Transmission of evanescent wave modes through a slab of negative-refractive-index material

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There has been a long-standing argument about Pendry's suggestion that a plane harmonic evanescent (surface) wave along the interface between free space and a slab of $\varepsilon = -1$, $\mu = -1$ double-negative (DNG) medium will emerge on the far side with recovery of phase and amplitude. While this is possible, it is subject to parameter restrictions. This work generalizes previous work and now gives analytical criteria for when to expect such a recovery in a Smith-Kroll DNG medium. Basically this requires, among other things, a relatively narrow bandwidth and relatively small transverse-mode component. There also is a very strong dependence on the ratio of slabwidth to plasma wavelength.

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I. INTRODUCTION

It is well known that the slab reflection and transmission coefficients of a monochromatic electromagnetic wave mode determined by wavenumber $\mathbf{K} = (K_x,\, 0,\, \sqrt{k^2 - K_x^2})$, where $k^2 = \omega^2\mu\varepsilon/c^2$ (ω is the angular frequency, c is the velocity of light *in vacuo*, and μ,ε are the dimensionless electromagnetic parameters relative to free space, with $\mu\varepsilon = n^2$ and n being the refractive index), are

$$R(\omega) = \frac{r(1-e^{2i\varphi})}{1-r^2e^{2i\varphi}}, \qquad T(\omega) = \frac{(1-r^2)e^{i\varphi}}{1-r^2e^{2i\varphi}},$$
$$r = \frac{\mu K_{0z} - K_{1z}}{\mu K_{0z} + K_{1z}}. \tag{1}$$

Here the subindices 0,1 indicate free space and the double-negative (DNG) medium, respectively, $\varphi = \sqrt{k^2 - K_{1x}^2}\,d$, where d is the slabwidth. In terms of a dimensionless wavenumber $q = K_{1x}/k_0$ the Fresnel coefficient becomes

$$r = \frac{\mu\sqrt{1-q^2} - \sqrt{n^2-q^2}}{\mu\sqrt{1-q^2} + \sqrt{n^2-q^2}}. \tag{2}$$

For evanescence, replace $\sqrt{n^2-q^2},\sqrt{1-q^2}$ by $\sqrt{q^2-n^2},\sqrt{q^2-1}$, respectively, in (2), and replace $i\varphi$ by $-\psi = -\sqrt{K_{1x}^2 - k^2}d = -kd\sqrt{q^2-n^2}$ in (1). Veselago [1] and colleagues [1,2] suggested curious properties for media with a negative refractive index, and specifically with relative parameters $\varepsilon < 0$, $\mu < 0$. More details can be found in Pendry [3]. Pendry and Smith [4] pointed out that $r \to \infty$ when $\varepsilon = -1, \mu = -1$ at a single frequency ω_m, as can be seen easily from (2), as a result of which one finds zero reflection and $T(\omega_m) = e^{-i\varphi(\omega_m)}$ (which indicates backwards evolution of phase), or $T(\omega_m) = e^{-\psi(\omega_m)}$ in the evanescent case (which indicates an increase of amplitude with slabwidth d).

This result has been criticized, or subjected to restrictive circumstances, by various researchers. Preceding work [5,6] by this author indicated, for a specific type of negative-refractive-index medium, that the recovery may occur only for signals with very small bandwidth. This work, in turn, was recently faulted [7] for not including a factor $e^{-i\omega t}$ in the needed integration over small bandwidth so that it would hold only for $\omega_m t \ll 1$. One purpose of this work is to redo the calculation with this factor and show more generally that the Pendry-suggested e^{ψ} factor will usually not manifest itself in a DNG medium. The obtained analytical results are confirmed by numerical evaluation of the relevant integrals. Another purpose is to try to find criteria for such a manifestation.

II. ANALYSIS

Let the band-limited input signal be $A(t)$ with Fourier transform $\widetilde{A}(\omega)$. The electric field of the mode in question is given by the inverse transform

$$E(x,d^+,q,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega T(\omega)\widetilde{A}(\omega)\mathsf{E}(q,\omega)\, e^{i(k_0qx-\omega t)}, \tag{3}$$

where $\mathsf{E}(q,\omega)$ is a strength factor only weakly dependent upon ω. For the purpose of this work, it suffices to study this integral in the immediate environment of $\omega = \omega_m$. Thus we need consider only the factor $\widetilde{A}(\omega)T(\omega)e^{-i\omega t}$, and we may replace $\mathsf{E}(q,\omega) \approx 1$ without significant error within the signal bandwidth $2\Delta\omega$.

