Matrix Schubert varieties for the affine Grassmannian

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(ABSTRACT)

Schubert calculus has become an indispensable tool for enumerative geometry. It concerns the multiplication of Schubert classes in the cohomology of flag varieties, and is typically conducted using algebraic combinatorics by way of a polynomial ring presentation of the cohomology ring. The polynomials that represent the Schubert classes are called Schubert polynomials.

An ongoing project in Schubert calculus has been to provide geometric foundations for the combinatorics. An example is the recovery by Knutson and Miller of the Schubert polynomials for finite flag varieties as the equivariant cohomology classes of matrix Schubert varieties. The present thesis is the start of a project to recover Schubert polynomials for the Borel–Moore homology of the (special linear) affine Grassmannian by an analogous process. This requires finitizing an affine Schubert variety to produce a matrix affine Schubert variety. This involves a choice of "window", so one must then identify a class representative that is independent of this choice. Examples lead us to conjecture that this representative is a k-Schur function. Concluding the discussion is a preliminary investigation into the combinatorics of Gröbner degenerations of matrix affine Schubert varieties, which should lead to a combinatorial proof of the conjecture.

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Contents

Introduction

This paper is the start of a project to exhibit a natural geometric construction of the symmetric function representatives of Schubert classes in the homology of the (Type A) affine Grassmannian. The project will proceed as follows, using $n \geq 2$ and w an affine Grassmannian permutation:

(1) Embed the affine Schubert variety $X_w \hookrightarrow \mathrm{Gr}_{h,m}$ into a finite Grassmannian. This is accomplished via an embedding $\mathrm{Gr}_{\mathrm{SL}_n} \hookrightarrow \mathrm{Gr}^0_\infty$ of the affine Grassmannian $\mathrm{Gr}_{\mathrm{SL}_n} = \mathrm{SL}_n(\mathbb{C}(\!(t)\!))/\mathrm{SL}_n(\mathbb{C}[\![t]\!])$ into the infinite Grassmannian $\mathrm{Gr}^0_\infty = \lim_{\infty \leftarrow h,m-h} \mathrm{Gr}_{h,m}$.

(2) Construct the *matrix affine Schubert variety* $Y_w^{h,m} = \overline{\pi^{-1}(X_w)} \subseteq M_{m\times h}$, where $\pi : M^\circ_{m\times h} \to \mathrm{Gr}_{h,m}$ is the column span map from the Stiefel manifold to the Grassmannian, and describe its ideal $\mathfrak{i}(Y_w^{h,m}) \subseteq \mathbb{C}[M_{m\times h}]$. (Note that $Y_w^{h,m}$ is stable under the column action of the maximal torus $T < \mathrm{GL}_h$.)

(3) Compute the torus-equivariant cohomology class $[Y_w^{h,m}]^T \in H_T^*(M_{m\times h}) \cong \mathbb{Z}[x_1, \ldots, x_h]^{S_h}$, identify a torus-equivariant homology representative via the rectangle complement bijection $\omega_{h\times(m-h)}$ on (the partition diagrams that index) the basis of Schur polynomials and verify that each representative is the localization of some common symmetric function $f_w^{(n-1)} = \lim_{\infty\leftarrow h,m-h} \omega_{h\times(m-h)}[Y_w^{h,m}]^T \in \Lambda$. We verify the existence of $f_w^{(n-1)}$ using the multidegree representation of equivariant cohomology.

(4) Perform a Gröbner degeneration of $Y_w^{h,m}$ into a union of coordinate subspaces and describe this union in combinatorial terms. We represent these subspaces as $m \times h$ arrays with ideal generators marked. We prove part of the claim, and conjecture the rest, that the collection of these arrays behave like the planar histories of a permutation.

(5) Use the Gröbner combinatorics to explicitly describe $f_w^{(n-1)}$ and its relationship to the known homological Schubert class representative, the k-Schur function $s_w^{(n-1)}$, using $k = n - 1$. We conjecture, having several cases empirically validated, that $f_w^{(n-1)} = s_w^{(n-1)}$. If true, this provides a purely geometric, monomial-by-monomial construction of k-Schur functions.

The proof of (2) involves the introduction of a class of *heralded spaces* in Gr^0_∞ that help us interface between Gr^0_∞ and $\mathrm{Gr}_{\mathrm{SL}_n}$. In particular, we use heraldedness to identify canonical bases in Gr_∞^0 for $V \in \mathrm{Gr}_{\mathrm{SL}_n}$ from which much of the combinatorics of the affine Weyl group can be recovered. Meanwhile, we observe in our examples, and clarify into a conjecture, a partial Gröbner degeneration of $Y_w^{h,m}$ into a union of GL_h-stable matrix varieties whose equivariant classes are Stanley symmetric polynomials. Up to these conjectures, this provides a geometrically natural expansion of $s_w^{(n-1)}$ into Stanley symmetric polynomials.

Notational conventions vary from chapter to chapter but in most cases remain steady within each chapter. Results are categorized as follows: Most sections contain a theorem, which represents the central result of that section. Propositions are borrowed from other sources and not proved. Lemmata and corollaries precede (respectively, follow) theorems and propositions (and corollaries lemmata) but range from highly technical miscellany to important results.

The document is organized as follows: Chapter 1 provides an overview of the classical finite flag varieties, with an emphasis on Grassmannians, upon which foundation Chapter 2 builds an introduction to the affine Grassmannian and its realization as an ind-subvariety of an infinite Grassmannian. Chapter 3 develops machinery (the heralded spaces) that interfaces between these infinite and affine Grassmannians and facilitates several proofs in Chapter 4. That chapter introduces matrix affine Schubert varieties, our main objects of interest, and proposes polynomial generators for their ideal. Finally, Chapter 5 constructs homological polynomial invariants of these varieties and examines their combinatorics.

Chapter 1

Finite flags

In this chapter we introduce two classical collections X — full flag varieties and Grassmannians — of subspaces of a finite-dimensional vector space satisfying certain intersectional criteria. Parametrized as a subspace of projective space, X inherits the topology of a manifold. We exhibit the embedding of the Grassmannian specifically into the projectivization of a certain vector space, the relations governing which grant it the structure of a projective variety. We then recover X as a quotient of the general linear group of the vector space, conferring smoothness on X as a manifold and leading to its stratification into orbits of the Borel subgroup called Schubert cells. The associated Schubert classes are central to the cohomology (alternatively, the intersection theory) of the Grassmannian, and to that of flags in general. We briefly recall the polynomial ring presentation of the cohomology of X and identify canonical polynomial representatives of the Schubert classes.

1.1 Flag and Grassmannian manifolds

Begin with an n-dimensional complex vector space V.

Definition 1.1.1. A *(k-step) partial flag in V* is a filtration of strict inclusions

$$F_\bullet : (\{0\} = F_0 \subset F_{i_1} \subset \cdots \subset F_{i_k} \subset F_n = V),$$

of subspaces of dimensions $\dim F_{i_j} = i_j$. $F_\bullet$ is a *full flag* if each $i_j = j$ and $k = n-1$. Given a dimension sequence $I = (i_1 < i_2 < \cdots < i_k) \subseteq \{1, 2, \ldots, n-1\}$, denote by $\mathrm{Fl}_I(V)$ the collection of flags arising from that sequence, which we call a *flag variety* because (as will be seen in the next section) it admits the structure of an algebraic variety.

For integers $a < b$, write $[a, b]$ for the sequence of integers from a to b, inclusive, and $[a] := [1, a]$. Write $\mathrm{Fl}_n(V) := \mathrm{Fl}_{[n-1]}(V)$ and $\mathrm{Gr}_{k,n}(V) = \mathrm{Fl}_{(k)}(V)$. The latter collection is a *Grassmannian.*

Pick a basis $(e_1, \ldots, e_n)$ for V. Then the *standard flag* $F_\bullet = F_\bullet^{\mathrm{std}}$ is defined by

$$F_i = \mathrm{Span}(e_1, \ldots, e_i).$$

We will also make use of the *opposite flag* $F_\bullet^{\mathrm{opp}}$ defined by

$$F_i^{\mathrm{opp}} = \mathrm{Span}(e_{n+1-i}, \ldots, e_n).$$

For the sake of following convention, in later chapters we will transition to a focus on opposite flags. ◇

The general linear group $G = \mathrm{GL}(V)$ exactly parametrizes the (ordered) bases of V by the assignment $g \mapsto (ge_1, \ldots, ge_n)$. The action of G on V induces an action on the ordered bases of V, and in turn on any flag variety $\mathrm{Fl}_I(V)$. This provides the projection

$$\begin{aligned} G &\to \mathrm{Fl}_I(V). \\ g &\mapsto (\mathrm{Span}(ge_1, \ldots, ge_{i_j})_{j=1}^{k}) \end{aligned} \tag{1.1}$$

Write $g_i = ge_i$ and denote the full flag $F_\bullet^{(g)} = (F_i^{(g)} = \mathrm{Span}(g_1, \ldots, g_i))$.

The next few paragraphs cover several basic facts about linear algebraic groups that establish a standard framework for what follows.

Definition 1.1.2. Given a finite linear algebraic group, any maximal connected solvable subgroup is *Borel*, and any subgroup containing a Borel subgroup is *parabolic*.

Explicitly, write

$$P_I = \{g \in G \mid \forall i,\ g_{i_{j-1}+1}, \ldots, g_{i_j} \in F_{i_j}^{\mathrm{std}}\}$$

for the *standard parabolic subgroups of* G, including the *standard Borel subgroup* $B = P_{[n-1]} < G$ and the maximal parabolic subgroups $P_k = P_{(k)} < G$. Designate the corresponding opposite subgroups B^- and P_k^-. ◇

By this definition, a Borel subgroup is a minimum parabolic subgroup.

Proposition 1.1.3 (?). *The stabilizer in G of a partial flag is a parabolic subgroup. In particular, the stabilizer of a full flag is a Borel subgroup. Moreover, the parabolic subgroups of G are precisely these stabilizers.*

Since the action of G is transitive, we have an equivalence of congruence classes

$$\mathrm{Fl}_I(V) \cong G/P_I. \tag{1.2}$$

Corollary 1.1.4. *Every parabolic subgroup of G is conjugate to a unique P_I.*

Without loss of generality, we may then focus only on the standard parabolic subgroups, which are the stabilizers of the standard partial flags.

Proof. Take a parabolic subgroup P. By Proposition 1.1.3, $P = \mathrm{Stab}_G(F_\bullet)$ for some $F_\bullet \in \mathrm{Fl}_I(V)$ (for some I). Find a basis $v_1, \ldots, v_n$ for V that recovers the partial flag $\mathrm{Fl}_I(V)$ under (1.1), and express this basis as a matrix $g \in G$ of column vectors. Thus, for all $j \in [n]$, $v_j = ge_j$. This provides the expansion

$$pv_j = \sum_{j' \leq j} a_{jj'} v_{j'}.$$

for any $p \in P$ and for each $j \in [n]$. Replacing each $v_{j'}$ with $ge_{j'}$ and multiplying on the left by g^{-1} yields

$$g^{-1}pge_j = \sum_{j' \leq j} a_{jj'} e_{j'}.$$

Since $g^{-1}pg$ ranges over the elements of $g^{-1}P_Ig$, and conversely since any element $g' \in F_\bullet^{\mathrm{std}}$ can be obtained this way from $p = gg'g^{-1}$, we have $P = g^{-1}P_Ig$. □

Definition 1.1.5. An *algebraic torus (over $\mathbb{C}$)* is an algebraic group isomorphic to the direct product of some number of copies of the multiplicative group $\mathbb{C}^*$. The *unipotent radical of an algebraic group* is the set of unipotent elements u in the radical of G; $u \in G$ is *unipotent* if $1 - u$ is nilpotent. ◇

Proposition 1.1.6 ([Ros02])**.** *Let G be any (complex) linear algebraic group. The stardard Borel subgroup $B < G$ contains a unique maximal torus $T < B$. The quotient $N_G(T)/T$ of the normalizer of T in G is a discrete group isomorphic to the Weyl group W associated with G.*

In our setting $W \cong S_n$ (see Appendix A), and we take W to act on $[n]$. The choice of standard basis defines an inner product $\langle e_i, e_j \rangle = \delta_{ij}$ and provides the faithful matrix representation

$$\begin{aligned} G &\to M_{n\times n}. \\ g &\mapsto (\langle g_j, e_i \rangle)_{ij} \end{aligned} \tag{1.3}$$

This representation sends B to the subgroup of upper triangular matrices, its unipotent radical U to the subgroup having 1s along the diagonal, and each P_I to the block–upper triangular matrices with blocks of sizes $i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k$. In particular,

$$P_k = \begin{pmatrix} \mathrm{GL}_k & * \\ 0 & \mathrm{GL}_{n-k} \end{pmatrix}.$$

The action of W on $\mathbb{C}^n$ given by $w(e_j) = e_{w(j)}$ defines a canonical embedding $W \hookrightarrow G$. Thought of as a point in $M_{n\times n}$, $w = (\delta_{w(j),j})$ is called a *permutation matrix.*

Lemma 1.1.7. *Each two-sided Borel orbit of G contains a unique permutation w. This delivers the Bruhat decomposition*

$$G = \bigsqcup_{w \in W} BwB.$$

See [Eme] for a basis-free proof. We will prove the stronger decomposition

$$G = \bigsqcup_{w \in W} U_-^w wB, \tag{1.4}$$

where $U_-^w = U \cap wU_-w^{-1} < B$. The proof will make use of one-parameter subgroups, which we define here in terms of a general group and will generalize in Section 2.2.

Definition 1.1.8. Let G be any (complex) linear algebraic group and $\theta : \mathbb{C} \to G$ an algebraic endomorphism, so that $\theta(\mathbb{C})$ is a nontrivial subgroup of G. We refer to θ and to its image in G as a *one-parameter subgroup*. The subgroup of G generated by the images of one-parameter subgroups $\theta_1, \ldots, \theta_m$ is said to be *generated by* $\theta_1, \ldots, \theta_m$. ◇

Returning to $G = \mathrm{GL}_n$, it is straightforward that the one-parameter subgroups

$$\begin{aligned} \theta_{ij} : \mathbb{C} &\to \mathrm{GL}_n \\ a &\mapsto I + aE_{ij} \end{aligned} \tag{1.5}$$

across $1 \leq i < j \leq n$ generate the unipotent radical $U < G$, and that those across $1 \leq j < i \leq n$ generate U_-.

Proof. For existence, pick $g \in G$. We shall identify $u \in U$ and $b \in B$ so that $u^{-1}gb^{-1} = w$ is a permutation matrix, then verify that $u \in wU_-w^{-1}$.

We construct u and b iteratively, using one-parameter subgroups, over the entries $(i, j) \in [n]^2$, proceeding from the bottom row to the top row and within each row from left to right. For the general step, suppose that all $(i', j')^{\text{th}}$ entries of g with $i' > i$, or with $i' = i$ and $j' < j$, are either 0 or 1, and no two such entries 1 lie in the same row or column.

- If some $g_{ij'} = 1$, $j' < j$, then take $\theta = \theta_{j'j}(-g_{ij}) \in B$ to get $(g\theta)_{ij} = 0$. Now $g_{ij} = 0$.
- If all entries of g leftward of (i, j) are zero but some $g_{i'j} = 1$, $i' > i$, then take $\theta = \theta_{ii'}(-g_{ij}) \in U$ to get $(\theta g)_{ij} = 0$. Now $g_{ij} = 0$.
- If all entries of g leftward of or below (i, j) are zero, then take $\theta = \theta_{jj}(-g_{ij}) \in B$ to get $(g\theta)_{ij} = 1$. Now $g_{ij} = 1$.

Any $(i,j)^{\text{th}}$ step that uses $\theta \in U$ changes g to θg by adding a row $i' > i$ with 1 in column j to row i. By the starting assumption now i' has at most one nonzero entry, so in fact only entry (i,j) is changed. On the other hand, any $(i,j)^{\text{th}}$ step that uses $\theta \in B$ either preserves the (non)zeroness of all entries (the third bullet) or changes g to $g\theta$ by adding a column $j' < j$ with 1 in row i to column j. In the latter case, by starting assumption column j' has at most one nonzero entry at or below row i, so only entries at or above (i,j) are changed. Thus, after each step, the starting assumption is extended to the entry after (i,j).

At the end of the iterative process, we arrive at a matrix $u^{-1}gb^{-1}$, where $u \in U$ and $b \in B$, having entries 0 and 1. Since $u^{-1}gb^{-1}$ is invertible, it must be a permutation matrix $w = (\delta_{iw(j)})_{ij} \in S_n$.

To see that $u \in wU_-w^{-1}$, observe first that, when $\theta_{ij}(a) \in U_-$,

$$w\theta_{ij}(a)w^{-1} = w(I + aE_{ij})w^{-1} = I + awE_{ij}w^{-1} = I + aE_{w(i)w(j)} = \theta_{w(i)w(j)}(a).$$

Conversely, a given one-parameter subgroup θ_{ij} maps to wU_-w^{-1} if and only if $w^{-1}(i) > w^{-1}(j)$. It will therefore be enough to show that each factor $\theta_{ii'}(-g_{ij})$ satisfies $w^{-1}(i) > w^{-1}(i')$. Note that each $(i,j)^{\text{th}}$ step that updates u requires that no 1 lies leftward of (i,j) and ends up changing entry (i,j) to 0—that is, $j' \not\leq j$ where $i = w(j')$—and that such a step requires that some 1 lies below (i,j)—that is, $w(j) > i$. Therefore $w^{-1}(i) = j' > j = w^{-1}(i')$, hence the factor $\theta_{ii'}(g_{ij})$ lies in wU_-w^{-1}.

For uniqueness, suppose that $u'wb' = uwb$, i.e. that $w^{-1}u^{-1}u'w = b(b')^{-1} \in B$. But $u, u' \in wU_-w^{-1}$, so $w^{-1}u^{-1}u'w \in U_-$. Since $B \cap U_- = \{\mathrm{id}\}$, we have $b = b'$ and $u = u'$. □

An analogous proof confirms the upcoming lemma, which delivers the central result upon which the rest of the work draws.

Definition 1.1.9. The Weyl group $W \cong S_n$ is generated by the *simple transpositions* $s_i = (i\ i+1)$ across $1 \leq i \leq n-1$, and these are among the *reflections* $t = ws_iw^{-1}$ having order 2. This allows us to define the *length* $\ell(w)$ of a permutation $w \in W$ as the minimum number ℓ of simple transpositions s_{i_j} required to write $w = s_{i_1} \cdots s_{i_\ell}$. The *Bruhat–Chevalley partial order* on W is then the transitive closure of the following covering relation: $w > tw$ when $t = vs_iv^{-1}$ is any reflection and $\ell(w) > \ell(tw)$.

Write $W_I < W$ for the subgroup generated by the reflections indexed by I. W_I is then the Weyl group of $P_{[n-1]\smallsetminus I}$. Write W^I for the collection of minimum-length coset representatives for W/W_I. These w satisfy $\ell(wv) = \ell(w) + \ell(v)$ for all $v \in W_I$.

When $I = [n-1] \smallsetminus \{k\}$ call $w \in W^I$ a k-*Grassmannian* or just *Grassmannian* permutation. A non-identity k-Grassmannian permutation has a unique *descent* $w(i) > w(i+1)$ *at* $i = k$. More generally, w is in W^I if it has an ascent at j for all and only j in I. ◇

The following lemma is immediate from the preceding.

Lemma 1.1.10. *Each two-sided $B \times P_{[n-1] \smallsetminus I}$-orbit of G contains a unique member of W^I. Consequently,*

$$G = \bigsqcup_{w \in W^I} BwP_{[n-1] \smallsetminus I}. \tag{1.6}$$

Theorem 1.1.11. *Let $B < P = P_{[n-1] \smallsetminus I} < G$ be the standard Borel and a parabolic subgroup. Then*

$$\mathrm{Fl}_I = \bigsqcup_{w \in W^I} BwP/P. \tag{1.7}$$

Proof. Take the quotient of (1.6) by P. □

Definition 1.1.12. The decomposition (1.7) is called the *Schubert decomposition of* Fl_I, and the subsets $\Omega_w(\mathrm{Fl}_I) := BwP/P$ are called *Schubert cells*. The general construction is outlined in Appendix A.2. ◇

Example 1.1.13. Take $n = 3$ and $I = (1, 2)$, so $W \cong S_3$ and $W^I = W$. Each right B-orbit of each Schubert cell of $\mathrm{Fl}_3 = G/B$ contains a unique B-reduced matrix as follows:

$$\Omega_{\mathrm{id}} \ni \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} B/B, \quad \Omega_{s_1} \ni \begin{pmatrix} * & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} B/B, \quad \Omega_{s_2} \ni \begin{pmatrix} 1 & & \\ & * & 1 \\ & 1 & \end{pmatrix} B/B,$$

$$\Omega_{s_2 s_1} \ni \begin{pmatrix} * & 1 & \\ * & & 1 \\ 1 & & \end{pmatrix} B/B, \quad \Omega_{s_1 s_2} \ni \begin{pmatrix} * & * & 1 \\ 1 & & \\ & 1 & \end{pmatrix} B/B, \quad \Omega_{s_1 s_2 s_1} \ni \begin{pmatrix} * & * & 1 \\ * & 1 & \\ 1 & & \end{pmatrix} B/B.$$

(Here and henceforth we take blank entries in a matrix to be zero.) These forms parametrize the cells; it is visually apparent in each case that $\Omega_w \cong \mathbb{C}^{\ell(w)}$, hence that $\dim \Omega_w = \ell(w)$.

Now take $I = (2)$ so that $W_I = \langle \mathrm{id}, s_2 \rangle$ and $W^I = \{\mathrm{id}, s_1, s_2 s_1\}$. The Schubert cells of $\mathrm{Gr}_{1,3} = G/P_1$ may be recognized by the P_1-reduced forms in each coset:

$$\Omega_{\mathrm{id}} \ni \left(\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) P_1/P_1, \quad \Omega_{s_1} \ni \left(\begin{array}{c|cc} * & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right) P_1/P_1, \quad \Omega_{s_2 s_1} \ni \left(\begin{array}{c|cc} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{array}\right) P_1/P_1.$$

The separator indicates the descent of w. Similarly, $W^{(1)} = \{\mathrm{id}, s_2, s_1 s_2\}$ and the P_2-stable cells consist of the cosets

$$\Omega_{\mathrm{id}} \ni \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) P_2/P_2, \ \Omega_{s_2} \ni \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & * & 1 \\ 0 & 1 & 0 \end{array}\right) P_2/P_2, \ \Omega_{s_1 s_2} \ni \left(\begin{array}{cc|c} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) P_2/P_2.$$

◇

In the general Grassmannian case, the Schubert decomposition provides that every $V \in \mathrm{Gr}_{k,n} = G/P_k$ has in its preimage a unique right-P_k-reduced matrix that has entries 1 at

$(w(j), j)$ for some $w \in W^{[n-1] \smallsetminus \{k\}}$ and 0 below and rightward of these. Since w has a unique descent at k, all nonzero entries at positions other than $(w(j), j)$ must be at or leftward of the k^{th} column. The permutation w identifies the Schubert cell $\Omega_w(\mathrm{Gr}_{k,n})$ to which V belongs.

We can also describe the Schubert decomposition of a Grassmannian using rank conditions. Typically we identify any integer subset with the increasing sequence of its distinct members, but this convention will be discarded in Chapter 4.

Definition 1.1.14. Given a k-Grassmannian permutation w, let $P = (p_1 < \cdots < p_k) = (w(1) < \cdots < w(k)) = \mathrm{piv}(w)$ denote the *pivots of* w. The leftmost k columns of the matrix w are then $e_{p_1}, \ldots, e_{p_k}$. Let $\Omega_P(\mathrm{Gr}_{k,n})$ consist of the subspaces $V \in \mathrm{Gr}_{k,n}$ subject to the *rank conditions*

$$\dim(V \cap F_{p_j}) = j. \tag{1.8}$$

Given k-element pivot sets $P, Q \subset [n]$, write $P \leq Q$ if $p_i \leq q_i$ across $1 \leq i \leq k$; write $P < Q$ if $P \leq Q$ with at least one strict inequality. $\diamond$

Lemma 1.1.15. *Given $w \in W^{[n-1] \smallsetminus \{k\}}$, take $P = \mathrm{piv}(w)$. Then $\Omega_w = \Omega_P$.*

These rank conditions provide a partition indexing scheme for Grassmannian Schubert cells. Note that, in $\mathrm{Gr}_{2,3}$, $\partial\Omega_{s_1 s_2} \supset \Omega_{s_2}$ and $\partial\Omega_{s_1} \supset \Omega_{\mathrm{id}}$. In general, the boundary conditions of the Schubert (respectively, opposite Schubert) cells recover (the dual of) the Bruhat–Chevalley order restricted to the $w \in W^I$. We will state a related, and stronger, result at Corollary 3.1.11.

Definition 1.1.16. A *partition of* N is a finite nonincreasing sequence of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$, with $\lambda_\ell \geq 0$, satisfying $|\lambda| = \lambda_1 + \cdots + \lambda_\ell = N$. Write $\lambda \vdash N$ and denote by Par the collection of all partitions of positive integers. We will often identify λ with its *Ferrers diagram*, the northwest-justified subset of positions (i, j) in the $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ grid (using the matrix indexing scheme, or the English convention for Ferrers diagrams) satisfying $j \leq \lambda_i$, which we denote as an arrangement of boxes at these positions and empty elsewhere. For example, the partition $(3, 1) \vdash 4$ is identical to the partition $(3, 1, 0, 0)$ and has Ferrers diagram $\ydiagram{3,1}$. Write

$$\lambda \subseteq \mu \quad \text{if } \forall i, \lambda_i \leq \mu_i$$

for containment of Ferrers diagrams. Write $\lambda \leq \mu$ for the reverse lexicographic order given by

$$\lambda < \mu \Leftrightarrow \exists j, \begin{cases} \lambda_i = \mu_i & i < j \\ \lambda_j < \mu_j. \end{cases}$$

Denote the *dominance order* on the partitions Par_N of a common number N by

$$\lambda \trianglelefteq \mu \Leftrightarrow \forall j,\ \lambda_1 + \cdots + \lambda_j \leq \mu_1 + \cdots + \mu_j.$$

If $\lambda \subseteq \mu$ then denote by $\nu = \mu/\lambda$ the *skew partition* identified with the diagram $\mu \setminus \lambda$ consisting of $|\nu| = |\mu| - |\lambda|$ boxes. If $\nu = \mu/\lambda$ consists of k boxes, no two in the same row, call ν a *vertical k-strip*. If ν has no two boxes in the same column, call ν a *horizontal k-strip*. Finally, note that if $\lambda \subseteq (a^b) = (a, \ldots, a)$ for some $a, b > 0$, then $(a^b)/\lambda$ is the rotation by π of another partition $\lambda^\vee \subseteq (a^b)$. Call (a^b) a *rectangle* and $\lambda^\vee$ the *rectangle complement of λ in (a^b)*. ◇

Pick $k \in [n-1]$, set $I = [n-1] \setminus \{k\}$, and pick $w \in W^I$. Suppose that $V \in \mathrm{Gr}_{k,n}$ satisfies (1.8). Consider a P_k-reduced matrix x in G. For $1 \leq j \leq k$, each j^{th} column of x includes $n-k$ entries outside pivotal rows, $n-k+i-p_j$ of which must be zero. Let $\lambda_j = n-k+j-p_j$ and $\lambda = (\lambda_1, \ldots, \lambda_k)$. Since the p_j are increasing, $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k)$ is a partition; and since each $\lambda_i \leq n-k$, we have $\lambda \subseteq ((n-k)^k)$. Thus we may restate (1.7) as

$$\Omega_\lambda = \Omega_P = \{V \in \mathrm{Gr}_{k,n} \mid \forall j,\ \dim(V \cap F_{n-k+j-\lambda_j}) = j\}. \tag{1.9}$$

The following example illustrates some combinatorial gadgets introduced in Appendix B.

Example 1.1.17. Take $n = 5$ and $k = 3$. Then $w = [2,3,5,1,4] \in S_5/\langle s_1, s_2 s_4 \rangle$ is 3-Grassmannian and factors as such into

$$[2,3,5,1,4] = [2,3,1,5,4]s_3 = [2,3,1,4,5]s_4 s_3 = [2,1,3,4,5]s_2 s_4 s_3 = s_1 s_2 s_4 s_3,$$

giving minimum word $w = s_1 s_2 s_4 s_3$ of length 4. This word recovers the partition $\lambda \subseteq ((n-k)^k) = (2^3)$ associated with w:

$$s_1 s_2 s_4 s_3 \cdot \begin{smallmatrix} 3 & 4 \\ 2 & 3 \\ 1 & 2 \end{smallmatrix} \rightsquigarrow s_1 s_2 s_4 \cdot \begin{smallmatrix} \boxed{3} & 4 \\ 2 & 3 \\ 1 & 2 \end{smallmatrix} \rightsquigarrow s_1 s_2 \cdot \begin{smallmatrix} \boxed{3} & \boxed{4} \\ 2 & 3 \\ 1 & 2 \end{smallmatrix} \rightsquigarrow s_1 \cdot \begin{smallmatrix} \boxed{3} & \boxed{4} \\ \boxed{2} & 3 \\ 1 & 2 \end{smallmatrix} \rightsquigarrow \begin{smallmatrix} \boxed{3} & \boxed{4} \\ \boxed{2} & 3 \\ \boxed{1} & 2 \end{smallmatrix} = \begin{smallmatrix} \square & \square \\ \square & \\ \square & \end{smallmatrix} = \lambda.$$

The Schubert cell $\Omega_w \subset \mathrm{Gr}_{3,5}$ consists of cosets of GL_5/P_3 having representatives

$$\left(\begin{array}{ccc|cc} * & * & * & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right),$$

where the free entries are arranged in the non-pivotal rows according to (a rotation by $-\frac{\pi}{2}$ of) λ, and therefore provide local coordinates to $\Omega_w \cong \mathbb{C}^4$. ◇

Remark 1.1.18. The rotation is needed because our matrices are transposed from most standard treatments, e.g. [Ful97]. We adopt this convention for consistency with the common vertical orientation of (column) vectors, of which we will make much use in Chapter 4, and with [KLMW07]. Coincidentally, it also aligns with the French convention for the two partitions canonically identified with an affine Grassmannian permutation w (see Remarks 3.2.5 and B.2.4). ◇

The above geometry and combinatorics on $F_\bullet^{\text{std}}$ extend, with appropriate tweaks to be briefly considered now, to $F_\bullet^{\text{opp}}$.

Definition 1.1.19. For $\lambda \subseteq ((n-k)^k)$, write $I_\lambda = (n-k+1-\lambda_1, \ldots, n-\lambda_k)$. Define the *opposite Schubert cells* $\Omega^w = \Omega^\lambda$ according to the rank conditions

$$\Omega^w = \{V \in \mathrm{Gr}_{k,n} \mid \forall j,\ \dim(V \cap F^{\text{opp}}_{k+1-j+\lambda_j}) = j\} = \{V \in \mathrm{Gr}_{k,n} \mid \forall j,\ \dim(V \cap F^{\text{opp}}_{n+1-p_j}) = j\},$$

since $n+1-p_j = k+1-j+\lambda_j$ when $p_j = n-k+j-\lambda_j$ as before. Write $\Omega^w = \Omega^Q$ where $Q = (n+1-p_k, \ldots, n+1-p_1)$, so that the pivotal rows of $\Omega_P = \Omega_\lambda$ and of $\Omega^P = \Omega^{\lambda^\vee}$ are the same (and their partitions are rectangle complements). $\diamond$

Example 1.1.20. Ignoring the superfluous $n-k$ columns of the reduced forms in Example 1.1.13 (and with it the second block of P_k), we have the reduced forms

$$\Omega^{\text{id}} \ni \begin{pmatrix} 1 \\ * \\ * \end{pmatrix} \mathbb{C}^*/\mathbb{C}^*, \quad \Omega^{s_1} \ni \begin{pmatrix} 0 \\ 1 \\ * \end{pmatrix} \mathbb{C}^*/\mathbb{C}^*, \quad \Omega^{s_2 s_1} \ni \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mathbb{C}^*/\mathbb{C}^*$$

for the opposite Schubert cells of $\mathrm{Gr}_{1,3}$ and

$$\Omega^{\text{id}} \ni \begin{pmatrix} 1 & \\ & 1 \\ * & * \end{pmatrix} \mathrm{GL}_2/\mathrm{GL}_2, \quad \Omega^{s_2} \ni \begin{pmatrix} 1 & \\ * & 0 \\ & 1 \end{pmatrix} \mathrm{GL}_2/\mathrm{GL}_2, \quad \Omega^{s_1 s_2} \ni \begin{pmatrix} 0 & 0 \\ 1 & \\ & 1 \end{pmatrix} \mathrm{GL}_2/\mathrm{GL}_2$$

for those of $\mathrm{Gr}_{2,3}$. $\diamond$

1.2 Projective varieties

In this section we imbue the Grassmannian manifold $\mathrm{Gr}_{k,n}$ with the structure of a projective variety. Slightly more complicated constructions achieve this for the other Fl_I, and the general fact can be stated without construction:

Proposition 1.2.1 ([Spr81] 6.2.7)**.** *A subgroup of a linear algebraic group is parabolic if and only if their quotient is a projective variety.*

Our construction embeds $\mathrm{Gr}_{k,n}$ into a projective space, then identifies a homogeneous ideal of the coordinate ring with the embedded image as its vanishing set. (That the ideal is radical we do not prove; see [Ful97].) Much of the following discussion derives from [MS05]. Here and henceforth we reserve the notation I for pivot sets, rather than for subsets of $[n]$ as in the previous section.

Definition 1.2.2. The *k^{th} exterior power of* $\mathbb{C}^n$, written $\bigwedge^k \mathbb{C}^n$, is the quotient of the k^{th} tensor power $T^k\mathbb{C}^n$ by the ideal R generated by the differences

$$v_1 \otimes \cdots \otimes v_k - v_{s(1)} \otimes \cdots \otimes v_{s(k)},$$

where $s \in S_k$ is any transposition. Write the cosets $v_1 \wedge \cdots \wedge v_k = v_1 \otimes \cdots \otimes v_k + R$. $\diamond$

It is immediate that $v_1 \wedge \cdots \wedge v_k = 0$ (only) whenever the vectors $v_1, \ldots, v_k$ are linearly dependent.

Lemma 1.2.3. *For each subset $I = (i_1 < \cdots < i_k) \subset [n]$, denote $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \in \bigwedge^k \mathbb{C}^n$. Then $\bigwedge^k \mathbb{C}^n$ is a complex vector space having basis the e_I. Consequently, $\bigwedge^k \mathbb{C}^n \cong \mathbb{C}^{\binom{n}{k}}$.*

Proof. Pick an arbitrary $v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathbb{C}^n$. Define constants a_{ij} by writing each $v_j = \sum_{i=1}^n a_{ij} e_i$ and take $A = (a_{ij}) \in M_{n \times k}$. Then

$$v_1 \wedge \cdots \wedge v_k = (\sum_{i=1}^n a_{i1} e_i) \wedge \cdots \wedge (\sum_{i=1}^n a_{ik} e_i) = \sum_{\substack{i_1, \ldots, i_k \in [n] \\ \text{all } i_j \neq i_{j'}}} a_{i_1 1} \cdots a_{i_k k} e_{i_1} \wedge \cdots \wedge e_{i_k} = \sum_{\substack{I \subset [n] \\ |I| = k}} \Delta_I^{[n]}(A) e_I,$$

where Δ_I^J returns the minor of a matrix at rows I and columns J. □

Given vectors $v_1, \ldots, v_k \in \mathbb{C}^n$, denote the matrix having the v_j as column vectors by $\begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix}$. Denote by $M^\circ_{n \times k} \subset M_{n \times k}$ the collection of full-rank matrices. (As sets, $M^\circ_{n \times n} = \mathrm{GL}_n \subset M_{n \times n}$.) As a topological space, $M^\circ_{n \times k}$ is called the *Stiefel manifold*. The map

$$\begin{aligned} \hat{\phi} : M^\circ_{n \times k} &\to \textstyle\bigwedge^k \mathbb{C}^n \\ v &\mapsto v_1 \wedge \cdots \wedge v_k \end{aligned}$$

factors into the column span map $\pi : M^\circ_{n \times k} \to \mathrm{Gr}_{k,n}$. Given a row subset I, denote by v_I the submatrix of v at rows I. The maximal minors of v are then $\det(v_I)$ across such I.

Lemma 1.2.4. *Take $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ to be linearly independent sets of vectors. Then $\mathrm{Span}(w_1, \ldots, w_k) = \mathrm{Span}(v_1, \ldots, v_k)$ if and only if there is a nonzero scalar γ such that $\det(w_I) = \gamma \det(v_I)$ for all I.*

Proof. Write W and V, respectively, for the subspaces spanned by the two sets of vectors; $\dim W = \dim V = k$. For the rightward implication, since matrices v and w have the same column span, we may identify $g \in \mathrm{GL}_k$ for which $w = vg$. Since g acts independently on the rows of v, this implies that $w_I = v_I g$ for all I.

For the converse, assume that we have such a $\gamma \in \mathbb{C}^*$ and suppose that $V \neq 0$. (Otherwise every $v_i = 0$, hence each $\det(v_I) = 0$, and we are done.) Find a row subset I for which $\det(v_I) \neq 0$. Set $\tilde{v} = v(v_I)^{-1}$ and $\tilde{w} = w(w_I)^{-1}$. (Note that w_I is invertible because $\det(w_I) = \gamma \det(v_I) \neq 0$.) Now $\mathrm{Span}(\tilde{v}_1, \ldots, \tilde{v}_k) = V$ and $\mathrm{Span}(\tilde{w}_1, \ldots, \tilde{w}_k) = W$ since the respective matrices are related by row operations, so it will suffice to show that these spans are equal. Instead of scaling by γ, however, we now have

$$\det(\tilde{w}_J) = \det((w_I)^{-1}) \det(w_J) = \tfrac{1}{\gamma} \det((v_I)^{-1}) \gamma \det(v_J) = \det((v_I)^{-1}) \det(v_I) = \det(\tilde{v}_J).$$

In particular, both $\tilde{v}$ and $\tilde{w}$ contain the $k \times k$ identity matrix I_k as a submatrix at rows I. We may thereby compute any specific entry $\tilde{w}_{ij}$ in another row $i \notin I$ via

$$\tilde{w}_{ij} = \det(\tilde{w}_{I \smallsetminus \{i_j\} \cup \{i\}}) = \det(\tilde{v}_{I \smallsetminus \{i_j\} \cup \{i\}}) = \tilde{v}_{ij}.$$

Thus $\tilde{w} = \tilde{v}$, so their column spans agree. □

Lemma 1.2.4 implies that, whenever the columns of v and of w span the same subspace, they have the same image under $\hat{\phi}$ up to a nonzero scalar. They therefore have the same image under the composition of $\hat{\phi}$ with the projectivization of $\bigwedge^k \mathbb{C}^n$. This delivers the *Plücker embedding*

$$\begin{aligned} \phi : \mathrm{Gr}_{k,n} &\hookrightarrow \mathbb{P}(\textstyle\bigwedge^k \mathbb{C}^n) \\ V &\mapsto [v_1 \wedge \cdots \wedge v_k]. \end{aligned} \tag{1.10}$$

when $V = \mathrm{Span}(v_1, \ldots, v_k)$. Whereas the e_I comprise a basis for $\bigwedge^k \mathbb{C}^n$, the dual basis of $(\bigwedge^k \mathbb{C}^n)^*$ consisting of the $p_I := e_I^*$ return homogeneous coordinates called *Plücker coordinates* for $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$, hence for $\mathrm{Gr}_{k,n}$. The Plücker coordinates allow us to express the embedded image $\mathrm{Gr}_{k,n} \subset \mathbb{P}(\bigwedge^k \mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{k}-1}$ as the vanishing set of a homogeneous ideal, to which end we now turn.

Definition 1.2.5. Let the E_{ij} (as in the previous section) comprise the basis of unit entries for $M_{n\times k}$ and $z_{ij} = E_{ij}^* \in M_{n\times k}^*$ their dual basis. If $I = (i_1 < \cdots < i_k) \subset [n]$ then call $[I] = \sum_{j=1}^k E_{i_j j}$ a *partial permutation matrix.* ◇

Lemma 1.2.6. *The Plücker coordinates of $v_1 \wedge \cdots \wedge v_k$ are the maximal minors of $\begin{pmatrix} v_1 & \cdots & v_k \end{pmatrix}$.*

Proof. It is enough to show the result for the e_I. Since $e_I = \hat{\phi}([I])$, this reduces to the claim that $\det([I]_J) = p_J(e_I)$ across suitable choices of I and J, both of which evaluate to δ_{IJ}. □

Write $\mathbb{C}[M_{n\times k}] \cong \mathbb{C}^{nk}$ for the coordinate ring generated by the z_{ij}; $\mathbb{C}[\bigwedge^k \mathbb{C}^n]$ is similarly generated by the p_I. Take $S_{k,n}$ to be the homogeneous coordinate ring for $\mathbb{P}(\bigwedge^k \mathbb{C}^n)$, also over the p_I. We then have the ring map

$$\begin{aligned} S_{k,n} &\to \mathbb{C}[M_{n\times k}]. \\ p_I &\mapsto \det(z_I) \end{aligned} \tag{1.11}$$

The kernel $\mathfrak{i}_{k,n} \subset S_{k,n}$ of this map consists of polynomials in the Plücker coordinates that vanish at points on the embedded Grassmannian. It is known, though we will not prove it, that this ideal vanishes precisely at these points. This leads to the following proposition.

Proposition 1.2.7 ([Ful97])**.** *Pick $I = (i_1 < \cdots < i_k)$, $J = (j_1 < \cdots < j_k) \subset [n]$ and fix $m \in [k]$. Then $\mathrm{Gr}_{k,n} \subset \mathbb{P}^{\binom{n}{k}-1}$ is a projective variety $V(\mathfrak{i}_{k,n})$ with ideal $\mathfrak{i}_{k,n}$ generated by the quadratic Plücker relations*

$$p_I p_J - \sum p_{I'} p_{J'},$$

where the sum is taken over the pairs I', J' obtained from I, J by exchanging $i_1, \ldots, i_m$ with any subset of J of size m, with $p_{I'}$ and $p_{J'}$ signed by the permutations that sort I' and J'.

Example 1.2.8. The Grassmannian $\mathrm{Gr}_{2,4}$ of 2-planes in $\mathbb{C}^4$ is the quotient of $M^\circ_{4\times 2}$ by the right action of GL_2. The "big" Schubert cell $\Omega_{\mathrm{id}} = \Omega_\varnothing \subset \mathrm{Gr}_{2,4}$ has a section in $M^\circ_{4\times 2}$ consisting of matrices of the form

$$\begin{pmatrix} a & c \\ b & d \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $a, b, c, d \in \mathbb{C}$ arbitrary. The big cell therefore has Plücker coordinates $[p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}] = [ad - bc : -c : a : -d : b : 1]$ (in lex order), at which the lone (up to a scalar) quadratic homogeneous polynomial generator $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}$ vanishes. $\diamond$

We will later make use of the inductive system of Grassmannians under the following closed embeddings. Given $n' > n$ with $k' \geq k$ and $n' - k' \geq n - k$, give $\mathbb{C}^n$ a standard basis of e_j across $1 - n + k \leq j \leq k$ and $\mathbb{C}^{n'}$ one of e'_j across $1 - n' + k' \leq j \leq k'$. The inclusion

$$\begin{aligned} \mathbb{C}^n &\to \mathbb{C}^{n'}, \\ e_j &\mapsto e'_j \end{aligned}$$

under which we say $V \mapsto V'$, then induces the natural embedding

$$\begin{aligned} \mathrm{Gr}_{k,n} &\hookrightarrow \mathrm{Gr}_{k',n'}, \\ V &\mapsto V' \oplus F^{\mathrm{opp}}_{k+1} \end{aligned} \tag{1.12}$$

where $F^{\mathrm{opp}}_\bullet$ is the opposite flag in $\mathbb{C}^{n'}$, by way of an induced embedding of exterior products. In particular, (1.12) takes each opposite Schubert cell $\Omega^\lambda \subset \mathrm{Gr}_{k,n}$ to $\Omega^\mu \subset \mathrm{Gr}_{k',n'}$, where the $((n'-k')^{k'})$-complement of μ equals the $((n-k)^k)$-complement of λ.

