two subgroups regarding five when shown briefly.	More sophisticated thinking structures are being developed or supporting a numerical expression of groups and subgroups. Thinking structures develop from item orientations that are clustered and/or patterned.
5	Conceptual Subitizer to 10 Verbally labels two sets of subgroups up to 6 and then one more subgroup when subitizing up to 10 items.	More sophisticated thinking structures are being developed or supporting a numerical expression of groups and subgroups. Thinking structures develop from item orientations that are clustered and/or patterned.

Note. Adapted from Sarama and Clements' (2009) "Table 2.1: A Developmental Progression for Recognition of Number and Subitizing" (p. 49-50).

As a result of the *Building Blocks* program, two general types of subitizing, *perceptual subitizing* and *conceptual subitizing* (Table 2.1) were described by Sarama and Clements (2009). Perceptual subitizing, an innate ability to discriminate different quantities, emerges in infants as young as three to five months of age and is limited to five items (Sarama & Clements, 2009). An indicator of individuals perceptually subitizing is their inability to describe subgroups relative to the composite group when subitizing (Sarama & Clements, 2009). Children younger than four do not always numerically name a group of objects as high as five, but can discern between quantities as small as two and three (Sarama & Clements, 2009). Perceptual subitizing can increase to a limited orientation of items when set size is as large as five (Sarama & Clements, 2009).

The research findings in the field of psychology contradict Sarama and Clements (2009) findings, which describes students' subitizing as limited to three or four items. However, Sarama and Clements (2009) describe subitizing that relies on perceptual processes which allows students to subitize sets as large as five items. The distinction between the two findings is in how subitizing is being defined. Subitizing in the field of psychology does not describe individuals as being limited by orientation of items because dependence upon item orientation is used by only a few researchers to describe why subitizing activity changes when subitizing five (Arp & Fagard, 2005; Mandler & Shebo, 1982). However, Sarama and Clements (2009) describe perceptual subitizers as being limited to five items in rectangular arrays, as shape of items is attached to a preliminary understanding of number.

Conceptual subitizing is when children begin to connect a more sophisticated notion of number to subitizing activity (Table 2.1) (Sarama & Clements, 2009). For instance, subitizing circular arrangements of five items is indicative of conceptual subitizing, as children would need

to conceptually group items to manage starting and stopping points when subitizing. This conceptual subitizing of five begins around the age of five (Sarama & Clements, 2009). As children develop more sophisticated understandings of number, more orientations and items can be subitized (Sarama & Clements, 2009).

These findings are explained by Sarama and Clements (2009) as children being capable of cognitively composing number with chunks of two or three items when subitizing (Table 2.1) (Sarama & Clements, 2009). For instance, children perceptually subitizing four items in a rectangular array, may need to depend upon the symmetrical orientation to subitize two and then compose this two with two more to connect the number name “four” to this orientation (Sarama & Clements, 2009). This action is an example of a perceptual subitizer to four or five as it is truly empirical, meaning that the child did not mentally iterate two, but visually attended to the symmetrical arrangement of two rows each composed of two (Table 2.1) (Sarama & Clements, 2009). However, four items displayed in a line may require a child to conceptually subitize this orientation, as a child may need to cognitively break this line in half, creating two subgroups of two, before recomposing these subgroups to construct the number four (Sarama & Clements, 2009). Thus, item orientation places less emphasis on conceptual subitizing, as children are capable of cognitively reorienting items to conceptually draw upon the notion of an understood number (Table 2.1).

Upon critical review, Sarama and Clements’ (2009) distinction between perceptual and conceptual subitizing is called into question, as it is not clear how conceptual change might support a child cognitively transforming from a perceptual subitizer to a conceptual subitizer. For instance, Sarama and Clements (2009) note that children perceptually subitizing four items might be relying upon a limited set of item orientations, indicating a reliance upon a low level of

number understandings (Sarama & Clements, 2009). However, children capable of subitizing five items in a circular or linear arrangement and then describing subgroups which make up five, may be drawing upon more sophisticated levels of number understandings (Sarama & Clements, 2009). With this understanding in place, what is it that transforms a child's learning trajectory as a child's capability shifts from perceptual subitizer towards conceptual subitizer, and how are children connecting this development of number understanding directly to subitizing activity?

Lastly, when considering the distinction between perceptual and conceptual subitizing activity, it is also important to consider how these types of subitizing might support mathematics instruction in the classroom? Clements and Sarama (2007) hypothesized in an analysis of their *Building Blocks* program, that children were either perceptual or conceptual subitizers due to significant changes in subitizing ability. Thus, it would important to consider instructional models that require students to rely upon both subitizing and counting to best support children making connections between informal mathematical tasks and conceptual subitizing (Clements & Sarama, 2007). Mathematics educational implications from the research discussed in this section are quite limited, as much of this research is being studied outside of general mathematics instruction.

Summary and Conclusion of Literature Review

Subitizing research tends to be investigated in the psychology field, explaining visual processing mechanisms that individuals rely upon when subitizing. Results from these studies indicate an individual's reliance upon object location when bridging subitizing acts to number understandings. Brain imaging, dual-task investigations, and the situated vision theory all indicate the importance of object location when an individual is numerically expressing a small set of objects through subitizing acts. Moreover, patterned and clustered object orientation

theories tend to support the notion that individuals rely upon objects orientated in subgroups composed of two or three.

These psychological research findings also suggest that individuals rely upon a combination of subitizing and counting when subitizing groups larger than three. Bridging subitizing activity with sets of items as large as five may begin connecting subitizing activity from an innate process to a learned process. Additionally, linking these findings to the Sarama and Clements (2009) learning trajectory (Table 2.1) it can be noted that clustered and patterned orientations influence perceptual subitizing ability and effectively increase connections between number development and subitizing activity. Also, for children to conceptually subitize, subitizing activity must begin with clustered and patterned orientations to connect to a more sophisticated understanding of number, and the construction of number must be considered as well.

These cognitive connections may also explain why both the parietal and occipital areas of the brain are engaged when an individual subitizes. Object activity and general mathematical activity reside in either the occipital or parietal area of the brain, respectively. Yet, some research describes subitizing activity as engaging only the occipital area of the brain and some research describes subitizing activity as engaging both the occipital and parietal area of the brain. These different findings, recognized at the neuroimaging level, question whether individuals were engaging with different types of subitizing activity, or combining subitizing and counting activity. Investigating the mental actions that children carry into subitizing activity might explain some of these conflicting results.

The next section will describe the theoretical framework which will better explain how subitizing activity may connect to a child's construction of number. Radical constructivism

philosophies, a larger umbrella in this discussion, will better explain learning philosophies related to scheme theory when describing how a child's construction of number may relate to subitizing activity.

Theoretical Framework

In order to build the theoretical framework of this study the following three topics will be discussed: 1) the *construction of mathematical knowledge*, 2) topological thinking structures that support children's number development, and 3) the radical constructivist philosophy and types of abstraction relative to *scheme* theory. Connections will be made at times to subitizing, but most often subitizing activity has not been discussed in relation to many of these theories. Therefore, an argument will be made that this theoretical framework can appropriately support this study.

Construction of Mathematical Knowledge

This study is grounded in the belief that individuals construct their own knowledge. Put simply, the underlying philosophy that informs this study on how mathematics is learned is based on the belief that mathematics is not "found" or passively acquired, but rather is mentally constructed (Clements & Battista, 2002). Clements and Battista (2002) describe the *construction of mathematical knowledge* as having the following characteristics:

- 1.) knowledge is actively created;
- 2.) [individuals] create new mathematical knowledge by reflecting on their physical and mental actions;
- 3.) No one true reality exists, only individual interpretations of the world; and
- 4.) Learning is a social process in which children grow into the intellectual life of those around them (p. 6).

More specifically, Clements and Battista (2002) describe the mental construction of knowledge as a necessary component for understanding quantitative relationships between concrete and conceptual objects, further expanding upon an individual's present mathematical scheme, or organized thinking structure. In other words, as an individual acts physically and mentally upon the external world new mathematical knowledge continues to be constructed (Clements & Battista, 2002).

The construction of mathematical knowledge, as described in mathematics education, is limited with regard to subitizing. For instance, Steffe et al. (1988) explain that subitizing activity may only be a child connecting number names to orientations. Moreover, Glasersfeld (1982) states that subitizing might only be dependent upon an innate process and visual scans, indicating connections are made to only low level mathematical understandings. However as Sarama and Clements (2009) state, subitizing activity may also directly relate to the construction of number, explaining subitizing development with regard to number development. To better understand how number develops, Piaget's (1941/1965) research needs to be considered.

The construction of number has been researched primarily by Jean Piaget (1941/1965) as relying heavily upon a child's physiological development. Basically, the formation of certain thinking structures have been described as supporting a child's construction of number (Piaget, 1941/1965). Therefore, this next section centers on Piaget's research to better understand how these necessary thinking structures contribute towards a child's understanding of number.

Topological Thinking Structures

Essentially, Piaget (1941/1965) describes a child's ability to conserve number as indicative of number being understood. Before children conserve number, they rely heavily upon the amount of space the objects take up, not the number of objects within a set (Piaget,

1941/1965). Piaget (1941/1965) found that for children to conserve number, two of the three larger thinking structures or *mother structures* (*serial* and *classification*) need to be coordinated simultaneously. Children capable of conserving number need to understand that number follows a serial order (e.g., 1, 2, 3, 4, 5,) and, at the same time, understand that number is composed of subgroups through the classification of number (e.g., 2 and 3 are contained in 5) (Piaget, 1941/1965). When both of these thinking structures are simultaneously coordinated, the child's expression of number is based on logical reasoning not intuitive reasoning as order and class size both describe number (Piaget, 1941/1965).

Topological thinking structures, the third mother structure (Piaget, 1968/1970) has not been believed to directly influence the construction of number, but has been thought to directly influence the development of the classification thinking structure (Piaget & Inhelder, 1948/1967). Children developing more sophisticated classification thinking structures initially rely on the spatial relations between items to decide if the items belong in a collection or not (Piaget & Inhelder, 1948/1967). This overreliance upon the location of items indicates that children early in this development are not capable of attending to abstract features of items when grouping items (Piaget & Inhelder, 1948/1967).

Piaget (1968/1970) defines topological thinking structures as mainly relating to geometry. For instance, the concept of a line is first understood as a visual trajectory, when infants look across a perceptual field at an object (Piaget & Inhelder, 1948/1967). As visual lines intersect, different dimensions are conceptualized, supporting children's later understandings of formal Euclidean Geometry (Piaget & Inhelder, 1948/1967). Thus the topological thinking structure research that Piaget and Inhelder (1948/1967) conducted is limited, as only indirect influences on number development have been described.

Topological thinking structures, as defined by Piaget and Inhelder (1948/1967), centers on spatial relations with regard to perceived dimensionality, and includes the following five areas of development: proximity (nearbyness), separation (betweenness), continuity (connecting objects in spatial fields), order (succession), and enclosure of shape (surrounding). Subitizing is influenced by an individual's perceived space between items suggesting that topological thinking structures may be an influential factor when individuals subitize small sets of items (Arp & Fagard, 2005). Patterned orientations of items are not included in Piaget's (1968/1970) definition of topological thinking structures, but have been found to decrease children's response time when subitizing (Mandler & Shebo, 1982). Therefore findings in this study will explore the idea of modifying the scope of the topological thinking structure definition in order to include the application of patterned orientations of items, defined as an individual's perceived amount of regular space between items constructing visual patterns.

These mother structures develop as a result of connections being made between schemes, which rise in levels of sophistication through a process of *equilibrium*, as new knowledge is either *assimilated* or *accommodated* (Piaget, 1968/1970). When an individual engages an action scheme, one of two things can occur. An individual may recognize perceptual data associated with a present thinking structure and an operative action elicits an anticipated result. If the anticipated result is similar to the actual result, then the scheme is assimilated, and relationships between thinking structures are established, further raising the level of the individual's mathematical understanding (Piaget, 1937/1954). However, if the anticipated result is quite different from the actual result then the scheme is accommodated.

Modifying present thinking structures to support new unanticipated results of operative activity is described as accommodation (Glaserfeld, 1995). An accommodation of a thinking

structure can change the thinking structure one of two ways (change in the recognition template or change in the result). Assimilation and accommodation will be described in more detail in the following section titled, “scheme theory.” Individuals establish equilibrium by accommodating present thinking structures and assimilating sensory data in order to bring regularity to perceived chaos by either changing what an individual should attend to or to expanding present schemes (Glaserfeld, 1995; Piaget, 1937/1954).

The accommodation of the schemes within the three mother structures establishes more sophisticated and rigorous constructs of knowledge, further *interiorizing* conceptual information to support more sophisticated ways to express, anticipate, and organize future experiences (Piaget, 1937/1954). Interiorizing conceptual information means that mental actions have become coordinated, which results in more sophisticated levels of operating, allowing for simultaneous coordination and reversibility of more than one mental action (Piaget, 1968/1970). In other words, interiorization is when individuals’ mathematical activity becomes an object in and of itself that can be acted upon mentally as if it were an object to be acted upon empirically (Piaget, 1968/1970).

In brief, further expansion of the definition for topological thinking structures, the third mother structure would allow for a child’s perceptual and operative development regarding perceived separation between items and patterned orientations to be considered as influential in both subitizing activity and early number understandings. Presently, topological thinking structures and number development are not believed to be directly related. Piaget’s (1968/1970) research regarding these three mother structures and how learning occurs as a result of their development aligns closely with radical constructivism philosophies. Thus, the theoretical

construct, radical constructivism, will be discussed in the next section to better explain mathematical learning.

Radical Constructivism

Piaget's (1968/1970) research findings first supported the notion that mathematics is constructed psychologically. These findings also began describing a philosophy of learning, and was later termed as radical constructivism (Glaserfeld, 1995). The notion of radical constructivism is evident in Piaget's (1968/1970) book, Genetic Epistemology, where "to know is to assimilate reality into systems of transformations. To know is to transform reality in order to understand how a certain state is brought about" (p. 15). Simply, an individual's ability to construct a reality may be distinctly different than others, as each individual engages with different types of cognitive change (Glaserfeld, 1995). These philosophies that Piaget (1968/1970) described also supported the philosophy that *knowledge* was regarded, not as "a passive copy" (Piaget, 1968/1970, p. 15) of reality, but an individual's active construction of her or his own reality. Constructing knowledge is a central tenet of the radical constructivism philosophy.

Knowledge, as defined by Piaget (1977/2001) can be considered as figurative or operative. *Figurative knowledge* is defined as knowledge gained through observable data, whereas *operative knowledge* is defined as the transformations in thinking that individuals make with regard to interactions with observable data (Piaget, 1977/2001). For example, figurative knowledge describes the location and feature of items that an individual considers, but operative knowledge describes the functional purpose for the location and feature of items as it is operated on (Piaget, 1977/2001). Individuals carry into experiences an operative "knowledge of what to do with something under certain possible conditions" (Campbell, 2001, p. 2) and "knowledge of

what that thing will do under different conditions” (Campbell, 2001, p. 2). The mixture of figurative and operative knowledge is described as being mediated by space (Piaget, 1977/2001). For example, an individual observes perceptual data, an example of figurative knowledge, and then perceives a purpose for engaging with the observed perceptual data, an example of operative knowledge (Piaget, 1977/2001). However when perceptual data is operated upon, the perceived spatial relations between the perceptual data mediates both the figurative and operative knowledge (Piaget, 1977/2001). The transformation of figurative knowledge towards operative knowledge occurs through multiple spatial experiences, which is described by Piaget (1977/2001) as relying upon different levels of *abstraction*.

Types of abstraction. *Abstraction* is defined by Glasersfeld (1995) as an individual’s ability to cognitively break experiences into distinct chunks in which to be *generalized*. Generalization, a type of transformation of experiences, groups and names these chunks of experiences to be used when anticipating results from future experiences (Glasersfeld, 1995). Generalization also raises the level of engagement with an experience (Glasersfeld, 1995), Individuals capable of generalizing experiences are able to “look down” upon an experience, which is described as raising the level of abstraction an individual relies upon (Glasersfeld, 1995). Piaget (1977/2001) describes two main types of abstraction, *empirical abstraction* and *reflective abstraction*, which children rely upon when developing mathematical ideas. Piaget (1977/2001) explains that operative knowledge is exercised when transformations, such as generalizations, are made to figurative knowledge. These mental transformations support the reorganization of children’s thinking structures. This reorganization of a thinking structure changes knowledge by raising it to a more sophisticated level (Piaget, 1977/2001). Thus, individuals may begin by relying upon *empirical abstractions*, but after learning occurs,

conceptual products change and individuals begin relying upon *reflective abstractions* (Piaget, 1977/2001).

Empirical abstractions are defined by Inhelder and Piaget (1964/1999) as a reorganizing and generalizing of the essential perceptual data extracted from their environment to construct anticipated results from basic empirical actions. For instance, when a young child first learns to count objects, a child relies upon empirical abstraction (Steffe et al., 1988). Thus, as children touch each item and coordinate their words to match these touches, these actions can be repeated to find out how many items are in future sets (Steffe et al., 1988). A child relying on empirical abstractions when counting objects can consider number as an iterative process of the physical act of counting perceptual items and cannot abstract these movements to re-present number another way (Table 2.2) (Steffe et al., 1988). For mathematical thinking to move beyond empirical activity, an individual must rely on a more sophisticated type of abstraction, allowing for empirical objects within an individual's perceptual field to become coordinated and transformed into perceived mental objects (Inhelder & Piaget, 1964/1999). This transformation of thinking supports the manifestation of reflective abstraction (Inhelder & Piaget, 1964/1999).

Reflective abstraction takes place when mental actions, resulting from empirical experiences, are isolated and generalized, and capable of being coordinated. This generalized mental action can then be considered as similar or different to other mental actions (Piaget, 1968/1970). For example, a child engaged in a counting activity might be able to coordinate some operative knowledge of number without depending upon the empirical act of touching items (Steffe et al., 1988). This transfer from reliance upon empirical actions to reliance upon more abstract cognitive operations allows a child to engage more sophisticated, coordinated, mental operations when numerically expressing a set of items (Steffe et al., 1988).

Table 2.2

Types of Abstraction

Abstraction	Mental Actions Resulting from Activity
Empirical Abstraction	An abstraction of perceptual dimensions. Still relying solely on empirical actions when re-presenting thinking.
Reflecting Abstraction	Capable of abstracting a property through a coordination of actions. Can re-present thinking with an abstract property (e.g., number).
Reflected Abstraction	Consciously reflecting on a mental action. Can re-present an abstract property and identify coordinated actions.
Metareflection	Consciously reflecting on reflected abstraction. Mental action becomes a mental object in itself for an individual to operate on mentally.

Note. Adapted from Campbell's introduction of Piaget's (1977/2001) Studies in Reflecting Abstraction.

However, some of these actions are not solely empirical, but do not rely solely on reflective abstractions either. Thus, Piaget (1977/2001) describes individuals engaging with the following three types of reflective abstraction when successively coordinating more abstract conceptual understandings: 1) *reflecting abstraction*, 2) *reflected abstraction*, and 3) *metareflection* (Table 2.2). Reflecting abstraction, a type of reflective abstraction, is when an abstract property is expressed from an empirical action (Table 2.2) (Piaget, 1977/2001). For instance, when a child visually scans a set of objects, Piaget (1977/2001) states that the child's mental actions, relying on a type of reflecting abstraction, would allow number to be re-presented, a property of the set of items. However, at this stage, the child may or may not be consciously aware of this mental operation attributed to this situation. Therefore, the coordination of the empirical actions with which the child engages, results in a more abstract property, such as number (Glaserfeld, 1995; Piaget, 1977/2001). Thus, as a child repeatedly experiences similar activity, more opportunities to transfer this property to a new situation would

solidify and make a child aware of how to coordinate these actions to again raise the level of abstraction (Glaserfeld, 1995; Piaget, 1977/2001).

Reflected abstraction, a type of reflective abstraction, is defined as an individual's awareness of particular mental actions or structures that captures more than one abstract property and is employed in new situations (Table 2.2) (Piaget, 1977/2001). This type of reflective abstraction with regard to counting is described by Steffe et al. (1988) as dependent upon patterns when re-presenting multiple properties in new situations. For instance, when a child counts a patterned arrangement of a set of items, and re-presents the number and orientation with finger patterns, this child is reflecting upon the mental actions used in counting, but may still rely on a pattern to assist her ability to re-present the number with different forms of experiential material (Steffe et al., 1988).

Metareflection, a type of reflective abstraction, describes an individual's ability to reflect upon reflected abstraction, so themes and series of operations can be considered conceptually (Table 2.2) (Piaget, 1977/2001). This type of reflective abstraction supports mathematical notions of number to be carried out in more abstract situations, without solely relying upon empirical actions or items. An example of metareflection would be when a child can count a set of items without having to touch or re-present items or rely upon nodding or pointing to arrive at the accurate total amount of items being counted (Steffe et al., 1988).

Furthermore, as individuals engage in operative activity with regard to conceptual understandings of number, reliance upon metareflection can be observed when individuals engage in *mental reversibility* (Piaget, 1977/2001). Mental reversibility simply describes an individual's ability to understand operatively the structure of the activity in which the individual engaged in so that the activity is a mental object itself (Piaget, 1941/1965). For instance, a child

capable of conserving number would need to understand both that $A + A' = B$, and that $B - A = A'$ (Piaget, 1941/1965). This understanding also allows the child to utilize B as an interiorized mental object, as the structure of the mental object is understood operatively.

Scheme theory. These different types of abstraction describe a spiraling set of mechanisms that build upon the construction of *schemes*. Schemes, defined by Glasersfeld (1995), are thinking models where individuals are capable of anticipating results from mental or physical actions. *Scheme theory*, a type of learning theory based on equilibrium, explains that for learning to occur, anticipated results need to inform individuals of which mental actions need to be put into action when engaging with perceived situations (Glasersfeld, 1995). For instance, when an individual perceives a situation, an operative mental action associated with this situation informs the individual of what to cognitively do and of the anticipated result from engaging with this action (Glasersfeld, 1995). If the anticipated results match the perceived results, then assimilation has occurred and the mental actions rises in levels of sophistication and connects to similar schemes (Glasersfeld, 1995). However, if the anticipated result does not match the actual result, then learning may occur in what Glasersfeld (1995) described as an expanding equilibration. Learning can occur in one of two ways, disappointment or surprise (Glasersfeld, 1995). If the unexpected result was disappointing, then what triggered the scheme changes to accommodate this new situation to a new action (Glasersfeld, 1995). However, if the unexpected result was a pleasant surprise, then the scheme will change to include this new result (Glasersfeld, 1995). Thus, scheme theory describes how learning takes place as schemes rise in level of abstraction with each new experience (Glasersfeld, 1995).

Changes in subitizing activity have been found to result from the implementation of a prek through grade 2 mathematics program, *Building Blocks* (Clements & Sarama, 2007).

Sarama and Clements (2009) described these subitizing changes with regard to scheme theory. To explain why these changes occurred using schemes theory, Sarama and Clements (2009) hypothesized a series of mental actions that children rely upon when subitizing. These mental actions can be described with different types of abstraction that children depend upon, further refining Sarama and Clements (2009) hypothesized learning trajectory (Table 2.3). As number schemes develop and become more sophisticated, children's subitizing activity connects to more abstract notions of number and a wider variety of orientations can be subitized (Table 2.3) (Sarama & Clements, 2009). Moreover, it seems that subgroups of orientations are mentally composed in parallel with each other to support conceptual subitizing activity that Sarama & Clements (2009) describe. As children's level of number understanding rises, it is hypothesized that they are more capable of conceptually grouping items without relying solely on perceived space between items or symmetry of items. Thus, through a synthesis of Piaget's (1977/2001) types of abstraction (Table 2.2) and Sarama and Clements (2009) hypothesized mental actions (Table 2.1), a hypothetical learning trajectory (Table 2.3) can be constructed for further study, suggesting that individuals rely upon different types of abstraction when subitizing.

To better understand what this hypothetical learning trajectory (Table 2.3) explains, it is best to consider subitizing ability as relying upon different types of actions (e.g., empirical, mental, coordinated). To better discern what type of actions a child is relying upon, one needs to first understand what the child can re-present of a subitized orientation with words or drawings. If a child considers only empirical actions when re-presenting what was attended to, such as a visual scan, the child would use words describing what the child's "eyes did" and then draw the items as if they were connected. However, if a child is capable of re-presenting an abstract property, such as number, and can isolate the empirical action as relevant for re-

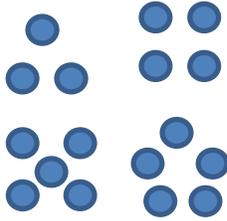
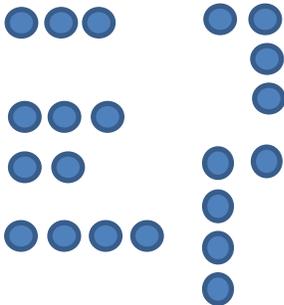
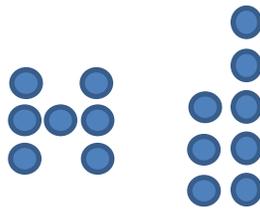
presenting the abstract property then the conceptual product of number can be described as resulting from a type of reflective abstraction, such as reflecting abstraction.

Once children can substitute necessary acts by acting on perceptual material which can stand in for the necessary acts then the child may be relying upon more sophisticated types of reflective abstraction, such as reflected abstraction, as coordinated mental actions are consciously reflected upon. Finally, a child capable of reflecting on the reflected abstraction to subitize is described as capable of mental activity without engaging in physical activity, which indicates a reliance on metareflective abstraction. This hypothetical learning trajectory (Table 2.3) suggests that actions, drawings, and words that children use when describing what they “saw” after subitizing can reveal the level of abstraction a child may be carrying into subitizing activity.

Also, children who conserve number do not rely upon the space that items take up when numerically expressing a group of items, so it would be necessary to compare how children who are *preconservers*, not capable of conserving number yet, and *conservers*, capable of conserving number, subitize different orientations. In order to consider this hypothetical learning trajectory (Table 2.3) with preconservers and conservers, it could be hypothesized that preconservers would be dependent upon empirical abstractions and rely on perceptual subitizing (Table 2.4). However, conservers might be dependent upon a type of reflective abstraction and rely upon conceptual subitizing. So, reconsidering the hypothetical learning trajectory (Table 2.4), two groups (preconservers and conservers) of four year-old children could be purposively chosen to investigate both perceptual and conceptual subitizing with regard to different types of abstractions.

Table 2.3

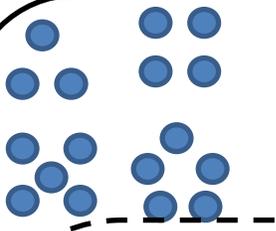
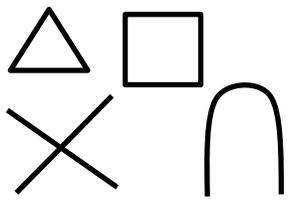
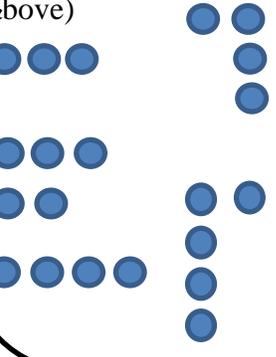
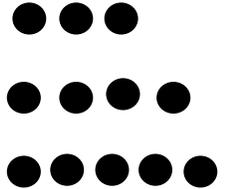
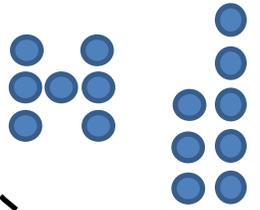
Hypothetical Learning Trajectory

Examples of Orientation of Items	Examples of Drawings of Items	Examples of Words Used to Describe Drawings
		<p>“My eyes did this.”</p> <p>“It looks like a _____ (shape name).”</p>
<p>(all orientations shown above)</p> 		<p>“I saw _____ (number name)”</p> <p>“I saw one, one, one...”</p>
<p>(all orientations shown above)</p> 	<p>Not Needed</p>	<p>“I saw three and two.”</p> <p>“I doubled three to make six.”</p>
	<p>Not Needed</p>	<p>“I saw three and three and one more.”</p> <p>“I saw three fives, which makes five, 10, and 15.”</p> <p>“I saw two less than 10, which makes eight.”</p>

Note. Synthesis of Levels of Abstraction (Piaget, 1977/2001) and Learning Trajectory for Recognition of Number and Subitizing (Sarama & Clements, 2009).

Table 2.4

Hypothetical Learning Trajectory with Preconservers and Conservers

Orientation of Items	Drawings of Items	Word Used to Describe Drawings
		<p>“My eyes did this.”</p> <p>“It looks like a _____ (shape name).”</p>
<p>(all orientations shown above)</p> 		<p>“I saw _____ (number name)”</p> <p>“I saw one, one, one...”</p>
<p>(all orientations shown above)</p> 	Not Needed	<p>“I saw three and two.”</p> <p>“I doubled three to make six.”</p>
	Not Needed	<p>“I saw three and three and one more.”</p> <p>“I saw three fives, which makes five, 10, and 15.”</p> <p>“I saw two less than 10, which makes eight.”</p>

Note. Learning Trajectory for Recognition of Number and Subitizing (Sarama & Clements, 2009).

In brief, children not capable of conserving number might rely more heavily upon perceptual items and empirical abstractions when subitizing (Table 2.4), as figurative knowledge would be primary and operative knowledge would result as secondary. Comparatively, children capable of conserving number would rely more heavily upon abstract properties of sets of items

when subitizing, revealing dependence upon reflected or metareflective abstraction (Table 2.4), as operative knowledge would be primary and figurative knowledge would be secondary.

Therefore, considering Table 2.4, it is hypothesized that preconservers will exhibit reliance upon empirical abstractions and reflective abstractions (solid line section). However, conservers may exhibit reliance upon one of the three types of reflective abstractions (dashed line section).

Comparing these different groups of students would offer different forms of development in subitizing ability to consider as well as the different types of knowledge that results from understanding number differently.

Summary and Conclusion of Theoretical Framework

Schemes either relate to other schemes, forming the boundary of a thinking structure, or become trimmed as unexpected results reveal unrelated components. When figurative and operative knowledge are constructed, schemes within thinking structures are said to change. Piaget and Inhelder (1964/1969), Glasersfeld (1995), and Steffe et al. (1988) explain scheme theory as a process of equilibrium, revealing different types of abstraction that children are relying upon.

Thus, children engaging in perceptual subitizing activity (Sarama & Clements, 2009) would be dependent upon rectangular orientations and rely on figurative knowledge of observable items (Piaget, 1977/2001). In turn, this would also indicate a child's reliance upon empirical abstractions (Glasersfeld, 1982) when subitizing. Comparatively, children engaging with conceptual subitizing activity (Sarama & Clements, 2009) would be capable of subitizing different circular or linear orientations as figurative and operative knowledge might be supporting this activity. This would indicate that a child would be relying upon a type of reflective abstraction (Piaget, 1977/2001).

Therefore, it would be important to investigate item orientation further, in coordination with subitizing ability, to explore connections between numerical scheme development and subitizing development, which might provide some insight as to how conceptual subitizing relates with the construction of number. For instance, how much does the spatial configuration play a role or indicate a child's interiorization of number evidenced with conceptual subitizing? Sarama and Clements (2009) describe hypothesized mental actions that children carry into subitizing activity, how might this relate to types of abstraction as described by Piaget (1977/2001)? Might these types of abstraction describe subitizing activity and influence the development of topological thinking structures? Would connections between the development of the topological thinking structure and changes within subitizing activity suggest that Piaget's (1968/1970) definition for topological thinking structures should be modified to increase the scope and allow for patterned arrangements to be considered when children engage in subitizing activity? Finally, if development of the topological thinking structures relate to subitizing activity, how might the other two thinking structures, classification and serial, relate to subitizing activity? Considering children who are not conserving number compared to children who are conserving number might provide an opportunity to explore the development of these thinking structures and the relationship they have with subitizing activity.

Summary and Conclusion of Chapter Two

This chapter contained two main sections. The first section reviewed the literature concerning subitizing in the psychology field and in the mathematics education field, and the second section outlined the theoretical framework. The purpose of the theoretical framework was to explain the relationship between subitizing and scheme theory by explaining different types of abstraction relative to understanding number. The relationships within the theoretical

framework suggest that if different subitizing activity is elicited by different orientations, there may be different types of abstractions that children rely upon in both number understanding and subitizing.

The following four theories explaining subitizing exist within the psychology research field: patterned orientations, density-based items, working memory resources, and the spatial indexing or FINST theory. All four theories explain how cognition influences subitizing. Although each theory explains different aspects of subitizing, how much each explains subitizing is still debatable. As each of the four theories continues to be investigated separately researchers struggle to conceptualize how they relate to each other. For instance, the FINST theory explains how preattentive mechanisms influence subitizing activity, and the density-based theory also supports the notion that the closer together the items, the shorter the saccades need to be. However, for number to be expressed, attentional mechanisms such as working memory would need to be employed. Furthermore, item location is encoded for working memory resources when visually processing items. Item location engages long-term memory schemes, as patterned orientations support perceived groups of items which eventually connects to the notion of one, two, or three.

Visual processing mechanisms investigated in the psychology field have also been used to distinguish differences between subitizing, estimating, and counting. In fact, conflicting findings describe subitizing as relying upon different areas of the brain, questioning whether there are different types of subitizing that an individual relies upon when numerically expressing a small set of items. Many researchers using neuroimaging methods have found evidence that subitizing, counting, and estimating rely on different perceptual mechanisms (Demeyere et al., 2012), but other evidence suggests that all three rely on the parietal area of the brain where

mathematical activity resides (Piazza et al., 2011). These findings indicate that individuals relying on different areas of the brain may be engaging in different types of subitizing activity.

Scheme theory can be used to describe counting as repeatable actions with perceptual data which results in reliance upon more abstract actions (Steffe et al., 1988). As an individual relies on different types of reflective abstractions her scheme raises to a higher plane, allowing an individual to rely upon metareflective abstractions (Piaget, 1977/2001). Once an individual is relying upon metareflective abstractions, the most sophisticated type of reflective abstractions, both the conceptual structure and mental reversibility of number can be considered. Specifically in regard to subitizing, Glasersfeld (1982) believes that subitizing relies upon empirical abstractions, as individuals rely on visual scans with small sets of items before expressing number. However, Sarama and Clements (2009) suggest that subitizing can rely on conceptual notions of number that allows children to subitize a wider variety of arrangements.

Thus, it would be important to specifically test the hypothetical learning trajectory (Table 2.4) which describes how level of number abstraction that children rely upon may describe children's subitizing activity as perceptual subitizing or conceptual subitizing (Sarama & Clements, 2009). Perceptual subitizing activity would indicate whether children are only relying upon figurative knowledge (Piaget, 1977/2001), resulting from empirical abstractions with regard to number understanding (Glasersfeld, 1982). Comparatively, children relying upon conceptual subitizing activity (Sarama & Clements, 2009) would be capable of subitizing circular or linear orientations. Investigating changes in subitizing activity relative to the type of knowledge children rely upon would suggest the conceptual changes children make when subitizing. Also, considering children who consistently conserve number compared to children not yet able to conserve number may also provide evidence as to how number development

influences subitizing activity with regard to the development of the serial and classification thinking structures.

Modifying Piaget's (1968/1970) definition of topological thinking structures to increase the scope of this theory to include patterned orientations would also be important. Piaget (1968/1970) describes topological thinking structures as primarily influencing an individual's understanding of Euclidean Geometry and early stages of development within a child's classification thinking structure. However, topological thinking structures may also directly influence the construction of number through subitizing activity. Piaget and Inhelder (1948/1967) suggest that a child's perceived separateness between items influences the development of the topological thinking structure, but does not specify the perceived regular space between items as influential. Item orientation has been described with separation or the space between items, but patterned orientations would increase the scope of the topological thinking structures definition to include *regular* space between items. Including patterned arrangements with Piaget's (1968/1970) notion of topological thinking structures affords the opportunity to directly relate the development of a child's construction of number to the development of a child's topological thinking structures through subitizing activity.

The research reviewed in this chapter rests primarily in a set of classic experimental approaches, suggesting that the constructed models of student thinking are separate from the mathematics classroom (Steffe & Thompson, 2000). Absent from this research is primarily the contextual component preventing educators and researchers from effectively transferring these models of student thinking into the mathematics classroom (Steffe & Thompson, 2000). Much of the subitizing research simply uses subitizing activity as a vehicle to investigate perceptual mechanisms, further separating this activity from the mathematics classroom. Thus, this present

study will consider the results with regard to the mathematics classroom to provide suggestions as to how early interventions and assessments might be used with subitizing activity.

The hypothesis put forth in this chapter (Table 2.4) suggests that children relying only upon perceptual subitizing activity (Sarama & Clements, 2009) would be limited to subitizing only patterned rectangular orientations. This would indicate that perceptual subitizing activity relies only upon empirical abstractions (e.g., visual scan) (Glaserfeld, 1982). If children only subitize patterned regular orientations this activity might also suggest a reliance upon an empirical template or structure, (e.g., shape) supporting the notion that the child has not developed an internal mental structure or template with regard to the number of items being subitized. Differences, such as this, between these two types of activity might refine perceptual subitizing activity already being described by Sarama and Clements (2009).

Questions still remain regarding how children's understanding of number relates to their subitizing activity. Number understanding is described as resulting from a coordination of the serial and classification thinking structures (Piaget, 1941/1965). Counting schemes have been found to change, indicating reliance on different types of abstractions. Thus, it would be important consider changes in subitizing activity relative to changes in number conservation to better explain why subitizing activity changes. Additionally, finding connections between number development and subitizing activity would also suggest that Piaget's (1968/1970) topological thinking structure definition should be modified to include patterned orientations. Modifying this definition would increase the scope of the application for topological thinking structures as including subitizing activity, and directly relating to number development through this activity.

Essentially, the research questions will address these questions that still remain.

Specifically this present study will address the following three questions:

1. Does item orientation (e.g., circular, linear, patterned) influence preschool age students' subitizing ability? If so, how might patterned orientations, as described by Sarama and Clements (2009), coordinate with level of number development?
2. How might preschool age students' empirical actions (e.g., counting items, re-presenting items) with a variety of item orientations influence their number development and types of abstraction carried into subitizing activity?
3. How might preschool age students' present thinking models change as item orientation that they subitize change? How might preschool age students' drawings of remembered subitized items transform in response to tasks, provoking a need to depend on a different stage of number understanding?

Addressing these questions and modifying Piaget's (1968/1970) definition for topological thinking structures to include patterned arrangements with the perceived amount of regular space between items would more closely align all three mother structures with the construction of number. Additionally, this would suggest the construction of number as more closely aligned to an individual's psychological development.

Chapter Three: Methodology

This study draws on qualitative research methods grounded in the radical constructivist paradigm, meaning that mathematics is not acquired passively, but actively constructed through interactions with our own reality (Glaserfeld, 1995). As described earlier, radical constructivism draws on Glaserfeld's (1995) basic notions that sophisticated mathematical knowledge relies largely upon reflective abstractions. The development of sophisticated schemes in mathematics is explained by this reliance upon reflective abstractions, because actions and operations need to be coordinated mentally for a student to be successful in abstract mathematical activity (Piaget, 1977/2001). The radical constructivism philosophy also considers mathematical learning as ontogenetic, meaning that humans are continually modifying their thinking structures to adapt for new experiences, establishing their own reality as unique (Glaserfeld, 1995).

In order to study changes in the student's scheme to understand ontogenetic evolution, a research framework flexibility, responding to the interactions between researchers and participants (Steffe, 1991). This flexibility prevents researchers from simply confirming predetermined mathematical knowledge carried by the researcher into an empirical study (Steffe, 1991). Therefore, this study will follow a constructivist *teaching experiment* which utilizes flexible tasks to respond to any changes in children's inferred thinking. Resulting from this study will be a mathematical thinking model describing students' subitizing activity relative to orientation and number development.