The relative electromagnetic parameters for a (theoretically lossless) Smith-Kroll [8] (SK) medium are

$$\varepsilon(\Omega) = 1 - \frac{1}{\Omega(\Omega + i\Omega_c)}, \quad \mu(\Omega) = 1 - \frac{F\Omega_0^2}{\Omega^2 - \Omega_0^2 + i\Omega\Omega_c}, \tag{4}$$

where all Ω refer to frequencies with respect to the plasma frequency ω_p, and $\Omega_c = \nu_c/\omega_p$ is a (vanishingly small) normalized collision frequency, which in effect will be taken to be infinitesimal. It is necessary that

$$1 = (F+2)\Omega_0^2 \tag{5}$$

in order for both μ and ε to be -1 at one and the same frequency Ω_m (this is not the case in Ref. [8]). It then follows to order $O(\Omega_c^2)$ that

$$\Omega_m \approx 1/\sqrt{2} - i\Omega_c/2. \tag{6}$$

If we define $\tau_m = (d\mu/d\omega)_{\omega_m}$ and $\tau_e = (d\varepsilon/d\omega)_{\omega_m}$ and note from energy considerations that for narrow bandwidth signals,

$d(\omega\mu)/d\omega > 1$, $d(\omega\varepsilon)/d\omega > 1$, so that $\tau_m, \tau_e > 1/\omega > 0$, it can then be seen that

$$r \approx \frac{2 - \left(\tau_m + \frac{1}{2}\frac{\tau_m+\tau_e}{q^2-1}\right)}{\left(\tau_m - \frac{1}{2}\frac{\tau_m+\tau_e}{q^2-1}\right)\delta\Omega} \approx \frac{2}{\left(\tau_m - \frac{1}{2}\frac{\tau_m+\tau_e}{q^2-1}\right)\delta\Omega} = \frac{\Omega_t}{\delta\Omega} \tag{7}$$

for $|\delta\Omega| \equiv |\Omega_m - \Omega| \ll \Omega_t$. Equations (1), (6), and (7) lead directly, for evanescence, to

$$T(\omega) \approx \frac{(\Omega - \Omega_m - \Omega_t)(\Omega - \Omega_m + \Omega_t)e^{-\psi_m}}{(\Omega - \Omega_b)(\Omega - \Omega_a)} \tag{8}$$

again in the vicinity of Ω_m,where $\psi \approx \psi_m = k_m d\sqrt{q^2 - n^2}$ is an adequate approximation within the very small bandwidth under consideration, and

$$\Omega_a = \Omega_m - \Omega_t e^{-\psi_m}, \qquad \Omega_b = \Omega_m + \Omega_t e^{-\psi_m}. \tag{9}$$

In Appendix B we show that $\Omega_t e^{-\psi_m} \ll \Omega_m$ for a number of SK models. A slabwidth of $d = (4/\pi)\lambda_p$ has been chosen to illustrate the effect. At this intermediate value of d it appears that $2\Omega_t e^{-\psi_m} \leqslant 10^{-6}$ so that enhancement would hold only for bandwidths less than $10^{-6}\omega_p$. With the experimental value used in Ref. [7], $\omega_p \sim 1.2 \times 10^{10}\ \mathrm{s}^{-1}$, bandwidths less than 2 kHz at frequencies of the order of 1.3 GHz would be needed to see the e^{ψ_m} growth factor. This critical bandwidth rapidly diminishes as slabwidth d is increased. We find the behavior of $T(\omega)$ for $F = 1.25$, $\Omega_0 = 1/\sqrt{3.25}$, and $d/\lambda_p = 4/\pi$ as shown in Fig. 1.

The spacing between Ω_a and Ω_b clearly tends to zero as slabwidth d and/or q increases and thus will establish below that, even though $T(\omega_m) = e^{\psi_m}$ will diverge as $d,q \to \infty$, a narrow-bandwidth signal around Ω_m resulting from (1) will not. Another set of divergences appear to occur when Ω is equal to or near the singularity frequencies Ω_a and Ω_b. However, examination of (5) shows that the singularities on either side of Ω_m are integrable so that even relatively narrow-bandwidth signals will not exhibit singular behavior upon transmission through the slab. In fact, the work described below also removes problems with these singularities.