Example 1.2.9. Recall Example 1.1.17, and consider instead the opposite Schubert cell $\Omega^{[2,3,5,1,4]}_{3,5} = \Omega^{\ydiagram{2,1,1}}_{3,5}$ having the same pivotal rows. The embedding $\mathrm{Gr}_{3,5} \hookrightarrow \mathrm{Gr}_{5,8}$ takes any subspace $V \in \Omega^{\ydiagram{2,1,1}}_{3,5}$ having a matrix of basis vectors

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & & \\ & 1 & \\ a & b & 0 \\ & & 1 \end{pmatrix}$$

to $V' \in \Omega_{5,8}^{\ydiagram{3,3,3,2,2}} \cong \Omega_{5,8}^{[3,4,6,7,8,1,2,5]}$ having basis

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & & & & \\ & 1 & & & \\ a & b & 0 & 0 & 0 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

The new permutation $w' = [3, 4, 6, 7, 8, 1, 2, 5] \in S_8$ is 5-Grassmannian, i.e. a minimum-length representative of a coset of W'/W'_I, where $I = (1, 2, 3, 4, 6, 7) \subset [7]$. $\diamond$

The embedding (1.12) takes place within the embedding of exterior powers given by

$$\begin{aligned} \bigwedge^k \mathbb{C}^n &\hookrightarrow \bigwedge^{k'} \mathbb{C}^{n'}, \\ e_I &\mapsto e_{I \cup [k+1,k']} \end{aligned} \tag{1.13}$$

remembering that now $I \subseteq [1-n+k, k]$. There is then a restriction map of coordinate rings

$$\rho_{k,n}^{k',n'} : S_{k',n'} \to S_{k,n} \tag{1.14}$$

that sends $p_I^{k',n'}$ (a Plücker coordinate of $\bigwedge^{k'} \mathbb{C}^{n'}$) to $p_{I \smallsetminus [k+1,k']}$ when $[1-n+k, k'] \supset I \supset [k+1, k']$ and to 0 otherwise.

Definition 1.2.10. Define the *Schubert varieties*

$$X_\lambda = X_{I_\lambda} = V(p_I \mid I \leq I_\lambda) \subset \mathrm{Gr}_{k,n}$$

and the *opposite Schubert varieties* $X^\lambda = X^{I_\lambda} = V(p_I \mid I \geq I_\lambda)$. From the correspondence between the Bruhat–Chevalley partial order and the boundary conditions on the Schubert cells it follows that $X_\lambda = \overline{\Omega_\lambda}$. $\diamond$

Under the embedding (1.12), each Ω_λ is open in X_λ (and Ω^λ in X^λ), i.e. locally closed. It is the opposites to which we now restrict our focus, in preparation for an embedding and parametrization of Schubert varieties from the affine Grassmannian.

Chapter 2

Infinite flags

The affine Grassmannian $\mathrm{Gr}_{\mathrm{SL}_n}$ [Kac90] admits an embedding into the infinite Grassmannian Gr^0_∞, an ind-variety, in terms of lattices, as shown in [KLMW07]. In this chapter we recover this embedding and specialize it to Schubert cells. We recover much of the combinatorics governing $\mathrm{Gr}_{\mathrm{SL}_n}$ by examining these embedded affine Schubert cells. In addition, we describe a broader class of "heralded" subspaces and cells that help bridge the embedded affine Grassmannian (lattices) and the entire infinite Grassmannian. This weakening of the notion of lattice prompts an important alternative construction of the matrix varieties we shall see in Chapter 4.

2.1 The infinite Grassmannian

Definition 2.1.1. Denote $\mathbb{C}^\infty = \{(a_i)_{i\in\mathbb{Z}} \mid \exists j,\ (i < j \Rightarrow a_i = 0)\}$ the vector space of infinite formal complex linear combinations of coordinate vectors $e_j = (\delta_{ij})_{i\in\mathbb{Z}}$ for $j \in \mathbb{Z}$ with support bounded below. There is a family of vector space isomorphisms τ_k, indexed by $k \in \mathbb{Z}$, given by

$$\begin{aligned} \tau_k : \mathbb{C}^\infty &\to \mathbb{C}^\infty. \\ (a_i) &\mapsto (a_{i-k}) \end{aligned} \tag{2.1}$$

Designate the subspaces $E_j = \{(a_i)_{i\in\mathbb{Z}} \mid i < j \Rightarrow a_i = 0\} \subset \mathbb{C}^\infty$. The *infinite Grassmannian* Gr_∞ is the collection of subspaces $V \subset \mathbb{C}^\infty$ satisfying $E_{1-k} \supset V \supset E_{1+k}$ for some $k \in \mathbb{Z}_{\geq 0}$. Each $V \in \mathrm{Gr}_\infty$ has virtual dimension $\operatorname{vdim} V$ given by

$$\operatorname{vdim} V = \dim(V/V \cap E_1) - \dim(E_1/V \cap E_1). \tag{2.2}$$

◇

The next few results establish a combinatorial framework for the infinite Grassmannian that extends that of the finite, and in particular decomposes Gr_∞ into connected components and each of these into (finite) Schubert cells.

Definition 2.1.2. Let $\mathcal{I}$ denote the set of *almost natural* subsets $I \subset \mathbb{Z}$, for which by definition $\mathbb{Z}_{\leq 0} \cap I$ and $\mathbb{Z}_{>0} \smallsetminus I$ are both finite. Each $I \in \mathcal{I}$ then has a *virtual cardinality*

$$\mathrm{vcard}(I) = |\mathbb{Z}_{\leq 0} \cap I| - |\mathbb{Z}_{>0} \smallsetminus I|. \tag{2.3}$$

For each I designate the subspace $E_I = \{(a_i) \mid i \notin I \Rightarrow a_i = 0\} \in \mathrm{Gr}_\infty$. ◇

Lemma 2.1.3. *Pick $I \in \mathcal{I}$. There is a unique minimum-length (finite) permutation $\sigma_I \in S_\mathbb{Z}$ that takes I as a set to $\mathbb{Z}_{>-k}$ for some $k \in \mathbb{Z}$. In particular, $k = \mathrm{vcard}(I)$.*

Proof. Pick $m \in \mathbb{Z}_{>0}$ so that $\mathbb{Z}_{-m} \supset I \supset \mathbb{Z}_{>m}$ and write $I \cap [1-m, m] = \{i_1 < \cdots < i_{m'}\}$. Taking $k' = m'$ to be the largest index for which $i_{k'} \leq 0$, we then have $\mathrm{vcard}(I)$ for any adjacent transposition s_i, hence for any finite permutation σ: If $i < -m$ or $i > m$ then $s_i(I) = I$ as sets. If $1 - m < i < m - 1$ then $|s_i(I) \cap [1-m, m]| = |I \cap [1-m, m]|$ so the calculation above recovers $\mathrm{vcard}(s_i(I)) = m' - m$. If $i = \pm(1-m)$ then pick a larger value of m, which reduces this case to the previous. It is straightforward that $\mathrm{vcard}(\mathbb{Z}_{\geq k}) = k$ for any $k \in \mathbb{Z}$, so this implies that if $\sigma(I) = \mathbb{Z}_{\geq k}$ as sets for some finite σ then $k = \mathrm{vcard}(I)$.

For existence, recall that there is a unique minimum-length permutation $\sigma \in S_{2m}$—the inverse of a Grassmannian—that takes $I \cap [1-m, m] + m$ to $[m - m' + 1, m]$. Define $\sigma_I \in S_\mathbb{Z}$ by $\sigma_I(i) = \sigma(i + m) - m$. Now σ_I has minimum length subject to the restriction $\sigma \in S_m$ or $\sigma_I \in S_{[1-m,m]}$. Specifically,

$$\sigma_I(i) = \begin{cases} m - m' + j & \text{if } i = i_j + m \\ i - |I \cap [1-m, i-m]| & \text{otherwise.} \end{cases}$$

For uniqueness and minimality in $S_\mathbb{Z}$, note that these are known in the finite setting (for permutations restricted to S). So suppose $\sigma'_I \in S_{[1-M,M]}$ has minimum length among $\sigma \in S_M$ for which $\sigma(I) = \mathbb{Z}_{>-k}$. Then σ'_I is Grassmannian in $S_{[1-M,M]}$, hence fixes indices outside $[1-m, m]$, hence is the image of σ_I under the natural inclusion $S_{[1-m,m]} \hookrightarrow S_{1-M,M}$. □

Definition 2.1.4. We define the *pivots* $\mathrm{piv}\, V = (i_1 < i_2 < \cdots) \subset \mathbb{Z}$ *of* $V \in \mathrm{Gr}_\infty$ to be the indices i for which $\dim(V/V \cap E_i) < \dim(V/V \cap E_{i+1})$ For a single vector $v \in E_m \smallsetminus E_{m+1}$, define $m = \mathrm{piv}(v)$. For $X = V \subset \mathbb{C}^\infty$ or $X = V \subset \mathbb{C}^m$, $X = x \in M$, or $X = v \in V$, let $\mathrm{lpiv}(X) = \min(\mathrm{piv}(X))$, denote the *leading pivot of* X. ◇

Henceforth, in the finite setting, for $V \in \mathrm{Gr}_{h,m}$ we take $\mathrm{piv}(V)$ to mean the pivot set of V with respect to the opposite flag $F_\bullet^{\mathrm{opp}}$.

Lemma 2.1.5. *Pick $V \in \mathrm{Gr}_\infty$. Then* *(a)* $\dim(V/V \cap E_{i+1}) - \dim(V/V \cap E_i) = \begin{cases} 1 & \textit{if } i \in \operatorname{piv}(V) \\ 0 & \textit{otherwise,} \end{cases}$

(b) $\dim(E_i/V \cap E_i) - \dim(E_{i+1}/V \cap E_{i+1}) = \begin{cases} 0 & \textit{if } i \in \operatorname{piv}(V) \\ 1 & \textit{otherwise,} \end{cases}$ *and*

(c) $\operatorname{vcard}(\operatorname{piv}(V)) = \operatorname{vdim}(V)$.

Proof. In proving (a) and (b) it will be helpful to know that

$$\dim(V \cap E_i/V \cap E_{i+1}) = \begin{cases} 1 & \text{if } i \in \operatorname{piv}(V) \\ 0 & \text{otherwise.} \end{cases}$$

We obtain the upper bound 1 because the quotient can only be generated by independent vectors $v \in V \cap E_i \smallsetminus V \cap E_{i+1}$, and if we take any two such vectors v and v' then we can find $\alpha \in \mathbb{C}^*$ for which $v - \alpha v' \in V \cap E_{i+1}$, i.e. $v + V \cap E_{i+1}$ and $v' + V \cap E_{i+1}$ are dependent. Thus

$$\dim(V \cap E_i/V \cap E_{i+1}) = 1 \Leftrightarrow V \cap E_i \neq V \cap E_{i+1} \Leftrightarrow V/V \cap E_i \neq V/V \cap E_{i+1} \Leftrightarrow i \in \operatorname{piv}(V).$$

Claim (a) follows from the observation that the projection $V/V \cap E_{i+1} \twoheadrightarrow V/V \cap E_i$, given by taking the quotient by $V \cap E_i/V \cap E_{i+1}$, is nonzero precisely when $V \cap E_i \neq V \cap E_{i+1}$ and therefore $i \in \operatorname{piv}(V)$; since $\dim(V \cap E_i/V \cap E_{i+1}) = 1$, the projection has nullity 1, hence $\dim(V/V \cap E_{i+1}) - \dim(V/V \cap E_i) = 1$.

For (b), consider the inclusion $E_{i+1}/V \cap E_{i+1} \subset E_i/V \cap E_{i+1}$ and the projection $E_i/V \cap E_{i+1} \twoheadrightarrow E_i/V \cap E_i$. By the same token, the projection has nullity 1 when $i \in \operatorname{piv}(V)$ and 0 otherwise. Meanwhile, $E_i/V \cap E_{i+1} = E_{i+1}/V \cap E_{i+1} \oplus \mathbb{C}(e_i + V \cap E_{i+1})$, so the inclusion has codimension 1. Thus

$$\begin{aligned} \dim(E_i/V \cap E_i) - \dim(E_{i+1}/V \cap E_{i+1}) = {} & (\dim(E_i/V \cap E_i) - \dim(E_i/V \cap E_{i+1})) \\ & + (\dim(E_i/V \cap E_{i+1}) - \dim(E_{i+1}/V \cap E_{i+1})) \end{aligned}$$

takes the value $-1 + 1 = 0$ when $i \in \operatorname{piv}(V)$ and $0 + 1 = 1$ otherwise, as desired.

Take $m \in \mathbb{Z}_{>0}$ so that $E_{-m} \supset V \supset E_m$. Note that this implies that $\dim(V/V \cap E_{i+1}) = \dim(V/V \cap E_i)$ for $i < 1 - m$ and $i > m$, i.e. $\mathbb{Z}_{>1-m} \supset \operatorname{piv}(V) \supset \mathbb{Z}_{>m}$. Recall the formulae (2.2) for virtual dimension and (2.3) for virtual cardinality. From (a) we know that $\dim(V/V \cap E_1) - \dim(V/V \cap E_{1-m})$ counts the pivots of V from $1 - m$ to 0, thus

$$\dim(V/V \cap E_1) - \dim(V/V \cap E_{1-m}) = |\mathbb{Z}_{\leq 0} \cap \operatorname{piv}(V)|.$$

From (b) we know that $\dim(E_1/V \cap E_1) - \dim(E_{1+m}/V \cap E_{1+m})$ counts the non-pivots from 1 to $1 + m$, thus

$$\dim(E_1/V \cap E_1) - \dim(E_{1+m}/V \cap E_{1+m}) = |\mathbb{Z}_{>0} \smallsetminus \operatorname{piv}(V)|.$$

The observations that $\dim(V/V \cap E_{1-m}) = \dim(V/V) = 0$ and that $\dim(E_{1+m}/V \cap E_{1+m}) = \dim(E_{1+m}/E_{1+m}) = 0$ deliver (c). $\square$

Definition 2.1.6. Let us denote by

$$\mathrm{Gr}_\infty^k = \{V \in \mathrm{Gr}_\infty \mid \operatorname{vdim} V = k\}$$

the subcollection of subspaces of virtual dimension k. It will follow from the ind-scheme structure on Gr_∞ that each Gr_∞^k is a connected component in the topology induced from that of the finite Grassmannians. As projective spaces, the Gr_∞^k are seen to be isomorphic via maps induced from (2.1). Hereafter we shall restrict to Gr_∞^0 (hence also to $\mathcal{I}_0$). $\diamond$

The following lemma invokes the permutation σ_I from Lemma 2.1.3.

Lemma 2.1.7. *There is a natural bijection*

$$\begin{aligned} \mathcal{I}_0 &\to \mathrm{Par} \\ I &\mapsto \lambda_I \end{aligned}$$

satisfying $|\lambda_I| = \ell(\sigma_I)$ *and* $I \leq J \Leftrightarrow \lambda_I \supseteq \lambda_J$.

Proof. Write $I = (i_1 < i_2 < \cdots)$. Across $j \in \mathbb{Z}_{>0}$ set $\lambda_j = j - i_j$. The injectivity is immediate and well-definedness follows from $\operatorname{vcard}(I) = 0$, which implies that all $i_j \leq j$ and that eventually $i_j = j$. For surjectivity, take any $\lambda \in \mathrm{Par}$, written as $(\lambda_1, \ldots, \lambda_\ell, 0, 0, \ldots)$ when λ has ℓ parts, and set $I = (1-\lambda_1, \ldots, \ell-\lambda_\ell, \ell+1, \ell+2, \ldots)$. Straightforward calculations verify the additional properties. $\square$

Definition 2.1.8. Given $I \in \mathcal{I}_0$, define the I^{th} *Schubert cell* of Gr_∞^0 to be

$$\Omega_I = \Omega_{\lambda_I} = \{V \in \mathrm{Gr}_\infty^0 \mid \operatorname{piv}(V) = I\}.$$

Provided $\lambda \subseteq ((m-h)^h)$, the preimage of Ω_λ under an embedding $\mathrm{Gr}_{h,m} \hookrightarrow \mathrm{Gr}_\infty^0$ is then the opposite Schubert cell $\Omega_{h,m}^{\lambda^\vee}$, where $\lambda^\vee = (m-h-\lambda_h, \ldots, m-h-\lambda_1)$ is the complement of λ in the $((m-h)^h)$ rectangle (Definition 1.1.16). Thus $\Omega_\lambda \cong \Omega_{h,m}^{\lambda^\vee} \cong \mathbb{C}^{(m-h)h-|\lambda^\vee|} = \mathbb{C}^{|\lambda|}$. Call $X_\lambda = \overline{\Omega_\lambda} \subset \mathrm{Gr}_\infty^0$ the I^{th} Schubert variety. $\diamond$

Recall the projective variety structure of $\mathrm{Gr}_{h,m}$ from Section 1.2. That construction will help us realize Gr_∞^0 as an ind-variety (the limit of an inductive system of varieties).

Definition 2.1.9. Consider the semi-infinite free vector space $F^{\infty/2}\mathbb{C}^\infty$ on $\mathbb{C}^\infty$ generated by formal sequences $(v_1, v_2, \ldots)$ of vectors $v_j \in \mathbb{C}^\infty$ subject to the condition that, for some sufficiently large $m \in \mathbb{Z}_{>0}$, $v_j = e_j$ when $j > m$. Then the *semi-infinite wedge space* $\bigwedge^{\infty/2} \mathbb{C}^\infty$ is the quotient of $F^{\infty/2}\mathbb{C}^\infty$ by the subspace generated by the vectors

- $(v_1, \ldots, v_i, \ldots) + (v_1, \ldots, v_i', \ldots) - (v_1, \ldots, v_i + v_i', \ldots)$,

- $c(v_1,\ldots,v_i,\ldots)-(v_1,\ldots,cv_i,\ldots)$, and
- $(v_1,v_2,\ldots)$ where $v_j=v_{j'}$ for some $j\neq j'$

across $j,j'\in\mathbb{Z}_{>0}$. $\diamond$

For fixed $h<m$, the proof of Lemma 2.1.3 suggests the embedding

$$\begin{aligned} \textstyle\bigwedge^h \mathbb{C}^m &\hookrightarrow \textstyle\bigwedge^{\infty/2}\mathbb{C}^\infty, \\ e_I &\mapsto e_{I\cup[h+1,\infty)} \end{aligned} \tag{2.4}$$

where for $\tilde{I}:=I\cup[h+1,\infty)=(\tilde{\imath}_1,\tilde{\imath}_2,\ldots)$ we mean $e_{\tilde{I}}=e_{\tilde{\imath}_1}\wedge e_{\tilde{\imath}_2}\wedge\cdots$. The image of $\bigwedge^h\mathbb{C}^m$ is then generated by the $e_{\tilde{I}}\in\bigwedge^{\infty/2}\mathbb{C}^\infty$ for which $\tilde{\imath}_1>m-h$ and $\tilde{\imath}_j=j$ for $j>m$. This makes $\bigwedge^{\infty/2}\mathbb{C}^\infty$ an ind-scheme under the following definition.

Definition 2.1.10. An *ind-scheme* is (the inductive limit of) an inductive system of schemes. Similarly, an *ind-variety* is an inductive system of varieties. For our purposes, let

$$X_1\to X_2\to\cdots\to X_i\to X_{i+1}\to\cdots$$

be closed embeddings of schemes (varieties). Then

$$X=\bigcup_{i=1}^{\infty}X_i \tag{2.5}$$

is an ind-scheme (ind-variety). $\diamond$

The embeddings (2.4) make $\bigwedge^{\infty/2}\mathbb{C}^\infty$ the limit of the inductive system (1.13).

Definition 2.1.11. Following the restrictions (1.14), we may take

$$S=\lim_{\infty\leftarrow h,m-h}S_{h,m},$$

a homogeneous polynomial ring over variables p_I with $I\in\mathcal{I}_0$ defined in terms of restrictions to the homogenous coordinate rings $S_{h,m}$ as follows: Write

$$\begin{aligned} \rho_{h,m}: S &\to S_{h,m}, \\ p_I &\mapsto p_{I\cap[1-m+h,h]} \end{aligned}$$

where we take $p_{I\cap[1-m+h,h]}$ to be zero unless $[1-m+h,\infty)\supset I\supset[1+h,\infty)$. $\diamond$

Whereas any vector (finite linear combination of wedge products) in $\bigwedge^{\infty/2}\mathbb{C}^\infty$ satisfies $v_j=e_j$ for $j\gg 1$; and since the e_I across $I\subset[1-m+h,h]$, $|I|=h$, are a basis for $\bigwedge^h\mathbb{C}^m$, every vector in $\bigwedge^{\infty/2}\mathbb{C}^\infty$ can therefore be expressed as a finite linear combination of e_I, $I\in\mathcal{I}_0$. Thus the e_I constitute a basis for $\bigwedge^{\infty/2}\mathbb{C}^\infty$. Definition 2.1.11 then provides that

$$p_I(e_J)=\delta_{IJ}.$$

In particular, the p_I evaluate at $V\in\mathrm{Gr}^0_\infty$ to the *infinite Plücker coordinates of* V.

Definition 2.1.12. Let
$$p_I = e_I^* \in (\textstyle\bigwedge^{\infty/2} \mathbb{C}^\infty)^*, \qquad I \in \mathcal{I}_0$$
comprise the dual basis.

Given $V \in \mathrm{Gr}_\infty^0$, identify $v_1, v_2, \ldots \in V$ so that, for sufficiently large m, V/E_{m+1} has basis $v_1 + E_{m+1}, \ldots, v_m + E_{m+1}$ and $v_i = e_i$ for $i > m$. Call $v_1, v_2, \ldots$ an *ind-basis of* V. ◇

Conversely, if $V \subset E_{m+1}$ and $v_1 + E_{m+1}, \ldots, v_m + E_{m+1}$ are a basis of V/E_{m+1} then $v_1, \ldots, v_m, e_{m+1}, e_{m+2}, \ldots$ constitute an ind-basis of V. The Plücker coordinate p_I of V agrees with the intuitive idea of taking the minor of a $\mathbb{Z} \times \mathbb{Z}_{>0}$ matrix (whose columns are an ind-basis of V) at rows I: Since the matrix is only non-identity in some upper-left submatrix, and since I eventually satisfies $e_j = j$, the infinite minor can be thought of as the minor of the $[m-h, m] \times [h]$ submatrix at rows $(i_1, \ldots, i_h)$ for suitable choices of m and h. This agrees with the finite Plücker coordinate at the preimage of V under (2.5).

Theorem 2.1.13. *For each $V \in \mathrm{Gr}_\infty^0$ identify an ind-basis $v_1, v_2, \ldots$. Then the map*
$$\begin{aligned} \mathrm{Gr}_\infty^0 &\to \mathbb{P}(\textstyle\bigwedge^{\infty/2} \mathbb{C}^\infty) \\ V &\mapsto v_1 \wedge v_2 \wedge \cdots \end{aligned} \tag{2.6}$$
is an embedding with image $\bigcup_{m>h>0} \mathrm{Gr}_{h,m}$.

Proof. The map is induced from the Plücker embeddings (1.10). □

2.2 The affine Grassmannian

The affine Grassmannian is a general construction for reductive linear algebraic groups, and ours is constructed for $G = \mathrm{GL}_n$. See [Gör10] for a more thorough treatment. Our first goal after constructing it will be to embed it in the infinite.

Definition 2.2.1. Take the ring $\mathbf{O} = \mathbb{C}[\![t]\!]$ of formal power series and the field $\mathbf{F} = \mathbb{C}(\!(t)\!)$ of formal Laurent series. The *order* $\operatorname{ord} f$ of a series $f \in \mathbf{F}$ will be the lowest degree of a nonzero term of f, with $\operatorname{ord}(0) = -\infty$. Take D to be the formal disk $\mathrm{Spec}(\mathbf{O}) = \{(0), (t)\}$ and $D^* = D \smallsetminus \{0\}$ the punctured formal disk $\mathrm{Spec}(\mathbf{F}) = \{(0)\}$. Given a reductive linear algebraic group $G(\mathbb{C})$, let $G(\mathbf{O})$ denote the group of algebra morphisms $D \to G(\mathbb{C})$, and $G(\mathbf{F})$ that of morphisms $D^* \to G(\mathbb{C})$. $G(\mathbf{F})$ is the *loop group of* G and $G(\mathbf{O}) < G(\mathbf{F})$ its *positive loop group.* ◇

The loop groups provide one of two important definitions for the affine Grassmannians; we instead take the following lattice description to be definitional and recover the loop group definition as a lemma.

Definition 2.2.2. A *lattice* $L \subset \mathbf{F}^n$ is an $\mathbf{O}$-submodule satisfying

(i) $t^{-m}\mathbf{O}^n \supset L \supset t^m\mathbf{O}^n$ for some $N \in \mathbb{Z}_{>0}$ and

(ii) $t^{-m}\mathbf{O}^n/L$ is finite-rank over $\mathbb{C}$ (i.e. L is full-rank in $\mathbf{F}^n$).

Call $E_1 = \mathbf{O}^n \subset \mathbf{F}^n$ the *standard lattice*. The *(general linear) affine Grassmannian* $\mathrm{Gr}_{\mathrm{GL}_n}$ is the collection of lattices. ◇

Each $g = \begin{pmatrix} g_1 & \cdots & g_n \end{pmatrix} \in \mathrm{GL}_n(\mathbf{F})$ yields a lattice $g_1\mathbf{O} \oplus \cdots \oplus g_n\mathbf{O}$. Observe further that $\mathbf{O}^n$ has generators $e_1, \dots, e_n \in \mathbf{F}^n$, so $\bigwedge^n \mathbf{O}^n = (e_1 \wedge \cdots \wedge e_n)\mathbf{O} \cong \mathbf{O}$.

Definition 2.2.3. Let
$$\mathrm{Gr}^r_{\mathrm{GL}_n} = \{L \in \mathrm{Gr}^r_{\mathrm{GL}_n} \mid \textstyle\bigwedge^n L \cong t^r\mathbf{O}\}.$$
The *(special linear) affine Grassmannian* is $\mathrm{Gr}_{\mathrm{SL}_n} = \mathrm{Gr}^0_{\mathrm{GL}_n}$. ◇

So $\mathbf{O}^n \in \mathrm{Gr}_{\mathrm{SL}_n}$. Now we come to the "$G/P$" description of the affine Grassmannians.

Lemma 2.2.4. *The projection*

$$\begin{aligned} \mathrm{GL}_n(\mathbf{F}) &\to \mathrm{Gr}_{\mathrm{GL}_n} \\ g = \begin{pmatrix} g_1 & \cdots & g_n \end{pmatrix} &\mapsto g_1\mathbf{O} \oplus \cdots \oplus g_n\mathbf{O} \end{aligned} \tag{2.7}$$

has fibers $g\mathrm{GL}_n(\mathbf{O}) \subset \mathrm{GL}_n(\mathbf{F})$*. Consequently,*

$$\mathrm{Gr}_{\mathrm{GL}_n} \cong \mathrm{GL}_n(\mathbf{F})/\mathrm{GL}_n(\mathbf{O}).$$

Similarly, $\mathrm{Gr}_{\mathrm{SL}_n} \cong \mathrm{SL}_n(\mathbf{F})/\mathrm{SL}_n(\mathbf{O})$.

Example 2.2.5. Take $n = 3$ and the three vectors

$$\begin{aligned} v_1 &= (te^{-2t} + 3t^2e^{-2t}, e^t\cos t + 2te^{-2t}, -e^t\cos t + te^t\sin t), \\ v_2 &= (-t^{-1}e^{-2t} - 3e^{-2t}, -2t^{-1}e^{-2t}, 0), \\ v_3 &= (e^{-2t} + 3te^{-2t}, 2e^{-2t} - e^t\sin t, e^t\sin t + te^t\cos t) \in \mathbf{F}^3, \end{aligned}$$

viewing each non-polynomial function as a power series. These generate an $\mathbf{O}$-module $L = v_1\mathbf{O} \oplus v_2\mathbf{O} \oplus v_3\mathbf{O} \subset \mathbf{F}^3$. In particular, $t^{-1}\mathbf{O}^3 \supset L \supset t^2\mathbf{O}^3$ and (it can be shown) $\dim(L/t^2\mathbf{O}^3) = 6 = \dim(\mathbf{O}^3/t^2\mathbf{O}^3)$, which puts $L \in \mathrm{Gr}_{\mathrm{SL}_n}$. Write

$$g = \begin{pmatrix} te^{-2t} + 3t^2e^{-2t} & -t^{-1}e^{-2t} - 3e^{-2t} & e^{-2t} + 3te^{-2t} \\ e^t\cos t + 2te^{-2t} & -2t^{-1}e^{-2t} & 2e^{-2t} - e^t\sin t \\ -e^t\cos t + te^t\sin t & 0 & e^t\sin t + te^t\cos t \end{pmatrix}.$$

We then have $L = g\mathrm{SL}_n(\mathbf{O})/\mathrm{SL}_n(\mathbf{O})$. ◇

Mark the coordinate vectors e'_j of $\mathbb{C}^\infty$. We may then identify $\mathbf{F}^n$ with $\mathbb{C}^\infty$ via the vector space isomorphism

$$\begin{aligned} \mathbf{F}^n &\xrightarrow{\sim} \mathbb{C}^\infty, \\ e_i t^c &\mapsto e'_{cn+i} \end{aligned} \tag{2.8}$$

with $c \in \mathbb{Z}$. The action of $\mathbf{O}$ thus passes to $\mathbb{C}^\infty$, prompting the following definition.

Hereafter less caution is used in viewing this action as a right action. We also abbreviate $\hat{\mathcal{G}} := \mathrm{GL}_n(\mathbf{F})$, $\hat{\mathcal{P}} := \mathrm{GL}_n(\mathbf{O})$, $\mathcal{G} := \mathrm{SL}_n(\mathbf{F})$, and $\mathcal{P} := \mathrm{SL}_n(\mathbf{O})$.

Definition 2.2.6. Write $u = 1 + t \in \mathbf{O}$. Call a vector space $V \subset \mathbb{C}^\infty$ *u-stable* if $u \cdot V = V$, or equivalently if $t \cdot V \subset V$. For any $\mathcal{C} \subset \mathrm{Gr}_\infty$, denote by

$$\mathcal{C}^u = \{V \in \mathcal{C} \mid uV = V\}$$

the subcollection of u-stable subspaces. Call an almost natural subset I *u-stable* if $I = \mathrm{piv}(V)$ for some u-stable V. It is straightforward that I is u-stable when $I + n \subset I$. ◇

Example 2.2.7. The matrix $g = (\, v_1\ v_2\ v_3 \,) \in \mathrm{SL}_3$ from Example 2.2.5 expands into Laurent series as

$$\begin{pmatrix} t + \cdots & -t^{-1} - 1 + 4t + \cdots & 4 - 8t + \cdots \\ 1 + 3t + \cdots & -2t^{-1} + 4 - 4t + \cdots & -t + \cdots \\ -1 - t + \cdots & 0 & 2t + \cdots \end{pmatrix},$$

so the identification $\mathbf{F}^3 \xrightarrow{\sim} \mathbb{C}^\infty$ (2.8) specializes to the vectors

$$v_1 = \begin{pmatrix} \vdots \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 1 \\ -1 \\ \hline 1 \\ 3 \\ -1 \\ \hline \vdots \end{pmatrix}, \quad v_2 = \begin{pmatrix} \vdots \\ \hline -1 \\ -2 \\ 0 \\ \hline -1 \\ 4 \\ 0 \\ \hline 4 \\ -4 \\ 0 \\ \hline \vdots \end{pmatrix}, \quad \text{and} \quad v_3 = \begin{pmatrix} \vdots \\ \hline 0 \\ 0 \\ 0 \\ \hline 4 \\ 0 \\ 0 \\ \hline -8 \\ -1 \\ 2 \\ \hline \vdots \end{pmatrix}$$

in $\mathbb{C}^\infty$, where the coordinates depicted are indexed -2 through 6. ◇

Theorem 2.2.8. *The isomorphism (2.8) induces an embedding*

$$\varphi : \mathrm{Gr}_{\mathrm{GL}_n} \hookrightarrow \mathrm{Gr}_\infty \tag{2.9}$$

having images

$$\varphi(\mathrm{Gr}_{\mathrm{GL}_n}) = {\mathrm{Gr}_\infty}^u \quad \textit{and} \quad \varphi(\mathrm{Gr}_{\mathrm{SL}_n}) = (\mathrm{Gr}^0_\infty)^u.$$

Proof. That (2.8) sends each $L \in \mathrm{Gr}_{\mathrm{GL}_n}$ to a u-stable subspace in Gr_∞ is straightforward. So consider any u-stable $V \subset \mathbb{C}^\infty$. For each $j \in [n]$ let $j + c_j n \in I = \mathrm{piv}(V)$ be the first pivot congruent to j, so that $I = \{j + c_j n\}_{j=1}^n + n\mathbb{Z}_{>0}$. Identify, for each j, $v_j \in V$ with $\mathrm{piv}(v_j) = j + c_j n$. For each $i, j \in [n]$, take $a_{ij} = \sum_{c=c_j}^\infty {e'_{i+cn}}^* v_j t^c \in \mathbf{F}$. Then each $\hat{v}_j := (a_{1j}, \dots, a_{nj}) \in \mathbf{F}^n$, and in each case $t^{c_j} e_j \in \hat{v}_j \mathcal{P}$, which means that the span of the $\hat{v}_j$ is full-rank.

For the image of $\mathrm{Gr}_{\mathrm{SL}_n}$, take $\hat{\mathcal{G}}_0$ to be the preimage under (2.9) of $\mathrm{Gr}^0_{\mathrm{GL}_n}$ (the identity component of $\mathcal{G}$). The composition $\mathcal{G} \hookrightarrow \hat{\mathcal{G}}_0 \to \hat{\mathcal{G}}_0/\hat{\mathcal{P}}$ sends precisely those morphisms $D^* \to \mathrm{SL}_n(\mathbb{C})$ in $\mathcal{G}$ to zero that are (restrictions of) morphisms $D \to \mathrm{GL}_n(\mathbb{C})$ in $\hat{\mathcal{P}}$; but these are precisely (restrictions of) the morphisms $D \to \mathrm{SL}_n(\mathbb{C})$ of $\mathcal{P}$. □

For the remainder of the text we treat (2.9) as a subspace inclusion. For instance, for each $h \in \mathbb{Z}$ the lattice $\bigoplus_{i=1}^n \mathbf{O}t^h e_i$ has image E_{hn+1}.

$\mathrm{Gr}_{\mathrm{GL}_n}$ has a connected component $\mathrm{Gr}^k_{\mathrm{GL}_n}$ for every integer k. The component $\mathrm{Gr}^k_{\mathrm{GL}_n} = \mathrm{Gr}_{\mathrm{GL}_n} \cap \mathrm{Gr}^k_\infty = (\mathrm{Gr}^k_{\mathrm{GL}_n})^u$ consists of the lattices of virtual dimension k, has basepoint E_{k+1}, and is isomorphic to $\mathrm{Gr}^0_{\mathrm{GL}_n}$ under the map induced from (2.1) by the quotient $\hat{\mathcal{G}}/\hat{\mathcal{P}}$.

Definition 2.2.9. Define the Iwahori subgroup $\hat{\mathcal{B}} = \{b = (b_{ij}) \in \hat{\mathcal{P}} \mid b(0) \in B_-\}$, where B_- is the opposite Borel group (Section 1.1); the subgroup $\hat{\mathcal{U}} = \{u = (u_{ij}) \in \hat{\mathcal{P}} \mid u(0) \in U_-\}$ of $\hat{\mathcal{B}}$ is the *prounipotent radical of* $\mathcal{B}$ [LM82]. Define $\mathcal{U} < \mathcal{B} < \mathcal{P}$ analogously.

$\hat{\mathcal{U}}$ contains the off-diagonal one-parameter subgroups

$$\begin{aligned} \theta^b_{ij} : \mathbb{C} &\to \hat{\mathcal{G}} \\ a &\mapsto I + at^b E_{ij} \end{aligned}$$

across *distinct* indices $i, j \in [n]$ satisfying $b \geq \left\lceil \frac{j-i}{n} \right\rceil$, which we take to generate the subgroup $\hat{\mathcal{U}}' \leq \hat{\mathcal{U}}$. The subgroup $\mathcal{U}' \leq \mathcal{U}$ is defined analogously. While we do not verify it here, it is intuitive on inspection that each subgroup is proper. We shall make use of these subgroups in Section 3.4. ◇

To avoid constructing affine and extended affine Weyl groups from scratch, we adopt the characterizations in Definition 2.2.11, proved in [KLMW07] and at [Kac90] Proposition 6.5, respectively, as definitions.[1] These require some preparation to state. The general statements involve the finite Weyl group $W \cong S_n$, the coroot lattice $Q^\vee \cong \mathbb{Z}^n_0 := \{c \in \mathbb{Z}^n \mid c_1 + \cdots + c_n = 0\}$, and the group of translations in the coweight lattice $P^\vee \cong \mathbb{Z}^n/\mathbb{Z}(1^n)$.

The representation (1.3) determines the representation

$$\begin{aligned} \hat{\mathcal{G}} &\to \mathrm{GL}_n(\mathbf{F}) \\ g &\mapsto (\, ge_1 \ \cdots \ ge_n \,) \end{aligned}$$

[1] Beware that the "extended affine Weyl group" discussed in [KLMW07] corresponds to our "auxiliary affine Weyl group", which is given the same label $\tilde{W}$.

in $M_{n\times n}(\mathbf{F})$. S_n embeds into $\hat{\mathcal{G}}$ via the natural embeddings $S_n \subset \mathrm{GL}_n \subset \hat{\mathcal{G}}$. The additive group action $c\cdot f = t^c f$ of $\mathbb{Z}$ on $\mathbf{F}$ extends to a diagonal embedding $\mathbb{Z}^n \hookrightarrow \hat{\mathcal{G}}$ by $(c_1,\ldots,c_n) \mapsto \mathrm{diag}(t^{c_1},\ldots,t^{c_n})$. For each $c \in \mathbb{Z}^n$ designate the $\mathbf{O}$-lattice $L_c = \bigoplus_{i=1}^n \mathbf{O}t^{c_i}$. Since then $\sigma e_{dn+i} = e_{dn+\sigma(i)}$ and $c e_{dn+i} = e_{(c+d)n+i}$ for $\sigma \in S_n$, $c \in \mathbb{Z}^n$, $d \in \mathbb{Z}$, and $i \in [n]$ under the map induced by (2.8), these embeddings also provide a group endomorphism

$$\mathbb{Z}^n S_n \hookrightarrow S_{\mathbb{Z}} \tag{2.10}$$

into the group of bijections on the integers via the action on the ind-basis of e_j. We then have $\tilde{S}_n$ acting on $\mathbb{Z}$ by

$$s_i(p) = \begin{cases} p+1 & \text{if } p \equiv i \mod n \\ p-1 & \text{if } p \equiv i+1 \mod n \\ p & \text{otherwise.} \end{cases}$$

Lemma 2.2.10. *We have semidirect products $\mathbb{Z}^n \rtimes S_n$ and $\mathbb{Z}_0^n \rtimes S_n$.*

A concept used in the proof is that of a *rook placement*, which we take to be a matrix having no two nonzero entries in the same row or column. That any product of rook placements is a rook placement, and that each nonzero entry of such a product is a (single) product of entries of the factor rook placements, is straightforward.

Proof. The generators of $\mathbb{Z}^n$ and of S_n are rook placements, which renders any $w \in \mathbb{Z}^n S_n$ a rook placement as well. Since $w \in \hat{\mathcal{G}}$ is invertible, every column must have a nonzero entry. So suppose that w has nonzero entries at $(\sigma(j), j)$, where $\sigma \in S_n$. Again by the properties of rook placements these entries must be powers of t; take the entry in row i to be t^{c_i}. Thus $w = (t^{c_{\sigma(j)}}\delta_{i,\sigma(j)})_{ij} = \mathrm{diag}(t^{c_1},\ldots,t^{c_n})(\delta_{i,\sigma(j)})_{ij} = c\sigma$ with $c \in \mathbb{Z}^n$ and $\sigma \in S_n$.

For uniqueness, suppose that $c\sigma = c'\sigma'$ with $c' \in \mathbb{Z}^n$ and $\sigma' \in S_n$. As rook placements, the nonzero entries of $c\sigma$ and $c'\sigma'$ agree, hence $\sigma = \sigma'$, and this then delivers $c = c'\sigma'\sigma^{-1} = c'$.

To see that $\mathbb{Z}^n$ is normal in $\mathbb{Z}^n S_n = \mathbb{Z}^n \times S_n$, it suffices since $\mathbb{Z}^n$ is abelian to see that

$$\sigma^{-1}c\sigma = (\delta_{\sigma(i),j})_{ij}(t^{c_i}\delta_{ij})_{ij}(\delta_{i,\sigma(j)})_{ij} = (\sum_{k,l}\delta_{\sigma(i),k}t^{c_k}\delta_{kl}\delta_{l,\sigma(j)})_{ij} = (t^{c_{\sigma(i)}}\delta_{ij})_{ij} \in \mathbb{Z}^n$$

for all $c \in \mathbb{Z}^n, \sigma \in S_n$.

The details are the same for $\mathbb{Z}_0^n \rtimes S_n$, with the additional observation that $|\det(c)| = |\det(\sigma)| = 1$ for all $c \in \mathbb{Z}_0^n, \sigma \in S_n$, a property conserved by multiplication.

The quotient semidirect product follows from $\mathbb{Z}(1^n) \lhd \mathbb{Z}^n$ (trivial since $\mathbb{Z}^n$ is commutative) and general properties. □

Definition 2.2.11. Designate the *affine Weyl group*

$$\mathcal{W} = Q^\vee \rtimes W \cong \mathbb{Z}_0^n \rtimes S_n < \mathcal{G}.$$

Let us also refer to the *auxiliary affine Weyl group* [KLMW07]

$$\tilde{\mathcal{W}} \cong \mathbb{Z}^n \rtimes S_n < \hat{\mathcal{G}},$$

from which we can get the *extended affine Weyl group*

$$\hat{\mathcal{W}} = P^\vee \rtimes S_n \cong \mathbb{Z}^n/\mathbb{Z}(1^n) \rtimes S_n.$$

Collect the minimum-length coset representatives of $\mathcal{W}/S_n$ into $\mathcal{W}^{\mathcal{P}}$ and those of $\tilde{\mathcal{W}}/S_n$ into $\tilde{\mathcal{W}}^{\mathcal{P}}$, and call these representatives the *Grassmannian elements of* $\mathcal{W}$ *and of* $\tilde{\mathcal{W}}$, respectively. In particular, when $w\mathcal{P}/\mathcal{P} \subset \mathrm{Gr}^k_\infty$ call w a *k-Grassmannian permutation.* $\diamond$

We may characterize $\mathcal{W}$, $\tilde{\mathcal{W}}$, $\mathcal{W}^{\mathcal{P}}$, and $\tilde{\mathcal{W}}^{\mathcal{P}}$, under the embedding (2.10), in terms of almost natural subsets.

Lemma 2.2.12. *We have Weyl group characterizations*

$$\begin{aligned}\tilde{\mathcal{W}} &= \{w \in S_{\mathbb{Z}} \mid \forall i \in \mathbb{Z},\ w(i+n) = w(i)+n\}\\ \mathcal{W} &= \{w \in \tilde{\mathcal{W}} \mid \mathrm{vcard}(w(\mathbb{Z}_{>0})) = 0\}.\end{aligned}$$

Moreover, we have bijections

$$\begin{aligned}\tilde{\mathcal{W}}^{\mathcal{P}} &\cong \mathbb{Z}^n \to \{I \in \mathcal{I} \mid I+n \subset I\}\\ \mathcal{W}^{\mathcal{P}} &\cong \mathbb{Z}^n_0 \to \{I \in \mathcal{I}_0 \mid I+n \subset I\}\end{aligned}$$

between Grassmannian permutations and u-stable almost natural subsets.

