

Left Orderable Residually Finite p -groups

Camron M. Withrow

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

Peter A. Linnell, Chair
Ezra A. Brown
Joseph A. Ball

December 11, 2013
Blacksburg, Virginia

Keywords: left orderable group, residually finite p -group
Copyright 2013, Camron M. Withrow

Left Orderable Residually Finite p -groups

Camron M. Withrow

(ABSTRACT)

Let p and q be distinct primes, and G an elementary amenable group that is a residually finite p -group and a residually finite q -group. We conjecture that such groups G are left orderable. In this paper we show some results which came as attempts to prove this conjecture. In particular we give a condition under which split extensions of residually finite p -groups are again residually finite p -groups. We also give an example which shows that even for elementary amenable groups, it is not sufficient for biorderability that the group be a residually finite p -group and a residually finite q -group.

Contents

1	Introduction	1
1.1	Motivation	1
1.2	Conjecture	1
2	Elementary Amenable Groups	3
2.1	Split Extensions of Groups	3
2.2	Construction of Elementary Amenable Groups	4
2.3	Examples	5
2.4	Remarks on the Conjecture	6
3	Residually Finite p-Groups	7
3.1	Topological Groups	7
3.2	Pro- p Completion	7
3.3	Definition of Residually Finite p -Groups	9
3.4	Elementary Amenable Residually Finite p -Groups	11
3.5	Generalization of Residually Finite p -Groups	14
4	Orderable Groups	17
4.1	Basic Definitions	17
4.2	Basic Propositions	17
4.3	Biorderability Results	18

Chapter 1

Introduction

1.1 Motivation

In [4], Linnell proves that if a group G is a residually finite p group for at least two distinct primes and G virtually solvable, then G is right (equivalently, left) orderable. It is natural to ask to what extent the hypotheses of the theorem are necessary. Theorem 1 in [4] shows that it does not suffice to consider groups which are residually finite p -groups for a single prime p . However, the condition that G be virtually solvable appears to be somewhat artificial.

The existence of a solvable subgroup of finite index allows Linnell to use induction and leverage the interaction of commutativity with pro p -completions to prove Theorem 2, [4]. Thus, it seems reasonable to consider the generalization to G being elementary amenable. The induction on the derived length of a maximal solvable subgroup of G would be replaced by transfinite induction on G . It also seems likely that pro p -completions could be leveraged, since G would be an extension of a group for which the induction hypothesis holds, by a finitely-generated abelian-by-finite group. These observations serve as the primary motivation and focus for this paper.

1.2 Conjecture

We would like to have had the main result of this paper be an extension of Theorem 2 in [4]. However, as of now this result is still a conjecture.

Conjecture. *Let p and q be distinct primes and let G be an elementary amenable group. If G is a residually finite p -group and a residually finite q -group, then G has a series*

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with $G_i \triangleleft G$ and G_{i+1}/G_i torsion free abelian for all i . In particular G is left orderable.

The results presented in what follows came as attempts at the conjecture above. As we shall see, one possible way to construct the elementary amenable groups makes it natural to consider how the property of being a residually finite p -group interacts with group extensions. It is also natural to consider to what extent the condition that G be residually finite p -group is used in the proof of Linnell's result (Theorem 2, [4]). These ideas are explored in sections 3.4 and 3.5 of this paper.

In Chapter 2 we present some background material on semidirect products of groups, and on elementary amenable groups (in our presentation, we are assuming standard first year graduate algebra). Chapter 3 begins with an introduction to residually finite p -groups in sections 3.1 - 3.3. The major results on the interaction of the property of being a residually finite p -group with group extensions and orderability of groups are contained in sections 3.4 and 3.5. Chapter 4 discusses the property of orderability showing which results in this paper imply orderability, along with a brief consideration of the question of biorderability.

Chapter 2

Elementary Amenable Groups

2.1 Split Extensions of Groups

As we shall see shortly, the definition of elementary amenable groups involves considering extensions of certain classes of groups. We will give two definitions, and some examples, of the simplest nontrivial type of group extension we wish to consider, a split extension of groups.

Recall that an exact sequence

$$1 \longrightarrow A \longrightarrow G \xrightarrow{f} B \longrightarrow 1$$

is called **split exact** provided there is a homomorphism $s : B \rightarrow G$ such that $f \circ s : B \rightarrow B$ is the identity homomorphism.

Definition 2.1.1. Let H, K, G be groups with a short exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

which is split exact. Then we write $G = K \rtimes H$, and say G is the **semidirect product** of K and H .

The definition above is useful when one wishes to prove something about a semidirect product of groups. However if one wishes to give examples, a constructive definition is useful.

Definition 2.1.2. Let H, K be groups and $\varphi : H \rightarrow \text{Aut}(K)$ be a homomorphism. Define $K \rtimes_{\varphi} H$ as $\{(k, h) : k \in K, h \in H\}$ with product given by

$$(k_1, h_1) \cdot (k_2, h_2) = (k_1(\varphi(h_1)(k_2)), h_1 h_2).$$

Then $G = K \rtimes_{\varphi} H$ is called the semidirect product of K with H .

Remark. The two definitions above are equivalent in the following sense: For $G = K \rtimes H$, let $\pi : H \rightarrow G$ be the splitting homomorphism. Define $\varphi : H \rightarrow \text{Aut}(K)$ by $\varphi(h)(k) = \pi(h)k(\pi(h)^{-1})$. Then $G \cong K \rtimes_{\varphi} H$.