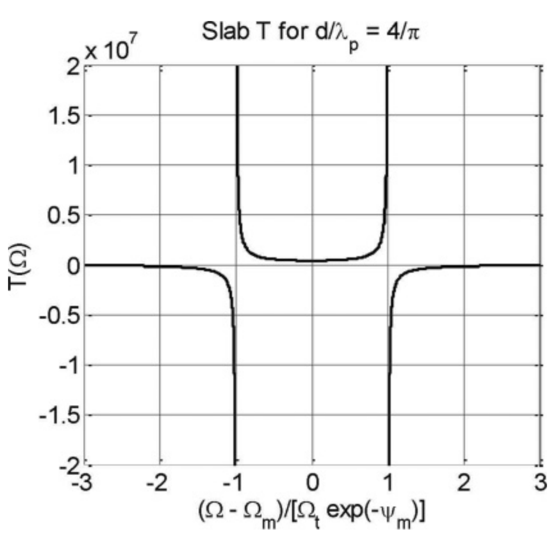

FIG. 1. Slab transmission coefficient vs. Ω near Ω_m.

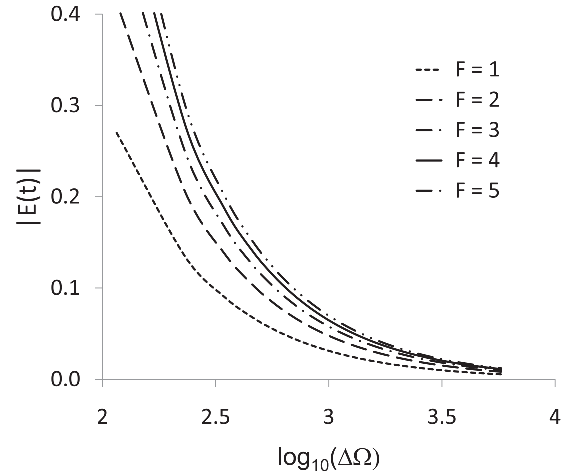

FIG. 2. $|E(t)|$ vs. bandwidth $\Delta\Omega$ in s^{-1}.

Other SK models (see Appendix B) give very similar results. It is straightforward to reduce (8) to

$$T(\Omega) = e^{-\psi_m} + \gamma\Omega_t e^{-\psi_m}\left(\frac{1}{\Omega - \Omega_a} - \frac{1}{\Omega - \Omega_b}\right) - \frac{\gamma^2\Omega_t^2 e^{-\psi_m}}{(\Omega-\Omega_b)(\Omega-\Omega_a)} \approx \frac{1}{2}\Omega_t\left(\frac{1}{\Omega - \Omega_a} - \frac{1}{\Omega-\Omega_b}\right), \tag{10}$$

where terms of $O(e^{-\psi_m})$ are ignored and $\gamma = 1 - e^{-\psi_m} \approx 1$ is set in the last right-hand side.

Let $\widetilde{A}(\omega) = e^{-i\omega t} = e^{-i\Omega\omega_p t}$ for $\Omega_m - \Delta\Omega < \Omega < \Omega_m + \Delta\Omega$ and let $\widetilde{A}(\omega)$ be zero outside the $2\Delta\Omega$ bandwidth. We then need to calculate from (1) and (10)

$$E(t) = \int_{\Omega_m-\Delta\Omega}^{\Omega_m+\Delta\Omega} d\Omega T(\Omega)e^{-i\Omega\varsigma}, \tag{11}$$

where $\varsigma \equiv \omega_p(t - qx/c)$ and where an inessential factor $\omega_p/2\pi$ is omitted in the calculations. This integral is further evaluated in Appendix A, leading to Eq. (A3).

The very narrow bandwidth $\Omega_2 - \Omega_1 = 2\Delta\Omega$ allows for further simplification of (A3) by exploiting the small size of $\Omega_t e^{-\psi_m}$ and ignoring negligible terms of order $O(\Omega_t e^{-\psi_m})$ compared to unity:

$$E(t) \approx 2\Omega_t e^{-i\Omega_m\varsigma}\cos(\Omega_t\varsigma e^{-\psi_m})\Omega_t e^{-\psi_m}\frac{\cos[(\Delta\Omega)\varsigma]}{\Delta\Omega} - 2\Omega_t e^{-i\Omega_m\varsigma}\sin(\Omega_t\varsigma e^{-\psi_m})\mathrm{Si}(\Delta\Omega\varsigma), \tag{12}$$

in which the second term is also negligible when $\Omega_t\varsigma e^{-\psi_m} \ll 1$ and $\Delta\Omega\varsigma \ll 1$. Expression (12) is the analytical result of this work and has been verified [9] by numerical integration of (11). It follows that $E(t) = |E(t)|e^{-i\Omega_m\varsigma}$. The factor $e^{-i\Omega_m\varsigma}$ is model independent in the context of (5), and the amplitude $|E(t)|$ is weakly dependent on model through some variation in Ω_t (as shown in Appendix B and Fig. 2), but is essentially independent of the value of ς and weakly dependent upon $\sqrt{q^2-1}$,d, until $\Omega_t\varsigma e^{-\psi_m}$ approaches unity (which, among other things, may require large values of ς).