Teaching Experiment Methodology

A teaching experiment is a research methodology used to construct a model of student's mathematical thinking through first-hand experiences of children's problem solving. (Cobb &

Steffe, 2011; Steffe & Thompson, 2000). In the 1970s, teaching experiment methodology was brought to the United States from the Soviet Union by Izaak Wirzup, at the University of Chicago. This new research approach filled a void in the mathematics educational research in the United States (Steffe & Thompson, 2000). At this time, mathematical thinking was being studied with traditional experimental design approaches. Limitations arose with these research designs because the data were not directly set in the classroom and did not directly support the development of effective classroom experiences (Steffe & Thompson, 2000).

Researchers utilizing a teaching experiment do not intend to teach students, but to assess student thinking and construct thinking models that explain schemes which students carry into different mathematical experiences (Steffe & Thompson, 2000). However, it is also the intent of a researcher, when conducting a teaching experiment, to observe the restructuring of schemes as students accommodate for different or new experiences (Steffe & Thompson, 2000). Thus, learning will most likely take place throughout the course of a teaching experiment (Steffe & Thompson, 2000). Teaching experiment time frames can range between six weeks to two years (Cobb & Steffe, 2011). The longitudinal nature of these approaches allows researchers time to design a model of a student's thinking, continually test and retest their hypotheses, and for students to experience cognitive change (Cobb & Steffe, 2011).

Cobb and Steffe (2011) explain that researchers engaged in teaching experiment approaches are given many opportunities to test and retest their hypothetical learning trajectories as learning occurs, in order to better understand how cognitive change influences students' thinking models (Cobb & Steffe, 2011). Moreover, in a teaching experiment, a researcher typically begins with a hypothetical learning trajectory in mind, narrowing the tasks given throughout the course of the teaching experiment and observing student responses (Cobb &

Steffe, 2011). However, it is not advised that a researcher narrow the tasks and results too much, as unexpected responses can reveal new unexpected findings (Cobb & Steffe, 2011).

Also, the connections made by students between activities, or within activities, are unpredictable and difficult to explain. Thus, situating activities within sessions, and solutions within activities, better supports the researchers as they infer the students' own logic with regard to the student's explanations (Cobb & Steffe, 2011). Teaching experiment methodologies also align closely with philosophies stemming from radical constructivism (Steffe & Thompson, 2000). It is understood that in a teaching experiment a student's own mathematical reality may be influenced by the reality of the researcher and that radical constructivist philosophies describe each individual's reality as being distinctly different (Steffe & Thompson, 2000). Being cognizant of the underlying notion that a researcher's mathematical reality is distinctly different from a student's mathematical reality, is foundational for teaching experiments, and purposively structures the researcher's role in this methodology.

Researcher Roles

Each teaching experiment includes a teacher-researcher, a witness for each teaching episode, at least one student, and a way to record student actions and words in each teaching session (Steffe & Thompson, 2000). These essential elements prevent researchers from making assumptions of students' mathematical conceptual understandings (Steffe & Thompson, 2000). For instance, many mathematics education researchers will focus on the mathematical knowledge that the students are exhibiting, rather than how the students are operating within their mathematical knowledge (Steffe & Thompson, 2000). This attention to only whether the students mathematical knowledge is correct or not is not recommended because the purpose is to understand how the student interacts and constructs her own mathematical logic. The purpose

of a witness is to provide an outsider's perspective of the students' actions and words, in coordination with the teacher-researcher's actions and words, so opportunities to assess or perturb some of these operative thinking structures is not lost (Steffe & Thompson, 2000). Recording devices also allow researchers time to reflect upon the operative strategies students are using, after the session is finished (Steffe & Thompson, 2000).

Therefore in this study, I am the teacher-researcher, and two fellow doctoral candidates alternated as the witness. Designing teaching experiments to include two witnesses supports fresh, rich perspectives, which one witness would be unable to offer. Findings from a pilot study, which informs this investigation, have been reviewed, analyzed and discussed with both of these doctoral candidates at length. These discussions centered on the observed interplay between the students' words, actions, drawings and general responses, and my words, actions, and general task designs. Moreover, similar biases regarding radical constructivism and mathematical learning will inform both the teacher-researcher and the witnesses, providing a similar theoretical grounding for this teaching experiment approach.

Purpose for Using a Teaching Experiment Methodology

Before an analysis of teaching experiment results can be considered, the purpose for conducting a teaching experiment needs discussion. There are two purposes for designing a teaching experiment: to create a hypothesis (exploratory teaching), or to build a model of student thinking with a hypothesis already in mind (experimental teaching) (Steffe & Thompson, 2000). This present study's hypotheses have been established with a pilot study; therefore this study was focused upon experimental teaching. The pilot study conducted in the fall of 2011 was an exploratory teaching experiment. An analysis of student responses resulting from this pilot study provided aspects to consider in task design, follow-up questions for students, and three

different types of subitizing activity. Thus, the purpose for using a teaching experiment was to design students' thinking models that explained the conceptual processes each student was relying upon when subitizing. Moreover, in regard to number or orientation of items, the learning trajectory was informed by cognitive changes which students made due to learning, with regard to number or orientation of items, along with changes to the student's thinking model.

Pilot study. Results from a previously conducted pilot study significantly influenced the hypothetical learning trajectory (Table 2.4) put forward in this research. Following a three-week exploratory teaching experiment with two four year-old students enrolled in a preschool program, an analysis of student responses indicated that the students relied upon different types of abstractions when subitizing. For instance, when both students were shown a circular arrangement of five dots each student agreed that they saw five dots. After a number was agreed upon, each student was asked to draw what they remembered. One student drew an arch, while another student drew a horizontal row of five dots. The student who drew the arch described the visual scan his eyes made, re-presenting an empirical action, and suggesting a reliance upon this empirical abstraction when subitizing. However, the other student, who drew the five dots in a row, was re-presenting the number of dots as a set of items. Thus, a more abstract property resulted from this subitizing activity suggesting reliance upon an early form of reflective abstraction.

These results indicated that reflecting on orientations and comparing patterned orientations may perturb present thinking models regarding number understanding, raising the level of the student's number scheme. Therefore, was hypothesized from this pilot study that item orientation and reflecting on item orientations may influence and indicate different types of abstraction regarding number understanding that children rely on when subitizing (see Table 2.4).

Research Questions

The purpose of this study was to determine how students between the ages of three and five years old were able to subitize (perceptual and conceptual), and how levels of number understanding (preconservers and conservers) related to subitizing ability with regard to item orientation (rectangular array, circular, or linear). Investigating students' subitizing activity at this age was necessary, as Sarama and Clements (2009) hypothesized that students typically rely on perceptual subitizing at around age four before shifting towards conceptual subitizing at age five. The three central research questions were as follows:

1. Does item orientation (e.g., circular, linear, patterned) relate to preschool age students' subitizing ability? If so, how might patterned orientations, as described by Sarama and Clements (2009), coordinate with level of number development?
2. How might preschool age students' empirical actions (e.g., counting items, re-presenting items) with a variety of item orientations relate to their number development and types of abstraction carried into subitizing activity?
3. How might preschool age students' present thinking models change as orientation of items that they subitize change? How might preschool age students' drawings and re-presentations of remembered subitized items change when shown different orientations of items and would these changes in activity be due to dependence upon different forms of number understanding?

Participants

In this study, subitizing activity was investigated with six students (three preconservers and three conservers). Fifteen students between the ages of three and five years old were recruited for the screening of this study. The 15 students were enrolled in a preschool located

near a university campus. The preschool was situated in the southeastern part of the United States and offered curriculum grounded in the understanding that preschool age students learn through play and social interactions. The 15 students ranged in age from 3 years, 11 months to 5 years, 5 months. Nine students were male and three students were female. Four students spoke a second language at home. Students between the ages of three and five years old were recruited because Sarama and Clements (2009) describe students at this age as typically capable of perceptually subitizing three to five items. Students from a range of ages were chosen to represent a range of subitizing activity and number understanding described in this study's hypothetical learning trajectory (Table 2.4).

More information about how and why the six students were chosen for this study is in the procedures section. However, in this section, the six students chosen for this study are described in detail regarding age, language, and general strengths. Six out of the fifteen students were chosen for this study (Amy, Ben, Craig, Diana, Ethan, and Frank). From this point forward, the students will have their age in months and years after their name as a number to indicate years with a period and a number to indicate months in age. At the onset of this research study, the age of each participant recruited was required for the screening interviews. Each participant's guardian or parent provided a demographics form on which was recorded the student's birth year and month. The researcher then calculated each participant's age in years and months.

Amy (4.2)

Amy was a female preschool aged student whose family was from India. She spoke only English, but in her home, a second language was spoken. Amy was in the prekindergarten class described as the "Yellow Room," and loved purple and Hello Kitty. Amy first was interviewed on May 28th and then again on May 29th. Throughout the interviews, Amy responded

confidently and seemed to enjoy showing me what she knew about math. Quite often when she was asked why she knew something, she would say, her “mind knows” or that she knew it because she knows “math.”

Ben (5.1)

Ben was a male preschool aged student whose family is from Spain. English and Spanish were spoken in his home, and English was spoken in school. Ben was in the same prekindergarten classroom as Amy, described as the “Yellow Room,” and playing with trucks and outside with his friends was an activity he enjoyed. Ben was initially interviewed on May 28th and then again on May 29th. Throughout the interviews, Ben responded hesitantly and seemed to be fearful of answering incorrectly. Quite often when he was asked why he knew something, he would say, “Because...I just knew it.”

Craig (5.1)

Craig was a male preschool aged student in the same prekindergarten program, but a student in the “Orange Room.” English was Craig’s only spoken language both in school and at home. Craig enjoyed building things with blocks, and specifically designing hollow structures that could hold his favorite toys. Craig was initially interviewed two times on June 4th. Throughout the interviews, Craig responded hesitantly but did not seem to be fearful of answering incorrectly. Quite often when he was asked to engage in difficult tasks, Craig asked if he could draw, build or re-create the arrangement of items before answering or to use this activity to replace a verbal response.

Diana (5.5)

Diana was a female preschool aged student enrolled in a prekindergarten program and also a student in the “Orange Room” with Craig. Diana was interviewed two separate times on

June 4th. Diana enjoyed writing and drawing during class and would often be found making books about bunny rabbits. English was Diana's only spoken language both at home and at school. Diana was hesitant to respond when tasks were too difficult for her, and seemed to concentrate for considerably long periods of time when engaged in these difficult tasks. Diana was verbally capable of explaining in great detail her logic when solving tasks.

Ethan (3.11)

Ethan was a male preschool aged student who spoke only English both at home and in school. Ethan was in the "Red" prekindergarten classroom and enjoyed sharing with us toys and items from both school and home. Ethan was initially interviewed two separate times on June 4th and June 5th. Throughout the interviews, Ethan responded with confidence and did not seem concerned whether his response was accurate or inaccurate. Ethan always relied on counting to explain or solve a task, which confused him at times, because his counting was inaccurate and differed from the logic he might have relied on when solving some tasks. Also, quite often when Ethan was asked why he knew something, he would say, "because, I just really know."

Frank (4.5)

Frank was a male preschool aged student whose family is from China. English was spoken by Frank at both home and school, and Mandarin was spoken at home. Frank was in the same prekindergarten classroom as Diana and Craig, described as the "Orange Room," and he enjoyed Angry Birds. Frank was initially interviewed two separate times on June 5th. Throughout the interviews, Frank responded confidently and seemed to not be afraid of answering incorrectly. Quite often when he was asked why he knew something, he would change the subject or use actions to explain his thinking. Frank was just learning how to write numerals, and often would write the number of dots he saw with both a drawing and by writing

the numeral. Frank also attended to the color of the items quite often and described them as matching other colors or his angry birds on his shirt. Throughout the teaching experiment sessions when Frank seemed to misunderstand a task objective, one of the witnesses (fluent in Mandarin) would restate the task's objective in Mandarin to be certain he understood what was required of him.

Only four of the students (Amy, Ben, Diana, and Frank) were considered for this particular study. However, all six students engaged in 16 to 22 teaching experiment sessions. Each student worked on tasks with the teacher-researcher individually, as the interaction between participants in the pilot study was not effective.

Procedures

Two screening interviews were used in this study to purposively select the six students who participated. The overall purpose for the screening interviews was three-fold. First, the interviews were used to determine which students were able or unable to conserve number. Second, to control for the chance that a student may have a visual processing disability, counting and subitizing were used to determine whether students were relying upon developmentally appropriate visual processing mechanisms (Demeyere et al., 2010). Third, initial types of subitizing were assessed to attempt to include students relying upon different types of abstraction. This section describes the tasks used within each of these clinical interviews to select participants who were either preconservers or conservers, exhibited developmentally appropriate counting and subitizing activity, and relied on different types of abstraction. This section also describes the criteria which were used to choose participants for this study.

Screening Interviews

All 15 students participated in two interviews to determine 1) whether children are able to conserve number or not conserve number, 2) whether children depend upon typical average visual processing mechanisms (Demeyere et al., 2010), and 3) whether students rely on a variety of different types of subitizing activity. The first series of interviews (interview A) assessed students' number conservation (Appendix A). In the second series of interviews (interview B) the child's type of subitizing activity and counting ability was initially assessed (Appendix B). Results from these screening interviews were used to choose the six participants for this study. Following a description of the tasks used in the screening interviews, the criteria used to make the choices will be described in more detail.

Interview A tasks. Interview A tasks had one aim for screening the students: to assess their ability or inability to consistently conserve number. To do so, three Piagetian (1941/1965) tasks were used with multiple follow up questions to best determine how changes in the space between perceptual data influenced students' level of correspondence and understanding of number. The first task consisted of a student describing the correspondence between two rows of cubes (each ranging between six and seven items) each row with the same number of items (Piaget, 1941/1965).

The second task consisted of a student describing the correspondence between water glasses and bottles (Piaget, 1941/1965). Bottles (ranging between five and seven) filled with water were placed in a row in front of the student. The student was asked to put out as many water glasses as there were bottles, so that each water bottle had one water glass. The researcher showed the student more glasses than bottles and asked the student to put on the table "enough glasses for the bottles, just one for each" (Piaget, 1941/1965, p. 43). Students capable of

arranging the glasses alongside the bottles, and also capable of corresponding one glass to one bottle were asked if “they are the same” after the glasses or bottles are bunched together in a shorter row (Piaget, 1941/1965). Students unable to arrange the glasses so that each bottle has exactly one glass corresponding with it were asked follow up questions such as, “Are there more here?” “Are they the same?,” and “Is there the same number of glasses and bottles?” (Piaget, 1941/1965, p. 43). These questions were used to be certain that the student understood the expected task.

The third conservation task was very similar to the glasses and bottles task, but involved flowers and vases (Piaget, 1941/1965). Vases (ranging between five and seven) were placed in a row in front of the student. The student was asked “what shall we put into these vases?” (Piaget, 1941/1965, p. 49). If the student did not know, then it was suggested that we use the flowers sitting beside the researcher. The researcher showed the student more flowers than vases and asked the student to put on the table “one flower for each vase, as many flowers as vases” (Piaget, 1941/1965, p. 49). Students capable of arranging the flowers alongside or in the vases, and are also capable of corresponding exactly one flower to one vase were asked if “they are the same” after the flowers or vases are bunched together in a shorter row (Piaget, 1941/1965). Students unable to arrange the flowers so that each vase has exactly one flower corresponding with it were asked follow up questions such as, “Are there more here?” “Are they the same?,” and “is there the same number of flowers and vases?” (Piaget, 1941/1965, p. 50). These questions were used to be certain that the child understood the expected task.

Students persistently unable to describe a consistent quantity when the space between the items changed were considered as preconservers. Comparatively, students capable of consistently understanding that the space between the items does not change the quantity were

considered conservers. Follow up questions were used as noted above to best determine whether students understood the task objectives throughout the conservation tasks.

Interview B tasks. These interviews consisted of students subitizing typical patterns, drawing what they remembered, and then counting five or more items in rows and in random orientations (Appendix B). Orientations used for the subitizing tasks ranged between two and five items and remained in patterned orientations. Students were asked to “draw what you remember seeing” in order to better understand the properties of the group to which they are attending, when subitizing. This present study required students to rely upon a variety of types of abstraction. Therefore, when students “drew what they saw,” different drawings were categorized by inferred types of abstraction.

The purpose for the assessing of students’ ability to keep track of items when counting was to determine if students’ perceptual field was of typical average development (Demeyere et al., 2010). In Demeyer et al.’s 2010 study, it was found that six year-old children with perceptual processing disabilities were unable to accurately count sets of items ranging from six to nine items. Due to the age of the students participating in this present study, the number of items ranged from five to seven items and were arranged in a row to appropriately support the students’ developmental needs (Schaeffer, Eggleston, & Scott, 1974). If the students were successful with this task, the items were pushed into a random arrangement or increased in number to better understand how the students’ ability to keep track might change without the row or when the items increased.

Criteria for selecting participants. Six students were chosen to represent two groups. The two groups were made up of three students who were unable to conserve number and three students able to conserve number. In addition, the six students were chosen to represent a

reliance upon varying types of abstraction when subitizing. After the 15 students were interviewed, transcripts were written and analyzed. Three student interviews were not considered, as the students' engagement with the task was greatly influenced by their affect. Basically, the students' responses were either too quiet or it appeared they did not engage with the tasks presented to them. Therefore, because these interviews were disregarded when considering students for this study; only 12 students' responses are considered in the main results (Appendix C).

Out of the 12 remaining students, six students were capable of conserving number and six students were considered as preconservers (Appendix C). When counting, nine children had one to one correspondence and the remaining three children were either skimming when counting, skipping a perceptual item when counting, or touching and verbally counting a perceptual item more than once (Appendix C).

When these 12 students subitized dots ranging from 3 to 5, five children seemed to primarily rely on thinking models which required perceptual subitizing, while the remaining seven children tended to rely on thinking structures which required conceptual subitizing. This was evident both in their drawings and in the words they used to describe what they saw and how many they saw (Appendix C). More specifically, four students grappled with re-presenting the orientation or the number when the number of dots reached five or more. Comparatively, four students drew rows of circles when asked to draw what they remembered seeing after subitizing, but were inconsistently accurate in their subitizing activity with regard to number. Two other students inconsistently re-presented orientation and number, and were accurate with both until shown a regular pentagonal arrangement of five dots. At this point both students either accurately named the number of objects but then drew circles until they filled a circular space, or

inaccurately stated the number of objects but then drew the accurate number of circles. Finally, the last two students were capable of attending to both the orientation and the number of dots when drawing what they remembered after subitizing.

The six students (i.e., 04, 05, 09, 10, 11, and 12) were chosen because they seemed to offer the most variety in terms of demographics, number understanding, and subitizing activity (Appendix C). Each student was given a pseudonym in an alphabetical order which aligned with the respective order of the numerical sequence. Therefore, student 04's pseudonym began with A and the student was named as Amy. Student 05's pseudonym began with a B and was Ben. Student 09's pseudonym began with a C was Craig. Student 10's pseudonym began with a D and was Diana. Student 11's pseudonym began with an E and was Ethan. Finally, student 12's pseudonym began with an F and was Frank.

In choosing the six students, three students were initially assessed to be preconservers (Ben, Craig, and Ethan) and three students were initially assessed to be conservers (Amy, Diana, and Frank). Two students were female (Amy and Diana) and four students were male (Ben, Craig, Ethan, and Frank). All six students' subitizing activity varied from what seemed to be a reliance upon empirical abstractions to reflected abstractions.

Also, interview footage was reviewed more closely to consider the order the dots and circles which were drawn by the students, to infer whether the dots were re-presenting a shape or subgroups. The four dots were re-presented, after subitizing, in the shape of a square by three of the six students; Craig, Ethan, and Frank. The dots were drawn in order by one student, Craig, making the outside edge of a square. However, two students, Ethan and Frank, drew two dots in a line and then two dots below these in a line. This may indicate a more abstract notion of four,

as symmetrical aspects might be influencing the notion of subgroups, whereas Craig's orientation is just mimicking what the student's eyes might have done.

Also, with regard to counting ability, two students, Ben and Ethan, were still skimming, while the remaining four students had 1:1 correspondence. The ages of students varied also, as Ethan was almost 4 years old, two students, Amy and Frank, were 4 years and a few months old, and three students, Ben, Craig, and Diana, were 5 years and a few months old. Lastly, three students, Amy, Ben, and Frank spoke two languages at home, offering diverse cultural perspectives.

As described earlier, initially, three students in this study were assessed to be conservers, and three students in this study were assessed to be preconservers. Purposive selection of students for this study was based on these two characteristics as it is hypothesized that subitizing activity will change when students are capable of conserving number. For instance, students conserving number may be able to subitize larger sets of orientations and a larger variety of orientations, as subsets of items will be considered by students as composing larger sets. This type of subitizing activity suggests that students may rely upon reflective abstractions when subitizing orientations as large as four, five and six. In comparison, students unable to consistently conserve number, preconservers, may only be able to subitize smaller sets and be limited to a smaller set of orientations, as placement of items relative to others will scaffold a child's ability to compose subsets of items. Thus, it was anticipated that preconservers relied upon empirical and possibly reflecting or reflected abstractions when subitizing orientations between three, four, and five.

Teaching Experiment Session Tasks

This 11-week teaching experiment was comprised of two-day per week sessions in which tasks were designed to either assess or provoke a student's present thinking model. Designed to assess a student's present thinking model tasks asked a student to subitize a set of items, draw or use counters to show what she remembered, and use words to justify why the number of items seen by the student was accurate. Drawings and counters were used to better understand what the students perceptually attended to when subitizing. Inhelder and Piaget (1964/1999) used drawing tasks to better determine what children would anticipate. Drawing tasks were considered an effective design in this study, as it informed the researcher of what properties of the perceived items the students were attending to or relying upon when subitizing. Aligning both sets of drawings with the language of the student describing how the number was composed, informed me as to student anticipations of knowing about number as well as the abstract or empirical properties perceptually attended to after subitizing the same number of items.

Tasks. Tasks were designed prior to the teaching experiment sessions, allowing for variance throughout so that the tasks could appropriately assess or perturb the students' thinking models. The four following types of games were designed prior to the teaching experiment sessions: 1) Draw what you saw, 2) Camera game, 3) Concentration, and 4) Board games (Appendix D). Two board games were designed for this study, the ice cream game, and the penny bug game. The ice cream game had an ice cream cone with six ice cream scoops. Each ice cream scoop had a different arrangement of dots, usually each with the same composite group of dots. The goal for the game was for the students to roll a die and find a match between the dots or numerals on the die and the dots in the ice cream scoops. If a match was justified, then the ice cream scoop could be colored in. The first player to color in all of their ice cream scoops

won the game. The dots and numerals on the die rarely matched the arranged dots on the ice cream scoops exactly, presenting an opportunity for a student to attend to subgroups that might be new to the student. The composite groups always matched the students' ice cream scoop composite groups. So, if the student had four dots arranged in a variety of ways on her ice cream scoops, then her die also had subgroups totaling to four on each face of the die. The die the teacher-researcher used had a range of two to seven dots on each face of the die, offering a much slimmer chance for the teacher-researcher to win the game.

The penny bug game was also a matching game, but more like a game board with a start and finish. The goal was to be the first to get the penny bug to the finish. Each space was designed to look like a leaf and the final space, the finish, looked like a flower. There were fifteen leaves in all, arranged in three rows. Some leaves had dot orientations, and some leaves did not have any dots on them. The flower always had an arrangement of dots, and was usually meant to elicit subgroups which the students had difficulty subitizing, or the composite group difficult for the students to subitize. When the students rolled the die, they had one of two choices. The student could hop their penny bug forward the number of dots that came up on the face of the die, or they could skip ahead to a space where their die face matched a leaf or the flower with the same number of dots on them. Similar to the ice cream game, the teacher-researcher's die never matched the game board as well as the student's die, offering the student a greater possibility of winning.

The concentration game was used once, and was adapted from Clements and Sarama's (2009) activities. Essentially, this game had six to eight cards with dots on each. The cards were faced down on the table. The student was required to find a match between cards, but the

dots on the cards did not match each other in orientation, but they did match in number.

Therefore, the students needed to rely on more abstract properties to even engage in the game.

The camera game was also adapted from Clements and Sarama's (2009) activities. Clements and Sarama's (2009) camera game used a computer program, but in this study the activity had a series of camera pictures on a three-ring notebook. The student was shown quickly an image of the back of a camera with dots arranged in the viewfinder. The student was told that my camera had taken a picture of some dots, and the student was asked how many dots were seen. Then the student drew or used counters to show me what the picture would look like when it came out of the camera. Variations of this game were also used in which the student was required to also choose from a set of cards a possible picture that might have come out of the camera, and to draw their favorite way to show a certain number of items that they saw in the camera's viewfinder.

The most basic activity, used frequently in this study, was for the student to simply subitize a set of dots or counters and then to draw or use counters to show me what they "saw" or "remembered." This activity was also followed up with "how do you know you saw _____?" Students were given either material to draw what they remembered or counters to re-present what they remembered.

Finally, two additional tasks were designed throughout the course of the study. One task offered to the student an opportunity to be the teacher, and the teacher-researcher to be the student. The teacher-researcher made *essential mistakes* to perturb student thinking and elicit a student's attention towards these mistakes. The second task was a hidden picture activity. The hidden picture was a coloring page on which the student needed to hunt for items that re-presented a number such as three, four, or five. Once the student found a section in the hidden

picture with dots that matched this number, the student then needed to color in this section.

When all of the sections with those matching dots were colored, the student would find a picture of a dinosaur, space ship, balloons, or acorn.

Tasks in which cards or die were used always had blank cards and a die set aside.

Providing these extra materials offered opportunities for the witness to design items in an ad-hoc situation. Also, the color red was frequently used with black, as an alternative color when eliciting attention towards subgroups of dots. Similarly, two color counters were offered to the students, with one side of the counter red, and the other side of the counter yellow. In this way, attention towards subgroups was supported, and activity connected from the red and black dots to re-presentations of the color of the dots.

Item arrangement was purposely designed prior to each teaching experiment session to reveal nuances in the types of abstraction which students rely upon relative to different levels of number understanding when subitizing these orientations. Many more arrangements were designed both ad-hoc and post-hoc, as unanticipated responses provided the teacher-researcher and witness opportunities to understand the conceptual processes the student might have been relying upon.

Forms of Analysis

The purpose for Steffe and Thompson's (2000) to description of a student's own mathematical reality was two-fold. As stated earlier, a student's mathematical reality can change as learning promotes a reorganization of a student's thinking model, which then can inform curriculum and instructional design. However, understanding a student's mathematical reality is for ontogenetic purposes also, meaning that the internal cognitive adaptations that are observed in the form of student responses informs researchers how human intelligence is

constructed in mathematics. The notion of mathematics as being constructed cognitively reveals mathematics as a product of human intellect, not a product of an objective world (Steffe & Thompson, 2000). Thus, the analysis of student responses in teaching experiments informs researchers of how humans construct mathematics presently to better understand how humans may have constructed mathematics centuries ago (Steffe & Thompson, 2000). For the sake of this study, two forms of analysis informed the results, a *conceptual analysis* and a *retrospective analysis*.

Conceptual analysis. Teaching experiments rely heavily upon students' actions and words to inform researchers of how students understand certain mathematical concepts (Steffe & Thompson, 2000). Moreover, it also becomes important to also explain students' actions and words in response to the teacher-researcher or witness's actions and words (Steffe & Thompson, 2000). Analysis of this type of interplay between the teacher-researcher and the student is described by Glaserfeld (1995) as *conceptual analysis*. Conceptual analysis involves the process of observing what students say and do as a result of what a teacher-researcher says and does to attempt to understand how each student's own mathematical realities change relative to each task (Steffe & Thompson, 2000).

Each session was video-taped with two video cameras to ensure words, gestures, and drawings were captured for the analysis. Transcriptions of purposively selected video-footage informed the teacher-researcher and witness as to how to best design details of subsequent teaching experiment tasks. The role of the witness in this stage of analysis was crucial as this perspective offered an outsider's view of the interaction between the teacher-researcher and student which could not be considered by the teacher-researcher during the session itself. Providing insight of unproductive interactions between the student and the teacher-researcher

prevented lost opportunities to elicit productive interactions throughout the teaching experiment session. The witness also assisted the teacher-researcher in the on-going analysis of the students' cognitive changes between tasks and sessions, as *essential mistakes* that each child made informed the teacher-researcher of the limitations of the student's present thinking model. Essential mistakes are student responses that may be construed as inaccurate, but which explain the boundaries of a student's thinking model. Conceptual analyses came in the form of both written communication with both witnesses and verbal communication with both witnesses. The conceptual analysis became more in-depth for the final six teaching experiment sessions, as the nuances of the students' thinking models were being investigated. Cognitive changes were slight between teaching experiment sessions and between tasks. However, larger changes to the students' thinking models were observed when overall analyses of the data were considered.

Retrospective analysis. Overall analysis or *retrospective analysis* reveals more broad transformations. These overarching conjectures were considered when a preliminary thinking model had been constructed after six teaching experiment sessions. Retrospective analysis occurred approximately six times (after teaching experiment sessions 6, 8, 13, 16, 18, and 22) throughout the course of the study. When a retrospective analysis occurred, the teacher-researcher considered all past sessions in the form of notes, transcripts, and video footage. These analyses were then shared with both witnesses in a group meeting. Three group meetings (after teaching experiment sessions 6, 13, and 18) between the teacher-researcher and witnesses took place allowing multiple perspectives to inform the analyses of the data.

Revisiting data in a retrospective analysis grounds conjectures made by researchers and provides further evidence as to general patterns that explain students' cognitive changes. Essentially after teaching experiment session 8, the teacher-researcher utilized retrospective

analysis to better understand how a student's thinking model reorganizes itself to successfully complete new tasks. The final retrospective analysis allowed the teacher-researcher time to reflect on recurring student responses in coordination with teacher-researcher tasks. Conjectures made following the course of the entire set of teaching experiment sessions informed the teacher-researcher at a deeper level of understanding, as findings were able to be generalized to depict typical conceptual changes necessary in the construction of the students' thinking models. This stage of analysis was essential in constructing typical learning trajectories that children traverse when connecting number development to subitizing activity.

Chapter Four: Analysis

This chapter is divided into four sections. First, to establish evidence of the students' initial thinking models, I offer evidence of the four students' responses in the screening interviews to better explain initial thinking structures regarding their number understanding and subitizing activity. The purpose for the second section is to both define and describe seven new types of subitizing activity, and then to focus in on the perceptual subitizing activity that Ben and Frank demonstrated throughout the course of the study. Perceptual aspects of orientations and color will structure these descriptions to explain changes Ben and Frank made to accommodate their thinking models relative to the subitizing tasks in which they engaged. In the third section I describe a chronological series of Amy's subitizing activity and some of Diana's subitizing activity relative to Amy's thinking model. The purpose of this section is to describe changes Amy and Diana made in order to accommodate different subitizing tasks. I also emphasize cognitive changes Amy made when bridging her perceptual subitizing activity to conceptual subitizing activity. Amy is the primary focus for this discussion, as her learning trajectory was distinctly different from Diana's learning trajectory and offered strong connections between her perceptual and conceptual subitizing activity. The purpose of the fourth section is to synthesize these discussions to suggest relationships between the different types of subitizing activity, certain topological influences, and how conservation of number may be indicative of certain subitizing activity. At this point, overall changes that Amy and Frank made to their thinking models will be discussed and compared to describe transformations that students made relative to their operative knowledge and the perceived topological influences.

Screening Interviews

In the following section, I describe how four students responded in the clinical interviews to establish evidence of the students' initial thinking structures regarding their understanding of number and subitizing. The protocols within the interview section are labeled as "Protocol [student's first letter of their name] [sequential number of the protocol for that student]" (e.g., Protocol B1 to indicate Ben's first sequential protocol). Similarly, within each transcription in a protocol, the first letter of the student's name will indicate the student's response, whereas the letter "T" indicates how I, the teacher-researcher, responded, and the letter "W" indicates how the witness responded. Also, descriptions in brackets indicate actions which are deemed by the teacher-researcher as important to identify. As described in chapter 3, the student's age in years and months at the onset of the research is shown immediately following their name. The first number represents the age of the student, followed by a decimal point and the number of months in the student's age.

Ben (5.1)

Ben was initially interviewed on May 28th and then again on May 29th. Throughout the interviews, Ben responded hesitantly and seemed fearful of answering incorrectly. Quite often when he was asked why he knew something, he would say, "Because...." Ben was inconsistently conserving number and skimmed items when counting, which usually resulted in a final amount described as "ten." When engaged in conservation tasks, Ben sometimes described the longer row of items as having more, the bunched up items as having more, or the two rows as having the same number of items. It was determined that Ben was a preconservator because he still had trouble even deciding when the two rows of blocks were the same length and that each row had the same number of items.

For example, in the following protocol, Ben inaccurately “counted” the six blocks in each row and stated that there were 10 in each row, which allowed Ben the logic that each row should have the same number of blocks in them. So, it seemed that counting was explanation enough as to why the rows could be the same. However, because the counting was inaccurate, it might be that he was taught that counting would help him understand they were the same, but Ben might not have been truly using this counting to conserve number.

Protocol B1.

T: Today Ben we’re going to just play with blocks [reaches over to the side and picks up twelve blocks], look at some vases, and do some counting. Okay? Alright, I am going to give you a row of blocks [places six blocks in a row in front of the student], and I’ve got a row of blocks [places six blocks in a row of the same length in front of the researcher]. Alright? So this is your row [places hands on student’s row of blocks] and this is my row [places hands on researcher’s row of blocks]. Do we have the same?

B: [Shakes head side to side to indicate “no”].

T: The same number?

B: [Shakes head side to side to indicate “no”]

T: No? How do you know?

B: Because...

T: Okay, let’s test it out. If you have this one [points to block in student’s row], do I have this one [points to block in researcher’s row directly across from the student’s block]?

B: No.

T: This is my row [points to the row of blocks in front of the researcher], and this is your row [points to the row of blocks in front of the student]. So, if you’ve got one here [points to block

in student's row], I've got one here [points to block in researcher's row directly across from the student's block].

B: [nods head up and down to indicate "yes"].

T: How about here. I've got one here [points to block in researcher's row], and you've got one here [points to block in student's row directly across from the researcher's block].

B: [nods head up and down to indicate "yes"]

T: How about here [points to block in student's row and then the block directly across from this block in the researcher's row]?

B: [Shakes head side to side to indicate "no"].

T: Yeah. You've got one here [points to block in student's row], and I've got one here [points to block in researcher's row directly across from the student's block]. Yeah? So, how many blocks do you have?

B: One, two, three, four, five, six, seven, eight, nine, 10 [points to the first block, but skims finger in front of the blocks and states the number words quickly to reach 10 before his finger reaches the end of the row of blocks].

T: Okay. So, how many do I have?

B: Ten.

T: Ten? How do you know?

B: 'Cause I counted mine and I know it.

T: You counted yours, and because I have some here, I have the same?

B: hmmm...mmm [indicating "yes"].

T: Okay. So we have the same, right?

B: hmmm...mmm [indicating "yes"].

T: Alright, now if I do this to them [spreads the row of six blocks in front of the student apart, so that the row is now much longer than the researcher's row], do you still have the same number of blocks that I have?

B: Yes.

T: Yes? How come?

B: I just counted them like that when they were together [puts hands together to indicate when the blocks were closer together], one, two, three, four, five, six, seven, eight, nine, 10.

T: Okay, but yours [points to student's row of blocks] takes up more space than mine [points to researcher's row of blocks]. Do we have the same number?

B: [pauses to look at each row] yes.

T: Yes? How do you know that?

B: 'Cause I counted mine.

T: And did mine change it all, and did yours change at all?

B: [Shakes head side to side to indicate "no"]

Again, this protocol is evidence that Ben seems to be accurately carrying into the conservation tasks an empirical counting act which allows him to appear as if he is conserving number and engaging in some reversibility of thinking, as Ben uses his past counting to explain what he did before as representing the number regardless of the orientation. However, this seems to be early in its development, as he was unable to use these counting acts when first shown the two equal length rows. Also, in the following protocol, Ben was given another conservation task and he was unable to conserve number when the flowers were bunched up. Also, it is interesting that Ben considers the bunched up flowers as being "more." This notion

that the density of the items may influence this conservation activity is one that we saw with other students and will be discussed in more detail later.

Protocol B2.

T: So, I am going to take these flowers out [takes flowers out of vases and places them in a pile in front of the middle two vases]. So, do we still have one flower for each vase?

B: No.

T: No? How come?

B: I just see it. I don't know.

T: You don't know [spreads vases out a bit]? Do we have enough flowers for each vase?

B: [nods head up and down to indicate "yes"]. Yes.

T: Yes? How come?

B: 'Cause there is [points to table] plenty for now.

T: There is plenty for ...what?

B: Flowers!

T: There is plenty of vases for flowers? How do you know?

B: Because I saw them.

T: You saw them where?

B: Here [points to group of flowers on the table].

T: Here [points to flowers on the table]? What about seeing the flowers here tells you that?

B: There is more of them [points to flowers] than them [points to vases].

These inconsistencies seemed to indicate that Ben was still a preconservers, and furthermore that for him, the flowers bunched together had more items than the vases, which

were spread apart. This seemed to be something that Ben was grappling with when subitizing as well and will be discussed further on in the analysis.

Ben's subitizing activity seemed to suggest reliance upon some reflecting abstractions and some empirical abstractions. For instance, Ben could subitize two, three, and five items when arranged in a row accurately. When Ben drew those items accurately in a row, he relied on his attention to number, as number had precedence when the subitizing was re-presented in his drawings. However, when Ben was shown three items in a triangular orientation, he stated that he saw "four" and drew four items in a row. Comparatively, when Ben was shown four items in a square orientation, he stated that he saw "three" and drew three items in a triangular orientation. This activity seemed to suggest that Ben was switching from attention towards the numerosity of the set towards part of the arrangement of items.

Furthermore, when Ben was shown the five dots shown in an "X" arrangement, similar to that on a die, Ben stated that he saw five dots, but when he drew what he remembered, he drew seven circles in two rows. The top row had three dots, and the bottom row had four dots. This also seemed to illustrate a reliance upon empirical abstractions when subitizing because when Ben had to choose between number and orientation he considered the orientation. This suggested Ben's reliance upon a visual scan or visual template when subitizing. Also, when Ben was shown five dots in a pentagonal arrangement, he stated that he saw "ten" dots and then drew eight circles in a circular arrangement. This is evidence that suggests that Ben was relying upon empirical abstractions when subitizing these two arrangements of five dots, but his statement that he saw "10" seems to revert back to his counting activity earlier when Ben was engaged in the conservation activities. Thus, these orientations seemed to elicit Ben's reliance upon empirical abstractions, with visual scans and counting activity.

Frank (4.5)

Frank was initially interviewed two separate times on June 5th. Throughout the interviews, Frank responded confidently and seemed to not be afraid of answering incorrectly. Quite often when he was asked why he knew something, he would change the subject or use actions to explain his thinking. Frank was just learning how to write numerals, and often would re-present the number of dots he saw by drawing dots and by writing the numeral. Frank also attended to the color of the items quite often and described them as matching other colors or his angry birds on his shirt.

Frank's engagement with the conservation tasks suggested that he was inconsistently conserving number, as when the blocks were spread apart in one task, Frank stated that the longer row of blocks had a fewer items compared to the shorter row of blocks. In comparison, when Frank matched a row of flowers to vases or a row of bottles to cups, he is able to conserve the number and even counted each row to explain why the rows each had the same amount. These actions initially suggested that Frank might be able to conserve number and misunderstood some of the questions asked because for him English was a second language. However, in subsequent sessions Frank's responses seemed to indicate he was still inconsistently conserving number.