Conversely, if $G = K \rtimes_{\varphi} H$, the sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

is exact, and $\pi : H \rightarrow G$ defined by $\pi(h) = (1, h)$ is a splitting homomorphism. Therefore, $G = K \rtimes H$ (as in the first definition).

Example.

1. For any semidirect product $G = K \rtimes_{\varphi} H$ where $\varphi = \text{id}_K$, $G \cong K \times H$; the direct product.
2. Consider $G = \mathbb{Z} \rtimes \mathbb{Z}$. There are two automorphism of \mathbb{Z} : the identity map, and the multiplication by -1 map. The first gives $G \cong \mathbb{Z} \times \mathbb{Z}$. The other is a nonabelian group; we reserve $\mathbb{Z} \rtimes \mathbb{Z}$ for this group.

There are several easy observations one can make from the definitions above. For example, if $G = K \rtimes H$, then H is isomorphic to a subgroup of G and K is a normal subgroup of H . The way in which we use the condition that G is the semidirect product of K and H (see Theorem 3.4.2) is that in the sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$$

the splitting homomorphism provides an action of H on K given by conjugation in G (see the remark above).

2.2 Construction of Elementary Amenable Groups

We present two ways to define the class of elementary amenable groups. The first is existential and the second constructive.

Definition 2.2.1. The class of **elementary amenable groups**, denoted by \mathcal{E} , is the smallest class of groups containing all abelian and all finite groups which is closed under the forming of group extensions and directed unions.

The only clear advantage of this definition is that it makes it clear what this class **must** contain (finite, abelian, solvable, polycyclic, nilpotent, etc..). Aside from that, the non-constructive nature of this definition makes it all but useless in any attempt to prove our conjecture by methods similar to those used in [4].

The second construction, found in [2], is much more convenient for us. First some notation: for a classes of groups \mathcal{A}, \mathcal{B} , the class \mathcal{AB} is the class of groups G which can be written as

$$1 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 1$$

where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The class $L\mathcal{A}$ is the class of groups which are *locally* in the class \mathcal{A} , that is all finitely generated subgroups are in the class \mathcal{A} . Lastly, we will reserve \mathcal{B} for the class of all virtually finitely generated abelian groups.

We are now ready to give the second characterization of the elementary amenable groups.

Definition 2.2.2. The class of elementary amenable groups can be constructed inductively as follows: let $\mathcal{E}_0 = \{1\}$, that is the class of groups consisting of only the trivial group. For α a successor ordinal, let $\mathcal{E}_\alpha = L\mathcal{E}_{\alpha-1}\mathcal{B}$. For α a limit ordinal, define $\mathcal{E}_\alpha = \bigcup_{\beta < \alpha} \mathcal{E}_\beta$.

The class of elementary amenable groups \mathcal{E} is then given by $\mathcal{E} = \bigcup_{\alpha \geq 0} \mathcal{E}_\alpha$.

So, for example, \mathcal{E}_1 is the class of all virtually finitely generated abelian groups.

2.3 Examples

In the examples below, we use the constructive definition to prove that groups of the specified type are elementary amenable (with the sole exception of the remark following example 2.).

1. Finite groups: Note that any finite group is abelian-by-finite, hence all finite groups are in the class \mathcal{E}_1 .
2. Solvable groups: By considering derived length, induction shows that all solvable groups are in \mathcal{E}_ω (where ω is the first infinite ordinal).

Remark. The first construction of elementary amenable groups given above shows that virtually solvable groups are elementary amenable.

3. An elementary amenable group which is not virtually solvable: Consider the group G of permutations of \mathbb{Z} with finite support. Note that we may write G as $\bigcup_{n=1}^{\infty} G_n$ where $G_n \cong S_n$ and $G_{n+1} \supset G_n$ for $n = 1, 2, \dots$. This implies G is locally finite and hence elementary amenable. Furthermore, suppose H were a solvable subgroup of finite index in G . Then since S_n contains no nontrivial abelian normal subgroup for $n > 4$, $S_n \cap H = 1$ and so S_n would embed in G/H for all n ; this is a contradiction since G/H was supposed to have been finite.

Remark. We can modify the group G above so that it is also finitely generated. Let $\tilde{G} = G \rtimes \mathbb{Z}$ where $1 \cdot g$ is the shift operator applied to g (that is, view G as sequences with i th component the difference between i and $g(i)$). Note that \tilde{G} is generated by $((1, 0, 0, \dots), 0)$ and $((0, 0, 0, \dots), 1)$.

2.4 Remarks on the Conjecture

Let us recall the main conjecture:

Conjecture. *Let p and q be distinct primes and let G be an elementary amenable group. If G is a residually finite p -group and a residually finite q -group, then G has a series*

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with $G_i \triangleleft G$ and G_{i+1}/G_i torsion free abelian for all i . In particular G is left orderable.

If we consider the construction of the elementary amenable groups given above, we see that it may be possible to approach the conjecture using transfinite induction on the classes \mathcal{E}_α . For α a limit ordinal, if $G \in \mathcal{E}_\alpha$, then $G \in \mathcal{E}_\beta$ for some $\beta < \alpha$, which by induction shows the result holds for G . Thus the successor ordinal case presents the only difficulty.

For α a successor ordinal, $G \in \mathcal{E}_\alpha$ means G has a subgroup A such that A is locally \mathcal{E}_α and G/A is finitely generated abelian-by-finite. As we shall see, if G is a residually finite p -group, then A is also a residually finite p -group. It does not necessarily follow that G/A is a residually finite p -group. Finding conditions under which it is true that G/A is a residually finite p -group is the main focus of the results in the next chapter.