Figure 2 shows plots of $|E(t)|$ versus bandwidth $\Delta\Omega$ [in s^{-1}as per Eq. (12)] in which expression the second term is negligible at the chosen parameter values ($q = 2.5$, $d/\lambda_p =$

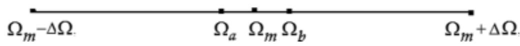

FIG. 3. Location of normalized frequencies.

$4/\pi$). The decrease with increasing $\Delta\Omega$ can be somewhat understood from Fig. 1.

III. DISCUSSION

This work demonstrates what to expect at the far side of a slab of DNG material when a surface or evanescent mode is created at the near side. Such a mode is defined by the extended sine parameter $q = K_x/k$. If the mode is created at the frequency ω_m at which the electromagnetic parameters μ and ε are -1, then the output mode has an amplitude exponentially proportional to slabwidth d, which would seem to represent a recovery of absorption loss inside the slab, but actually is an enhancement of the input mode because the enhancement factor $\exp(k_m d\sqrt{q^2-1})$ multiplies the input amplitude and appears to grow to unbounded strengths, as previously noted by various authors. The result (12) shows that such exponential growth does not occur for even relatively small bandwidths $\Delta\omega$. The minimum bandwidth at which (12) holds is $\Delta\Omega \sim \Omega_t e^{-\psi_m}$, and Table I in Appendix B demonstrates that this can be a very small bandwidth. However, for any fixed value of q and bandwidth, a decrease in slabwidth d will enhance the exponential growth in evanescent amplitude. The major result of this work is that the exponential growth (compensating for evanescent loss) does not take place in signals with a bandwidth that exceeds $2\Omega_t e^{-\psi_m}$ around the frequency Ω_m, which determines $\mu = \varepsilon = -1$. In Appendix B we discuss the conditions under which this critical bandwidth is obtained for an SK medium.

Other mechanisms that inhibit exponential growth have been proposed [10,13].

APPENDIX A

To avoid difficulties with the integrable singularities, the integral in (11) needs to be split into three pieces:

$$E(t) = I_1 + I_2 + I_3 = \frac{1}{2}\Omega_t\left[\int_{\Omega_1}^{\Omega_a} d\Omega T(\Omega)e^{-i\Omega\varsigma} + \int_{\Omega_a}^{\Omega_b} d\Omega T(\Omega)e^{-i\Omega\varsigma} + \int_{\Omega_b}^{\Omega_2} d\Omega T(\Omega)e^{-i\Omega\varsigma}\right], \tag{A1}$$

where $\Omega_1 = \Omega_m - \Delta\Omega$ and $\Omega_2 = \Omega_m + \Delta\Omega$ (see Fig. 3).

TABLE I. Parameter values for five SK models

Key parameters	τ_e	τ_m	ψ_m	Ω_t	$\Omega_t e^{-\psi_m}$
$F = 1$	5.657	16.971	12.961	0.1350	3.171×10^{-7}
$F = 2$	5.657	11.314	12.961	0.2062	4.845×10^{-7}
$F = 3$	5.657	9.428	12.961	0.2503	4.324×10^{-7}
$F = 4$	5.657	8.485	12.961	0.2802	6.582×10^{-7}
$F = 5$	5.657	7.920	12.961	0.3018	7.090×10^{-7}