Proof. For the characterization of $\tilde{\mathcal{W}}$, check that $\sigma(i+n) = \sigma(i)+n$ for $\sigma \in S_n$ and that $c(i+n) = c(i)+n$ for $c \in \mathbb{Z}^n$. That of $\mathcal{W}$ then follows from Definition 2.2.11.

The latter bijection sends $w \in \mathcal{W}^{\mathcal{P}}$ to $w(\mathbb{Z}_{>0}) \in \mathcal{I}_0$. To check that it is one-to-one, observe that $\sigma([n]) = [n]$, and therefore that as sets $w\sigma(\mathbb{Z}_{>0}) = w(\mathbb{Z}_{>0})$, for any $\sigma \in S_n$.

To check that it is onto, pick any u-stable set $I \in \mathcal{I}_0$. For each $s \in [n]$ let $\hat{\imath}_s$ be the first member of I congruent to s modulo n, and take $\hat{I} = (\hat{\imath}_1, \dots, \hat{\imath}_n)$. As sets, $I = \hat{I} + n\mathbb{Z}_{>0}$; by virtual cardinality zero, if we write $\hat{\imath}_s = s + c_s n$ then we must have $\sum_{s=1}^n c_s = 0$. Therefore, if we let $w' = [\hat{\imath}_1, \dots, \hat{\imath}_n]$ then we get $w' \in \mathcal{W}$, and also $I = w'(\mathbb{Z}_{>0})$ as sets by the aforeshown characterizations. Find $\sigma \in S_n$ for which $\hat{\imath}_1 < \cdots < \hat{\imath}_n$ in order to get $w := w'\sigma \in \mathcal{W}^{\mathcal{P}}$.

The former bijection obtains by translating the argument for the latter along $\mathbb{Z}$.

Still need to prove the bijections. $\square$

Proposition 2.2.13 ([Kum02] Theorem 6.2.8)**.** *We have the Birkhoff decomposition*

$$\hat{\mathcal{G}} = \bigsqcup_{w \in \tilde{\mathcal{W}}^{\mathcal{P}}} \hat{\mathcal{U}} w \hat{\mathcal{P}}. \tag{2.11}$$

Definition 2.2.14. The quotient by $\hat{\mathcal{P}}$ produces the Schubert decomposition

$$\hat{\mathcal{G}}/\hat{\mathcal{P}} = \bigsqcup_{w \in \tilde{\mathcal{W}}^{\mathcal{P}}} \hat{\mathcal{U}} w \hat{\mathcal{P}}/\hat{\mathcal{P}}, \tag{2.12}$$

into *affine Schubert cells* $\Omega_w := \hat{\mathcal{U}} w \hat{\mathcal{P}}/\hat{\mathcal{P}}$. Define also the *affine Schubert varieties* $X_w = \overline{\Omega_w}$. $\diamond$

Example 2.2.15. Retrieve g from Example 2.2.5 under $n = 3$, the 0-Grassmannian permutation $w = [-2, 2, 6]$, and the matrix

$$p = \begin{pmatrix} t^2 e^{-2t} & -e^{-2t} & te^{-2t} \\ e^t \cos t & 0 & -e^t \sin t \\ e^t \sin t & 0 & e^t \cos t \end{pmatrix} \in \mathcal{P}.$$

Matrix multiplication yields $wp = (\, e_{-2} \;\, e_2 \;\, e_6 \,)p = (\, t^{-1}e_1 \;\, e_2 \;\, te_3 \,)p = g$, so $L = g\mathcal{P}/\mathcal{P} \in \hat{\mathcal{U}} w\mathcal{P}/\mathcal{P} = \Omega_{[-2,2,6]}$. $\diamond$

The Bruhat–Chevalley order on $\tilde{\mathcal{W}}$ induces an order on $\tilde{\mathcal{W}}^{\mathcal{P}}$ as in the finite cases. Also as before, the boundary conditions among the Ω_w respect this order: $\Omega_v \subset \partial\Omega_w \Leftrightarrow v < w$. (Note that this generalizes the boundary relations among finite Schubert cells.)

The simple reflections $s_1, \ldots, s_{n-1}$ generate $W = S_n$ as a Coxeter group. We may realize $\mathcal{W}$ as a Coxeter group using the additional reflection $s_0 = t_\theta s_\theta = (1, 0, \ldots, 0, -1) \cdot [n, 2, \ldots, n-1, 1]$ as in Example A.2.4. $\mathcal{W}$ then has matrix representation

$$s_0 = \begin{pmatrix} t & & \\ & I_{n-2} & \\ & & t^{-1} \end{pmatrix} \begin{pmatrix} & & 1 \\ & I_{n-2} & \\ 1 & & \end{pmatrix} = \begin{pmatrix} & & t \\ & I_{n-2} & \\ t^{-1} & & \end{pmatrix}.$$

Example 2.2.16. Still using $n = 3$, the permutation $w = [-2, 2, 6] \in \mathcal{W}^{\mathcal{P}}$ factors as

$$[-2, 2, 6] = [3, 2, 1]s_0 = [3, 1, 2]s_2 s_0 = [1, 3, 2]s_1 s_2 s_0 = s_2 s_1 s_2 s_0$$

with matrix representation

$$w = \begin{pmatrix} t^{-1} & & \\ & 1 & \\ & & t \end{pmatrix} = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \begin{pmatrix} & & t \\ & 1 & \\ t^{-1} & & \end{pmatrix}.$$

$\diamond$

In the next section we shall refine the embeddings in [KLMW07].

Chapter 3

Heralded spaces

This chapter defines a collection of "heralded subspaces" of $\mathbb{C}^\infty$ that soften the transition between Gr_∞ and $\mathrm{Gr}_{\mathrm{GL}_n}$. Each section extends concepts from the previous chapters to the heralded setting: Section 3.1 introduces the framework of heralding to characterize the affine Grassmannian, and Section 3.2 connects affine Schubert cells to the combinatorics of the affine symmetric group. Section 3.3 uses the framework thus established to describe a broader class of subspaces, which have in common with u-stable subspaces canonical bases that respect this property and a decomposition into embedded affine spaces (cells) in Gr_∞. The formulations here will aid the proofs in Chapter 4.

3.1 Lattices as heralded spaces

This section introduces the framework of "heralding", a loosening of u-stability, by which we characterize spaces, their pivots, and their ind-bases in the embedded affine Grassmannian.

Lemma 3.1.1. *For all $m \in \mathbb{Z}$, $\hat{\mathcal{U}} E_m = E_m$.*

Proof. $\hat{\mathcal{U}}$ is generated by the one-parameter subgroups θ^b_{ij} for which $b \geq \left\lceil \frac{j-i}{n} \right\rceil$. Taking $m' \geq m$ so that the $e_{m'}$ form an ind-basis of E_m, and writing $m' = j' + c'n$ with $j' \in [n]$, we have

$$\theta^b_{ij}(a)\cdot e_{m'} = (I + at^b E_{ij}) e_{m'} = e_{m'} + at^b E_{ij} t^{c'} e_{j'} = e_{m'} + at^{b+c'} \delta_{j=j'} e_j = e_{m'} + a\delta_{j=j'} e_{m'} + i - j + bn,$$

which can be seen to lie in E_m with the observation, assuming $j = j'$, that $m' + i - j + bn \geq m' + i - j + \frac{j-i}{n} n = m' \geq m$, hence that $e_{m'+i-j+bn} \in E_m$. $\square$

Thus $\mathbf{O}$ acts on the quotient $E_{1-m+h}/E_{1+h} \cong \mathbb{C}^m$ as follows:

$$t \cdot e_i = \begin{cases} e_{i+n} & \text{if } i \leq h-n \\ 0 & \text{otherwise.} \end{cases}$$

Our goal in this section is to provide local coordinates for $\Omega_w \subset \mathrm{Gr}_{h,m}$. We do this by way of a standard basis for any $V \in \Omega_w$ called a heralded basis, which will be determined by the pivot indices of V and a fixed (for fixed w) number of free coordinates. The heralded basis will provide a canonical preimage of V under the column-span map

$$\pi : M^\circ_{m\times h} \to \mathrm{Gr}_{h,m}.$$

The free entries in this matrix form will then determine local parametrizations $\mathbb{C}^{\ell(w)} \to \Omega_w$.

Definition 3.1.2. Call a finite pivot set $I \subseteq [m]$ *u-stable* if whenever $i \in I$ either $i+n > m$ or $i+n \in I$. If I is u-stable and $i \in I$ but $i-n \notin I$ then call i a *heralding index of* I; if $I = \mathrm{piv}(V)$ for $V \in {\mathrm{Gr}_{h,m}}^u$ then call i a *heralding pivot of* V. Denote the collection of (at most n) heralding pivots by $\hat{I} \subseteq I$. $\diamond$

Every lattice $V \in \mathrm{Gr}_{\mathrm{GL}_n}$ has u-stable pivot set $\mathrm{piv}(V)$. Corollary 3.1.3 verifies that $\mathrm{piv}(V)$ is constant for all $V \in \Omega_w$; we designate this pivot set $\mathrm{piv}(w)$. Note that $w\hat{\mathcal{P}}/\hat{\mathcal{P}} \in \mathrm{Gr}_{\mathrm{SL}_n}$ is the lattice $E_{w(\mathbb{Z}_{>0})}$; denote this lattice E_w.

Corollary 3.1.3. *As an affine Schubert cell and a Schubert cell, respectively,* $\Omega_w \subseteq \Omega_{w(\mathbb{Z}_{>0})}$. *Consequently* $X_w \subseteq X_{w(\mathbb{Z}_{>0})}$.

Proof. Since $\Omega_w = \hat{\mathcal{U}}w\hat{\mathcal{P}}/\hat{\mathcal{P}} = \hat{\mathcal{U}}E_{w(\mathbb{Z}_{>0})}$, it is enough to invoke Lemma 3.1.1. $\square$

Example 3.1.4 ([KLMW07] Section 4.1)**.** With $n > 1$ fixed and $s \in \mathbb{Z}_{>0}$, designate

$$w_s = \begin{pmatrix} t^{-s(n-1)} & & & \\ & t^s & & \\ & & \ddots & \\ & & & t^s \end{pmatrix} = [1 - sn(n-1), 2 + sn, \ldots, (s+1)n] \in \mathcal{W}^{\mathcal{P}}.$$

Then $X_{w_s} \hookrightarrow \mathrm{Gr}_{sn,sn^2}$, in fact $X_{w_s} = (\mathrm{Gr}_{sn,sn^2})^u$, under (2.9). This gives $\mathrm{Gr}_{\mathrm{SL}_n} = \bigcup_{s=1}^\infty X_{w_s} \subset \mathrm{Gr}^0_\infty$ the structure of an ind-subvariety. $\diamond$

Next we obtain local coordinates for ${\Omega_{w(\mathbb{Z}_{>0})}}^u$, which imply the converse of the preceding lemma as well as provide a natural canonical form for the matrices in $\pi^{-1}({\mathrm{Gr}_{h,m}}^u)$. It will help to first state some pivot set arithmetic for subspaces in Gr_∞.

Lemma 3.1.5. *Suppose* $U, V \in \mathrm{Gr}_\infty$ *with* $U \subset V$. *Then*

(a) $\mathrm{piv}(U) \subset \mathrm{piv}(V)$ *with* $\dim(V/U) = |\mathrm{piv}(V) \setminus \mathrm{piv}(U)|$; *and*

(b) if V *is u-stable then* $\mathrm{piv}(U) + n\mathbb{Z}_{\geq 0} \subseteq \mathrm{piv}(V)$.

Proof. For (a), the negation of the containment would imply the existence of a vector $v \in U$ having $\operatorname{piv}(v) \notin \operatorname{piv}(V)$, hence $v \notin V$. Take $m \in \mathbb{Z}$ so that $E_m \subset U$. Then $\dim(V/U) = \dim((V/E_m)/(U/E_m))$ and $|\operatorname{piv}(V) \smallsetminus \operatorname{piv}(U)| = |\operatorname{piv}(V/E_m) \smallsetminus \operatorname{piv}(U/E_m)|$, and the dimension count follows from finite subset arithmetic on $U/E_m \subseteq V/E_m$.

For (b), pick a pivot $i \in \operatorname{piv}(U)$ and $c \in \mathbb{Z}_{\geq 0}$. Then there is a vector $v \in U$ with $\operatorname{piv}(v) = i$, and $v \in V$ implies that $t^c v \in V$. Thus $i + cn = \operatorname{piv}(t^c v) \in \operatorname{piv}(V)$. □

Lemma 3.1.6. *If* $V \in {\Omega^{h,m}_{w(\mathbb{Z}_{>0})}}^u$ *and* $I = w(\mathbb{Z}_{>0}) \cap [1-m+h, h] = (i_1 < \cdots < i_h)$ *then* V *has a unique basis* $v_1, \ldots, v_h$ *satisfying*

(i) $\operatorname{piv}(v_j) = i_j$,

(ii) if $i_{j'} \notin \hat{I}$ *then* $v_{j'} = tv_j$ *for* $i_{j'} = i_j + n$, *and*

(iii) if i_j *is a heralding pivot then* $e^*_k(v_j) = \delta_{i_j k}$ *for all* $k \in I$,

where e^*_k *returns the* k^{th} *entry of a column vector.*

Definition 3.1.7. We will call the $v_1, \ldots, v_h$ in the proposition the *heralded basis for* V and say that the matrix $\begin{pmatrix} v_1 & \cdots & v_h \end{pmatrix}$ is in *heralded form.* ◇

Proof of Lemma 3.1.6. For existence, pick $V \in {\Omega_{w(\mathbb{Z}_{>0})}}^u$. Assume that $v_1, \ldots, v_{j-1} \in V$ so that $v_1 + V \cap E_{i_j}, \ldots, v_{j-1} + V \cap E_{i_j}$ comprise a basis for $V/V \cap E_{i_j}$ that satisfies (i), (ii), and (iii) for the pivot set $I \cap [1-m+h, i_j - 1]$. Take $i_0 = -m+h$ and $i_{h+1} = h+1$ for notational convenience, and notice that the empty subset of V satisfies these conditions for $V/V \cap E_{i_0} \cong \mathbb{C}^0$ and $I \cap \varnothing = \varnothing$. We shall identify a vector $v_j \in V$ so that these conditions are satisfied for $V/V \cap E_{i_{j+1}}$ and $I \cap [1-m+h, i_{j+1} - 1]$. How v_j is chosen depends on whether i_j is heralding.

- If i_j is heralding then we may identify $v'_j \in (V \cap E_{i_j}) \smallsetminus E_{i_j+1}$, which satisfies (i). Since $e^*_{i_j}(v'_j) \neq 0$, we may take $v''_j = \frac{v'_j}{e^*_{i_j}(v'_j)}$, which preserves (i) and provides that $e^*_{i_j}(v''_j) = 1$. We may furthermore locate, for any $j' > j$, $v''_{j'} \in (V \cap E_{i_{j'}}) \smallsetminus E_{i_{j'}+1}$ similarly. By finding $v''_{j'}$ and taking $v^{(j')}_j - e^*_{i_j}(v^{(j'-1)}_j) v''_{j'}$ sequentially over $j+1 \leq j' \leq h$, we produce $v^{(h)}_j$ satisfying (i) and (iii). Set $v_j = v^{(h)}_j$.
- If instead $i_j = i_{j'} + n$ for some $j' < j$ then since V is u-stable we must have $tv_{j'} \in (V \cap E_{i_j}) \smallsetminus E_{i_j+1}$. Set $v_j = v_{j'}$, which satisfies (ii) and therefore (i) and (iii) by u-stability.

In each case we get $\operatorname{Span}\{v_1 + V \cap E_{i_{j+1}}, \ldots, v_j + V \cap E_{i_{j+1}}\} = V/(V \cap E_{i_{j+1}})$ since no pivots of V lie between i_j and i_{j+1}.

For uniqueness, suppose that $v_1', \ldots, v_h' \in V$ also satisfy the conditions of the proposition. It will suffice by (ii) to show that the heralding vectors in each basis agree at each e_k^*. So suppose that at least some $v_j' \neq v_j$ and take k to be minimum for which $e_k^*(v_j) \neq e_k^*(v_j')$ at some j. We then have $v_j - v_j' \in V \cap E_k \smallsetminus E_{k+1}$, hence that $k \in I$. This runs contrary to (iii), rendering the supposition false. $\square$

Corollary 3.1.8. *If* $y \in \pi^{-1}(X_w^{h,m})$ *then* $\mathrm{Stab}_{\mathrm{GL}_h}(y) = \mathrm{id}_{\mathrm{GL}_h}$.

Example 3.1.9. Take $n = 3$, $m = 6$, $h = 3$, and $w = [-2, 2, 6]$. Then

$$\mathrm{piv}(w) = \{-2, 2, 6\} + n\mathbb{Z}_{>0} = (-2, 1, 2, 4, 5, \ldots),$$

which contains $\mathbb{Z}_{>3}$ and restricts to $I = (-2, 1, 2) \subset [-2, 3]$. A matrix $\pi^{-1}(\Omega_w^{3,6}) \subset \mathrm{Gr}_{3,6}$ has the heralded reduced form

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \\ c & b & d \end{pmatrix}$$

$\diamond$

The following theorem generalizes Example 3.1.4 and facilitates the key construction in the next chapter.

Theorem 3.1.10. *Pick* $w \in \mathcal{W}^{\mathcal{P}}$ *and set* $I = \mathrm{piv}(w)$. *Then* $\Omega_w = (\Omega^I)^u = (\Omega^{\gamma'})^u$ *and* $X_w = (X^I)^u = (X^{\gamma'})^u$.

Proof. The heralded basis of any $V \in \Omega_w$ provides an explicit construction of V via the action of a finite product p of generating one-parameter subgroups of $\hat{\mathcal{U}}$ on $E_{w(\mathbb{Z}_{>0})}$, hence as an element of $\hat{\mathcal{U}} E_{w(\mathbb{Z}_{>0})} = \Omega_w$. To construct p, we first construct p^{-1}: Let i be the smallest heralding pivot of $w(\mathbb{Z}_{>0})$ having heralding vector v ($\mathrm{piv}(v) = i$) not equal to e_i. Pick the smallest index $i' > i$ at which v has a nonzero coordinate a_1. Then we may set $b_1 = \left\lfloor \frac{i'-i}{n} \right\rfloor$ and locate $i_1, j_1 \in [n]$ with $i_1 \equiv i$ and $j_1 \equiv i'$ so that $V_1 = \theta_{i_1 j_1}^{b_1}(a_1)V$ is generated by the same heralded basis as V except with the i'^{th} coordinate of v set to zero. Iterating this process produces $E_w = V_{\ell'} = \theta_{i_{\ell'} j_{\ell'}}^{b_{\ell'}}(a_{\ell'}) \cdots \theta_{i_1 j_1}^{b_1}(a_1)V$, so that $V = pE_w \in \hat{\mathcal{U}} w \hat{\mathcal{P}} / \hat{\mathcal{P}}$ where $p = \prod_{\nu=1}^{\ell'} \theta_{i_\nu j_\nu}^{b_\nu}(a_\nu)^{-1}$.

The above provides that

$$X_w = \overline{\Omega_w} = \overline{(\Omega^I)^u} = \overline{\Omega^I}^u = (X^I)^u$$

because $\mathrm{Gr}_{\mathrm{SL}_n}$ is closed in Gr_∞^0. $\square$

A consequence of Corollary 3.1.3 and Theorem 3.1.10 will prove useful in the next chapter, as we characterize when a span of vectors can be "augmented" to a subspace corresponding to a point in an embedded affine Schubert variety.

Corollary 3.1.11. *If $w, v \in \mathcal{W}^{\mathcal{P}}$ with $X_w^{h,m}, X_v^{h,m} \subset \mathrm{Gr}_{h,m}$, $P = \mathrm{piv}(w)$, and $Q = \mathrm{piv}(v)$, and if $p_\nu \leq q_\nu$ for $\nu = 1, \ldots, h$, then $X_v^{h,m} \subseteq X_w^{h,m}$.*

Together with Lemma 2.1.7, Theorem 3.1.10 identifies, for a minimum-length coset representative $w \in \tilde{\mathcal{W}}^{\mathcal{P}}$, the maximum partition λ (equivalently, the minimum Schubert cell $\Omega_{h,m}^\lambda$) such that $X_w^{h,m} \subseteq X_{h,m}^\lambda$.

3.2 Parametrization of heralded spaces

We shall now use the heralded basis to parametrize the embedding $\Omega_w^{h,m} \hookrightarrow \Omega_{h,m}^\lambda$. For this section fix $n \geq 2$, $w \in \mathcal{W}^{\mathcal{P}}$, and $\Omega_w^{h,m} \subset \mathrm{Gr}_{h,m}$.

Definition 3.2.1. The *n-core $\gamma(w)$ of w* is the conjugate of the partition $\lambda_{\mathrm{piv}(w)}$ associated via Lemma 2.1.7 with the pivot set $w(\mathbb{Z}_{>0})$. Explicitly, $\gamma'_j = j - i_j$ for $j \geq 1$, which gives $\gamma'_j = 0$ for $j > h$. (Since (2.9) preserves indices, this also works in the finite setting.) ◇

Since $\Omega_w = \Omega_{w(\mathbb{Z}_{>0})}{}^u$, and since each u-stable $I \not\geq w(\mathbb{Z}_{>0})$ carries a standard lattice $E_I \in (\Omega_I^{h,m})^u \smallsetminus (\Omega_{w(\mathbb{Z}_{>0})}^{h,m})^u$, $\gamma(w)$ is the smallest partition γ (under containment) for which $\Omega_w^{h,m} \subset \Omega_{h,m}^{\gamma'}$.

Definition 3.2.2. Given any Ferrers diagram λ, the *hook of λ* is the subdiagram $(\lambda_1, 1, \ldots, 1)$ having λ'_1 parts, and the *hook length of λ* is the number $\lambda_1 + \lambda'_1 - 1$ of boxes in the hook. ◇

Nakayama [Nak41a, Nak41b] introduced partitions having no n-hooks in order to classify irreducible representations of S_n; these have since been termed *n-cores* [JK81]. Lapointe and Morse [LM05] defined the n-core associated with an affine permutation in $\tilde{S}_n$, which definition we now recover. For a primer on fillings and tableaux, and analogous statements to the following in the finite setting, see Appendix B.2.

Lemma 3.2.3. *Let γ be the n-core of w. Label each box (i, j) of γ by its residue $j - i$ modulo n. Take steps $k = 1, 2, \ldots$ as follows: So long as the partition remains nonempty, pick a residue i_k of some corner of γ and remove all corners of this residue. Then the process terminates at $k = \ell(w)$ and $s_{i_1} s_{i_2} \cdots s_{i_{\ell(w)}}$ is a reduced word for w.*

Proof. We note that the result is immediate for $w = \mathrm{id}$ and proceed by induction on $\ell(w)$. Let $I = w(\mathbb{Z}_{>0}) \cap [1 - m + h, h]$, the pivots of w preserved under E_{1-m+h}/E_{1+h}. Observe that, for $\lambda \subset ((m - h)^h)$, the condition on λ (equivalently, on λ') of having no n-hooks is equivalent to the condition on I of u-stability. Furthermore, the condition on λ' of having a corner of residue i is equivalent to the condition on I of having a pivot in some row $cn + i \in [1 - m + h, h - 1]$ but not in row $cn + i + 1$. From this it follows by u-stability that the collection of integers c for which this this holds is contiguous (broken above when some $cn+i$ is (also) not a pivot and below when some $cn+i+1$ is (also) a pivot). Thus $s_i w(\mathbb{Z}_{>0}) = s_i(w(\mathbb{Z}_{>0}))$ consists of the same pivots as $w(\mathbb{Z}_{>0})$ except with those pivots $cn+i$ corresponding to corners of γ replaced by pivots of $cn+i+1$. This condition corresponds to the new partition having dual corners of residue i—that is, being obtained from γ by removing the corners of this residue. $\square$

Example 3.2.4. Retrieve $w = [-2, 2, 6]$ from Example 3.1.9 and its reduced word

$$w = [-2,2,6] = [3,2,1]s_0 = [2,3,1]s_1s_0 = [2,1,3]s_2s_1s_0 = s_1s_2s_1s_0$$

from Example 2.2.16. Using this word and Lemma 3.2.3, we can construct the 3-core of w on an array of residues:

$$s_1s_2s_1s_0 \cdot \begin{array}{cccc} 0&1&2&0 \\ 1&2&0&1 \\ 2&0&1&2 \\ 0&1&2&0 \end{array} \rightsquigarrow s_1s_2s_1 \cdot \begin{array}{cccc} 0&1&2&0 \\ 1&2&0&1 \\ 2&0&1&2 \\ \boxed{0}&1&2&0 \end{array} \rightsquigarrow s_1s_2 \cdot \begin{array}{cccc} 0&1&2&0 \\ 1&2&0&1 \\ 2&0&1&2 \\ \boxed{0}&\boxed{1}&2&0 \end{array} \rightsquigarrow s_1 \cdot \begin{array}{cccc} 0&1&2&0 \\ 1&2&0&1 \\ \boxed{2}&0&1&2 \\ \boxed{0}&\boxed{1}&\boxed{2}&0 \end{array} \rightsquigarrow \begin{array}{cccc} 0&1&2&0 \\ \boxed{1}&2&0&1 \\ \boxed{2}&0&1&2 \\ \boxed{0}&\boxed{1}&\boxed{2}&0 \end{array}$$

The 3-core is $\gamma = (3, 1, 1)$. $\diamond$

Remark 3.2.5. The bijection between h-Grassmannian permutations in S_m and Ferrers diagrams contained in the $h \times (m-h)$ box is an immediate corollary, proved by choosing n greater than the maximum absolute value $\max(h, m-h)$ of the residue of any box. This implies the special case Corollary B.2.12 familiar from the finite setting. Take care to remember that γ is transposed as a Schubert index due to mismatched notational origins. $\diamond$

It remains to parametrize this embedding in terms of an affine coordinatization of $\Omega_w^{h,m}$.

γ is completely determined by the numbers of the diagonals in which the first right step in each congruence class occurs, i.e. when γ is the n-core of w, by the heralding pivots of $\mathrm{piv}(w)$. This motivates the following definitions, which will lead to an answer to the second problem.

Definition 3.2.6. The free entries in the $\hat{v}_i$ across $1 \leq i \leq n$ are nonincreasing in number, hence form a partition of at most $n-1$ parts. Denote this partition λ' and call its conjugate λ the $(n-1)$-*bounded partition associated with* w. $\diamond$

λ is "$(n-1)$-bounded" in the sense that $\lambda_1 \leq n-1$. Lemma 3.1.6 then provides the parametrization $\mathbb{C}^{|\lambda|} \to \Omega_w$, though we will illustrate it in detail using the upcoming Lemma 3.2.9 (see Corollary 3.2.12).

Example 3.2.7. Recall Example 3.1.9. The 3-core $\gamma = (3, 1, 1)$ and 2-bounded partition $\lambda = (2, 1, 1)$ of w appear in the heralded form of x:

$$\begin{pmatrix} 1&0&0 \\ a&0&0 \\ b&0&0 \\ 0&1&0 \\ 0&a&1 \\ c&b&d \end{pmatrix} \rightsquigarrow \gamma = \begin{array}{ccc} \boxed{a} && \\ \boxed{b} && \\ \boxed{c}&\boxed{b}&\boxed{d} \end{array}, \quad \lambda = \begin{array}{cc} \boxed{a} & \\ \boxed{b} & \\ \boxed{c}&\boxed{d} \end{array}$$

Note that the a in column 2 does not appear in γ because it lies in a pivotal row. $\diamond$

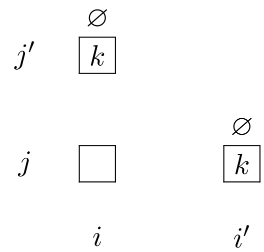

Figure 3.1: Box (i, j) is a top box of γ° when (i, j') and (i', j) are top boxes of γ.

Lapointe and Morse [LM05] define λ from γ, a definition we recover by way of the boundary decomposition due to Chen [Che10].

Definition 3.2.8. Take γ to be an n-core. The boxes of γ having hook length $< n$ comprise the skew shape $\partial\gamma \subset \gamma$, called the *boundary of* γ. Also denote $\gamma^\circ = \gamma \setminus \partial\gamma$.

Label the boxes of $\partial\gamma$ with their residues (modulo n). Let i_1 be the residue in the top-left box of $\partial\gamma$; let $i_2, \ldots, i_{n'}$ be the next distinct residues, in order, to appear at the top of any column. (Call this a column's *top residue*.) For each $j = 1, \ldots, n'$, construct the diagram $\lambda^{(j)}$ from the j^{th} occurrences of columns with a given top residue. Then the collection $\{\lambda^{(j)}\}_j$ constitutes the *boundary decomposition of* γ. ◇

Lemma 3.2.9. *For each heralding vector $\hat{v}_i$ in a heralded basis, partition the indices of the free coordinates into batches numbered $j = 1, 2, \ldots$ according as they are between $(j-1)n$ and jn coordinates below the vector's pivot. Then the height of the i^{th} column of $\lambda^{(j)}$ is the number of indices in the j^{th} batch of $\hat{v}_i$.*

Proof. Designate $i_1, \ldots, i_{n'}$ as in Definition 3.2.8. We will exhibit, across $2 \le k \le n'$ a height-preserving bijection between the columns of γ of top residue i_k, excluding the tallest, and the columns of γ° below top residues i_k in γ. Since the columns of the $\lambda^{(j)}$ are the columns of $\partial\gamma$, i.e. the columns of γ less those of γ°, this implies that each first column of γ of residue i_k has height $(\lambda^{(1)})'_k + (\lambda^{(2)})'_k + \cdots$, the sum of the heights of the k^{th} columns of the $\lambda^{(j)}$. But the height of this column of γ is the number of free entries of $\hat{v}_k$ and the batches therein are separated by east moves of residue i_k in γ—that is, by breaks between the columns of $\partial\gamma$ that constitute the column of γ.

Here is the bijection: Match each top box of γ° to the top box (i, j) of γ in row j with the same residue as the top box of γ in column i. We need to check that it is well-defined, injective, and surjective.

The map is surjective because each pair of consecutive top boxes (i, j') and (i', j) of γ of a given residue are directly above and rightward of a top box of γ°. Indeed, (i, j) necessarily has hook length $> n$, so is in γ°; while, since $(i', j+1) \notin \gamma$, $(i, j+1)$ has hook length $< n$, so is in $\partial\gamma$, leaving (i, j) a top box of γ°. (See Figure 3.1.) This also makes the proposed

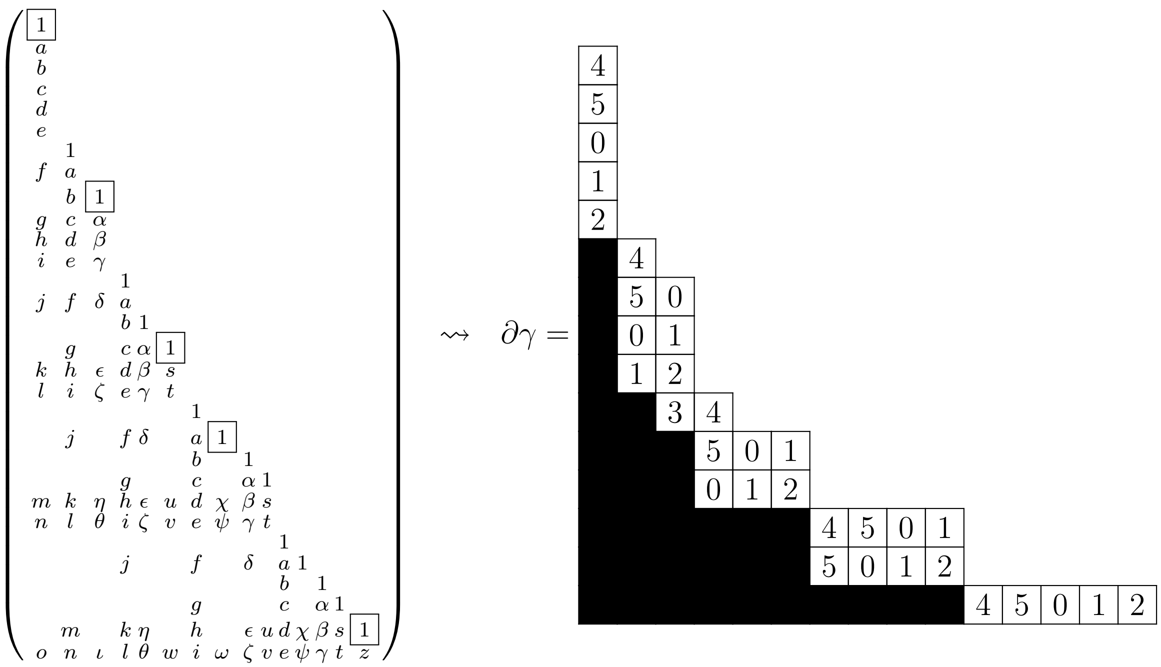

Figure 3.2: Boundary of the 6-core of w and heralded form for a matrix in $\pi^{-1}(\Omega_w)$.

bijection well-defined at these specific top boxes of γ°. It is therefore sufficient to observe that these constitute *all* such boxes; indeed, every top box of γ° has a top box of γ above it, and the next top box of γ of the same residue must share its row with a top box of γ° in the same column as its predecessor; and this top box must therefore be that with which we started. $\square$

Remark 3.2.10. Lemma 3.2.9 shows that λ is the Ferrers diagram obtained from γ by either (a) collapsing the skew shape $\partial\gamma$ leftward (the definition given in [LM05]) or (b) selecting for each residue the tallest column of γ with that top residue, which corresponds to selecting the heralding vectors (respectively, columns) in a heralded basis (heralded matrix). Of course $|\partial\gamma| = |\lambda| = \ell(w)$. $\diamond$

Example 3.2.11. Take $n = 6$. The reduced word

$$w = s_4s_5s_0s_1s_2s_3s_5s_4s_0s_5s_1s_0s_2s_1s_4s_3s_2s_5s_4s_3s_1s_0s_5s_4s_2s_1s_0s_5s_4s_3s_2s_1s_0 \in \tilde{S}_6/S_6$$

has 6-core $\gamma = (15, 10, 10, 6, 6, 4, 3, 3, 3, 2, 1, 1, 1, 1, 1)$ with the boundary depicted in Figure 3.2. The numerical entries are the n-residues. The boundary decomposition produces the 5-bounded partitions

$$\begin{array}{lllll} 4 & & & & \\ 5 & 0 & & & \\ 0 & 1 & & & \\ 1 & 2 & 1 & 5 & \\ 2 & 3 & 2 & 0 & 2 \end{array}, \quad \begin{array}{llll} 4 & & & \\ 5 & & & \\ 0 & 0 & 1 & \\ 1 & 1 & 2 & 5 \end{array}, \quad \begin{array}{lll} 4 & & \\ 5 & 0 & \\ 0 & 1 & 1 \end{array}, \quad \begin{array}{ll} 4 & \\ 5 & 0 \end{array}, \quad \begin{array}{l} 4 \end{array},$$

filled by the $(n-1)$-residues from $\partial\gamma$. The columns of each 5-bounded partition correspond to a batch of free entries within the heralding column of the heralded form for a matrix $\pi^{-1}(\Omega_w) \subset \mathrm{Gr}_{15,30}$; the partitions are listed in the order of their associated heralding pivots. For instance, the first column of $(\lambda^{(1)})$, of height 5, corresponds to the 5 entries a, b, c, d, e in the first column of x. $\diamond$

Corollary 3.2.12 follows from Lemma 3.1.6 and Lemma 3.2.9.

Corollary 3.2.12. *If a heralded basis of $V \in \Omega_w^{h,m}$ has heralding vectors $\hat{v}_1, \ldots, \hat{v}_{n'}$ with $n' \leq n$ then these have $\lambda'_1, \ldots, \lambda'_{n'}$ free entries, respectively, where λ is the $(n-1)$-bounded partition of w and $\lambda'_k = 0$ for $k > n'$.*

3.3 Stable closure and heralded closure

In this section we characterize the collection of subspaces V for which $\mathbf{O}V$ lies in a given affine Schubert cell Ω_w. (This characterization underlies a crucial matrix decomposition in the next chapter.) The collection is inelegant from the vantage point of u-stability. Instead we introduce the weaker property of "heraldedness" by relaxing the criteria for $V \subset \mathbb{C}^m$ to have a heralded basis (thereby generalizing the definition). This involves first extending the notion of heralded basis to that of heralded ind-basis in $\mathbb{C}^\infty$. We then describe a closure operation on Gr_∞ that factors through the assignment $V \mapsto \mathbf{O}V$ and the collection of whose closed subspaces exhibits a Schubert-like decomposition into cells according to pivot set.

Definition 3.3.1. Let $V \in \mathrm{Gr}_\infty^k$ with $\mathrm{piv}(V) = (i_1 < i_2 < \cdots) \in \mathcal{I}_k$. Call V *heralded* if

$$\dim(\mathbf{O}V/V) = |(\mathrm{piv}(V) + n\mathbb{Z}_{\geq 0}) \smallsetminus \mathrm{piv}(V)|.$$

Separately, if $v_1, v_2, \ldots \subset V \in \mathrm{Gr}_\infty^k$ are an ind-basis for V satisfying

(i) $v_1 + E_{m+1}, \ldots, v_m + E_{m+1}$ comprise a basis of V/E_{m+1} for all sufficiently large m,

(ii) $\mathrm{piv}(v_j) = i_j$,

(iii) if $i_{j'} = i_j + cn$ with $c > 0$ then $v_{j'} = t^c v_j$, and

(iv) if i_j is a heralding pivot then $e_k^*(v_j) = \delta_{i_j k}$ for all $k \in I$

then call $v_1, v_2, \ldots$ a *heralded ind-basis for V*. ◇

One then can obtain a heralded subspace W with pivots $I \in \mathcal{I}$ from any subspace $V \in {\Omega_{I+n\mathbb{Z}_{\geq 0}}}^u$ by taking a heralded basis for V, deleting the basis vectors with pivots in $(I + n\mathbb{Z}_{\geq 0}) \smallsetminus I$, and taking W to consist of (possibly infinite) sums of the remaining. Naturally then $\mathbf{O}W = V$, since no heralding vectors would have been deleted.

If an ind-basis $w_1, w_2, \ldots$ for V satisfies (i), (ii), and (iii) then a heralded ind-basis for V can be directly obtained by constructing from each heralding basis vector $\hat{w}_i$ with $\operatorname{piv}(\hat{w}_i) = \hat{p}_i$ the vector $\hat{v}_i = \frac{1}{e^*_{\hat{p}_i}(\hat{w}_i)} \hat{w}_i$, then including every $t^c \hat{v}_i$ in the new ind-basis whenever $t^c \hat{w}_i$ is in the old. In the proof of Lemma 3.3.2 we only deal with this equivalent but conciser condition.

Lemma 3.3.2. *Let $V \in \mathrm{Gr}_\infty$. Then the following are equivalent:*

(i) *V is heralded;*

(ii) $\operatorname{piv}(\mathbf{O}V) = \operatorname{piv}(V) + n\mathbb{Z}_{\geq 0}$*;*

(iii) $W := \bigoplus_{j=1}^n \mathbf{O}\hat{v}_j = \mathbf{O}V$ *for some (equivalently, any) heralding vectors $\hat{v}_1, \ldots, \hat{v}_n \in V$;*

(iv) $B := \{t^c \hat{v}_j \mid j \in [n], c \in \mathbb{Z}_{\geq 0}, \operatorname{piv}(t^c \hat{v}_j) \in \operatorname{piv}(V)\}$ *is a heralded ind-basis for V for some (equivalently, any) heralding vectors $\hat{v}_1, \ldots, \hat{v}_n$;*

(v) *V has a heralded ind-basis.*

Proof. Take $I = \operatorname{piv}(V)$ as in Definition 3.3.1.

(i) ⇔ (ii): Observe from Lemma 3.1.5 that $\dim(\mathbf{O}V/V) = |\operatorname{piv}(\mathbf{O}V) \smallsetminus I|$ and that $I + n\mathbb{Z}_{\geq 0} \subseteq \operatorname{piv}(\mathbf{O}V)$. Thus V is heralded if and only if $|\operatorname{piv}(\mathbf{O}V)| = |I + n\mathbb{Z}_{\geq 0}|$, i.e. when $\operatorname{piv}(\mathbf{O}V) = I + n\mathbb{Z}_{\geq 0}$.

(ii) ⇔ (iii): By construction $\operatorname{piv}(\mathbf{O}V) \subseteq I + n\mathbb{Z}_{\geq 0} = \operatorname{piv}(W)$ while $W \subseteq \mathbf{O}V$, and again from Lemma 3.1.5 we have $\dim(\mathbf{O}V/W) = \operatorname{piv}(\mathbf{O}V) - |I + n\mathbb{Z}_{\geq 0}|$. Thus $\operatorname{piv}(\mathbf{O}V) = I + n\mathbb{Z}_{\geq 0}$ if and only if $\dim(\mathbf{O}V/W) = 0$, i.e. when $\mathbf{O}V = W$.

(iii) ⇔ (iv): For the forward direction, find $m \in \mathbb{Z}$ such that $E_m \subset V$ $(\subseteq \mathbf{O}V = W)$. Pick any $v \in V$, so that $v + E_m \in V/E_m \subset W/E_m$. Write $v + E_m = \sum_{j,c} a_{jc} t^c \hat{v}_j + E_m$ with $\hat{\imath}_j + cn < m$. Then $v + V = \sum_{t^c \hat{v}_j \notin V} a_{jc} t^c \hat{v}_j + V \in W/V$. But $v \in V$, so $v + V = 0 + V$, so $a_{jc} = 0$ when $t^c \hat{v}_j \notin V$. Since $\operatorname{piv}(t^c \hat{v}_j) \in \operatorname{piv}(V)$ when $t^c \hat{v}_j \in V$, and since m and v were arbitrary, it must be that B is a heralded ind-basis for V. For the backward direction, by construction B contains heralding vectors $\hat{v}_1, \ldots, \hat{v}_n$ for W and $B \subseteq \mathbf{O}\operatorname{Span}(\hat{v}_1, \ldots, \hat{v}_n)$, so we have $\mathbf{O}V = \mathbf{O}\operatorname{Span}(\hat{v}_1, \ldots, \hat{v}_n) = W$.

(iv) $\Leftrightarrow$ (v): The forward direction is trivial. For the backward, choose heralding vectors $\hat{v}_1, \ldots, \hat{v}_n$ from a heralding set for V; the construction of B from these $\hat{v}_j$ recovers the heralding set itself. $\square$

Definition 3.3.3. Given $I \in \mathcal{I}$, let $\hat{\Omega}_I \subset \mathrm{Gr}_\infty$ consist of the heralded subspaces V with $\mathrm{piv}(V) = I$. If $\mathrm{vcard}\, I = k$ and $I' = I \cap [1-m+h-k, h-k]$ let $\hat{\Omega}^{h,m}_{I'} = \{V + E_{1+h-k} \subset E_{1-m+h-k}/E_{1+h-k} \mid V \in \hat{\Omega}_I\} \subset \Omega^{h,m}_{I'} \subset \mathrm{Gr}_{h,m}$. (Only the indexing of $\hat{\Omega}^{h,m}_{I'}$ depends on k.) Call the $\hat{\Omega}^{h,m}_I \cong \hat{\Omega}_I$ *heralded cells* and their closures $\hat{X}_I = \overline{\hat{\Omega}_I}$ *heralded varieties*. We observe in Remark 3.3.11 that heralded cells are indeed cells but in Example 3.3.13 that heralded varieties are not made up only of heralded cells. $\diamond$

Equipped with a network of heralded cells that includes the embedded affine Schubert cells, we describe an operation that takes each V into some $\hat{\Omega}_{k,I}$ such that $\mathbf{O}V \in \mathbf{O}\hat{\Omega}_{k,I'}$.