In the following protocol, Frank is engaged in a counting task in which he was expected to count some blocks. His counting was quite strong so I asked him to tell me how many would we have if we added two more.

Protocol F1.

T: I want to see how high you can count. You were counting just now, and you did a great job [places a pile of blocks on the table in front of Frank]. Let me see here. I am going to put these all in a row [orders nine blocks into a vertical row directly in front of Frank].

F: Yeah.

T: Can you count these for me?

F: One, two, three, four, five, six, seven, eight, nine [as Frank states each number word in order, he also touches each block with his finger].

T: Okay, [places two more blocks on the end of the row of blocks that Frank just counted] can you count these now?

F: 10, 11 [points from his seat in the general direction towards the blocks in a patterned manner while stating each number word].

T: My goodness. That is very good. What if I do this [moves blocks around so that they are in an unordered arrangement]? Can you count them now?

F: One, two, three, four, five, six, seven, eight, nine, 10, 11 [as Frank states each number word in order, he also touches each block with his finger].

T: Very good. What if I...what if I add two more? How many do we have now?

F: One, two, three, four, five, six, seven, eight, nine, 10, 11, 12, 13 [as Frank states each number word in order, he also touches each block with his finger].

Frank relied most often on reflecting types of abstraction, as each time he was shown an orientation of two, three, Frank accurately stated the number of dots that he saw, and drew a row of dots that matched that number. Comparatively, when Frank was shown four dots in a square

arrangement, he stated he saw four, but then drew the four dots in a square arrangement as well. He described this four as needing a space for the numeral four to “go inside.”

Amy (4.2)

Amy was first interviewed on May 28th and then again on May 29th. Throughout the interviews, Amy responded confidently and seemed to enjoy showing me what she knew about math. Quite often when she was asked why she knew something, she would say, her “mind knows” or that she knew it because “she knows math.” Also, she told me that she loved math and that is why she knew.

Amy’s strengths, noted early on, included her ability to conserve number, count items accurately, and accurately subitize as high as five items when arranged as an array. In fact, Amy relied on counting quite a bit to explain why the number of items had not changed when the items were spread apart or pushed together. For example, in the following protocol, Amy was shown two rows of six blocks. Amy was capable of using her counting activity to explain why the blocks are the same when the rows are same length, but had no logical explanation as to why the rows have the same number of blocks when the rows are different lengths.

Protocol A1.

T: Do we have the same number of blocks?

A: [Nods head up and down to indicate “yes”].

T: We do? How do you know?

A: One, two, three, four, five, six [points to each block while stating each number word]. One, two, three, four, five, six [points to each block while stating each number word – this indicates that each row is comprised of six blocks].

T: Oh, okay. So you can count and tell. Alright, I'm going to do this [spreads the blocks apart so the row is longer compared to the other row]. Do you still have the same number of blocks as me, or do you have more than me?

A: Same as you.

T: How do you know?

A: Because, I know.

T: Yea? What do you know? What tells you that you know for sure?

A: Mind.

T: Hmmm?

A: Mind.

T: Your mind tells you that? What does your mind say?

A: These [points to all of six blocks in her row] just are the same as that.

T: So, they are the same as this [points to the blocks in researcher's row]? Okay, very good [scoops blocks up and sets them aside].

This protocol provides evidence that Amy consciously understands that counting can explain why groups of items have the same number, but she may not necessarily consciously rely on this logic to understand that the two groups of items have the same number of items. This was evident when Amy stated that when the rows that are the same length they have the same number of blocks because she can count. However, when her row of blocks is spread apart and is now longer than my row, she could state that the blocks have the same number of blocks, but could not tell me how she knew. It seems as if she was unconsciously carrying into this activity a logic which may indicate reliance on some reflecting abstractions when conserving number, as

she had some early reasoning that was grounded in number, but cannot put into words why the number of blocks remained the same.

When Amy engaged in subitizing activity, it seemed that she also relied on reflecting types of abstraction, as she subitized accurately two and three dots when the dots were in a row or a triangular arrangement, and only draws the dots in a row. However, when Amy was shown four dots arranged in a square formation, she stated that she saw three, and drew three dots in a row. Just before Amy was shown the orientation of four dots in a square, Amy was shown the orientation of three dots in a triangular arrangement. This seems to suggest that Amy carried into this activity an empirical abstraction, as each orientation is similar, because they take up a two-dimensional amount of space. Comparatively, when Amy is shown five dots in a row, or in an “X” arrangement, she could accurately subitize these orientations, but again can only represent through drawing the items in a row. In the following protocol, it was evident that Amy was required to engage in subitizing activity, that she was not capable of accomplishing this, as she was shown five dots in a pentagonal arrangement, and reverted to some quiet counting out loud.

Protocol A2.

T: Good, last one. This is a hard one, okay? [Student is shown a card with five dots on it. The five dots are arranged in a Pentagonal orientation.]

A: [student points her finger towards the card and quietly counts the items that she sees.]

T: [the teacher researcher quickly turns the card facedown.] We are not counting, okay?

A: [student looks to the side] five.

This protocol is evidence that when Amy is shown a circular arrangement of five dots, she reverts to counting, as she is not able to subitize them accurately. Also, this protocol seems

to indicate that when Amy cannot subitize items, she reverts to some perceptual counting, but seems to use pointing from a distance in coordination with her verbal utterance of the counting sequence, and does not seem to need to depend on touching the perceptual items. This type of activity was found quite often throughout the study when students believed they were unable to subitize items, and will be discussed in more detail in subsequent sections.

Diana (5.5)

Diana was interviewed two separate times on June 4th. Diana was able to conserve number and offered a clear logic as to why the items' quantity did not change when they were spread out by stating, "because, you didn't take any away." This seemed to indicate that she was basing her decision on a logical understanding of quantity. In fact, in the following protocol, Diana is engaging in the conservation task involving flowers and vases. It seems evident that Diana relies on this logic throughout, as when she is asked if the arrangement of the flowers should change the number of flowers, she states there is not a change in the number of flowers and that there are still the same number of flowers as vases.

Protocol D1.

T: I am going to pull these back a bit, and I am going to lay them down [takes flowers out of vases and places them in front of each vase]. Now, do we still have the same flowers as vases?

D: hmmm...mmm [nods head up and down to indicate "yes"].

T: Yes? Okay [bunches flowers together so they are in front of the two middle vases]. Do you still have the same number of flowers as vases?

D: [nods head up and down to indicate "yes"].

T: We do? How come?

D: Because you didn't take one away again.

T: I didn't? But look this looks like it is all squished together [points at flowers in a pile], and this is much longer [points to row of vases]. Does that make a difference for how many there are?

D: [nods head up and down to indicate "yes"].

T: It does? How so?

D: Because these are squished together [places hands on either side of the flowers in a pile], and these are out [places hands on either side of the vases].

T: Do we still have the same number of flowers as vases?

D: [nods head up and down to indicate "yes"].

This protocol seemed to indicate that Diana was able to consistently conserve number regardless of the changes made to the flowers. Also, when Diana was offered an argument that she can adopt regarding the space the items take up, she was not willing to change her mind completely and still stated that the flowers and vases were the same number. Similarly, Diana's counting activity also illustrated her ability to count blocks accurately in both a row and in a random arrangement. Diana's subitizing activity was quite sophisticated, as she was able to accurately subitize all arrangements except a row of five dots. In addition, Diana drew the dots in a similar orientation and with the accurate number of dots, indicating reliance upon more sophisticated abstractions. Comparatively, in the following protocol, Diana was shown the row of five dots, and it seemed evident that Diana reverts to an empirical abstraction, as she consciously could only attend to the dots as a row of many that was dependent upon her visual scan of the items, and stated that she saw eight dots. When asked to draw what she remembered, Diana drew a row of eight dots.

Protocol D2.

T: Okay last one [teacher-researcher shows Diana a card with five dots in a row.]

D: [pauses]

T: Do you want to see it again?

D: [Diana nods her head up and down to indicate yes.]

T: Okay. [Teacher-researcher shows Diana of the card with the five dots in a second time.]

D: [pauses and looks off to the side] Eight.

T: Eight? Very good. Draw what you remember. [Teacher-researcher gives Diana paper for her to draw on.]

D: [Diana draws eight dots in a row.]

It seems that her need to pause and look away from the table or the card before she responded might also be her need to re-imagine her visual scan to determine what number she saw. Also, in earlier sessions, Diana was able to show us with words and with fingers how many dots she saw. However in this task, Diana could only tell us with words how many dots she remembered seeing.

Ben and Frank

Ben and Frank were two students that I worked with individually, totaling 22 teaching experiment sessions each. Both students will be discussed in this same section, as they each were considered preconservers, and demonstrated some similar subitizing activity. This section is two-fold first offering an audit trail, describing how five different types of perceptual subitizing activity (Initial Perceptual Subitizing, IPS; Perceptual Subgroup Subitizing, PSS; Perceptual Ascending Subitizing, PAS; Perceptual Descending Subitizing, PDS; and Perceptual Counting Subitizing, PCS) and two different types of conceptual Subitizing activity (Rigid Conceptual Subitizing, RCS; Flexible Conceptual Subitizing, FCS) resulted from analysis of the

data in this study. Second, this section will more robustly describe and define the five different types of perceptual subitizing activity, which seemed to be influenced by reliance upon three different types of orientations or attributes of experiential items (space, symmetry, and color).

Categories Describing Seven Different Subitizing Activity

It is noteworthy that perceptual and conceptual subitizing have both already been defined and described relative to students' actions by Sarama and Clements (2009) and Clements and Sarama (2009) earlier in chapter 2. However, it is important to revisit these definitions for this discussion. Perceptual subitizing is defined as an innate ability to discriminate different quantities, emerges in infants as young as three to five months of age and is limited to five items (Sarama and Clements, 2009). In comparison, conceptual subitizing is when children begin to connect the notion of number to subitizing activity and can describe utilizing subgroups when subitizing (Table 2.1) (Sarama & Clements, 2009). These two definitions were consistently used by the teacher-researcher and witnesses to both explain and to describe student responses when the data were being analyzed. However, after the sixth teaching experiment session, a retrospective analysis supported the notion that perceptual and conceptual subitizing may have subgroups, further refining Sarama and Clements' (2009) descriptions of students' subitizing activity.

Additionally, the pilot study conducted in the fall of 2011 provided insight as to how subitizing activity may differ, as three different types of activity were noted after a retrospective analysis took place. The following three types of activity informed researchers as to how present activity might be considered: 1) activity where students' described the movement that their eyes made, 2) activity where the number of items re-presented or drawn are shown as one

item and one item and etc., or 3) activity where the students described the total number of items as composed of subgroups of items.

As described in chapter 3, after the sixth teaching experiment session, a group meeting was held on July 3rd by the teacher-researcher and both witnesses in order to discuss the students' preliminary thinking models. Sections from transcripts were discussed and several possible mental actions were inferred through an analysis of the students' language and actions. At this time, data were brought forward both in the form of video footage and tables which described the number and orientation of items a student subitized and how their subitizing action was represented. Interaction between the teacher-researcher and the students was also discussed at this time to better understand the activity resulting from each of these interactions. Thus, the discussion was supported with the witnesses' valuable outside perspectives.

Early on in the teaching experiment, Ben and Frank seemed to rely upon empirical abstractions when subitizing, as responses such as, "it is four because it looks like a square," "it looks like a kid," or "it looks like a 'y'" seemed to result when they were each subitizing orientations with four items. Similarly, Frank also described five items as a "K" when subitizing. This type of activity seemed quite different when Frank subitized subgroups two and three separately. Several examples from other students also came forward at this meeting, describing activity where subgroups of two, two, and one were used to build up to five, and five was also being decomposed as two, two, and one. Also, some students would describe seeing an orientation as five because it had "four and five" in it which seemed distinctly different than describing an orientation as being five because it was composed of "four and one." Thus, to better understand what this different subitizing activity might be dependent upon, it became necessary to categorize the observed subitizing activity and name it appropriately.

Descriptions of students' actions became categorized, but descriptions of these categories changed as retrospective analyses continued throughout the course of this study. Initially the analysis of the data described three different types of perceptual subitizing activity when a student subitized three items (*perceptual subitizer of three, perceptual counter of three, perceptual composer of three*). Perceptual subitizer of three was described as a student subitizing three items and dependent upon empirical abstractions, as the shape of items were often described by students as eliciting a reason for a set of three to be named as "three." Also, the space between items disrupted the student's notion of "three" when subitizing, as students engaging in this activity seemed to also rely heavily upon visual scans. Perceptual counter activity was named as such because it seemed as if there was reliance upon serial thinking activity, as orientation or color were used to elicit attention towards one subgroup, and one more or one less. When observing students engaging in this activity, students would explain that they saw three items because there were "two and three." It seemed as if students were subitizing two and then counting forward one more to understand that there were three items in a group. Finally, perceptual composers of three explained that three dots in a line were three because the two dots on the outside edges of the line and the one dot in the middle make three.

Similarly, there were three different types of perceptual subitizing activity for four (*perceptual subitizer of four, perceptual counter of four, perceptual composer of four [2 and 2]*). These types of activity were similar to the perceptual subitizing for three, but, as noted above, the subgroups considered when a student engaged in perceptual composer of four were two and two. However, when the subitizing activity for five items was grouped, four different types of perceptual subitizing activity were found (*perceptual subitizer of five, perceptual counter of five [subgroups only], perceptual counter of five [whole group only], perceptual composer of five [2,*

2, and 1]). This increase in the types of categories resulted from a distinction that was made between students capable of subitizing five items as “four...and five” or only subitizing the subgroups, “two and three.” This seemed to describe a cognitive choice students had to make between reliance upon their serial thinking structures or their classification thinking structures. Perceptual subitizing activity with regard to six items had similar categories (*perceptual subitizer of six*, *perceptual counter of six [subgroups only]*, *perceptual counter of six [whole group only]*, *perceptual composer of six [3 and 3]*).

As retrospective analyses continued throughout the course of the study, more differences and similarities were found when observing the student responses that pushed for both more depth and refinement of the students’ perceptual subitizing activity. For instance, Frank began composing “two plus one plus two” to subitize five in his seventh teaching experiment session. This seemed to indicate that Frank is composing five as two, one, and two. Comparatively, in Frank’s 17th teaching experiment session, Frank subitized five and then decomposed it as “two down and one up” to explain how he knew it to be five. The second subgroup of two was not initially considered, and will be discussed in more depth in the latter portion of this section. This activity indicated that Frank was beginning to decompose five into subgroups. Thus, the perceptual composer category was divided into two different types of activity, perceptual ascending subitizing (PAS) and perceptual descending subitizing (PDS). Also, it was confusing for the perceptual counter activity to be named in this manner, as Steffe et al. (1988) has a type of counting named as “perceptual counter.” Therefore, this activity’s name was changed to emphasize the subitizing activity more and became known in this study as perceptual counting subitizing (PCS). Similarly, “perceptual subitizing” is already described by Sarama and Clements (2009) and titling a subgroup of this activity with the same name was confusing, so

this activity became known as initial perceptual subitizing (IPS). Finally, perceptual counter (subgroups only) seemed to be confusing as a title because the student seemed to rely more heavily upon their classification thinking structures, not their serial thinking structures, as they subitized only the subgroups. Thus, this activity was renamed to suggest this reliance and became known in this study as perceptual subgroup subitizing (PSS).

The following section will offer an in-depth analysis of five of these different types of activity regarding the topological influences, and utilizing Ben and Frank's thinking models with regard to this activity. Before this analysis, each activity is described to frame this discussion. Also, it is important that the distinction be made between perceptual subitizing activity and conceptual subitizing activity in this study. Sarama and Clements (2009) describe a student capable of decomposing an entire set of items after subitizing accurately into subgroups, which is evidence of a student engaging in conceptual subitizing. This is true in this study as well, but there were tasks where students engaged in PAS or PDS activity because the perceptual material offered support to the student's subitizing ability. However, there is reliance upon some conceptual understandings of number, as the student would then need to rely on the conceptual understanding that these subgroups can either compose or decompose a number. Conceptual subitizing is described in more detail in Amy and Diana's section.

Initial perceptual subitizing (IPS) is defined as a student's reliance upon empirical activity when subitizing, as the student explained or justified that he knew the number of items he saw to be accurate because it looked like a shape or he described what his eyes did. This activity seems to be initial because it is grounded in his physical activity, and the student relies on perceptual material when explaining his thinking. *Perceptual subgroup subitizing (PSS)* is defined upon a student's ability to consider only subgroups when subitizing. This activity also

relies on empirical activity, but there is a chunking in this activity, supporting a student's ability to consider chunks of visual scans as re-presenting subgroups of a set of items. *Perceptual ascending subitizing (PAS)* and *perceptual descending subitizing (PDS)* are defined as a student relying on the orientation or color of items to either subitize subgroups and then compose a composite group (PAS) or subitize a composite group and then decompose this group into subgroups (PDS). Finally, *perceptual counting subitizing (PCS)* describes a student who is subitizing and then counting up one or down one. Thus, students engaged in this activity might be described as ascending PCS or descending PCS.

Evidence of Ben and Frank's Initial Perceptual Subitizing Activity (IPS)

Ben and Frank's actions suggested early on that they were engaged in IPS activity in the teaching experiment sessions, and also revisited IPS activity in subsequent sessions. Most often this type of activity indicates the students' reliance upon empirical abstractions, and was found to be evident when students described either a shape or a motion that helped them state or re-present the number or the orientation that they remember.

Shape and visual motion. For instance, in the first teaching experiment session, Ben was shown five dots arranged in two rows. The bottom row had three dots, and the top row had two dots. After being shown the dots, he could clearly state that he saw five dots, but when asked to draw what he remembered, he reverted to the empirical nature of the arrangement, as he drew two rows identical to what he saw, and then added one more circle at the top left portion of the orientation. This activity seemed to indicate reliance upon the space the items take up, not necessarily upon the numerosity of the items.

Protocol B3.

T: [teacher-researcher shows student five dots. Two dots are in the top row three dots are in the bottom row directly below the two dots]

B: Five.

T: Five. How many... er... Can you draw what you remember?

B: [student draws two circles in the bottom row, three circles in the row directly above the two circles, and one final circle above the row of three dots. At this point Ben has drawn six circles]

T: Okay? [Researcher covers the last dots drawn with a card leaving five dots visible] how many is hidden underneath here [researcher points to card]?

B: One.

T: [researcher slides card to cover two dots leaving four dots visible.] How about now?

B: Two

T: [researcher slides card to cover three dots leaving three dots visible.] How about now?

B: Three

T: [researcher slides card to cover four dots leaving two dots visible.] How about now?

B: Four

T: [researcher slides card to cover five dots leaving one dot visible.] How about now?

B: Five

T: [researcher slides card to cover all six dots leaving zero dots visible.] How about now?

B: Five

T: Five? Can I have five underneath here [researcher slides card back to reveal one dot leaving five dots covered], and underneath here [researcher slides card back covering all six dots and leaving zero dots visible]?

B: [student shakes head side to side to indicate no]

T: No. [Researcher slides card back to reveal all six dots.] How many are there?

B: Five.

Two types of activity seemed to be evident in this protocol when Ben was subitizing. First, when an individual engages in subitizing, two general processing mechanisms are engaged, a preattentive processing mechanism (e.g., spatial indexing) and an attentive processing mechanism (e.g., number scheme). Ben seemed to be relying on these two different processing mechanisms when he stated an accurate number, but then drew an inaccurate number of circles. Ben preattentively processed or encoded the space and location of the items, but relied upon his number scheme regarding what he processed preattentively. Many times, the students' drawings or re-presentations, and verbal number word did not align, as educators would expect only the notion of number to result from this activity. Though empirical attributes regarding the items being subitized can also result from this activity. A filling of space, a "shape" made up of dots, a visual motion, a cluster of dots, or group attributes such as color might also result when students engage in subitizing. When the students were engaged in perceptual subitizing activity of any kind, what resulted from the preattentive processing mechanism did not necessarily need to align with what resulted from their attentive processing mechanism, as their number schemes may still be grounded in the space or density of items.

Second, in this protocol, it is evident that Ben was relying on an empirical abstraction when subitizing this orientation of five dots, as he re-presented six circles which filled the space and aligned with results from his preattentive processing. This is also evident when Ben described five circles being covered up, but it does not perturb Ben's thinking structure when one more circle was covered up and the entire set still represented five. Even when this activity was

brought to Ben's attention and he agreed that he cannot have five and then five with one more circle it did not bother Ben to still describe the entire set of circles as five. This activity also suggests that "one more" circle does not necessarily influence the collection of the set, as each time the set has to be reconsidered as number is not a permanent attribute of the set itself.

In this same teaching experiment session, Ben was shown three dots in a triangular orientation early on, and he stated that he saw "three" and drew three dots from memory in the same exact orientation. However, later on, when Ben was asked to draw three circles and design his own card, he drew three circles in a row. At this point, Ben was comparing both of his drawings to better describe some attributes that might be the same in both drawings and some attributes that might be different in both drawings. The purpose of this task was to better understand what attributes Ben was attending to when describing three, and if he could begin to flexibly describe three when items are arranged differently.

Protocol B4.

T: What if I said to you I'm going to show you three? Can you make a card for me that has three on it? Can you do that?

B: [student nods head up and down to indicate yes]

T: What would it look like? Go ahead and draw. You design it. How do you want the three to look?

B: [student draws three dots in a row]

T: Is it three?

B: [student nods head up and down to indicate yes]

T: It is. Is there any other way that we could have a card with three on it?

B: [student shakes head side to side to indicate no]

T: No? [Researcher reaches down and takes a drawing that the student made earlier in the teaching experiment session. This drawing had three dots in a triangular orientation.] You did another way to draw three earlier, didn't you?

B: [student nods head up and down to indicate yes]

T: What is the same about both of these?

B: Three.

T: They both have three. What's different about them?

B: I drew one like a rocket, and one like a train.

T: Oh, okay. Which one is the rocket?

B: [student points to three dots in a triangular orientation]

T: This one? That's a really neat rocket. Can you show me where it is headed? Is it headed this way or this way?

B: [student points to the top of the paper]

T: This way? That's what I thought. And what direction is the train headed?

B: [student points the train away from him]

This protocol seems to indicate that each orientation is visualized as an object rather than three discrete circles. The "rocket," which was the triangular orientation of three circles and the "train" which was the row of three dots, seemed to allow Ben a way of holding on to three circles visually and suggests reliance upon empirical abstractions. Thus, reliance upon empirical abstractions is one aspect of IPS activity found in this study.

Frank also relied upon empirical abstractions early on in the first teaching experiment session, which illustrated that what resulted from his subitizing activity was either a visual scan or a shape. However, Frank was not even willing to state a number at first. His initial response

was to describe the motion his eyes made or to quickly state what shape the dots would make if they were connected.

Protocol F2.

T: Okay, what about... All right, here is another one. Are you ready? [Researcher shows student a card with five dots. Two dots on the left-hand side in a column and three dots in a triangular orientation on the right-hand side.]

F: Hey! I can't see it. It was like this. [Student re-creates the orientation in the air with his marker.]

T: Draw what you saw. How many do you think that was?

F: [student begins to draw dots again in an upside down L formation. When student finishes drawing dots, there are seven dots on the paper.]

T: Do you think seven?

F: Let me see, count.

At this point, I attempted to scaffold this subitizing experience by showing the card to Frank in parts. I covered up one section and then showed Frank the card with three dots shown, and then covered up the other section with only two dots shown. The purpose for this was to allow Frank to attend to the subgroups two and three when describing the number. Frank was able to describe the orientation as having three and two respectively, but then reverted back to reliance upon empirical abstractions when shown the entire card immediately afterwards.

Protocol F3.

T: [researcher shows card with all five dots visible. Again, two dots on the left are in a column and three dots on the right are in a triangular orientation.]

F: K

T: What?

F: K

T: Oh, it looks like a K?

F: Yeah.

T: Yeah? So, how many do you think were there?

F: Umm... Eight.

T: You think eight? And is that what you drew there? [Researcher points to original seven dots drawn in an upside down L formation.]

F: Seven.

T: That seven [points to original drawing of seven circles]?

F: I made this... [waves the end of the marker over his original drawing of seven dots to indicate what "this" is.]

Frank's final statement here clearly demonstrates what resulted from his subitizing activity is the orientation or the motion, not the number. When asked if this is the seven, he does not commit to the result of this activity as being "seven," as his drawing resulted from this activity. Also, in each of these subitizing activities, the space between the dots shown to Ben and Frank seemed to influence this activity. For instance in protocol B3, Ben seems to use the space the items together take up when re-presenting what resulted from the subitizing activity. Comparatively, in both of the previous protocols Frank uses the space to describe both the visual motion and the orientation formed when the dots joined together from that visual motion, suggesting the visual scan as necessary when numerically expressing the set. However, in all protocols, Ben and Frank are not bothered when the number used to describe the set changes, or the subgroups of the set do not build towards the number of items in the entire set, as number

does not yet hold permanence when students engage in IPS activity. Both Ben and Frank's thinking models will be described early on in this activity to explain how IPS activity changes when students experience cognitive change.

Perceived space between items. In the following section, space between items will be the main focus, as both Ben and Frank seem to be influenced by the space between items when both subitizing and re-presenting what they remembered. In this section and subsequent sections Frank and Ben's thinking model is described relative to IPS activity, and then cognitive changes made by both Ben and Frank are described within IPS activity.

Frank's thinking model with regard to IPS activity and the perceived space between items. The purpose for the following protocols is two-fold; it is an exemplar of the IPS activity with regard to the space between items, but it also presents some evidence towards Frank's thinking model with regard to IPS activity.

In this next protocol Frank was working in his eighth teaching experiment session and was shown pictures with the back of a camera window, with different dot arrangements in the camera window. After Frank was shown this camera for approximately 1 to 2 seconds, he was asked to state how many dots he saw and use circular counters to make the picture he thought would come out of the camera. Specifically, Frank was shown four dots in a diamond arrangement and asked how he knew there were four counters on his mat and how he knew there were four dots in the camera window. The IPS activity that Frank relied upon suggests that for him to describe this orientation as "four," he needed to recreate his visual actions, and when the space between the items changed, he did not necessarily consider both groups to represent the same number.

Protocol F4.

T: Okay, you ready? [Teacher-researcher shows Frank the back of camera with four dots in the window arranged in a diamond orientation. The four dots are not touching and there is a clear space in the center of the orientation.]

F: Four.

T: You saw four? Can you make what you saw? How do you know there's four there?

F: [Frank picks up the counters one at a time and places them on a piece of white cardstock which served as a working mat. Once all four counters are arranged in a diamond orientation and Frank takes his hands off of the counters and looks at what he created. The four counters are all touching which leaves only a small space in the center of the orientation.]

T: How do you know there is four there?

F: Ummm... [rotates mat to the left and then back to the right.] Like a diamond.

T: Yeah, it looks like a diamond, and you know a diamond is four?

F: Yeah.

T: Yeah, you were making those yesterday weren't you...or Tuesday?

F: My scout movie goes just like this. [Frank places four counters on his mat next to the diamond of four. The four counters are spread apart and take up much more space than the four counters arranged in the original "diamond." One extra counter spills onto his mat, and when he rearranges his counters to make the first diamond again, he takes two counters away. Then, he looks back on his mat at his second diamond and realizes that he took one too many away and replaces this counter.] Like...like this.

T: What is this again?

F: [Shifts the second diamond to the right, providing more space between the two diamonds.] A diamond.

T: Right...but what is...Right, so this is a diamond too? [Points to the second diamond.]

F: Yes.

T: Are there the same number in each diamond?

F: Umm...no. This is covered [points to the smaller diamond he created originally] and this is putted out [points to the larger diamond he created the second time.] Just like my scout movie.

T: Do they have the same....Your scout movie?

F: Hmmm...mmm [indicating yes].

T: Oh! Okay.

F: And we draw something.

T: You drew something with that?

F: [nods head up and down to indicate yes.]

T: Oh, are you in boy scouts or cub scouts?

F: Scouts.

T: Just scouts. Okay. Do you...Are there the same counters here [researcher covers the smaller diamond with her hand to indicate this group of counters] that are here [researcher covers the larger diamond with her hand to indicate this group]?

F: [pauses for about 2 seconds] I can do like...this out... [spreads the smaller diamond counters apart so there is the same amount of space between the counters in both orientations.]

T: Hmmm...mmm [indicating agreement].

F: same diamond.

T: Now, are they the same number?

F: Yes.

Here, it is evident that Frank subitized four accurately, because it looked like a diamond that he saw from a scout movie, and that he remembered drawing that diamond before. Also, it is interesting that Frank did not need to count the items when one extra counter was removed from his mat. However, the perceived patterns that the counters create informed him more of how many, rather than the number the items have as a collection. This was evident, as Frank did not realize there was one missing or one extra until he rearranged the counters to recreate this perceived pattern. Also, Frank could only consider one orientation at a time. He took the extra counters off his mat, but as he did this he took one counter from his second orientation. Once these were removed, Frank then could consider this second orientation, which was when he realized that this orientation was missing a counter.

More importantly, this protocol also illustrates the influence which space between items has on his notion of number. Frank was not willing to consider each diamond as having the same number even though he made the shape originally to explain why he knew a diamond to be four. This lack of permanence with regard to the number cannot be rectified until Frank spreads the counters further apart and both diamonds had the same amount of space between the counters.

With regard to Frank's thinking model, Frank seemed to be relying upon empirical activity when subitizing four items in a diamond shaped orientation, as the perceived space between the items and the orientation of the items influenced what Frank knew about number when subitizing this set of four. Salient aspects of Frank's thinking model to attend to at this point, is his ability to coordinate both the shape and space between items when subitizing four. This is evident when Frank decided that "this is four because it looks like a diamond," and then attended to the space between the items in which to re-present what he remembered about the

diamond. This physical spacing of items also seemed to disrupt his original notion of what he believed to be four, as the diamond with more space may not necessarily have had the same quantity as the diamond with less space. Thus, it seems that Frank initially understood that four counters constitutes what he believes four to be, but this is disrupted when he relies upon the space between items. Changes in Frank's thinking model are found in the section titled, "IPS activity with regard to the perceived symmetrical aspects of the items," as Frank's reliance upon empirical activity changed quite drastically with regard to IPS activity throughout the course of this study, and this reliance upon space between items is not evident in latter sessions.

Ben's thinking model with regard to IPS activity and the perceived space between items. The purpose for the following protocol is two-fold, as it is an exemplar of the IPS activity with regard to the space between items, but it also presents some evidence towards Ben's thinking model with regard to IPS activity.

Ben was also strongly influenced by the space between the counters when subitizing and re-presenting what he remembered. In protocol B5, Ben was in his fifth teaching experiment session and was shown counters on a mat and asked to tell me how many counters he saw. He was then asked to make what he remembered with the same size counters on his mat. Each mat was an eight by 11 and half piece of cardstock. Ben was first shown an arrangement with four counters. Three counters were in a triangular arrangement on the teacher-researcher's left-hand side, and one counter was in the right-hand corner of the mat. The teacher-researcher's triangular arrangement had space in the middle and between each of the counters. Ben stated that he saw "five" and proceeded to make what he saw. Ben placed three counters with no space between them in the same triangular arrangement on the right-hand side of his mat. He then placed one more counter in the bottom left-hand portion of his mat.

Protocol B5.

B: [Ben places three counters one at a time in a triangular arrangement on his right-hand side of the mat, and then places one counter on the bottom left-hand portion of his mat.]

T: Nice job! So, how many are here? [Points to the top counter in his triangular arrangement.]

B: One.

T: How many altogether? [Waves finger over the triangular collection of counters on Ben's mat.]

B: [Pauses and counts silently while looking with his eyes.] Three.

T: And how many here? [Points to the counter off to the side and separated from the cluster of three dots.]

B: One.

T: One and how many does that make altogether?

B: [Counts silently while looking at all of the counters.] Five.

T: Okay let's...

W: How do you know five?

B: [Looks up at Witness (W).]

T: Yes, how do you know five? Thank you for keeping me to my questions. [to the witness]

B: There's five.

T: What...wha...

W: Just show it on your mat. How do you know there is five over there?

B: [Ben looks at his mat.]

T: Can you count these for me?

B: [Points at each counter one at a time and quietly counts.] Four.

T: Four. Let's see if four is the way it should be, or if we need to add another, okay? [Teacher-researcher lifts top piece of cardstock covering the original orientation shown to Ben so Ben can see it again.]

B: [Ben looks at the teacher-researcher's mat again and then back at his own mat.]

T: [The teacher-researcher lifts the top piece of cardstock to show Ben the original arrangement.]

Do you think your mat matches mine?

B: [student nods head up and down to indicate yes.]

T: I think so too. I think you did an excellent job Ben. Now....so would you like to change it to four, you think? Your answer?

B: [Shakes his head side to side to indicate no.]

T: Do you want to keep it to five?

B: [Shakes his head up and down to indicate yes.]

T: Okay, so what could you add to this [teacher-researcher points to Ben's mat] to make it five?

B: [Ben slides each counter in the triangular arrangement out so there is now an empty space between the three counters.]

T and W: Ohhhh.

T: Okay, that's interesting. So now do we have five here? [Teacher-researcher points to all of the counters on Ben's mat.]

B: [Nods head up and down to indicate yes.]

T: We do? Can we count? Here, let's count mine first. [Teacher-researcher lifts the top piece of cardstock up so that the original arrangement is shown to the student.]

B: [Ben looks at the teacher-researcher's mat and quietly counts items without pointing.]

T: Point to the counters so that you can make sure because sometimes our eyes kind of move around a bit, and we don't keep track.

B: [Ben uses his finger to point at the counters, but still does not touch each counter as he states the number words.] Four.

T: Four and how many do you have here?

B: [Ben uses his finger to point at the counters, but still does not touch each counter as he states the number words.] Four.

T: Do they match?

B: [Nods head up and down to indicate yes.]

T: They do. You did a great job.

Ben's response when asked "what could you add to this" suggests that Ben is considering not only the space between items, but also moving the items apart from each other as a method to use when increasing the quantity of the set. It seems as if the set of three counters, compared to its previous arrangement, needed to take up more space to increase the quantity. Ben's reflection upon the teacher-researcher's original arrangement might also have influenced what he thought five should look like, as once he moved the counters apart from each other, his orientation more closely aligned with the teacher-researcher's orientation. This also suggests that what Ben is attending to when subitizing is not discrete experiential items, but the experiential items relative to the space.

At this point, Ben's thinking model with regard to IPS activity tends to illustrate a dependence upon the space between items. As described in protocol B5, Ben considers the larger the space between items comparatively as increasing the quantity of the set. Also, Ben seems to consider the dots and the counters as connected to form shapes. Thus, subgroups could

not be considered, as Ben's continuous visual scan prevented him from breaking items into chunks when subitizing. Also, Ben relied on counting items when he was unable to subitize, suggesting that Ben is what Sarama and Clements (2009) describes as a *maker of four*. A maker of four is a student who is not able to consistently subitize four items, and then needs to rely upon making a group or counting a group to consider it as four.

Comparatively, when the space between groups of items is smaller, Ben's subitizing activity with regard to four items tended to be much more accurate. For instance, in the fourth teaching experiment, Ben was shown four dots in a diagonal line, with a larger space between the third dot and the fourth dot. Ben was able to accurately state that he saw four, and was capable of re-presenting an orientation similar to the one shown to him, as he drew three circles in a row with one circle just above the middle circle. Thus, it seems as if Ben could scan items closer together and numerically describe them, but when the space between the counters, and between the two groups of items increased, Ben was not able to scan the entire set of items, perturbing his subitizing ability.

Changes in Ben's thinking model with regard to IPS activity and the perceived space between items. At this point in teaching experiment 15, it was advantageous to consider how Ben's thinking model might change with regard to the space between items, as Ben seemed to be considering subgroups two and two when subitizing four items. This change in attention towards subgroups elicited the notion that Ben's perceived space between items in a triangular arrangement of items might also be changing what Ben knew about number, as Ben may have been chunking his visual scans to attend to subgroups when subitizing.

In Ben's 15th teaching experiment session, Ben was shown in Protocol B6 the same configuration. His attention towards the density of the items began to change with regard to his

IPS activity. Ben was shown the same arrangement from protocol B5 with the same counters in the fifth teaching experiment, and he scanned the objects while engaging in quiet counting. Ben typically reverted to this quiet counting when shown items that he was not confident he could subitize. After Ben engaged in this quiet counting, he stated that he saw “four.” Four is accurate, but Ben did not engage in subitizing activity to numerically express this amount. Therefore, what Ben showed me with counters may be more closely aligned with how Ben encoded the orientation of counters on his mat through the serial act of counting.

Ben placed three counters on his mat in a triangular arrangement with about 2 inches of space between each counter. After Ben was shown the original orientation again, he added a counter to his mat and rearranged all of the counters so their placement was similar to the way in which the teacher-researcher’s counters are placed. At this point, Ben was asked, “I want to make my mat look like five. What could you do to your mat to make it look like five?” Ben waited for several seconds, obviously unsure of what to do to make his mat look like “five.” At this point I offered a solution that Ben has used in the past, in hopes to perturb this notion that space would increase the quantity of items. I spread my three counters apart so there is about three inches of space between each counter, and asked Ben, “If I do this, will I have more, or less, or the same?” Ben said there will be “less.” This is the opposite of what Ben has stated in the fifth teaching experiment session, and was unexpected. However, when Ben is questioned further, he reverted back to his argument that things “more spread apart” have more.

Protocol B6.

T: If I go like this [teacher-researcher spreads the three counters arranged in a triangular orientation further apart so that there are about three inches between each counter] do I have more now, or less, or the same?

B: Less.

T: I have less? Okay, how do you know?

B: Because I spread it out [Ben spreads his counters further apart at this point so that his counters are arranged similar to the teacher-researcher's counters.]

T: You spread it out, so that makes it less?

B: [Ben nods his head up and down to indicate yes.]

T: I thought it would make it more. It makes it less though?

B: [Ben nods his head up and down to indicate yes.] That's more bigger.

T: Oh, so because it is bigger it is less?

B: [Ben nods his head up and down to indicate yes.]

T: Interesting. Alright, we are going to just do one and then you can go back to your class, okay?

B: [Ben nods his head up and down to indicate yes.]

At this point, Ben has shifted his logic from the notion that the larger the space a collection of items has, as having “more,” towards the larger the space a collection of items covers, as having “less.” This shift was not one that was expected by me or the witness, and I wondered if it may or may not be influencing the IPS activity. This next protocol further extends this discussion between Ben and myself with regard to his attention towards the space between the items. At this point, the witness asked questions of Ben to better determine what Ben was attending toward and how this might be influencing the mental activity he was engaging in throughout this task.

Protocol B7.

W: [Speaking to the teacher-researcher] can I ask a quick question while you are setting that up?

T: Yeah.

W: Ben can I ask you a really quick question? I am going to make two things here that I want you to compare [places three counters in a triangular orientation with almost no space between the three counters on the left-hand portion of Ben's mat and places three counters in a triangular orientation with about two inches of space between the three counters on the right-hand portion of Ben's mat.] Okay. Which is more? This [witness points to left-hand side orientation.] or this? [Witness points to the right-hand side orientation.]

B: This. [Ben points to the right-hand side orientation, which takes up more space.]

W: This is more?

B: [nods head up and down to indicate yes.]

W: Why?

B: Because...because it is more spread out.

W: I thought you said spread out would be less. Is spread out more or less?

B: Less.

W: So, which one is more? This one or this one? [Witness points to each orientation.]

B: This one. [Points again to the orientation that takes up more space on the right-hand side of the mat.]

W: And how many is here? [Witness points to the orientation of the right-hand side.]

B: [Ben quietly counts while his eyes scan the orientation.] Four.