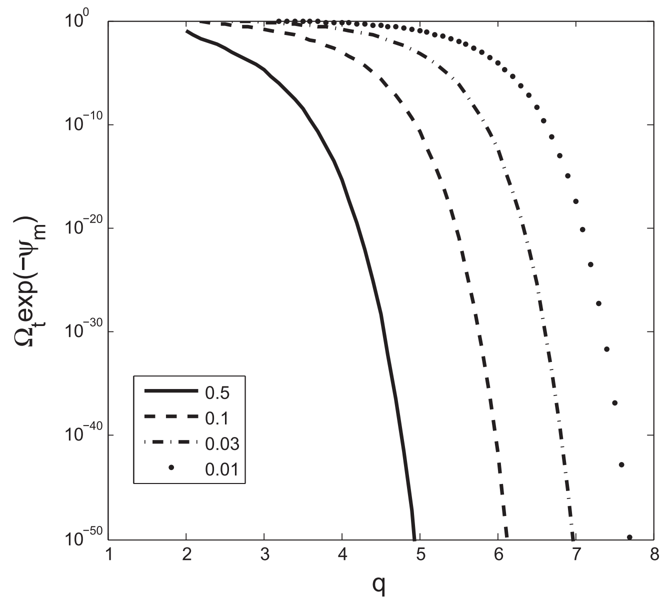

FIG. 4. $\Omega_t e^{-\psi_m}$ vs. normalized wavenumber $q = K_x/k_0$ for various d/λ_p.

After insertion of (10) and some algebraic manipulation one obtains

$$\begin{aligned} E(t) \approx{}& \frac{1}{2}\left(e^{-i\Omega_a\varsigma}\int_{(\Omega_a-\Omega_1)\varsigma}^{(\Omega_2-\Omega_a)\varsigma} dx\frac{\cos x}{x} + e^{-i\Omega_b\varsigma}\int_{(\Omega_2-\Omega_b)\varsigma}^{(\Omega_b-\Omega_1)\varsigma} dx\frac{\cos x}{x}\right) \\ &- \frac{1}{2}(ie^{-i\Omega_a\varsigma}\{\mathrm{Si}[(\Omega_2-\Omega_a)\varsigma] + \mathrm{Si}[(\Omega_a-\Omega_1)\varsigma]\}) \\ &+ \frac{1}{2}(ie^{-i\Omega_b\varsigma}\{\mathrm{Si}[(\Omega_2-\Omega_b)\varsigma] + \mathrm{Si}[(\Omega_b-\Omega_1)\varsigma]\}), \end{aligned} \tag{A2}$$

where $\mathrm{Si}(x) = \int_0^x dx'(\sin x'/x')$. This can be rewritten to exhibit the possibility of further possible approximations as

$$\begin{aligned} E(t) \approx{}& \Omega_t e^{-i\Omega_m\varsigma}\cos(\Omega_t\varsigma e^{-\psi_m})\int_{-\Omega_t\varsigma e^{-\psi_m}}^{\Omega_t\varsigma e^{-\psi_m}} d\Omega\frac{\cos[(\Omega+\Delta\Omega)\varsigma]}{(\Omega+\Delta\Omega)\varsigma} \\ &- \Omega_t e^{-i\Omega_m\varsigma}\sin(\Omega_t\varsigma e^{-\psi_m})\{\mathrm{Si}[(\Delta\Omega+\Omega_t e^{-\psi_m})\varsigma] \\ &+ \mathrm{Si}[(\Delta\Omega-\Omega_t e^{-\psi_m})\varsigma]\}. \end{aligned} \tag{A3}$$

APPENDIX B

Table I lists parameter values for five SK models, with $q = 2.5$, $d/\lambda_p = 4/\pi$ so that the slabwidth d is comparable to the plasma wavelength λ_p. The parameter $\Omega_t e^{-\psi_m}$ decreases rapidly in value with increasing q,d (see also Fig. 4), but the main purpose here is to show the weak dependence upon F.

In all of these, both Ω_a and Ω_b lie extremely close to Ω_m because $\Omega_t e^{-\psi_m}$ is very small in comparison. The sensitivity of spacing $\Omega_t e^{-\psi_m}$ to parameters q,d is more aptly illustrated in Fig. 4 for $F = 2$.The values of d/λ_p are indicated in the legend (the dependence upon F is too weak to be shown on this scale). It is apparent that $\Omega_t e^{-\psi_m} < 10^{-10}$ for $d/\lambda_p = (d/\lambda)(\lambda/\lambda_p) < 0.5$ and for $q > 6.6$. In that case, if $\lambda > \lambda_p$ for lossless evanescence, it follows that $d < \lambda$ (which includes the near-field cases of interest in contemporary metamaterial-lens applications) in order that $d/\lambda_p < 0.5$. Thus the present analysis indicates, for near-field applications with $0.01 < d/\lambda_p < 0.5$, that $q < 6.6$ is required for potential exponential increase of amplitude in the emerging evanescent wave. Further criteria can be obtained from the above analysis for other parameter values. For example, if much smaller values of d/λ_p are required, it follows that the critical value for q must be larger.

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