Chapter 3

Residually Finite p -Groups

3.1 Topological Groups

Definition 3.1.1. A group G with a topology is a **topological group** provided the operations $G \rightarrow G : g \mapsto g^{-1}$ and $G \times G \rightarrow G : (g, h) \mapsto gh$ are continuous (where $G \times G$ is given the product topology).

Equivalently, G is a topological group provided the map $G \times G \rightarrow G : (g, h) \mapsto gh^{-1}$ is continuous.

For any finite group G it will be assumed that G has the discrete topology.

For a good survey of the basic topological properties of topological groups, see [8]. These properties are at work in the background in the material that will be presented in the next few sections. However, we will not need to consider these purely topological properties for our purposes.

The next two sections define the pro- p completion functor and prove the basic properties of residually finite p -groups.

3.2 Pro- p Completion

We begin with a brief discussion of inverse limits in the category of groups. In this section we closely follow [8].

Definition 3.2.1. Let I be a directed set (i.e. (I, \leq) is a partially ordered set with the additional property that for any pair $\alpha, \beta \in I$, there is $\gamma \in I$ such that $\alpha, \beta \leq \gamma$). An **inverse system** of (topological) groups is a collection $\{G_\alpha : \alpha \in I\}$ of (topological) groups

along with (continuous) homomorphisms $\varphi_{\alpha\beta} : G_\beta \rightarrow G_\alpha$ for each pair (α, β) where $\alpha \leq \beta$ such that for $\alpha \leq \beta \leq \gamma$: $\varphi_{\alpha\gamma} = \varphi_{\alpha\beta} \circ \varphi_{\beta\gamma}$.

Definition 3.2.2. Let G be a (topological) group and $\{G_\alpha : \alpha \in I\}$ an inverse system of (topological) groups. A family of (continuous) homomorphisms $\{\varphi_\alpha : G \rightarrow G_\alpha : \alpha \in I\}$ will be called **compatible** provided for each $\beta \geq \alpha$, $\varphi_\alpha = \varphi_{\alpha\beta} \circ \varphi_\beta$.

Definition 3.2.3. An **inverse limit** of the inverse system $\{G_\alpha : \alpha \in I\}$ is a (topological) group G along with a family of compatible maps $\{\varphi_\alpha : \alpha \in I\}$ with the following universal property: for any family of compatible maps $\{\psi_\alpha : H \rightarrow G_\alpha : \alpha \in I\}$ there is a unique (continuous) homomorphism $\psi : H \rightarrow G$ such that $\psi_\alpha = \varphi_\alpha \circ \psi$ for each $\alpha \in I$.

It is an easy exercise to show that any two inverse limits of a given inverse system are (topologically) isomorphic. With this result in mind, we will make a slight abuse of language and talk about *the* inverse limit of a given inverse system of topological groups. This slight oversight should not cause any confusion in what follows.

The pro- p completion of a group G will be the inverse limit of a particular inverse system of groups: let (I, \preceq) be the collection of all normal subgroups of G which have p th-power index in G ordered by reverse inclusion (hence, I is a directed set). Consider the collection $\{G/N : N \in I\}$ with the canonical maps, for $N \preceq K$, $\varphi_{NK} : G/K \rightarrow G/N$ defined by $\varphi_{NK}(gK) = gN$. Then $\{G/N : N \in I\}$ is an inverse system of groups. Giving each group G/N the discrete topology, and G the weak topology induced by the canonical maps $\phi_N : G \rightarrow G/N$, we may view $\{G/N : N \in I\}$ as an inverse system of topological groups.

Definition 3.2.4. The **pro- p completion** of a group G , denoted \widehat{G} , is the inverse limit of the inverse system $\{G/N : N \in I\}$ described above.

Remark. Notice, in our notation we do not indicate the prime p . This should not cause confusion since it will be clear from context which prime p is being considered.

The definition above may be troubling since we have not shown that inverse limits always exists in the category of (topological) groups, however in this case we can construct the inverse limit explicitly.

Consider the set $\{x \in \prod_{N \in I} G/N : \varphi_{NK} \circ \pi_K(x) = \pi_N(x) \text{ for all } N \preceq K\}$ where $\pi_H : \prod_{N \in I} G/N \rightarrow G/H$ (for $H \in I$) is the standard projection map. Notice that the set above is nonempty since $(\phi_N(g))_{N \in I}$ is an element of the set for all $g \in G$. Furthermore, all the maps $\varphi_{NK}, \pi_N, \pi_K$ above are continuous homomorphisms, therefore the set above is a topological group with product inherited from $\prod_{N \in I} G/N$ and given the subspace topology.