W: And how many is here? [Witness points to the orientation on the left-hand side.]

B: Three...Four. [Ben states this quickly without counting.]

Protocol B7 illustrates the inconsistent logic that Ben was carrying into his number understanding, as he continued to switch from one orientation as having more, but used his language to describe the smaller orientation as having more. Forcing Ben to make a decision

comparatively between the two orientations may have elicited these conflicting statements, as Ben was unable to state that they are both larger and smaller simultaneously.

This protocol also seems to indicate Ben's inability to subitize the larger orientation, as he reverts to a quiet counting, but Ben was able to subitize the smaller portion, further providing existence of reliance upon a visual scan, as the space between the orientations seems to perturb the amount of space a visual scan may be dependent upon. At first, he states that he saw "three," but then changes this to "four." It is not clear here why he changed it, but it may be that Ben wanted the number to be the same, even though one orientation was supposed to have "more" in comparison. Regardless, this notion of more or less with regard to space seemed to be changing Ben's understanding of number, as an attention towards density may be a shift in Ben's cognitive processes.

Ben's thinking model was changing at this point, as the space between items may have been competing with the density of the items. Ben was also more consistent in his attention towards subgroups as noted in his re-presentation of the three counters in protocol B6, and earlier in the same teaching experiment session. Thus, these changes may be indicative of Ben shifting towards an ability to coordinate subgroups and composite groups more flexibly when subitizing, as density relative to space seems to have been considered by Ben when numerically expressing the number of items.

Perceived symmetrical aspects of items. In these final protocols, which describe the IPS activity, the students' activity with regard to symmetry will be described. Symmetry seemed to be effective when students were attending to subgroups, but when students were attending to the space taken up by the items, symmetry both influenced what the child re-presented and explained certain essential mistakes, allowing me to better understand the

boundaries of the students' thinking models. For example, in protocol B8, Ben was in his fifth teaching experiment session. This was the same teaching experiment session as described in protocol B5 when Ben considered three and one to be five, simply spreading the three items apart to make this orientation the same as five. Ben was shown four counters in a square orientation with about two inches of space between each counter and stated that he saw "five," but reverted to the symmetry of the orientation when re-presenting what he remembered.

Protocol B8.

T: Ready, I am going to show you a new one. Ready Ben? [Teacher-researcher lifts the cardstock covering the counters on the mat to show Ben four dots arranged in a square orientation.]

B: Five.

T: You saw five? Okay, go ahead and make what you saw. Good job.

B: [Ben takes six counters out of his pile and places them one at a time on his mat in two vertical columns each made up of three counters. He places one counter on the right-hand portion of his mat, and then places another counter directly across from this counter on the left-hand portion of his mat. Ben does this again below with the next two counters, making a square-like orientation. Ben then considers his orientation again before adding a counter in the middle section of his right-hand column, and another counter in the middle section of his left-hand column.]

T: Do you want to see if it matches? [Lifts the top piece of cardstock to reveal the original orientation of four counters to Ben.] Does it match? [Replaces the top piece of cardstock, covering the orientation again.]

B: [Nods head up and down to indicate yes.]

In protocol B8, Ben stated that he saw five counters, but when asked to re-present what he saw, he seemed to want to attend to both number and symmetry, or the space the counters took up with regard to symmetry. Regardless, the symmetrical aspects of the orientation seemed to have a dominating influence over the action, which resulted from Ben's re-presentation. This protocol suggests the importance symmetry had with regard to Ben's IPS activity, as he even placed the counters down in a symmetrical fashion, so that if one counter was placed on one side of his mat, he was careful to place another counter directly across from this counter on the opposite side of his mat.

Preattentively, Ben's encoding also seemed to rely on the symmetrical aspects of the items. When Ben relied on the symmetry, he was not consciously attending to subgroups, but seemed to want the counters to re-present two rows of equal length. Thus, symmetry with regard to the orientation seemed to strongly relate to Ben's IPS activity.

Changes in Frank's thinking model with regard to IPS activity and the perceived symmetrical aspects of items. As stated earlier, Frank's thinking model indicated that Frank's reliance upon the space between items was transitioning towards a variety of new activity. In fact, in Frank's 13th teaching experiment session, Frank was still re-presenting an orientation with three counters in a diagonal row and one counter just above and to the right of this diagonal row as a diamond of four counters, indicating that the space between items was not a primary influence, and that Frank might be utilizing this diamond-like structure to conceptually match the four counters in such a way to consider the orientation as four. This activity might also indicate that Frank was not consciously attending to subgroups, but still relied on them when subitizing, and simply drawing on the fact that he knew four to be a diamond.

Regardless, Franks' attachment to four still utilized the diamond, but was not solely dependent upon this when subitizing four items. Also in the 13th teaching experiment session, Frank was shown five counters arranged so that four counters are in a square orientation and one counter was just above this square orientation. Frank engaged in PCS activity, as he described this orientation as "four...five." This activity was described in more detail later in protocol F15. Frank's attention seemed to have undergone a change towards subgroups. To test Frank's reliance upon the IPS activity when subitizing, an orientation was shown to Frank that had been designed in order to push him to choose between one activity or another when subitizing. Thus, this next task was designed to better understand how Frank's thinking model was changing with relation to the perceived symmetrical aspects of the number of items with regard to subgroups.

In protocol F5 Frank was in his 15th teaching experiment session; was shown five counters, but named them as "six." The orientation of five counters was arranged to mimic the typical die orientation with two vertical columns of three dots in each column. However, this orientation of five had one column of three dots, and one column of two dots. If Frank was relying upon the symmetrical aspects of the items, then Frank would numerically express this as six. If he could attend to the subgroups when subitizing, he might have been able to describe this orientation as five, as two-three would be built up to consider five as a composite group.

Protocol F5.

T: Alright, are you ready? Okay, how many do you see? [Teacher-researcher lifts the top piece of cardstock to reveal five red counters. Four counters are arranged in a square formation with one more counter directly above the column on the teacher-researcher's right-hand side. The top piece of cardstock is dropped down to cover the orientation shown to Frank.]

F: Six.

T: You saw six? How do you know it was six?

F: Umm...you put two in down, and you put three in down.

T: Okay, go ahead and make what you saw. Very good.

F: [Frank picks up two counters and places them in a column on the right-hand portion of his mat. Frank picks up two more counters and places one at the top of his column of two dots on the right-hand portion of his mat, and the other is placed on the left-hand portion of his mat directly across from the bottom counter in his first column. Frank picks up one more counter and places it above the last counter placed on his mat creating a column with two counters and a column with three counters. The counters at the bottom and the one counter at the top of his orientation are red, but the counters in the center of his orientation are yellow.] Make different [indicating there are different colors in his orientation.]

T: Yeah, so how many are they there again?

F: Six

T: Yeah? Let's do this. [Teacher-researcher covers up the one counter at the top of his orientation leaving four counters visible.]How many are there here?

F: One.

T: There's one underneath here?

F: Hmm...mmm and there's four.

T: Oh, so there's one and four?

F: Yeah.

T: Yeah? How about now? [Places cardstock over two and then switches it so it is still covering two, but the column of three counters is visible and the column of two counters is hidden.] Let's do this, how many are hidden underneath there now?

F: Two.

T: Two. [Places cardstock over the column with three counters and leaves two counters visible.]

How many are hidden underneath here now?

F: Three.

T: Yeah, and now? [Places cardstock over the square orientation of four counters, leaving one counter visible.]

F: Hmmm...three?

T: There's three?

F: Four.

T: Four? You want to check? [Teacher-researcher lifts the cardstock up revealing the four counters hidden by the cardstock.]

F: Four!

T: How about now? [Teacher-researcher covers up all five counters.]

F: Umm...six?

T: Six? [Teacher-researcher shifts cardstock towards Frank covering up the four counters in a square orientation, revealing one counter.] How about now?

F: Four.

T: Yeah? [Teacher-researcher covers up all five counters.]

F: Six.

T: Yeah? [Teacher-researcher shifts cardstock towards Frank covering up the four counters in a square orientation, revealing one counter.] Look at this, how many is there here? [Teacher-researcher points to the one counter visible.]

F: One.

T: And how many is underneath here? [Teacher-researcher points to the cardstock covering up the four counters in the square orientation.]

F: Umm...Four.

T: Four. [Teacher-researcher covers all five counters.]

F: Six.

T: [Removes cardstock and sets it aside.] Alright, do you want to count?

F: Yeah. One, two, three, four, five. [Points to each counter while stating each number word.]

Five.

T: Five. Very good.

As evidenced in protocol F5, Frank was capable of attending to subgroups, which might indicate a more conceptual type of subitizing, but when asked how many counters Frank saw, both as subgroups and as an entire group, Frank was not able to use these subgroups to subitize. He seemed to revert to the symmetrical aspects of the orientation. Interestingly, he did not represent the symmetrical aspects when he made the orientation with his counters, suggesting a transition towards what was described in the study as perceptual subgroup subitizing activity (PSS Activity) and will be further defined in the next section.

These cognitive changes that Frank was making, relative to his thinking model, suggested a shift towards reliance upon the subitizing of subgroups, though not necessarily composition of the entire group with these subgroups. This seems evident when Frank was not perturbed by the notion that four counters under the mat and one counter visible would explain this orientation as “five.” Thus, it seemed that Frank’s inability to rely on the abstract notion of “+1” or “-1” limited his ability to consider subgroups and the composite group conceptually.

Synthesis of IPS Activity

IPS activity is defined as a reliance upon empirical orientations which seemed to be influenced by the visual motion which the students engaged in when subitizing. In addition, when children rely on IPS activity, the space between items seems to influence their understanding of number, which then indicates that students rely upon similar thinking structures when both subitizing and when conserving number. In fact, when Ben and Frank were asked if orientations that differed only in the space between the items were in fact the same quantity, they could not state that the quantities are the same until the objects are transformed so that the counters have the same space between them. The notion that students lack reversibility of mental action, required when conserving number (Piaget, 1941/1965), is supported by these actions, as the students in this study had to physically move the objects to match the orientation of the objects on the teacher-researcher's mat. If Ben and Frank were able to engage in mental reversibility, the orientation of the items would not need to change for the number to remain the same.

Symmetry also seemed to influence this IPS activity and tended to be more noticeable in this activity, as an explanation of essential mistakes made by students when relying upon IPS activity. In the next section, some students' early attention towards subgroups when subitizing is explained as being elicited through a reliance upon symmetry. Attention towards subgroups should not be a sole indicator for more conceptual forms of subitizing, as indicated by Frank in protocol F5, as Frank attended to two subgroups when describing what he "saw." Frank relied heavily upon symmetry, as only one group is attended to when subitizing, or the entire orientation itself is attended to when subitizing. Regardless, the two subgroups do not inform

Frank of the total number. This becomes more evident in the perceptual ascending subitizing (PAS) activity section when Frank was shown this orientation again.

Evidence of Ben and Frank's Perceptual Subgroup Subitizing Activity (PSS)

Early on in the teaching experiment sessions, both Ben and Frank's subitizing activity suggested that there may be an unconscious attention towards subgroups, however, this activity did not necessarily afford Frank and Ben the opportunity to subitize the entire group. In fact, it seemed evident that Frank and Ben could subitize these smaller groups later on, but could not coordinate this with an understanding of the entire composite group. Perceived shape of the items, space between items, color of items, and symmetry seemed to be influential factors that provided the students some perceptual scaffolding in which to engage in PSS activity. Similar to the IPS activity, students with weaker understanding of number tended to engage in PSS activity, as their attention to subgroups only allowed them the opportunity to subitize these subgroups. The subgroups at this juncture remained at one, two, or three items per subgroup, as this activity was grounded in perceptual subitizing activity.

Perceived symmetrical aspects of items. The perceived symmetrical aspects of items seemed to elicit some early forms of attention towards subgroups, which rarely prevented students from numerically expressing the entire orientation. The students did not rely on symmetry as often when engaging in PSS, as it seemed more evident in PAS and PDS activity. In protocol F6, Frank was in his first teaching experiment session, and was shown a card with four dots arranged in a square orientation. Frank stated that he saw "T...Four," but when asked about almost stating "two," he does not remember seeing "two." However, he became distracted, and when asked to draw what he remembered, he drew two dots and wrote the numeral two beside them. When he was shown the orientation a second time, he then stated that he saw four

and drew the four dots in the same square-like orientation. This activity seems to suggest that he was unconsciously attending to the two in which to build up towards four. Symmetry tended to influence this activity, as well as perceptual ascending subitizing (PAS) activity, which will be discussed further in a later portion of this analysis.

Protocol F6.

T: Alright. We're going to start with one that I thought you did a good job with last time. Are you ready? [Teacher-researcher shows student a card with four dots on it the dots are arranged in a square orientation.]

F: T...four.

T: Four? You almost said two. Why did you almost say two?

F: [Frank shrugs his shoulders to indicate that he does not know why he almost said two.]

T: Did you see two also?

F: Also four. [Frank drops the marker top.]

T: Also four? Okay, so draw what you remembered. [Teacher-researcher attempts to pick up marker top.] Whoops, here you go. I can't reach that.

W: [Witness picks up marker top and hands it to Frank.] Here you go. Do you still remember what you saw?

F: I swam with Mr._____. [Uncertain of name.]

T: Oh, do you still remember what you saw?

F: [Frank draws two dots and writes the numeral two.]

T: Okay, you saw two, but you also saw four, because you told me that. Do you want me to show you one more time?

F: Uh huh [indicating yes].

T: What is interesting is I'm going to show it to you in parts. Are you ready?

F: Uh huh [indicating yes]

T: [Teacher-researcher shows student a card with two dots visible. Two dots below these two dots are covered up.] Is that what you just drew?

F: Two.

T: Okay, now watch this. [Teacher-researcher shows Frank the entire card with all four dots.]

F: Four.

T: Okay, now draw four.

F: [Frank draws four dots in a square orientation, and then writes the numeral four.]

It seemed as if Frank could not consider the two as a subgroup of the four, but was beginning to attend towards these subgroups unconsciously. Symmetry also seemed to elicit this type of activity, as it seems that if symmetry offered a “free” subgroup that did not need to be consciously considered at first. In fact, in later protocols in the PAS activity section, Frank began to attend to two, two, and one to describe five. However, when depending on these subgroups, Frank only described it as two and one at times, which suggests a similar type of activity. The distinction between PSS activity and PAS activity is that students need to be able to describe the subgroups and then the entire orientation. Here Frank may be beginning to rely on the symmetrical aspects of this orientation to attend to the subgroup “two,” but he does not state that he saw two and two, which makes four.

Ben also relied on symmetry when engaging in PSS activity, but this task had to be explicitly designed so that the space between the items and the frames around the dots introduced the notion of subgroups early on. In his third teaching experiment session, Ben and I played what I described in Chapter 3 as the “Ice Cream” game. Essentially, the purpose of the Ice

Cream game is for students to roll a die and to find matches between their die and the game mat. When a match is found, and the students can justify the match, the students can then color in the ice cream scoops on their mat. The first person to color in all of their ice cream scoops wins the game. Ben's mat had six different dot orientations, and each one totaled to "four." Ben's die had six different dot orientations, also totaling to four, but some of the orientations were drawn to look like two die faces. For instance the die face that Ben was looking at in protocol B9 has two squares drawn on it with an addition sign in between the two squares. Each square has two dots drawn in, to elicit Ben's attention towards the subgroups "two and two." In protocol B9, Ben was not playing the game yet, he was reviewing the characteristics of his die relative to his mat. This discussion before the game is played was put in place to allow the student time to build some strategies before playing the game.

Protocol B9.

T: Super. How about this right here? [Teacher-researcher shows Ben the face of a die with two squares drawn on the die and an addition sign drawn in between the two squares. In each square there are two dots drawn in a diagonal line from the right top corner of each square to the bottom left corner of each square.] What do you see here? [Teacher-researcher points to one square of two dots.]

B: Two.

T: and... [Teacher-researcher points to the dots in the other square.]

B: Two.

T: Do you see two and two over here?

B: [Pauses before pointing to different ice cream scoops, and then finally landing on the top ice cream scoop. The four dots in this scoop are arranged as a row of three dots with one dot just above the middle dot.]

T: Right here? [Teacher-researcher points to this ice cream scoop.] Can you show me two and two?

B: [Ben points to the middle column of two dots.] One, two. [Ben points to the dots on either side of the column.] One, two.

T: Oh...neat. I never would have seen it like that. I like that.

Protocol B9 indicated that Ben was relying upon the symmetrical aspects of the orientation of four to describe subgroups. His attention towards these subgroups did not necessarily provide him with the understanding of what the composite group was. In fact, the distinction between the PSS activity and the PAS activity, which is discussed in the next section, was that students did not use these subgroups to describe the composite group. If Ben were attending to these subgroups and then stating that this two and two made four, then Ben would be engaged in PAS activity. One more thing of note is that Ben's reliance upon these subgroups was conscious, which is slightly different than Frank's reliance on the subgroup two. However, Ben was only cognizant of the groups two and two because the purpose of the game was to find a match from two and two with the orientations of four. This activity was not one in which Ben was carrying into his subitizing activity independently, whereas Frank seemed to be unconsciously carrying this activity into some early subitizing activity with regard to four in a square orientation. The distinction between the two types of activity suggested that Ben relied upon empirical abstractions, whereas Frank was relying upon some reflecting abstractions. Thus, it does not seem that attending to subgroups automatically raises the students' thinking

structures to more sophisticated levels. In fact it seems as if PSS activity in which students engage would transition students from IPS activity towards PAS and PSC activity.

Shape of the items. The shape of items also seemed to elicit an attention towards subgroups, but did so in only one protocol. The shape of items seemed to be more prevalent in IPS activity. However, there were times when the shape of the items offered to students provided two shapes for the students to utilize when subitizing subgroups.

Ben's thinking model with regard to PSS activity and the perceived shape of the items. This section is, again, offering two discussions. PSS activity, with regard to the perceived shape of the items, will be described relative to Ben's responses. Also discussed will be Ben's thinking model, with regard to PSS activity. Ben's thinking model, regarding IPS activity, suggests reliance upon the space between the items and visual scans when numerically describing a set of items. Cognitive changes that occurred indicated Ben's cognitive shift towards attention, regarding the density of items and subgroups when subitizing. Considering this trajectory in IPS activity, relative to the following description of PSS activity, Ben's thinking model becomes a bit more defined.

In protocol B10, Ben was in his first teaching experiment session and he was asked to tell me how many dots he saw, and then draw what he had seen. This was near the end of his session and Ben was shown three dots in a column and one dot directly to the left-hand side of the middle dot in this column to make four dots total. This orientation seemed to suggest the two subgroups, three and one. When Ben was shown this orientation, he stated that he saw three, and then drew a row of three circles. Ben was given the opportunity to see the card a second time, and then realized that he has missed a dot, and he added one circle directly above the

middle circle. Here, he was capable of subitizing both subgroups at two different points but was not able to use these subgroups to tell him how many dots he saw altogether.

Protocol B10.

T: Okay. Good. All right, we're going to do this one here. [Teacher-researcher shows Ben a card with four dots on it. Three dots are in a column and one dot is to the left of the middle dot in the column.]

B: Three

T: Three? Can you draw what you saw?

B: [Student draws three circles in a straight row.]

T: Okay, I'm going to show you again. Watch one more time Ben, okay? [Teacher-researcher shows Ben the card again.]

B: Three.

T: Now, look at your draw... Let's look at it this way [Teacher-researcher shows Ben the card and rotates it so his row of "three" matches the card's row of three dots] so that your three matches my three.

B: [Ben nods his head up and down to indicate yes.]

T: Okay, is there something that you can add to your drawing? [Pauses] What could you add?

B: [Ben draws one more circle above the middle circle in the bottom row. Now Ben's drawing is very similar to the card with the four dots on it.]

T: Yeah, look at that. You could add one more couldn't you? So, what does it make now?

[Teacher-researcher points to all of the circles on Ben's piece of paper.]

B: Three.

T: Okay, count and check.

B: One, two, three, four. [Ben points to each circle with his marker as he says each number word.]

T: Four.

This protocol was very similar to Frank's F5 protocol in the IPS activity section. In comparison, Frank was subitizing the entire orientation, but then describing the column two and three. Yet, Ben, in protocol B10, was subitizing three, and then later one; he does not consider these groups as helpful when subitizing the entire orientation. Therefore, it seems there is a slight distinction, which I feel is necessary to emphasize here. The subgroups which Ben subitized in protocol B10 is dominating his subitizing activity and more directly results from encoding the column of three, as compared to the entire orientation. However, the shape or the symmetry that is influencing Frank's subitizing activity seems to have a dominate presence in the encoding process, preventing Frank from first subitizing the subgroups. Frank's activity could also be confused as Perceptual Descending Subitizing (PDS) activity, as Frank may have been subitizing the composite group by relying upon the subgroups three and two. PDS activity will be described in more detail later on in the analysis.

Another important aspect to discuss here is Ben's shift towards reliance upon reflecting types of abstraction with regard to this orientation. This reliance seems evident, as number is attended to unconsciously, indicating a more sophisticated type of activity. If Ben were relying on the shape or orientation of the items, then this might suggest reliance upon empirical abstractions. This seems evident as Ben describes seeing three, which I believe resulted from attending to the larger column of dots. I do not believe Ben anticipates relying upon a subgroup of three because he states that he only saw three. Therefore this activity is not consistently utilized early in his subsequent teaching experiment sessions, but may begin providing Ben a

notion of subgroups when he looks back at the original orientation as needing one more. This attention towards “one more” or “subgroups” was utilized by the teacher-researcher to try and promote conceptual subitizing activity.

Therefore, Ben’s thinking model at this point suggests that Ben is shifting from reliance upon shape and space between items to some early attention towards clusters of items, which he was more cognizant of in protocol B6 and B7. However, because Ben still seems to rely upon the items being broken into two groups, with one group in a row and one dot beside this row suggests reliance upon some empirical activity. Furthermore, the two groups are each composed of a number of items that can be subitized effectively without relying upon conceptual understandings of number. Regardless of this reliance upon perceptual material, Ben’s visual scan was chunked to allow for accurate subitizing of a subgroup. Also, when subgroups were shown to Ben for use when subitizing other orientations, he was capable of using symmetry to find one dot on either side of the orientation as making two, and the column of two dots in the middle of the orientation as making two. This seems to illustrate a slight shift in Ben’s subitizing activity, as this was Ben’s third session; the session in which Ben could subitize only the three dots in the row, was Ben’s first session. Broader changes in Ben’s thinking model relative to PSS activity occurred when color and symmetry, or color and the space between items, were introduced into Ben’s subitizing tasks, and will be discussed in more detail, later in this section.

Space between items. Space between items seemed to largely influence this PSS activity because it seemed that this space offered a student an opportunity to engage in two scans. However, because the student seemed to scan the groups twice, the students who did not have number permanence grappled with how to combine the two groups. So, these protocols provide

an insight as to how the space between items elicited attention towards only subgroups when subitizing.

Frank's thinking model with regard to PSS activity and the perceived space between items. Specifically, protocol F7 is also meant to illustrate aspects of Frank's thinking model relative to PSS activity. At this point, Frank's thinking model relative to IPS activity suggested that Frank began relying on the space between items, and continued to revert to a re-presentation of diamond, but was shifting to an attention towards subgroups. In fact, in Frank's first teaching experiment session, there was an unconscious attention towards the subgroup of two, but Frank had difficulty considering this when re-presenting the composite group.

In protocol F7, Frank was in his fourth teaching experiment session, playing what I described in chapter 3 as the "Camera Game." In the Camera Game, students were shown an image of the back of a camera for just a second or two with some dots in the viewfinder. After being shown the image, the students needed to say how many they saw, and draw what they thought would come out of the camera. Frank is shown in protocol F7 an image with four dots in a square formation in the left-hand portion of the view finder and one dot in the top right corner of the view finder. This space between the square and the one dot is not easily rectified and disrupts Frank's ability to subitize the composite group.

Protocol F7.

T: [Teacher-researcher shows Frank an image with four dots in a square formation in the left-hand portion of the view finder and one dot in the top right-hand corner of the view finder.]

How many did you see?

F: [Smiles and giggles.]

W: I don't think he saw that.

T: Did you see that? I am going to give you another second or two. [Teacher-researcher shows Frank the back of the camera again.]

F: One, two...

T: [Teacher-researcher covers up the orientation with hand.] Oh, no don't count. Just look. [Takes hand off of orientation and shows one more time for about half a second.]

F: Four, one.

T: You saw what? [Hands Frank a piece of paper.]

F: Four and one.

T: You saw four and one? How much is that altogether?

F: [Frank begins drawing what he saw.] Hmmmm...make it. [Frank then returns to his paper to draw what he remembered.]

T: You want to make it first?

F: [Frank draws four dots in a square and one dot in the upper right-hand corner of his paper.]

W: Whoa, good.

T: Very good.

F: [Frank writes next to each orientation, the numerals four and one respectively.]

T: So, what is that altogether?

F: Four and one.

T: Yeah?

F: It's fourteen.

Frank responded to the subgroups only in protocol F7 which seems to suggest that the space elicited his attention to both subgroups as separate actions, which could only be referred to as four and one. Once Frank wrote the numerals "one" and "four" next to the orientations, he

began to employ his number schemes, which involve two digit numbers. This happened in subsequent sessions with Frank. Frank did not always need to write out numerals to draw on these notions of number, which offered some questions as to how numbers at home and/or school were being taught to Frank. Frank's reliance upon his drawing actions also indicated that for Frank to numerically express the entire set, he needed to revert to some empirical actions or have perceptual material in front of him to stand in for the subgroups he subitized. Also, the space between the two subgroups may have elicited two visual scans, supporting the argument that Frank cannot combine the two groups because they resulted from two different types of empirical activity.

Changes in Frank's thinking model with regard to PSS activity and the perceived space between items. So, to control for Frank's attention towards the dots and not the numerals, I offered more activities which allowed him to use circular counters. Each counter had a red side and a yellow side. In protocol F8 Frank was in his seventh teaching experiment session and was shown a total of five counters. The counters had a large space between them, to elicit Frank's attention toward these subgroups. However, this orientation had two counters arranged in a vertical column on the teacher-researcher's left-hand portion of her mat, and three counters arranged in a triangular orientation on the teacher-researcher's right-hand portion of her mat. By removing Frank's ability to write a numeral and offering Frank a chance to count the counters he used to re-present the orientation, I hoped to elicit an understanding that the two subgroups could be considered relative to the composite group.

Protocol F8.

T: You ready Frank?

F: Uh...huh [indicating yes].

T: Okay. [Teacher-researcher lifts the top piece of cardstock, revealing two counters on the left-hand portion of her mat in a column, and three counters on the right-hand portion of her mat in a triangular arrangement.]

F: Two plus two equals three.

T: Yeah? How many did you see altogether?

F: Two plus...Two plus three.

T: Uh...huh [indicating yes], and how many is that altogether?

F: Umm...twenty-three.

T: Twenty-three? Can you make what you saw?

F: [Nods head up and down to indicate yes. Frank places three counters on the left-hand portion of his mat, and two counters on the right-hand portion of his mat.]

T: Nice job!

W: Good job Frank!

T: Look at that. [Teacher-researcher lifts cardstock to reveal the original orientation so that Frank can compare his orientation to the one shown to him.]

F: [Frank changes his counters slightly so that the two counters on the right-hand portion of his mat match the two counters on the left-hand portion of the teacher-researcher's mat.]

T: Very much the same, isn't it? [Covers the original orientation with the top piece of cardstock.]

F: Hmm...mm [indicating agreement].

T: Yes, you did a great job. How many is there altogether?

F: Twenty-three.

T: Do you want to count and check?

F: One, two, three [points to each counter in the triangular arrangement while stating each number word.]

T: How about these guys? [Points to the two counters sitting off to the side.]

F: Two.

When Frank was shown the cluster of two counters and three counters, he not only subitized them separately, but also carried into the activity the notion that each set of counters represented different place values. Similarly, when asked to count, Frank still counted each group separately. These groups were not meant to be combined, and represent Frank's over-reliance upon his PSS activity, as it prevented him from considering the composite group when the space between the items is large. However, there was a slight change in Frank's reliance upon the perceptual items. Earlier in protocol F7, Frank could not tell us how many dots he saw altogether until he "makes it" by drawing what he saw. However, this reliance on a drawing or counters, as an explanation of the subgroups Frank subitized is absent, as Frank was able to describe both the subgroups and what he considered to be the composite group, without any perceptual material in front of him. Thus, there seemed to be a slight shift away from reliance upon perceptual material when subitizing.

Color of items combined with space between items and symmetry. Frank depended heavily upon the space between items when engaging in PSS activity. However, Ben relied heavily upon color and symmetry or color and space between items when engaging in PSS activity. For instance, when Ben was shown counters or dots of different colors that emphasized groups, the space between the groups of items or the symmetrical aspects that the arrangement of items held, seemed to support Ben's ability to consciously consider subgroups. In a similar fashion the color of items also disrupted Ben's ability to subitize accurately.

Changes in Ben's thinking model with regard to PSS activity and the perceived color of items combined with space between items and symmetry. Ben's thinking model relative to PSS activity described him relying heavily upon symmetry when subitizing subgroups, as he was capable of finding two and two in an orientation of four by describing the dots on the left and right side of the orientation as combined to be two, and the column in the middle of these two dots as two. This is a change in Ben's thinking model, because earlier Ben was not able to subitize two subgroups. However, it could be argued that Ben was subitizing one subgroup of two, and then the remainder simply happens to be two.

When Ben subitized an orientation of three, it seemed that his attention was drawn towards the comparatively larger cluster of items. Ben seemed to be successful with this subitizing because it required him only to rely upon perceptual subitizing, as it was less than four. At this point, Ben's ability to consider two subgroups when subitizing needed to be introduced by the teacher-researcher, and was not carried into subsequent sessions unless introduced in a similar manner. Symmetry seemed to be supporting some students' ability to subitize subgroups and color also seemed to support students' ability to subitize subgroups. Thus, this next task in protocol B11 was designed to better understand which one Ben primarily relied upon.

In protocol B11, Ben was working in his 15th teaching experiment session and was being shown a square-like orientation with three red counters and one yellow counter. At this point, Ben had been successful at subitizing four counters when two were red and two were yellow. He was even more successful when the two red counters were in a column on one side of the square orientation, and the two yellow counters were in a column on the other side of the square orientation. The purpose for showing Ben this orientation with these color combinations was to better understand which Ben would attend to when he had to choose between color or the

symmetrical aspects of the orientation. If Ben relied upon the symmetrical aspects of the items, then he would describe seeing four, but if he was relying upon the color of the items, he would describe seeing three.

Protocol B11:

T: Okay, are you ready? Alright, watch this. [Teacher-researcher lifts the top piece of cardstock revealing four counters in a square orientation. Three counters are red, and the teacher-researcher's top left-hand counter of the orientation is yellow. The teacher-researcher drops the top piece of cardstock hiding the orientation again.] How many do you see?

B: Three.

T: You see three? Can you make what you saw? You are so good at this.

B: [Ben places two red counters down on the mat in a horizontal row before picking up two more red counters and placing them in another horizontal row just below the top two.]

T: So how do you know you have three there?

B: Because I know [states this very quietly].

T: Watch this. Does it match? [Teacher-researcher lifts the top piece of cardstock to reveal the original orientation.] How can you change it to make it match?

B: [Shakes head side to side to indicate no, and picks up the red counter in the bottom right-hand portion of his mat and turns it over so it is yellow like the researcher's orientation.]

T: Okay.

W: It's not three is it?

T: Is it three?

B: [shakes his head side to side to indicate no.]

T: How much is it altogether?

B: Four.

T: Oh, four!

Ben begins to rely on PSS activity by stating he saw three, but when Ben re-presents what he “saw,” he is able to construct the orientation as four. This is an example of PSS activity where Ben is relying upon color when considering subgroups, but preattentively encodes this orientation as four, which I believe is dependent upon the symmetrical aspects of the orientation. For instance, even though he states that he saw three, he picked up counters two at a time and placed them on his mat, suggesting that he did not encode this as three ones, or even four ones, but two groups of two. The rest of the task is one that we will revisit in the PAS and PDS sections, as Ben begins to be able to describe subgroups relative to this orientation of four without pausing or counting. This protocol illustrates Ben’s ability to engage in PSS activity and the way in which color seemed to elicit the notion of number that he carried into his subitizing activity.

In this final protocol for this section, Ben was shown four counters and was in his seventeenth teaching experiment session. Two red counters were in a diagonal row in the teacher-researcher’s top right-hand portion of her mat, and two yellow counters were in a diagonal row with the same slope as the red counters, but in the teacher-researcher’s bottom left-hand portion of her mat. The whole orientation looked as if it could represent a diagonal line with four counters, but with a large space between the two groups of counters. This protocol was evidence of how symmetry, space, and color supported Ben’s ability to attend towards multiple subgroups when engaging in PSS activity.

Protocol B12.

T: Okay, are you ready? [Teacher-researcher shows Ben four counters in a diagonal row with a space between the middle two counters. Two red counters are in a diagonal row in the teacher-researcher's top right-hand portion of her mat, and two yellow counters are in a diagonal row with the same slope as the red counters, but in the teacher-researcher's bottom left-hand portion of her mat. The whole orientation looks as if it could represent a diagonal line, but with a large space between the two groups of counters.] How many do you see?

B: Two and two.

T: You saw two and two? How much is that?

B: [pauses.]

T: Do you want to make it and then count?

B: [Nods head up and down to indicate yes.]

T: Yeah? Okay you can.

B: [Ben takes the counters one at a time and places them red side up on his mat in a horizontal row. As Ben places each counter on his mat, he counts them quietly.] Four.

T: Four? Holy cow! That is awesome! Do you think it is how it looks on my mat, or do you want to move them around a bit?

B: [Ben places both of his hands on two counters and slides both groups to the corners of the mat.]

T: Oh, nice. Look and see. [Lifts top piece of cards stock up so Ben can see the original orientation.]

This protocol illustrates Ben's first time consciously attending to two subgroups, and then using these subgroups to count and numerically describe the composite group. It is important to

note that for Ben to count the four items, he needed to place them on the mat one at a time. However, when asked if his orientation aligned with my orientation, he quickly slid the two dots away from each other, indicating that even though the row of four re-presented the composite group, the two groups needed to be separated to re-present the two and two that he saw. This reliance upon empirical activity was evidence of Ben's inability to conceptually consider two and two as composing four, as it seemed for Ben to know this orientation as two and two, the counters need to be physically changed.

Color and symmetry, or color and space between items, seemed to scaffold some tasks for Ben and support his ability to engage in PSS activity. The color used was usually red and black, or red and yellow. It was found in teaching experiment session two that when Ben was shown cards with groups of two or three dots the color red and two or three dots the color black, Ben's subitizing ability and re-presentations more accurately aligned with the card that was shown to Ben. So, color seemed have a strong influence upon Ben's subitizing activity early on. However, once color was combined with large spaces between these same groups of two or three, and combined with symmetrical groups, where one vertical column was one color and one vertical column was another color, Ben was able to consciously attend to subgroups and engage in more effective PSS activity, as it elicited an internal attention towards these subgroups without having to be told explicitly to attend to these subgroups.

Synthesis of PSS Activity

When Ben and Frank engaged in PSS activity, the strongest influential characteristic was the space between the items. However, once space, symmetry, and color were all three utilized, then Ben's PSS activity was supported, as the smaller subgroups were emphasized, freeing up a need to consider the composite group. As interesting as it is to observe students engaged in PSS

activity, an over-reliance upon this activity can result in limiting results. For example, Frank typically relied on subgroups when subitizing, but could not use this to inform him of the composite group. In fact, the PSS activity that Frank relied on seemed limited, because his number schemes were not grounded in quantities, but in reading two-digit numerals. So, whenever Frank was able to find two quantities to subitize, he attended to this because it aligned more closely with what he knew about two-digit numerals. However, helping Frank link this activity with the construction of the composite group would allow Frank to depend on his subitizing activity to numerically describe the composite group.

Evidence of Ben and Frank’s Perceptual Ascending Subitizing Activity (PAS) and Perceptual Descending Subitizing Activity (PDS)

As Ben and Frank engaged in more PSS activity, there were moments when Ben and Frank were able to describe the subgroups and then the composite group that the subgroups composed. This activity was described in this study as perceptual ascending subitizing (PAS) activity. Comparatively, Ben and Frank sometimes described the composite group and then decomposed the subgroups. This activity is described in this study as perceptual descending subitizing (PDS).

The terms “ascending” and “descending” originate from Inhelder and Piaget’s (1964/1999) description of how classification thinking structures develop. As children engage in sorting activities, they can initially consider the subgroups and then in ascending order, build towards the composite group, or they can consider the composite groups first and in a descending order sort to find the subgroups (Inhelder & Piaget, 1964/1999). Thus, this section will consider perceptual ascending subitizing (PAS), where students attend to subgroups and then use this to describe the composite group, and then perceptual descending subitizing (PDS), where students

attend to the composite group and then use this to describe the subgroups. The distinction between PAS and PDS with conceptual subitizing activity is that the students are still relying upon the perceptual material when engaging in this activity. In other words, if the topological characteristics that are influencing this activity are removed, the students would not still be able to rely on PAS or PDS activity.

Symmetry between items. When students were shown orientations that were symmetrical, they were able to consider one subgroup and then build towards a composite group without having to consider both groups at once. For instance, Frank was very close to this type of activity in protocol F6 when he was described as engaging in PSS activity. However, Frank was not consciously describing any of the subgroups when subitizing four, so one aspect of this activity is to be cognizant of the subgroups one is using to build towards the composite group. This type of activity is more sophisticated than PSS activity because students need to consider the subgroups and then flexibly switch to consider the composite group. It is important to note here that when the students in this study initially began engaging in PAS activity, symmetry seemed to support this activity because they were able to consider two groups, as one group was a “free” unit. This notion of how one unit did not need to be coordinated when engaging in this activity will be described in more depth in this section.

In protocol B13, Ben is in his third teaching experiment session and is playing the Ice Cream scoop game described earlier in both chapter 3 and in this chapter. Ben has a mat with six different orientations of dots which total to equal four. In an earlier portion of this teaching experiment session, Ben had already circled two dots twice in the row of four dots on his ice cream scoop mat to indicate this orientation has two and two. He then rolled his die and found

the same row of four dots on his die. He was now looking for an orientation on his ice cream scoop mat that would match the face of his die.

Protocol B13.

T: What did you get?

B: [Ben rolled his die and the orientation with four dots in a row is facing upwards.] Two.

T: You got two?

W: Show the two.

T: Show me the two.

B: [Ben points to the two on the right-hand side of the row of four dots.]

T: Right there? [Teacher-researcher points to the same two dots Ben pointed to.]

B: [Ben points to the two other dots on the left-hand side of the row of four dots.]

T: ...and right there? Two and these two?

B: [Ben nods his head up and down to indicate yes.]

T: So, I don't see one two, I see two twos, right?

B: [Ben nods his head up and down to indicate yes.]

T: What are two twos?

B: Four.

T: Yeah! Do you see four somewhere down here? [Points to Ben's mat.] That has not been colored in?

B: [Pauses.] Points to the ice cream scoop with four dots in a random orientation.

T: Okay, can you circle the four dots to show me? Where you see the four? [Teacher-researcher points to the orientation to redirect the students' attention.] Circle the four dots.

B: [Ben begins to circle just one dot.]

T: Remember we do one big circle? Can you show me with one big circle what it looks like?

B: [Ben circles two dots that are closest together.]

T: There's two, where is the other two?

B: [Ben then circles two more dots that are closest together.]

T: Nice job Ben! Go ahead and color that all in. How many is that altogether?

B: Four.