Theorem 3.3.4. *Pick $V \in \mathrm{Gr}_\infty$. Under the partial order $\leq$ on $\mathcal{I}$, there is a minimum $J \subset \mathcal{I}$ for which there exists $W \in \hat{\Omega}_J$ with $V \subseteq W$. Furthermore, W is unique in $\hat{\Omega}_J$.*

Definition 3.3.5. Refer to $\mathbf{O}V$ as the *stable closure of* V. When V and W are as in the lemma, call $\mathrm{hc}\, V = W$ the *heralded closure of* V. Note that if $V \in \mathrm{Gr}^k_\infty$ and $\mathrm{hc}\, V \in \mathrm{Gr}^{k'}_\infty$ then $k' \leq k$ with equality if and only if $\mathrm{hc}\, V = V$. $\diamond$

Proof of Theorem 3.3.4. Let $I = \mathrm{piv}(V)$ and pick m so that $E_{-m} \supset V \supset E_m$. Iterating on leading pivots, we will construct a heralded basis for some $W \subset \mathbb{C}^\infty$, using vectors from V at heralding pivots but shifts by t (which generally will not be in V) at non-heralding pivots. We will then check that $V \subset W$ and that if W' is any heralded subspace that contains V then W' contains this basis. These observations imply the lemma.

For the construction, begin with an empty basis. For the iterative step, suppose that basis vectors $w_1, \ldots, w_b$ have accrued so far. Identify the first pivot i of $(V + \mathrm{Span}(w_1, \ldots, w_b)) \cap E_j$. If (A) $\mathrm{piv}(w_l) = i - cn$ for some $l \in [b]$, $c > 0$, then take $w_{b+1} = t^c w_l$. Otherwise (B) identify any vector $w \in (V + \mathrm{Span}(w_1, \ldots, w_b))$ with $\mathrm{piv}(w) = i$ and take $w_{b+1} = w$. Continue the process until the pivot $i \geq m$, and take $W = \mathrm{Span}(w_1, \ldots, w_b) + E_m$ and $J = \mathrm{piv}(W)$.

To verify that $V \subset W$, pick any vector $v \in V$ and let $j = \mathrm{piv}(v)$. If $j \geq m$ then we are done, since $E_m \subset W$. If, on the other hand, $j < m$, then since $I \subset \mathrm{piv}(W)$ we may identify $w_l \in W$ with $\mathrm{piv}(w_l) = j$ and take $\alpha_l = e^*_j(v)/e^*_j(w_l) \in \mathbb{C}$ so that $j' := \mathrm{piv}(v - \alpha_l w_l) > j$ with $j' > j$. By construction, we may proceed until $v - \alpha_l w_l - \alpha_{l'} w_{l'} - \cdots - \alpha_{l(k)} w_{l(k)} \in E_m$ to reveal that $v = \alpha_l w_l + \cdots + \alpha_{l(k)} w_{l(k)} \in W$.

Now suppose that W' is heralded and $V \subset W'$ and pick any ind-basis vector w_l. If $l = 1$ then $w_l \in V \subset W'$; if $l > b$ then $w_l \in E_m \subset V \subset W'$. So suppose that $1 < l \leq b$. We proceed by forward induction on l; assume that $w_1, \ldots, w_{l-1} \in W'$. Now w_l was selected in case (A) or (B) above. If (B), $w_l \in V + \mathrm{Span}(w_1, \ldots, w_b)$, while $V \subset W'$ and $w_1, \ldots, w_b \in W'$ by assumption and induction. If (A), $w_l = t^c w_{l'}$ for some $l' < l$; since W' is heralded by assumption, Definition 3.3.1 (iii) implies that $w_l \in W'$. $\square$

$$
\begin{pmatrix} 1 & & & & & \\ a & & & & & \\ \hline & 1 & & & & \\ b & f & & & & \\ c & g & & & & \\ & & 1 & & & \\ d & h & j & & & \\ e & i & k & & & \\ & \vdots & & & & \end{pmatrix}
\xrightarrow{\mathrm{hc}}
\begin{pmatrix} 1 & & & & & \\ a & & & & & \\ \hline & 1 & & & & \\ b & f & & & & \\ c & g & & & & \\ & & 1 & & & \\ d & h & f & 1 & & \\ e & i & g & a & 1 & \\ & & \vdots & & & \end{pmatrix}
\xrightarrow{\mathrm{GL}}
\begin{pmatrix} 1 & & & & & \\ a & & & & & \\ \hline & 1 & & & & \\ b & \alpha & & & & \\ c & \beta & & & & \\ & & 1 & & & \\ & & \alpha & 1 & & \\ & & \beta & a & 1 & \\ & & \vdots & & & \end{pmatrix}
\xrightarrow{\mathbf{O}}
\begin{pmatrix} 1 & & & & & \\ a & & & & & \\ \hline & 1 & & & & \\ b & \alpha & 1 & & & \\ c & \beta & a & & & \\ & & & 1 & & \\ & & b & \alpha & 1 & \\ & & c & \beta & a & 1 \\ & & \vdots & & & \end{pmatrix}
\xrightarrow{\mathrm{GL}}
\begin{pmatrix} 1 & & & & & \\ a & & & & & \\ \hline & 1 & & & & \\ & & 1 & & & \\ j & \gamma & a & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & j & \gamma & a & 1 \\ & & \vdots & & & \end{pmatrix}
$$

Figure 3.3: (Ind-)bases discussed in Example 3.3.9, first paragraph.

Corollary 3.3.6. *If $V \in \mathrm{Gr}_\infty$ then $\mathbf{O} \cdot \mathrm{hc}\, V = \mathbf{O}V$. Moreover, the heralding pivots of $\mathrm{hc}\, V$ are precisely those of $\mathbf{O}V$.*

Corollary 3.3.7 extends Lemma 3.1.6 to heralded subspaces. The proofs are entirely analogous.

Corollary 3.3.7. *If $V \in \hat{\Omega}_I$ then V has a unique basis $v_1, \ldots, v_h$ satisfying, for all j, j',*

(i) $\mathrm{piv}(v_j) = i_j$,

(ii) if $i_{j'} = i_j + cn$ with $c > 0$ then $v_{j'} = t^c v_j$, and

(iii) if $i_j \in \hat{I}$ then $e^*_{i_{j'}}(v_j) = \delta_{jj'}$.

Again we call $v_1, \ldots, v_h$ the *heralded basis for* V and say that the matrix $(\, v_1 \ \cdots \ v_h \,)$ is in *heralded form*. Any basis $w_1, \ldots, w_h$ for $V \subset \mathbb{C}^m$, up to reordering, can consequently be expanded as

$$w_j = \sum_{j'=1}^{h} C^j_{j'} v_{j'}, \qquad C^j_{j'} \in \mathbb{C},\ C^j_j \neq 0, \tag{3.1}$$

where the v_j comprise the heralded basis for V.

Corollary 3.3.8. *If $v_1, v_2, \ldots$ comprise the heralded ind-basis for $V \in \mathrm{Gr}^k_\infty$, $\mathrm{piv}(v_{h+1}) = h + 1 - k$, and $V \supset E_{1+h-k}$ then $v_1 + E_{1+h-k}, \ldots, v_h + E_{1+h-k}$ comprise the heralded basis for V/E_{1+h-k}.*

Example 3.3.9. Figure 3.3 depicts, leftmost, the first few vectors (those outside E_7) of a heralded ind-basis for some $V \in \hat{\Omega}_{(-1,1,4)\cup\mathbb{Z}_{>6}} \subset \mathrm{Gr}^3_\infty$ (or, by ignoring the '$\vdots$'s, the heralded form of a matrix x whose columns span some $V \in \Omega^{3,8}_{(-1,1,4)} \subset \mathrm{Gr}_{3,8}$). Moving rightward, we then see the (first few) vectors of $\mathrm{hc}\, V \in \hat{\Omega}_{(-1,1,4,5,6)} \subset \mathrm{Gr}^1_\infty$; of its heralded (ind-)basis; of $\mathbf{O}V \in \Omega_{s_2 s_1 s_0} \subset \mathrm{Gr}_{\mathrm{SL}_n}$; and of its heralded (ind-)basis. (We use a horizontal bar to

$$\left(\begin{array}{ccccc} 1 &&&& \\ a &&&& \\ \hline & 1 &&& \\ b & f &&& \\ c & g &&& \\ && 1 && \\ d & h & j && \\ e & i & k && \\ && \vdots && \end{array}\right) \xrightarrow{\text{hc}} \left(\begin{array}{ccccc} 1 &&&& \\ a &&&& \\ \hline & 1 &&& \\ b & f &&& \\ c & g &&& \\ &&& 1 & \\ d & h & f & 1 & \\ e & i & g & a & \\ &&\vdots && \end{array}\right) \xrightarrow{\text{GL}} \left(\begin{array}{ccccc} 1 &&&& \\ a &&&& \\ \hline & 1 &&& \\ b & \alpha &&& \\ c & \beta &&& \\ && 1 && \\ && \alpha & 1 & \\ d & \gamma & \beta & a & \\ && \vdots && \end{array}\right) \xrightarrow{\mathbf{O}} \left(\begin{array}{ccccc} 1 &&&& \\ a &&&& \\ \hline & 1 &&& \\ b & \alpha & 1 && \\ c & \beta & a && \\ &&& 1 & \\ && b & \alpha & 1 \\ d & \gamma & c & \beta & a \\ && \vdots && \end{array}\right) \xrightarrow{\text{GL}} \left(\begin{array}{ccccc} 1 &&&& \\ a &&&& \\ \hline & 1 &&& \\ && 1 && \\ b & \alpha & a && \\ &&& 1 & \\ &&&& 1 \\ c & \beta & b & \alpha & a \\ && \vdots && \end{array}\right)$$

Figure 3.4: (Ind-)bases discussed in Example 3.3.9, second paragraph.

demarcate the rows indexed by $\mathbb{Z}_{\leq 0}$ and by $\mathbb{Z}_{>0}$.) The cells $\Omega_{(-1,1,4)\cup\mathbb{Z}_{>6}}$, $\hat{\Omega}_{(-1,1,4,5,6)\cup\mathbb{Z}_{>7}}$, and $\Omega_{[-1,1,6]}$ have dimensions 11, 5, and 3, respectively.

The above assumes, however, both that $j \neq f$ and that $k \neq -af + aj + g$. If instead we have $j \neq f$ but $k = -af + aj + g$ then the heralded closure of V lies in $\hat{\Omega}_{(-1,1,4,5)\cup\mathbb{Z}_{>6}} \subset \mathrm{Gr}^2_\infty$ and its u-stable closure lies in $\Omega_{[-1,1,9]} \subset (\mathrm{Gr}^1_\infty)^u$, as in Figure 3.4. From the reduced forms we can tell that $\dim \Omega_{(-1,1,4,5)\cup\mathbb{Z}_{>6}} = 7$ and that $\dim \Omega_{[-1,1,9]} = 5$. This singular case complicates proofs in the next chapter but will be handled by Lemma 4.2.10. $\diamond$

Corollary 3.3.10. *If* $y \in \pi^{-1}(X^{h,m}_w)$ *then* $\mathrm{Stab}_{\mathrm{GL}_h}(y) = I_h$.

Remark 3.3.11. As with lattices, a heralded subspace V is determined by the coordinates of the heralding vectors in its heralded basis vectors and the set $(\hat{I} + n\mathbb{Z}_{>0}) \smallsetminus I$ of "missing" indices. A heralded cell is therefore by the free coordinates in the heralding members of an arbitrary heralded basis, which number

$$\nu(I) := \sum_{i \in \hat{I}} |[i, \infty) \smallsetminus I|.$$

(If $\mathrm{vcard}\, I = k$, $I \supset \mathbb{Z}_{>h-k}$, and $I' = I \cap [1 - m + h - k, h - k]$ then write $\nu(I') = \nu(I)$.) Thus $\hat{\Omega}_I \cong \mathbb{C}^{\nu(I)}$. $\diamond$

Remark 3.3.12. The closure operations define two stratifications

$$\mathrm{Gr}_\infty = \bigsqcup_{w \in \mathcal{W}^{\mathcal{P}}} \mathcal{S}_w = \bigsqcup_{I \in \mathcal{I}} \mathcal{T}_I$$

according to which cell contains the (heralded or stable) closure of a given subspace; since the heraled closure factors into the stable, the "heralded stratification" into $\mathcal{T}_I$ is a refinement of the "stable stratification" into $\mathcal{S}_w$, in the sense that each $\mathcal{T}_w$ is a disjoint union of $\mathcal{S}_I$. Moreover, if we denote, say, $\mathcal{S}^k_I = \mathcal{S}_I \cap \mathrm{Gr}^{k-\mathrm{vcard}(I)}_\infty$ for $k \geq 0$ (so that $\mathcal{S}^0_I = \hat{\Omega}_I$) then each $\mathcal{S}^{k+1}_I$ exists in the stratum of subspaces that are contained in a subspace in $\mathcal{S}^k_I$. The geometry of these strata is not as elegant as that of $\mathrm{Gr}_{\mathrm{GL}_n} \subset \mathrm{Gr}_\infty$; even the union of heralded cells of common virtual dimension is not closed, as Example 3.3.13 points out. $\diamond$

Example 3.3.13. Take $n = 3$, $m = 6$, and $h = 2$, and let $I = (1, 4)$ and $J = (2, 4)$. It can be seen from the heralded forms

$$y_I = \begin{pmatrix} 1 & \\ a & \\ b & \\ & 1 \\ c & a \\ d & b \end{pmatrix} \quad \text{and} \quad y_J = \begin{pmatrix} & \\ 1 & \\ a & \\ & 1 \\ b & d \\ c & e \end{pmatrix}.$$

for matrices in $\pi^{-1}(\hat{\Omega}_I^{3,6})$ and in $\pi^{-1}(\hat{\Omega}_J^{3,6})$ that $\dim \hat{\Omega}_I^{3,6} = 4$ while $\dim \hat{\Omega}_J^{3,6} = 5$. The generalization of Corollary 3.1.11 to non–u-stable heralded varieties is therefore false. It may be possible to generalize Corollary 3.1.11 to containment relations on heralded varieties using a more complicated condition than just $\leq$, for instance that $I + n\mathbb{Z}_{>0} \leq J + n\mathbb{Z}_{>0}$.

Moreover, the observation that

$$\lim_{d \to \infty} y_J G/G = \begin{pmatrix} & \\ 1 & \\ a & \\ & \\ & 1 \\ c & \end{pmatrix} G/G \in \Omega_{2,6}^{\ydiagram{3,1}} \setminus \hat{\Omega}_{(2,5)}^{2,6}$$

confirms that the heralded subspaces do not constitute a subvariety, or even a Euclidean-closed subset, of Gr_∞. $\diamond$

3.4 Lattice-respecting orbit decompositions

The previous chapter generalized affine Schubert varieties to the larger collection of heralded varieties. Chapters 4 and 5 concern matrix varieties obtained from Schubert and heralded varieties. These varieties are acted upon by a representation $\mathcal{U} \to \mathrm{GL}_m$. The special elements $\theta_{rd}^{\mathrm{aff}} \in \mathrm{GL}_m$ developed here generate this representation, and will prove handy in the later chapters. Here we illustrate the control they afford over heralded spaces and use them to prove orbit decompositions of $M_{m\times h}$ and of $X_w \subset \mathrm{Gr}_{h,m}$ with canonical orbit representatives explicitly described.

This section requires only concepts from Chapter 2 and may be viewed as a continuation of the discussion in Section 2.2. However, much of the discussion should be clearer having been through the previous two sections. At the finite level, GL_m acts naturally on $\mathrm{Gr}_{h,m}$, as well as on the matrx space $M_{m\times h}$ that houses $\pi^{-1}(\mathrm{Gr}_{h,m})$. At the affine level, recall the prounipotent radical $\mathcal{U} < \mathcal{B}$ and its subgroup $\mathcal{U}'$ generated by the one-parameter subgroups $\theta_{ij}^b : \mathbb{C} \to \mathcal{U}$, where $b \geq 0$ and if $i \leq j$ then $b > 0$. The action of $\mathcal{U}$ on Gr_∞^0 stabilizes every standard $\mathbf{O}$-lattice, and this induces a natural action of $\mathcal{U}$ on E_{1-m+h}/E_{1+h}. For convenience of notation, we return to the usual coordinate indices $[m]$ for $M_{m\times h}$.

Definition 3.4.1. The action of $\mathcal{U}$ on E_{1-m+h}/E_{1+h} induces a homomorphism $\mathcal{U} \to \mathrm{GL}(\mathbb{C}^m)$ of algebraic groups. This homomorphism is given explicitly, at the generating one-parameter subgroups θ^b_{ij}, as

$$\begin{aligned} \Theta = \Theta_{h,m} : \mathcal{U}' &\to \mathrm{GL}_m \\ \theta^b_{ij}(a) &\mapsto \prod_{\substack{j'-m+h \equiv j \\ i'-j' = i+bn-j}} (I + aE_{i'j'}) = \prod_{\substack{j'-m+h \equiv j \\ i'-j' = i+bn-j}} \theta_{i'j'}(a), \end{aligned} \tag{3.2}$$

where the $\theta_{ij} : \mathbb{C} \to U_-$ are the generating one-parameter subgroups of U_-. Write $U^{\mathrm{aff}} = \Theta(\mathcal{U}') < U_-$ when m and h are understood. ◇

The images are products of one-parameter subgroups of U_-, and as generators of $\mathcal{U}'$ they generate U^{aff}. They are sometimes unwieldy, however; in order to more efficiently tinker with matrices of heralded column vectors we introduce the elements

$$\theta^{\mathrm{aff}}_{rd}(a) = I + \sum_{\substack{i \equiv r \\ i-j=d}} aE_{ij} \in \mathrm{GL}_m. \tag{3.3}$$

These are generally not one-parameter subgroups, as will be demonstrated in Example 3.4.3. As an action on M, $\theta^{\mathrm{aff}}_{rd}(a)$ adds to each row $i \equiv r$ (modulo n) a copy of the row d indices prior, scaled by a. (Liken d, the "difference", to $i + bn - j$ in (3.2).) The ensuing examples will expose a caveat that Lemma 3.4.4 will resolve.

Example 3.4.2. Take $n = 3$ and $\theta^2_{13} : a \mapsto I + at^2E_{13}$, and consider $\Theta_{49} : \mathcal{G} \to \mathrm{GL}_9$. We get

$$\theta^2_{13}(a) = \begin{pmatrix} 1 & & at^2 \\ & 1 & \\ & & 1 \end{pmatrix} \mapsto \Theta_{49}(\theta^2_{13}(a)) = \left(\begin{array}{cccc|ccccc} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ &&& 1 &&&&& \\ \hline a &&&& 1 &&&& \\ &&&&& 1 &&& \\ &&&&&& 1 && \\ &&& a &&&& 1 & \\ &&&&&&&& 1 \end{array}\right) = \theta^{\mathrm{aff}}_{54}(a).$$

If we take $w = s_0s_1s_2s_0s_1s_0$, $m = 9$, and $h = 5$, one step on the way to a heralded matrix might invoke $\theta^{\mathrm{aff}}_{54}(a)$ to remove an unwanted entry c in row 5 and the same column as pivot 1 as follows:

$$\theta^{\mathrm{aff}}_{54}(-c) \cdot \left(\begin{array}{ccccc} 1 &&&& \\ a &&&& \\ b &&&& \\ & 1 &&& \\ \hline c & a & 1 && \\ d & b & g && \\ &&& 1 & \\ e & c && a & 1 \\ f & d & h & b & g \end{array}\right) = \left(\begin{array}{ccccc} 1 &&&& \\ a &&&& \\ b &&&& \\ & 1 &&& \\ \hline & a & 1 && \\ d & b & g && \\ &&& 1 & \\ e &&& a & 1 \\ f & d & h & b & g \end{array}\right).$$

◇

We hit the caveat at θ^b_{jj}, i.e. when $i = j$.

Example 3.4.3. Take $n = 4$ and $\theta^1_{11} : a \mapsto I + atE_{11}$, but again choose $m = 9$ and $h = 4$. Then $\Theta_{49} : \mathcal{G} \to \mathrm{GL}_9$ provides

$$\theta^1_{11}(a) = \begin{pmatrix} 1+at & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \mapsto \Theta_{49}(\theta^1_{11}(a)) = \left(\begin{array}{cccc|ccccc} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ \hline a & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ a^2 & & & & a & & & & 1 \end{array}\right) = \theta^{\mathrm{aff}}_{54}(a)\theta^{\mathrm{aff}}_{98}(a^2).$$

We can recover $\theta^{\mathrm{aff}}_{54}(a)$ through Θ by using another one-parameter subgroup as

$$\theta^{\mathrm{aff}}_{54}(a) = \Theta(\theta^1_{11}(a)) + \Theta(\theta^2_{11}(-a^2)),$$

lest we doubt that it is indeed an element of U^{aff}. However, $\theta^{\mathrm{aff}}_{54}$ is itself not a one-parameter subgroup, being not closed under multiplication: $\theta^{\mathrm{aff}}_{54}(a)^2 = \theta^{\mathrm{aff}}_{54}(2a)\theta^{\mathrm{aff}}_{98}(a^2)$. ◇

Lemma 3.4.4. *$U^{\mathrm{aff}} < \mathrm{GL}_m$ is generated by the $\theta^{\mathrm{aff}}_{rd}(a)$.*

Proof. Let $\tilde{U}_- \leq U^{\mathrm{aff}}$ be the subgroup generated by the $\theta^{\mathrm{aff}}_{rd}$. Example 3.4.3 illustrates a helpful factorization of a $\theta^{\mathrm{aff}}_{rd}$. If $n \nmid d$ then $\theta^{\mathrm{aff}}_{rd}(a) = \theta_{rc}(at^b) \in U^{\mathrm{aff}}$, where $d = r + bn - c$ with $c \in [n]$. If $n \mid d$ then, using $b = \frac{d}{n}$ and $c = r$, we have the alternating decomposition

$$\theta^{\mathrm{aff}}_{rd}(a) = \Theta(\theta^b_{rr}(a))\Theta(\theta^{2b}_{rr}(-a^2)) \cdots \Theta(\theta^{kb}_{rr}(\pm a^k)), \tag{3.4}$$

where $k \geq \left\lfloor \frac{m-1}{bn} \right\rfloor$. (If k is unnecessarily large then some factors in the product are I.) The factorization gives $\theta^{\mathrm{aff}}_{rd}(a) \in U^{\mathrm{aff}}$, hence $\tilde{U}_- \leq U^{\mathrm{aff}}$.

It remains to show that $U^{\mathrm{aff}} \leq \tilde{U}_-$, which is provided by a decomposition dual to (3.4): Given $\theta^b_{ij}(a) \in \mathcal{U}'$, if $i \neq j$ then $\Theta(\theta^b_{ij}(a)) = \theta^{\mathrm{aff}}_{id}$, where $d = i + bn - j$; otherwise, as in the example with k as above,

$$\Theta(\theta^b_{ij}(a)) = \theta^{\mathrm{aff}}_{id}(a)\theta^{\mathrm{aff}}_{i,2d}(a^2) \cdots \theta^{\mathrm{aff}}_{i,kd}(a^k).$$

□

Before we examine the two-sided action of $U^{\mathrm{aff}} \times \mathrm{GL}_h$, it will help to understand the one-sided orbits of U^{aff}. The following terminology is borrowed from [Ful97].

Definition 3.4.5. Call the i^{th} row of a matrix *essential* if it is nonzero and every $(i - cn)^{\mathrm{th}}$ row is zero for $c \in \mathbb{Z}_{>0}$. Say that an entry of a matrix is *essential* if it is the leftmost nonzero entry in an essential row. (If no row $i \equiv k$ (modulo n) is essential, we may take the first row index $i > h$ with $i \equiv k$ to be essential.) ◇

Theorem 3.4.6. *Each U^{aff}-orbit of M contains a unique matrix y such that all entries below essential entries are zero. Call y abacus-reduced.*

For a fixed set of essential entries, then, the locus of abacus-reduced matrices is parametrized by $(\mathbb{C}^*)^{n'} \times \mathbb{C}^b$, where $n' \leq n$ is the number of essential entries and b is the number of free entries, given as the number of entries neither below an essential entry nor wrap-leftward of one, where we take (i', j') to be "wrap-leftward" of (i, j) if either $i' = i$ and $j' < j$ or $j' = j - cn$ for some $c \in \mathbb{Z}_{>0}$. Consequently, the theorem provides a U^{aff}-orbit decomposition

$$M = \bigsqcup_{n' \leq n} \left(\bigsqcup_{|I| = |a| = n'} U^{\mathrm{aff}}[I, a] \right)$$

across possible collections $I = \{(\hat{\imath}_1, \hat{\jmath}_1), \ldots, (\hat{\imath}_{n'}, \hat{\jmath}_{n'})\}$ of essential entries and choices $a = (a_1, \ldots, a_n) \in (\mathbb{C}^*)^{n'}$ of the cornerstone entries, and taking $[I, a]$ to be the abacus-reduced matrix having $(\hat{\imath}_k, \hat{\jmath}_k)^{\mathrm{th}}$ entries a_k.

Example 3.4.7. Taking $n = 3$, $m = 7$, and $h = 5$, the matrix

$$y = \begin{pmatrix} a_1 & * & * & * & * \\ & & & & a_2 \\ & & & & \\ & * & * & * & \\ & * & * & * & \\ & & a_3 & * & \\ & * & & * & \end{pmatrix}$$

is abacus-reduced for any nonzero values of the a_i (and any values of the asterisks). Note also that for any $m' < m$ the top m' rows of y comprise an abacus-reduced matrix. $\diamond$

Proof of Theorem 3.4.6. For existence, pick any $x \in M$, and initialize $\phi = \mathrm{id}_{U^{\mathrm{aff}}}$, $E = \varnothing$, and $y = (y_{ij}) = x$. We shall operate by $\theta^{\mathrm{aff}}_{rd}$ to make y abacus-reduced. If y is the zero matrix then we are done with $E = \varnothing$. Otherwise iterate the following procedure until the rows of y of every congruence class either begin with an essential entry in E or are all zero: Take $\hat{\imath}$ to be the topmost nonzero row of y not in the congruence class of some topmore essential row, and take $\hat{\jmath}$ to be the column of the first nonzero entry in row $\hat{\imath}$. Append $(\hat{\imath}, \hat{\jmath})$ to E, reset

$$\phi = \theta^{\mathrm{aff}}_{\hat{\imath},1}(y_{\hat{\imath}+1,\hat{\jmath}}) \cdots \theta^{\mathrm{aff}}_{\hat{\imath},m-\hat{\imath}}(y_{m\hat{\jmath}})\phi,$$

and operate by this factor on y to clear column j below row $\hat{\imath}$. By selection each entry wrap-leftward of $(\hat{\imath}, \hat{\jmath})$ is zero, by construction each entry below $(\hat{\imath}, \hat{\jmath})$ is zero, and by the downward progression the construction does not disrupt the selection.

For uniqueness, suppose that x and $y = \theta x$ are abacus-reduced, where $\theta \in U^{\mathrm{aff}}$. Pick $\tilde{\theta} = (a_{ij}) \in {\Theta_{h,m}}^{-1}(\theta) \subset \mathcal{U}'$ so that $y = \tilde{\theta} \cdot x$. We shall show that, up to the choice of preimage, $\tilde{\theta} = \mathrm{id}_{\mathcal{U}}$, hence that $\theta = \mathrm{id}_{U^{\mathrm{aff}}}$ and $y = x$.

Take $(\hat{\imath},\hat{\jmath})$ to be any essential entry of x and identify $j \in [n]$ with $j \equiv \hat{\imath}$. Thus the $(i,j)^{\text{th}}$ entries of $\tilde{\theta}$ operate on M by adding multiples of row $\hat{\imath}$ (possibly among others) to rows below. Consider such an entry $a_{ij} \in \mathbf{O}$. Let $k = \left\lfloor \frac{m-\hat{\imath}}{n} \right\rfloor + \delta_{i<j}$, so that $\tilde{\theta}' = \tilde{\theta} + t^{k+1} f E_{ij}$ exerts the same action $\tilde{\theta}' \cdot x = \tilde{\theta} \cdot x$; the additional shift has run out of rows of x to contribute to. However, for $\hat{\imath}$ suitably large, $\Theta(\tilde{\theta}')$ may not be equal to θ. Up to the choice of $\tilde{\theta}$, then, we may assume that $\deg(a_{ij}) \leq k$.

We proceed to show that each essential entry $(\hat{\imath},\hat{\jmath})$ of x is an essential entry of y. Since $\tilde{\theta}(0) \in U_-$ is unidiagonal, this implies that $x_{\hat{\imath}\hat{\jmath}} = y_{\hat{\imath}\hat{\jmath}}$; and since $\tilde{\theta}$ acts by adding rows downward, the condition that $x_{i\hat{\jmath}} = 0$ for $i > \hat{\imath}$ implies that $y_{i\hat{\jmath}} = 0$ also.

Take $(\hat{\imath},\hat{\jmath})$ to be the topmost essential entry of x, so that all topmore rows of x are zero. No topmore essential entry $(\hat{\imath}',\hat{\jmath}')$ exists with $\hat{\jmath}' < \hat{\jmath}$ to contaminate entries of row $\hat{\imath}$ leftward of $\hat{\jmath}$, so $(\hat{\imath},\hat{\jmath})$ remains essential in $y = \theta x$. Proceed by induction on the next topmost essential row $\hat{\imath}$, assuming that all essential entries $(\hat{\imath}',\hat{\jmath}')$ of x with $\hat{\imath}' < \hat{\imath}$ are also essential entries of y. For the essential entry $(\hat{\imath},\hat{\jmath})$ of x to not be an essential entry of y, θ must have added a multiple of either (A) a topmore essential row $\hat{\imath}'$ or (B) a topmore nonessential nonzero row i' to either row $\hat{\imath}$ or some row $i = \hat{\imath} - cn$, with some entry leftward of $\hat{\jmath}$ nonzero in the case that the target row is $\hat{\imath}$. In case (A) there remains a nonzero entry below essential entry $(\hat{\imath}',\hat{\jmath}')$, so y is not abacus-reduced. In case (B), by n-periodicity, not only is a multiple of row i' added to row i but the same (nonzero) multiple of some essential row $\hat{\imath}' = i' - c'n$, $c' \in \mathbb{Z}_{>0}$, is added to row $i - c'n$. In either case, by the inductive hypothesis only the added essential row can be used to clear the nonzero entry above $\hat{\imath}$ in the associated column $\hat{\jmath}$; so neither can result in an abacus-reduced y. □

Corollary 3.4.8. *Every $U^{\text{aff}} \times \mathrm{GL}_h$-orbit of $\pi^{-1}(X_w)$ contains a unique matrix of distinct coordinate column vectors in increasing order. Consequently every U^{aff}-orbit of X_w contains a unique E_I, with $I \subset [m]$ u-stable.*

Proof. For existence, pick $x \in \pi^{-1}(X_w)$ and take $y = xg$ to be in heralded form. Since every column of y with pivot in congruence class c is, modulo n and up to the (h,m) window, the same as every other, the operations from the proof of Theorem 3.4.6 that clear the heralding columns v will also clear their shifts tv.

The consequence follows from restricting this result to full-rank matrices and taking the quotient by GL_h. □

Chapter 4

Ideal generators

In this chapter we exhibit the central construction and prove the central result of this document. The construction is of an ideal in $\mathbb{C}[M_{m\times h}]$ that vanishes at the preimage under π of an embedded affine Schubert variety $X_w \subset \mathrm{Gr}_{h,m}$. The result is that this ideal has vanishing set the closure of $\pi^{-1}(X_w)$ in $M_{m\times h}$. Both extend the celebrated results of Knutson and Miller [KM05]. The analogous statements connecting the equivariant cohomology of the matrix variety to the Schubert polynomials are made precise and proved in Chapter 5. It turns out that the combinatorics are less tractable; we draw only some preliminary results. For convenience, in this chapter we adopt the coordinates (respectively, row indices) $[m]$ for $\mathbb{C}^m$ (respectively, $M_{m\times h}$), rather than $[1-m+h, h]$ as in the previous chapter.

4.1 Matrix varieties

The following definition, adapted from [Ful97], specializes to Grassmannians a broader collection of matrix Schubert varieties, defined analogously from the projections of $\mathrm{GL}_n \subset M_{n\times n}$ to the flag varieties (1.1). Recall the Stiefel manifold $M^\circ_{m\times h} \subset M_{m\times h}$ of full-rank matrices and the column-span quotient map $\pi : M^\circ_{m\times h} \to \mathrm{Gr}_{h,m}$. The preimages $\pi^{-1}(X^{h,m}_\lambda)$ constitute a Birkhoff-like orbit decomposition of $M^\circ_{m\times h}$:

$$M^\circ_{m\times h} = \bigsqcup_{\lambda\subseteq((m-h)^h)} \pi^{-1}(X^{h,m}_\lambda) = \bigsqcup_{\substack{I\subset[m]\\ |I|=h}} U[I]\mathrm{GL}_h,$$

where $[I] = (\, e_{i_1} \ \cdots \ e_{i_h} \,)$ as in Definition 1.2.5.

Definition 4.1.1. Let $\lambda \subseteq ((m-h)^h)$, so that $X^{h,m}_\lambda \subseteq \mathrm{Gr}_{h,m}$ is a Schubert variety. Then $Y^{h,m}_\lambda := \overline{\pi^{-1}(X^{h,m}_\lambda)} \subseteq M_{m\times h}$ is the *matrix Schubert variety associated with* λ. The *opposite matrix Schubert varieties* $Y^\lambda_{h,m} = \overline{\pi^{-1}(X^\lambda_{h,m})}$ are defined analogously. $\diamond$

Because we will need to discuss pivot sets and multiple other row subsets in the same breath, let us reserve the letters P and Q for pivot sets and I and J for other row subsets.

Example 4.1.2. Recall Example 1.1.17, in which $m = 5$, $h = 3$, and $\lambda = (1,1)$, and retrieve the Schubert variety $X_\lambda^{3,5} = \overline{UE_{235}\mathrm{GL}_3/\mathrm{GL}_3}$ and opposite Schubert variety $X_{3,5}^\lambda = \overline{U_-E_{134}\mathrm{GL}_3/\mathrm{GL}_3}$. Then $\Omega_\lambda^{3,5} = UE_{235}\mathrm{GL}_3$ and $\Omega_{3,5}^\lambda = U_-E_{134}\mathrm{GL}_3$ consist of matrices having column-reduced forms

$$\begin{pmatrix} * & * & * \\ 1 & & \\ & 1 & \\ 0 & 0 & * \\ & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & \\ * & 0 & 0 \\ & 1 & \\ & & 1 \\ * & * & * \end{pmatrix},$$

respectively, and $Y_\lambda^{3,5} = \overline{\Omega_\lambda^{3,5}}$ and $Y_{3,5}^\lambda = \overline{\Omega_{3,5}^\lambda}$ consist of matrices in the respective closures of these collections. $\diamond$

Henceforth we restrict our attention to opposite matrix Schubert varieties unless explicitly stated otherwise, though we omit the qualifier "opposite".

It has been shown (see [Ful97]) that the preimages under (1.11) of the rank conditions of Definitions 1.1.14 and 1.1.19 generate the ideal of a matrix Schubert variety.

Example 4.1.3. Recall the notation z_I for the $h \times h$ submatrix on rows I. Write Δ_I^J for the minor on rows I and columns J. The matrix Schubert variety $Y_{2,4}^{\ydiagram{2,1}} \subset M_{4\times 2}$ has ideal generated by Δ_{12}^{12}, while its subvariety $Y_{2,4}^{\ydiagram{2}}$ has ideal $(\Delta_{12}^{12}, \Delta_{13}^{12}, \Delta_{23}^{12})$. $Y_{3,5}^{\ydiagram{1,1}}$ from Example 4.1.2 has ideal $\mathfrak{i}(Y_{3,5}^{\ydiagram{1,1}}) = (\Delta_{12}^{12}, \Delta_{12}^{13}, \Delta_{12}^{23})$. Then $Y_{2,4}^{\ydiagram{1}} \subset M_{4\times 2}$ has ideal $\mathfrak{i}(Y_{2,4}^{\ydiagram{1}}) = (\Delta_1^1, \Delta_1^2, \Delta_{23}^{12}) = (z_{11}, z_{12}, z_{21}z_{32} - z_{22}z_{31}$. $\diamond$

The embedding $X_w^{h,m} \subset X_{h,m}^P \subset \mathrm{Gr}_{h,m}$ for $P = \mathrm{piv}(w)$ and suitable h, m (Definition 2.1.12) suggests an analogous construction.

Definition 4.1.4. Let $w \in \mathcal{W}^\mathcal{P}$ (a 0-Grassmannian affine permutation), let γ be the n-core of w and pick $m \geq h > 0$ so that $\gamma_1 \leq h$ and $\gamma_1' \leq m - h$. Then $Y_w^{h,m} = \overline{\pi^{-1}(X_w^{h,m})} \subset M_{m\times h}$ is the *matrix affine Schubert variety associated with w.* $\diamond$

It is immediate that $Y_w^{h,m} \subset Y_{h,m}^P$, and we get the diagram

$$\begin{array}{ccccccc} & & M_{m\times h} & \supset & M_{m\times h}^\circ & \twoheadrightarrow & \mathrm{Gr}_{h,m} \\ & & \cup & & \cup & & \cup \\ Y_w^{h,m} & = & \overline{\pi^{-1}(X_w^{h,m})} & \leftarrow & \pi^{-1}(X_w^{h,m}) & \rightarrow & X_w^{h,m} \end{array}.$$

We may then use matrix affine Schubert varieties to approach the homology and cohomology of $X_w \subset \mathrm{Gr}_{\mathrm{SL}_n}$ by way of equivariant cohomology. Note that $X_w = \overline{\mathcal{U}w\mathcal{P}/\mathcal{P}}$ provides that $Y_w = \overline{\mathcal{U}[w]\mathrm{GL}} = \overline{U^{\mathrm{aff}}[w]\mathrm{GL}}$, where $[w] = E_{(\mathrm{piv}(w)+m-h)\cap[m]}$. Hence Y_w is $U^{\mathrm{aff}} \times \mathrm{GL}_h$-stable.

Lemma 4.1.5. $Y_w^{h,m} \cap M^\circ_{m\times h} = \pi^{-1}(X_w^{h,m})$.

Proof. The leftward containment follows from construction. The rightward follows from the continuity of π, the closedness of $X_w^{h,m}$, and that $M^\circ_{m\times h}$ is the domain of π. Specifically, if $x \in Y_w^{h,m}$ then there is a sequence $\{x_i\}_{i=1}^\infty \subset \pi^{-1}(X_w^{h,m})$ such that $\lim_{i\to\infty} x_i = x$, and if $x \in M^\circ_{m\times h}$ then $\pi(x) = \lim_{i\to\infty} \pi(x_i) \in X_w^{h,m}$. $\square$

For the remainder of this chapter, let $\mathrm{GL} = \mathrm{GL}_h$, $\mathrm{Gr} = \mathrm{Gr}_{h,m}$, $M = M_{m\times h}$, and $M^\circ = M^\circ_{m\times h}$, and write the cells and varieties without the superscripts h, m when these parameters m and h are understood. All discussion in this chapter will take place within this finite window of Gr and M unless otherwise indicated. We take the remainder of this section to adapt the notions of stable and heralded closure to the matrix setting. We will use the resulting framework to determine relations for Y_w as an affine variety.

Definition 4.1.6. Take $x \in M$ and suppose that $V = \mathrm{Span}(x) \in \mathrm{Gr}_{h',m}$ with $h' \le h$. Call x *w-augmentable* if there exists $V' \in X_w \subset \mathrm{Gr}$ such that $V \subseteq V'$. Denote by $Z_w \subset M$ the set of w-augmentable matrices.

Given a heralded cell $\hat\Omega_I \subset \mathrm{Gr}$, let $\hat Y_I = \overline{\pi^{-1}(\hat\Omega_I)}$ be the associated *matrix heralded variety*. Call a matrix x *I-augmentable* if there exists $V \in \hat\Omega_I$ such that $\mathrm{Span}(x) \subset V$, and let $\hat Z_I$ denote the collection of such matrices. $\diamond$

If $I = \mathrm{piv}(w)$ then $\hat Z_I = Z_w$.

Lemma 4.1.7. *Z_w is stable under the two-sided action of $U^{\mathrm{aff}} \times \mathrm{GL}$.*

Proof. Let x be w-augmentable, $V = \mathrm{Span}(x)$, and $V' \in X_w$ such that $V \subseteq V'$. Pick $u \in U^{\mathrm{aff}}$ and $g \in \mathrm{GL}$ and find $\tilde u \in \Theta^{-1}(u) \subset \mathcal{U}'$. Then

$$uxg\mathrm{GL}/\mathrm{GL} = ux\mathrm{GL}/\mathrm{GL} = \tilde u V \subseteq \tilde u V' \in \mathcal{U} X_w = X_w.$$

$\square$

Lemma 4.1.8. *If x is I-augmentable and $h' = \mathrm{rank}(x)$ then there exists an I-augmentation $x' \in \pi^{-1}(V')$ of x consisting of h' columns from x and $h - h'$ new columns.*

Proof. We need only identify a minimal spanning set from among the columns of x and a basis from among V'/V. $\square$

Example 4.1.9. Adapting the first case from Example 3.3.9 with $m = 8$ and $h = 5$, a matrix in $Y^{\ydiagram{5,4,2}}$ may be augmented

$$\left(\begin{array}{ccccc} 1 & & & & \\ a & & & & \\ \hline & 1 & & & \\ b & f & & & \\ c & g & & & \\ & & 1 & & \\ d & h & j & & \\ e & i & k & & \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc} 1 & & & & \\ a & & & & \\ \hline & 1 & & & \\ b & f & & & \\ c & g & & & \\ & & 1 & & \\ d & h & j & 1 & \\ e & i & k & a & 1 \end{array}\right)$$

to a matrix in $\hat{Y}_{(-1,1,4,5,6)}$, whose heralded form may in turn be augmented

$$\left(\begin{array}{ccccc} 1 & & & & \\ a & & & & \\ \hline & 1 & & & \\ b & \alpha & & & \\ c & \beta & & & \\ & & 1 & & \\ & & \alpha & 1 & \\ & & \beta & a & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccccc} 1 & & & & & \\ a & & & & & \\ \hline & 1 & & & & \\ b & \alpha & 1 & & & \\ c & \beta & a & & & \\ & & & 1 & & \\ & & b & \alpha & 1 & \\ & & c & \beta & a & 1 \end{array}\right)$$

to $Y_{[-1,1,6]}$. Similarly, in the second case a matrix augments from $Y^{\ydiagram{5,4,2}}$ to $\hat{Y}_{(-1,1,4,5)}$ to $Y_{[-1,1,9]}$. $\diamond$

The following result is one third of Theorem 4.3.17.

Theorem 4.1.10 (C)**.** $\hat{Z}_I \subseteq \hat{Y}_I$.

Proof. Pick $x \in \hat{Z}_I$ and find an I-augmentation $x' \in \pi^{-1}(\hat{\Omega}_I)$. Then

$$x \in x' M_{m \times h} = \overline{x' \mathrm{GL}} \subset \hat{Y}_I.$$

$\square$

4.2 Matrix shuffles

Recall the homogeneous coordinate rings $S_{h,m} = \mathbb{C}[p_I^{h,m} \mid I \subset [m], |I| = h]$ and their inverse image $S = \varprojlim S_{h,m}$ under restriction, which is expressed in the infinite Plücker coordinates. Recall in particular that $p_I = 0$ when I contains duplicates and $p_{\sigma(I)} = \operatorname{sgn}(\sigma) p_I$ for $\sigma \in S_h$.

Definition 4.2.1. For any *finite* index vector $K = (k_1, \ldots, k_\ell)$, let $\tilde{K} = (k_1 - n, \ldots, k_\ell - n)$. The *(infinite) shuffle at I of weight k* is the polynomial

$$\mathrm{sh}_I^k = \sum_{\substack{K \subset I \\ |K| = k}} p_{I \smallsetminus K \cup \tilde{K}} \in S.$$

(The sum is finite because $I \setminus K \cup \tilde{K}$ contains a duplicate index for almost every choice of K.) Its image under restriction is then

$$\mathrm{sh}^k_{I'} = \rho_{h,m}(\mathrm{sh}^k_I) = \sum_{\substack{K \subset I' \\ |K|=k}} p_{I' \setminus K \cup \tilde{K}},$$

the *(finite) shuffle at* $I' = (I + m - h) \cap [m]$ *of weight* k. ◇

Shuffles were used in the service of the following result, which we recover as a consequnece of an analogous result for matrices (see Remark 4.4.11).