Proposition 3.2.1. *The group \mathcal{G} defined above along with the family of compatible maps $\{\pi_N : N \in I\}$ is the inverse limit of the inverse system $\{G/N : N \in I\}$ with continuous homomorphisms $\varphi_{NK} : G/K \rightarrow G/N$ whenever $N \preceq K$.*

Proof. We need only check that the pair $(\mathcal{G}, \{\pi_N : N \in I\})$ satisfies the universal property. Let $\{\psi_N : G' \rightarrow G/N\}$ be a family of compatible maps. Note that $(\psi_N(g))_{N \in I} \in \mathcal{G}$ for all $g \in G'$. Define $\psi : G' \rightarrow \mathcal{G}$ by $\psi(g) = (\psi_N(g))_{N \in I}$. Then ψ is a continuous homomorphism since each ψ_N is a continuous homomorphism. Furthermore, if $\psi' : G' \rightarrow \mathcal{G}$ is another map for which $\pi_N \circ \psi' = \psi_N$, then

$$\pi_N(\psi'(g)) = \psi_N(g)$$

Hence, $\psi'(g) = (\psi_N(g))_{N \in I} = \psi(g)$. Therefore ψ is the unique continuous homomorphism with the property $\pi_N \circ \psi = \psi_N$. \square

Proposition 3.2.2. *The pro- p completion functor is right-exact; that is, given a short exact sequence of topological groups*

$$1 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 1$$

there is a corresponding exact sequence for the pro- p completions

$$\widehat{A} \longrightarrow \widehat{G} \longrightarrow \widehat{B} \longrightarrow 1$$

Proof. We will require [1, Corollary 10.3]: Let

$$1 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 1$$

be an exact sequence of groups, and G_n be a sequence of normal subgroups of G . Give A and H the topologies induced by the G_n . Then

$$1 \longrightarrow \widetilde{A} \longrightarrow \widetilde{G} \longrightarrow \widetilde{H} \longrightarrow 1$$

(where \widetilde{A} is the inverse limit of $\{A/(A \cap G_n)\}$, \widetilde{G} and \widetilde{H} defined similarly) is exact.

We note that if we let $\{G_n\}$ be the sequence of normal subgroups of G with p th-power index, then \widetilde{G} is the pro- p completion of G , \widehat{G} . Similarly, $\widetilde{H} = \widehat{H}$. However, \widetilde{A} is not in general \widehat{A} since $\{A \cap G_n\}$ will be only some of the normal subgroups of p th-power index in A . Therefore, there is a map $\widehat{A} \rightarrow \widetilde{A}$ (specifically, a continuous homomorphism). Since \widehat{A} is compact, its image, I , in \widetilde{A} is compact. Furthermore, \widetilde{A} is Hausdorff, therefore I is closed in \widetilde{A} . Furthermore, since I contains the image of A in \widetilde{A} , I is dense in \widetilde{A} . Thus, $\widehat{A} \rightarrow \widetilde{A}$ is surjective which implies

$$\widehat{A} \longrightarrow \widehat{G} \longrightarrow \widehat{H} \longrightarrow 1$$

is exact. \square

3.3 Definition of Residually Finite p -Groups

In the definition of the inverse limit of an inverse system of (topological) groups, there is a universal property which asserts the existence of a (continuous) homomorphism from a group H to the inverse limit, provided H is equipped with a compatible family of maps.

Specializing to the pro- p completion of a group G , we observe that G itself has the compatible family of maps $\{\phi_N : G \rightarrow G/N : N \in I\}$, the canonical maps. Therefore, there is a continuous homomorphism $j : G \rightarrow \widehat{G}$.

Proposition 3.3.1. $j(G)$ is dense in \widehat{G} .

Remark. This proposition justifies the use of the word “completion” in the topological sense.

Proof. Note that by construction $\phi_N(j(G)) = G/N$ for each $N \in I$. Therefore, by, [8, Proposition 1.1.6], $j(G)$ is dense in \widehat{G} . \square

Definition 3.3.1. A group G is called a **residually finite p -group** if the map $j : G \rightarrow \widehat{G}$ defined above is an injection.

It is quite easy to see subgroup inheritance from the definition above:

Proposition 3.3.2. Let G be a group, $H \leq G$. Then G is a residually finite p -group implies H is a residually finite p -group.

Proof. Consider the following diagram of maps:

$$\begin{array}{ccccc} 1 & \longrightarrow & H & \xrightarrow{i} & G \\ & & \downarrow j_H & & \downarrow j_G \\ & & \widehat{H} & \xrightarrow{\widehat{i}} & \widehat{G} \end{array}$$

where i is the inclusion map, and j_H, j_G are the natural maps from H and G to their pro- p completions. Since $j(H)$ is dense in \widehat{H} , we can lift the continuous map $j_G \circ i : H \rightarrow \widehat{G}$ to a map, denoted \widehat{i} , from \widehat{H} to \widehat{G} such that the diagram commutes. Then $\widehat{i} \circ j_H$ is injective since $j_G \circ i$ is, which implies j_H is injective. That is, H is a residually finite p -group. \square

There is a group theoretic characterization of the residually finite p -group property; this is the form we will usually use in what follows.

Proposition 3.3.3. A group G is a residually finite p -group if and only if the intersection of all normal subgroups with p th-power index in G is trivial.

Proof. Write \widehat{G} as \mathcal{G} defined in section 3.2 above. Then $j : G \rightarrow \widehat{G}$ is defined by $j(g) = (\phi_N(g))_{N \in I}$. Suppose $\bigcap_{N \in I} N = 1$. Then $j(g) = 1$ implies $\phi_N(g) = 1$ for all $N \in I$, from which it follows that $g = 1$.

Conversely, suppose j is injective and let $g \in \bigcap_{N \in I} N$. Then $\phi_N(g) = 1$ for all $N \in I$ which implies $j(g) = 1$. However, j injective implies $g = 1$. \square

We now list some examples of residually finite p -groups:

Example.