So, in this protocol, after Ben was shown two and two and engaged in PSS activity; then he circled and described orientations similar or identical to the four dots in a row, Ben used this PSS activity to transition towards PAS activity. The symmetrical aspects of the orientation allowed Ben to engage in this activity, as he did not even need to consider both twos at first. He clearly stated that he saw a group of two, but when asked to show the group of two, he then described two groups of two. This provided us with a glimpse of both what he needed to encode and what he drew upon numerically. For instance, for Ben to preattentively encode this orientation, he simply needed to see one two. This is evident when he stated that he saw two. However, this drew upon the PSS activity, which connected with his notion of two and two. He then had to reconsider the whole orientation again and subitize it again. This resulted in the accurate numerical expression "four." The most salient part of this activity was when Ben considered the two groups of two as matching his row of four. Comparatively, Ben did not choose the four dots as matching. When Ben circled the two dots twice, this activity seemed to suggest that for Ben to consider this as four, he needed to ascend towards it with two groups of two.

Frank's thinking model with regard to PAS and PDS activity and the perceived symmetrical aspects of the items. Attention to students' transitions is necessary as they utilize the PAS activity when subitizing four to subitizing five. One example is protocol F9, when Frank was in his seventh teaching experiment session. In protocol F9, Frank utilized space and symmetry when subitizing five, which illustrated how the symmetrical aspects of the items and the space between items can elicit more conceptual understandings of five. Earlier, Frank's thinking model relied on space to elicit an attention towards subgroups when Frank engaged in some PSS activity. However, once symmetry was introduced, Frank was capable of using what he knew about four to subitize five. Frank's ability to subitize four was evident in protocol F7 when Frank described seeing four and one. However, Frank may not be considering four as a mental object, but building towards four when subitizing orientations that are in diamond or square shapes. To better test Frank's reliance upon the subgroups, as noted in protocol F6 (when Frank described seeing "T...Four),” or Frank's reliance upon the shape of the items or the space between items (as noted in protocol F4), the counters are spread apart to disrupt the shape of the items, but support the notion of subgroups two, two, and one.

In protocol F9, Frank was being shown counters on a cardstock mat and was being asked how many he saw, and also to make what he saw. This protocol began with Frank being shown five counters and follows immediately protocol F8, in which Frank was shown five dots separated into two groups (two and three), and Frank described this as twenty-three. Next, I showed Frank three groups of counters (two, one, and two). The two sets of two were symmetrically balanced so that there were two rows of two on the teacher-researcher's bottom left-hand and top right-hand portion of her mat. The one counter was placed directly in the

middle of the mat. This design was meant to use symmetry and space to support five, as one of the groups of two may not need to be considered as something for Frank to have to coordinate.

Protocol F9.

T: Alright, you ready? [Teacher-researcher places the five counters on her mat where there are two counters on the teacher-researcher's bottom left-hand and two counters on the top right-hand portion of her mat. The one counter is placed directly in the middle of the mat. The top piece of cardstock is lifted up to reveal the orientation to Frank.] Alright, let's clear our mat, so that it is fresh and we can be ready when we see it.

F: [Frank pushes his counters off of his mat so his mat does not have any counters on it.]

T: [Teacher-researcher shows the orientation to Frank for about one second before covering the orientation up again.] How many did you see?

F: Two plus one plus two equals...five.

T: Okay, go ahead and make that. Very good.

F: [Frank picks up two counters and places them together in a row on the right-hand side of his mat. Frank picks up two more counters and places them together in a row on the left-hand portion of his mat. Frank then gets one more counter and places it above the two counters, making the left-hand side orientation a triangular arrangement of three. Frank then picks up one more counter and places it in the middle between the two orientations. The total number of counters is now six.]

T: Okay, where do you...you said three numbers. You said, two plus [pauses] one...

B: One...

T: plus...two. Right? Equals ... what?

F: Five.

Frank is still grappling with how to coordinate subgroups to construct a composite group, but it seems evident here that Frank is capable of considering two, and two, and one when subitizing five dots. However, Frank's subitizing has obviously shifted here, as he considers the subgroups as composing the entire group. As Frank re-presented this orientation, he seemed to want to build this orientation into a number sentence, as he ended up constructing two plus one equals three on his mat, which he explained later. More importantly, two, one, and two are initially considered as composing five, whereas two and three were too difficult for Frank to consider as composing five.

Changes in Frank's thinking model with regard to PAS and PDS activity and the perceived symmetrical aspects of the items. In protocol F10, Frank was in his 17th teaching experiment session and was shown a similar orientation in which the teacher-researcher has shown him the red and yellow counters, and asked him how many he saw. Frank had brought his stuffed mouse to this teaching experiment session, and he talked through the mouse at one point. The teacher-researcher had arranged the counters so that two counters were on the left-hand portion of the mat, and two counters were on the right-hand portion of the mat. In the middle there was one counter. This subitizing task was near the end of the teaching experiment session, and just before this task Frank was shown five counters which he determined were four and one, which then made fourteen. Once Frank counted these five counters he determined the group was five. So again, Frank was having difficulty with subgroups greater than two, but was having very little difficulty with three subgroups that were made up of two or less. It is important to note how Frank initially described this arrangement, as it illustrated PDS activity. Also, as Frank described the subgroups which made up the composite groups, it seems evident that symmetry supported this activity, as only one two is mentioned.

Protocol F10.

T: Alright, are you ready?

F: Yeah.

T: Okay mousey sit aside. Sit aside, so Frank can look at the mat. We only have a couple of these, Frank.

F: [Frank continues to bounce his stuffed mouse on his mat.]

T: Okay, set mousey aside. Okay, one, two, three. [Teacher-researcher lifts the top piece of cardstock revealing counters arranged so that two counters are on the left-hand portion of the mat, and two counters are on the right-hand portion of the mat. In the middle there is one counter.]

F: Five. [Frank talks in a squeaky “mouse-like” voice.]

T: Five? How did you know that so fast?

F: Mousey says that.

W: How did mousey know it so fast?

T: Yeah, how did mousey know it that quickly? Do you agree with mousey?

F: Yeah.

T: Yeah? Why?

F: Because mousey said five [says five again in a squeaky “mouse-like” voice.] But mousey wins.

T: He did win, but why did mousey know it was five? I don’t know why that’s five.

F: But you put it...you put two and down [motions with both his hands to show two on his left hand and right-hand portion on the bottom portion of his mat] and one and up [motions one in the middle top portion of his mat.]

T: Oh, okay good. Go ahead and make it just like that. Okay?

F: [Frank uses mousey to pick up counters. This becomes cumbersome and takes a few minutes.]

T: You may have to help mousey with your fingers and mousey can tell you if you are right or wrong because mousey doesn't have fingers like you do.

F: [Frank turns all of the counters over so that the yellow side is facing up and sorts them by clean yellow counters and dirty yellow counters.]

T: How many do you need again? How many mousey? How many do you need?

F: I need lots of yellows. [Says again in a squeaky "mouse-like" voice.]

T: How many yellows?

F: One, two, three, four, five, six, seven. [Says again in a squeaky "mouse-like" voice while pointing mousey's paw at the pile of clean yellow counters.]

T: Is that what you saw? That is not what you told me mousey. [Teacher-researcher picks up the top piece of cardstock to peek under and look at her original orientation.] I don't see seven on my mat. How many did you see again?

F: Five. [Says again in a squeaky "mouse-like" voice.]

T: Oh, okay. Let's show me five.

F: Mousey's going to do it. [Uses mousey's paw to pick up two counters one by one and sets them on his mat.]

T: No, you need to do it. We can't let mousey do all of your work for you. That's not going to help. Okay.

F: [Picks up two more counters one at a time and arranges four counters in a row.]

T: Nice, that looks good Frank.

F: I am going to get one more. [Says again in a squeaky “mouse-like” voice. Frank uses mousey’s paw to pick up one more counter and place it in the middle of his mat and just above the row of four.]

T: Okay, you landed it there?

F: Yes. [Says again in a squeaky “mouse-like” voice.]

T: Okay, let’s check mousey [Teacher-researcher lifts the top piece of cardstock so that Frank can compare his orientation to the original orientation.] Does it match?

F: Yes. [Says again in a squeaky “mouse-like” voice.]

T: Yes, it does. Good job mousey! Very good. [Teacher-researcher drops the top piece of cardstock hiding the original orientation.] Okay, so where was the two and the one you said you saw?

F: Right here. Right here. Right here. [Says again in a squeaky “mouse-like” voice.]

T: Where? Can you make the two red?

F: [Frank turns over the middle counter above the row.]

T: Is that the two?

F: No that’s one.

T: Oh, that’s the one. Okay.

F: [Turns over the remaining counters in the row so all five counters are red.]

T: Okay, which ones were the two then?

F: [Turns over the top middle counter and the two counters on the right-hand side of the row.]

Two reds and...

T: Is that two?

F: Yeah.

T: Okay, and where was the other two?

F: Right here. [Says again in a squeaky “mouse-like” voice while placing mousey in front of the two yellows at the right-hand side of his row of four counters.]

T: Right here? [Points to the two yellows at the left-hand side of his row of four counters.]

F: ...and one. [Mousey points with his nose towards the top middle counter.]

T: Okay, good. Let’s see if it matches [lifts the top piece of cardstock revealing the original orientation.] Look at that mousey!

This protocol strongly illustrates PDS activity because Frank is capable of seeing the orientation as five and also that it is composed of two, two, and one. However, I would argue that Frank is still relying upon the symmetrical aspects of the orientation to describe this as five and then two, two, and one because he only has to consider the orientation as having two and one. The symmetrical aspects of the orientation offer a free group of two to support Frank’s PDS activity, as Frank initially describes five as having two and one, not two, two, and one. Changes made by to his thinking model, with regard to reliance upon PDS activity, suggests that when the perceptual items are grouped to elicit subgroups two, two, and one, Frank now has the ability to subitize five items. The distinction being made between protocol F9 and this protocol is that Frank is not building up to the composite group of five, but decomposing five, which suggests Frank is beginning to rely on some reversibility.

Also, similar to changes Frank made between protocols F7 and F8, Frank does not need to “make” the orientation to describe the subgroups that helped him know that it was “five.” This step away from the perceptual material is an important one; it seems as if Frank relied upon some reflected abstractions here, as he was consciously considering a composite group and the subgroups that composed this composite group. Also, there seems to have been a dependence

upon more conceptual understandings of number, but for Frank to even consider the subgroups, the perceptual orientations need to exist to elicit attention towards these subgroups for Frank to subitize them. Thus, perceptual subitizing activity still roots this activity. The connections between PDS and PAS activity with regard to conceptual subitizing activity will be discussed with Amy and Diana's subitizing activity later in this chapter.

The symmetrical aspects of the orientations, with regard to PAS and PDS activity seemed to allow for more orientations. When symmetry was used to support PSS activity, the items were always in two columns, and may have had one item or a column of two items in the middle which broke the symmetry. When Frank relied on symmetry in the PAS and PDS activity, he was able to rely on both vertical and diagonal lines of symmetry. Also, the items were in rows and columns. So, it seems that as the students transitioned towards PAS and PDS activity, their reliance upon symmetry became more flexible. A reason for this shift in the utilization of symmetry is due to the increasing sophistication of the students' thinking models as they rely on more refined understandings of number. Thus, their reliance on symmetry is shifted as the students' conceptual understandings are increasing and influencing their subitizing activity.

Shape of items. Similar to the symmetrical aspects of an orientation, the shape of the items also influenced both the PAS and PDS activity early on in the students' development. This activity was one in which students were able to engage when they could visualize two orientations that were "known" as re-presenting a number. Similar to the IPS activity, students could explain they saw "these two parts" which helped them know the number. The parts were rarely described as numbers, because this activity was still dependent upon empirical types of abstractions. However, it was more sophisticated because it was supporting a student's ability to flexibly engage with subgroups and composite groups when subitizing.

In protocol F12, Frank illustrated PAS activity. Here Frank was in his 19th teaching experiment session and was pretending to be the “teacher” and I am the “student.” The witness was providing suggestions for Frank as to which card with dots he needed to show me. Prior to the teaching experiment session I chose approximately 10 to 12 cards from which Frank and I were to choose. The cards chosen for Frank were meant to perturb his present thinking models with regard to how he was attending to the subgroups and the composite group. Prior to this task, I planned how I was to respond to each card, and what type of misconception I could present to Frank. These misconceptions would be ones that either pushed Frank to explain his logic with regard to subgroups or the composite group. In this specific task, Frank chose a card with four dots in a square orientation and one dot in the middle and just above the square orientation. In the past teaching experiment sessions, Frank described this orientation as “six,” or that there was four and one which made fourteen. So my response was similar to his past responses and was meant to both understand the boundaries of Frank’s thinking model and to push Frank to consider the composite group of five as being composed of four and one.

Protocol F12.

W: What do you think? Which one would be a good one?

F: [Chooses the card on top which has four dots in a square orientation and one dot in the middle and just above the square orientation.] This one.

W: How many is that?

F: Six.

W: Okay, do you want to show her?

F: [Flips card towards the teacher-researcher to show her the card.]

T: Oh, I think that is four and one, that’s fourteen

F: No, it's six.

T: It's six? [Teacher-researcher takes a bag of counters out and begins to make the orientation shown to her with the counters.]

F: Where did you get the chips from?

T: Where did I get the chips from? I borrowed them from a classroom. They are not mine, that's why we have to keep them really nice. Okay, so this is what I saw, four [places one hand over the four counters in the square orientation.] and one [lifts hand up and points to the one counter directly above the four counters.] That doesn't make fourteen?

F: Yeah...no.

T: No?

F: That's five [states this without counting].

T: It's five, but you told me...how many did you say you saw altogether?

F: One, two, three, four, five [points to each dot on his card while stating each number word.]

W: Oh, I thought you said it was six?

T: You told me six.

F: Umm...I think there needs to be one more dot in there, and that makes that six.

So, protocol F12 illustrates Frank's thinking model was transitioning towards more PAS activity with regard to four and one to compose five, because once it was pointed out to him that this orientation has four and one, he reverted to PAS-like activity and described this orientation as "five." Initially, Frank might have been relying upon IPS activity when he described this as "six," but as soon as the teacher-researcher described this orientation as four and one, Frank began to agree that it was fourteen and then stopped himself and argued that it was five. He was not able to state logically that four and one made five, as this activity was still early in his

development, so he reverted to counting to explain why the orientation was composed of five dots. As I stated before, more often the shape of the items tended to only support IPS activity, but Frank's responses in this protocol suggest that when the teacher-researcher covered both the portions up and named them, Frank was able to break this orientation into two separate orientations which allowed him to consider a square and one more as five.

It is also important to note here that Frank carried into this activity the conceptual understanding that five is one less than six. This type of activity begins to describe some early PCS activity, which will be described in more detail in subsequent sessions regarding Frank and Amy's subitizing activity. This early notion of "+1" or "-1" is more conceptual and tends to transition students from one activity to another as students begin to carry into their subitizing activity more sophisticated understandings of number .

In protocol F13, the shape of the items was carried in to Frank's PDS activity, suggesting that Frank's reliance upon the perceptual items may be shifting towards reliance upon more abstract items. In this protocol, Frank was playing what I have described both in chapter 3 and earlier in this chapter as the Camera Game, in which I showed him a picture of the back of the camera and some dots in the viewfinder window. Frank was then asked how many dots the camera took a picture of, and if he can make the picture that came out of the camera with his counters.

This was Frank's eighth teaching experiment session and his last task for the session. In previous tasks within this session, Frank had been successful in either subitizing the composite group or the subgroups when shown five dots, but had not been able to engage in any PAS or PDS activity. Frank was shown five dots in the viewfinder window. One dot was in the top left-hand side of the camera window, two dots were in a row in the middle and on the right-hand

side of the camera window, and two more dots were in a row in the center bottom portion of the camera window. Frank engaged in PDS activity here, as he initially stated that he saw five, but when asked how he knew it was five, he described the subgroups that he saw. At first, Frank did not numerically describe these subgroups, but when asked how many in each he was able to state the numerical value of each.

Protocol F13.

T: Alright. [Teacher-researcher shows Frank the back of a camera picture with five dots in the view finder. One dot is in the top left-hand side of the camera window, two dots are in a row in the middle and on the right-hand side of the camera window, and two more dots are in a row in the center bottom portion of the camera window.]

F: Five.

T: Five? Can you make what you saw?

F: [Frank begins to take the counters out one at a time and places them on his mat.]

T: Five? How do you know five?

F: [Frank finishes placing all five counters on his mat with four counters in a square formation and one counter in the top left-hand portion of his mat.]

T: Very good. How did you know it was five?

F: Umm...This one goes here [points to the formation of four counters in a square], and this one goes here [points to the one counter at the top left of his mat.]

T: Yeah?

W: So, this one is how much? [Witness points to the counters in a square arrangement.]

T: This one [places hand on four counters in a square arrangement] is how much?

F: One.

T: All together.

F: Four

T: Okay and this one? [Points to the one dot in the top left of his mat.]

F: One.

W: How did you know that it is four?

T: How did you know this was four? [Places hand on top of four dots in square arrangement.]

F: Umm...I put together.

T: You put together...what?

F: Umm...I put together four.

T: Oh, okay.

It seems evident here that Frank is carrying in an orientation that he knows to be four as a subgroup. However, the interesting aspect to this activity, and the reason why it is described as a type of PDS activity, is that the original four counters were not arranged in a square orientation. However, both the original orientation and Frank's notion of four as a square were similar enough that Frank carried this arrangement in to his subitizing activity with this particular task. Also, even though he could state that there were two groups that helped him see this as five, he was not consciously stating that four and one are five. This seems evident when asked how many were in the square arrangement first, and he had to pause before answering.

Both of these activities suggest that he was probably relying on some reflecting abstractions, as I am not convinced here that he was consciously composing four and one to make five and then decomposing five as four and one. This was also evident, in that he needs to make the orientation and have the perceptual material in his perceptual view in order to consider the subgroups that were utilized to compose five. This was quite different from protocol F10, in

which Frank could explain the subgroups without the perceptual material. However, he may have been considering both groups and that does seem to influence his subitizing ability because the counters were re-presented in the subgroups four and one.

The shape of the items did not influence the bulk of the PAS and PDS activity, but did seem to support students transitioning toward more abstract versions of PAS and PDS activity, as it seems that these the students are relying upon empirical and reflecting types of abstraction in this activity. Students able to engage in IPS and PSS activity seem to be coordinating this activity to utilize it within PAS and PDS activity. Students needed to rely on shapes that had been utilized when subitizing in prior IPS activity, and on students understandings about number. For example, when I first showed Frank the orientation in protocol F13 in protocol F7, he could describe the two groups as four and one, but his number scheme did not include the concept of five as being composed of four and one. Until Frank had this understanding in place, the shape of the items would have continued to be expressed numerically as fourteen or six. So, the shape of items needs to extend back to the student's prior IPS and PSS activity, and forward to the student's understanding of number.

Color of items. As the students relied on more sophisticated understandings of number in their subitizing activity, they seemed to rely on color to describe subgroups and compose these subgroups to describe the composite group. Therefore, in this last section as I describe how color influenced PAS and PDS activity I will be drawing upon protocols which occurred later in the teaching experiment sessions. Throughout the teaching experiment sessions, the color red was introduced to scaffold the students' ability to attend to subgroups, as it was noticed in teaching experiment session two that students were more accurate in both their re-presentations and their numerical expressions. Additionally, red was used in combination with symmetry and

space between items to support students engaging in PSS activity. However when students engaged in PAS and PDS activity, the color of the items was enough to elicit attention towards subgroups and composite groups. Also, combining color with another topological attribute seemed to offer a rigid attention towards only subgroups, preventing students from considering flexibly both the subgroups and the composite group. Thus, the color of items will be described in this section with regard to PAS and PDS activity.

Ben's thinking model with regard to PAS and PDS activity and the perceived color of the items. At this point in the analysis, Ben's thinking model has been established as relying heavily upon empirical activity and that differing color of items elicits an attention towards subgroups. However, for Ben to begin to consider both the subgroups and the composite group, Ben needed to begin considering two as a subitized group in which to group with one or group with two when engaging in PAS and PDS activity. In protocol B13, the perceived symmetrical aspects of items allowed Ben to consider two as composing four, but Ben did not need to consider two twos at first, as the symmetry of the items gave Ben a "free" group when building towards four. Thus, at this juncture, Ben's composite group had shrunk, allowing him to use the notion of two, but with one more when subitizing a composite group of three. Again, knowing that Ben relies heavily upon the perceived color of the items, the orientations have red and black dots.

In protocol B14, Ben was in his ninth teaching experiment session and was playing the Ice Cream Scoop game. His mat had one scoop with an orientation of two dots, and the rest of the scoops had orientations of three dots. As noted earlier, color was used here, so combinations of two and one are the emphasis for today. Ben was capable of describing the orientations as two and one or as three, but at this point could not use the logic that two and one make three to

explain why he knows an orientation is three. Just before this task, Ben was engaging in PSS activity, where he was able to find two dots and one dot, but did not numerically name this orientation as “three.” This PSS activity was prompted as he rolled a die with two squares drawn. In one square there was one dot, and then in the other square there were two dots. There is an addition symbol between the two squares. Ben stated that he saw two and one; he found this in an orientation of three on his ice cream scoop mat, and justified this for me by circling the two and the one. So in protocol B14, Ben rolled his die and got the same two squares on his die, but was able to bridge his PSS activity toward PAS activity.

Protocol B14.

T: Are you ready to roll?

B: [Ben picks up the die and rolls it. Face up on Ben’s die there are two squares drawn. In one square there is one dot, and then in the other square there are two dots. There is an addition symbol between the two squares.]

T: How many did you get?

B: One, two.

T: One and two? How much is that altogether?

B: Three.

T: Three. Okay, can you find where you see one and two making three somewhere?

B: [Quickly points to the ice cream scoop at the bottom of his page. The orientation on this ice cream scoop has a row of three dots. Two dots are red and they are on each end of the row. The one black dot is in the middle of the row.]

T: Right there? Where is the two? Where do you see the two?

B: [Ben points to the two red dots with the end of his marker.]

T: The red dots are two?

B: [Ben nods his head up and down to indicate yes.]

T: Okay, and where is your one?

B: [Ben points to the one black dot in the center of the row.]

T: The black dot in the center? Okay, you've convinced me, you can color it in.

In this protocol Ben relied on the frames to see two and one as three initially, but when he matched it to the ice cream scoop orientation, Ben relied on the color of the items when describing the two and one. Interestingly, Ben could have also been relying on symmetry here, but this would mean that Ben would need to consider the group of two as one and one, and the group of one as breaking the symmetry as one and one. This numerical description did not enter Ben's PAS activity. This was the first time Ben was able to flexibly consider two and one as composing three indicating an engagement with PAS activity.

Changes in Ben's thinking model with regard to PAS and PDS activity and the perceived color of the items. So, already it seems as if Ben's thinking model is changing, as he is now capable of not only describing two subgroups, but using this to describe three dots. Interestingly, for Ben to engage in this activity, it seems as if Ben needed to first consider two when subitizing, and understanding that the remaining dot was one less than two, resulting in three. Thus, this dependence upon symmetry helped elicit attention towards two, but the color of the items supported further attention towards two and one while the student remained able to describe two and one as three. To consider this design aspect, it would be advantageous to rely on the shape of items and the color of items at this point so that Ben could subitize the composite group and then understand the subgroups with the use of color. Thus, in the subsequent protocol, Ben is engaging in an orientation of this nature. The protocol was designed to help Ben utilize

the shape of the items to subitize, but it was not originally considered in eliciting PDS activity, as I hoped that this would simply allow Ben to compose three with the subgroups shown with differing colors.

In this final protocol, B15, Ben was still in the ninth teaching experiment session when he engaged in PAS activity, as described in protocol B14, and nearing the end of playing the Ice Cream Scoop game. Ben rolled his die and saw that he got three dots arranged in a triangular orientation. The bottom two dots were the color red and the top dot was black. In protocol B2, Ben's first teaching experiment session, this orientation was described as a rocket. Comparing the two descriptions also introduces shifts in Ben's thinking model, both from one task to another, and retrospectively from the first session to the ninth session.

Protocol B15.

B: [Ben rolls his die and looks at the face of the die facing upwards.]

T: What did you get?

B: Three.

T: Three? Can you tell me... are there other ways to think about that three?

B: [Ben takes a moment to look at his mat. After looking at it, he quickly points with his marker to the ice cream scoop that is located second from the top on his mat. The orientation of three in this ice cream scoop is two red dots in a vertical column and one black dot directly to the left of the bottom red dot. The entire orientation looks like a backwards facing capital L.]

T: These two match? Why do these two match?

B: 'Cause they are three.

T: They are three in each of these. What is the same about these dots and these dots though?

B: [Ben touches the dots on his die and quickly looks at his mat.] Two and one.

T: Where do you see two and one?

B: [Points to the two red dots in a vertical column with the end of his marker. Ben then points to the black dot with the end of his marker.]

T: Can you put a circle around two and a circle around one, so I can see?

B: [Ben immediately takes the cap off of his marker and draws a circle around each of the three dots.]

T: There is your one? [Indicating the first circle that Ben draws.] So, I see three circles. Which one is your two?

B: [Ben points to the two red dots with the end of his marker.]

T: Those two. Okay. And this is your one? [Points to the one black dot.]

B: [Nods head up and down to indicate yes.]

T: Alright, I just wanted to make sure. Okay, you've convinced me. Go ahead and color that in.

Ben had difficulty at first considering the two orientations as anything other than three. However, once Ben touched the dots on the die, he could quickly see that they each have two and one. This response suggests that he knew the orientation on his ice cream scoop was two and one, but he was confirming this with the orientation on the face of his die. What is interesting also is that his circles are each around a single dot. This might indicate that Ben was still building two up when decomposing his three and with his PDS activity. Comparatively, in protocol B13, Ben did not have this difficulty when he engaged with his PAS activity. This might suggest that Ben is still struggling with the reversibility of his mental action in which to engage fully in this PDS activity, meaning that Ben is not yet capable of knowing both that two

and one are three, and three is built up with two and one. Understanding both directions in the structure of three might promote conceptual subitizing.

Color did not directly influence Frank's ability to engage in PAS or PDS activity. In fact, it seemed that the shape of items and symmetry had a greater influence on this activity than color. This may suggest that the color of the items limited Frank's ability to consider the subgroups relative to the composite group, or that this was a property that was too abstract for Frank to rely on. I would lean toward the former argument, as Frank was attending toward color when engaged in PSS activity. Thus, grouping items by color may prevent the entire group from being considered.

The color red seemed to be attended to initially when Ben used groups to compose or decompose three. Interestingly, when the color red was used with the larger cluster of two, Ben engaged more often in the PAS and PDS activity. Whereas, when red was the cluster of one, and black was the cluster of two, Ben only described the orientation as one, two, and one suggesting this slight difference in color may have influenced Ben's ability to engage in PAS or PDS versus PSS.

Synthesis of PAS and PDS Activity

PAS and PDS activity seems to lead towards conceptual subitizing activity. However, there are still limiting factors, which need to be in place before students can engage in conceptual subitizing activity. As explained earlier, students engaged in PAS and PDS activity rely more on the conceptual understanding of number to engage in a composition of number or a decomposition of number. This type of activity requires students to begin coordinating these topological influences to best coordinate their classification and serial thinking structures regarding number. Most often the subitizing activity described up to this point indicated that

students either relied on their classification thinking structures to subitize or simply counted items. However, there is one final perceptual subitizing activity found that required students to begin coordinating both their classification and serial thinking structures. This activity seems to offer a transition towards more conceptual subitizing activity, and was named as the Perceptual Counting Subitizing (PCS) activity. Because the PCS activity is a transitional activity with regard to conceptual subitizing activity, this activity will be described with Ben and Frank, but also with Amy and Diana.

Evidence of Frank's Perceptual Counting Subitizing Activity (PCS)

In these protocols students were able to engage in PCS activity when they were shown dot orientations while relying upon the symmetrical aspects of an orientation and the shape of the items. Interestingly, students either ascended in their PCS activity or descended in the PCS activity. Also, Frank once engaged in PCS activity that was very similar to PSS activity. So, when PCS activity was used in a PSS manner, the students did not necessarily use this activity initially to describe the composite group, and it elicited reliance upon the serial structure more so than the classification thinking structure. Thus, PCS activity describes a variation in different types of activity which might suggest a student's attempt to coordinate the two thinking structures in order to describe the number of items conceptually. Also, Ben did not exhibit evidence of any PCS activity unless the teacher-researcher covered portions of an orientation up with a card or a finger so that Ben might consider the subgroup and then the composite group. However, this activity was not carried into the tasks independent of this adult support. Due to this limited discussion, PCS activity will also be discussed further with regard to Amy and Diana in the subsequent section.

Visual scan of items. It was evident that Frank was engaged in this activity earlier than many of the other students, as he seemed to have a lot of experience counting, and attempted to carry his counting into his subitizing activity. In Amy and Diana's section, it is observed that Diana was engaging in this activity early on as well, but was able to use it more effectively when subitizing.

In protocol F14, Frank was in his first teaching experiment session and was shown cards with dots on them and asked "how many" and to "draw what he saw." Frank was near the end of his session and seemed to be engaged in a lot of IPS activity, as the letter "K" is used to describe what he saw, and he was struggling with describing groups of dots. The teacher-researcher showed Frank a card with five dots on it. Two dots were in a horizontal row in the center of the card, and three dots were in a row in the bottom center of the card. Frank was asked to state how many he saw and then to draw what he saw when he was shown the card.

Protocol F14.

T: You ready?

F: Uh huh [indicating yes]

T: I am just going to put this down here. [Researcher slides paper off of the table.] Here's another one that is fun. I like this one. [Researcher shows student a card with five dots on it. There are two dots in a row and just below these two dots there are three dots in a row.]

F: Forty-five.

T: What? Forty-five? Let's look at that again. [Researcher shows that the student the card with five dots on it for approximately 2 seconds.]

F: Forty-five.

T: How many did you see?

F: I didn't see... I didn't see how many the number was.

It seemed in Frank's initial teaching experiment session, as if he was trying to coordinate his serial and classification thinking structures, as he did not count on, or consider four and one to make a number, but he seemed to be doing a bit of subitizing and counting in this activity. Evidence of this is when Frank states that he saw forty-five. The only logical explanation for this is that he saw four and then five. He then combined both of these activities in a counting-like activity to describe it as forty-five instead of four and then five. In the end, he could not say how many he saw, but it seemed as if he was not using four and then five to find a composite group. However, he seemed to be relying on IPS activity for the four and then PSS activity to state the forty-five. Frank did not rely on the shape of the image, but probably instead his visual scan, to encode four dots; then preattentively he saw one more, making this orientation have a numerical value of forty-five. Regardless, it seems that the PCS activity here may be transitioning subitizing activity from IPS activity towards PSS activity.

Shape of items. PCS activity in this activity is influenced by the shape of the items in the same way it influences PAS and PDS activity, as it seems as if the students rely mainly upon empirical abstractions when initially engaging in PCS activity, before relying upon reflecting abstractions. It seems also as if there is a conceptual aspect to this activity, when considering the shape inside of the entire shape as an object to count on from. Even though this activity at the surface level appears to be a counting-on activity, in which the student is given a set of objects, and can count on from this set of objects without having to recount, it is not. The students are still engaging in an empirical or mental activity each time there is a re-visitation of the collection of items. So, as the next protocols are described, the salient aspects of the

protocols are how the students use the shape of items to ascend or descend when engaging in PCS activity.

Frank was attempting to coordinate both the serial and classification thinking structures early on in an ascending order. This seemed evident in his 13th teaching experiment session described below, in protocol F15, when Frank was shown an orientation with four counters arranged like a square and one counter in the middle and just above this square. Frank did not originally describe this as five, but mentally counted back from six toward five. In the 19th teaching experiment session, Frank counts forward to describe this orientation again as “4...5.”

Protocol F15.

T: Alright, are you ready? Tell me how many you see. Frank.

W: Frank look at Ms. Beth.

T: Look. [Teacher-researcher shows student an orientation with five dots. Four dots are in a square formation and one dot is placed just above the square and in the center top portion of the card.] How many did you see?

F: Four. [Begins making the orientation with counters. Frank makes the square formation.]

T: You saw four again?

F: Five. [Finishes making the orientation by adding one more counter above the square.] Like this.

T: Oh!!

W: Wow!

T: That is exactly right. Good job.

W: Good job Frank.

This protocol seems to indicate the strong reliance Frank had on recreating the orientation to scaffold his notion that one more would change the dots from four to five. This protocol seems to have had a heavy reliance upon these empirical actions as well because Frank needed to create a reference, which was the counters in a square orientation, before understanding that it was one away from being five. Furthermore, in protocol F16, it seemed evident that Frank carried the result from this experience forward in a similar task.

In protocol F16, Frank was relying on the shape of items to engage in ascending PCS activity. This was Frank's 19th teaching experiment session and Frank was pretending to be the "teacher" and I was pretending to be the "student." This protocol follows protocol F12, when Frank initially engaged in PAS activity due to the shape of the items. This time, Frank chose a card with four dots on it. Two dots were black and in a vertical column, and two dots were red, in a symmetrically oriented vertical column. Just before showing me the card, Frank was able to say there were four dots on the card. As the session progressed, Frank struggled with using subgroups to explain why he knew there were four dots on the card. In my attempt to refine his thinking, Frank began to engage in ascending PCS activity.

Protocol F16.

W: Before you show her you need to tell me what you think that is.

F: [Flips card around to show the teacher-researcher. The teacher-researcher is reading a Dinosaur book and appears to not be paying attention.]

W: [Places hand down in front of the card to block the teacher-researchers view of the dots on the card.]

F: Four.

W: Okay. [Removes hand.]

T: Okay.

F: That's a silly dinosaur [referring to the book the teacher-researcher was reading.]

T: [Laughs.] It is a silly dinosaur. I was reading the book too. I think that is two and two...that's 22.

F: No! It's four.

T: [Places four counters on her mat. Two red counters are in a horizontal row and two yellow counters are in a horizontal row just below the red counters.] But look, I saw two and two. Is that 22?

F: No, you have to put some red and blacks.

T: I don't have black. I will match it to this [indicating Frank's card] though. [Teacher-researcher picks up counters and places them in two vertical columns beside each other. This forms a square just like before.] So why is that four?

F: [Places his card face down over the teacher-researcher's counters. This covers all of the counters.] There is four under here.

T: How can I remember that as four?

F: [Lifts up the front edge of the card, revealing to Frank two counters.] There's two in there.

T: Because there's two in there? So, [picks up one yellow counter, leaving three counters on her mat altogether] is this four?

F: [Looking at his card.] Yeah.

T: This [points to the counters on the mat] is four?

F: Hmm...mm [indicating agreement.] No, three!

T: But there's two in there [points to the two red counters.]

F: There's tthhrrrrreeeeee.

T: But there's two in there. You told me that all I needed to know to know four was to know two. There's two in there.

F: There's three. [Ben covers up one dot with his finger showing the same three dots on his card that the teacher-researcher has re-presented on her mat.]

T: But I see two also [points to the two red counters on her mat.]

F: There's no two in there. There's one!

T: Oh, [points to the yellow counter to indicate understanding] but I see two here [points to the two red counters.]

F: There's three.

T: There's three. What if I did this? [Teacher-researcher places the yellow counter back in place, but also adds a third yellow counter to the left-hand side of the square.] Now there's two in there.

F: There's four. That makes five.

T: Four and what makes five? [Teacher-researcher pick up the extra yellow counter and then replaces it to scaffold the discussion.]

F: This one. [Frank points to the square arrangement within the orientation.]

T: Yeah, so this makes four and this [points to the extra yellow counter]?

F: Five.

T: Six.

F: Wait...no!

T: No [laughs]?

In protocol F16, Frank did not rely upon his recreation of the items to consider this orientation as five, but in a sense had to view the arrangement first, both as counters arranged as

a square and then in the orientation of a square with one dot above it, before engaging in PCS ascending activity. So, the teacher-researcher's actions seemed to stand in for Frank's actions that he needed to engage in PCS activity in protocol F16. Thus, it seemed as if Frank was transitioning in his ability to coordinate both his notion of subgroups and his counting scheme, as he seemed to be relying on more abstract actions to support this PCS activity.

The shape of the items seems to be built up by students through empirical actions. Students may rely solely upon empirical actions like a visual scan or using counters to re-present their visual scan. As different shapes can be used to stand in for these empirical actions, students began to use these objects to re-present composite groups and the subgroups to ascend towards the construction of composite groups. Thus, it seems as these activities became more and more aligned with conceptual understandings, students needed to re-visit their empirical actions to accommodate for new subitizing activity.

Symmetrical aspects of items. Changes have been noted in how student-perceived symmetry has been described in earlier portions of the analysis of these data. For instance, in prior protocols that describe symmetry in the PSS activity, compared to the protocols that describe symmetry in the PAS and PDS activity, student-perceived symmetry changed to allow for a dependence upon more orientations and lines of symmetry. However, when students engaged in PCS activity, students revisited some of the IPS activity in order to use shape as an object from which to either compose or decompose. Thus, perceived student symmetry is utilized in an IPS like manner, and how students perceived the symmetrical aspects of the items changed in this PCS activity so that is perceived in the same way it was in the IPS activity.

When symmetry influenced PCS activity, students typically engaged in descending PCS activity, where the student described a larger number first and then the sequential backwards

counting number next. This seemed to occur when shown two vertical columns of three. For example, in protocol F17, Frank was in his 20th teaching experiment session and was shown cards with dots on them and asked how many he saw. He then used the counters to make what he saw. This was Frank's second task in the session. In the task immediately before this one, Frank was shown two columns of three dots, with a total of six dots. Frank accurately told the teacher-researcher how many he saw, and he was able to also make a similar orientation. His orientation differed by color in that he placed four yellows out on the top and bottom portion of his columns, while the two middle counters are red. He stated that he knew it was six because he saw four yellows and two reds. So in this protocol, Frank was shown two vertical columns of dots. One column had three dots, and the other column had two dots in it.

Protocol F17.

T: Alright. [Teacher-researcher shows Frank a card with two vertical columns. The column on Frank's left-hand side has three dots in it. The column directly across from the first column and on the right-hand side has two dots in it. The total number of dots is five.]

W: Next one.

F: Six.

T: Six? Can you make...

F: Five.

T: Five? Can you make what you saw?

F: [Frank picks up six counters one at a time and places them on his mat in two vertical columns. Four counters are yellow and two counters are red. The column on Frank's left-hand portion of his mat has four in them. The column on Frank's right-hand portion of his mat has two in them.]
Like that.

T: Like that? So, how do you know that this is...uh... five?

F: There are five yellows and two reds.

T: Oh, I see...uh...I don't know about five yellows. Check again. How many yellows?

F: One, two, three, four [points to each yellow counter while stating each number word.]

T: Four yellows and two reds.

T: But, you told me last time that was six. [Teacher-researcher picks up yellow counter on the end of his longer column and places it on the end of the shorter column, making both columns equal in length and quantity.]

F: Yeah.

T: But is that five or six then?

F: Six... [slides counter back to the left-hand column] back to the five.

T: You saw...now is that five or six? That's five now?

W: Why? I still see six.

T: Why is that...why is that five?

F: I think it is...I can take out one [slides the extra yellow counter on the end of the longer column off of the mat entirely.]

W: Oh...

T: Is that six now?

F: No, that's five.

T: How do you know it is five?

F: Umm...I don't know.

This activity is interesting because it so closely resembles IPS activity, as Frank continues to re-present other topological influences, not just symmetry. For instance, it seemed

evident that Frank was relying upon the length that the counters took up for the left-hand side column, but was able to attend to the shorter column on the right-hand side. However, the length seemed to be considered for one row, though Frank had no difficulty coordinating the two counters in the shorter row. It seems as if the PCS activity is being re-presented as two and “some,” as Frank’s number scheme cannot coordinate five as being one less than six, as well as a group of two and a group of three. So, Frank seemed to hold on to this row being described as “some” and then adding the two on. Also, Frank began to consider the location of the counter as influencing the quantity, but when asked to reflect on his actions, Frank realized that he could not simply slide the counter to change the quantity and he took that counter off the mat to make five. Thus, this activity seems to marry the IPS activity with the PAS and PDS activity. Frank’s reversibility in actions seemed to be transitioning from physical actions towards some mental actions, indicating a shift towards more sophisticated subitizing activity.