Proposition 4.2.2 ([KLMW07]). *The infinite shuffles have vanishing set* $\mathrm{Gr}_{\mathrm{GL}_n} \subset \mathrm{Gr}_\infty$.

The coordinate ring map $S_{h,m} \to \mathbb{C}[M]$ (1.11) sends each finite shuffle to an alternating sum of maximal minors. The following definition generalizes these full-rank "matrix shuffles" to sums of partial-rank minors taken over a larger collection of shifted index sets.

To simplify notation in the remainder of this discussion, take $\mathbb{N} = \mathbb{Z}_{\geq 0}$. Analogously to $\mathbb{Z}^n_k$, we write $\mathbb{N}^\ell_k$ for the set of ordered ℓ-tuples from $\mathbb{N}$ having sum k. Also recall the notation $\Delta^J_I \in \mathbb{C}[M]$ for the minor at rows I and columns J (or zero if $I \not\subseteq [m]$ or $J \not\subseteq [h]$).

Definition 4.2.3. Pick row and column subsets $I = (i_1, \ldots, i_\ell) \subseteq [m]$ and $J = (j_1, \ldots, j_\ell) \subseteq [h]$ and a nonnegative integer k, and denote by $\mathbb{N}^\ell_k$ the nonnegative integer ℓ-tuples that sum to k. The *(matrix) shuffle at* (I, J) *of weight* k, or k*-shuffle at* (I, J), is the finite sum

$$\mathrm{sh}^k_{I,J} = \sum_{R \in \mathbb{N}^\ell_k} \Delta^J_{I-nR} \in \mathbb{C}[M].$$

We'll refer to the Δ^J_{I-nR} over which the sum is taken as the *minor summands*. Note that 0-shuffles are just minors. ◇

For the remainder of the section fix $w \in \mathcal{W}^{\mathcal{P}}$, γ its n-core, $h \geq \gamma_1$, and $m - h \geq \gamma'_1$. We will always take $P = (p_1 < \cdots < p_y) \subset [m]$ to be a pivot set and $\hat{P}$ the heralding pivots of P, though P may not be of the form $\mathrm{piv}(w)$ or even u-stable. I and J will be ordered row and column subsets, respectively, of the same cardinality ℓ.

Below we give names to the shuffles at I of maximum weight that fail to vanish on $\pi^{-1}(\Omega_w)$. The rest of this section is spent identifying these shuffles. We apply this proposition to define our candidate ideal for Y_w (Definition 4.3.14) and to check that it is set-theoretic (Proposition 4.4.5).

Definition 4.2.4. Pick P and I. Define the *shuffle threshold* $k_P(I)$ *from* I *to* P, or just *of* I when P is understood, to be the maximum nonnegative integer k for which $\mathrm{sh}^k_{I,J}|_{\pi^{-1}(\hat{\Omega}_P)} \not\equiv 0$ for some (equivalently, any) J. Note that $k_P(I) \leq \lfloor \frac{m}{n} \rfloor \ell$. Write $k_w(I) = k_{\mathrm{piv}(w)}(I)$. If $\mathrm{sh}^k_{I,J} = 0$ on $\pi^{-1}(\hat{\Omega}_P)$ for all $k \geq 0$ then set $k_P(I) = -1$, so that shuffles are always well-defined when $k > k_P(I)$. ◇

Example 4.2.5. Let $n = 3$ and $w = s_1 s_2 s_0 s_1 s_2 s_0 = [-5, 5, 6]$. Then w has 3-core $\gamma = (3^2, 2^2, 1^2) \subset (3^6)$. We may take $m = 9$ and $h = 3$ to get $\Omega_w \cong \Omega_w^{3,9} \subset \mathrm{Gr}_{3,9}$. Consider an arbitrary u-reduced matrix

$$x = \left(\begin{array}{ccc} 1 & & \\ a & 0 & 0 \\ b & 0 & 0 \\ & 1 & \\ c & a & 0 \\ d & b & 0 \\ \hline & & 1 \\ e & c & a \\ f & d & b \end{array}\right) \in \pi^{-1}(\Omega_w^{3,9})$$

(Note again that the parts of γ correspond to the non-pivotal rows of x.) Then $\mathrm{sh}^k_{89,12}(x) = 0$ for $k = 2$, though not for $k = 1$:

$$\mathrm{sh}^2_{89,12}(x) = \Delta^{12}_{29}(x) + \Delta^{12}_{56}(x) + \Delta^{12}_{83}(x) = \begin{vmatrix} a & 0 \\ f & d \end{vmatrix} + \begin{vmatrix} c & a \\ d & b \end{vmatrix} + \begin{vmatrix} e & c \\ b & 0 \end{vmatrix} = 0,$$

while

$$\mathrm{sh}^1_{89,12}(x) = \Delta^{12}_{59}(x) + \Delta^{12}_{86}(x) = \begin{vmatrix} c & a \\ f & d \end{vmatrix} + \begin{vmatrix} e & c \\ d & b \end{vmatrix} \neq 0.$$

At columns $(1, 3)$ and $(2, 3)$ the same or a smaller value suffices. Since every matrix in $\pi^{-1}(\Omega_w^{3,9})$ is in the GL-orbit of some u-reduced x, this implies that $k_{[-5,5,6]}(8, 9) = 1$. (This does not result from the vanishing of all minor summands, which only occurs for $k \geq 4$.) $\diamond$

The expansion (3.1) allows us to expand each $(I, J)^{\mathrm{th}}$ minor of x multilinearly into minor summands:

$$\begin{aligned} \Delta_I^J(x) &= \Delta_I(\, x_{k_1} \ \cdots \ x_{k_\ell} \,) \\ &= \sum_Q \Delta_I(\, C^1_{q_1} v_{q_1} \ \cdots \ C^\ell_{q_\ell} v_{q_\ell} \,) \\ &= \sum_Q C^J_Q \Delta_I(\, v_{q_1} \ \cdots \ v_{q_\ell} \,), \end{aligned} \tag{4.1}$$

where the sum ranges over index sets $Q = (q_1, \ldots, q_\ell)$ satisfying $q_{\ell'} \leq \ell'$, each $C^j_j \neq 0$, and $C^J_Q = \prod_{j=1}^{\ell} C^j_{q_j}$. The following lemma then provides a geometric interpretation of shuffles that simplifies the calculations and proofs to follow. For simplicity of notation, let $x_1, \ldots, x_\ell$ denote the J^{th} column vectors of a matrix $x \in M$, as opposed to $x_{j_1}, \ldots, x_{j_\ell}$. Write $(x_j)_{j=1}^{\ell} = (\, x_1 \cdots x_\ell \,)$ for the $m \times \ell$ submatrix, then $\Delta_I(x_\nu)_{\nu=1}^{\ell} = \Delta_I^J(x)$ and $\mathrm{sh}_I^k(x_\nu)_{\nu=1}^{\ell} = \mathrm{sh}^k_{I,J}(x)$ for the minors and shuffles.

Lemma 4.2.6. *For $x \in M$ and $R = (r_1, \ldots, r_\ell)$, write $\text{þ}^R_{I,J}(x) := \Delta_I(t^{r_1} x_1 \cdots t^{r_\ell} x_\ell)$.*[1] *Then*

$$\mathrm{sh}^k_{I,J}(x) = \sum_{R \in \mathbb{N}_k^\ell} \Delta_I(t^{r_1} x_1 \cdots t^{r_\ell} x_\ell) = \sum_{R \in \mathbb{N}_k^\ell} \text{þ}^R_{I,J}(x).$$

[1] We adopt the common practice of approximating the Bactrian letter *sho* by the Old English *thorn*.

Let us also call the $\mathfrak{p}^R_{I,J}$ "minor summands".

Proof. Let R range over $\mathbb{N}^\ell_k$ and σ over the symmetric group S_ℓ.

$$\begin{aligned}
\mathrm{sh}^k_{I,J} = \mathrm{sh}^k_I(\, x_1 \ \cdots \ x_\ell \,) &= \sum_R \Delta_{I-nR}(\, x_1 \ \cdots \ x_\ell \,) \\
&= \sum_R \sum_\sigma \mathrm{sgn}(\sigma) x_{i_1 - nr_1, \sigma(1)} \cdots x_{i_\ell - nr_\ell, \sigma(\ell)} \\
&= \sum_R \Delta_I(\, t^{r_1} x_1 \ \cdots \ t^{r_\ell} x_\ell \,).
\end{aligned}$$

□

Example 4.2.7. The calculations on columns 1 and 2 of x in Example 4.2.5 also imply that $\mathrm{sh}^2_{89,J}|_{\pi^{-1}(\Omega^{3,9}_{(1,4)})} \equiv 0$ for the remaining column subsets J of size 2:

$$\begin{aligned}
\mathrm{sh}^2_{89,13}(x) &= \mathfrak{p}^{20}_{89,13} + \mathfrak{p}^{11}_{89,13} + \mathfrak{p}^{02}_{89,13} \\
&= \Delta_{89}(\, t^2 x_1 \ x_3 \,) + \Delta_{89}(\, t x_1 \ t x_3 \,) + \Delta_{89}(\, x_1 \ t^2 x_3 \,) \\
&= \begin{vmatrix} e & 0 \\ f & 0 \end{vmatrix} + \begin{vmatrix} c & 0 \\ d & 0 \end{vmatrix} + \begin{vmatrix} a & a \\ b & b \end{vmatrix} \\
&= 0,
\end{aligned}$$

and $\mathrm{sh}^2_{89,23}(x) = 0$ similarly. ◇

We can leverage this formulation to obtain an expansion of $\mathrm{sh}^k_{I,J}(x)$ multilinearly into shuffles on the members of a heralded basis for $\mathbf{O}\,\mathrm{Span}(x)$ (Corollary 4.2.10), which in turn will be much easier to compute or to test for (non)vanishing. This comes from an expansion of the columns of x into the heralded basis of $\mathbf{O}\,\mathrm{Span}(x)$ (Theorem 4.2.8). Under the right conditions, the expansion into "shuffle summands" will yield a single "leading shuffle" that is the last to vanish as the shuffle weight reaches $k_P(I)$.

Theorem 4.2.8. *Pick $x \in M$. Let $V = \mathrm{Span}(x)$, $v_1, \ldots, v_{m'}$ the heralded basis for $\mathrm{hc}\, V$ with heralding vectors $\hat{v}_1, \ldots, \hat{v}_{n'}$, and $P = \mathrm{piv}(\mathrm{hc}\, V)$. Write x_j for the j^{th} column of x. Then there are columns $J = (j_1, \ldots, j_{m'}) \subset [h]$ so that each*

$$x_{j_\nu} = \sum_{\nu=1}^{m'} C^\xi_\nu v_\xi, \tag{4.2}$$

where if $\xi > \nu$ and $v_\nu = \hat{v}_\zeta$ for some $\zeta \in [n']$ then $C^\xi_\nu = 0$, while if $\xi = \nu$ and $v_\nu = \hat{v}_\zeta$ then $C^\xi_\nu \neq 0$.

Proof. The proof iterates along the pivots of $\operatorname{hc} V$. Recall lpiv from Section 2.1.

Suppose that we have selected $v_1, \ldots, v_{\nu-1}$. Take x' to be the matrix obtained from x by using $v_1, \ldots, v_{\nu-1}$ to clear rows $p_1, \ldots, p_{\nu-1}$, in order. Necessarily $p_\nu = \operatorname{lpiv}(\operatorname{Span}(x'))$.

- If $p_\nu \notin \hat{P}$ then find $p_\xi \in \hat{P}$ with $p_\xi \equiv p_\nu$ modulo n; necessarily $\xi < \nu$. Take $c = \frac{p_\nu - p_\xi}{n}$ and $v_\nu = t^c v_\xi$.
- If $p_\nu \in \hat{P}$ then say $p_\nu = \hat{p}_\xi$ and find $j_\xi \in [h]$ at which $\operatorname{piv}(x'_{j_\xi}) = \operatorname{lpiv}(\operatorname{Span}(x'))$. Take $v_\nu = \frac{1}{e^*_{p_\nu}(x'_{j_\xi})} x'_{j_\xi}$.

Stop when $\operatorname{Span}(x') \subset \operatorname{Span}(v_1, \ldots, v_\nu)$.

By construction, and by including all $p_j \in P$, we arrive at the heralded basis for some $W \supseteq V$. The order of the steps in the cases that $p_\nu \in \hat{P}$ implies the triangularity condition on the coefficients. Since each $x_{j_\nu} \in V$, and since each shift $t^c \hat{v}_\nu$ is necessary to obtain some $x_j \in V$, we must have $W \subseteq \operatorname{hc} V$. Since W is heralded, $\operatorname{hc} V \subseteq W$. Thus $W = \operatorname{hc} V$. $\square$

Example 4.2.9. Set $n = 3$ and

$$x = \left(\begin{array}{ccc} 1 & & \\ a & & \\ \hline & 1 & \\ b & f & \\ c & g & \\ & & 1 \\ d & h & j \\ e & i & k \end{array}\right)$$

(Fig. 3.3) and assume that the entries satisfy $j \neq f$ and $k \neq -af + aj + g$ (being careful to keep local coordinates distinct from indexing variables). The proof of Theorem 4.2.8 proceeds as follows, depicting each matrix after the j^{th} step with shifted basis vectors $v_1, \ldots, v_\nu$ to the left and a complementary basis to the right:

$$\left(\begin{array}{ccc} 1 & & \\ a & & \\ & 1 & \\ b & f & \\ c & g & \\ & & 1 \\ d & h & j \\ e & i & k \end{array}\right) \xrightarrow{v_1 = x_1} \left(\begin{array}{c|cc} 1 & & \\ a & & \\ & 1 & \\ b & f & \\ c & g & \\ & & 1 \\ d & h & j \\ e & i & k \end{array}\right) \xrightarrow{v_2 = x_2} \left(\begin{array}{cc|c} 1 & & \\ a & & \\ & 1 & \\ b & f & \\ c & g & \\ & & 1 \\ d & h & j \\ e & i & k \end{array}\right) \xrightarrow{v_3 = t v_2} \left(\begin{array}{ccc|c} 1 & & & \\ a & & & \\ & 1 & & \\ b & f & & \\ c & g & & \\ & & 1 & \\ d & h & f & 1 \\ e & i & g & \frac{k-g}{j-f} \end{array}\right) \xrightarrow{v_4 = t^2 v_1} \left(\begin{array}{cccc|c} 1 & & & & \\ a & & & & \\ & 1 & & & \\ b & f & & & \\ c & g & & & \\ & & 1 & & \\ d & h & f & 1 & \\ e & i & g & a & 1 \end{array}\right),$$

concluding with $v_5 = \hat{v}_3 = \frac{Ke_8}{K} = e_8$, where $K = k - g - aj + af \neq 0$. This provides $x_1 = \hat{v}_1$, $x_2 = \hat{v}_2$, and $x_3 = K\hat{v}_3 + v_3 + (j - f)v_4$, in satisfaction of (4.2). $\diamond$

The example suggests that (4.2) might be stated fully triangularly, not just partially so as in Theorem 4.2.8, but this is an artifact of our choice of x.

Corollary 4.2.10. *Let $x \in M$ with* $\mathrm{Span}(x) = V$*,* $v_1, \ldots, v_{m'}$ *the heralded basis for* $\mathrm{hc}\, V$*, and* $\hat{v}_1, \ldots, \hat{v}_{n'}$ *the heralding basis vectors. Take* $J = (j_1, \ldots, j_\ell)$ *as in Theorem 4.2.8 and pick* I *and* k *with* $|I| = \ell$*. Let* $\Xi = (\xi_1 < \cdots < \xi_\ell)$ *range over the subsets of* $[m']$*. Then*

$$\mathrm{sh}^k_{I,J}(x) = \sum_{\Xi} C_\Xi \mathrm{sh}^k_I(\, v_{\xi_1} \ \cdots \ v_{\xi_\ell}\,), \qquad C_{\mathrm{N}} \neq 0,$$

where N *consists of the* $\xi \in [m']$ *for which* $v_\xi = \hat{v}_\nu$ *for some* $\nu \in [n']$*.*

Proof. The expansion

$$\begin{aligned}
\mathrm{sh}^k_{I,J}(x) &= \mathrm{sh}^k_I(\, x_{j_1} \ \cdots \ x_{j_\ell}\,) \\
&= \mathrm{sh}^k_I\left(\sum_{\xi=1}^{m'} C^\xi_1 v_\xi \ \cdots \ \sum_{\xi=1}^{m'} C^\xi_\ell v_\xi \right) \\
&= \sum_{\xi_1,\ldots,\xi_\ell \in [m']} C_{(\xi_1,\ldots,\xi_\ell)} \mathrm{sh}^k_I(\, v_{\xi_1} \ \cdots \ v_{\xi_\ell}\,) && C_{(\xi_1,\ldots,\xi_\ell)} := \prod_{\nu=1}^{\ell} C^{\xi_\nu}_\nu \\
&= \sum_{\Xi} C_\Xi \mathrm{sh}^k_I(\, v_{\xi_1} \ \cdots \ v_{\xi_\ell}\,) && C_\Xi := \sum_{\sigma \in S_\ell} C_{(\xi_{\sigma(1)},\ldots,\xi_{\sigma(\ell)})}
\end{aligned}$$

follows from $V \subseteq \mathrm{hc}\, V = \mathrm{Span}(v_1, \ldots, v_y)$, (4.2), and the multilinearity of $\mathrm{sh}^k_{I,J}$. In particular, any shuffle summand of the multilinear expansion in which some $\xi_i = \xi_j$ necessarily vanishes, due to a repeated column; it is enough to consider only distinct indices $\xi_1, \ldots, \xi_\ell \in [m']$. The coefficient

$$C_{\mathrm{N}} = \sum_{\sigma \in S_\ell} \mathrm{sgn}(\sigma) \prod_{\nu=1}^{\ell} C^{\xi_{\sigma(\nu)}}_\nu$$

uses $v_{\xi_\nu} = \hat{v}_\nu$ (see the qualification of (4.2)) and is therefore the determinant of a triangular matrix with nonzero diagonal entries $C^{\xi_\nu}_\nu$, hence itself nonzero. □

4.3 Abacus slides

In Appendix C.1 we compute shuffle thresholds for several illustrative cases. The following definitions and discussions can be motivated by these cases, and in particular Lemma C.1.4 introduces a rudimentary notion of straddling (Definition 4.3.2) that signals the vanishing of certain minor summands in a shuffle. For further space savings, we abbreviate the matrix $(x_\nu)^\ell_{\nu=1}$ to (x_ν).

Definition 4.3.1. Given P and I, let $\mathcal{A} = \mathcal{A}(I, P)$ denote the collection of maps $A : I \to P$ satisfying the following:

(i) for all i, $Ai \leq i$;

(ii) if $i < i'$ and $Ai \equiv Ai'$ modulo n then $Ai < Ai'$;

(iii) if $Ai - cn \in P$ for some $c > 0$ then $Ai - cn \in \operatorname{img} A$.

We call these maps *abacus slides* after the abacus presentation of $\mathrm{SL}_n(\mathbf{F})$ [KLMW07]. Partition P into

$$P^{(1)} = (p_1^{(1)} < \cdots < p_{y_1}^{(1)}), \ldots, P^{(n)} = (p_1^{(n)} < \cdots < p_{y_n}^{(n)})$$

where each $p_j^{(i)} \equiv i$ modulo n, and I into

$$I^{(1)} = (i_1^{(1)} < \cdots < i_{\ell_1}^{(1)}), \ldots, I^{(n)} = (i_1^{(n)} < \cdots < i_{\ell_n}^{(n)})$$

where each $Ai_j^{(i)} \equiv i$ modulo n. It follows from (ii) and (iii) that each $Ai_j^{(i)} = p_j^{(i)}$. Call $(\ell_1, \ldots, \ell_n)$ the *bead composition* of A. Additionally let $\mathcal{A}' = \mathcal{A}'(I, P) \supset \mathcal{A}$ be the collection of *slides*, which by definition satisfy only (i). Associate with each (abacus) slide A the *(abacus) buffer* $B : I \to [m]$ defined as follows: Let $Bi \in (Ai + n\mathbb{N}) \cap [m]$ be maximum so that $\{Ai, Ai + n, \ldots, Bi\} \cap \operatorname{img} A = \{Ai\}$. $\diamond$

Recall that the shuffle threshold is bounded above by $\left\lfloor \frac{m}{n} \right\rfloor \ell$, the sum of a copy of the maximum power of t any vector in $\mathbb{C}^m$ can sustain before vanishing for each column in J ($|J| = \ell$). This is clearly a coarse estimate. It is improved somewhat by taking a matrix $x \in \pi^{-1}(\hat{\Omega}_P)$ in heralded form, whose column vectors have pivots $p_1, \ldots, p_h$. In this case one might act on the leftmost ℓ columns by powers of t until just before they vanish, which gives the upper bound $\sum_{j=1}^{\ell} \left\lfloor \frac{m - p_j}{n} \right\rfloor$. The following definition and lemma begin the process of using (abacus) slides to improve further upon these bounds.

Definition 4.3.2. Let $A : I \to P$ be an injective slide with buffer B. Given $R \in \mathbb{N}_k^\ell$, say that R *straddles* A *(at ν)* if

$$Ai_\nu + r_\nu n > \min(i_\nu, Bi_\nu) \tag{4.3}$$

(for some ν). $\diamond$

Lemma 4.3.3. *Fix P and I. Let $A : I \to P$ be an injective slide with buffer B, and denote by* $\operatorname{strad}(A) \subset \mathbb{N}_k^\ell$ *the subset of vectors R that straddle A. Then*

$$\sum_{R \in \operatorname{strad}(A)} \mathfrak{b}_{I,J}^R \,|_{\hat{Y}_P} \equiv 0 \tag{4.4}$$

for any $J \subseteq [m]$, $|J| = \ell$.

The proof exhibits an involution on $\operatorname{strad}(A)$ that induces a sign-reversing bijection on the minor summands of $\mathrm{sh}_I^k \underline{(t^{r_\nu} v_\nu)}$, where $v_1, \ldots, v_\ell$ is a heralded basis for $V \in \hat{\Omega}_P$. The result then follows from $\hat{Y}_P = \pi^{-1}(\hat{\Omega}_P)$.

Proof. Pick $R \in \mathrm{strad}(A)$ and identify the smallest index $\nu = \nu'$ at which R straddles A. If $Ai_{\nu'} + r_{\nu'}n \leq Bi_{\nu'}$ then set $\tilde{R} = R \in \mathrm{strad}(A)$. This assignment is trivially involutive. Our choice of ν' implies that $Ai_{\nu'} + r_{\nu'}n > i_{\nu'}$, which means that $\Delta_I(t^{r_\nu}v_\nu) = 0$ by the vanishing of column ν'. Thus $\Delta_I(t^{\tilde{r}_\nu}v_\nu) = 0 = -\Delta_I(t^{r_\nu}v_\nu)$.

Otherwise we have $Ai_{\nu'} + r_{\nu'}n \leq i_{\nu'}$. Identify the index $\nu'' > \nu'$ for which $Ai_{\nu''} = Bi_{\nu'} + n$ and define $\tilde{R} = (\tilde{r}_1, \ldots, \tilde{r}_\ell)$ by

$$\begin{aligned}
\tilde{r}_{\nu'} &= (r_{\nu''}n + Ai_{\nu''} - Ai_{\nu'})/n \\
\tilde{r}_{\nu''} &= (r_{\nu'}n + Ai_{\nu'} - Ai_{\nu''})/n \\
\tilde{r}_{\nu} &= r_\nu \text{ for } \nu \notin \{\nu', \nu''\}.
\end{aligned}$$

Observe that each $\tilde{r}_\nu \in \mathbb{N}$ and that $\sum_\nu \tilde{r}_\nu = \sum_\nu r_\nu$, so $\tilde{R} \in \mathbb{N}^\ell_k$.

In the remaining case, the matrices $(t^{r_\nu}v_\nu)$ and $(t^{\tilde{r}_\nu}v_\nu)$ differ by the transposition $(\nu'\ \nu'') \in S_\ell$ of columns ν' and ν'', so their associated pure minor summands differ by $\mathrm{sgn}((\nu'\ \nu'')) = -1$. Since the choice of ν'' depends upon ν' but not upon R, involutivity will follow if $\tilde{R}$ straddles A at minimum index ν'. Indeed, At $\nu = \nu'$,

$$\begin{aligned}
Ai_{\nu'} + \tilde{r}_{\nu'}n &= Ai_{\nu'} + r_{\nu''}n + Ai_{\nu''} - Ai_{\nu'} \\
&= Ai_{\nu''} + r_{\nu''}n \\
&= Bi_{\nu'} + n + r_{\nu''}n \\
&> Bi_{\nu'} \geq \min(i_{\nu'}, Bi_{\nu'}),
\end{aligned}$$

while at $\nu < \nu'(< \nu'')$ we have $Ai_\nu + \tilde{r}_\nu n = Ai_\nu + r_\nu n \leq \min(i_\nu, Bi_\nu)$ by our choice of $\tilde{R}$. $\square$

Our goal is to slide the indices of I as far back as possible (meaning, to choose a shuffle of maximum weight) without allowing the minor summands to cancel. The following definition and lemma use the straddling condition (4.3) (after solving for r_ν) to connect abacus slides to shuffle weights. These are the formulae by which $\mathcal{A}$ governs $k_P(I)$.

Definition 4.3.4. Pick P and I. Let $A : I \to P$ be a slide and B its buffer. Define the *slide weight of A* to be

$$k_P(I, A) = \sum_{i \in I} \left\lfloor \frac{\min(i, Bi) - Ai}{n} \right\rfloor. \tag{4.5}$$

$\diamond$

Corollary 4.3.5. *Pick P and I. Then $k_P(I) \leq \max_{A \in \mathcal{A}'} k_P(I, A)$.*

Proof. Pick $k > \max_{A \in \mathcal{A}'} k_P(I, A)$. By design, for every $R \in \mathbb{N}^\ell_k$ and every $A \in \mathcal{A}'$ there is an index ν at which R straddles A. By Lemma 4.3.3, if $x \in \hat{Y}_P$ then the vanishing sum of minors (4.4) accounts for every minor summand of $\mathrm{sh}^k_{I,J}(x)$ (for any J). $\square$

Example 4.3.6. Recall Example 4.2.5. There $n = 3$ and $P = (1, 4, 7)$ with $\hat{P} = (1)$, while $I = (8, 9)$. The only possible abacus slide $A : I \to P$ is given by $A(8) = 1$ and $A(9) = 4$; $\mathcal{A} = \{A\}$. A has buffer B given by $B(8) = 1$ and $B(9) = 7$. The slide weight of A is therefore

$$k_{147}(89, A) = \left\lfloor \tfrac{\min(8,1)-1}{3} \right\rfloor + \left\lfloor \tfrac{\min(9,7)-4}{3} \right\rfloor = 0 + 1 = 1,$$

which is also the maximum shuffle weight k for which $\mathrm{sh}^k_{89,J}\,|_{\Omega_w} \not\equiv 0$.

Another (non-abacus) slide A' is given by $A'(8) = 7$ and $A'(9) = 1$ and has buffer B' given by $B'(8) = 7$ and $B'(9) = 4$. Now

$$k_{147}(89, A') = \left\lfloor \tfrac{\min(8,7)-1}{3} \right\rfloor + \left\lfloor \tfrac{\min(9,4)-1}{3} \right\rfloor = 0 + 1 = 1$$

has the same slide weight. If instead we take $A'(9) = 4$ then A' we get slide weight 0. ◇

In Example 4.2.5 the maximum slide weight (for a nonvanishing shuffle) is attained by both an abacus slide and another slide. Other examples of maximizing the slide weight may be found in Appendix C.1 (see in particular Example C.1.6). In our search for the maximum slide weight, Lemma 4.3.7 allows us to narrow our focus to abacus slides.

Lemma 4.3.7. *Pick P and I and let A be an abacus slide with buffer B. Partition P and I as in Definition 4.3.1. Then*

(a) $k_P(I, A) = \sum_s \left(\left\lfloor \dfrac{i^{(s)}_{\ell_s} - p^{(s)}_{\ell_s}}{n} \right\rfloor + \sum_{\nu=1}^{\ell_s - 1} \left\lfloor \dfrac{\min(i^{(s)}_\nu, Bi^{(s)}_\nu) - p^{(s)}_\nu}{n} \right\rfloor \right)$ *and*

(b) $\max_{A \in \mathcal{A}} k_P(I, A) = \max_{A \in \mathcal{A}'} k_P(I, A)$.

Note that if $P = \mathrm{piv}(w)$ for some $w \in \mathcal{W}^{\mathcal{P}}$ then $Bi^{(s)}_\nu = p^{(s)}_\nu$ for all s, ν with $\nu < \ell_s$. Since each $p^{(s)}_\nu \le i^{(s)}_\nu$, this means that the numerator of each floored fraction, except the last, vanishes:

$$k_w(I, A) = \sum_s \left\lfloor \frac{i^{(s)}_{\ell_s} - p^{(s)}_{\ell_s}}{n} \right\rfloor = \sum_s \left\lfloor \frac{\max(I^{(s)}) - p^{(s)}_{\ell_s}}{n} \right\rfloor. \tag{4.6}$$

The proof makes use of the following partial order on slides having the same bead decomposition. Both definition and proof refer to the enumerated criteria of Definition 4.3.1.

Definition 4.3.8. For $A_1, A_2 \in \mathcal{A}'$ having the same bead composition, write $A_1 \prec A_2$ if

- there are fewer pairs (i, i') at which A_1 violates (ii) than A_2 or
- A_1 and A_2 violate (ii) at the same number of pairs but $A_1 I <_{\mathrm{lex}} A_2 I$.

Notice that if $A \in \mathcal{A}$ then A is minimal with respect to $\prec$. $\diamond$

Proof of Lemma 4.3.7. For (a), take $A \in \mathcal{A}$ and partition I and P as in Definition 4.3.1. Expanding the nested sum, it is enough to check that $Ai_\nu^{(s)} = p_\nu^{(s)}$, which follows from the definition of $I^{(s)}$ and $P^{(s)}$.

For (b), suppose that $A' \in \mathcal{A}' \smallsetminus \mathcal{A}$, so that A' fails to satisfy (ii) or (iii). Write $k'_P(I,A)$ for the formula in (a). It will suffice to find $A \in \mathcal{A}$ with $k'_P(I,A') \leq k'_P(I,A)$. To this end, we will identify $A'' \in \mathcal{A}'$ such that $A'' \prec A'$ and $k'_P(I,A') \leq k'_P(I,A'')$. Since $A' \in \mathcal{A}' \smallsetminus \mathcal{A}$ was arbitrary, this process terminates in $\mathcal{A}$, which implies the result.

If A' fails (ii), find two indices $i < i'$ with $p = A'i$ and $p' = A'i'$ satisfying $p \equiv p'$ but $p' < p$. Take $A''i = p'$ and $A''i' = p$, and otherwise $A''i'' = A'i''$. Since $p' < p \leq i < i'$, A'' satisfies (i). Let B' and B'' be the buffers of A' and A'', respectively. To see that $k'_P(I,A') \leq k'_P(I,A'')$ we need only consider the adjusted summands, i.e. to check that

$$\left\lfloor \frac{\min(i,B'i)-p}{n} \right\rfloor + \left\lfloor \frac{\min(i',B'i')-p'}{n} \right\rfloor \leq \left\lfloor \frac{\min(i,B''i)-p'}{n} \right\rfloor + \left\lfloor \frac{\min(i',B''i')-p}{n} \right\rfloor . \tag{4.7}$$

We have $\min(i',B'i') = B'i' = B''i = \min(i,B''i)$ because $B''i = B'i' < p \leq i < i'$. We have $\min(i,B'i) \leq \min(i',B''i')$ because $B'i = B''i'$ and $i < i'$. This leaves $\min(i',B'i') - p' = \min(i,B''i) - p'$ and $\min(i,B'i) - p \leq \min(i',B''i') - p$, which together give (4.7).

Free up the symbol i' from the previous part of the proof.

If A' satisfies (ii) but fails (iii), pick $i \in I$ and $c > 0$ for which $A'i - cn \in P \smallsetminus \operatorname{img} A'$. Set $p = A'i$ and take $p' = p - cn \in P$ to be as large as possible; that is, let no members of P congruent to p lie between p' and p. Identify $i' \in I$ for which $A'i'$ is the largest index in $\operatorname{img} A'$ before $A'i$ and in the same congruence class; this means that $A'i' < p' < p \leq i$. Take A'' so that $A''i'' = A'i''$ except for $A''i = p'$. Since $p' < p < i$, A'' satisfies (i). By assumption, there are no $i' \neq i$ with $p' \leq A'i' \leq p$. Thus the order of $\operatorname{img} A'$, restricted to any one congruence class, has not changed, so A'' satisfies (ii). Again only two summands differ between A' and A''; with B' and B'' as before, we must show that

$$\left\lfloor \frac{\min(i,B'i)-p}{n} \right\rfloor + \left\lfloor \frac{\min(i',B'i')-A'i'}{n} \right\rfloor \leq \left\lfloor \frac{\min(i,B''i)-p'}{n} \right\rfloor + \left\lfloor \frac{\min(i',B''i')-A''i'}{n} \right\rfloor . \tag{4.8}$$

$B''i = B'i$ so $\min(i,B''i) - p' = \min(i,B'i) - p + cn$. Meanwhile, $A''i' = A'i'$ but $B''i' = p' - n = p - (c+1)n$, so $\min(i',B''i') - A''i' = \min(i', p-n-cn) - A''i' \geq \min(i',p-n) - A''i' - cn$. Take floors of the quotients of these terms by n, which preserve weak inequalities, to produce (4.8). $\square$

Example 4.3.9. Let $n = 2$. Then $\gamma(w) = (\ell(w), \ell(w)-1, \ldots, 1)$ and we may choose $h = \ell(w)$ and $m = 2\ell(w)$. Now pick I with members distinct modulo 2; I has size 1 or 2. If $I = (i)$ then $k_w(I) = \left\lfloor \frac{i-1}{2} \right\rfloor$. Suppose that $I = (i_1 < i_2)$. Since 1 is the only heralding pivot in $M_{2\ell(w)\times\ell(w)}$, each abacus slide A takes I into $\{1,3,\dots,2\ell(w)-1\}$. As in Example C.1.2, we may maximize $k_w(I,A)$ by either $A_1:(i_1,i_2)\mapsto(1,3)$ or $A_2:(i_1,i_2)\mapsto(3,1)$. The former gives

$$k_w(I,A_1)=\left\lfloor\frac{\min(i_1,1)-1}{2}\right\rfloor+\left\lfloor\frac{\min(i_2,2\ell(w)-1)-3}{2}\right\rfloor=0+\left\lfloor\frac{i_2-3}{2}\right\rfloor=\left\lfloor\frac{i_2-1}{2}\right\rfloor-1,$$

and the latter gives $k_w(I,A_2)=\left\lfloor\frac{i_1-1}{2}\right\rfloor-1$. Since $i_1<i_2$ we have $\max_{A\in\mathcal{A}'}k_w(I,A)=\left\lfloor\frac{i_2-1}{2}\right\rfloor-1$. ◇

Example 4.3.10. Take $n=3$ and $w=s_0s_1s_2s_1s_0=[-3,2,7]$, and use $m=8$ and $h=4$. This provides the u-reduced form

$$x=\begin{pmatrix}1&&&\\a&&&\\b&&&\\&1&&\\c&a&&\\&b&1&\\&&&1\\d&c&\alpha&a\end{pmatrix}\in\pi^{-1}(\Omega_w^{4,8}),$$

with $P=(1,4,6,7)$ and $\hat{P}=(1,6)$. Take $I=(6,7,8)$. To maximize the slide weight, we need only consider the abacus slides. These send $(6,7,8)$, in order, to $(1,4,7)$ (single congruence class), $(1,4,6)$, $(1,6,4)$, and $(6,1,4)$ (both congruence classes). Index these slides by the congruence classes to which the ordered members of I are assigned (which uniquely determines the slides). Then

$$\begin{aligned}
k_P(I,A_{111})&=\left\lfloor\tfrac{\min(6,1)-1}{3}\right\rfloor+\left\lfloor\tfrac{\min(7,4)-4}{3}\right\rfloor+\left\lfloor\tfrac{\min(8,7)-7}{3}\right\rfloor=0+0+0=0\\
k_P(I,A_{113})&=\left\lfloor\tfrac{\min(6,1)-1}{3}\right\rfloor+\left\lfloor\tfrac{\min(7,7)-4}{3}\right\rfloor+\left\lfloor\tfrac{\min(8,6)-6}{3}\right\rfloor=0+1+0=1\\
k_P(I,A_{131})&=\left\lfloor\tfrac{\min(6,1)-1}{3}\right\rfloor+\left\lfloor\tfrac{\min(7,6)-6}{3}\right\rfloor+\left\lfloor\tfrac{\min(8,7)-4}{3}\right\rfloor=0+0+1=1\\
k_P(I,A_{311})&=\left\lfloor\tfrac{\min(6,6)-6}{3}\right\rfloor+\left\lfloor\tfrac{\min(7,1)-1}{3}\right\rfloor+\left\lfloor\tfrac{\min(8,7)-6}{3}\right\rfloor=0+0+0=0.
\end{aligned}$$

Meanwhile,

$$\mathrm{sh}^1_{678,123}(x)=\begin{vmatrix}b&0&0\\0&0&0\\d&c&\alpha\end{vmatrix}+\begin{vmatrix}0&b&1\\0&1&0\\d&c&\alpha\end{vmatrix}+\begin{vmatrix}0&b&1\\0&0&0\\c&a&0\end{vmatrix}=-d$$

but

$$\mathrm{sh}^2_{678,123}(x)=\begin{vmatrix}b&0&0\\0&1&0\\d&c&\alpha\end{vmatrix}+\begin{vmatrix}b&0&0\\0&0&0\\c&a&0\end{vmatrix}+\begin{vmatrix}0&b&1\\1&0&0\\d&c&\alpha\end{vmatrix}+\begin{vmatrix}0&b&1\\0&1&0\\c&a&0\end{vmatrix}+\begin{vmatrix}0&b&1\\0&0&0\\a&0&0\end{vmatrix}=b\alpha-b\alpha+c-c=0.$$

Therefore shuffles at I of weight >1 vanish at x. ◇

We conclude this section with the observation that specific shuffle equations are not U^{aff}-stable. (It is straightforward that partial-rank shuffles are not GL-stable.)

Example 4.3.11. If $n=2$, $w=s_1s_0s_1s_0$, $m=8$, and $h=4$, then

$$[w]=\begin{pmatrix} 1 & & & \\ 0 & & & \\ & 1 & & \\ & 0 & & \\ & & 1 & \\ & & 0 & \\ & & & 1 \\ & & & 0 \end{pmatrix}\in M_{8,4}$$

with pivots $(1,3,5,7)$. Then $\mathrm{sh}^0_{57,12}=\Delta^{12}_{57}(x)=0$ but if we take $u=\theta^{\mathrm{aff}}_{1,4}(1)$ then we get

$$\Delta^{12}_{57}(ux)=\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}=1.$$

It is similarly easy to demonstrate this phenomenon for shuffles of positive weight. ◇

It turns out that the only full-rank shuffles that *don't* vanish at Y_w are minors.

Lemma 4.3.12. *If* $|I|=h$ *then* $k_w(I)\le 0$.

Proof. Our proof requires both full-rank and u-stability: Every pivot is the image of an index in I, including those in the last window $[m-n+1,m]$, so the formula (4.6), which reduces the slide weight to information about last pivot in each congruence class, vanishes.

Explicitly, set $P=\mathrm{piv}(w)\in[m]$ and pick any abacus slide $A\in\mathcal{A}$ of maximum slide weight (Lemma 4.3.7 (b)). Adopt the partitions $I^{(s)}$ and $P^{(s)}$ of Definition 4.3.1. By full-rank every $p^{(s)}_{\ell_s}=\max(P^{(s)})$, and by u-stability each $\max(P^{(s)})\in[m-n+1,m]$. So $p^{(s)}_{\ell_s}\le i^{(s)}_{\ell_s}\le m<p^{(s)}_{\ell_s}+n$ by Definition 4.3.1 (i). Thus each term $\left\lfloor\frac{i^{(s)}_{\ell_s}-p^{(s)}_{\ell_s}}{n}\right\rfloor<1$ in (4.6), which leaves A having slide weight at most zero (if any terms are negative then $k_w(I,A)=-1$ by definition). By Corollary 4.3.5, then, $k_w(I)\le 0$. □

Remark 4.3.13. It comes as little surprise that the Grassmannian shuffles map via (1.11) only to matrix shuffles of weight 1; that is, that no higher-weight finite (Plücker) shuffles are necessary to define the vanishing set X_w. In particular, for the calculations summarized in Appendix C.2, we generate our test ideal $\mathfrak{i}_w$ with only the shuffles having weight $k_w(I)+1$. The higher-weight finite shuffles

$$\sum_{R\in\mathbb{N}^h_k}p_{I-nR}\in S$$

must expand into weight-1 shuffles, and an interesting problem is to develop a standard monomial theory to show how. ◇

The infrastructure is now in place to define our candidate ideal.

Definition 4.3.14. Let $\mathfrak{i}_P = \mathfrak{i}_P^{h,m} \subset \mathbb{C}[M]$ be the ideal generated by the shuffles $\mathrm{sh}_{I,J}^k$ where $I \subseteq [i-n+1, i]$ for some i and $k > \max_{A\in\mathcal{A}} k_P(I, A)$. Write $\mathfrak{i}_w = \mathfrak{i}_{\mathrm{piv}(w)}$. ◇

The condition that I is contained in a window of length n (hence contains indices that are pairwise distinct modulo n) accommodates the shuffles we will construct for Theorem 4.4.5, which exclude matrices outside the P-augmentable variety $\hat{Z}_P$. In view of Lemma 4.3.12, it should at least not seem extortionate. The absence of vanishing minors *per se* from the generating set is made less mysterious by Example 4.4.7.

Lemma 4.3.15. *Given pivots $P \subset [m]$, write $d_j = \left\lfloor \frac{m-\hat{p}_j}{n} \right\rfloor$ across $1 \leq j \leq |\hat{P}|$. Then $\mathfrak{i}_P$ is generated by $\mathrm{sh}_{I,J}^k$ across $I \in [i-n+1, i]$ for $i \in [n, m]$, all J, and $k_P(I) < k \leq \sum_{j=1}^{\ell} d_j$. In particular, $\mathfrak{i}_P$ is finitely generated.*

Proof. If $k > \sum_{j=1}^{\ell} d_j$ then for every $R \in \mathbb{N}_k^{\ell}$ there is an index j at which $i_j + r_j n > m$, so that every minor summand of $\mathrm{sh}_{I,J}^k$ is zero by definition. □

Example 4.3.16. For the specific instance $\ell = 2$ of Example 4.3.9, $i_{s_1 s_0}^{2,4}$ is generated by $\mathrm{sh}_{12,12}^0 = \Delta_{12}^{12}$ and $\mathrm{sh}_{34,12}^1 = \Delta_{14}^{12} - \Delta_{23}^{12}$; the other generators implied by Lemma 4.3.15 are zero. See Appendix C for the cases $\ell = 3$ and $\ell = 4$. ◇

Theorem 4.3.17. *$\hat{Y}_w$ is the vanishing set of $\mathfrak{i}_w$.*

Proof. We invoke the containments

$$Y_w \overset{\mathrm{A}}{\subseteq} V(\mathfrak{i}_w) \overset{\mathrm{B}}{\subseteq} Z_w \overset{\mathrm{C}}{\subseteq} Y_w.$$

The proof of inclusion B comprises Section 4.4; we proved inclusion C at Theorem 4.1.10; and earlier discussions have prepared us to address inclusion A, wherewith we conclude this section. □

Unfortunately, only two of the three containments is proved for the more general heralded varieties and the ideals defined for them. The reason, which was spotlighted in Example 3.3.13, becomes clear in the proof of Theorem 4.4.5.