1. Finite p -groups: The trivial group is a normal subgroup of p th-power index, so clearly the intersection over all such normal subgroups is trivial. Furthermore, a finite p -group is a residually finite q -group exactly when $q = p$.
2. Abelian groups: For any abelian group A , the collection $\{p^n A : n \in \mathbb{N}\}$ is a collection of normal subgroups of p th-power index in A with $\bigcap_{n \in \mathbb{N}} (p^n A) = 1$. Thus A is a residually finite p -group for all primes p .
3. $G = \mathbb{Z} \rtimes \mathbb{Z}$ where G is generated by a, b with $bab^{-1} = a^{-1}$: Note that G is abelian by finite. Indeed, the subgroup generated by $(1, 2)$ has index 2 and is isomorphic to \mathbb{Z} (and is characteristic in G). Therefore G is a residually finite 2 group. However, G is not a residually finite p group for any other prime p (this follows from Theorem 3.4.1 and Proposition 3.3.4).

The next result is an easy application of the results above. However, later we will note that a necessary condition for a group to be orderable is that it is torsion free. In this regard, the proposition has some interest.

Proposition 3.3.4. *Let G be a group which is a residually finite p -group and a residually finite q -group for distinct primes p and q . Then G is torsion-free.*

Proof. Suppose G has torsion and let $1 \neq H \leq G$ with $|H| < \infty$. Then H is also a residually finite p -group and a residually finite q -group. However, as we saw in example 1. above, H must be a p group and a q group which implies $H = 1$; a contradiction. \square

3.4 Elementary Amenable Residually Finite p -Groups

This section, it can be reasonably argued, is rather poorly named; we do not prove a single result about general elementary amenable groups. Instead, we are content to prove some theorems concerning solvable groups with the additional property that they are residually finite p -groups. We hope that some of these results can be extended to elementary amenable groups in general.

The following results, while basic, will be used to great effect throughout the rest of this paper.

Theorem 3.4.1. *Let G be a residually finite p -group for some prime p , and let A be a maximal abelian subgroup of G . Then G/A is a residually finite p -group*

Proof. Since $\widehat{}$ is right-exact, we have the following exact sequence:

$$\widehat{A} \longrightarrow \widehat{G} \longrightarrow \widehat{G/A} \longrightarrow 1$$

Let B denote the image of \widehat{A} in \widehat{G} . Then since A is maximal abelian, $B \cap G = A$, and so G/A is isomorphic to a subgroup of $\widehat{G/A}$. Therefore, G/A is a residually finite p -group. \square

Corollary 3.4.1. *Let G be a solvable, residually finite p -group for some prime p . Then \widehat{G} is solvable.*

Proof. Let A be a maximal abelian subgroup of G . Then by the theorem above, G/A is a residually finite p -group with Hirsch number strictly less than G . So by induction, G/A is solvable, which implies that G is solvable (since the class of solvable groups is extension closed). \square

Corollary 3.4.2. *Let G be a residually finite p -group for some prime p , and let A be a maximal solvable subgroup of G . Then G/A is a residually finite p -group.*

Proof. Note, in the theorem above, the only property of A which was used was \widehat{A} was in the same class of subgroups as A (i.e. A abelian implied \widehat{A} also abelian). Thus, by the corollary above, A solvable implies \widehat{A} solvable (since A is a residually finite p -group). With this replacement, the proof of the theorem above is left otherwise unchanged. \square

Before stating the next results, we recall the definition of a group ring.

Definition 3.4.1. Let G be a group and R a ring. The group ring RG is an R -algebra with basis given by the elements of G , and multiplication induced by the multiplication in G . Namely

$$\left(\sum_{i=1}^n r_i g_i \right) \left(\sum_{j=1}^n r_j g_j \right) = \sum_{i,j=1}^n (r_i r_j) (g_i g_j).$$

The theorem that we present gives a way to “bootstrap” the property of being a residually finite p -group from subgroups and quotients given the group G and its subgroup/quotient structure is “nice” (this will be made more precise below).

Theorem 3.4.2. *Let $G = A \rtimes H$ where A, H are finitely generated free abelian, and let p be a prime. Suppose $\mathbb{Q}A$ is irreducible as a $\mathbb{Q}H$ -module, and that $p \mid |A/[A, H]|$. Then there exist $A_i \triangleleft G$ with $A_0 = A$, A_i/A_{i+1} a finite p -group for all i , $\bigcap A_i = 1$, and $[A_i, H] \subseteq A_{i+1}$ for all i . In particular, G is a residually finite p -group.*

Remark. The hypothesis that $\mathbb{Q}A$ be an irreducible $\mathbb{Q}H$ -module cannot be removed. For example, consider a group $G = (A \oplus B) \rtimes H$ with $p \mid |A/[A, H]|$ and $B = [B, H]$. Then if it were possible to find groups $(A \oplus B)_i$ such that $[(A \oplus B)_i, H] \subseteq (A \oplus B)_{i+1}$, each $[(A \oplus B)_i, H] \supseteq [B, H] = B$, so $\bigcap_i (A \oplus B)_i \supseteq B \neq 1$.

Proof. Consider

$$A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \supseteq [A, H] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \supseteq [A, {}_2H] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cdots \quad (3.1)$$

First note that each of the quotients $[A, {}_iH] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is finite. Indeed, suppose one of the quotients were infinite. Then

$$[A, {}_iH] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \subseteq [A, H] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \subsetneq A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)},$$

and $[A, {}_iH] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \neq 1$ (note that the condition $p \mid |A/[A, H]|$ implies $A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \neq [A, H] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$). Furthermore, the quotient $[A, {}_iH] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} / [A, {}_{i+1}H] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is not torsion (it is finitely generated as a \mathbb{Z} -module, and hence has nonzero $\mathbb{Z}_{(p)}$ -rank.) Therefore, $[A, {}_iH] \otimes_{\mathbb{Z}} \mathbb{Q}$ is a proper $\mathbb{Q}H$ -submodule of $\mathbb{Q}A$; a contradiction.