Symmetry does not have a large influence on this activity, but it did seem that at times students who understood the orientation of six as being two columns with three in each column, or that there were two columns that were the same length, could then comparatively consider five when the columns two and three are lined up. Interestingly, Amy, who will be discussed in the next section, also used this orientation of six as a physical object in which to consider 5 and 7. Symmetry also seemed to influence this activity. Therefore, symmetry is described here as having two columns of dots with an equal length. Disrupting the equal length of each column can elicit a student’s PCS activity, which seems to scaffold coordination of the classification and serial thinking structures.

Synthesis of PCS Activity

PCS activity seems to offer students an opportunity to revisit IPS types of activity and PAS or PDS types of activity, to coordinate serial and classification thinking structures when subitizing. It seems to differ slightly from PAS and PDS activity, as two subgroups are not considered when composing or decomposing the composite group. PCS activity seems to consider a “+1” or a “-1” types of understanding with regard to number, whereas PAS and PDS activity simply is a logic on which students rely when subitizing, and explaining how they know a number which they have subitized. Therefore PCS activity seems to support students transitioning from a reliance on one activity toward reliance on different types of activity.

Transformations in thinking structures which allowed Amy and Diana to engage in conceptual subitizing will be the focus of the next section. In this section PCS activity will be discussed in more depth, as the activity will look different when students have begun coordinating serial and thinking structures when subitizing.

Amy and Diana

Amy and Diana were two students that I worked with individually totaling 22 teaching experiment sessions for Amy and 16 teaching experiment sessions for Diana. Both students will be discussed in this same section, as they each were considered conservers, and demonstrated some similar subitizing activity. However, this section will primarily focus in on Amy’s responses within the teaching experiment. The purpose of this section is to attend to transformations in Amy’s thinking model, which supported her ability to engage first in Rigid Conceptual Subitizing (RCS) and then Flexible Conceptual Subitizing (FCS) activity. Essentially, both RCS and FCS activity are not primarily dependent upon perceptual material, but on conceptual understandings of number. Sarama and Clements (2009) describe this activity

simply as Conceptual Subitizing. However, findings from this study suggest that there are two different types of Conceptual Subitizing, which is described in this study as RCS and FCS activity.

In this section, examples of Amy's and Diana's RCS and FCS activity will be used to describe what this activity relies on and how transformations from one activity to another occur. However, this section will begin with Amy's initial thinking models, with Diana's activity as a comparison. Next, protocols will be used to illustrate the transformations in Amy's thinking models. The reason for the consideration of these transformations in this section is threefold; first, these protocols will further illustrate the different forms of perceptual subitizing activity described and defined in the previous section; second, these protocols will describe in depth both RCS and FCS activity; and third, a focus on these transformations will outline relationships between activity in order to refine the proposed hypothetical learning trajectory described in chapter 3.

Amy's Initial Thinking Model

In Amy's first teaching experiment session, she was capable of subitizing up to five dots in both patterned and unpatterned orientations. When asked how she knew the number, she simply stated that her "mind" told her or that she knew because she was "good in math." When drawing what she remembered, Amy always drew circles in a row, and two times was inaccurate in her re-presentation. Essentially, Amy's actions and responses suggested that Amy was primarily relying on types of reflecting abstractions, as she may have engaged in some PSS activity, but she was not cognizant of any subgroups when she re-presented what she remembered. However, Amy's drawings may simply have been inaccurate because she was shifting her attention to her drawings and away from what she encoded when she subitized.

So, in the second teaching experiment session, Amy was shown cards with dots on them, but asked to use cubes to re-present what she saw. Also, some subgroups of dots were red or the size of some of the subgroups of dots was relatively larger. The size and the color of the dots seemed to influence Amy's subitizing activity, allowing her to consider subgroups and accurately re-present the number of dots and numerically express how many dots she was able to see. Without the color and size of dots introduced, Amy was inaccurately stating that five dots were 10 dots. This activity seemed to suggest reliance upon symmetry when bridging preattentive encoding towards Amy's understanding of number. So, in protocol A3, Amy was shown a diagonal row of five dots, and stated that she saw "ten." However, when she re-presented what she saw with the cubes, she created two rows of five, suggesting she encoded five dots and doubled them, or that her visual scan of up and down had twice the number of eye movements as compared to a horizontal or vertical row of dots.

Protocol A3.

T: Okay, you ready? [Teacher-researcher shows Amy a card with five dots on it in a diagonal row.]

A: Ten.

T: Ten? Can you show me what you saw?

A: [Amy takes the 10 cubes off of the table and places them one at a time on her mat. She creates a row of five and then below that another row of five.]

T: Wow. Awesome. Did you see it in two lines like that?

A: [Nods head up and down to indicate yes.]

T: You did? So, when you broke it up, was there an equal number on either side?

A: [Nods head up and down to indicate yes.]

T: That's awesome.

So, it is not clear whether symmetry influenced this activity or if the orientation offered more visual scans than Amy was used to experiencing. It simply could have been too many counters for Amy to consider and 10 was a number she might just have known and considered when there are a lot of counters. However, one thing to note here is this is the first time Amy had re-presented anything in more than one row. In both the initial interviews and the first teaching experiment session, Amy only re-presented her subitizing activity in single rows. However, she was drawing those re-presentations and in this session she is using counters. In the next protocol, Amy is shown this same orientation, but four of the dots, two on each end of the row are relatively larger than the center dot. This card was shown to Amy in hopes that she may consider subgroups and be capable of subitizing this set of dots accurately. At first, Amy stated that she saw 10 again, and placed a row of six and a row of four counters on her mat. However, as you will notice in protocol A4, Amy was asked how many big dots; she stated four and placed the four counters down on her mat, adding one more for the smaller dot.

Protocol A4.

T: Now, I am going to show you what I showed you before. This time I want you just to tell me how many big dots you see. Did you see there were some big dots in there? They were weird weren't they? Okay, so how many big dots do you see? [Teacher-researcher shows Amy a card with five dots in a diagonal row. Four dots are relatively larger than one dot. The smaller dot is in the center and the four larger dots are broken into two groups where two dots are on either end of the row.]

A: Four.

T: Oh four? Okay, I am going to erase your board [slides counters off of the paper serving as Amy's working mat] and can you show me the big dots that you saw?

A: [Amy places four counters on her mat, one at a time.]

T: Alright so...do you remember how many little dots there were?

A: [Amy nods her head up and down to indicate yes.] One.

T: One. Will you put something out there to make that one?

A: [Amy places one more counter on her mat.]

T: So, how many are there altogether now?

A: [Looks at her mat, but quickly states her response.] Five.

So, when the dots differed in size and Amy was told to attend to those differences she was able to consider the subgroups four and one, and then she could reconsider the entire orientation as five. This seems to suggest that the size of the dots supported Amy's ability to engage in PSS activity. This was evident in a subsequent task, as once some time had passed, I showed Amy that same orientation of dots, and at that point she stated that she saw four dots. This suggests that Amy was still unable to understand four and one as five, but could encode one of the subgroups when subitizing.

As interesting as this was, it still did not answer the question of whether symmetry was influencing Amy's thinking model with regard to subitizing five. So in Amy's third teaching experiment session described in protocol A5, Amy was playing the Ice Cream game, which was described in more detail in chapter 3 and in Ben and Frank's section. It was suspected that if Amy was relying on symmetry, she would carry into her subitizing of four the subgroups two and two, and if she was going to subitize five, she might have to carry into her subitizing activity the subgroups two, two, and one. So, one of the faces that Amy is shown on her die has a three

square on it. One square had two dots inside of it, another square had two dots inside of it, and the third square had one dot inside of it. Interestingly, Amy was able to subitize these three sets of dots by engaging in some PAS activity, and carried these subgroups into almost every subitizing activity with regard to five after this session. At this point in the session, Amy was looking at her die and describing the orientations and groups that she saw, and then she matched this to her ice cream scoop mat.

Protocol A5.

T: So how many do you see here? [Teacher-researcher points to a face on the die where there are five dots in a diagonal row, very similar to the second teaching experiment.]

A: [Amy points at each dot with her pinky finger while whispering number words.] Five.

T: Five. How many do you see...where do you see that there [indicating the ice cream scoop mat]?

A: [Amy points to the second ice cream scoop where a similar orientation is shown.]

T: Right there? Okay. Good. Do you see that anywhere else?

A: [Amy scans her game board.] No.

T: No. Okay, how about here? What's this one? [Teacher-researcher points to a face on the die where there are three squares on it. One square has two dots inside of it, another square has two dots inside of it, and the third square has one dot inside of it.]

A: Do you know that two plus two and one makes five? [As Amy states this she holds up her fingers to re-present what she is describing. She begins with two fingers on her left-hand and then two fingers on her right-hand, and then adds one more finger to her right-hand.]

T: Really?

A: [Amy smiles and nods her head to indicate yes.]

T: Wow. Do you see that here [indicating the face of the die pointing towards Amy]?

A: [Nods head up and down to indicate yes.]

T: Tell me where...where you see two plus two plus one.

A: [Amy places two fingers on one group of two dots, two fingers on a second group of two dots, and then one finger on a group of one dot.]

T: Two plus two...plus one! Yea! Do you see two plus two plus one somewhere here [points toward the game board]?

A: [Points to the ice cream scoop at the top of her game board where there is an orientation of five dots. Four dots are in a square formation and one dot is directly above this orientation.]

Two [points to two dots in a vertical column in the square part of the orientation] plus two [points to the remaining two dots in a vertical column in the square part of the orientation] plus one [points to the dot directly above the square formation].

T: Very good. That's nice! Okay, you ready? What's this one [turns the die so that Amy is looking at two squares with four dots in one square and one dot in another square. The four dots are in a square formation.]

A: [Amy points at each dot with her pinky finger while whispering number words.] Five.

T: Yes, but how do you see five here?

A: Because it's really hard and I think.

T: Right, you have to think. This is hard. But how many do you see here? [Points to the square with four dots in it.]

A: Four.

T: What do you see here? [Points to the square with one dot in it.]

A: One.

So Amy seemed to be engaging in PAS activity when shown the die face with two, two, and one, but reverted to counting when shown four and one, or a diagonal row of five. This offered some strong claims regarding Amy's thinking model. For instance, when Amy considered the four and one, she may have been only be able to compose the four as two and two. Therefore, for her to rely on subitizing activity to explain four and one means five, she needed to engage in PDS activity to decompose four as two and two, and then PAS activity to compose two and two and one to understand it as five. This was too much for Amy to coordinate so she reverted to counting the dots or subitizing it as four and one. Another piece of evidence, which supports this claim, is when Amy was in the second teaching episode (protocol A4) and needed to subitize the four large dots and the smaller dot separately.

In protocol A6, Amy was still in her third teaching experiment session and playing the ice cream scoop game. At this point, Amy was rolling the die and trying to match orientations on her die to the orientations in the ice cream scoops on her mat. When she rolled her die the orientation with two, two, and one was facing up. Amy was quickly able to subitize this as five, but struggled in finding an orientation to match these subgroups.

Protocol A6.

T: Okay, go ahead and roll your die. Let's see if we can get a match first. [Teacher-researcher rolls die and finds seven dots are face up.]

A: [Amy rolls her die and finds two, and two, and one are face up.]

T: Oh, what did you get?

A: Five.

T: Okay, how did you know it was five? What does it say on the die?

A: Because it's two and two and one.

T: Okay, where do you see two, two, and one?

A: Two and two [Amy points to an ice cream scoop with two dots in a row at the top of the frame and three dots in a triangular arrangement near the bottom of the frame. Amy first tries to consider one dot while stating each word ‘two’. Upon realizing that she’s made a mistake she switches to a whole new orientation.] Two and one [Amy is now pointing to the orientation of five that is an “X” arrangement and points to two dots in a column and one dot in the center of the arrangement. There are two dots remaining in the orientation that Amy has not considered.]

T: There is two and one, and we need another two.

A: Two [student points to two remaining dots that had not been intended to previously.]

This PDS activity was common for students relying on symmetry early on with the doubles plus one. As noted in the previous section regarding Ben and Frank in protocols B13 and F9, Frank and Ben did not state a symmetrical subgroup, as if stating one subgroup symmetrical to another offered a “free” subgroup that did not need to be coordinated. Amy did this one more time in the same teaching experiment session, as noted below in protocol A7. In this protocol, Amy had rolled her die and again gotten the face with two dots, two dots, and one dot. She was now trying to match this to the diagonal row of five dots that she had struggled with for the past two teaching experiment sessions.

Protocol A7.

T: Can you match that to something over here? [Teacher-researcher points to the ice cream mat, indicating that Amy needs to match the two and two and one to an orientation on the ice cream mat.]

A: [Amy quickly points to the diagonal row of five dots.]

W: Could you circle the two, two, one?

T: Yes, if you can find the two, two, one, circle the two and circle the two and circle the one for me.

A: [In the diagonal row of five dots, the student circles two dots at one end of the diagonal row and one dot in the middle of the diagonal row. Amy then looks up to indicate she is finished, but has left two dots in the diagonal row without a circle.]

T: There is one, two and one, one. Where is the other two?

A: [Amy circles the remaining two dots at the other end of the diagonal row.]

Amy's thinking model with regard to subitizing five items. This protocol again indicates Amy's reliance upon the symmetrical aspects of the orientation, which I believe allows her to only have two units to consider when subitizing the subgroups. It should be noted that each time Amy only considered the orientations in which symmetry could be utilized when considering the two, two, and one. In this protocol, she was not given the opportunity to name the composite group, so it is not certain if Amy was engaged in PSS, PAS, or PDS activity, but it is certain that Amy was depending upon symmetry to utilize the subgroups two, two, and one when subitizing. However, earlier, Amy was capable of engaging in PAS activity with regard to the subgroups two, two, and one "making five." So, at this point, in order for Amy to subitize orientations which elicit the subgroups four and one, she needed to decompose four before composing two, two, and one which required Amy to engage in bidirectional thinking. Essentially she needed to mentally "unpack" four to understand four as two and two and then "repack" two and two and one which she was not capable of coordinating.

A Comparative Analysis of Amy and Diana's Thinking Models

Diana also relied on the subgroups two, two, and one when engaged in PDS activity. In Diana's fifth teaching experiment session, Diana was shown a diagonal row of five counters. In

previous sessions Diana has also had difficulty subitizing this diagonal row of five dots or counters. So, Diana was originally shown five counters with a large space between the fourth and fifth counter. This task was the first task in this session, and after she was shown five counters on a mat she was asked, “how many she saw,” and “how she knew that she saw five.”

Protocol D3.

T: Okay, are you ready? [Teacher-researcher lifts the top piece of cardstock to reveal five counters in a diagonal line moving from the bottom left-hand portion of the teacher-researcher’s mat to the top right-hand portion of the teacher-researcher’s mat. There is a large space between the fourth and fifth counter.] How many did you see?

D: Five.

T: Five? Can you show me what it looked like on your mat, so that your mat matches mine?

D: [Diana picks up the five counters one at a time and places them on her mat in a very similar arrangement as the teacher-researcher’s mat.]

T: Alright, you ready? Let’s check. [Teacher-researcher lifts the top piece of cardstock up to reveal the original orientation.] Did it match?

D: [Diana nods her head up and down to indicate yes.]

T: How did you know there was five here?

D: Because I saw four here [points to the row of four], and number five right here [points to the fifth counter in the corner of the mat.]

T: Oh, and one more was like number five for you?

D: [Nods her head up and down to indicate yes.]

This protocol is a wonderful example of a different type of PCS activity. Here, it is not even evident she engaged in PCS activity until she was asked how she “knew she saw five”.

Compared to Ben and Frank's PCS activity, this seemed to show more sophistication, as Diana was conserving number already and she seemed to be transitioning from PSS or PAS activity towards PDS activity with regard to this orientation. If Diana had stated four...five before she said five, then she might have still been participating in the PCS activity and relying on this activity in less sophisticated manner. So, it seems that a large space between a row of four and one dot allowed Diana to engage in a type of PCS activity when subitizing five dots. Questions still surrounded what Diana knew about five and if she understood five as four and one, or if she and Amy had been operating in a similar fashion, a regarding five as two, two, and one when subitizing.

This activity is also important to consider relative to the next protocol, D4. In protocol D4 Diana was still in her fifth teaching experiment session, in which she is shown orientations as described in protocol D3, and two other orientations with six and eight counters. So, this is the fourth orientation shown to Diana and it was very similar to the first orientation (from protocol D3), as there were five counters in a diagonal line, with only a slightly larger space between the fourth and fifth counter. Diana needed to be shown this orientation twice, but then numerically described it as five and then two, two, and one.

Protocol D4.

T: Alright. [Teacher-researcher lifts the top piece of cardstock revealing five counters in a diagonal row. There is a slightly larger space between the fourth and fifth counter.] How many did you see?

D: [pauses.]

T: What do you think? [Pauses] Do you want to look at it a little longer?

D: [Nods head up and down to indicate yes.]

T: Yeah? Okay. [Teacher-researcher lifts the top piece of cardstock to reveal the original orientation.]

D: Five.

T: You saw five? Can you make what you saw?

D: [Diana places five counters on her mat one at a time until she creates a diagonal row composed of five counters.]

T: Okay, how do you know that there is five here?

D: Two, two, and one more makes five.

In this protocol, Diana grappled with subitizing this orientation, further suggesting the importance of the space between the items when engaging in PCS activity. However, upon reflection Diana did not state that she knew it to be five because she knew four and one make five, but because two, two, and one make five. These subgroups had not been introduced to Diana as they had been for Amy; Diana was given a different die and ice cream scoop mat in her third teaching experiment session. This response from Diana seemed to suggest that symmetry perceptually offered students an opportunity to engage in some PAS and PDS activity before they could conceptually subitize.

So, this notion that symmetry may support students engaging in PAS and PDS activity pushed me to wonder if this activity could transition students toward conceptual activity, where the topological aspects of the orientations did not elicit the students' subitizing activity. An indication that students were making this transition would be that students' attention towards subgroups would be primarily dependent upon their conceptual understanding of number, not the topological aspects of the orientation.

In protocol A8, Amy was in her fifth teaching experiment session and was being shown counters on a white piece of cardstock and asked “how many she saw,” and if she could “make what she saw.” This protocol was near the end of the session and Amy had struggled with orientations in which there were five counters in a diagonal row and a large space between the fourth and fifth counter (similar to Diana’s orientation in protocol D3). Amy stated that she saw “four,” and placed seven counters on her mat, suggesting Amy’s ability to only build up toward four with two and two, but then she relied on some empirical abstractions when re-presenting what she saw. In approximately the middle of the session, Amy was shown this orientation again, but with a slightly smaller space between the fourth and fifth counter. This time, Amy stated five, but then re-presented four with a large space between the third and fourth counter. So, it seemed as if she was having difficulty managing what this orientation should look like with regard to the number of counters she was subitizing. In the final two tasks, Amy was shown five counters each time. The first orientation was a diagonal row of five counters. Space between each counter was equal. The second orientation was a square orientation of four counters with one counter off to the top right-hand side of the mat. This orientation was one she had had to count in protocol A5.

Protocol A8.

T: Alright, you ready? [Teacher-researcher lifts the top piece of cardstock revealing five counters in a diagonal row.] How many did you see?

A: Five.

T: Can you show me...how do you know there was five there?

A: ‘Cause I saw two [uses two of her fingers to point on her mat], two [uses two of her fingers to point on her mat], and one [uses one finger to point on her mat].

T: Okay, go ahead and show me what you saw there.

A: [Amy places two counters on her mat, and then two more counters on her mat, and then one counter on her mat. The entire orientation is a diagonal row similar to what she was shown.]

T: We're going to do just one more.

A: I'm tired.

T: Okay, well we are going to do just one more. [Looks at Amy's mat.] That looks just like my mat. Excellent. You ready?

A: [Nods head up and down to indicate yes.]

T: Clear your mat off.

A: [Amy slides her counters off of her mat.]

T: Okay. [Teacher-researcher lifts up the top piece of cardstock revealing five counters. Four counters are in a square formation on the left-hand portion of the teacher-researcher's mat, and one counter is placed at the top right-hand portion of the teacher-researcher's mat.] How many did you see there?

A: Five, five, and five.

T: Yeah? How did you know there was five that time?

A: Umm...because there's two here [uses two fingers to point to her mat], two here [uses two fingers to point to her mat], and one there [uses one finger to point to her mat].

T: Oh. Excellent. Interesting.

So, when Amy re-presented what she knew, she did not even need the counters. She could use her fingers and a blank mat to stand in for what she saw. This activity seems to suggest a transition towards what I have described in this study as Rigid Conceptual Subitizing (RCS) activity. RCS activity is a type of conceptual subitizing in which students carry in their

conceptual understandings of number, but are limited in what they know a number to be composed or decomposed as. It is evident that Amy was not relying on the perceptual material when she subitized the diagonal row, as there was not a larger or smaller space between the items, and relying on symmetry would mean that she would explain seeing two counters on each end and one counter in the middle of the diagonal row. However, Amy was not considering any other groups at this point when subitizing, so the groups are rigid and limit Amy's number scheme with regard to what she knows about five. These actions all indicate a transition toward RCS activity. The previous tasks with the five counters in a diagonal row may have supported Amy's subitizing ability, so at this point I am hesitant to say that Amy was relying on RCS activity, but Amy was definitely transitioning away from a primary dependence upon the perceptual material.

Changes within Amy's Thinking Model

At this point it is known that Amy can engage in PAS and PDS activity with regard to subitizing, but can only engage in this activity with patterned arrangements of five items and that she carries into each of these tasks the notion of two, two, and one. It is believed that Amy may be transitioning towards RCS activity, but that is not clear at this point. Therefore, I wanted to go back to three and four to build some PAS and PDS activity with regard to four. I believed that if Amy could engage in some PAS or PDS activity with three and one as subgroups, Amy might be able to build towards using three as a mental object when subitizing five, and then eventually four as a mental object when subitizing five.

In protocol A9, Amy was in her eighth teaching experiment session, and was playing the Camera game described in more detail in chapter 3, and in the earlier section regarding Ben and Frank's subitizing activity. Amy was being shown dots in the camera window totaling three,

four, and five. Prior to protocol A9, Amy was shown orientations each with three counters at different times and each time she could accurately state that she saw “three,” but when asked how she knows it is “three,” she stated that she saw “one, one, and one.”

This was evident in the next three tasks, when Amy was shown four dots in the camera window. The first task had three dots in a horizontal row and one dot just above the dot on the left side of the row. Amy stated that she saw “three” and made an “L” like arrangement with the three counters. Amy looked at the camera window again, and realized she needed one more counter. Amy was then shown a pushed together version of the same orientation and stated that she saw “three” again and made the same “L” arrangement with three counters. After looking back at this arrangement, she again changed it to match the camera image. This seems to suggest that Amy was engaging in an ineffective PDS activity where the subgroups are all “ones.” This would be ineffective and limiting, just as always using two, two, and one is limiting. As described earlier in chapter 2 with regard to spatial indexing, visual scans can usually only capture so many perceptual items, and when students only consider these small groups, this prevents them from subitizing a larger number of items.

Thus, Amy’s inability to subitize four dots accurately at this point makes sense, as she is not carrying in larger groups of items, like two or three. Also, the perceptual arrangement of the dots is similar to what three dots sometimes looks like. For instance, the “L” arrangement could be construed as three dots because of the “ends” and the “corner,” which support a visual scan’s starting, turning, and stopping point. Amy is shown four dots, with two dots in a row at the top middle portion of the camera window, and one dot in each of the bottom corners of the camera window. This orientation suggests a triangular arrangement that is usually formed with three dots, but also engages three points where Amy might be starting, turning, and stopping her

visual scan. So, even though Amy was able to describe these arrangements as one, one, and one, it might also be that Amy was relying on IPS type of activity in which the visual scan still had a dominant influence on Amy's subitizing ability, with regard to these arrangements. Regardless, it was evident at this point that Amy was not using three to subitize arrangements with four dots. In protocol A9, Amy was shown five dots altogether. Four dots were in a square formation and one dot was off to the right-hand side of the camera window.

Protocol A9.

T: [Teacher-researcher shows Amy a picture of the back of a camera. In the camera window there are five dots. Four dots are in a square formation, and one dot is off to the right-hand side of the camera window.]

A: Five.

T: You saw five? How do you know there is five there?

A: Because two plus two plus two plus one equals five.

T: Can you show me? You said two plus two plus two plus one?

A: Two plus two and one equals five. [Amy places four counters in a square in the extreme top left-hand corner of her mat and then places one counter in the far right-hand bottom corner of her mat.]

T: Oh, okay. Wow. Where did you see the two plus two plus one?

A: [Begins sliding the counters off of her mat.]

W: Oh, don't clean it.

T: Where do you see... [places counters back on Amy's mat in the same arrangement] you made it like this, right? Where do you see your two plus two plus one?

A: Two, two [points to the top two counters in the square formation] two, two [points to the bottom two counters in the square formation] one [points to the one counter in the bottom right-hand corner of her mat.]

T: Two, two [points to the two top counters in the square formation]? Or two [uses two fingers to point to the top two squares in the square formation]?

A: [Amy nods her head up and down to indicate agreement.]

T: Which one?

A: These are two [points to the two counters grouped earlier in the square formation] and this is the one [points again to one counter in the corner of the mat].

T: Oh, okay, I just wanted to be sure. Okay, clear your board.

A: [Amy slides her counters off of her mat.]

T: Ready? [Teacher-researcher shows Amy five dots in the back of the camera window. Three dots are in a triangular arrangement in the bottom left-hand corner of the camera window. Two dots are in a diagonal row in the top right-hand corner of the camera window.] How many did you see?

A: Five.

T: How did you know five?

A: [giggles.] I just know it.

T: You just knew it? Go ahead and make what you saw.

A: [Amy picks up five counters one at a time and places them on her mat. She arranges them very much like she did in the previous task where there are four counters in a square formation and one counter in the far top right-hand corner of her mat.]

So, it seems that when Amy was subitizing five that she was not cognizant of a strategy or she was relying on the shape of the items. However, it is clear that she is not yet able to use three when subitizing and even reverts back to the “four and one,” which is in her reality, the “two, two, and one” orientation that she had relied on in the previous task when re-presenting what she saw. She eventually looked again at the camera window and decides she needed to rearrange her counters to match the dot arrangement, but then described this new arrangement as “five” because it is “two, one, and two.”

Comparatively, Diana’s subitizing activity was changing to allow for six and seven items to be subitized accurately. She seemed to be engaging in PDS and FCS activity which suggested that her conceptual number understandings were supporting this transition. FCS activity has not been described in full detail. Essentially, when students engage in FCS activity they have flexible understandings for number and can use this to subitize, regardless of orientation. For example, in protocol D6, Diana is in her eighth teaching experiment session and is also playing the “Camera Game.” Seven dots are shown to her. There were two vertical columns with three dots in each and one dot in the center of this arrangement. She stated that it was seven because “three and three make six and one more makes seven.” Next, she was shown seven dots in a diagonal column. She again stated that it was seven. When asked how she knew that she had seven on her mat, she said that she saw four and three more, and that would make seven.

Protocol D6.

T: I am going to show you some tough ones now, okay? [Teacher-researcher shows Diana the back of a camera with seven dots in the camera viewfinder. There are two vertical columns of

dots with three dots in each column and one dot between both of those columns.] How many did you see there?

D: Seven.

T: Seven? Okay, do you want to make what you saw?

D: [Diana takes one counter off of the table at a time and places seven counters on her mat in an arrangement very similar to the camera arrangement.]

T: Wow. I think that looks just like my picture. [Teacher-researcher shows Diana camera picture so she can check.] Is it exactly like it?

D: [Diana looks up at the camera picture and back at her arrangement. She slides some of the counters apart to be sure there is the same proportional space between her items as is on the camera picture.]

T: Yeah? I think you just had that space. Very good. How did you know you had seven there?

D: [slides counters together into a pile.] Because three and three make six and one more makes seven.

T: Oh, okay. Nice. Go ahead and clear your board.

D: [slides the pile of counters off of her mat.]

T: Alright. [Teacher-researcher shows Diana a diagonal row of seven dots.] How many did you see?

D: Seven again.

T: You saw seven again? Can you make a picture of what my camera would look like?

D: [Diana places seven counters in a diagonal row one at a time.]

T: So, how do you know you have seven there?

D: [Shrugs her shoulders to indicate that she was unsure.]

T: Did you just count them when you put them on your board?

D: No, I just put four on and then I put three more, and I knew that was seven.

T: Oh, okay. Very good.

This protocol is evidence of the existence of Diana's flexible understandings of seven at this point. She was able to rely on perceptual material or rely primarily on her conceptual understanding to explain or justify why she had seven counters on her board. It is still not certain here how much of this relates to her subitizing activity because she had always re-presented the orientation, then told me what she thought when she thought of seven. When Diana created the two vertical rows of three with one counter in the middle, she did not need it in front of her to describe how she knew that orientation to be seven, so her subitizing ability may be closely aligned to this understanding.

The last task in Diana's eighth teaching experiment session began to suggest that Diana was engaging in FCS when subitizing five items, as she was shown an arrangement, and was able to group items effectively to support her subitizing activity. This was conceptual subitizing, because Diana was able to re-present them in a more effective way that aligned with what she conceptually knew about five. This conceptual subitizing is considered to be flexible because she could have offered a two, two, and one explanation, but was able to state that she saw four and one, which makes five. This is also important to note because Diana was engaging in descending PCS activity earlier in protocol D3 by describing "four and five," not four and one make five. Thus, it seems as if the PCS activity was indicative of Diana's transition from PDS activity towards FCS activity.

In protocol D7, Diana is shown five dots. Two dots were in a row in the bottom left-hand corner of the viewfinder, and three dots were each placed in the remaining three corners.

Diana stated quickly that she saw five and then re-presented it with four counters divided into two equal groups of two and two. Two of the counters were in the top left-hand corner of her mat, and two of the other counters were in the bottom left-hand corner of her mat. The fifth counter was in the bottom right-hand corner of her mat. When asked, she stated that she saw five because there were four and one.

Protocol D7.

T: Last one. You ready?

D: Hmm...mmm [indicating yes.]

T: [Teacher-researcher shows Diana a picture of the back of a camera. In the viewfinder, there are five dots. Two dots are in a row in the bottom left-hand corner of the viewfinder, and three dots are each placed in the remaining three corners.]

D: Five.

T: You saw five? Can you make what you saw?

D: [Diana places one counter in the bottom right-hand corner of her mat. She then picks up the remaining four counters, two at a time and places them directly across from each other on the top left and bottom left of her mat.]

T: [As Diana is picking up her final two counters.] How did you know you saw five? You said that pretty quickly.

D: Because four and one more make five.

T: Okay good.

So, this protocol seems to illustrate that Diana's thinking model is changing and transitioning towards FCS type of activity. I initially considered that Amy should be given opportunities to subitize arrangements where space, symmetry and color might evoke a three and

one combination for four. The purpose for this would be to change Amy's thinking model with regard to engagement in FCS activity, as she would be able to use three and one or three and two when subitizing four and five items. It seemed that once Amy could consider three and one when subitizing four, Amy would be able to consider flexible groups with regard to five. However, for several sessions Amy engaged in games centered on four and five items, where symmetry, space and color were in place to help elicit some attention towards the subgroup "three." However, what resulted from these attempts to perturb Amy's present thinking model was Amy transitioning towards what is described in this study as RCS activity.

Changes in Amy's thinking model regarding subitizing four and five flexibly. In protocol A10, Amy was playing the Ice Cream Scoop game, and had been given a game mat with multiple orientations showing four dots. As stated earlier, color, symmetry, and space were used here to emphasize the subgroups three and one. Before playing the game Amy was shown her mat and her die, and asked to describe the different number of dots she had seen. In this protocol, Amy quickly looked at an orientation in the top scoop of her game mat. The orientation had three black dots in a row and one red dot directly above the middle dot in this row. Amy was drawing upon her only understanding of three and some more as being five. This was different from carrying in to her subitizing activity the idea that three and two were five, which was evident in the next few protocols. What also resulted from this activity was a clear indication that Amy was accommodating her subitizing activity to allow her success regardless of the topological orientations. Amy continually did not carry into her activity the three and one, but regardless of orientation carried into her activity the notion that two and two make four.

Protocol A10.

T: Okay, what do you have here? [Teacher-researcher points to the top ice cream scoop on Amy's game mat. The four dots are arranged with three black dots in a row and one red dot directly above the middle dot in this row.]

A: Five.

T: You see five here? You want to count and check?

A: [Amy points to each dot and whispers number words.] Four.

T: Four. How many do you see here that are in black? [Teacher-researcher points to row of three black dots and covers up the red dot when doing so.]

A: [Amy skims her finger over the row of three dots.] Three.

T: Three, and how about right here? [Teacher-researcher points to the red dot above the row.]

A: One.

T: One. So, we could say that we have three and one, which makes [pauses] four.

A: Four.

T: Yeah. How about down here? [Teacher-researcher points to four dots in the ice cream scoop second from the top. The dots here are arranged where two black dots are in a row at the top middle portion of the scoop and two red dots are in the bottom corners of the scoop.]

A: Four.

T: Four? How did you know that so quickly? You did not need to count that one.

A: Because there's two here [places two fingers from each hand on the two symmetrical groups of dots.]

T: So, you see two here [points and slides finger from top dot to the bottom dot on Amy's right-hand side] and two here [points and slides finger from top dot to the bottom dot on Amy's left-

hand side.] Okay, very cool. Alright, how about here? [Teacher-researcher points to the scoop third from the top. These four dots are arranged in one row of three red dots with one black dot directly above the red dot at the left-hand side of the row.]

A: Four.

T: How did you know that?

A: Uh...I know it.

T: Yeah? What do you...do you see something that is the same between this one [points to the ice cream scoop third from the top] and this one [points to the top ice cream scoop.]?

A: 'Cause they each have a dot.

T: They each have one dot [points to the black dot in the orientation third from the top] one color [points to the red dot in the top orientation], and they have how many [points to the row of three red dots in the orientation third from the top] in another color [points to the row of three black dots in the top orientation]?

A: Three.

T: Three. Yeah. How about right here? [Teacher-researcher points to the ice cream scoop fourth from the top. The four dots are arranged in a row. There are three red dots and one black dot at the end of the row.]

A: Red dots.

T: One black and three red? So, how many is altogether here?

A: Four.

T: Four. [Teacher-researcher points to the ice cream scoop second from the bottom. The four dots are arranged in a more random fashion. Three red dots are in a triangular arrangement and one black dot is directly above this arrangement.] How about here?

A: Four.

T: How did you know that?

A: I know it.

T: Yeah? What parts do you see here? [Teacher-researcher points to the ice cream scoop second from the bottom.]

A: Two [places two fingers on the black dot and the red dot at the top of the triangular arrangement] and two [places two fingers on the two red dots at the side and bottom of the triangular arrangement.]

This protocol illustrates Amy transitioning away from the topological aspects of the orientations with regard to four items. She was not perturbed by color or shape. In fact, after exploring these orientations, when she is shown the die, she saw a die face with two squares on it. One square had three dots in a diagonal row, and the other square had one dot in it. She subitized this and quickly stated that it was five. She then needed to count to check, and to discover that it was four. This was very similar to what she did with the first scoop. She was not able to connect three and one to her number scheme for four. Therefore, this activity begins to suggest that Amy was transitioning toward what is described as RCS activity, as she had a limited or rigid understanding of four, and carried this conceptually into all subitizing activity with regard to four. The issue here was that Amy's thinking model was unable to change as her reliance upon the topological aspects of the orientations influenced her subitizing activity.

Regardless of this early set back regarding changes in Amy's subitizing ability, Amy was still transitioning towards conceptual subitizing. However, this transition was not one I had anticipated. Through about two weeks of games and activities, Amy still carried two and two into her subitizing of four and two, two, and one into her subitizing of five. This was noted in

protocol A11, in which Amy was in her 13th teaching experiment session and was shown some cards with dots on them and then asked to use counters to make what she saw. Amy was shown a card with five dots in a diagonal row. There was a large space between the fourth and fifth dot. It is understandable that Amy described this as two, two, and one, but it was interesting that she also saw five as two, two and one when she was shown a card with three dots in the top row and two dots in the bottom row.

Protocol A11.

T: Alright, this one is tricky. [Teacher-researcher shows Amy a card with five dots on it. The dots are in a diagonal row and there is a large space between the fourth and fifth dot.]

A: Five.

T: You saw five? How did you know you saw five?

A: Because I saw two here [points to two imaginary dots in a row on the table], two here [points to two more imaginary dots in a row next to the first imaginary dots on the table.] and one here [points to one final imaginary dot next to the previous imaginary dots].

T: Good. Go ahead and make that.

A: [Amy places two counters out in a row, and the two more counters out next to this row, and then one more counter far above and just to the right of the row of four dots.]

T: You want to check [shows Amy the card again]?

A: [Nods head up and down to indicate agreement.]

T: Alright, you ready? [Teacher-researcher shows Amy a card with five dots on it. This time the five dots are arranged in two rows. One row has three dots in it, and one row has two dots in it.]

A: Five.

T: You saw five? How did you know you saw five?

A: I saw two here [places two counters in front of her in a row and then pushes it back with the pile of counters] two here [places two more counters in a row and then pushes it back with the pile of counters] and one here [places one counter off to the right of her other supposed rows].

T: Is that how you saw it?

A: Hmm...mmm [indicating agreement].

T: Here, put it back together so I can see it again. Ms. Beth needs it very clear.

A: [Amy places two counters in a row, and then two more counters in a row next to that row, and then one counter on the right-hand end of the row. Amy then slides two of the counters from the left-hand side of the row so they are resting just below the middle two counters.]

T: So, there is two [points to the two counters in what is now Amy's top row] and one [points to the one counter at the end of the top row], and then two [points to the two counters in what is now the bottom row]?

A: [Nods head up and down to indicate agreement.]

T: Okay. Good.

So, at this point it seemed clear that Amy was engaging in RCS activity, as she was unable to think about the groups flexibly when subitizing. Furthermore, in a subsequent task in this teaching experiment session, Amy seemed to still be grappling with how to coordinate three and one to compose four. She saw a diagonal row of four dots with a large space between the third and fourth dot. After Amy saw this, she switched to some ascending PCS activity, as she stated that she saw "three...four," but then was unable to make this arrangement as she wanted to re-present what she saw as two and two and one more. This activity suggests that Amy may

have been beginning to transition toward a flexible understanding of four, but was having difficulty coordinating her grouping and counting structures.

Using Symmetry and Space to Evoke Changes within Amy's Thinking Model

To evoke change within Amy's thinking model, one of the witnesses suggested we use symmetry with six dots. The logic here was that Amy's thinking model relied on symmetry when composing four with two and two, so why not carry three into Amy's subitizing activity by composing six with three and three. Also, if Amy was engaged in RCS activity with four and five, she might have still been engaged in some form of perceptual activity with six. Last, Amy was shown a typical orientation of six dots similar to that on a die. Upon seeing the dots she stated that three and three were six, indicating PAS or PDS activity with regard to symmetry or space. However Amy was not able to use this understanding of six to engage in PAS or PDS activity with five or four.