Corollary 4.3.18 (A)**.** *$\hat{Y}_P \subseteq V(\mathfrak{i}_P)$.*

Proof. By the construction of $\mathfrak{i}_P$, the definition of $k_P(I)$, Corollary 4.3.5, and Lemma 4.3.7 (b), we have $\pi^{-1}(\hat{\Omega}_P) \subseteq V(\mathfrak{i}_P)$. Since π is continuous, we get $\pi^{-1}(\hat{X}_P) \subset \overline{\pi^{-1}(\hat{\Omega}_P)}$. Thus

$$\hat{Y}_P = \overline{\pi^{-1}(\hat{X}_P)} = \overline{\pi^{-1}(\hat{\Omega}_P)} \subseteq V(\mathfrak{i}_P).$$

□

4.4 Set-theoretic equations

In the previous section we introduced a candidate ideal for Y_w by way of identifying a natural collection of generators. Here we leverage the machinery of heralded linear algebra and shuffle–slide combinatorics to prove that $V(\mathfrak{i}_w) \subseteq Z_w$, which tops off Theorem 4.3.17. First up are some further results (Lemmata 4.4.1 and 4.4.2) that tie together shuffle thresholds and slide weights, in the spirit of Lemma 4.3.15. Then come some supplementary combinatorics of pivot sets (Definition 4.4.3 and Lemma 4.4.4) that facilitate the application of these and earlier general results to the case of a w-inaugmentable matrix. Finally, given such a matrix $x \notin Z_w$, Theorem 4.4.5 produces a shuffle that fails to vanish at x, and Lemma 4.4.6 verifies that this shuffle lies in $\mathfrak{i}_w$.

In this section, since we will be dealing with partial shuffles and hence column subsets, we adopt the indices ν, ξ, ζ for, e.g., $I = (i_\nu)_{\nu=1}^{\ell}$, as needed to avoid confusion with J.

Lemma 4.4.1. *Pick P and $I \subset [m-n+1, m]$. Suppose that there exists an abacus slide that contains $\hat{P}$ in its image. Then there exists an abacus slide of maximum slide weight that contains $\hat{P}$ in its image.*

Proof. Write $\hat{P} = (\hat{p}_1 < \cdots < \hat{p}_y)$ and begin with any abacus slide $A : I \to P$ of maximum slide weight. From A we will obtain a map $A' : I \to P$ satisfying (ii) and (iii) of Definition 4.3.1, at which (4.6) evaluates to at least $k_P(I, A)$, and such that the collection of $i_{\ell_s}^{(s)} \in I$ in the notation of Definition 4.3.1 comprise $(i_{\ell-y+1} < \cdots < i_\ell)$. These must contain the $i_{\ell_s}^{(s)}$ of our supposed abacus slide A^* containing $\hat{P}$ in its image. Using this fact, we will be able to reassign the $i_{\ell_s}^{(s)}$ of A' in order to satisfy (i) while preserving (ii) and (iii). We will then observe that $\hat{P} \subseteq \operatorname{img}(A'')$. Let $d_j = \left\lfloor \frac{m-\hat{p}_j}{n} \right\rfloor$ across $1 \leq j \leq y$ and write $d^{(s)} = d_j$ when $s \equiv \hat{p}_j$.

The obtenance of A' from A is iterative. Each step begins with A satisfying (i) and (iii). If, under A, the $i_{\ell_s}^{(s)}$ comprise the largest y members of I then the obtenance is done. Otherwise either some $i_{\ell_s}^{(s)} < i_{\ell_{s'}-1}^{(s')}$ or some $I^{(s)} = \varnothing$ with $s \equiv \hat{p} \in \hat{P}$.

- If $i := i_{\ell_s}^{(s)} < i' := i_{\ell_{s'}-1}^{(s')}$ then obtain A' as

$$\begin{aligned} A'i &= Ai' \\ A'i' &= Ai \\ A'i'' &= Ai'' \quad \text{for } i'' \notin \{i, i'\}. \end{aligned}$$

  If this results in any violations of (ii) then reorder the images of $I^{(s)}$ and of $I^{(s')}$ accordingly. Since $\operatorname{img}(A') = \operatorname{img}(A)$, A' satisfies (iii). This step does not change the summand

$$\left\lfloor \frac{\max(I^{(s')}) - p^{(s')}_{\ell_{s'}}}{n} \right\rfloor$$

indexed by s' since $i' > i$ by assumption and does not change the summand indexed by s in any case.

- If $I^{(s)} = \varnothing$ with $s \equiv \hat{p} \in \hat{P}$ then find s' with $\ell_{s'} \geq 2$, write $i = i^{(s')}_{\ell_{s'}}$, and obtain A' as

$$\begin{aligned} A'i &= \hat{p} \\ A'i' &= Ai' \quad \text{for } i' \neq i. \end{aligned}$$

This preserves (ii). Since $Ai = \max(AI^{(s')})$ and $A'I^{(s)} = \{A'i\}$, A' satisfies (iii). This step introduces a summand in (4.6) indexed by s, which can be at least -1 since $\hat{p} \leq m < \min(I) + n$. The existing summand indexed by s' is unchanged if $i^{(s')}_{\ell_{s'}-1} < \hat{p} + \lfloor \frac{m-\hat{p}}{n} \rfloor n \leq i^{(s')}_{\ell_{s'}}$ (notation for the original slide A) and increased by 1 otherwise. Since $\hat{p} > i^{(s')}_{\ell_{s'}} \Rightarrow \hat{p} + \lfloor \frac{m-\hat{p}}{n} \rfloor n \nleq i^{(s')}_{\ell_{s'}}$, this increase occurs only when the -1 appears to counter it.

Stop when the $i^{(s)}_{\ell_s}$ comprise $(i_{\ell-y+1} < \cdots < i_\ell)$. At the map $A' : I \to P$ we obtain, (4.6) evaluates to at least $k_P(I, A)$.

The map $A' : I \to P$ is not yet an abacus slide if some $A'i^{(s)}_{\ell_s} > i^{(s)}_{\ell_s}$; though note that $A'i^{(s)}_j \leq A'i^{(s)}_{\ell_s} - n < m - n + 1 \leq i^{(s)}$; whenever $j < \ell_s$. By supposition there exists an abacus slide A^* whose image contains $\hat{P}$; necessarily the $i^{(s)}_{\ell_s}$ under A^* constitute a subset of $(i_{\ell-y+1} < \cdots < i_\ell)$ that consists of the $i^{(s)}_{\ell_s}$ under A'. To obtain A'' from A', then, reassign the $i^{(s)}_{\ell_s}$ to the $p^{(s)}_{\ell_s}$ in such a way that each $p^{(s)}_{\ell_s} \leq i^{(s)}_{\ell_s}$. This achieves (i) of Definition 4.3.1, and preserves (ii) and (iii) since if $i < i^{(s)}_{\ell_s}$ then $i = i^{(s')}_{\ell_{s'}}$ for some s'. Each transposition

$$\begin{aligned} A''i^{(s)}_{\ell_s} &= A'i^{(s')}_{\ell_{s'}} \\ A''i^{(s')}_{\ell_{s'}} &= A'i^{(s)}_{\ell_s} \end{aligned}$$

with $A'i^{(s)}_{\ell_s} > i^{(s)}_{\ell_s}$ increases the summand indexed by s in (4.6) by 1 but only decreases the summand indexed by s' by 1 if $A'i^{(s')}_{\ell_{s'}} \leq i^{(s')}_{\ell_{s'}} < A''i^{(s')}_{\ell_{s'}}$. This leaves A'' having slide weight at least the evaluation of (4.6) at A', hence at least the slide weight of A, which was maximal to begin with.

Since $I^{(s)} \neq \varnothing$ under A'' whenever $s \equiv \hat{p}_j \in \hat{P}$, and since A'' satisfies (iii), we have $\hat{P} \subseteq \operatorname{img}(A)$. $\square$

Next up we have a stronger result about a more constrained case.

Lemma 4.4.2. *Pick P. Let $D = (d_1, \ldots, d_{n'})$ where $d_\nu = \left\lfloor \frac{m-\hat{p}_\nu}{n} \right\rfloor$, and take $I = \hat{P} + nD$. Then $k_P(I) = \max_{A\in\mathcal{A}'} k_P(I,A)$. Moreover, $\mathfrak{p}^D_{I,J} \mid_{\hat{\Omega}_P}$ is the unique nonvanishing minor summand of $\mathrm{sh}^k_{I,J} \mid_{\hat{\Omega}_P}$, and the unique slide $\hat{A}$ of maximum slide weight is given by $\hat{A}(\hat{p}_\nu + d_\nu n) = \hat{p}_\nu$ across $1 \le \nu \le n'$.*

Proof. Set $k = \max_{A\in\mathcal{A}'} k_P(I,A)$ and recall the partial permutation matrix $[P] \in \hat{Y}_P$. It will be enough to check that $\mathrm{sh}^k_{I,J}([P]) \neq 0$. Define the abacus slide $\hat{A} : I \to P$ by $\hat{A}(\hat{p}_\nu + d_\nu n) = \hat{p}_\nu$. By Lemma 4.3.7 (a), $k_P(I,\hat{A}) = \sum_s \left\lfloor \frac{i^{(s)}_1 - p^{(s)}_1}{n} \right\rfloor = \sum_\nu d_\nu$. By Lemma 4.3.3, $\mathrm{sh}^k_{I,J} \mid_{\hat{Y}_P} = \mathfrak{p}^D_{I,J} \mid_{\hat{Y}_P}$. Thus

$$\mathrm{sh}^k_{I,J}([P]) = \mathfrak{p}^k_{I,J}([P]) = \Delta_I(t^{d_\nu} e_{\hat{p}_\nu}) = \Delta_I(e_{\hat{p}_\nu + d_\nu n}) = 1.$$

For uniqueness, let $A \in \mathcal{A}$ such that $k_P(I,A)$ is maximum. Suppose first that $\mathrm{img}(A) \neq \hat{P}$. Fix an arbitrary heralding pivot $\hat{p}_\nu$ and consider $I^{(s)}$ from Definition 4.3.1, where $s \equiv \hat{p}_\nu$. To check that $k_P(I,A) < k_P(I)$, it will be enough to show that the summands of Definition 4.3.4 featuring indices $i \in I^{(s)}$ sum to at most d_ν, and in at least one instance to strictly less.

First suppose that $I^{(s)} = \varnothing$. (This may not ever be the case.) Then certainly $\sum_{i\in I^{(s)}} \left\lfloor \frac{\min(i,B'i)-Ai}{n} \right\rfloor = 0$ is no larger than d_ν. Next suppose that $I^{(s)} = (i^{(s)}_1)$. This obtains the single summand

$$\left\lfloor \frac{i^{(s)}_1 - Ai^{(s)}_1}{n} \right\rfloor = \left\lfloor \frac{i^{(s)}_1 - (\hat{p}_\nu + cn)}{n} \right\rfloor = \left\lfloor \frac{i^{(s)}_1 - \hat{p}_\nu}{n} \right\rfloor - c$$

for some $c \ge 0$. Since $i_\nu \equiv \hat{p}_\nu$ and $i_{\pi(\nu)} < i_\nu + n$, we have $\left\lfloor \frac{i_{\pi(\nu)}-\hat{p}_\nu}{n} \right\rfloor \le \left\lfloor \frac{i_\nu-\hat{p}_\nu}{n} \right\rfloor = d_\nu$.

Now suppose, without loss of generality, that $I^{(s)} = (i^{(s)}_1 < \ldots < i^{(s)}_{\ell_s})$ with $\ell_s \ge 2$, as must at least once be the case. By Definition 4.3.1 (ii), $Ai^{(s)}_1 < \cdots < Ai^{(s)}_{\ell_s}$. Note that ν may or may not lie in $[\ell_s]$. As in Lemma 4.3.7 we then get

$$\begin{aligned}
\sum_{i\in I^{(s)}} \left\lfloor \frac{\min(i,Bi)-Ai}{n} \right\rfloor &= \left\lfloor \frac{i^{(s)}_{\ell_s} - (\hat{p}_\nu + c_{\ell_s} n)}{n} \right\rfloor + \left\lfloor \frac{\min(i^{(s)}_{\ell_s-1}, Bi^{(s)}_{\ell_s-1}) - (\hat{p}_\nu + c_{\ell_s-1} n)}{n} \right\rfloor + \\
&\quad \cdots + \left\lfloor \frac{\min(i^{(s)}_1, Bi^{(s)}_1) - (\hat{p}_\nu + c_1 n)}{n} \right\rfloor \\
&\leq \left\lfloor \frac{i^{(s)}_{\ell_s} - (\hat{p}_\nu + c_{\ell_s} n)}{n} \right\rfloor + \left\lfloor \frac{(\hat{p}_\nu + (c_{\ell_s}-1)n) - (\hat{p}_\nu + c_{\ell_s-1} n)}{n} \right\rfloor + \\
&\quad \cdots + \left\lfloor \frac{(\hat{p}_\nu + (c_2-1)n) - (\hat{p}_\nu + c_1 n)}{n} \right\rfloor \\
&= \left\lfloor \frac{i^{(s)}_{\ell_s} - \hat{p}_\nu}{n} \right\rfloor - c_{\ell_s} + (c_{\ell_s} - c_{\ell_s-1} - 1) + \cdots + (c_2 - c_1 - 1) \\
&= \left\lfloor \frac{i^{(s)}_{\ell_s} - \hat{p}_\nu}{n} \right\rfloor - c_1 - (\ell_s - 1),
\end{aligned}$$

where $\left\lfloor \frac{i^{(s)}_{\ell_s} - \hat{p}_\nu}{n} \right\rfloor \leq d_\nu$ as above and $c_1 \geq 0$, hence $\sum_{i\in I^{(s)}} \left\lfloor \frac{\min(i,Bi)-Ai}{n} \right\rfloor \leq d_\nu - 0 - (2-1) < d_\nu$. We conclude that $\operatorname{img}(A) = \hat{P}$ when $k_P(I,A)$ is maximum.

Now suppose that $k_P(I,A)$ is maximum and $\operatorname{img}(A) = \hat{P}$, but that some $A(\hat{p}_\nu + d_\nu n) \neq \hat{p}_\nu$. Let $\pi \in S_\ell$ be the permutation given by $Ai_{\pi(\nu)} = \hat{p}_\nu$. Without loss of generality, suppose that $(1\,2\,\cdots\,\ell)$ is a nontrivial cycle of π (of which at least one exists by assumption). The summands of Definition 4.3.4 associated with $\hat{p}_1, \ldots \hat{p}_\ell$ are then given, in the case of A, by $\left\lfloor \frac{i_{\pi(\nu)} - \hat{p}_\nu}{n} \right\rfloor$. As above, this provides that $\left\lfloor \frac{i_{\pi(\nu)} - \hat{p}_\nu}{n} \right\rfloor \leq d_\nu$. Moreover, at least some $i_{\pi(\nu)} < i_\nu$, hence $\left\lfloor \frac{i_{\pi(\nu)} - \hat{p}_\nu}{n} \right\rfloor < d_\nu$. Since the cycle was arbitrarily chosen, this implies that $k_P(I,A) < k_P(I,\hat{A})$. □

Given a w-inaugmentable matrix x, in order to employ the preceding results to produce a shuffle that fails to vanish at x we shall need to strategically situate the window $[i-n+1, i]$ from Definition 4.3.14. The following definition and lemma are motivated by this need.

Definition 4.4.3. Say that an index set P is *u-stable up to index i* if whenever $p \in P$ with $p + n \leq i$ we have $p + n \in P$, so that P is u-stable if P is u-stable up to m. If ν is an index such that $p_\nu > q_\nu$ but $p_\xi \leq q_\xi$ whenever $\xi < \nu$, say that *P breaches Q at ν*. ◇

Lemma 4.4.4. *Let $P, Q \subseteq [m]$ and suppose that P breaches Q at ν while both P and Q are u-stable up to q_ν. Then fewer heralding pivots of P than of Q are less than or equal to q_ν.*

Proof. For $R \in \{P, Q\}$, recall that $\hat{R}$ denotes the heralding pivots of R and write $R_\nu = R \cap [q_\nu]$, $\hat{R}_\nu = \hat{R} \cap R_\nu$, and $R' = R \cap [q_\nu - n + 1, q_\nu]$. Note that $|P_\nu| + 1 = |Q_\nu| = \nu$, and that $|\hat{P}_\nu| = |P'|$ and $|\hat{Q}_\nu| = |Q'|$ by u-stability up to q_ν.

Denote the injective map $F : p_\xi \to q_\xi$ from P_ν to Q_ν. F is nondecreasing since P only breaches Q at ν, so $F(P') \subseteq Q'$, but $p_\nu \notin P'$ while $q_\nu \in Q'$, so $F(P') \subsetneq Q'$ (a proper subset). Thus $|\hat{P}_\nu| = |P'| = |F(P')| < |Q'| = |\hat{Q}_\nu|$. $\square$

We are now prepared to prove the final inclusion of Theorem 4.3.17.

Theorem 4.4.5 (B)**.** $V(\mathfrak{i}_w) \subseteq Z_w$.

Proof. Pick $x \in M \setminus Z_w$. The proof involves identifying a shuffle $\mathrm{sh}^k_{I,J} \in \mathfrak{i}_w$ that fails to vanish at x. We first construct the shuffle, choosing I, J, and k to leverage the fact that no subspace containing the u-stable closure of $\mathrm{Span}(x)$ lies in the closure of Ω_w. This allows us to show that $\mathrm{sh}^k_{I,J}(x) \neq 0$. We leave the verification that $\mathrm{sh}^k_{I,J} \in \mathfrak{i}_w$ to Lemma 4.4.6.

Set $P = (p_1 < \cdots < p_h) = \mathrm{piv}(w)$, $V = \mathrm{Span}(x)$, and $Q = \mathrm{piv}(\mathbf{O}V)$. By choice of x, there exists no subspace $W \in X_w$ for which $\mathbf{O}V \subseteq W$. This means that $P \not\leq Q$; by Corollary 3.1.11, P breaches Q at some index ν. (It is here that we lose the ability to draw analogous conclusions for $\hat{Z}_P$ and $\mathfrak{i}_P$.) Define P_ν, Q_ν, $\hat{P}_\nu$, and $\hat{Q}_\nu$ as in the proof of Lemma 4.4.4, so that $\ell := |\hat{P}_\nu| < \ell' := |\hat{Q}_\nu|$. Set $d_\xi = \left\lfloor \frac{q_\nu - \hat{q}_\xi}{n} \right\rfloor$ across $1 \leq \xi \leq \ell'$ and then take $I = (\hat{q}_1 + d_1 n, \ldots, \hat{q}_{\ell'} + d_{\ell'} n)$.

We now restrict to quotients of subspaces by their intersections with $E_{q_\nu+1}$, i.e. to the row indices $[q_\nu]$. Note, however, that $k_{Q_\nu}(I) = k_Q(I)$.

By Theorem 4.2.8 we may identify ℓ' columns J of x and the heralded basis $v_1, \ldots, v_z$ of $\mathbf{O}V$, so that the heralding vectors $\hat{v}_\xi$ satisfy

$$x_{j_\xi} = C_\xi \hat{v}_\xi + \sum_{q_\zeta \neq \hat{q}_\xi} C_\xi^\zeta v_\zeta$$

with $C_\xi \neq 0$. Set $k = k_Q(I)$. By Corollary 4.2.10, then,

$$\mathrm{sh}^k_{I,J}(x) = C_\Xi \mathrm{sh}^k_I(\hat{v}_\xi) + \sum_{\substack{\text{not all} \\ q_\zeta = \hat{q}_\xi}} C_\Xi^{\mathrm{Z}} \mathrm{sh}^k_I(v_\zeta)$$

with $C_\Xi \neq 0$. Lemma 4.4.2 provides that $\mathrm{sh}^k_I(\hat{v}_\xi) = \mathfrak{b}^D_I(\hat{v}_\xi) \neq 0$, where $D = (d_1, \ldots, d_{\ell'})$, using Q_ν in place of P.

To check that the other $\mathrm{sh}^k_I(v_\zeta)$ vanish, let Q'_ν consist of the pivots of $\mathrm{Span}(v_\zeta)$ and $D' = (d'_1, \ldots, d'_{\ell''}$ where $\ell'' = |Q'_\nu|$. Since $Q'_\nu \subset Q_\nu$, any slide $A' \in \mathcal{A}'(I, Q'_\nu)$ yields a slide $A \in \mathcal{A}(I, Q_\nu)$ of the same slide weight that cannot be $\hat{A}$ from Lemma 4.4.2, hence has slide weight

$k_{Q'_\nu}(I, A') = k_{Q_\nu}(I, A) < k_{Q_\nu}(I)$ by the uniqueness of $\hat{A}$. Combined with Corollary 4.3.5, we get $k_{Q'_\nu}(I) \leq \max_{A \in \mathcal{A}'(I, Q'_\nu)} k_{Q'_\nu}(I, A) < k_{Q_\nu}(I)$.

It remains to check that $\mathrm{sh}^k_{I,J} \in \mathfrak{i}_w$, which is done by Lemma 4.4.6, using Lemma 4.4.1 as a foundation. □

For an integer p, write "$p\%n$" for the member of $[0, n-1]$ congruent to p modulo n. It is easy to show, for integers p and q, that

$$(p+q)\%n \leq p\%n + q\%n \leq n + (p+q)\%n, \tag{4.9}$$

with equality on the left when the middle term is $< n$—in particular, if either $p\%n = 0$ or $q\%n = 0$—and on the right when the middle term is $\geq n$.

Lemma 4.4.6. *Take $x \in M$ and I, J, k as in the proof of Theorem 4.4.5. Then $\mathrm{sh}^k_{I,J} \in \mathfrak{i}_w$.*

Proof. Using Lemma 4.4.1 we may find an abacus slide $A : I \to P$ of maximum slide weight satisfying $\hat{P} \subseteq \mathrm{img}(A)$. Retrieve P' and Q' from the Proof of Lemma 4.4.4 and note that $I = Q'$ as sets.

For each $\xi \in [\ell]$, if $p_\zeta = \hat{p}_\xi + cn \in P'$ then $p_\zeta \leq q_\zeta < q_\nu$: By the definition of P', $p_\zeta \leq q_\nu$. Since $\zeta \geq \nu$ would imply that $p_\zeta > p_\nu > q_\nu$ by the location of the breach, this must mean that $\zeta < \nu$. Thus $p_\zeta \leq q_\zeta$, again by the location of the breach, and $q_\zeta < q_\nu$ by the ordering on the members of Q. Picking up from the proof of Lemma 4.4.1, we may assume that A satisfies $\hat{p} = p_\xi = Aq_\xi$ when $\hat{p} \in P'$, i.e. that A sends the largest members of I to the largest members of $\hat{P}$ *in order*.

Using the formula (4.6), it follows that $\sum_s \frac{p^{(s)}_{\ell_s} - p^{(s)}_1}{n} = \ell' - \ell$, and consequently

$$\begin{aligned} k_P(I) &= \sum_s \left\lfloor \frac{\max(I^{(s)}) - p^{(s)}_{\ell_s}}{n} \right\rfloor \\ &= -\ell' + \ell + \sum_{\xi \in \mathrm{img}\, A} \left\lfloor \frac{i_{A^{-1}\xi} - \hat{p}_\xi}{n} \right\rfloor \\ &= -\ell' + \ell + \sum_{\xi \in \mathrm{img}\, A} \left\lfloor \frac{\hat{q}_{A^{-1}\xi} + d_{A^{-1}\xi} n - \hat{p}_\xi}{n} \right\rfloor \\ &= -\ell' + \ell + \sum_{\xi \in \mathrm{img}\, A} \left(\left\lfloor \frac{\hat{q}_{A^{-1}\xi} - \hat{p}_\xi}{n} \right\rfloor + d_{A^{-1}\xi} \right). \end{aligned}$$

Since $k \geq \sum_\xi d_\xi$, it will suffice to show that $\sum_\xi d_\xi > k_P(I)$, i.e. that

$$\ell' - \ell + \sum_{\xi \notin \mathrm{img}\, A} d_\xi > \sum_{\xi \in \mathrm{img}\, A} \left\lfloor \frac{\hat{q}_{A^{-1}\xi} - \hat{p}_\xi}{n} \right\rfloor. \tag{4.10}$$

Consider that the numbers of pivots in P and in Q up to *but not including* q_ν are equal. Since both pivot sets are u-stable up to q_ν, each number can be expressed as

$$\sum_{\xi=1}^{\ell'}\left(1+\left\lfloor\frac{q_\nu-1-\hat{q}_\xi}{n}\right\rfloor\right)=\sum_{\xi=1}^{\ell}\left(1+\left\lfloor\frac{q_\nu-1-\hat{p}_\xi}{n}\right\rfloor\right),$$

the sum of the numbers of shifts of each heralding pivot not more than $q_\nu - 1$. In light of the previous paragraph, this is equivalent to

$$\begin{aligned}\ell'-\ell+\sum_{\xi\notin\operatorname{img}A}\left\lfloor\frac{q_\nu-1-\hat{q}_\xi}{n}\right\rfloor &= \sum_{\xi\in\operatorname{img}A}\left(\left\lfloor\frac{q_\nu-1-\hat{p}_\xi}{n}\right\rfloor-\left\lfloor\frac{q_\nu-1-\hat{q}_\xi}{n}\right\rfloor\right)\\ &= \sum_{\xi\in\operatorname{img}A}\left(\left\lfloor\frac{q_\nu-\hat{p}_\xi}{n}\right\rfloor+\left\lfloor\frac{\hat{q}_\xi-q_\nu}{n}\right\rfloor+1\right),\end{aligned}$$

where $p_{\nu-1}\leq q_{\nu-1}<q_\nu<p_\nu$ implies that $q_\nu\notin P$, so we may drop -1 from the summands with $\hat{p}_\xi$. (The rest of the equality is arithmetic.) Using (4.9), the right side equals

$$\begin{aligned}&\sum_{\xi\in\operatorname{img}A}\frac{\hat{q}_{A^{-1}\xi}-\hat{p}_\xi}{n}+\ell-\frac{1}{n}\sum_{\xi\in\operatorname{img}A}((\hat{q}_{A^{-1}\xi}-q_\nu)\%n+(q_\nu-\hat{p}_\xi)\%n)\\ &=\sum_{\xi\in\operatorname{img}A}\left\lfloor\frac{\hat{q}_{A^{-1}\xi}-\hat{p}_\xi}{n}\right\rfloor+\frac{1}{n}\sum_{\xi\in\operatorname{img}A}(n+(\hat{q}_{A^{-1}\xi}-\hat{p}_\xi)\%n-(\hat{q}_{A^{-1}\xi}-q_\nu)\%n-(q_\nu-\hat{p}_\xi)\%n)\\ &\geq\sum_{\xi\in\operatorname{img}A}\left\lfloor\frac{\hat{q}_{A^{-1}\xi}-\hat{p}_\xi}{n}\right\rfloor.\end{aligned}$$

Now

$$\sum_{\xi\in\operatorname{img}A}\left\lfloor\frac{\hat{q}_{A^{-1}\xi}-\hat{p}_\xi}{n}\right\rfloor\leq\ell'-\ell+\sum_{\xi=\ell+1}^{\ell'}\left\lfloor\frac{q_\nu-\hat{q}_\xi-1}{n}\right\rfloor\leq\ell'-\ell+\sum_{\xi=\ell+1}^{\ell'}\left\lfloor\frac{q_\nu-\hat{q}_\xi}{n}\right\rfloor,$$

and, in fact, the overall inequality is strict: Identify $\zeta\in[m]$ so that $q_\nu\in\hat{q}_\zeta+n\mathbb{N}$. If $\zeta\in\operatorname{img}A$ then $(\hat{q}_\zeta-q_\nu)\%n=0$, so the summand

$$n+(\hat{q}_\zeta-\hat{p}_\xi)\%n-(q_\nu-\hat{p}_\xi)\%n-(\hat{q}_\zeta-q_\nu)\%n$$

is positive, which makes the left inequality strict. Otherwise $\left\lfloor\frac{q_\nu-\hat{q}_\zeta-1}{n}\right\rfloor<\left\lfloor\frac{q_\nu-\hat{q}_\zeta}{n}\right\rfloor$ makes the right inequality strict. This proves (4.10). $\square$

In addition to Theorem 4.3.17, this establishes that the augmentable variety $\hat{Z}_P$ gives an alternative characterization of the matrix heralded variety $\hat{Y}_P$.

Example 4.4.7. Let $n = 2$, $m = 4$, $h = 2$, and $w = s_0$, so that $[\mathrm{piv}(w)] = (\, e_2 \; e_4 \,) \in M_{4\times 2}$. Take $x = (\, e_1 \; e_3 \,)$. Using the notation of the proposition and its proof, we get $P = (2,4)$, $Q = (1,3)$, and $\nu = 1$ since $q_1 < p_1$. Then $Q_1 = (1) = \hat{Q}_1$ while $P_1 = \varnothing = \hat{P}_1$. We choose $I = (1 + \lfloor \frac{1-1}{2} \rfloor 2) = (1)$, which yields $k_Q(I) = k_{13}(1) = \sum_\xi d_\xi = 0$. Only the first column of x lies in $E_1 \setminus E_2$, so $J = (1)$. Our shuffle is $\mathrm{sh}^0_{(1),(1)} = z_{11}$. Indeed, $z_{11} \mid_{\pi^{-1}(\Omega^{2,4}_w)} \equiv 0$ but $z_{11}(x) = 1$. $\diamond$

Example 4.4.8. Retrieve $x \in M_{8\times 4}$ from Fig. 3.3 (where $n = 3$ and $k \neq -g - af + aj$) and take $w = s_1 s_0 s_2 s_1 s_0$. Then a matrix $y \in \pi^{-1}(\Omega^{5,8}_w)$ in heralded form looks like

$$y = \begin{pmatrix} 1 & & & & \\ a & & & & \\ & 1 & & & \\ & & 1 & & \\ b & \alpha & a & & \\ & & & 1 & \\ & & & & 1 \\ c & \beta & b & \alpha & a \end{pmatrix}.$$

Following the proof of the proposition, $P = (1,3,4,6,7)$ and $Q = (1,3,4,6,7,8)$, so that we must invoke $p_6 = 9$ in order for P to breach Q, at index $\nu = 6 = h+1$ with $q_6 = 8 < p_6$. Still, $\hat{P}_6 = (1,3)$ and $\hat{Q}_6 = (1,3,8)$ lie within $[m]$. We choose $I = (1,3,8) + (\lfloor \frac{8-3}{3} \rfloor, \lfloor \frac{8-8}{3} \rfloor, \lfloor \frac{8-1}{3} \rfloor)3 = (1,3,8) + (2,1,0)3 = (7,6,8)$ and $k_Q(I) = 2+1+0 = 3$. As obtained in Example 4.2.9, $J = (1,2,3)$. The resulting shuffle expands into minors and evaluates at x and y as follows, omitting J from the notation and recognizing that, in general, all column subsets of size ℓ from a matrix y in heralded form would need to be tested to confirm that the shuffle vanishes at $\pi^{-1}(\Omega^{h,m}_w)$:

$$\begin{array}{lcccccccccccc} \mathrm{sh}^3_{768} & = & \Delta_{138} & + & \Delta_{165} & + & \Delta_{435} & + & \Delta_{462} & + & \Delta_{732} & & \\ \mathrm{sh}^3_{768}(y) & = & b & + & 0 & - & b & + & 0 & + & 0 & = & 0 \\ \mathrm{sh}^3_{768}(y) & = & k & - & g & + & 0 & - & af & + & aj & \neq & 0 \end{array}$$

$\diamond$

While we have not shown that $\mathfrak{i}_w$ is radical, we suspect that the answer may be obtained via the identification of a Gröbner basis under a suitable term order, taking advantage of the "shuffle" symmetry of its generators. However, the generators alone do not necessarily provide a Gröbner basis, as Example 5.4.7 will illustrate.

Conjecture 4.4.9. $\mathfrak{i}_w$ *is radical.*

The computer calculations discussed in Appendix C.2 sometimes involve ideals too large to radicalize under our computational constraints. The exceptions support the conjecture. In particular, under $n = 2$ the ideals $\mathfrak{i}^{2,4}_{[-1,4]}$, $\mathfrak{i}^{3,6}_{[-2,5]}$, and $\mathfrak{i}^{4,8}_{[-3,6]}$ are radical; while under $n = 3$ we have verified radicality for the ideals $\mathfrak{i}^{h,m}_w$, for minimal and other choices of m and h, across the Bruhat closure of $w = [-4,4,6]$, $w = [-1,0,7]$, and $w = [-2,2,6]$.

Example 4.4.10. Take $n = 2$ and $w = s_0 s_1 s_0$. Then w has 2-core $\gamma = \ydiagram{3,2,1}$ and $\pi^{-1}(\Omega_w^{3,6}) \subset M_{6\times 3}$ has GL_3-orbit representatives of the form

$$\begin{pmatrix} 1 & & \\ a & & \\ & 1 & \\ b & a & \\ & & 1 \\ c & b & a \end{pmatrix}.$$

In Plücker coordinates we obtain the ideal

$$\mathfrak{i}(X_w^{3,6}) = (p_{123}, p_{124}, p_{125}, p_{126}, p_{134}, p_{234}, \mathrm{sh}_{356} = p_{156} - p_{345}, \mathrm{sh}_{456} = p_{256} - p_{346}) \subset S_{3,6}.$$

The induced map (1.11) on coordinate subspaces takes $\mathfrak{i}(X_w^{3,6})$ to

$$(\Delta_{123}^{123}, \Delta_{124}^{123}, \Delta_{125}^{123}, \Delta_{126}^{123}, \Delta_{134}^{123}, \Delta_{234}^{123}, \mathrm{sh}^1_{356,123}, \mathrm{sh}^1_{456,123}) \subset \mathbb{C}[M_{6\times 3}],$$

though clearly missing from these generators is the triplet of 2×2 minors in rows 1 and 2, from which the first four 3×3 minors may be recovered. Less immediately evident is that the highest-dimensional component of $V(\mathfrak{i})$ has in its radical ideal the triplet of partial-rank shuffles $\mathrm{sh}^2_{56,J}$ across $J \subset [3]$ of size 2—which is satisfied on $\pi^{-1}(\Omega_w^{3,6})$. In fact,

$$\mathfrak{i}_w^{3,6} = (\Delta_{12}, \Delta_{134}, \Delta_{234}, \mathrm{sh}^1_{356}, \mathrm{sh}^1_{456}, \mathrm{sh}^2_{56}) = \mathfrak{i}(Y_w^{3,6}),$$

where consistent with GL_3-stability and for ease of notation we assume J in each generator to range over the column index subsets of the same size as the associated row subset. $\diamond$

Remark 4.4.11. The map of coordinate rings (1.11) sends $\mathfrak{i}(X_w)$ into $\mathfrak{i}(Y_w)$ (it misses the partial-rank shuffles), which means that the radicality of $\mathfrak{i}_w$—or of a larger ideal having no additional full-rank generators—would imply the radicality of the conjectured ideal for X_w [KLMW07].[2] Conversely, the radicality of that ideal would lend epistemic support for the radicality of $\mathfrak{i}_w$. $\diamond$

[2] Thanks to Peter Magyar for bringing this to my attention.

Chapter 5

Schubert classes

The last chapter concerned the algebraic geometry of matrix affine Schubert varieties. This chapter concerns their connection to algebraic topology. In Section 5.1 we exhibit linear maps $H_T^*(M_{m\times h}) \to H_*(\mathrm{Gr}_{h,m})$ from the torus-equivariant cohomology of M to the singular homology of $\mathrm{Gr}_{h,m}$, which maps send the equivariant class of $Y_w^{h,m}$ to the singular class of $X_w^{h,m}$. Each such class therefore has a unique image in the Borel–Moore homology of Gr_∞^0, specifically the class of X_w under the embedding of $\mathrm{Gr}_{\mathrm{SL}_n}$. We conclude with an explicit, geometric procedure called the multidegree for calculating the (polynomial) equivariant class of a matrix variety. We use this procedure in Section 5.3 to show that the classes of $Y_w^{h,m}$ are compatible across different choices of m and h, in the sense that their (suitably defined) homological counterparts have a projective limit $f_w^{(n-1)}$ in the ring of symmetric functions Λ. We state the natural conjecture that $f_w^{(n-1)}$ equals, up to an involutive automorphism on Λ, the symmetric function representative of $[X_w]_\bullet \in H^{BM}(\mathrm{Gr}_{\mathrm{SL}_n})$ identified in [Lam08]. Section 5.4 contains a preliminary investigation into the Gröbner geometry of matrix affine Schubert varieties. A full Gröbner degeneration, under certain natural expectations and in several computed examples, takes Y_w to a union of coordinate subspaces of M whose single monomial generators behave like planar histories. In addition, we exhibit in several cases a partial Gröbner degeneration of Y_w into Schubert-like varieties whose classes are Stanley symmetric polynomials.

5.1 Equivariant cohomology

Equivariant cohomology is the natural cohomological setting for our T-stable matrix varieties. The theory is motivated by the need for a meaningful cohomology theory for the quotient bundle $X \to X/G$ of a space X with an action of a linear algebraic group G that is not necessarily free. For a thorough introduction to this topic, see [FA], from which most notational conventions here are borrowed or derived.

Definition 5.1.1. Let G be a topological group. Identify a contractible space EG with a free left action of G, and call EG a *mixing space for* G. (For our cases $G = \mathrm{GL}$ and $G = T < \mathrm{GL}$ the standard maximal torus, we will construct EG explicitly.) Call $BG \cong G\backslash EG$ the *classifying space for* G. Now take X to be a topological space with a right G-action, which we call a G-*space*. Take G to act (freely) on $X \times EG$ via $g(x,e) = (xg^{-1}, ge)$ and let

$$X_G = X \times^G EG \tag{5.1}$$

be the quotient by this action. Write $[x\!:\!e]$ for the equivalence class of (x,e). Then $H^*_G(X) = H^*(X_G)$ is the G-*equivariant cohomology ring of* X. ◇

Example 5.1.2. The multiplicative group $\mathbb{C}^*$ acts freely on $\mathbb{C}^{m+1} \smallsetminus \{\mathbf{0}\}$ by scaling:

$$z \cdot (p_1, \ldots, p_{m+1}) = (z^{-1}p_1, \ldots, z^{-1}p_{m+1}).$$

The quotient by this action yields the principal bundle $\mathbb{C}^{m+1} \smallsetminus \{\mathbf{0}\} \to \mathbb{P}^m$. The cohomology ring of $\mathbb{P}^m$ is generated by the equatorial class $[\mathbb{P}^{m-1}]^\bullet \cong [V(p_1)]^\bullet$, which intersects with itself inductively as $[\mathbb{P}^k]^\bullet \smile [\mathbb{P}^{m-1}]^\bullet = [\mathbb{P}^{k-1}]^\bullet$. This gives

$$H^*(\mathbb{P}^m) \cong \mathbb{Z}[x]/(x^{m+1}) \tag{5.2}$$

The inductive limit $\bigcup_{m=0}^\infty \mathbb{C}^{m+1} \smallsetminus \{\mathbf{0}\} = \mathbb{C}^\infty \smallsetminus \{\mathbf{0}\}$ is contractible, so we may pick $E\mathbb{C}^* = \mathbb{C}^\infty \smallsetminus \{\mathbf{0}\}$ as a mixing space for $\mathbb{C}^*$. This yields the bundle

$$\begin{array}{rcl} E\mathbb{C}^* & = & \mathbb{C}^\infty \smallsetminus \{\mathbf{0}\} \\ \downarrow & & \\ B\mathbb{C}^* & = & \mathbb{P}^\infty \end{array}$$

over the classifying space $\mathbb{P}^\infty := \bigcup_{m=0}^\infty \mathbb{P}^m$. Then

$$H^*_{\mathbb{C}^*}(\mathrm{pt}) = H^*(\mathbb{P}^\infty) = \lim_{\infty \leftarrow m} H^*(\mathbb{P}^m) \cong \lim_{\infty \leftarrow m} \mathbb{Z}[x]/\langle x^{m+1}\rangle = \mathbb{Z}[x]. \tag{5.3}$$

The algebraic torus $T \cong (\mathbb{C}^*)^h$ acts similarly on $(\mathbb{C}^{m+1} \smallsetminus \{\mathbf{0}\})^h$ as a bundle over $(\mathbb{P}^m)^h$. In the limit we obtain

$$\begin{array}{rcl} ET & = & (\mathbb{C}^\infty \smallsetminus \{\mathbf{0}\})^h \\ \downarrow & & \\ BT & = & (\mathbb{P}^\infty)^h \end{array}$$

and $H^*_T(\mathrm{pt}) \cong \mathbb{Z}[x_1, \ldots, x_h]$. ◇

Example 5.1.3. Take $G = \mathrm{GL}$. Consider the principal GL-bundle $M^\circ_{h\times m} \to \mathrm{Gr}_{h,m}$, which is not contractible. In the limit $m \to \infty$ we obtain the mixing space $E\mathrm{GL} = M^\circ_{h,\infty}$ and the classifying space $B\mathrm{GL} = \mathrm{GL}\backslash M^\circ_{h,\infty} =: \mathrm{Gr}_{h,\infty}$. We then have

$$H^*_{\mathrm{GL}}(\mathrm{pt}) = H^*(\mathrm{Gr}_{h,\infty}) = \lim_{\infty \leftarrow m} H^*(\mathrm{Gr}_{h,m}) \cong \lim_{\infty \leftarrow m} (\mathbb{Z}[x_1, \ldots, x_h]^{S_h}/\mathfrak{i}_{h,m}) = \mathbb{Z}[x_1, \ldots, x_h]^{S_h},$$

where $\mathfrak{i}_{h,m} = (s_\lambda(x_1, \ldots, x_h) \mid \lambda \not\subseteq (m^h))$ is as in (A.6). ◇

Proposition 5.1.4 outlines conditions under which the cohomology ring of a fiber bundle may be factored into those of its fibers and its quotient. This delivers Lemma 5.1.8 above and Corollary 5.1.6, which confirms a motivating property of equivariant cohomology, namely that it perceives free actions as quotients.

Proposition 5.1.4 (Leray–Hirsch, [Hat02] Proposition 4D.1)**.** *Let $p : E \to B$ be a fiber bundle with fibers $i : F \hookrightarrow E$. Suppose each $H^j(F)$ is a free $\mathbb{Z}$-module finitely generated by classes in $i^*(H^*(E))$. Then there is a ring isomorphism*

$$H^*(B) \otimes_{\mathbb{Z}} H^*(F) \xrightarrow{\sim} H^*(E).$$

Corollary 5.1.5. *Suppose $E \to B$ and $E' \to B'$ are principal G-bundles with $H^i(E) = 0$ for $0 < i < k$. Then $H^i(E \times^G X) \cong H^i(E' \times^G X)$ for $i < k$.*

Proof. Take G to act diagonally on $E \times E'$. Then the square

$$\begin{array}{ccc} X \times (E \times E') & \to & X \times E' \\ \downarrow & & \downarrow \\ X \times^G (E \times E') & \to & X \times^G E' \end{array}$$

consists of G-bundles (downward) and E-bundles (rightward). By Proposition 5.1.4 and the restriction on i, we have

$$H^i(X \times^G E') \otimes_{\mathbb{Z}} \mathbb{Z} = \bigoplus_{p+q=i} H^p(X \times^G E') \otimes_{\mathbb{Z}} H^q(E) \cong H^i(X \times^G (E \times E')).$$

□

Proof of Lemma 5.1.8. For any desired homology group $H^i_G(X) = H^i(X \times^G EG)$, apply Corollary 5.1.5 to the square

$$\begin{array}{ccc} X \times EG & \to & X \times E'G \\ \downarrow & & \downarrow \\ X \times^G EG & \to & X \times^G E'G. \end{array}$$

□

Corollary 5.1.6. *If G acts freely on X then $H^*_G(X) \cong H^*(X/G)$.*

Proof. Apply Corollary 5.1.5 to the square

$$\begin{array}{ccc} X \times EG & \to & X \\ \downarrow & & \downarrow \\ X \times^G EG & \to & X/G. \end{array}$$

□

Example 5.1.7. Take $G = \mathrm{GL}$. The action on M° is free, so by Lemma 5.1.6 we have, for all i, the natural isomorphism $H^i_G(M^\circ) \xrightarrow{\sim} H^i(\mathrm{Gr})$. ◇

The finite-dimensional but not contractible spaces in Examples 5.1.2 and 5.1.3, whose limits produced the mixing spaces EG, suggest the following result, which is a special case of Corollary 5.1.5. It allows us to compute G-equivariant cohomology groups, of arbitrary degree, using sufficiently high-dimensional approximations to EG.