In order to show that the containments in equation (3.1) are all proper, we introduce additive notation for $[A, H]$. Let I be the ideal (of $\mathbb{Z}_{(p)}$) generated by elements $h - 1$ for $h \in H$, and p . Then $[A, {}_mH] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = AI^m$. Suppose, for some n , $AI^n = AI^{n+1}$. Then there is a k such that $p^k A \subseteq AI^n$ and by Nakayama's lemma, there is an $\alpha \in \mathbb{Z}_{(p)}H$ congruent to 1 mod I such that

$$(AI^n)\alpha = 0.$$

Therefore $(p^k A)\alpha = 0$ which implies $A\alpha = 0$. Therefore $A\alpha \subseteq AI$. But $\alpha \equiv 1 \pmod{I}$ implies $\alpha = i + 1$ for some $i \in I$. Therefore

$$a\alpha = ai + a \equiv a \pmod{I}$$

for all $a \in A$ which implies $A = 0$; a contradiction.

Let $N = \bigcap_i [A, {}_iH] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Then by the observations above, we have $[A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} : N] = \infty$. Suppose $N \neq 1$, then $N \otimes_{\mathbb{Z}} \mathbb{Q}$ is a nontrivial $\mathbb{Q}H$ -submodule of $\mathbb{Q}A$; a contradiction. Therefore $N = 1$. Intersect each term in equation (3.1) with A to obtain a sequence

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots, \quad (3.2)$$

with A_i/A_{i+1} a finite p -group, $A_i \triangleleft G$, $[A_i, H] \subseteq A_{i+1}$, and $\bigcap A_i = 1$.

In order to show that G is a residually finite p -group, consider the quotient G/A_1 . Note that $G/A_1 \cong A_0/A_1 \times G/A_0$ is abelian (since $[A_0, H] \subseteq A_1$) and so $G/A_1 \cong A_0/A_1 \times G/A_0$. Therefore, there exists $N \triangleleft G$ with p th power index in G and so $G'G^{p^n} \subseteq G$ (where G^{p^n} is the p^n divisible subgroup of G .) Then $G/G'G^{p^n}$ is an abelian p -group with $G'G^{p^n}$ char G . Since $N \cap A_0 = A_1$, we have that $N/A_1 \cong H$ and $A_1G'G^{p^n}/A_1$ has finite index in N/A_1 . Therefore $A_1G'G^{p^n}/A_1 \cong H$. Hence $A_1G'G^{p^n}/A_2 \cong A_1/A_2 \times H$ and so the process above can be repeated (applied to $A_1G'G^{p^n}$.) Hence we arrive at a chain of subgroups, each characteristic in the preceding, all of p th-power index in G . Since the chain in equation (3.2) intersects trivially, this chain will also intersect trivially and so G is a residually finite p -group. □

Remark. The theorem above can be extended so that one need only assume H is torsion-free abelian. Combining the proof above with the following lemma yields the general result:

Lemma 3.4.1. *Let A be a finitely generated free abelian group, and let H be an abelian group which acts on A . Let K denote the kernel of the action of H on A . Then H/K is a finitely generated subgroup of $\text{Aut}(A) \cong \text{GL}_n(\mathbb{Z})$.*

This shows that, even in the case where H is not finitely generated, the quotient H/K , where K is the kernel of the action of H on A , is finitely generated.

3.5 Generalization of Residually Finite p -Groups

In this section, we consider a generalization residually finite p -group, and prove some theorems analogous to earlier results concerning orderability of these “generalized residually finite p -groups.”

Definition 3.5.1. We define $\rho_p(G)$ to be the intersection of all normal subgroups N of G with the index of N in G a power of p , i.e.

$$\rho_p(G) = \bigcap \{N : N \triangleleft G, [G : N] = p^\alpha \text{ for some } \alpha \in \mathbb{N}\}$$

By $\rho_p^n(G)$, we mean the n -fold composition of ρ_p with itself:

$$\rho_p^n(G) = \bigcap \{N : N \triangleleft \rho_p^{n-1}(G), [\rho_p^{n-1}(G) : N] = p^\alpha \text{ for some } \alpha \in \mathbb{N}\}.$$

It is clear from the definition that G is a residually finite p -group if and only if $\rho_p(G) = 1$. Furthermore, $G/\rho_p(G)$ is always a residually finite p -group. In the theorems that follow, we assume that G is expressible in terms of groups from a particularly nice class (free abelian and nilpotent specifically) and that $\rho_p^n(G) = 1$ for some positive integer n , and conclude that G has a normal series of the form $1 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G$ with torsion-free abelian factors. In this sense, these results directly generalize earlier results by Linnell, and show to what extent the hypothesis of being a residually finite p -group is used.

First we state (without proof) a technical lemma which will be used in the proof of the next theorem.