In protocol A12, Amy was in her 14th teaching experiment session and was coloring in a hidden picture. The hidden picture activity was designed so that a rectangle was divided up into unusual shaped sections. Inside each of the sections there were rectangles with different orientations totaling two, three, four, five, six, or seven. Amy needed to find all of the sections that had six dots in them. When she found these sections with six dots in them, she needed to color them in. Once all of the sections with six dots in them were colored in, Amy would see that they created a picture of an acorn that the squirrel, Skip, had lost. If Amy found a section without six dots in it, she could cover these up with a counter. This was added to the activity to allowed students an opportunity to find sections through a process of elimination.

Prior to protocol A12, Amy had already found two sections with six dots in them. One section had two vertical columns with three dots in each column, and the other section had two

horizontal rows of three dots. The rows were not directly above or below each other, as one had been slid off to the side so that the first dot in the row was aligned with the third dot in the other row. Each time Amy engaged in PDS activity, she could state that she saw six here, because she saw three and three. In protocol A12, Amy found a section with five dots in a diagonal row and she pointed to it and stated it was six, but in showing me the two groups of three, Amy realized it was three and two.

Protocol A12.

T: Do you see any other sixes?

A: [Amy points to the section with five dots in a diagonal row.]

T: This one? [Teacher-researcher points to the section Amy indicated]. Okay, how do you know that is six?

A: Because there is three and... [points to three dots in the row, but stops]. Oh.

T: How many is this?

A: Five.

T: Five. How do you know that is five?

A: Because there's three here and two here [points to three dots in the row and then two dots in the row.]

T: Oh, okay. Very good. Do you see it somewhere else?

A: [points to a section with seven dots arranged in the section. Three dots are in two vertical columns and one more dot is placed between the two columns.] If this dot wasn't here this would be six.

T: Wow. So what does it make it when we add that dot?

A: Seven.

T: Seven. That's right. Good for you.

Amy's responses to these tasks suggested that Amy was engaging in some ascending and descending PCS activity with regard to five and seven, as she was using what she knew about six to subitize five and seven comparatively. Furthermore, her responses suggest a reliance still on perceptual material when engaging in this activity, as she was unable to accurately subitize five when she attended to the diagonal row of dots. With this transition in mind, Amy's next teaching experiment session was designed to assess and further support her understanding with regard to five and four when subitizing. So, in protocol A13, Amy was in her 15th teaching experiment session and was shown counters and asked how many she saw and if she could make what she saw. Prior to protocol A13, Amy was shown five counters arranged in a diamond of four and one counter off to the side. She accurately stated that there were five there, but could not tell me how she knew. Amy was then shown two vertical columns of three, and accurately stated that she saw six. Again, Amy could not tell me how many she saw, but, as described in protocol A13, when I covered up groups, she was able to describe the covered up groups accurately relative to the symmetrical aspects of the orientation. For instance, Amy could say that there were two covered up, and three, but when I skipped to five counters, she said there were four counters covered up. However, this was easily perturbed by covering up four, which she said was three, and then covering up the vertical column of three, which she said was three again. This change was evident in that Amy was then able to accurately state there were five covered up. I believe this activity supported Amy's ability to attend to the subgroups three and two when subitizing five in subsequent tasks.

Protocol A13.

T: How many are hidden under there? [Teacher-researcher covers up two counters leaving a square like orientation of four counters visible.]

A: Two.

T: Two? How about now? [Teacher-researcher covers up a vertical column of three counters leaving the other column of three counters visible.]

A: Three.

T: Yeah? How about now? [Teacher-researcher covers up four counters leaving two counters in one column visible.]

A: Four.

T: How about now? [Teacher-researcher covers up a vertical column of three counters leaving the other column of three counters visible.]

A: Two...Three.

T: [Laughs] Two...three? How about now? [Teacher-researcher covers up five counters leaving one counter in one column visible.]

A: [pauses.] Four.

T: Four? [Teacher-researcher covers up four counters leaving two counters in one column visible.]

A: [pauses.] Three.

T: Three? [Teacher-researcher covers up a vertical column of three counters leaving the other column of three counters visible.]

A: Three. [Whispers and smiles.]

T: Three again? Yeah? How about now? [Teacher-researcher covers up five counters leaving one counter in one column visible.]

A: Five.

T: You think five? Good. [Lifts cardstock to reveal all counters.] Were you right?

A: [Nods head up and down to indicate yes.]

T: Yeah. Alright, you ready? Okay clear your mat. [Teacher-researcher lifts edge of top piece of cardstock to rearrange counters.]

A: [Amy slides the counters off of her mat.]

T: Alright, you ready?

A: Hmm...mm [indicating yes.] Why do you put that paper there?

T: So, you don't see what I am doing to my board.

A: Oh.

T: [Teacher-researcher lifts the top piece of cardstock to reveal five counters on her mat in two vertical columns. One column has three and directly across from the top two counters are two more counters in a second column.] How many did you see?

A: Five.

T: You saw five? What helped you see five?

A: 'Cause there's three here and two here [Amy recreates the vertical columns of three and two with her finger.]

Here Amy was beginning to engage in PDS activity with five. However it is important to first consider the task described just before she is shown five counters. In this task she was given an opportunity to describe the subgroups hidden and by looking at the visible counters and the teacher-researcher's actions, she seemed to understand that five is one less than six. So the

very next orientation shown to Amy was meant to continue this one less notion and see if this also supported Amy's awareness of four being composed of three and one. In protocol A14, Amy was still in her 15th teaching experiment session, and was shown three counters in a vertical column and one counter directly next to the top counter in that vertical column. She accurately stated this as four, but could not tell me why she knew that it was four.

Protocol A14.

T: Alright. [Teacher-researcher lifts the top piece of cardstock, revealing four counters. Three counters are in a vertical column and one counter is directly next to the top counter in that vertical column.] How many did you see?

A: Four.

T: You saw four? Can you make what you saw?

A: [Amy places all four counters on her mat, one at a time. The final counter is placed with the yellow side facing up.]

T: Did you see any groups that helped you?

A: [Shakes head side to side to indicate no.]

T: No? [Looks at the yellow counter placed face up.] Did you see some groups?

A: [Shakes head side to side to indicate no.]

T: You made this a different color. Did you mean to do that?

A: [Nods head up and down to indicate yes.]

T: Yeah? Does that mean you saw this one different from these?

A: [Nods head up and down to indicate yes.]

T: So, are those the two groups you saw?

A: [Nods head up and down to indicate yes.]

T: Okay, so what are the two groups?

A: Four.

T: You see four? I see four too. [Places a piece of cardstock over the vertical column of three counters.] How many are hidden underneath that paper?

A: Three.

T: Yeah? [Slides paper over to cover the bottom two counters in the vertical column.] How about now?

A: Two.

T: Two. [Slides paper over to cover up all counters.]

A: Thre...four

T: And there's four. How many reds are there?

A: Three [holds up three fingers with her left hand.]

T: Yeah, and how many yellows?

A: One [holds up one finger with her right hand.]

T: Yeah, look at that. How much is that altogether?

A: Six.

T: That's six? Yeah? Let's see. Let's count your fingers again.

A: [Puts fingers up one at a time as teacher-researcher counts aloud and points towards Amy's fingers.]

T: One, two, three... [pauses] four.

A: Four.

T: Yeah.

A: If it was five and this [indicating all five fingers and one finger.] it would be six.

T: It would. You're right. Very good.

Notable from this protocol is Amy's response, suggesting that she may have been using three and one to compose four, but it is unclear whether this is true. When Amy placed the yellow counter on her mat, she may or may not have been able to consider this orientation as three and one, as is evident when she was asked what the groups were-- she simply stated four. Also, when re-presenting her orientation, Amy used her fingers instead of having to touch the mat or the counters. This activity seems to suggest Amy may be transitioning towards a more sophisticated number scheme with regard to numbers, as Amy was beginning to use her fingers more consistently to stand in for the experiential items she was describing when subitizing. It seems though, that this confused her notion of the whole number in the end, because she considered five and one to be six when using her fingers, so that when she held up one finger to re-present that one yellow counter, she thought for a moment that she had one and some more which made six. How Amy used her fingers will be discussed near the end of this section.

One thing that is clear from protocol A12 is Amy was engaging in PDS activity with five, but only following a subitizing task which showed her two columns of three. So, it is not clear if Amy could break a bit from these orientations and the subitizing activity of six just beforehand. After protocol A13, it was not clear upon what activity Amy was relying when subitizing four. It is clear that she may have been unconsciously attending to subgroups three and one. However, it may also have been that Amy was describing five relative to six, and therefore could only subitize four relative to five.

Therefore, in the next protocol, A14, Amy was in her 17th teaching experiment session and was being shown counters and asked how many she saw and to make what she saw. This was Amy's first task in the teaching experiment session; she was shown four counters similar to

the orientation in protocol A13. There was one vertical column of three counters and one counter directly beside the top counter. Amy described this as four, and after she made it, described the subgroups that she saw.

Protocol A14.

T: Okay. You ready? [Teacher-researcher lifts the top piece of cardstock revealing four red counters. There is one vertical column of three counters and one counter directly beside the top counter.]

A: Four.

T: Four? How did you know it was four?

A: Because I thought first.

T: Hmmmm...mmm [indicating agreement]. You want to make what you saw?

A: [Amy places four counters on her mat one at a time. They are arranged in an orientation similar to what was on the researcher's mat. One counter at the top of the vertical column is yellow and the rest are red.]

T: What did you think about? Do you think that matches exactly?

A: [Nods head to indicate yes.]

T: Did you think about it in groups?

A: [Nods head up and down to indicate yes.] Yeah.

T: Yeah? What groups did you think about?

A: One yellow here [points to yellow counter] and three reds there [points to red counters].

T: Yeah, well that's interesting [flips top piece of cardstock up to reveal original orientation to Amy.] Because I didn't have a yellow there did I?

A: [Shakes head side to side to say, no, there was not a yellow there.]

T: No. So you thought of it as two, and one and one, or three and one?

A: Three and one.

T: Three and one? Oh, okay interesting. Go ahead and wipe off your board.

A: [Amy slides counters of her board and flips all of her counters over so all of them have the red side facing up.]

T: [Teacher-researcher is rearranging the counters behind the top piece of cardstock.] Alright, you want to try another one?

A: Hmm...mmm [indicating yes.]

T: [Teacher-researcher shows Amy a mat with five red counters on it. The counters are arranged were three counters are in a vertical column on the right-hand portion of the mat and two counters are in a vertical column on the left-hand portion of the mat.]

A: Four...

T: Four?

A: Five.

T: Five? Can you make what you saw?

A: Hmm...mmm [indicating agreement. Amy places three counters on the left-hand side of her mat in a vertical column. These three counters are red. Amy then picks up two counters at the same time and places them yellow side up in a vertical column on the right-hand side of her mat.]

T: How did you know it was five?

A: Because I know it.

T: Yeah? You made these into different colors. Was that for reason?

A: Yeah?

T: Yeah? Is that how you thought about it?

A: [Amy nods her head up and down to indicate yes]

T: The reds like this [points to the vertical column of three counters with the red side facing up] and the yellows [points to the vertical column of two counters with the yellow side facing up]?

A: [Amy nods her head up and down to indicate yes]

T: What are the two groups that you have here then? [Pauses] how many are there here [points to the vertical column of three counters with the red side facing up]?

A: Three.

T: Yeah? And here [points to the vertical, two counters with the yellow side facing up]?

A: Two.

T: How much is that all together then?

A: Three and two... Five

T: Five... Nice. Okay.

Here it seems as if Amy was considering subgroups after subitizing, and was engaging in ascending PCS activity when she was subitizing. It is not clear if Amy used three and two to subitize, or used the orientation of four shown to her just before the orientation of five.

However, the subgroups two and three may have influenced her preattentive encoding activity, as she re-presented her arrangement into the two subgroups three and two, and even used color to emphasize these subgroups. Interestingly, Amy was engaging in more PCS activity regarding four and five, which suggests that she was beginning to transition toward a new type of activity. If she was engaged in PDS activity when subitizing four and five, as Amy's response within this protocol suggests, her PCS activity may have been supporting a transition towards FCS activity, but at this point, it is unclear.

In protocol A15, Amy is in her 17th teaching experiment session, and her response to this task seems to support the fact that Amy's activity may have been transitioning towards FCS activity. Amy was shown 5 red counters, arranged so that four formed a square formation,, with one counter placed off to the right-hand side of this formation. Amy easily stated that she saw five, and then made an orientation similar to what was shown, but explained that she noticed they were grouped as three and two.

Protocol A15.

T: Okay, are you ready? [Teacher-researcher shows Amy five red counters. Four are arranged in a square formation, and one counter is placed off to the right-hand side of this formation.]

A: Five.

T: Oh! How did you know that so quickly?

A: I thought about it.

T: You thought about it? Do you want to make what you saw that helped you think about it?

A: [Amy picks up five counters one at a time and places them on her mat. Two are quickly placed in a vertical row and three are placed quickly in a triangular orientation.]

T: What groups did you see...were there groups in that? [Points to Amy's mat.]

A: Hmm...mmm [indicating yes.]

T: Yeah? What groups?

A: There's two here [uses two fingers to point towards the column of two counters.] and three here [points to the triangular orientation of three counters].

T: Nice.

W: Two and three.

T: Okay [Lifts the top piece of cardstock so Amy can compare her orientation with the original orientation].

W: Do you see two and three in Ms. Beth's paper? Or do you see other groups in her paper?

A: One... [picks up one counter and moves it over to the triangular orientation. Amy tries to make the square formation with her counters.]

T: Are there other groups on my paper that could help you think about it too?

A: There's four here and one here.

T: Oh, so you could think about it two ways, huh?

A: [Nods head up and down to indicate yes.]

T: I remember a long time ago you would have said I saw, two, two, and one. [Points to two counters, two counters, and one counter.] So there's that way too. Very good.

This seems to show evidence of the existence of Amy's transition towards more FCS activity with regard to five. In retrospective, Amy was only able to count this orientation in protocol A5, and was only able to use two, two, and one when subitizing this orientation in protocol A8. However, she was now relying more on her conceptual understanding of five to describe both the composite group and the subgroups. Her orientations clearly illustrate this. Teaching experiment sessions after this, offered Amy opportunities to subitize seven and eight in hopes to elicit more perceptual subitizing activity and to attempt to support some PAS or PDS activity. Amy relied mainly on IPS activity, but at times utilized her IPS activity to engage with some PCS activity where she said there were four [pointing toward a square formation], and then five, six, seven, eight. Unfortunately, there was not enough time to observe any real transitions in this area.

There was one final protocol that will be a part of this discussion with regard to Amy. This was Amy's final teaching experiment session, session 22, and the purpose for this session was to reassess her conservation of number and have her count some objects. I had been assessing only Amy's ability to count all versus her ability to count on, but I was curious as to what type of counting she was relying upon, as described by Steffe et al., (1988). So, tasks from Steffe et al.'s (1988) study were duplicated with Amy to better ascertain if she was a perceptual counter, figural counter, or motor patterned counter. In protocol A14, Amy was using fingers to describe parts, and Diana was using fingers to re-present how many she saw as early as her screening interviews. This activity became a salient aspect to consider, as counting schemes described by Steffe et al., (1988) consider students reliance upon finger patterns as standing in for perceptual material as a type of transitioning activity from perceptual material towards spatio-motor unit counting. Thus, connections between how number was being re-presented when reflecting on subitizing activity and how number was being re-presented when counting became the focus for Amy's final teaching experiment session.

Prior to protocol A16, Amy was shown seven counters in a row. Four counters were covered with a piece of cardstock. Amy was that told this was her candy and there were four pieces of candy under this piece of paper. Amy was then asked how many pieces of candy she had altogether. Amy's response was surprising, as she used fingers to hold onto her counting, but did not look at them. Also, when she explained what she knew with her fingers, she was able to draw on what she knew about four being composed as two and two in order to re-present the problem with regard to her finger patterns. After this, which begins protocol A16, Amy was shown three counters and six counters were placed under my hand. She again used her fingers

to re-present what she knew, and decomposed the six into three and three in order to solve the problem.

Protocol A16.

T: Alright, watch this. We are going to change it up now, okay? [Slides nine counters out in a row in front of Amy.] Now this time there are six under my hand [doubles a row of three counters and places this array under the teacher-researcher's hand] and there are three right there. How many do we have altogether now?

A: [Amy counts up with three on one hand, then counts up with three more, two on one hand, and one on her other hand. Finally, Amy counts up two more on her other hand.] Eight.

T: Eight? How did you figure that out?

A: I just counted with my hands.

T: Show me what you did with your hands to find there was six and three altogether.

A: Because there was three here. [Shows three fingers with one hand] three here [shows two fingers with the same hand and one finger with the other hand] and three here [adds three fingers to the hand with one finger]. Three [shows three fingers again with one hand], and there's three [shows three fingers with two from one hand and one finger with the other hand], and there's three [shows three more fingers with the other hand. There are a total of nine fingers shown.]

T: So, is there eight?

A: [Amy reviews all of her fingers again.] Nine!

T: Nine? Yeah?

A: Yeah [Nods head to indicate yes.]

T: Now, you said three, three, and three, but I told you six and three.

A: No.

T: So, where did you ... what did you do with your six?

A: Three [shows three fingers with one hand], three [shows three more fingers with her other hand], and three [shows nine fingers with both hands].

T: Oh! So, is three and three, six? Do you want to count and see?

A: Yeah, one, two, three, four, five, six, seven, eight, nine [touches each counter with her finger while stating each number word].

T: You were right! Good for you! Alright, you ready? We're going to do another one.

So, protocol A16 does a nice job of illustrating some of Amy's counting strategies, and it seems that she is carrying in to her counting activity some very similar number schemes with regard to subitizing. Amy decomposed the numbers into doubles and then reorganized this with regard to her finger patterns. This might be interesting to consider with future studies, as this activity suggests that Amy is relying on some similar thinking structures when counting and conceptually subitizing.

At this point, I wondered what I might do to elicit an essential mistake so that the boundaries of Amy's thinking model might be more apparent with counting. Therefore, the next task shown in protocol A17 both went beyond Amy's fingers (the total would be 11), and required her to use four and seven as subgroups. Seven was chosen because Amy was not engaging in any PAS or PDS activity with regard to seven, so I wondered if she would be able to revert to straight counting, or what her response might be. If she was able to successfully engage in this task in which seven was attempted, then it might be that her counting schemes and subitizing activity were not relying on similar thinking structures or understanding of number.

Protocol A17.

T: [Teacher-researcher places seven counters on the table and four counters on the table. As these counters are being added to their respective groups, Amy uses her fingers to try and capture the number of items. Before she can finish, the teacher-researcher covers up the group of counters with a piece of cardstock.] Now, there's seven under here and ...

A: There's four.

T: There's four here...right. So, how many do we have altogether now?

A: Umm...I don't know.

T: You don't know? Do you want to try it with your fingers like you did before?

A: Yeah.

T: Yeah, okay.

A: There's four [holds up four fingers], and seven [holds up two fingers].

T: Yeah.

A: [Shows me her fingers as if six fingers was her solution.]

T: Where is your seven?

A: Seven [bends her two fingers to indicate that these two fingers represent her seven].

T: Yeah?

A: And this is my four [bends her four fingers to indicate that these fingers represent her four].

T: This is your seven here [points to Amy's two fingers]?

A: [Nods her head up and down to indicate yes.]

T: Yeah? So how many is there altogether do you think?

A: I don't know.

T: You showed me that this is seven [holds up two fingers] right?

A: [At first nods head up and down to indicate yes.] It's two!

T: That's two. Oh, okay, and this is four?

A: [Nods head up and down to indicate yes.]

T: How can you make this...here's your four [holds up four fingers]. Where do you see...how can you add seven to that?

A: [pauses]

T: What if you just counted on with your fingers? Can you do that?

A: [Looks at her hands and makes four and three with her fingers.] Seven.

T: How much is that altogether then you think?

A: Eight.

T: You think eight? You want to count and see?

A: Yeah. One, two, three, four, five, six, seven, eight, nine, 10, 11 [points to each counter with her finger while stating each number word].

T: Whoa! That was high!

So based on Amy's responses in this protocol, Amy was unable to decompose seven conceptually so she could utilize her finger patterns. She was able to decompose seven as four and three, and she knew that five and two would make seven, but organizing this in relation to four was too much for her to coordinate. Amy's response aligned with her subitizing activity, as she was mainly relying on her IPS activity when subitizing seven, and at times her ascending PCS activity. However, she was not engaging in FCS activity with regard to seven, and I believed she would need that to know how to do this, or at least RCS activity, so she could engage in reversible mental thinking regardless of perceptual material she was utilizing when counting.

Synthesis of Subitizing Activity

This section will be divided into three subsections centered on the following three main themes: the influence of symmetry on subitizing activity, conservation of number as an indicator for subitizing activity, and the relationships of the seven different subitizing activities to each other. As the analysis of these data is discussed it is important to note the limitations of this study. In no way can the previous protocols be any more than inferred mental actions. Suggesting these students' physical actions result in these mental actions is not the aim of this analysis. Furthermore, the data resulting from this study were rich, selected protocols were chosen so that they pertained only to the purpose of this study. With a different theoretical paradigm or research purpose, these data might be analyzed quite differently.

Relationship Between Symmetry and Subitizing Activity

Symmetry seemed to provide Frank and Amy a “free” unit when subitizing even numbered items that were symmetrically oriented. Early on, Frank was attending to subgroups, as noted in the previous section, in which Frank clearly stated “T...Four” when subitizing four. This was described as PSS activity, but transitioned Frank towards PAS activity. It could also be argued that Frank may have also been engaged in ascending PCS activity which supported his PAS and PDS activity when subitizing four. What is necessary to consider is the fact that both students relied on symmetry to subitize five items and both described five as two, two, and one, further emphasizing the link symmetry has with early PAS and PDS activity when subitizing five items.

In transitioning from PAS and PDS activity, Amy seemed to rely solely on the subgroups two, two, and one when subitizing five, which resulted in her engagement with RCS activity. This activity seemed to limit Amy's ability to flexibly rely on subgroups when conceptually

subitizing. If Amy could not use three as an object when subitizing, it seemed as if her conceptual subitizing would be inefficient and limiting, preventing her from re-presenting number in a variety of different subgroups. The reliance upon the perceptual material changed her thinking model, as subitizing six items grouped both symmetrically and with space between two groups of three, offered three as a unit for Amy to subitize. It seemed that Amy was able to capitalize on three as a “free” subgroup so she could focus on the subgroup two. This “free” subgroup began to support Amy’s engagement with FCS, as three and two now supported Amy’s ability to subitize five.

Not only is it important to note how symmetry influenced this activity, but for Amy to conceptually grasp the notion that five was composed of three and two, she needed to re-visit perceptual subitizing activity. This trajectory simply emphasizes the importance the relationship topological thinking structures have with number development. Thus, further modifying Piaget’s (1968/1970) definition of topological thinking structures to increase the scope of the application, and include patterned orientations and symmetrical aspects of two groups of items would offer perceptual subitizing activity as a vehicle in which a direct connection can be made between this third mother structure and early number development.

Conservation of Number as an Indicator for Subitizing Activity

For students like Ben and Frank, who were not yet conserving number consistently, subitizing activity was limited to perceptual activity. This seemed evident all throughout the study, as even though these students could describe subgroups, which either composed (ascended) or decomposed (descended) a composite group, this was due to their primary reliance upon perceptual items. This connection between conservation of number activity and conceptual subitizing activity suggests an individual’s reliance upon the simultaneous coordination of both

the serial and classification thinking structures. That is also evident when students engaged in PCS activity, as it seemed as if students could subitize a group and then count forward from this activity, or subitize a group and count backward from this activity. In no way does analysis of PCS activity suggest that these students were counting on, as described by Steffe et al. (1988), as the number was not an object, but a result from an activity. However, the subitizing activity seemed to rely more on the students' classification and topological thinking structures and the PCS activity allowed students reliance upon their serial thinking structures.

Also, for children to conserve number, Piaget (1941/1965) explains that children would need to be capable of engaging in what is described as mental reversibility. This mental reversibility is also an influential factor for children engaging in conceptual subitizing activity. For example, for Amy to state that she knew an orientation to be two, two, and one, she needed to subitize an orientation of items by conceptually grouping them and then state the composite group. For her to use this activity as an explanation, she would then need to reverse her mental actions to describe how the items were grouped. This would necessarily mean that Amy needs to understand that if $A + A' = B$, then $B - A = A'$. This is, essentially, what conceptual subitizing relies on, and which is also what children conserving number rely on.

Furthermore, it seemed as if Ben's thinking structure was beginning to change as he seemed to grapple with the space between the items, making the group of items larger or smaller. It is still unclear as to how this activity influenced his reliance upon different forms of subitizing activity, but Ben's attention towards density when describing which group of items had "more" may have influenced his ability to engage in PSS activity. However, Ben's learning trajectory was difficult to analyze because previous tasks strongly influenced present tasks. For instance if Ben was shown three dots with very little space between them, and he says he saw "three," then,

if the next task is the same, but with a large space between the items, he may say “four,” as comparatively, the space between the items seemed to influence his perceptual subitizing ability. There is not enough information regarding this phenomenon from this study, and can only be described as an explanation as to why Ben’s responses were not always based solely on the perceptual properties of the items.

How the Seven Different Subitizing Activities Relate to Each Other

The most surprising result from this study was discovery of relationships between perceptual and conceptual processing, noted with the seven different types of subitizing activity. Initially, these different types of activity seemed like a variety of cognitive tools, which explained different types of abstraction which the students relied on when subitizing. However, throughout the analysis, these activities all seemed to have purpose. Table 4.1 captures the different protocols used for Amy, Ben, and Frank throughout the analysis to describe the students’ shift from one subitizing activity to another. This table effectively grouped protocols used in the analysis into five different time frames (e.g., A = TE 1-4. B = TE 5-8). The number in the parenthesis describes the composite group of items the students were attempting to subitize. Considering this table allowed for a more in-depth analysis regarding transformations that students made in activity relative to time and composite group.

Frank and Amy engaged in the material in a cyclical manner, as it seemed that they each had to re-visit IPS or PSS activity before moving forward. For example, Frank seemed to undergo considerable growth with regard to number and subitizing activity when engaging in PCS activity. This activity seemed to merge some of Frank’s counting strengths with Frank’s subitizing ability and attention toward subgroups. At first, this activity did not support Frank’s understanding of the composite group, which resulted in a bit of PSS and a bit of PCS, but once

his subitizing ability developed, it seemed as if Frank had begun to transition in subitizing ability by utilizing more PAS and PDS activity.

Table 4.1

Synthesis of Subitizing Activity by Time Range and Number Subitized

	IPS	PSS	PAS	PDS	PCS	RCS	FCS
Amy	A(5)	A(5) D(4), D(5) E(4)	A(5)	A(5),A(5) B(5) D(6),D(5)	D(5), D(7) E(5)	B(5) C(5) D(5)	E(5)
Ben	A(3),A(5), B(4),B(4), D(3),	A(4),A(4), E(4),	A(4), C(3)	D(3)			
Frank	A(5), B(4), D(5)	A(4),A(5), B(5), D(4)	B(5), E(5)	B(5), E(5)	A(5), E(5),E(5),E(5)		

Note. A = TE 1-4; B = TE 5-8; C = TE 9-12; D = TE 13-16; E = 17-22. Number in parentheses represents number of dots shown.

Amy had the most growth comparatively, as she was able to quickly transition from IPS activity towards PDS and then RCS activity. However, for Amy to engage in FCS activity, she needed to revisit PCS activity with regard to five and seven relative to six. This supported Amy's engagement with PDS and PCS activity when subitizing five, which then allowed Amy to flexibly conceptually subitize, which is described as FCS activity. This cyclical activity suggests an overwhelming importance that topological aspects of items have on early subitizing activity and understanding of number.

It is important to note that Ben's activity, as described earlier, was strongly influenced by the space between the items comparatively from one task to another. However, it is important to

note that even though Ben was early in his development, this activity may be foundational for Ben's future development, as it allowed Ben a forum in which to consider subgroups and begin composing and decomposing small numbers. Each time Ben was able to engage in PAS activity, color or symmetry was used to evoke this activity. Furthermore, it was noted that the one task in which Ben engaged in PDS activity color was used to elicit this activity. Therefore, it seemed that color served to counter the space between the items and was an influential factor regarding Ben's ability to attend to both the subgroups and the composite groups.

Finally, this section will conclude with an explanation of the different types of activity and how it supported transformations within Amy's thinking model (Figure 4.1). Amy began the first teaching experiment session relying heavily upon IPS activity, but when the size of dots was shown to Amy, she was able to attend to subgroups more efficiently. This influenced future behavior related to five, as Amy was able to begin relying on symmetry from four when subitizing five, and could consider four as two and two which supported the PAS and PDS activity related to five. However, since two, two, and one were Amy's only known conceptual understanding related to five, she was unable to engage in FCS activity and remained at RCS activity for a few sessions.

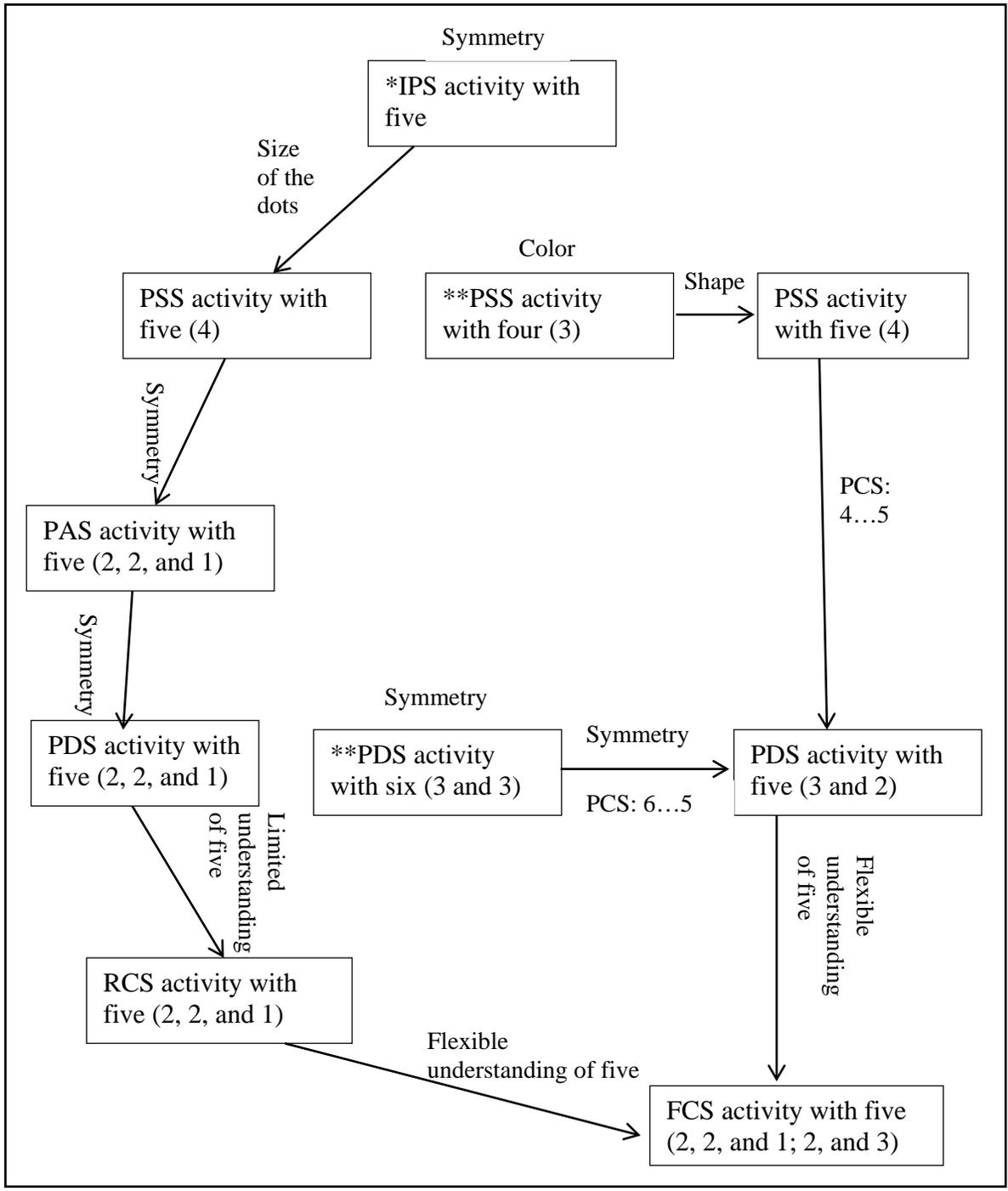


Figure 4.1. Transformations to Amy’s Thinking Model. The * indicates Amy’s initial teaching experiment session, and the ** indicates two different sessions where task design was meant to support FCS activity relative to five. This figure is an overview of the subitizing activity that Amy engaged in relative to the number of items being subitized.

Therefore, Amy was engaged in activities centered on three and four, but she found little success, as Amy could only build to three with single units, not use three as a subgroup. Once Amy was engaged in subitizing six, symmetrical aspects of the subitizing activity allowed Amy to attend to three as a subgroup. This attention towards three then offered Amy opportunities to engage in descending PCS activity with five from six. Continually holding on to the symmetrical aspects of five as being similar to the two vertical columns of three supported Amy's ability to conceptually decompose five as three and two, and two, two, and one, evidence of FCS activity.

Considering transitions in the model, it seems as if both symmetry and PCS activity offered Amy a scaffold when subitizing five. The key to this activity was to re-visit the perceptual subitizing activity with six, to allow Amy to consider three as a unit. This logic stemmed from Amy's ability to extend her subitizing from four to five through the use of symmetry. Thus, symmetry influenced transformations within subitizing and provided Amy with a "free" unit when transitioning towards different conceptual understandings for five. Also, PCS activity seemed to offer Amy opportunities to subitize orientations that are one less or one more than they had subitized previously. This PCS activity seems important also because it appeared to allow Amy an opportunity to engage in a combination of counting and subitizing, which seems to be necessary for conceptual subitizing.

Considering Frank's cognitive changes (Figure 4.2) throughout the study, it seems as if the student's perceived symmetrical aspects of the items influences the changes in subitizing activity. Considering Frank's transformations, specific to his thinking model, it also seems evident that Frank relied on the shape of items when moving from IPS activity to PSS activity and between PAS and PDS activity. Symmetry seemed to influence transitions within PDS

activity and early PSS activity. The changes made by Frank within PDS activity were important because Frank was able to describe and explain the subgroups he considered, relative to the composite group without the perceptual material within his perceptual field. This type of activity suggests reliance upon more abstract notions of number, as Frank was capable of re-imagining the material and seemed to rely upon the space the material “took up” as Frank described the counters relative to their location.

Comparing Frank and Amy’s learning trajectories, it seems clear that even though Frank was not conceptually subitizing, his notion of subgroups relative to the composite groups was beginning to inform his subitizing activity. This foundational aspect to his perceptual subitizing activity may further support his conceptual subitizing activity in the future. Even though Amy was capable of conceptual subitizing, for her to be able to conceptually subitize, she needed to explicitly connect what she understood about number to this activity.

It is hypothesized that Frank was not able to conceptually subitize throughout the study because he was not able to conserve number. Piaget (1941/1965) describes conservation of number as dependent upon physiological development, which influences the coordination of the serial and classification thinking structures. Interestingly, near the end of the study, in the 20th teaching experiment session, Frank was able to engage in some PCS activity with regard to six and then five, suggesting some early reliance upon three as a subgroup. Considering Amy’s trajectory relative to this activity, it is also hypothesized that with more time, Frank may have begun engaging in some PSS, and then PAS and PDS activity with regard to three and three, three and two, and three and one.

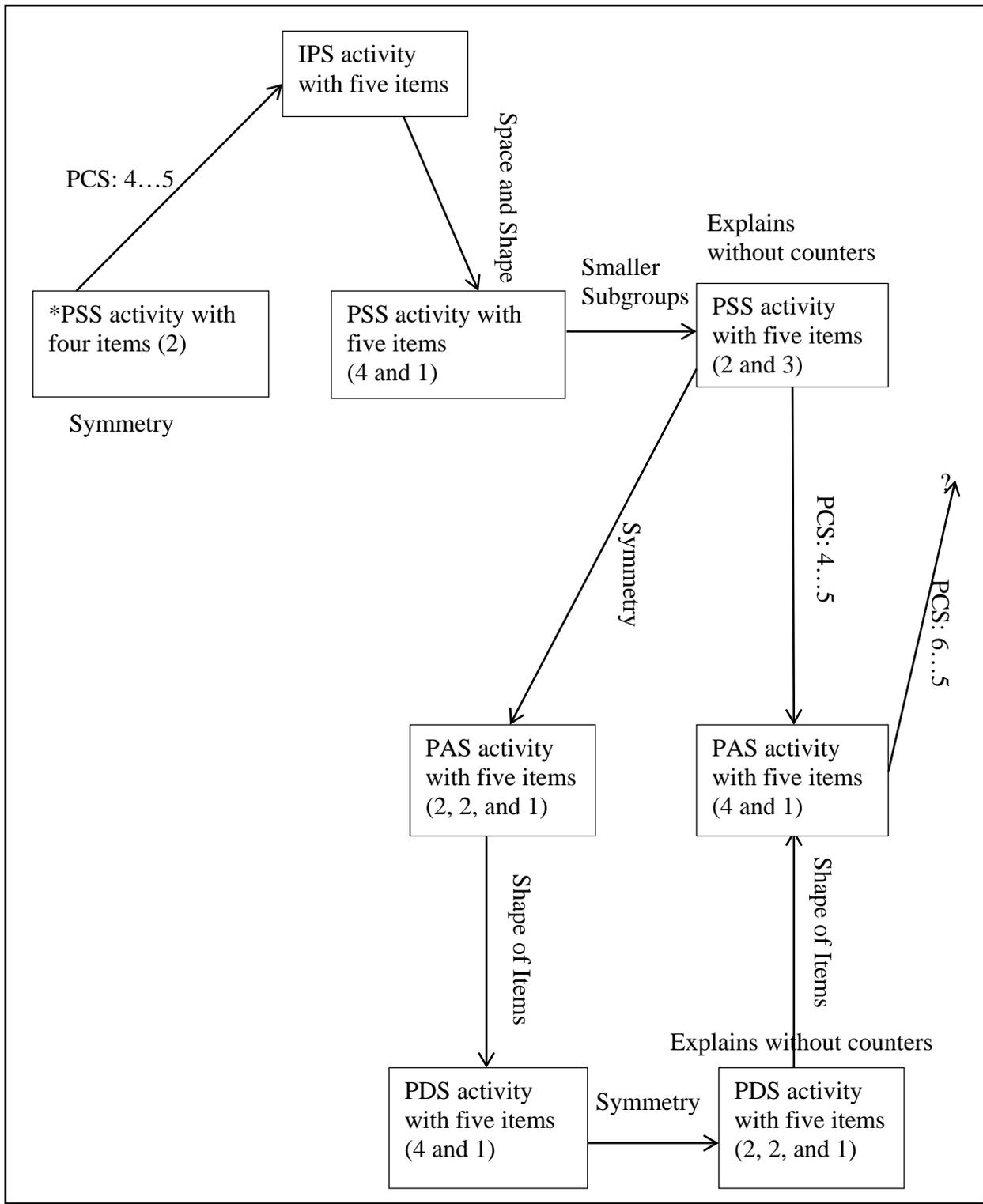


Figure 4.2. Transformations in Frank's Thinking Model. The * indicates Frank's initial teaching experiment session. This figure is an overview of the subitizing activity that Frank engaged in relative to the number of items being subitized.

Chapter Five: Conclusion

The purpose for this study was to determine the relationship between preschool aged children's subitizing activity (perceptual and conceptual) and number understanding (preconservation and conservation). This study was also meant to increase the scope of Piaget's (1968/1970) definition for topological thinking structures to allow the application of this thinking structure to include patterned orientations or regular space between items. The purpose for increasing the scope for this definition was to better understand how the development of topological thinking structures may directly influence number development when children subitize.