Lemma 5.1.8. *Let $E'G$ be a connected space with a free G-action satisfying $H^i(E'G) = 0$ for $0 < i < k$. Then there are natural isomorphisms*

$$H^i(X \times^G E'G) \cong H^i(X \times^G EG) \tag{5.4}$$

for any space X with a G-action and $0 < i < k$.

Definition 5.1.9. The space $E'G$ in Lemma 5.1.8 is called an *approximation space for EG*. ◇

Computing the G-equivariant cohomology of arbitrary degree thus reduces to finding a family E_mG of approximation spaces satisfying (5.4) for $i >$ some $k = k(m)$, where $k \to \infty$ as $m \to \infty$. Let us build upon the examples, identifying explicit approximation spaces and formulae for $k(m)$.

We use indices and asterisks in the superscript position to indicate cohomology, while the group occupies the subscript position. For equivariant classes, we put the group in the superscript position. For example, if $Z \subset X$ are G-stable with $\operatorname{codim} Z = d$ then $[Z]^G \in H^{2d}_G(X)$.

Example 5.1.10. Let $T \cong (\mathbb{C}^*)^h$ and retrieve $E_mT = (\mathbb{C}^{m+1} \setminus \{0\})^h$ from Example 5.1.2. We have $H^i(E_mT) = 0$ for $0 < i < 2m$, so we may take $k(m) = 2m$. Therefore $H^i_T(X) = H^i(X \times^G E_mT)$, provided $m > \lfloor \frac{i}{2} \rfloor$. ◇

Proposition 5.1.11. *If $V \subset \mathbb{C}^N$ is Zariski closed then $H^i(\mathbb{C}^N \setminus V) = 0$ for $0 < i \leq 2 \operatorname{codim} V - 2$.*

Example 5.1.12. Let $G = \mathrm{GL}$ and $E_m\mathrm{GL} = M^\circ_{h\times(m+1)}$.[1] Note that $M^\circ_{h\times(m+1)} = M_{h\times(m+1)} \setminus V(\mathfrak{d})$, where

$$\mathfrak{d} = (\Delta_I^{[1,h]} \mid I \subset [1, m+1], |I| = h) \subset \mathbb{C}[M].$$

The independent contiguous minors (where $I = [i, i+h-1]$ across $1 \leq i \leq m-h+1$) generate $\mathfrak{d}$, so $V(\mathfrak{d})$ has codimension $m - h + 1$. From the proposition we then have $H^i(E_mG) = 0$ for $0 < i \leq 2(m - h + 1) = k$. ◇

[1]This is transpose to the convention elsewhere in order that $E_m\mathrm{GL}$ is a left GL-space.

Example 5.1.13. The column action of GL on M is not free. (For example, the zero matrix is a fixed point.) However, given a mixing space $E\mathrm{GL}$, the space

$$M \times E\mathrm{GL} \cong (\, M^t \mid E\mathrm{GL}\,) = (\, M_{h\times m} \mid M^\circ_{h\times\infty}\,) \hookrightarrow M_{h\times\infty}$$

(where the superscript t denotes transpose) is contractible because M and $E\mathrm{GL}$ are contractible. We may then think of $M \times E\mathrm{GL}$ as a mixing space for GL, where $g \cdot (\, x^t \mid e\,) = (\,(g^{-1}x)^t \mid ge\,)$. Thus

$$H^*_{\mathrm{GL}}(M) = H^*(\mathrm{GL}\backslash(M \times E\mathrm{GL})) = H^*(M \times^{\mathrm{GL}} E\mathrm{GL}) = H^*_{\mathrm{GL}}(\mathrm{pt}) \cong \mathbb{Z}[x_1, \ldots, x_h]^{S_h}. \quad (5.5)$$

Write $M_{\mathrm{GL}} := M \times^{\mathrm{GL}} E\mathrm{GL}$. Similarly,

$$H^*_T(M) = H^*(M_T) = H^*_T(\mathrm{pt}) \cong \mathbb{Z}[x_1, \ldots, x_h]. \quad (5.6)$$

Note that (5.5) and (5.6) preserve degree in equivariant cohomology, though this may differ from the corresponding polynomial degree. $\diamond$

The generator classes may be obtained from the bundle structure as Chern classes, and this fact will prove useful in the proof of Lemma 5.2.6.

Example 5.1.14. Specialize Example 5.1.10 to $h = 1$ to obtain $E_m\mathbb{C}^* = \mathbb{C}^{m+1} \smallsetminus \{\mathbf{0}\}$ and $B_m\mathbb{C}^* = \mathbb{P}^m$. The top Chern class $c_1(\mathcal{O}(1)) \in H^1(\mathcal{O}(1))$ of the line bundle $\mathcal{O}(1) \to \mathbb{P}^m$ dual to the tautological line bundle is a projective subspace of codimension 1, and the isomorphism (5.2) can be realized as

$$H^*(\mathbb{P}^m) = \mathbb{Z}[c_1(\mathcal{O}(1))]/(c_1(\mathcal{O}(1))^{m+1}).$$

The tautological line bundle $\mathcal{O}(1) \to \mathbb{P}^m$ then has top Chern class $c_1(\mathcal{O}(1)) = x \in \mathbb{Z}[x]/(x^{m+1})$. In the inductive limit, this provides that $\mathcal{O}(1) \to \mathbb{P}^\infty$ has top Chern class $c_1(\mathcal{O}(1)) = x \in \mathbb{Z}[x] = H^*(\mathbb{P}^\infty)$.

Now take $\mathbb{C}^*$ to act on $\mathbb{C}$ by multiplication and write $\pi : \mathbb{C}_{\mathbb{C}^*} \to B\mathbb{C}^*$. Then the pullback bundle

$$\begin{array}{ccccc} \pi^*\mathcal{O}(1) & \to & \mathbb{C}_{\mathbb{C}^*} & = & \mathbb{C} \times^{\mathbb{C}^*} (\mathbb{C}^\infty \smallsetminus \{\mathbf{0}\}) \\ & & \downarrow & & \\ \mathcal{O}(1) & \to & B\mathbb{C}^* & = & \mathbb{P}^\infty \end{array}$$

has top Chern class $c_1(\pi^*\mathcal{O}(1)) = \pi^*c_1(\mathcal{O}(1)) = \pi^*x$ under the isomorphism (5.6), hence is a generator for $H^*(\mathbb{C}_{\mathbb{C}^*}) = H_{\mathbb{C}^*}(\mathbb{C})$. $\diamond$

Proposition 5.1.15. *In the setting of Example 5.1.14, $[V(z)]^T = c_1(\pi^*\mathcal{O}(1))$ in $H^*_T(\mathbb{C} \times^T ET) = H^*(\mathbb{C}_{\mathbb{C}^*})$, where z parametrizes $\mathbb{C}$ as in Example 5.1.2.*

This proposition can be partly understood in terms of the section

$$\begin{aligned} s : \mathbb{C}_{\mathbb{C}^*} &\to \pi^*\mathcal{O}(1), \\ (z,p) &\mapsto z \end{aligned}$$

which is transverse to the zero section and meets it precisely on $\{0\} \times^T ET$. Our use of multidegrees in the next section will be justified in part by the generalization of the proposition to T-modules of arbitrary rank.

Definition 5.1.16. Let V be a T-module. Call $v \in V$ a *weight vector*, and an algebraic homomorphism $\chi : T \to \mathbb{C}^*$ a *character of* T, if, for all $t \in T$, $t \cdot v = \chi(t)v$. For character χ, call

$$V_\chi = \{v \in V \mid \forall t,\ t \cdot v = \chi(t)v\}$$

the χ*-weight space of* V. ◇

Proposition 5.1.17 ([Hal03], B.8)**.** *If V is a T-module then*

$$V \cong \bigoplus_{\chi \in \mathrm{Hom}(T,\mathbb{C}^*)} V\chi.$$

That is, V has a basis of weight vectors.

Corollary 5.1.18. *Denote the weights of T*

$$\begin{aligned} \chi_j : T &\to \mathbb{C}^*. \\ t &\mapsto t_j \end{aligned}$$

(Not all weights are being considered.) Let V be a T-module with weight space decomposition $V = \bigoplus_{i=1}^k \mathbb{C}v_i$, where $t \cdot v_i = \chi_{j_i}(t)v_i$. Then the vector bundle $\pi : V_T = V \times^T ET \to BT = (\mathbb{P}^\infty)^h$ splits as

$$\begin{array}{ccc} V_T & \cong & \bigoplus_{i=1}^k L_{\chi_{j_i}} \\ \downarrow & & \\ BT & & \end{array}$$

where each $L_{\chi_j} \to BT$ is the tautological line bundle on the j^{th} copy of $\mathbb{P}^\infty$. Moreover, $c_1(\pi^* L_{\chi_{j_i}}) = [V(v_i)]^T \in H^*(V_T)$.

Proof. Proposition 5.1.17 implies that V splits into one-dimensional T-modules. The Chern class identification then follows from Proposition 5.1.15. □

Example 5.1.19. $M_T = M \times^T ET$ from Example 5.1.13 splits as $M_T = \bigoplus_{(i,j)\in[m]\times[h]} L_{\chi_j}$ into line bundles, whose Chern classes $c_1(L_{\chi_j}) = [V(z_{ij})]^T$ (i arbitrary) generate $H^*(M_T) = H_T^*(M)$. ◇

Our goal, the long map (5.12), requires some categorical (morphism) knowledge of equivariant cohomology. Consider an open embedding $f : Y \hookrightarrow X$. In the traditional setting, f induces a map $f^* : H^*(X) \to H^*(Y)$ on singular cohomology given by $f^*([Z]) = [f^{-1}(Z)]$, where each subvariety $Z \subset X$ determines a class $[Z]^\bullet \in H^{\operatorname{codim}_X(Z)}(X)$. For the open embedding $M^\circ \hookrightarrow M$, the composite map at our destination requires an analogous construction in equivariant cohomology, for which we make use of approximation spaces.

Definition 5.1.20. Let X and Y be G-spaces. Call a map $f : X \to Y$ *G-equivariant* if $f(x \cdot g) = f(x) \cdot g$ for all $x \in X$ and $g \in G$. ◇

Lemma 5.1.21. *Let $f : X \to Y$ be G-equivariant. Then f induces a degree-preserving ring homomorphism $f_G^* : H_G^*(Y) \to H_G^*(X)$ that evaluates as $f_G^*([Z]^G) = [f^{-1}(Z)]^G$.*

Proof. We may use approximation spaces to reduce the claim to singular cohomology. Let us adopt homogeneous coordinates $[x:e]$ on $X \times^G EG$ for X. If we use $X_{G,m} = X \times^G E_mG$ and $Y_{G,m} = Y \times^G E_mG$ then we get an induced map $f_{G,m} : X_{G,m} \to Y_{G,m}$ defined by $f_{G,m}([x:e]) = [f(x):e]$, which is well-defined by the G-equivariance of f.

Now take a subvariety $Z \subset X$ and set $d = \operatorname{codim}_X Z$. The inclusion $Z \times^G EG \hookrightarrow X_G$ induces, for any G-subvariety $Z \subset X$ of an algebraic space, a fundamental class

$$[Z \times^G EG]_\bullet = [Z]_G \in H_*^{BM}(X_G)$$

in the Borel–Moore homology of X_G. This provides (see Appendix B.3) the *equivariant cohomology class*

$$[Z \times^G EG]^\bullet = [Z]^G \in H_G^*(X).$$

The construction is similar using approximation spaces. The subvariety $Z \subset X$ has equivariant class $[Z]^G = [Z \times^G E_mG]^\bullet$, and $Z_{G,m} = Z \times^G E_mG \subset X_{G,m}$ also has codimension d in $X_{G,m}$. We may therefore find m sufficiently large for $d < k(m)$, whereby

$$\begin{array}{ccc} f_G^*([Z]^G) & & [(f^{-1}(Z)) \times^G E_mG]^\bullet \;=\; [f^{-1}(Z)]^G \\ \updownarrow & & \updownarrow \\ f_{G,m}^*([Z_{G,m}]^\bullet) & = & [{f_{G,m}}^{-1}(Z_{G,m})]^\bullet \end{array}$$

□

Example 5.1.22. The inclusion $i : M^\circ \to M$ induces a degree-preserving ring homomorphism $H_{\mathrm{GL}}^*(M) \to H_{\mathrm{GL}}^*(M^\circ)$ given by $[Z]^{\mathrm{GL}} \mapsto [i^{-1}(Z)]^{\mathrm{GL}}$. ◇

Suppose now that $H < G$ is a topological subgroup. Then the subgroup inclusion induces the G/H-bundle

$$X \times^H EG \to X \times^G EG,$$

which by the commonality of mixing spaces can be computed on approximation spaces. This bundle induces the injective map on equivariant cohomology

$$H_G^*(X) \to H_H^*(X). \tag{5.7}$$

In the case that $H = T$ is a split maximal torus in a connective reductive group G, Edidin and Graham [EG98] showed that (5.7) takes the form

$$H_G^*(X) \xrightarrow{\sim} H_T^*(X)^W, \tag{5.8}$$

where W is the Weyl group $N_G(T)/T$. (G is *reductive* if its unipotent radical is trivial.) This is a degree-shifting ring isomorphism; the maps on homology take $H_{i+g}^G(X) \xrightarrow{\sim} H_{i+t}^T(X)^W$, where $g = \dim G$ and $t = \dim T$, so on cohomology we have

$$H_G^{i-g}(X) = H_{\dim X - i + g}^G(X) \xrightarrow{\sim} H_{\dim X - i + t}^T(X)^W = H_T^{i-t}(X)^W. \tag{5.9}$$

In the case $G = \mathrm{GL}$, (5.8) specializes as the inclusion

$$H_{\mathrm{GL}}^*(M) \cong \mathbb{Z}[x_1, \dots, x_h]^{S_h} \hookrightarrow \mathbb{Z}[x_1, \dots, x_h] \cong H_T^*(M). \tag{5.10}$$

We can now state and prove the linear map connecting torus-equivariant classes of matrix affine Schubert varieties to Borel–Moore homology classes of affine Schubert varieties. Borel–Moore homology allows us to speak meaningfully about homology on noncompact spaces, which is of especial help in the context of the affine matrix space M, the Stiefel manifold M°, and the infinite-dimensional Grassmannians $\mathrm{Gr}_{\mathrm{SL}_n} \subset \mathrm{Gr}_\infty^0$. See Appendix B.3 for the definition of Borel–Moore homology.

Theorem 5.1.23. *There is an injective linear map*

$$\begin{aligned} H_T^*(M)^W &\to H_*^{BM}(\mathrm{Gr}_\infty^0). \\ [Y^{\lambda^\vee}]^T &\mapsto [X_\lambda]_\bullet \end{aligned} \tag{5.11}$$

Proof. It is known ([KM05]) that the full-rank matrix Schubert varieties provide a linear basis for $H_T^*(M)^W$, so the map is injective provided that it is defined. The component maps

$$H_T^*(M)^W \overset{(5.8)}{\cong} H_{\mathrm{GL}}^*(M) \overset{5.1.21}{\to} H_{\mathrm{GL}}^*(M^\circ) \overset{5.1.7}{\cong} H^*(\mathrm{Gr}) \overset{(A.5)}{\cong} H_*(\mathrm{Gr}) \overset{2.1.8}{\hookrightarrow} H_*^{BM}(\mathrm{Gr}_\infty^0) \tag{5.12}$$

evaluate these basis elements as follows:

$$[Y_{h,m}^{\lambda^\vee}]^T \mapsto [Y_{h,m}^{\lambda^\vee}]^{\mathrm{GL}} \mapsto [\pi^{-1}(X^{\lambda^\vee})]^{\mathrm{GL}} \leftrightarrow [X_{h,m}^{\lambda^\vee}]^\bullet \leftrightarrow [X_{h,m}^{\lambda^\vee}]_\bullet \mapsto [X_\lambda]_\bullet.$$

As every component is linear, so is their composition. □

This is a natural analog to the diagram in the proof of Corollary 2.3.1 in [KM05] and will later justify the analogous Corollary 5.3.13.

Corollary 5.1.24. *Take $N = |\lambda^\vee|$, hence $|\lambda| = h(m-h)-N$. Then $[Y^{\lambda^\vee}]^T \in H_T^{2(h^2-h+N)}(M)^W$ and $[X^\lambda]_\bullet \in H^{BM}_{2(h(m-h)-N)}(\mathrm{Gr}^0_\infty)$.*

Proof. Take $i = \frac{N}{2}$ and specialize (5.12) to fixed degree:

$$H_T^{2(h^2-h)+i}(M)^W \cong H^i_{\mathrm{GL}}(M) \to H^i_{\mathrm{GL}}(M^\circ) \cong H^i(\mathrm{Gr}) \cong H_{2h(m-h)-i}(\mathrm{Gr}) \hookrightarrow H^{BM}_{2h(m-h)-i}(\mathrm{Gr}^0_\infty). \tag{5.13}$$

□

5.2 Multidegrees

We now introduce the multidegree, a purely geometric calculation whose evaluation at $X \subset M$ may be identified with $[X]^T \in H^*_T(M)$. It requires a structure theorem for T-modules, which precedes the definition. The inductive definition itself is borrowed from [KZJ07]; for the full context and definition of multidegree (the top-degree part of a K-polynomial), see [MS05], Chapter 8.

Definition 5.2.1 ([KZJ07] Section 1.3)**.** The possible weights $\chi : T \to \mathbb{C}^*$ comprise the *weight lattice* $\mathfrak{h}^*_\mathbb{Z} \subset \mathfrak{h}^*$, where $\mathfrak{h} = \mathrm{Lie}(T)$ is the Cartan subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$. If we identify the characters $x_j := \chi_{E_{ij}} : T \to \mathbb{C}^*$ across the coordinate subspaces $\mathbb{C}E_{ij} \subset M$ then we may decompose the weight lattice as $\mathfrak{h}^*_\mathbb{Z} = \bigoplus_{j=1}^h \mathbb{Z}x_j$. Denote by $\mathrm{Sym}^\bullet(\mathfrak{h}^*_\mathbb{Z})$ the symmetric algebra $\mathbb{Z}[x_1, \ldots, x_h]$.

Let V be a finite-dimensional T-module and $Y \subset V$ a T-stable variety. Define the *multidegree* $\mathrm{mdeg}_V Y \in \mathrm{Sym}^\bullet(\mathfrak{h}^*_\mathbb{Z})$ *of Y in V* to be the element uniquely determined by the following inductive properties:

(i) If $Y = V = \{0\}$ then $\mathrm{mdeg}_V Y = 1$.

(ii) If $H \subset V$ is a T-stable hyperplane and $Y \subseteq H$ then $\mathrm{mdeg}_V Y = \chi_{V/H}\, \mathrm{mdeg}_H Y$.

(iii) If $H \subset V$ is a T-stable hyperplane and $Y \not\subseteq H$ then $\mathrm{mdeg}_V Y = \mathrm{mdeg}_H(Y \cap H)$.

(iv) If Y has top-dimensional components Y_i across $1 \le i \le k$ with multiplicities m_i then $\mathrm{mdeg}_V Y = \sum_{i=1}^k m_i\, \mathrm{mdeg}_V Y_i$.

Since the weights of the T-action on H are (all but at most one of) the weights on V, we may view any $\mathrm{mdeg}_H Z$ as a polynomial in the same ring. ◇

The method of the following example underlies our multidegree calculations later in the section and in Appendix C.2.

Example 5.2.2. Let $T \cong (\mathbb{C}^*)^n$ act on $V \cong \mathbb{C}^n$ coordinatewise, i.e. by $(t_1, \ldots, t_n) \cdot (v_1, \ldots, v_n) = (t_1 v_1, \ldots, t_n v_n)$. Then the standard basis is a basis of weight vectors; set $x_i = \chi_{e_i}$. Pick any subspace $W \subseteq V$. We may use the definitional properties to compute the multidegree of W in V by taking intersections and quotients of $W \subset V$ sequentially by the standard flag $F_n = V, \ldots, F_0 = \{0\}$, starting with $\operatorname{mdeg}_V W = \operatorname{mdeg}_{F_n}(F_n \cap W)$.

For the inductive step, note that F_i/F_{i-1} is a 1-dimensional T-module of weight x_i. If $F_i \cap W \subseteq F_{i-1}$ then $\operatorname{mdeg}_{F_i}(F_i \cap W) = \operatorname{mdeg}_{F_i}(F_{i-1} \cap W) = x_i \operatorname{mdeg}_{F_{i-1}}(F_{i-1} \cap W)$. If instead $F_i \cap W \nsubseteq F_{i-1}$ then $\operatorname{mdeg}_{F_i}(F_i \cap W) = \operatorname{mdeg}_{F_{i-1}}(F_{i-1} \cap W)$. The process terminates at $F_0 \cap W$, which has $\operatorname{mdeg}_{F_0}(F_0 \cap W) = 1$.

Altogether we get

$$\operatorname{mdeg}_V W = \prod_{F_i \cap W \subseteq F_{i-1}} x_i = \prod_{\dim(F_{i-1} \cap W) = \dim(F_i \cap W)} x_i = \prod_{i \notin \operatorname{piv}(W)} x_i,$$

which depends exactly on $k = \deg(\operatorname{mdeg}_V W) = \operatorname{codim}_V W$ and on what Schubert cell $\Omega_{\operatorname{piv}(W)} \subset \mathrm{Gr}_{n-k,n}$ contains W. ◇

Definition 5.2.3. Given a matrix space $M_{a \times b}$ and a subset of positions $D \subseteq [a] \times [b]$, define the *weight of* D as

$$x^D = \prod_{(i,j) \in D} x_j \in \mathbb{Z}[x_1, \ldots, x_n] \tag{5.14}$$

and the coordinate subspace $L_D = V(z_{ij} \mid (i,j) \in D) \subseteq M_{a \times b}$ associated with D. ◇

These diagrams will become our combinatorial objects of interest in the coming sections.

Example 5.2.4. Given $D \subseteq [m] \times [h]$, we can compute the multidegree of L_D as in Example 5.2.2, via any full flag of coordinate hyperplanes in M, to get $\operatorname{mdeg}_M L_D = x^D$. ◇

Proposition 5.2.5 (Self-intersection formula, [Ful98], following Corollary 6.3)**.** *Let $i : X \to Y$ be an embedding of smooth projective varieties of codimension d and normal bundle $N = N_X(Y)$. Pick any class $\alpha \in H^*(X)$. Then*

$$i^*(i_*(\alpha)) = \alpha \smile c_d(N) \in H^*(X),$$

where $i_* : H^*(X) \to H^*(Y)$ is the pushforward in Corollary A.2.8, $i^* : H^*(Y) \to H^*(X)$ is the usual pullback, and $c_d(N) \in H^*(X)$ is the top Chern class of N.

Lemma 5.2.6. *Take $M_{T,m} = M \times^T E_m T$ and for any T-stable variety $Y \subseteq M$ write $[Y]^{T,m}$ for the class of $Y \times^T E_m T$ in $H^*(M_{T,m})$. Then there is a surjective ring homomorphism*

$$\begin{aligned} \operatorname{Sym}^\bullet(\mathfrak{h}_{\mathbb{Z}}^*) &\to H^*(M_{T,m}) \\ \operatorname{mdeg}_M Y &\mapsto [Y]^{T,m} \end{aligned} \tag{5.15}$$

with kernel $({x_j}^{m+1} \mid 1 \leq j \leq h)$.

The result generalizes to any T-module V, but we state and prove it in terms of M for simplicity and to prompt the following corollary, which is the real goal.

Theorem 5.2.7. *The multidegree of Y in M returns the representative of $[Y]^{T,m}$ in $\mathbb{Z}[x_1,\dots,x_h]$ under (5.15).*

Proof of Lemma 5.2.6. The homomorphism is provided by Example 5.1.2, identifying $\mathrm{Sym}^\bullet(\mathfrak{h}_\mathbb{Z}^*)$ with $\mathbb{Z}[x_1,\dots,x_h]$. Any T-stable variety $Y \subseteq M$ has multidegree a sum of terms that divide $x^{(m^h)} = (x_1,\dots,x_h)^m$; and Corollary 5.1.18 establishes a correspondence of generators. It will therefore suffice to show that the definitional properties of the multidegree are satisfied in equivariant cohomology.

Properties (i) and (iv) correspond to definitional properties of cohomology, so are immediate. For (ii) we invoke the self-intersection formula: Take $Y = H$ and $X = H' = V(z_{ij})$ as a subvariety of H. The normal bundle $N = N_{H'}(H)$ is then a line bundle on H' of weight χ_j, so by Corollary 5.1.18 its top Chern class is $c_1(N) = x_j \in H_T^*(H)$. Now viewing $Y \subset H'$, so that $[i(Y)]^{T,m} = i_*[Y]^{T,m} \in H^*(H_{T,m})$, we get

$$i^*[i(Y)]^{T,m} = i^*(i_*[Y]^{T,m}) = [Y]^{T,m} \smile c_1(N) = x_j[Y]^{T,m} \in H_*(H_{T,m}),$$

as desired.

If $Y \nsubseteq H$ then since $\operatorname{codim}_M H = 1$ we must have $\operatorname{codim}_Y(Y \cap H) = 1$ (a transverse intersection). This provides that $i^*[Y]^{T,m} = [Y \cap H]^{T,m}$, which is the property (iii). $\square$

The original K-polynomial definition confers degeneracy on the multidegree. The calculation can therefore be performed (very conveniently) on the union of coordinate subspaces that arises from a Gröbner degeneration, rather than on the original variety.

Proposition 5.2.8 ([MS05] Corollary 8.47)**.** *The multidegree is preserved under flat deformations.*

The variety of the initial ideal of $\mathsf{i}(Y)$ is the result of such a (Gröbner) deformation.

Corollary 5.2.9. *For any T-stable variety $Y \subset V$, $\mathrm{mdeg}_V\, V(\mathsf{in}(\mathsf{i}(Y))) = \mathrm{mdeg}_V\, Y$.*

Example 5.2.10. The 3-Grassmannian finite permutation $\sigma = [1,3,4,2] = s_3 s_2$ has matrix presentation

$$\sigma = (\delta_{\sigma(j),j}) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

marking the descent as in Section 1.1. Obtain $\lambda = \square \subset \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ from σ using Corollary B.2.12, in the opposite setting. The Schubert cell $\Omega_{3,4}^\sigma \subset \mathrm{Gr}_{3,4}$ consists of subspaces spanned by column vectors of the form

$$\begin{pmatrix} 1 & & \\ * & 0 & 0 \\ & 1 & \\ & & 1 \end{pmatrix}.$$

The GL_3-orbits of these basis matrices have closure in $M_{4\times 3}$ the (opposite) matrix Schubert variety $Y^{\square}_{3,4} = \overline{\pi^{-1}(\Omega^{\sigma}_{3,4})} = \overline{U[P]\mathrm{GL}_3}$, where $P = (1,3,4)$. The matrix variety has ideal $\mathfrak{i}(Y^{\square}_{3,4}) = (\Delta^{12}_{12}, \Delta^{13}_{12}, \Delta^{23}_{12})$ generated by the 2×2 minors at the top two rows of $M_{4\times 3}$.

$M_{4\times 3}$ is a module under the right action of $T = (C^*)^3$ and decomposable into weight spaces as

$$V = \bigoplus_{i=1}^{4} \mathbb{C}E_{i1} \oplus \bigoplus_{i=1}^{4} \mathbb{C}E_{i2} \oplus \bigoplus_{i=1}^{4} \mathbb{C}E_{i3}$$

with basis of weight vectors E_{ij}. Denote $x_j = \chi_{E_{ij}}$. The ideal $\mathfrak{i}(Y^{\square}_{3,4}) = (\Delta^{12}_{12}, \Delta^{13}_{12}, \Delta^{23}_{12})$ then provides the multidegree

$$\begin{aligned}
[Y^{(1)}_{3,4}]^T &= \mathrm{mdeg}_{M_{4\times 3}} Y^{(1)}_{3,4} \\
&= \mathrm{mdeg}_{M_{2\times 3}} V(\Delta^{12}_{12}, \Delta^{13}_{12}, \Delta^{23}_{12}) && \text{(iii)} \\
&= \mathrm{mdeg}_{(z_{21})} V(z_{11}z_{22}, z_{11}z_{23}, \Delta^{23}_{12}) && \text{(iii)} \\
&= \mathrm{mdeg}_{(z_{21})} V(z_{11}, \Delta^{23}_{12}) + \mathrm{mdeg}_{(z_{21})} V(z_{22}, z_{23}, \Delta^{23}_{12}) && \text{(iv)} \\
&= \mathrm{mdeg}_{(z_{21},z_{22})} V(z_{11}, z_{12}z_{23}) + \mathrm{mdeg}_{(z_{21})} V(z_{22}, z_{23}) && \text{(iii)} \\
&= \mathrm{mdeg}_{(z_{21},z_{22})} V(z_{11}, z_{12}) + \mathrm{mdeg}_{(z_{21},z_{22})} V(z_{11}, z_{23}) \\
&\quad + x_2 \cdot \mathrm{mdeg}_{(z_{21},z_{22})} V(z_{23}) && \text{(iv),(ii)} \\
&= \mathrm{mdeg}_{(z_{21},z_{22},z_{23},z_{24})} V(z_{11}, z_{12}) + x_3 \cdot \mathrm{mdeg}_{(z_{21},z_{22},z_{23},z_{24})} V(z_{11}) \\
&\quad + x_2x_3 && \text{(iii),(ii),(i)} \\
&= x_1 \cdot \mathrm{mdeg}_{(z_{21},z_{22},z_{23},z_{24},z_{11})} V(z_{12}) + x_1x_3 + x_2x_3 && \text{(ii),(i)} \\
&= x_1x_2 + x_1x_3 + x_2x_3 && \text{(ii),(i)}
\end{aligned}$$

In each line following the second, we take the ambient space to be the coordinate space of $M_{2\times 3}$ determined by the ideal of the monomial subscripts of "mdeg". The arguments are then taken to be subspaces *of this coordinate subspace*, not of $M_{2\times 3}$ itself. The calculation identifies $[Y^{\square}]^T \in H^*_T(M_{4\times 3})$ with $s_{\boxminus}(x_1, x_2, x_3) = \mathfrak{S}_{s_3s_2} \in \mathbb{Z}[x_1, x_2, x_3]$. $\diamond$

Example 5.2.11. Take $n = 2$ and $w = s_1s_0 = [-1, 4]$, so that $\mathrm{piv}(w) = \{-1, 4\} + 2\mathbb{Z}_{>0}$ as sets. The Schubert variety $X^{\boxplus}_{2,5} \subset \mathrm{Gr}_{2,5}$ contains the affine Schubert variety $X^{2,5}_{[-1,4]}$, which yields the matrix affine Schubert variety $Y^{2,5}_{[-1,4]} \subset M_{5\times 2}$ obtained as the closure of the GL_2-orbits of matrices in heralded form having pivots $(2, 4)$:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ a & 0 \\ 0 & 1 \\ b & a \end{pmatrix}.$$

We know (see Section 5.4) that its ideal is generated by shuffle relations:

$$\mathfrak{i}^{2,5}_{[-1,4]} = (\Delta^1_1, \Delta^2_1, \Delta^{12}_{23}, \mathrm{sh}^1_{45,12}) = (z_{11}, z_{12}, z_{21}z_{32} - z_{22}z_{31}, z_{21}z_{52} - z_{22}z_{51} + z_{41}z_{32} - z_{42}z_{31}).$$

From this we can compute $[Y^{2,5}_{[-1,4]}]^T$ under the action of $T \cong (\mathbb{C}^*)^2$ directly using Definition 5.2.1. We slice by coordinate hyperplanes, starting at the bottom-left corner of $M_{5\times 2}$ and proceeding rightward, then upward:

$$\begin{aligned}
[Y^{2,5}_{[-1,4]}]^T &= \operatorname{mdeg} Y^{2,5}_{[-1,4]} \\
&= \operatorname{mdeg}_M V(z_{11}, z_{12}, z_{21}z_{32} - z_{22}z_{31}, z_{21}z_{52} - z_{22}z_{51} + z_{41}z_{32} - z_{42}z_{31}) \\
&= \operatorname{mdeg}_{z_{51}} V(z_{11}, z_{12}, z_{21}z_{32} - z_{22}z_{31}, z_{21}z_{52} + z_{41}z_{32} - z_{42}z_{31}) \\
&= \operatorname{mdeg}_{z_{51},z_{52}} V(z_{11}, z_{12}, z_{21}z_{32} - z_{22}z_{31}, z_{41}z_{32} - z_{42}z_{31}) \\
&= \operatorname{mdeg}_{z_{51},z_{52},z_{41}} V(z_{11}, z_{12}, z_{21}z_{32} - z_{22}z_{31}, z_{42}z_{31}) \\
&= \operatorname{mdeg}_{z_{51},z_{52},z_{41}} (V(z_{11}, z_{12}, z_{21}z_{32} - z_{22}z_{31}, z_{42}) \cup V(z_{11}, z_{12}, z_{21}z_{32} - z_{22}z_{31}, z_{31})) \\
&= x_2 \operatorname{mdeg}_{z_{51},z_{52},z_{41},z_{42}} V(z_{11}, z_{12}, z_{21}z_{32} - z_{22}z_{31}) \\
&\quad + \operatorname{mdeg}_{z_{51},z_{52},z_{41},z_{42}} V(z_{11}, z_{12}, z_{21}z_{32} - z_{22}z_{31}, z_{31}) \\
&= x_2 \operatorname{mdeg}_{z_{51},z_{52},z_{41},z_{42},z_{31}} V(z_{11}, z_{12}, z_{21}z_{32}) \\
&\quad + x_1 \operatorname{mdeg}_{z_{51},z_{52},z_{41},z_{42},z_{31}} V(z_{11}, z_{12}, z_{21}z_{32}) \\
&= x_2 \operatorname{mdeg}_{z_{51},z_{52},z_{41},z_{42},z_{31}} (V(z_{11}, z_{12}, z_{21}) \cup V(z_{11}, z_{12}, z_{32})) \\
&\quad + x_1 \operatorname{mdeg}_{z_{51},z_{52},z_{41},z_{42},z_{31}} (V(z_{11}, z_{12}, z_{21}) \cup V(z_{11}, z_{12}, z_{32})) \\
&= x_2 (\operatorname{mdeg}_{z_{51},z_{52},z_{41},z_{42},z_{31}} V(z_{11}, z_{12}, z_{21}) + \operatorname{mdeg}_{z_{51},z_{52},z_{41},z_{42},z_{31}} V(z_{11}, z_{12}, z_{32})) \\
&\quad + x_1 (\operatorname{mdeg}_{z_{51},z_{52},z_{41},z_{42},z_{31}} V(z_{11}, z_{12}, z_{21}) + \operatorname{mdeg}_{z_{51},z_{52},z_{41},z_{42},z_{31}} V(z_{11}, z_{12}, z_{32})) \\
&= x_2({x_1}^2 x_2 + x_1 {x_2}^2) + x_1({x_1}^2 x_2 + x_1 {x_2}^2).
\end{aligned}$$

This identifies $[Y^{2,5}_{[-1,4]}]^T$ with $s_{(2,2)}(x_1, x_2) + s_{(3,1)}(x_1, x_2) \in \mathbb{Z}[x_1, x_2]$. ◇

5.3 Compatibility of rectangle complements

This multidegree depends greatly on our choice of m and h. Compatibility across choices within the inductive system (1.12) amounts to equivalent expansions of multidegrees into the corresponding bases of Schur polynomials (the multidegrees of h-Grassmannian matrix Schubert varieties). We therefore require, when $m' > m$ with $h' \geq h$ and $m' - h' \geq m - h$, that $\operatorname{mdeg} Y^{h,m}_w$ be the specialization of $\operatorname{mdeg} Y^{h',m'}_w$ to $x_{h+1} = \cdots = x_{h'} = 0$. To prove this we invoke the machinery of Kempf collapsings. In the following discussion we reverse the order of multiplication from [KS06] in order to conform with our matrix convention (see Remark 1.1.18) and the mixing space convention in the previous section.

Definition 5.3.1. Let $P < G$ be a parabolic subgroup of a complex semisimple Lie group. Pick a representation V under a right action of G and a P-stable subvariety $Y \subset V$. Define $V \times^P G$ and $Y \times^P G$ as the quotients under the relation $\sim$ given by

$$(v, g) \sim (vp, p^{-1}g) \qquad p \in P,$$

as in Section 5.1, and write $[v\!:\!g]$ for the equivalence class of (v, g). Then the fiber bundle

$$\begin{aligned} \kappa : Y \times^P G &\to V \\ [v\!:\!g] &\mapsto vg \end{aligned} \tag{5.16}$$

is a *Kempf collapsing of* $Y \times^P G$. ◇

We will use a specially-designed Kempf collapsing involving two sizes of Y_w in order to invoke a theorem of Knutson and Shimozono to verify the compatibility of multidegrees.

Example 5.3.2. Set $\mathrm{GL}' := \mathrm{GL}_{h'}$ and $M' := M_{m' \times h'}$. Consider the embedding

$$\begin{aligned} i : M &\hookrightarrow M' \\ x &\mapsto \begin{pmatrix} 0 & 0 \\ x & 0 \\ 0 & I_{h'-h} \end{pmatrix}. \end{aligned} \tag{5.17}$$

Take the parabolic subgroup

$$P_h^- = \begin{pmatrix} \mathrm{GL} & 0 \\ * & \mathrm{GL}_{h'-h} \end{pmatrix} < \mathrm{GL}'$$

corresponding to the h^{th} reflection. If a variety $Y \subset M$ is GL-stable then we may construct the P_h^--stable subvariety

$$\tilde{Y} := \overline{i(Y)P_h^-} \subset M', \tag{5.18}$$

which admits the Kempf collapsing

$$\begin{aligned} \kappa : \tilde{Y} \times^{P_h^-} \mathrm{GL}' &\to M'. \\ [\tilde{y}\!:\!g] &\mapsto \tilde{y}g \end{aligned} \tag{5.19}$$

We will return to this setup several times in this section, so we fix the notation GL', M', and $\tilde{Y}$ for the remainder, along with the standard maximal torus $T' < \mathrm{GL}'$. ◇

Lemma 5.3.3 connects the setup in Example 5.3.2, in particular the unmotivated choice of $\tilde{Y}$, with our main objects of interest. Corollary 5.3.5 and Lemma 5.3.3 leverage a theorem from [KS06] to give an explicit calculation of $[Y_w^{h,m}]^T$ from $[Y_w^{h',m'}]^{T'}$. Theorem 5.3.11 then uses this calculation to verify compatibility.

Lemma 5.3.3. *Take $Y = Y_w^{h,m}$ in Example 5.3.2. Then $\tilde{Y}\mathrm{GL}' = Y_w^{h',m'}$. Moreover, if $\tilde{y} \in \tilde{Y} \subset Y_w^{h',m'}$ has rank h' then $\tilde{y}P_h^-$ contains the matrix of $\tilde{y}\mathrm{GL}'$ in heralded form.*

Hereafter take $\tilde{Y} = \overline{i(Y_w^{h,m})P_h^-}$.

Proof. The latter claim will aid us in the proof of the former. For the latter, pick any full-rank

$$\tilde{y} = \begin{pmatrix} 0 & 0 \\ y & 0 \\ B & C \end{pmatrix}. \tag{5.20}$$

Find (the unique) $g \in \mathrm{GL}$ so that yg is in heralded form. Necessarily $\operatorname{rank} C = h' - h$, so we may set $B' = -C^{-1}Bg$. Then take

$$p = \begin{pmatrix} g & 0 \\ B' & C^{-1} \end{pmatrix} \in P_h^-$$

to get $\tilde{y}p$ in heralded form.

Now for the former claim. Since both $\tilde{Y}\mathrm{GL}'$ (being a continuous group orbit of $\tilde{Y}$) and $Y_w^{h',m'}$ are closed and $\pi^{-1}(\Omega_w^{h',m'})$ is open in $Y_w^{h',m'}$, it will be enough for the leftward containment to consider $y' \in \pi^{-1}(\Omega_w^{h',m'})$. Let $y'g \in y'\mathrm{GL}'$ be (uniquely) in heralded form. Then Corollary 3.1.11 requires that $\mathrm{piv}(y'g) = \mathrm{piv}(y')$ contains $[m' - h' + h + 1, m']$, hence that

$$y'g = \begin{pmatrix} 0 & 0 \\ y & 0 \\ 0 & I_{h'-h} \end{pmatrix}$$

for some full-rank $y \in M_{m \times h}$. Since $\mathrm{piv}(y) \geq \mathrm{piv}(w)$ in $[m]$, another invocation of Corollary 3.1.11 confirms that $y \in Y_w^{h,m}$. Thus $y'g \in i(Y_w^{h,m})$ so $y' = (y'g)g^{-1} \in \tilde{Y}\mathrm{GL}'$.

For the rightward containment, pick any full-rank $\tilde{y}$ as in (5.20). Again $C \in M_{(h'-h)\times(h'-h)}$ is invertible, so the heralded representative of $\tilde{y}\mathrm{GL}'$ clearly lies in $Y_w^{h',m'}$. □

The following heavy machinery converts the geometric setup of Example 5.3.2 to a verifiable statement in algebraic combinatorics. Divided difference operators are defined at Definition B.1.1.

Proposition 5.3.4 ([KS06] Theorem 1, special case)**.** *Let $\kappa : Y \times^P G \to V$ be a Kempf collapsing in which every negative root space lies in* $\mathrm{Lie}(P)$ *and every weight of T that acts on V lies in an open half-space of* $\mathrm{Lie}(T')^*$. *Further suppose that κ is generically finite, and let d be the number of points in a general fiber of κ. Let $w_0 w_0^P$ be the product of the long elements of the Weyl groups W of G and W_P of P, respectively. Denote by W^P the set of minimal coset representatives in W/W_P and by Δ and Δ_P the sets of roots of G and of P, respectively. Then*

$$d[YG] = \partial_{w_0 w_0^P}[Y].$$

Corollary 5.3.5. *Where $W < \mathrm{GL}'$ and $W_P < P_h^-$ are the respective Weyl groups,*

$$[\tilde{Y}\mathrm{GL}']^{T'} = \partial_{w_0 w_0^P}[\tilde{Y}]^{T'}.$$

Proof. Designate $\mathfrak{h}' = \mathrm{Lie}(T')$ and $\mathfrak{b}^- = \mathrm{Lie}(B^-)$ in $\mathfrak{g}' = \mathrm{Lie}(\mathrm{GL}')$. We must establish the three preconditions and one specification of the theorem, specifically

(a) that $\mathrm{Lie}(P_h^-)$ contains the negative root spaces of $\mathfrak{g}'$,

(b) that the weights of the action of T' on $\tilde{Y}$ lie in an open half-space of $(\mathfrak{h}')^*$,

(c) that κ is generically finite, and

(d) that a general fiber of κ consists of a single point (i.e. that $d = 1$).

For (a), $\mathfrak{b}^- \subset \mathrm{Lie}(P_h^-)$ follows from $B^- < P_h^-$.

Recall for (b) that the weights of T' on M' are the coordinate vectors $x_j \in (\mathfrak{h}')^*_{\mathbb{Z}} \subset (\mathfrak{h}')^*$ across $1 \leq j \leq h'$. These lie in any half-plane of $(\mathfrak{h}')^*$ containing the first orthant.