Lemma 3.5.1. *Let G be a residually finite p -group and let H be a finite normal subgroup of G . Then G/H is also a residually finite p -group.*

Theorem 3.5.1. *Let p, q be distinct primes, and let G be a group with a finitely generated abelian normal subgroup with finite index in G (that is, $A \triangleleft G$ with A finitely generated abelian, and $[G : A] < \infty$). Suppose there is a positive integer n such that $\rho_p(G) = \rho_q^n(G) = 1$. Then G has a series $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ with $G_i \triangleleft G$ and G_{i+1}/G_i torsion-free abelian for all i .*

Proof. We may assume G/G' is finite. Indeed, suppose that G/G' were infinite with torsion subgroup T/G' . Then T is characteristic in G , so we may as well start our series above with $G \triangleright T \triangleright \cdots$. Note that $\rho_p(T) = 1 = \rho_q^n(T)$, and $A \cap T$ is an abelian subgroup of finite index in T with strictly smaller free rank than that of A . Since A is finitely generated, after finitely many steps we arrive at a group N such that $G \triangleright T \triangleright \cdots \triangleright N$ with each subgroup normal in G , each quotient torsion-free abelian, and N/N' finite.

Let n be the smallest positive integer such that $\rho_q^n(G) = 1$ and set $Q = \rho_q^{n-1}(G)$; without loss of generality $n > 1$. Then $\rho_p(Q) = \rho_q(Q) = 1$ which implies Q is abelian. Indeed, since G is polycyclic-by-finite, G has the maximal condition on subgroups. Therefore, since Q has a normal abelian subgroup of finite index, namely $A \cap Q$, Q has a maximal normal abelian subgroup M with finite index in Q . By theorem 3.4.1, Q/M is a residually finite p -group and a residually finite q -group which by proposition 3.3.4 implies Q/M is torsion-free. Therefore $Q/M = 1$ which implies Q is abelian.

Let M be a maximal abelian subgroup of G containing Q . Since G is a residually finite p -group, we see that G/M is also a residually finite p -group. Let T/Q denote the torsion subgroup of M/Q . Then G/T is a residually finite q -group by Lemma 3.5.1. Furthermore M/T is a finitely generated torsion-free abelian group, and is therefore in particular a residually finite p -group. Therefore $\rho_p^n(G/T) = 1 = \rho_q(G/T)$ (in fact, $\rho_p^2(G/T) = 1$ since $\rho_p(G) \subseteq M$). G/T cannot be finite (in fact, it is torsion free by Proposition 3.3.4), so the result follows by induction. \square

This result can be generalized to the case G nilpotent-by-finite using a similar argument. It is important to note that in this case, we need the added condition that G is polycyclic-by-finite so that G has the maximal condition on subgroups.

Theorem 3.5.2. *Let p, q be distinct primes, and let G be a polycyclic-by-finite group with a finitely-generated nilpotent subgroup of finite index. Suppose there is a positive integer n such that $\rho_p(G) = \rho_q^n(G) = 1$. Then G has a series $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ with $G_i \triangleleft G$ and G_{i+1}/G_i torsion-free abelian for all i .*

Proof. As in the proof above, we let A denote a nilpotent subgroup of G with finite index in G . Again, we may assume G/G' is finite. In this case, if T/G' is the torsion subgroup of G/G' , $A \cap T$ is a nilpotent subgroup of finite index in T with strictly smaller Hirsch length than that of A .

Let n be the smallest positive integer such that $\rho_q^n(G) = 1$ and set $Q = \rho_q^{n-1}(G)$. Then Q is nilpotent; this follows since nilpotent groups are solvable, and so Corollary 3.4.1 applies.

Let N be a maximal nilpotent subgroup of G containing Q . Then G/N is a residually finite p -group. Let T/Q denote the torsion subgroup of N/Q . Then G/T is a residually finite q -group by Lemma 3.5.1 (here we needed that all subgroups of G are finitely generated, in particular that T is finitely generated). Furthermore, N/T is torsion-free nilpotent, and so in

particular is a residually finite p -group (see [3]). Therefore $\rho_p^n(G/T) = 1 = \rho_q(G/T)$ (in fact, $\rho_p^2(G/T) = 1$ since $\rho_p(G) \subseteq N$). G/T cannot be finite, so the result follows by induction. \square

Remark. In the next chapter we will show that any group G with a series of the form

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

such that G_{i+1}/G_i torsion free abelian for all i is left orderable.

Chapter 4

Orderable Groups

4.1 Basic Definitions

Definition 4.1.1. A group G is **left orderable** means there is a total order \leq on G for which $a \leq b$ implies $ga \leq gb$ for all $g \in G$. In this case we say the order “respects left multiplication in G .”

A group G is **right orderable** if there is a total order \leq on G which respects right multiplication in G .

A group G is **biorderable** if there is a total order \leq on G which respects left and right multiplication in G .

A group for which any of the above holds is called **orderable** (left/right/bi- used for emphasis), and an orderable group G with a specified such order \leq is called an **ordered group**.

It is clear that any group which is left orderable is also right orderable (and vice versa). Namely, let \leq be a left order on G , then the order \preceq on G defined by $a \preceq b$ means $a^{-1} \leq b^{-1}$ is a right order on G .

Example. \mathbb{Z} with the usual ordering is a biordered group. Similarly, \mathbb{Q}, \mathbb{R} are also biordered under the usual ordering. If A, B are ordered groups, $A \times B$ can be ordered using the lexicographic ordering; it is clear that the lexicographic ordering respects left and right multiplication if the orders on each factor do.

4.2 Basic Propositions

Proposition 4.2.1. *Let G be a (left) orderable group. Then G is torsion free.*

Proof. Suppose G were not torsion free, and say $a \in G$ had order n . Then, without loss of generality, $1 \leq a$ which implies $a^{-1} \leq 1$. However $a^{n-1} = a^{-1}$, and since the order is a left order we have

$$a^{n-1} \leq a^n = 1$$

a contradiction since $a \neq 1$. □

Proposition 4.2.2. *Let G be a group with series $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ with the properties $G_i \triangleleft G$ and G_{i+1}/G_i is (left) orderable. Then G is (left) orderable.*

For a proof, see [6], Theorem 7.3.2.