Before discussing the analysis relative to the research questions and purpose, it is important to refine Sarama and Clement's (2009) definition for perceptual and conceptual subitizing. Furthermore, seven different types of subitizing activity resulted from the analysis of the data in this study. Thus, it is important to again define these types of activity to ensure this discussion is framed appropriately.

Subitizing Definitions

Subitizing comes from the Latin term *subitus* and describes a quick apprehension of numerosity of a small set of items. Initially subitizing was named and defined by Kaufman et al. (1949) and more often became a vehicle in the literature with which to investigate perceptual mechanisms individuals depend on when encoding their perceptual environment. Researchers in the field of psychology describe subitizing as limited by three or four items. However, Sarama and Clements (2009) describe subitizing that relies on perceptual processes as limited by five items. The distinction between the two perspectives is in how subitizing is being defined. Subitizing in the field of psychology is not limited by orientation of items because dependence

upon item orientation is used by only a few researchers to describe why subitizing activity changes when subitizing five (Arp & Fagard, 2005; Mandler & Shebo, 1982). However, Sarama and Clements (2009) describe perceptual subitizers as being limited to five items in rectangular arrays, as shape of items is attached to a number name. Thus, allowing for only a limited set of orientations allows for Sarama and Clements (2009) to include five items when describing subitizing that is dependent upon perceptual processing.

Sarama and Clements (2009) included subitizing in their preschool mathematics program, titled *Building Blocks*. Results from an evaluation of the *Building Blocks* program indicated that students' subitizing activity increased and grew in both number and orientations as they participated in this preschool curriculum. Also, resulting from this study were hypothesized mental actions that students relied upon as their subitizing activity changed. To better describe these hypothesized mental actions Sarama and Clements (2009) coined the terms perceptual and conceptual subitizing.

Perceptual subitizing is defined by Sarama and Clements (2009) as an innate ability to discriminate different quantities, emerges in infants as young as three to five months of age, and is limited to five items. In comparison, conceptual subitizing is observed when children begin to connect the notion of number to subitizing activity and can describe utilizing subgroups when subitizing (Table 2.1) (Sarama & Clements, 2009). Results from this study indicate some refinement of these two definitions. In this study it was important to consider nuances in students' subitizing activity, which suggested reliance upon either perceptual or conceptual processes. For students to be described as engaging in conceptual subitizing, it was not enough for students to simply describe subgroups relative to composite groups, as students may be relying on the space between items, symmetry, or color. Also, students' conceptual

understanding of number needs to be sophisticated enough to support an understanding of the subgroups relative to the composite group. Thus, the distinction between perceptual and conceptual subitizing was considered through both observed student activity and reliance upon inferred mental activity.

Essentially, the distinction between a student engaged in perceptual subitizing versus a student engaged in conceptual subitizing is that a student's activity can be perturbed with space between items, symmetry or color of items. If a student describes seeing a composite group and subgroups which were decomposed from this composite group, regardless of the topological aspects or attributes of the items, then the student is relying only on her number understanding when subitizing. This is evidence of conceptual subitizing activity. The mental activity that a student requires when conceptual subitizing is an ability to engage in mental reversibility. This mental reversibility is also required for a student to conserve number. Thus, a student capable of conceptual subitizing must also be capable of conservation of number. However, this does not mean that a student capable of conservation of number is capable of conceptual subitizing, as a student's number schemes may not always be sophisticated enough or her subitizing activity may not trigger dependence upon a number scheme.

With these definitions stated, it is also important to name and define the refinements of these two types of activity as seven different types of subitizing activity (*initial perceptual subitizing, perceptual subgroup subitizing, perceptual ascending subitizing, perceptual descending subitizing, perceptual counting subitizing, rigid conceptual subitizing, and flexible conceptual subitizing*) described the students' activity when analyzing data in this study (Table 5.1). These seven different types of subitizing activity resulted from retrospective analysis throughout this study and were important to name and define to describe perceptual and

conceptual processes students engaged in when subitizing. It is important to note that these different types of activity are neither linear nor hierarchical, as students tended to revisit perceptual types of activity after engaging in conceptual subitizing activity (Table 4.1).

It is also important to note that the definitions for perceptual and conceptual subitizing, as defined by Sarama and Clements (2009), were altered slightly as a result of this analysis. Sarama and Clements (2009) state that students capable of describing subgroups relative to the composite group are capable of conceptually subitizing. However, in this study it was important to consider whether the subgroups were described relative to the orientation shown to the student (perceptual subitizing), or if the student was capable of describing the subgroups relative to the composite regardless of the orientation (conceptual subitizing).

The distinction between the two types of subitizing activity is that engagement in conceptual subitizing requires a student to rely upon mental reversibility and more sophisticated number schemes whereas perceptual subitizing is elicited from reliance upon perceptual material. A student's reliance upon mental reversibility is evident when composite groups that a student conceptually subitizes can be composed with subgroups, regardless of orientation, and then decomposed into these conceptual subgroups. A student's reliance upon more sophisticated number schemes is evident when the subgroups described by the student relative to the composite group are not clearly grouped through symmetry, color, or space between items. This definition for conceptual subitizing also suggests that for a student to conceptually subitize she would need to conserve number, but a student capable of conserving number may not necessarily be capable of conceptually subitizing. The remainder of this section will define and describe these seven different types of subitizing activity (Table 5.1).

Table 5.1

Seven Different Types of Subitizing Activity Resulting from this Study

Type	Description
Initial Perceptual Subitizing (IPS)	<ul style="list-style-type: none"> • Students describe the visual motion or the shape of the dots. • Typically a student engaging in IPS activity will not numerically name the items, but describe seeing a “K,” or a “kid” when looking at five dots.
Perceptual Subgroup Subitizing (PSS)	<ul style="list-style-type: none"> • Students numerically subitize small subgroups of two or three, but cannot subitize the entire composite group. • Typically a student engaging in PSS activity will state that she saw “two and three,” or “two plus three,” but does not use this to accurately describe the composite group.
Perceptual Ascending Subitizing (PAS)	<ul style="list-style-type: none"> • Students describe the subgroups and then the composite group. • Typically a student engaging in PAS activity will state that she saw “two and three,” and then state the total composite group.
Perceptual Descending Subitizing (PDS)	<ul style="list-style-type: none"> • Students describe the composite groups and then describe the subgroups. • Typically a student engaging in PDS activity will state that she saw “five” because she saw “two and three.”
Perceptual Counting Subitizing (PCS)	<ul style="list-style-type: none"> • Students initially describe seeing one more or one less than the composite group, and then counts down or up, respectively to the composite group. • Typically a student engaging in PCS activity will state that she saw “4...5” or “6...5.” Once this activity becomes more sophisticated, a student will state that she knows this number to be five because she saw “4...5,” suggesting a coordination of their serial and classification thinking structures.
Rigid Conceptual Subitizer (RCS)	<ul style="list-style-type: none"> • Students describe seeing the composite unit and then one set of subgroups that always remain the same, regardless of the orientation or color of the items. • Typically a student engaging in RCS will always state that she knows a composite group to be four because she saw “two and two.”
Flexible Conceptual Subitizing (FCS)	<ul style="list-style-type: none"> • Students describe seeing the composite unit and then two or more sets of subgroups in different tasks regardless of the orientation or color of the items. • Typically a student engaging in FCS activity will state that she knows a composite group to be five because she saw “two and “three,” but previously she explained the same orientation to be five because she saw “two, two, and one.”

Note. These seven different types of subitizing activity categorically represent the observed student responses both between students and throughout the duration of the study with each student.

Initial perceptual subitizing (IPS) described students who were solely dependent upon empirical activity, as their visual scans or shapes were described by the students when asked to justify or further explain why they thought their solution was accurate. Most often students may not even be aware of why they think their answer is justified, they “just know” the set of items is the same as this number (Table 5.1).

Perceptual subgroup subitizing (PSS) is described as activity in which students are capable of subitizing subgroups as small as one, two, or three, but do not consciously use this activity to numerically describe the composite group (Table 5.1). Activity such as this may indicate reliance upon perceptual subitizing activity, but differs from IPS activity, as the student is capable of chunking visual scans to subitize two groups.

Perceptual ascending subitizing (PAS) and perceptual descending subitizing (PDS) are described as activity in which students’ rely upon orientation or color of items, but are capable of subitizing subgroups and composing a composite group (PAS) or subitizing the composite group and decomposing the composite group into subgroups (PDS) (Table 5.1).

Perceptual counting subitizing (PCS) is described as activity in which students subitize and then count. This activity can be observed as ascending towards a composite group of items or descending towards a composite group of items. As the activity becomes more it offers students a “+1” or a “-1” notion of number (Table 5.1). Rigid conceptual subitizing (RCS) is defined as students who no longer rely on perceptual material when subitizing, but rely on one set of subgroups to compose or decompose number when Subitizing (Table 5.1). Flexible conceptual subitizing (FCS) is defined by a student’s ability to rely on more than one set of subgroups when composing or decomposing number when subitizing (Table 5.1). The

relationships between these various types of activity relative to student activity will be described in latter sections of this chapter.

Addressing the Research Questions

The analysis described in Chapter 4 revealed a strong relation between subitizing activity and students' ability to coordinate their serial and classification thinking structures that students rely on simultaneously when conserving number. As mentioned in Chapter 4, not only did students described as preconservers rely heavily upon perceptual material when subitizing, but also upon the space between items. Also, when Ben, a preconservers, subitized items, it seemed as if the space between the items from one task to another strongly influenced his subitizing ability, as he either stated that he saw relatively one more or one less when the quantity did not change, but the space did change.

Reliance upon items and their orientation relative to the space seemed to influence students' numerical expression when they were not capable of conserving number. This reliance suggests that students not able to conserve number who are subitizing, drew upon number schemes relying heavily upon empirical activity. When students could effectively chunk items into meaningful subgroups, more space was available within their working memory, allowing more accurate subitizing to occur. This is not to say that subitizing solely relies on attentive mechanisms within the working memory. However, for a number scheme to be drawn from long-term memory, encoded material, which results from spatial indexing (Pylyshyn, 2001), it needs to be held in temporary storage in the working memory. So, when students said one number, but re-presented an orientation that did not align with the number stated, this usually occurred because the student preattentively encoded only the space and location of their number

schemes were not fully developed preventing them from effectively utilizing the chunks of items attended to in their working memory.

This study was also meant to increase the scope of Piaget's (1968/1970) definition of topological thinking structures to allow for the application of this thinking structure to include patterned orientations or regular space between items. Sarama and Clements' (2009) hypothetical learning trajectory described students engaging first with patterned sets of items arranged in arrays. The importance of the regular space between these items was not given purpose. However, findings from this study suggest that students initially engage with items arranged in arrays due to the benefits gained when relying on symmetry. The symmetrical aspects of items effectively influenced students in this study when initially chunking items in PAS or PDS activity. This influences both ascending and descending activity at the empirical level, but also raises the students' number schemes to a higher plane. Thus, it is important to consider modifying Piaget's (1968/1970) definition of topological thinking structures to increase the scope of the application to include patterned orientations that allows for students to rely on the symmetrical aspects of these items. The purpose for modifying this thinking structure was to consider how development of the topological thinking structure and number understanding may directly relate to each other.

Symmetry was defined a few different ways when the students engaged in different types of perceptual subitizing activity. Essentially, students relying primarily upon perceptual material when subitizing are more effective when perceiving items arranged in two vertical columns. As students step further from the perceptual material and rely more on different types of reflective abstractions, the symmetrical aspects of the items can be more flexible. For instance, it was noted in the IPS activity that students can only rely on symmetry when there is a

vertical line of symmetry and the items are arranged in two vertical columns. However, when students engage in PAS or PDS activity, symmetry can have a vertical or a diagonal line of symmetry and the items can be grouped as rows, columns, or rows and columns. Thus, findings from this study indicated that as students' perceptual subitizing activity changed from IPS activity towards PAS or PDS activity, they utilized more sophisticated forms of symmetry, suggesting reliance upon more sophisticated forms of abstraction.

The remainder of this chapter will be divided into four sections. Three sections will address the research questions, and the fourth section will be devoted to secondary findings:

1. Does item orientation (e.g., circular, linear, patterned) relate to preschool-age children's subitizing activity? If so, how might patterned orientations, as described by Sarama and Clements (2009), coordinate with number development?
2. How might preschool age children's empirical actions (e.g., counting items, re-presenting items) with a variety of item orientations relate to children's number development and type of abstraction carried into subitizing ability?
3. How might children's present thinking models change as item orientation that children subitize change? How might preschool age children's drawings of remembered subitized items transform due to tasks provoking a need to depend on a more sophisticated understanding of number?

Relationship Between Item Orientation and Subitizing Activity

This section addresses the first research question: Does item orientation (e.g., circular, linear, patterned) relate to preschool-age children's subitizing activity? If so, how might patterned orientations, as described by Sarama and Clements (2009), coordinate with number development? Item orientation (rectangular array, circular, or linear) seemed to relate to

subitizing activity, as students who could conserve number were much more capable of subitizing items in a circular arrangement, but diagonal rows with five or more were difficult for all students who could not engage in some sort of conceptual subitizing activity. This suggests that students are primarily relying upon visual scans, as visual scans tend to move vertically and horizontally. Also, students who subitized circular arrangements inaccurately re-presented the circular orientation, but students who subitized circular arrangements accurately re-presented groups of two, two, and one. Thus, a diagonal row or a circular arrangement of items shown to students would require them to chunk items when subitizing or students would need to simply guess a large number. So, when students accurately subitized four or five items that were in diagonal rows or in circular arrangements they were required to engage any type of activity except IPS and PSS activity.

Students were also most effective when engaging in RCS or FCS activity with items that were in circular arrangements because they needed to effectively chunk items, as engaging in a visual scan required a starting and stopping point. As described earlier, when students were shown rectangular arrays smaller than five, all students could effectively subitize these items regardless of the type of subitizing activity they were engaging in. However, color could effectively support or disrupt students' subitizing activity when orientation emphasized attention towards certain subgroups. For instance, when students relied on symmetry to subitize four dots in a square formation, this could be disrupted when the color did not emphasize the subgroup two. This was illustrated in protocol B11, when Ben was unable to attend to the three and the one as composing four, as he was relying unconsciously on the two and two that the symmetry offered when he subitized four. Thus, Ben had to choose which attribute to attend to and the color seemed to dominate this activity.

This finding also aligns with findings from Whelley's (2002) dissertation study, as the stroop effect was found to disrupt younger students' subitizing activity, as evidenced by the students' inability to coordinate numerical symbols that did not express the number of items in a set. Results from Whelley's (2002) study were explained as younger students' executive functioning being too immature and students being unable to coordinate the items' different attributes. Thus, results from this present study regarding color could be explained as a student's immature executive functioning, and that may support or disrupt perceptual subitizing activity relative to the arrangement of the items the students were subitizing.

So, item orientation seems to be strongly related to subitizing activity. Subitizing activity was also influenced by the students' own understanding of the numbers being subitized. Most often students who were not conserving number could not consciously carry into their subitizing a "+1" or a "-1" logic, but may have been able to engage in PCS activity without being cognizant of a "+1" or "-1" logic. This allowed students to consider an object of four and one more dot to be six, as described earlier by Frank in protocol F5. Once Frank describes this orientation as six, he is not perturbed by "+1" or "-1" type of activity, as his conscious logic for this orientation being six is not based on four and one more or six and one less, but that there are two equal columns that look like what should be six.

Sarama and Clements (2009) describe students' reliance upon orientation when subitizing, as their ability subitizes transitions from rectangular arrays towards circular and linear patterns. Also described in their learning trajectory is the notion that a relatively larger space between items can elicit attention towards subgroups. These aspects of Sarama and Clements (2009) hypothetical learning trajectory were all found to be true in this study, but when space was used to elicit students attention towards subgroups it also prevented some students from

perceptually subitizing the composite group. This was only found to occur in students who had limited number understandings. For instance if Ben could subitize two dots and two more dots because of a large space between the orientations, he would need to know that two and two composed four to be able to conceptually subitize. Similarly, Frank could subitize two and three, but when asked how many he had altogether, he described seeing 23. Thus, item orientation was quite influential in students' subitizing ability, but the students' number schemes needed to also be sophisticated enough to allow for the subitized items to be effectively composed.

Students' Empirical Actions Related to Types of Abstractions Carried Into Subitizing Activity

This section addresses the second research question: How might preschool age children's empirical actions (e.g., counting items, re-presenting items) with a variety of item orientations relate to children's number development and type of abstraction carried into subitizing ability? When students were not able to subitize a set of items, they reverted to either their counting or they said they wanted to make "it." This natural opportunity for students to physically engage with the experiential items supported students with weak number understandings when finding success within that particular task by giving them an opportunity to empirically build an understanding of number.

Throughout this study, the teacher-researcher covered up subgroups of items to scaffold students' early dependence upon types of reflective abstractions. This reliance upon empirical abstractions shifted towards reflective abstractions when students were able to carry their understanding of how subgroups can build toward composite groups in subsequent subitizing activity. This was evidenced in PAS and PDS activity. This transition from students primarily relying upon empirical abstractions towards reflective types of abstractions occurred

more often when students had a more sophisticated counting scheme. This seemed evident when subgroups of items were covered for some students, as they were able to consider subgroups as composing the whole group. However, Ben was not able to use this to help him. Ben only relied on the color of items when engaging in PAS and PDS activity. It seemed as if Ben's relatively weaker counting schemes may have influenced his limited subitizing activity, as color was a dependent factor that supported Ben's PAS and PDS activity.

Also, whenever students were unsure, they reverted to counting out loud quietly. This seemed to improve their counting scheme because they had to point with their finger at the items to count at the beginning of the study, but near the end of the study they seemed to be relying upon some more abstract actions (e.g., verbal utterances, visual scan) when counting and looking at the perceptual material.

Students usually did not need to be prompted to count items, but tended to revert to counting when unsure of their subitizing activity. When students counted out loud, they used items to re-present what they knew, or described covered-up subgroups, students were subsequently able to later subitize similar arrangements and number of items. Thus, it seemed this activity allowed students a physical activity that was more concrete than a visual scan to support their PAS and PDS activity later.

Students who did not need to count as often attempted to use fingers to re-present their subitizing activity, and at times, composed new subgroups when using finger patterns to stand in for number patterns that were subitized. Similarly, these same students were more able to show without counters what they saw when they subitized. Thus, reliance on more abstract actions (e.g., finger patterns) when subitizing or explaining thinking after subitizing was indicative of a

student's reliance upon more reflective types of abstractions, which related to the students' number understanding.

Relationship between Students' Re-presentations of Remembered Subitized Items and Item Orientations

This third section addresses the third research question: How might children's present thinking models change as item orientation that children subitize changed? How might preschool age children's drawings of remembered subitized items transform due to tasks provoking a need to depend on a more sophisticated understanding of number? As described in Chapter 4, students' numerical expression did not always align with what they re-presented, nor did students consistently state an accurate number of items. This misalignment between students' numerical expressions and re-presentations illustrates a number scheme that relies heavily upon the empirical aspects of the items. Many times this was also demonstrated when the students engaging in perceptual subitizing activity simply attempted to fill up the space the items appeared to take up, which also explains reliance upon the empirical aspects of the perceptual material.

When students attended towards subgroups, this activity seemed to also relate directly with the development of a student's classification thinking structure (Inhelder and Piaget, 1964/1999). Whelley (2002) found that conflicting attributes slowed subitizing progress, and prevented students from attending to number when subitizing. This finding is understandable, as Inhelder and Piaget (1964/1999) also describe attributes such as color, shape, and space between items as influential in a child's development of their classification thinking structures. When students consider attributes of items (e.g., color, shape) or topological features between items (e.g., space, symmetry), disrupting one perceived subgroup with a competing attribute can

be distracting to the student when engaging in perceptual subitizing, as the student is not able to coordinate the topological properties in order to effectively group items.

Similarly, students early in their perceptual subitizing activity have difficulty flexibly understanding a subgroup to be part of a composite group or for the composite group to be composed of subgroups. This ascending and descending activity needs to begin by utilizing the empirical attributes or arrangement of the items to emphasize these subgroups. Again, symmetry seemed to substantially influence the students ability to engage in this type of subitizing activity early on, as one subgroup might not need to be coordinated at first for a student to find success in PAS or PDS activity. However, color can also support early attention towards subgroups, as was evident in the case of Ben.

Students' reliance upon item orientations is also indicative of reliance upon a perceptual processing mechanism quite different from counting, as counting does not rely solely upon perceived item orientation. The distinction between these two types of activities were also made visible when students counted items and then subitized items and were not bothered that a different number was reached with each activity. However, students were capable of combining both counting and subitizing when engaged in some of the tasks. This was evident in the PCS activity and that students demonstrated.

Secondary Findings

The most striking finding, which resulted from this study was the seven different types of subitizing activity. These seven different types of subitizing activity supported much of Sarama and Clements's (2009) hypothetical learning trajectory and effectively explained connections between students' subitizing activity and number understanding. These types of

subitizing activity also explained more precisely the boundaries of students' thinking models and changes made to their thinking model as their number understanding became more sophisticated.

Basically, students begin subitizing one, two, and three items which have significantly fewer orientations compared with sets larger than four by engaging in IPS activity. IPS activity explained students' engagement with an empirical activity that supported future subitizing activity. When three items were arranged in a triangular formation or four items were arranged in a square formation, the students unconsciously attended towards the subgroups two and one, or two and two respectively. This PSS activity transitions students towards an awareness of subgroups and is indicative of a students' number understanding. Students may also rely on some PCS activity at this point and combine IPS activity with their counting scheme. This was evident when students stated a number because it triggered a shape that they knew this number to represent, but then, they considered it further and realized they needed to count down or up from that number. Revisiting IPS activity to use the shape of items as a subgroup can further scaffold early PAS activity. Once students are capable of utilizing subgroups to ascend towards the composite group, they will then begin to descend from the composite group to describe subgroups in what is described as PDS activity. This activity engages physical reversibility as students still rely on perceptual material to describe the subgroups.

However, students may transition from PDS towards RCS activity if they have only engaged in ascending PCS activity, as their notion of four or five may only be understood from the symmetrical aspects of four. So, for students to engage in FCS activity they need to revisit PSS, PAS, and PDS activity with symmetrical arrangements of six items. Re-visiting this activity provides students the opportunity to rely on an empirical abstraction of three relative to six, which then supports reliance upon an empirical abstraction of three as relative to five and

four. This type of cyclical activity offers students opportunities to flexibly rely on subgroups when conceptually subitizing.

Some findings resulted from this study were not anticipated. For instance, students capable of conserving number were capable of combining some of their counting and subitizing activity when engaged in PCS activity. This activity further establishes counting and subitizing as two distinct sets of mental activity. For students to conceptually subitize, one activity needs to be leveraged to support the other activity. Students' counting seems to relate to their subitizing activity, but development in subitizing activity seems also to offer students a stronger set of strategies from which to draw when solving counting activities. This PCS activity changed when students transitioned from preconservation of number to conservation of number. When students were not yet able to conserve number, they participated in PCS activity only to reach the composite group. However, when students were capable of conserving number, they used PCS activity to explain or justify why a group of items were accurate. These two different types of PCS activity suggest that when students are not yet able to conserve number, they need to subitize and then count because they are not yet able to coordinate their serial and classification thinking structures. However, when students are capable of conserving number, they have already simultaneously coordinated their serial and classification thinking structures and they use this to explain how they were able to subitize certain items.

Thus, findings from this study suggest that there is a relationship between subitizing activity and number understanding that can be explained more explicitly with different types of perceptual and conceptual subitizing activity. This relationship was often observed when students revisited perceptual subitizing activity relative to the orientation and their understanding of number. Changes in the PCS activity also more explicitly described how students' reliance

upon their ability to coordinate their serial and classification thinking structure may directly relate to subitizing activity.

Remaining Questions and Implications

Some questions resulted from this study that are discussed in this next section. One student's attention to subgroups occurred at the same time that density influenced his conservation of number. Piaget (1941/1965) describes this "density" logic as a rarity, simply a different logic that prevents students from conserving number. However, upon review of the four assessments of Frank's conservation of number, he always considered the pushed together items as having "more." Interestingly, Frank also had a strong affinity towards subgroups and was able to capitalize on this activity to engage in more PAS and PDS activity as compared to Ben. These findings raise some interesting questions, such as: is there a connection between attention towards subgroups and attention towards density when conserving number, or is this logic that students rely on when conserving number simply a rarity as Piaget (1941/1965) describes it to be?

Finally, Frank was unable to conceptually subitize, but was able to count very well, write his numbers, and recognize two digit numbers. Even though Frank was offered multiple advanced strategies with regard to counting objects, counting aloud, and writing his numbers, he struggled with many subitizing activities. It seemed as if Frank was given strategies that did not align with his zone of potential construction (Norton & Ambrosia, 2008), frustrating him at times when he was not sure what to do empirically to solve a task. As the study continued, more in-depth understandings of number were developed by Frank, as compared to Ben, when subitizing. The aspect with which he was still struggling was his inability to engage in any conceptual subitizing activity. However, even though Frank's growth was not as linear as Amy's, it offered

Frank a sound foundation for future mathematical activity. These findings lead to questions, such as the following: did his sophisticated counting and number recognition strategies further this development, or were the differences between their strategies due to physiological development? When is it appropriate to engage students with early counting and when is it inappropriate? How much of this early counting and subitizing influences students' number understanding?

Educational Implications

Implications of this study could inform curriculum and instructional design. Students who were capable of conceptually subitizing needed to be able to conserve number. Yet, students who were conserving number were not necessarily able to conceptually subitize. Thus, it is important to make connections from perceptual subitizing activity to early number understanding in order to explicitly support conceptual subitizing activity in students capable of conserving number. To bridge this gap, students in this study engaged in subitizing activity with symmetrical arrangements and covering up portions of items to use PCS activity to transition from IPS activity towards PSS, PAS, PDS, and FCS activity. With this understanding, it would be important to bridge students' subitizing activity from perceptual to conceptual by designing games that elicit attention towards subgroups with symmetry. When students are given items to subitize, it could be important to consider how to design arrangements which depend upon vertical lines of symmetry toward diagonal lines of symmetry and how these arrangements may change the type of number understanding that students carry into their subitizing activity. These topological transitions also change relative to students' conceptual understandings of number. Also, covering one item to show a student a known arrangement with "one more" could scaffold early PCS activity to introduce a coordination of both counting and subitizing activity, which is

necessary for students to coordinate simultaneously when engaging in conceptual subitizing activity later.

Similarly, designing arrangements of dice and dominoes to elicit attention towards subgroups with color and space between items would further support students' ability to compose and eventually decompose number through subitizing activity. Thus, placing games, and tasks where students need to match subgroups to composite groups, similar to those used in this study, would allow students' subitizing activity to be further developed as they engage in PAS activity. An example would be for students to roll dice and quickly name the number of items seen and re-present these orientations with counters. This type of activity might provoke attention towards subgroups and then the composite group. Students who may be ascending towards a composite group through a naming of subgroups could then play games that require them to begin with a composite unit and find matches that show subgroups. These instructional tools could allow students opportunities to engage in mental reversibility of thinking and begin coordinating their serial and classification thinking structures prior to their ability to conserve number.

Educators more often are limited by the amount of time available when working with young children, as lengthy assessment tools typically reduce time for instruction. Thus, considering how to use game play for both dynamic assessment and intervention tools could inform educators of appropriate instructional strategies that could best challenge and support early construction of number. Utilizing the seven types of subitizing activity (Table 5.1) to categorize games relative to student thinking could appropriately support student interventions and allow educators more effective ways to discuss students' mathematical activity. Naming the type of activity young children engage in when subitizing explicitly informs teachers as to how

best to design activities and games that are meant to support a more thorough comprehension of number.

Subitizing activity is typically underutilized in the classroom, preventing students from constructing an understanding of number through the development of multiple forms of empirical and mental activity. Leveraging subitizing activity in coordination with counting activity might benefit students with limited sets of meaningful educational experiences prior to entering kindergarten. Bridging perceptual subitizing activity to conceptual subitizing activity could then support counting activity. Similarly, scaffolding the reversible mathematical activity that students need to engage in to support later operative thinking in mathematics can happen through PAS and PDS types of activity. Embedding dynamic assessment models in the form of games can inform classroom instruction and prevent students from becoming frustrated in the kindergarten classroom. Appropriately challenging students in mathematics in K-2 programs could better differentiate instruction in mathematics and support students with multiple needs prior to upper elementary formal assessments. Activities designed effectively can both assess students' number understandings, and appropriately challenge student to transition their subitizing activity from perceptual subitizing to conceptual subitizing as they continue to learn and develop.

Research Implications

Results from this study regarding how counting schemes and subitizing activity relates to students' engagement with conceptual subitizing were not directly investigated, but reliance upon number schemes may allow students to leverage subitizing activity to support counting activity. Future research investigating relationships between preschool age students'

development in subitizing and counting would offer a more in-depth understanding of when and how students begin to rely on both of these activities.

Some preliminary connections were also found between students' attention towards density and attention towards subgroups. Investigating these connections might better explain how students develop an ability to conserve number. As an example, Ben was conflicted when choosing whether the triangular orientation of three counters that took up more space had more or less items in it compared to the triangular orientation of three counters that took up less space. This conflict arose in the study at the same time that Ben was also attending towards subgroups and beginning to engage in PSS activity. However Ben's ability to conserve number was not observed frequently enough, and limited the findings in this way.

Summary

By interpreting the findings from this study, it is possible to effectively describe the relationship between students' subitizing activity and students' number understanding through a description of the seven different types of subitizing activity (Table 5.1). Symmetry and PCS activity were found to transition students within perceptual subitizing types of activity to allow students an opportunity to both compose and decompose number prior to their ability to conserve number. As students stepped further and further from the perceptual material towards RCS or FCS activity, it was found to be advantageous for students to revisit perceptual subitizing activity when subitizing larger numbers of items such as six or seven. When students revisited perceptual subitizing activity, they were more capable of FCS activity as a more comprehensive understanding of number results.

Finally, it is important to note that when Piaget's definition for topological thinking structures is modified to increase the scope of the application, allowing for patterned orientations

and symmetry, all three mother structures (classification, serial, and topological) directly influence students' understanding of number when subitizing. However, there seems to be a stronger correlation between children's psychological development and mathematical thinking structures, than earlier noted in Piaget's (1968/1970) *Genetic Epistemology* providing a forum in which to begin describing mathematics as a true psychological expression of each of our own realities. Thus, findings resulting from this study may begin to explain students' cognition and learning in mathematics at an ontogenetic level.

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Appendices

Appendix A: Screening Interview A Sample of Questions

- 1.) Circular counters (ranging between three and seven) were placed in a row in front of the student. Another row with the same number of counters will be placed next to the first row. The row of counters is stretched out so that it is longer than the first row.

The student was told, “this is my row (pointing at the shorter row), and this is your row (pointing at the longer row). “Are they the same?,” “Who has more?”

- 2.) Bottles (ranging between three and seven) filled with water were placed in a row in front of the student.

The student was asked “What shall we need if we want to drink this water.”

(If the student does not know or does not think that we need water glasses, then it was suggested that we use the water glasses sitting beside the researcher. The researcher showed the student more glasses than bottles.)

The student was asked to put on the table “Enough glasses for the bottles, just one for each” and then after the glasses or bottles are bunched together in a shorter row “Are they the same?.”

(Students unable to arrange the glasses so that each bottle has exactly one glass corresponding with it were asked follow up questions such as, “Are there more here?,” “Are they the same?,” and “Is there the same number of glasses and bottles?”)

- 3.) The second conservation task is very similar to the glasses and bottles task, but involves flowers and vases. Vases (ranging between three and seven) were placed in a row in front of the student.

The student was asked “what shall we put into these vases?.”

(If the student did not know or did not think that we needed flowers, then it was suggested that we use the flowers sitting beside the researcher. The researcher showed the student more flowers than vases.)

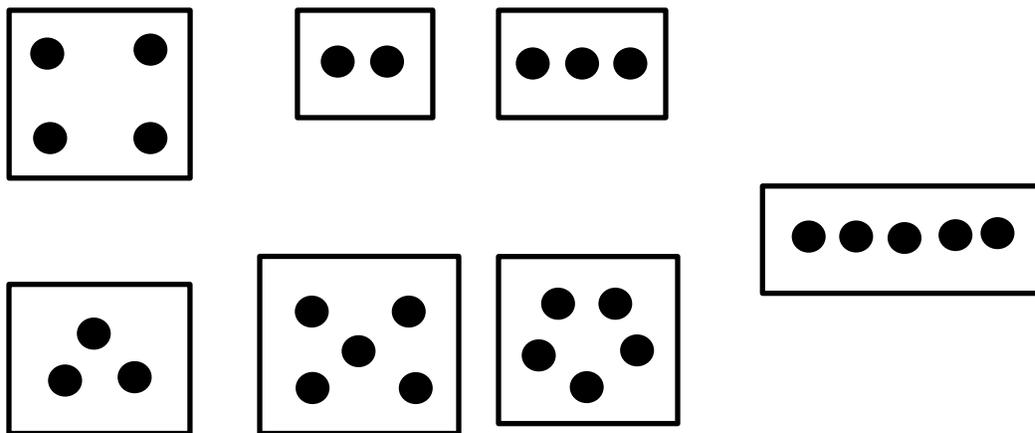
The student was asked to put on the table “One flowers for each vase, as many flowers as vases.”

Students capable of arranging the flowers alongside the vases and also capable of corresponding exactly one flower to one vase were required to describe if “They are the same” after the flowers or vases are bunched together in a shorter row.

(Students unable to arrange the flowers so that each vase has exactly one flower corresponding with it will be asked follow up questions such as, “Are there more here?,” “Are they the same?,” and “Is there the same number of flowers and vases?”)

Appendix B: Screening Interview B Sample of Questions

- 1.) Students were randomly shown one of the following seven arrangements on 4" X 4" cardstock for about 1 second. Each arrangement was framed with a rectangle drawn around the outside of the dots, and shown one at a time. After each card was shown, the student was asked to draw what he/she remembered seeing. The following represent similar orientations of dots shown to each student in this interview.



Students were asked “How many did you see,” and then asked to “Draw what you saw.” Each of the four arrangements were mixed up and shown to the student one at a time. Each time the student was shown an arrangement, she/he was asked to state “How many did you see” and asked to “Draw what you saw.”

- 2.) Students were shown three to seven blocks arranged in a row and asked, “Can you count these _____ (name of items) for me?,” and “Show me.”

If the student was inaccurate in counting due to their verbal counting sequence or their inability to correspond their number words to their pointing, then the student was given a smaller set of blocks (three or four) and asked, “Can you count these _____ (name of items) for me?, and “Show me.”

If the student was accurate in counting the items, then the items were rearranged into a random grouping or increased in quantity.

Appendix C: Overall Analysis of Screening Interviews

ID	G	PreCserv	Cserv	I:I	PerSub	ConsSub	3- line	3-tri	4- square	5- line	5-X	5- circle	Age
01	M		X			X		5 Y. 1 M.
02	F	X		X		X	three	four	5 Y. 2 M.
03	M		X	X		X							5 Y. 5 M.
04 *	F		X	X		X	three	4 Y. 2 M.
05 *	M	X			X		three	five	ten	5 Y. 1 M.
06	M		X	X		X	five	5 Y. 5 M.
07	M		X	X	X		four	four	four	five	5 Y. 5 M.
08	M	X		X		X	three	three	4 Y. 2 M.
09 *	M	X		X	X			three	four	don't know	5 Y. 1 M.
10 *	F		X	X		X	eight	5 Y. 5 M.
11 *	M	X			X		X	four	four	X	3 Y. 11 M.
12 *	M		X	X		X	six	4 Y. 5 M.

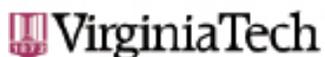
Appendix D: Teaching Experiment Tasks

	Directions	Materials
Draw what you saw	<p>After showing a child a card of dots to be subitized, ask the child to “draw what you saw.”</p> <p>Ask the child, “why” this is what she saw.</p> <p>*Variation: use counters instead of drawing.</p>	<p>*Cards with dots. *Markers/Crayons *Paper *Counter (var.)</p>
Camera game	<p>Explain to the child that you “took a picture of some dots.” (Show the dots in the camera frame to the child for no more than 2 seconds).</p> <p>Ask the child “how many dots she saw.”</p> <p>Then explain that your “camera is broken and that the picture of dots has the same number of dots, but the dots have been moved around.”</p> <p>Ask the child “which picture could be the one you took of the dots.”</p>	<p>*Camera frame *Dot cards</p>
Concentration	<p>Explain to the child that “today we are going to play concentration.”</p> <p>Ask the child to “find all of the matches before we begin to play.”</p> <p>Have six to eight cards facing up and have the child find matches.</p> <p>Ask the child “why the cards match and why they do not match.”</p> <p>Then, flip over the cards and have the child play concentration.</p> <p>*Variation: Use domino cards and have the child match the cards.</p>	<p>*Dot cards *Domino cards (var.)</p>

Board games	Explain to the child that “today we are going to play _____, some dice.”	*Game Board *Two different dice (one for student/one for researcher-teacher).
	(Make sure you have a die with smaller numbers so the child wins the game.)	
	Game board should include some matching spaces.	

Note. Games were either adapted from Clements and Sarama’s (2009) exemplars of activities or developed for this study by the teacher-researcher.

Appendix E: Institutional Review Board Approval Letter



Office of Research Compliance
 Institutional Review Board
 North End Center, Suite 4120, Virginia Tech
 300 Turner Street NW
 Blacksburg, Virginia 24061
 540/231-4606 Fax 540/231-0959
 email irb@vt.edu
 website <http://www.irb.vt.edu>

MEMORANDUM

DATE: May 14, 2013
TO: Jesse L Wilkins, Beth Loveday MacDonald
FROM: Virginia Tech Institutional Review Board (FWA00000572, expires April 25, 2018)
PROTOCOL TITLE: Subitizing Activity: Item Orientation with regard to Number Abstraction
IRB NUMBER: 13-427

Effective May 13, 2013, the Virginia Tech Institutional Review Board (IRB) Chair, David M Moore, approved the New Application request for the above-mentioned research protocol.

This approval provides permission to begin the human subject activities outlined in the IRB-approved protocol and supporting documents.

Plans to deviate from the approved protocol and/or supporting documents must be submitted to the IRB as an amendment request and approved by the IRB prior to the implementation of any changes, regardless of how minor, except where necessary to eliminate apparent immediate hazards to the subjects. Report within 5 business days to the IRB any injuries or other unanticipated or adverse events involving risks or harms to human research subjects or others.

All investigators (listed above) are required to comply with the researcher requirements outlined at:

<http://www.irb.vt.edu/pages/responsibilities.htm>

(Please review responsibilities before the commencement of your research.)

PROTOCOL INFORMATION:

Approved As: **Expedited, under 45 CFR 46.110 category(ies) 6,7**
 Protocol Approval Date: **May 13, 2013**
 Protocol Expiration Date: **May 12, 2014**
 Continuing Review Due Date*: **April 28, 2014**

*Date a Continuing Review application is due to the IRB office if human subject activities covered under this protocol, including data analysis, are to continue beyond the Protocol Expiration Date.

FEDERALLY FUNDED RESEARCH REQUIREMENTS:

Per federal regulations, 45 CFR 46.103(f), the IRB is required to compare all federally funded grant proposals/work statements to the IRB protocol(s) which cover the human research activities included in the proposal / work statement before funds are released. Note that this requirement does not apply to Exempt and Interim IRB protocols, or grants for which VT is not the primary awardee.

The table on the following page indicates whether grant proposals are related to this IRB protocol, and which of the listed proposals, if any, have been compared to this IRB protocol, if required.

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