Consider the last proposition in terms of theorems 3.5.1, 3.5.2, and our conjecture: we frequently ended up with a series $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ for G , with the property G_{i+1}/G_i torsion-free abelian. It can be shown (and it follows from results stated earlier with the additional assumption G finitely generated) that torsion-free abelian groups are left orderable.

4.3 Biorderability Results

We need a result of Linnell, Rhemtulla, and Rolfsen (Proposition 2.2 in [5]):

Proposition 4.3.1. *Let A be a torsion-free abelian group of finite rank and let θ be an automorphism of A . Then θ preserves an order if and only if for each eigenvalue of θ , at least one of its Galois conjugates is a positive real number.*

Example. As we saw earlier, $\mathbb{Z} \rtimes \mathbb{Z}$ is left (hence right) orderable. However, this result shows that $\mathbb{Z} \rtimes \mathbb{Z}$ is not biorderable (the automorphism $n \mapsto -n$ has only -1 as an eigenvalue).

In [7], Rhemtulla proves that groups which are residually finite p -groups for infinitely many primes p are biorderable. Here we provide a means of producing examples of groups which are residually finite p -groups for any finite collection of primes where checking if the group is biorderable is relatively easy given the proposition above.

First, we will need a restatement of Theorem 3.4.2 which is more convenient to apply in conjunction with Proposition 4.3.1 above:

Proposition 4.3.2. *Let $G = A \rtimes \mathbb{Z}$ where A is finitely generated free abelian of rank n , and \mathbb{Z} generated by α as a subgroup of G . Let M be the matrix in $GL_n(\mathbb{Z})$ corresponding to the action of α on A and suppose that the characteristic polynomial of M is rationally irreducible. Let p be a prime and suppose that $M - I$ is nilpotent modulo p (and nonzero). Then G is a residually finite p -group.*

Proof. First, note that since the characteristic polynomial of M is rationally irreducible, $\mathbb{Q}A$ is irreducible as a $\mathbb{Q}\alpha$ -module. If $M - I$ is nilpotent modulo p , then $(M - I)a = npa$ for some n (depending on the $a \in A$). That is, for each $a \in A$, there is a multiple of p such that $npa = [a, \alpha]$. This implies that $p \mid |A/[A, \mathbb{Z}]|$, which by Theorem 3.4.2 implies G is a residually finite p -group. \square

Now we see that to find an example of a group which is a residually finite p -group for some finite collection of primes which is not biorderable, it suffices to find a matrix $M \in GL_n(\mathbb{Z})$ such that $M - I$ is nilpotent modulo p for all primes p in the finite collection (the determinant of the matrix being the product of said primes is sufficient provided the characteristic polynomial of M is rationally irreducible), and for which no eigenvalue of M has a positive Galois conjugate (over \mathbb{Z}).

Theorem 4.3.1. *Let \mathcal{P} be a finite collection of distinct primes. Then there exists a poly-(infinite cyclic) group G that is a residually finite p -group for all primes $p \in \mathcal{P}$, but which is not biorderable.*

Proof. Consider the polynomial

$$p(n; x) = x^3 + nx + 1$$

where n is a positive integer. Notice that $p(n; x)$ is rationally irreducible since any rational root would have to divide -1 (an impossibility since $p(n; -1) \neq 0$ for any integer $n > 0$). Furthermore, $p(n; x)$ is an increasing function with only one real root that must be negative. Then the matrix

$$M_n = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -n \\ 0 & 1 & 0 \end{pmatrix}$$

is a matrix in $GL_3(\mathbb{Z})$ with $\det(M_n - I) = n + 2$. Let $n = \prod_{p \in \mathcal{P}} p - 2$. Then $M_n - I$ is nilpotent modulo p for all primes $p \in \mathcal{P}$. Therefore, by Proposition 4.3.2, $G = \mathbb{Z}^3 \rtimes \mathbb{Z}$ with action given by M_n is a residually finite p -group for all primes $p \in \mathcal{P}$. Furthermore, by Proposition 4.3.1, G is not biorderable. \square

Bibliography

- [1] Michael F. Atiyah and Ian G. Mcdonald. *Introduction to commutative algebra*. Addison-Wesley series in mathematics. Addison-Wesley Pub. Co., Reading, Mass., 1969.
- [2] F. Thomas Farrell and Peter A. Linnell. K-theory of solvable groups. *Proc. London Math. Soc.*, 87(2):309–336, 2004.
- [3] K. W. Gruenberg. Residual properties of infinite soluble groups. *Proc. London Math. Soc.*, s3-7(1):29–62, 1957.
- [4] Peter A. Linnell. Right orderable residually finite p-groups and a Kourovka notebook problem. *J. Algebra*, 248(2):605–607, 2002.
- [5] Peter A Linnell, Akbar H. Rhemtulla, and Dale P.O. Rolfsen. Invariant group orderings and galois conjugates. *J. Algebra*, 319(12):4891–4898, 2009.
- [6] Roberta B. Mura and Akbar H. Rhemtulla. *Orderable groups*, volume 27 of *Lecture Notes in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1977.
- [7] A. H. Rhemtulla. Residually F_p -groups, for many primes p , are orderable. *Proc. Amer. Math. Soc.*, 41:31–33, 1973.
- [8] John S. Wilson. *Profinite Groups*. Oxford University Press, New York, 1998.