It suffices for (c) to show that $\dim(\tilde{Y} \times^{P_h^-} \mathrm{GL}') = \dim(\mathrm{img}\,\kappa)$. This is confirmed from

$$\begin{aligned}
\dim(\tilde{Y} \times^{P_h^-} \mathrm{GL}') &= \dim \tilde{Y} + \dim(P_h^- \backslash \mathrm{GL}') \\
&= (\dim Y_w^{h,m} + \dim M_{(h'-h)\times h'} + \dim \mathrm{GL}') \\
&\quad - (\dim \mathrm{GL} + \dim \mathrm{GL}_{h'-h} + \dim M_{(h'-h)\times h}) \\
&= ((\ell(w) + h^2) + (h'-h)h' + (h')^2) - (h^2 + (h'-h)^2 + (h'-h)h) \\
&= \ell(w) + (h')^2 \\
&= \dim Y_w^{h',m'}
\end{aligned}$$

and, from Lemma 5.3.3, that $\mathrm{img}\,\kappa = Y_w^{h',m'}$.

For (d), pick any $\tilde{y}g \in \tilde{Y}\mathrm{GL}'$. From Lemma 4.1.5, $Y_w^{h,m} = M^\circ \cap \pi^{-1}(X_w^{h,m})$, so $\tilde{Y} = i(\pi^{-1}(X_w^{h,m}))P_h^-$. We may therefore assume for a general fiber that $\mathrm{rank}\,\tilde{y} = h'$.

We now want to show that if $\tilde{y}g = \tilde{y}'g'$ then necessarily $[\tilde{y} : g] = [\tilde{y}' : g']$. By assumption we have $\tilde{y} = \tilde{y}'g'g^{-1}$. By Lemma 5.3.3 we can find $p, p' \in P_h^-$ so that $\tilde{y}p$ and $\tilde{y}'p'$ are in heralded form. But then $\tilde{y}p = \tilde{y}'g'g^{-1}p \in \tilde{y}'\mathrm{GL}'$, so by Corollary 3.3.10 the heralded forms must agree and $g'g^{-1}p = p'$. This means that $g'g^{-1} = p'p^{-1} \in P_h^-$. Altogether, we get

$$[\tilde{y} : g] = [\tilde{y}'p'p^{-1} : g] = [\tilde{y}' : p'p^{-1}g] = [\tilde{y}' : g'g^{-1}g] = [\tilde{y}' : g'],$$

as desired. □

Example 5.3.6. Take $n = 2$, $w = s_1 s_0$, $m = 4$, $h = 2$, $m' = 5$, and $h' = 3$; we are growing the 4×2 matrix window to the 5×3 window. In this setting

$$P_2^- = \begin{pmatrix} \mathrm{GL}_2 & 0 \\ * & \mathbb{C} \end{pmatrix}$$

has dimension $\dim P_h^- = \dim \mathrm{GL}_2 + \dim M_{1\times 2} + \dim \mathbb{C} = 4 + 2 + 1 = 7$. The heralded-form coset representatives in $Y_w^{2,4}$ take the form

$$\begin{pmatrix} 1 & & \\ a & & \\ & 1 & \\ b & a & \\ & & 1 \end{pmatrix}$$

with $\ell(w) = 2$ free entries, so the P_h^--orbit closure of $i(Y_w^{2,4})$ has dimension $\ell(w) + \dim P_h^- = 9$. The canonical form of a point in $\tilde{Y} \times^{P_h^-} \mathrm{GL}_3$ is

$$\left[\tilde{y} : \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right],$$

which reveals that $\dim(\tilde{Y} \times^{P_h^-} \mathrm{GL}_3) = \dim \tilde{Y} + 2 = 11$. Consistent with this, $\dim Y_w^{3,5} = \ell(w) + 3^2 = 11$. ◇

The following lemma appropriates Corollary 5.3.5 toward obtaining the relation

$$[Y_w^{h',m'}]^{T'} = \partial_{w_0 w_0^P}([Y_w^{h,m}]^{T'} p), \tag{5.21}$$

which links the smaller and larger matrix affine Schubert varieties through the calculation provided by Proposition 5.3.4. This sets the stage for the compatibility statement of Theorem 5.3.11. The lemma views $H_*^T(M) \subset H_*^{T'}(M')$ via the ring endomorphism

$$\begin{aligned} \mathbb{Z}[x_1, \dots, x_h] &\to \mathbb{Z}[x_1', \dots, x_{h'}'] \\ x_i &\mapsto x_i' \end{aligned}$$

induced from (5.17).

Lemma 5.3.7. *In the setting of Example 5.3.2,*

(a) $[\tilde{Y}\mathrm{GL}']^{T'} = [Y_w^{h',m'}]^{T'}$ *and*

(b) $[\tilde{Y}]^{T'} = [Y_w^{h,m}]^T p$, *where*

$$p = (x_1 \cdots x_{h'})^{l'-l} (x_{h+1} \cdots x_{h'})^m.$$

For the proof and for the remainder of the section, take $l = m - h$ and $l' = m' - h'$.

Proof. Lemma 5.3.3 implies (a).

The copy of M under (5.17) is recovered as the coordinate subspace $i(M) = V(\mathfrak{i}_M) \subset M'$, where

$$\mathfrak{i}_M = (z_{ij} \mid (i,j) \notin [l'-l+1, l'-l+m] \times [h]) + (z_{ij} - 1 \mid i = j \in [h+1, h']).$$

Given any variety $Y \subset M$, by Definition 5.2.1 we may therefore write

$$\operatorname{mdeg}_{M'}(i(Y)) = \operatorname{mdeg}_M(Y) \cdot \prod_{z_{ij}-a \in \mathfrak{i}_M} x_j,$$

where a is taken to be 0 or 1 as needed. In contrast, $\tilde{Y}$ is contained in none of the "bottom" coordinate hyperplanes $V(z_{ij})$ with $i > l' - l + m$, so by Definition 5.2.1 (iii) we have the stronger statement

$$\operatorname{mdeg}_{M'}(i(Y_w^{h,m})) = \operatorname{mdeg}_M(Y_w^{h,m}) \cdot \prod_{\substack{z_{ij} - a \in \mathfrak{i}_M \\ i \leq l' - l + m}} x_j = \operatorname{mdeg}_M(Y_w^{h,m})p.$$

From this Lemma 5.2.6 obtains (b). $\square$

For a subvariety $X \subset \mathrm{Gr}$, it is the homology class $[X]_\bullet = [X]^\bullet \frown [\mathrm{Gr}]_\bullet$ that is preserved by the long map (5.12) through the inductive system (1.12). In the particular case $X = X^\lambda$, the indexing partition λ stores, respectively, the degree and the codegree of the cohomology and the homology classes of X:

$$[X^\lambda]^\bullet \in H^{2|\lambda|}(\mathrm{Gr}), \qquad [X^\lambda]_\bullet \in H_{2(((m-h)h)-|\lambda|)}(\mathrm{Gr}) \tag{5.22}$$

We wish to define here a homological statistic "dual" to the multidegree, in the sense that the statistic is preserved under, or can be recovered after, taking the limit $h, m - h \to \infty$. In lieu of a full representation-theoretic discussion, we sketch the construction of interest and use it to motivate Definition 5.3.8.

The compact algebraic torus $T' = T \cap \mathrm{SL}_h$ has characters

$$\begin{aligned} \chi_\alpha : T' &\to \mathbb{C}^*. \\ t &\mapsto {t_1}^{\alpha_1} \cdots {t_h}^{\alpha_h} \end{aligned}$$

These generate the ring of regular functions $R(T') = \mathbb{C}[{x_1}^{\pm 1}, \ldots, {x_h}^{\pm 1}]/(x_1 \cdots x_h - 1)$, where $x_j = \chi_{e_j}$. View M as an SL_h-module under the column action. Then any GL-invariant subvariety $Y \subset M$ is an SL_h-submodule. As such Y has a decomposition $Y = \bigoplus_{\alpha \in \mathbb{Z}^h/\mathbb{Z}(1,\ldots,1)} V_\alpha$ into irreducible SL_h-modules from which the character $\operatorname{ch} Y = \sum_\alpha (\dim V_\alpha) x^\alpha \in R(T')^{S_h}$ may be computed. In particular, each α is a congruence class modulo $(1,\ldots,1)$, the characters of Y and of Y^* are only determined up to multiplication by $x^{(1,\ldots,1)} = x_1 \cdots x_h$.

Now, the dual space $Y^* = \mathrm{Hom}(Y,\mathbb{C})$ to Y has an SL_h-module structure given by $(g\cdot\phi)(v) = \phi(g^{-1}\cdot v)$ for $\phi \in Y^*$ and $v \in Y$. This yields, for $\phi \in V_\alpha$, $t\cdot\phi = t_1{}^{-\alpha_1}\cdots t_h{}^{-\alpha_h}\phi$, hence if $\mathrm{ch}\, V_\alpha = \overline{f(x_1,\ldots,x_h)}$ then $\mathrm{ch}\, V_\alpha^* = \overline{f(x_1{}^{-1},\ldots,x_h{}^{-1})}$. Thus if $\mathrm{ch}\, Y = \overline{f(x_1,\ldots,x_h)} \in R(T')^{S_h}$ then $\mathrm{ch}\, Y^* = \overline{(x_1\cdots x_h)^{m-h} f(x_1{}^{-1},\ldots,x_h{}^{-1})} \in R(T')^{S_h}$.

Definition 5.3.8. Given $h, l > 0$, write

$$\mathcal{V} = \bigoplus_{\alpha\in\mathbb{Z}^h_{\geq 0}} \mathbb{C}x^\alpha \subset H_T^*(M),$$

and view $\mathcal{V} \subset \mathbb{C}[x_1{}^{\pm 1},\ldots,x_h{}^{\pm 1}]$. For $f \in \mathcal{V}$, let $\mathrm{PD}_{h\times l}(f) = (x_1\cdots x_h)^l f(x_1{}^{-1},\ldots,x_h{}^{-1}) \in \mathcal{V}$ be the *rectangle-complement of f in (l^h)*. ◇

Remark 5.3.9. If $f = \sum_{\lambda\subseteq(l^h)} c_\lambda s_\lambda(x_1,\ldots,x_h) \in \mathcal{V}$ (i.e. f is symmetric) then

$$\mathrm{PD}_{h\times l}(f) = \sum_{\lambda\subseteq(l^h)} c_\lambda s_{\lambda^\vee}(x_1,\ldots,x_h),$$

where $\lambda^\vee$ is the (l^h)-complement of λ. Given $\mu \subseteq (l^h)$, let $\hat{\mu}$ be the concatenation of $(l')^{h'-h}$ and $(l'-l)^h + \mu$. Observe that $\hat{\mu} \subseteq ((l')^{h'})$ and that

$$\mathrm{PD}_{h'\times l'}(s_{\hat{\mu}}(x_1,\ldots,x_{h'})|_{x_{h+1}=\cdots=x_{h'}=0}) = \mathrm{PD}_{h\times l}(s_\mu(x_1,\ldots,x_h)). \tag{5.23}$$

◇

Remark 5.3.10. The rectangle complement corresponds to Poincaré duality in Gr, in the sense that the class of X^λ in $H_*(\mathrm{Gr})$ lands on $s_{\lambda^\vee}$ under the composition $H_*(\mathrm{Gr}) \to H_*^{BM}(\mathrm{Gr}^0_\infty) \cong \Lambda$ (see Appendix B). Specifically, Poincaré duality (A.4) on the Grassmannian is a linear map

$$\begin{aligned} H^*(\mathrm{Gr}) &\to H_*(\mathrm{Gr}), \\ \alpha &\mapsto \alpha \frown [\mathrm{Gr}]_\bullet \end{aligned} \tag{5.24}$$

and there are (ring and graded module) isomorphisms

$$\mathbb{Z}[x_1,\ldots,x_h]^{S_h}/\mathfrak{i}_{h,m} \cong H^*(\mathrm{Gr}) \qquad \text{and} \qquad \bigoplus_{\lambda\subseteq((m-h)^h)} \mathbb{Z}s_\lambda \cong H_*(\mathrm{Gr})$$

which allow (5.24) to be expressed

$$\begin{aligned} \mathbb{Z}[x_1,\ldots,x_h]^{S_h}/\mathfrak{i}_{h,m} &\to \bigoplus_{\lambda\subseteq((m-h)^h)} \mathbb{Z}s_\lambda. \\ s_\lambda(x_1,\ldots,x_h) &\mapsto s_{\lambda^\vee}(x_1,\ldots,x_h) \end{aligned}$$

◇

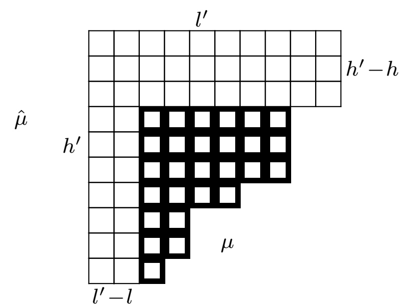

Figure 5.1: $\mu \subset (l^h) = (8^7)$ as a subdiagram of $\hat{\mu} \subset ((l')^{h'}) = (10^10)$.

Theorem 5.3.11. *Pick $m' > h' > 0$ so that $h' \geq h$ and $m' - h' \geq m - h$. Write $f_w^{h,m}(x_1, \ldots, x_h) = \mathrm{PD}_{h\times(m-h)}(\mathrm{mdeg}_M Y_w^{h,m})$ and $f_w^{h',m'}(x_1, \ldots, x_{h'}) = \mathrm{PD}_{h\times(m-h)}(\mathrm{mdeg}_{M'} Y_w^{h',m'})$. Then*

$$f_w^{h',m'}|_{x_{h+1}=\cdots=x_{h'}=0} = f_w^{h,m}.$$

We may therefore define

$$f_w^{(n-1)} := \lim_{\infty \leftarrow h,m-h} f_w^{h,m} \in \Lambda.$$

Write $\lambda + \mu = (\lambda_1 + \mu_1 \geq \lambda_2 + \mu_2 \geq \cdots)$ for two partitions λ and μ.

Proof. Our objective is to show that the rectangle complements of the polynomials $[Y_w^{h,m}]^T$ and $[Y_w^{h',m'}]^{T'}$ have the same Schur expansion. We know from GL'-stability that the polynomials are symmetric. We have acquired

$$[Y_w^{h',m'}]^{T'} = [\tilde{Y}\mathrm{GL}']^{T'} = \partial_{w_0 w_0^P}[\tilde{Y}]^{T'} = \partial_{w_0 w_0^P}[Y_w^{h,m}]^T p,$$

which provides a direct, calculable connection between them. It therefore suffices to check that

$$s_{\hat{\mu}}(x_1, \ldots, x_{h'}) = \partial_{w_0 w_0^P}(s_\mu(x_1, \ldots, x_h)p), \tag{5.25}$$

for any Schur function s_μ. On the right, p factors as $(x_1 \cdots x_{h'})^{l'-l} q$ into a symmetric monomial and $q = (x_{h+1} \cdots x_{h'})^m$. Provided the divided difference does not exhaust $s_\mu(x_1, \ldots, x_{h'})q$, it commutes with the monomial (and by the end of the proof we will have seen this to be the case). On the left, while it could be stated using the Pieri Rule, the product

$$s_\lambda(x_1, \ldots, x_{h'}) \cdot s_{(1^{h'})}(x_1, \ldots, x_{h'}) = s_{\lambda+(l')^{h'}}(x_1, \ldots, x_{h'})$$

obtains for any $\lambda \subseteq ((m' - h')^{h'})$ because in the h' variables $s_{(1^{h'})}(x_1, \ldots, x_{h'}) = x_1 \cdots x_{h'}$. We may therefore decompose

$$s_{\hat{\mu}}(x_1, \ldots, x_{h'}) = s_{\tilde{\mu}}(x_1, \ldots, x_{h'}) \cdot (x_1 \cdots x_{h'})^{l'-l},$$

where $\tilde{\mu} = \hat{\mu} - ((l' - l)^{h'})$. (Each $\hat{\mu}_i \geq l' - l$ by construction.) Now divide (5.25) by $(x_1 \cdots x_{h'})^{l'-l}$ to obtain

$$s_{\tilde{\mu}}(x_1, \ldots, x_{h'}) = \partial_{w_0 w_0^P}(s_\mu(x_1, \ldots, x_h)q), \tag{5.26}$$

the goal of the remainder of the proof.

Give the names $y_1 > \cdots > y_h$ to $x_1 > \cdots > x_h$ and $z_1 > \cdots > z_{h'-h}$ to $x_{h+1} > \cdots > x_{h'}$, but let $z_{h'-h} > y_1$. Write the alphabet $z = \{z_1 > \cdots > z_{h'-h} > y_1 > \cdots > y_h\}$. This change of ordering does not affect (5.26) since w_0^P an be viewed as the product $w_0' w_0''$ of the longest elements of the Weyl groups of GL and of $\mathrm{GL}_{h'-h}$, respectively, which commute. From Proposition B.1.3 we get

$$s_\mu(y_1, \ldots, y_h) = \partial_{w_0'}(y_1 \cdots y_h)^{\mu+\delta'} \quad \text{and} \quad q = s_{\mu'}(z_1, \ldots, z_{h'-h}) = \partial_{w_0''}(z_1 \cdots z_{h'-h})^{\mu'+\delta''},$$

where $\mu' = (m^{h'-h})$ and δ' and δ'' are the respective staircase partitions. This brings us to

$$\begin{aligned}
\partial_{w_0 w_0^P} s_\mu(x_1, \ldots, x_h)q &= \partial_{w_0 w_0^P}(\partial_{w_0'} y^{\mu+\delta'} \partial_{w_0''} z^{\mu'+\delta''}) \\
&= \partial_{w_0 w_0^P} \partial_{w_0'} \partial_{w_0''}(z^{\mu'+\delta''} y^{\mu+\delta'}) \\
&= \partial_{w_0 w_0^P} \partial_{w_0^P} z^{\nu+\delta} \\
&= \partial_{w_0} z^{\nu+\delta} \\
&= s_\nu(z),
\end{aligned}$$

again using Proposition B.1.3, where ν is the concatenation $(m - h \geq \cdots \geq m - h \geq \mu_1 \geq \cdots \geq \mu_h)$ of $((m-h)^{h'-h})$ and μ. (That $\mu \subseteq ((m-h)^h)$ ensures that $\mu_1 \leq m - h$.) Since Schur polynomials are symmetric, $x = z$ as sets, and $\nu = \tilde{\mu}$ (see Fig. 5.1), this obtains (5.26). $\square$

Example 5.3.12. Consider $X_{2,5}^{(2,1)} \subset \mathrm{Gr}_{2,5}$. The corresponding matrix Schubert variety $Y_{2,5}^{(2,1)} \subset M_{5\times 2}$ is the closure of the collection of GL_2-orbits of matrices of the form

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ * & 0 \\ 0 & 1 \\ * & * \end{pmatrix}.$$

This variety has multidegree $s_{(2,1)}(x_1, x_2) = {x_1}^2 x_2 + x_1 {x_2}^2$ having rectangle complement $\mathrm{PD}_{2,3}(s_{(2,1)}(x_1, x_2)) = s_{(2,1)}(x_1, x_2)$. The embedding $\mathrm{Gr}_{2,5} \subset \mathrm{Gr}_{3,7}$ takes $X_{2,5}^{(2,1)}$ to $X_{3,7}^{(4,3,1)}$, which has preimage in $M_{7\times 3}^\circ$ the closure of the GL_3-orbits of

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ * & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The operative parabolic subgroup is $P = P_2$ and gives us $w_0 w_0^P = (s_1 s_2 s_1)(s_1) = s_1 s_2$. Applying Proposition 5.3.4, we get the $(\mathbb{C}^*)^3$-equivariant cohomology class

$$\begin{aligned}
[Y_{3,7}^{(4,3,2)}]^{(\mathbb{C}^*)^3} &= \partial_{w_0 w_0^P} [Y_{2,5}^{(2,1)}]^{(\mathbb{C}^*)^2} ({x_1}^{4-3} {x_2}^{4-3} {x_3}^{2+4}) \\
&= \partial_1 \partial_2 s_{(2,1)}(x_1, x_2)(x_1 x_2 {x_3}^6) \\
&= {x_1}^2 {x_2}^2 {x_3}^2 \, \partial_1 \partial_2 (x_1 {x_3}^4 + x_2 {x_3}^4) \\
&= {x_1}^2 {x_2}^2 {x_3}^2 \, \partial_1 (x_1 {x_2}^3 + x_1 {x_2}^2 x_3 + x_1 x_2 {x_3}^2 + x_1 {x_3}^3 + {x_2}^3 x_3 + {x_2}^2 {x_3}^2 + x_2 {x_3}^3) \\
&= {x_1}^2 {x_2}^2 {x_3}^2 ({x_1}^2 x_2 + {x_1}^2 x_3 + x_1 {x_2}^2 + x_1 x_2 x_3 + x_1 {x_3}^2 + {x_2}^2 x_3 + x_2 {x_3}^2) \\
&= s_{(1,1,1)}(x_1, x_2, x_3)^2 s_{(2,1)}(x_1, x_2, x_3) = s_{(4,3,2)}(x_1, x_2, x_3),
\end{aligned}$$

again with rectangle complement $\mathrm{PD}_{3\times 4}(s_{(4,3,2)}(x_1, x_2, x_3)) = s_{(2,1)}(x_1, x_2, x_3)$, as expected. ◇

The proof of the theorem constructs $\mathrm{mdeg}_{M'} Y_w^{h',m'}$, rather than the other way around. This is significant because it restricts the Schur expansion of $f_w^{h',m'}$, hence that of $f_w^{(n-1)}$, to partitions in the $h \times (m-h)$ rectangle. Under (5.12), it therefore gives the expansion of the class in $H_*^{BM}(\mathrm{Gr}_\infty^0)$ of the affine Schubert variety into the classes of finite Schubert varieties (which comprise a basis):

Corollary 5.3.13. *Write*

$$f_w^{(n-1)} = \sum_{\lambda \in \mathrm{Par}} c_w^\lambda s_\lambda \in \Lambda.$$

Then $\lambda \not\subseteq ((m-h)^h) \Rightarrow c_w^\lambda = 0$ *and*

$$[X_w]_\bullet = \sum_{\lambda \subseteq ((m-h)^h)} c_w^\lambda [X_\lambda]_\bullet \in H_{2\ell(w)}^{BM}(\mathrm{Gr}_\infty^0).$$

That the finite Schubert class expansion of $[X_w]_\bullet$ does not escape the $h \times (m-h)$ rectangle can also be seen from the intermediate expansion

$$[X_w]_\bullet = \sum_{\lambda \subseteq ((m-h)^h)} c_\lambda [X^\lambda]_\bullet \in H_{2\ell(w)}(\mathrm{Gr}).$$

The corollary states that, for sufficiently large m and h, the rectangle-complemented Schur expansion of the equivariant class of the matrix affine Schubert variety gives the expansion of the Borel–Moore class of the affine Schubert variety into classes of finite Schubert varieties.

Remark 5.3.14. This observation is consistent with what is known (Proposition B.4.5) about the Schur expansions of $(n-1)$-Schur functions. In particular, an $(n-1)$-Schur function $s_w^{(n-1)}$ expands into Schur functions indexed by partitions contained in the smallest box $\gamma_1 \times \gamma_1'$ containing the transpose of the n-core γ of w [LLMS13]. ◇

As mentioned in Remark 3.2.5, the embedding $X_w \subset X^{\gamma'}$, where γ is the n-core of w, exhibits a "transpose partition" filter through which the finite and affine Schubert varieties interface. This filter is encoded in the following conjecture by way of the involution $\omega : \Lambda \to \Lambda$ of Proposition B.3.4, which permutes the Schur functions by $\omega(s_\lambda) = s_{\lambda'}$.

Conjecture 5.3.15. *For all suitable choices of m and h,*

$$f_w^{h,m}(x_1, \ldots, x_h) = s_{\lambda'}(x_1, \ldots, x_h) + \sum_{\mu \lhd \lambda} d_{\lambda\mu}^{(n-1)} s_{\mu'}(x_1, \ldots, x_h),$$

where λ is the $(n-1)$-bounded partition associated with w and the $d_{\lambda\mu}^{(n-1)}$ expand $s_\lambda^{(n-1)}$ into s_μ. Consequently,

$$\lim_{\infty \leftarrow h, m-h} f_w^{(n-1)} = \omega(s_w^{(n-1)}).$$

Example 5.3.16. Recall that $\operatorname{mdeg} Y_{[-1,4]}^{2,5} = s_{(2,2)}(x_1, x_2) + s_{(3,1)}(x_1, x_2)$ from Example 5.2.11. The 2×3 rectangle complement is

$$\mathrm{PD}_{2\times 3}(s_{(2,2)}(x_1, x_2) + s_{(3,1)}(x_1, x_2)) = s_{(1,1)}(x_1, x_2) + s_{(2)}(x_1, x_2).$$

In Appendix C.2 we obtain $\operatorname{mdeg} Y_{[-1,4]}^{2,6} = s_{(3,3)}(x_1, x_2) + s_{(4,2)}(x_1, x_2)$ and $\operatorname{mdeg} Y_{[-1,4]}^{3,6} = s_{(4,2,2)}(x_1, x_2, x_3) + s_{(3,3,1)}(x_1, x_2, x_3)$. As a further illustration, we may compute that of $Y_{[-1,4]}^{3,7}$ under $T' = (\mathbb{C}^*)^3$:

$$\begin{aligned}
[Y_{[-1,4]}^{3,7}]^T &= \partial_1\partial_2({x_1}^3 x_2 + 2{x_1}^2{x_2}^2 + x_1{x_2}^3)(x_1 x_2 {x_3}^6) \\
&= {x_1}^2{x_2}^2{x_3}^2\partial_1\partial_2({x_1}^2{x_3}^4 + 2x_1x_2{x_3}^4 + {x_2}^2{x_3}^4) \\
&= {x_1}^2{x_2}^2{x_3}^2\partial_1({x_1}^2{x_2}^3 + {x_1}^2{x_2}^2x_3 + {x_1}^2x_2{x_3}^2 + {x_1}^2{x_3}^3 \\
&\quad + 2x_1{x_2}^3x_3 + 2x_1{x_2}^2{x_3}^2 + 2x_1x_2{x_3}^3 + {x_2}^3{x_3}^2 + {x_2}^2{x_3}^3) \\
&= {x_1}^2{x_2}^2{x_3}^2(s_{(2,2)} + s_{(2,1,1)})(x_1, x_2, x_3) \\
&= s_{(4,4,2)}(x_1, x_2, x_3) + s_{(4,3,3)}(x_1, x_2, x_3).
\end{aligned}$$

Consequently

$$\mathrm{PD}_{3\times 4}([Y_{[-1,4]}^{3,7}]^T) = \mathrm{PD}_{3\times 4}(s_{(4,4,2)}(x_1, x_2, x_3) + s_{(4,3,3)}(x_1, x_2, x_3)) = s_{(2)}(x_1, x_2, x_3) + s_{(1,1)}(x_1, x_2, x_3).$$

The rectangle complements of all four multidegrees are polynomial specializations of the same symmetric function $s_{(1,1)} + s_{(2)} \in \Lambda$. And, indeed, this is the involuted (1)-Schur function $\omega(s_{[-1,4]}^{(1)})$. ◇

We used the algebraic software package `Macaulay 2` to compute the multidegrees of the matrix affine Schubert varieties associated with several 0-Grassmannian permutations for several choices of m and h each. We used the software system `Sage`, specifically the combinatorics patches of `*-combinat`, to compute the Schur expansions of the corresponding $(n-1)$-Schur functions, which we checked against the $f_w^{(n-1)}$. The procedure and results comprise Appendix C.2 and are, with one exception still under investigation, consistent with the conjecture.

5.4 Gröbner geometry

The Schubert polynomials $\mathfrak{S}_\sigma$ for finite flags have been used to perform Schubert calculus since [LS82]. Knutson and Miller [KM05] recovered the $\mathfrak{S}_\sigma$ geometrically as torus-equivariant cohomology classes $[Y^\sigma]$ (or as multidegrees) of matrix Schubert varieties Y^σ. Furthermore, under an appropriate term order, the planar histories of σ [BJS93, FK96] can be recovered from the Gröbner degeneration of Y^σ. See Appendix B.1 for a primer on Schubert polynomials and planar histories.

We showed in the previous section that the inverse limit across $m > h > 0$ of the rectangle-complements of equivariant classes $[Y_w]^T$ is a symmetric function and conjectured that it is in fact the $(n-1)$-Schur function $s_w^{(n-1)}$. This symmetric function represents $[X_w]_\bullet$ under the Hopf isomorphisms (B.10), lending plausibility to the conjecture. In this section we shall explore the Gröbner geometry of several small matrix affine Schubert varieties and the combinatorics that arise therefrom, and make some predictions as to how the geometry and combinatorics play out in general. First we review some results from [KM05].

Definition 5.4.1. Retrieve the coordinate generators z_{ij} of $\mathbb{C}[M_{m\times n}]$ from Section 1.2. Call a term order on $\mathbb{C}[M_{m\times n}]$ *antidiagonal* if, for every $I = (i_1 < \cdots < i_\ell) \subseteq [m]$ and $J = (j_1 < \cdots < j_\ell) \subseteq [n]$,

$$\mathrm{LT}(\Delta_I^J) = \prod_{k=1}^{n} z_{i_k j_{n+1-k}}.$$

$\diamond$

Proposition 5.4.2 ([KM05] Theorem B)**.** *Pick $\sigma \in S_n$. Let $B, B_- < \mathrm{GL}_n$ be the standard and opposite Borel subgroups and take $Y^\sigma = \overline{B_-\sigma B} \subseteq M_{n\times n}$. Under an antidiagonal term order, a Gröbner basis for $\mathfrak{i}(Y^\sigma)$ is given by the vanishing minors*

$$\mathcal{M} = \{\Delta_I^J \mid I \subseteq [p], J \subseteq [q], |I| = |J| = r > r_{pq}\},$$

where $r_{pq} = \mathrm{rank}(\sigma_{[p],[q]})$.

Let $\mathfrak{j}_\sigma =\in (\mathfrak{i}(Y^\sigma))$ be the initial ideal, generated by antidiagonals from $\mathcal{M}$, and let $\mathcal{L}_\sigma$ be the collection of coordinate subspaces obtained as the varieties of the monomial ideals comprising the primary decomposition of $\mathfrak{j}_\sigma$. As sets, $V(\mathfrak{j}_\sigma) = \bigcup_{L\in\mathcal{L}_\sigma} L$, and the primary decomposition is given by $\mathfrak{j}_\sigma = \bigcap_{L\in\mathcal{L}_\sigma} \mathfrak{i}(L)$. Write $D = D_L \subset [n]^2$ when $L = L_D$ (Definition 5.2.3).

Let $\mathcal{R}_\sigma$ be the collection of planar histories D of σ, viewed as subsets of $[n]^2$ indicating the locations of crosses. Then the assignment

$$\begin{aligned} \mathcal{L}_\sigma &\to \mathcal{R}_\sigma \\ L &\mapsto D_L \end{aligned} \tag{5.27}$$

is well-defined and bijective.

The following consequence recalls the weight (5.14) and was known previously to [KM05].

Corollary 5.4.3. *Pick* $D \in \mathcal{R}_\sigma$. *Then* $\sum_{D \in \mathcal{R}_\sigma} x^D = \mathfrak{S}_\sigma$.

Let us reconcile our two types of matrix Schubert variety. Pick a k-Grassmannian permutation $\sigma \in W^{P_k}$, and take $Y^\sigma \subseteq M_{n\times n}$ as above. As in Section 1.1, if we write $I = (\sigma(1) < \cdots < \sigma(k))$ and $I' = (\sigma(k+1) < \cdots < \sigma(n))$ then we may write $Y^\sigma = (\, Y^I_{k,n} \; M_{n\times(n-k)} \,)$ as matrices, or $Y^\sigma = Y^I_{k,n} \times M_{n\times(n-k)}$ as vector spaces. Write Y^σ_{Fl} for $Y^\sigma \subset M_{n\times n}$ and Y^σ_{Gr} for $Y^I_{k,n} \subseteq M_{n\times k}$ when necessary to avoid confusion. By Proposition 5.3.4, then, $\operatorname{mdeg}_{M_{n\times k}} Y^\sigma_{\mathrm{Gr}}$ is obtained from $\operatorname{mdeg}_{M_{n\times n}} Y^\sigma_{\mathrm{Fl}}$ by localizing $x_{k+1}, \ldots, x_n$ to zero.

Example 5.4.4. Take $\sigma = [2,3,5,1,4,6] \in S_6$. Using $B, B_- < \mathrm{GL}$, we then have $Y^\sigma_{\mathrm{Fl}} = \overline{B_-\sigma B} \subseteq M_{6\times 6}$ and $Y^\sigma_{\mathrm{Gr}} = \overline{B_-(\, e_2 \; e_3 \; e_5 \,)\mathrm{GL}_3} \subseteq M_{6\times 3}$. In $M_{6\times 3}$, Y^σ_{Gr} is the GL_3-orbit closure of the matrices having reduced column echelon form

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & & \\ & 1 & \\ * & * & 0 \\ & & 1 \\ * & * & * \end{pmatrix}.$$

Meanwhile, the top-justified planar history for σ (depicted as an rc-graph) is

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & + & + & + & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & + & \cdot & \cdot & \\ 3 & \cdot & \cdot & \cdot & \cdot & & \\ 4 & \cdot & \cdot & \cdot & & & \\ 5 & \cdot & \cdot & & & & \\ 6 & \cdot & & & & & \end{array},$$

from which it is easy to check using chute moves that all planar histories are confined to the first three columns of the grid—that is, that no term of $\operatorname{mdeg} Y^{[2,3,5,1,4,6]}_{3,6}$ has a factor x_4 or x_5e. We may then confine the planar histories of σ on 6×3 grids to get the diagrams

$$\begin{array}{ccc} \square & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \quad \begin{array}{ccc} \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \square & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \quad \begin{array}{ccc} \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \square & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}.$$

In Definition 5.4.8 we name these objects “punchcards”. ◇

Definition 5.4.5 offers another interpretation of this fact.

Definition 5.4.5. Given a collection $\mathcal{P} \subseteq 2^S$ of subsets of a finite set S, call $E \subset S$ a *minimal poisoning of* $\mathcal{P}$ if

(i) $D \cap E \neq \varnothing$ for every $D \in \mathcal{P}$ (E *poisons* $\mathcal{P}$) and

(ii) E is minimal subject to (i).

Write $\mathcal{P}^*$ for the collection of minimal poisonings of $\mathcal{P}$. $\diamond$

Lemma 5.4.6. *If $\mathcal{P} \subset 2^S$ such that $\bigcap_{D \in \mathcal{P}} D = \varnothing$ and no subset in $\mathcal{P}$ contains any other then every subset in $\mathcal{P}$ minimally poisons $\mathcal{P}^*$.*

Proof. By construction every $D \in \mathcal{P}$ poisons $\mathcal{P}^*$. For minimality, pick any $D \in \mathcal{P}$ and any $s \in D$. Since no elements of S are common to all subsets in $\mathcal{P}$, we may identify $D_s \in \mathcal{P}$ with $s \notin D_s$. Since every $D' \smallsetminus D$ is nonempty, we may identify a minimal poisoning E' for $\{D' \smallsetminus D \mid D' \in \mathcal{P} \smallsetminus \{D\}\}$. Take $E = E' \cup \{s\}$, which poisons $\mathcal{P}$. To check the minimality of E, pick $s' \in E$. If $s' = s$ then $E \smallsetminus \{s'\} \cap D = \varnothing$. If $s' \neq s$ then by the minimality of E' there exists $D' \in \mathcal{P} \smallsetminus \{D\}$ for which $E' \cap (D' \smallsetminus D) = \{s'\}$, which then implies that

$$\begin{aligned}
(E \smallsetminus \{s'\}) \cap D' &= (E \cap D') \smallsetminus \{s'\} \\
&= ((E \cap (D' \smallsetminus D)) \cup (E \cap (D' \cap D))) \smallsetminus \{s'\} \\
&= (\{s'\} \cup (D' \cap \{s\})) \smallsetminus \{s'\} \\
&= D' \cap \{s\}.
\end{aligned}$$

Take $D' = D_s$ to find that $E \smallsetminus \{s'\}$ fails to poison $\mathcal{P}$. $\square$

The antidiagonal momomials that generate $\mathfrak{j}_\sigma$ and the rc-graphs of σ, both viewed as subsets of $[m]^2$, minimally poison each other ([KM05] Remark 1.5.5). We build upon this interpretation in our approach to the combinatorics of matrix affine Schubert varieties, though we do not yet know ideal generators of $\mathfrak{i}(Y_w)$ having squarefree monomial ideal.

Adopt the "downward–rightward" reverse-lexicographic order (d–r revlex) on the z_{ij}. For example, the generic entries of $M_{6\times 2}$ fall in d–r order as

$$z_{11} > z_{12} > z_{21} > z_{22} > z_{31} > z_{32} > z_{41} > z_{42} > z_{51} > z_{52} > z_{61} > z_{62}$$

and the shuffle $\mathrm{sh}^2_{56,12}$ on $M_{6\times 2}$ under $n = 2$ expands in revlex order as

$$z_{11}z_{62} - z_{12}z_{61} - z_{21}z_{52} + z_{22}z_{51} + z_{31}z_{42} - z_{32}z_{41}.$$

We make this choice partly for consistency with [KM05], though by virtue of column GL-stability we could dispense with the antidiagonal property and take columns in any order; and partly, for reasons that will emerge later, for compatibility with the order on minor summands in a shuffle induced from the revlex order on Plücker coordinates.

The following example shows that the shuffle generators identified in Definition 4.3.14 are insufficient to identify the leading monomials of $\mathfrak{i}(Y_w)$.

Example 5.4.7. Take $n = 3$ and $w = [-2, 3, 5]$, so that $\pi^{-1}(\Omega_w^{2,5}) \subset M_{5\times 2}$ has orbit representatives of the form

$$\begin{pmatrix} 1 & \\ a & \\ b & \\ & 1 \\ c & a \end{pmatrix}.$$

We then get a candidate ideal $\mathfrak{i}_w^{2,5}$ generated by Δ_{12}^{12}, Δ_{13}^{12}, Δ_{23}^{12}, and $\mathrm{sh}_{45,12}^{1}$; these generators have antidiagonal leading terms $z_{12}z_{21}$, $z_{12}z_{31}$, $z_{22}z_{31}$, and $z_{22}z_{41}$. However, the initial ideal $\mathsf{in}(\mathfrak{i}_w^{2,5})$ also contains

$$-\Delta_{13}z_{51} + \Delta_{23}z_{41} + \mathrm{sh}_{45}^{1}z_{31} = -z_{21}\Delta_{34} + z_{11}\Delta_{35},$$

which has the additional leading term $z_{21}z_{32}z_{41}$. ◇

Definition 5.4.8. Write $\mathfrak{j}_w = \mathsf{in}(\mathfrak{i}(Y_w^{h,m})$, and write $\textsc{lt}(f)$ for the leading term of f. Given a minimal Gröbner basis F, denote by $\mathcal{P}_w = \mathcal{P}_w^{h,m}$ the collection of subsets $E_f \subset [m] \times [h]$ described by

$$(i, j) \in E_f \Leftrightarrow z_{ij} | \textsc{lt}(f)$$

across $f \in F$. Call the $E_f \in \mathcal{P}_w$ *(reduced) dual punchcards of* w. Call the minimal poisonings $D \in \mathcal{P}_w^* = (\mathcal{P}_w^{h,m})^*$ *(reduced) punchcards of* w. (We might call each not-necessarily-minimal poisoning a *punchcard of* w, should a K-theoretic analysis, for example, call for it.) To each punchcard D associate the coordinate space

$$L_D = V(z_{ij} \mid (i, j) \in D).$$

◇

Conjecture 5.4.9. $\mathfrak{j}_w$ *is squarefree (generated by monomials having powers at most* 1*) and* $V(\mathfrak{j}_w)$ *is equidimensional (a union of coordinate subspaces of equal dimension). Moreover,*

$$[Y_w]^T = \sum_{D \in \mathcal{P}_w^*} x^D.$$

Example 5.4.10. The ideal $\mathfrak{i}_w^{2,5} = \mathfrak{i}(Y_w^{2,5})$ from Example 5.4.7 has minimal Gröbner basis

$$\{\Delta_{12}, \Delta_{13}, \Delta_{23}, \mathrm{sh}_{45}^{1}, z_{11}\Delta_{35} - z_{21}\Delta_{34}\}$$

with dual punchcards

$$\begin{array}{cc} \cdot & \square \\ \square & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array},\quad \begin{array}{cc} \cdot & \square \\ \cdot & \cdot \\ \square & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array},\quad \begin{array}{cc} \cdot & \cdot \\ \cdot & \square \\ \square & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array},\quad \begin{array}{cc} \cdot & \cdot \\ \cdot & \square \\ \cdot & \cdot \\ \square & \cdot \\ \cdot & \cdot \end{array},\quad \begin{array}{cc} \cdot & \cdot \\ \square & \cdot \\ \cdot & \square \\ \square & \cdot \\ \cdot & \cdot \end{array},$$

arising from leading terms. The only three-punch punchcards that poison each of these are

$$\begin{array}{cc} \cdot & \square \\ \square & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array},\quad \begin{array}{cc} \cdot & \cdot \\ \square & \square \\ \square & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array},\quad \begin{array}{cc} \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \end{array},\quad \begin{array}{cc} \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \square & \cdot \\ \cdot & \cdot \end{array},\quad \begin{array}{cc} \cdot & \square \\ \cdot & \cdot \\ \square & \cdot \\ \square & \cdot \\ \cdot & \cdot \end{array},\quad \begin{array}{cc} \cdot & \cdot \\ \square & \cdot \\ \square & \cdot \\ \square & \cdot \\ \cdot & \cdot \end{array},$$

and no two-punch punchcards do. ◇

Example 5.4.10 is representative of our data in that the collection of punchcards derived from $\mathfrak{i}_w$ is closed under chute moves within the $m \times h$ grid; see Appendix C.2 for several examples. (This is contingent upon either an antidiagonal or a diagonal term order, though by virtue of GL_h-stability the columns are interchangeable.) We formalize this observation as follows:

Conjecture 5.4.11. *The punchcards $\mathcal{P}_w^*$ are closed under chute moves.*

The conjecture heralds a stronger geometric interpretation of punchcards, to which we turn shortly with Definition 5.5.1. It is also combinatorially and morally compatible with Conjecture 5.3.15. However, it imposes some constraints on the combinatorics of punchcards, as Lemma 5.4.13 will elucidate.

Lemma 5.4.12. *A right-justified punchcard D in the $\mathbb{Z}_{>0} \times [h]$ grid generates, via chute moves, a component with a unique left-justified punchcard.*

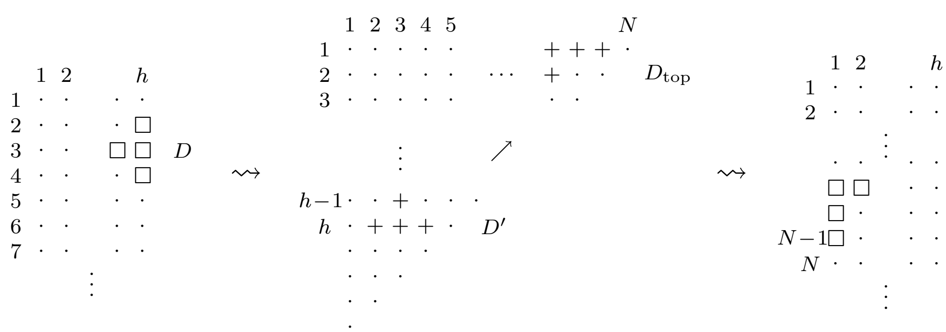

Figure 5.2: Proof of Lemma 5.4.12.

Proof. Reflect D over a northwest–southeast mirror to get D'. Pick $N \in \mathbb{Z}_{>0}$ and $\sigma \in S_N$ so that D' is a planar history of σ. Then, as shown in Fig. 5.2, $\mathcal{R}_\sigma$ has a unique top planar history D_{top} obtainable from D' via inverse chute moves. Reflect D_{top} back to get the unique right-justified punchcard in $\mathbb{Z}_{>0} \times [h]$ chute-connected to D. $\square$

From [Sta84, BB93] we know that a connected component of punchcards in $\mathbb{Z} \times [h]$ has generating function a (symmetric) *Stanley polynomial*, taken to mean the specialization of a Stanley symmetric function (see Definition B.3.7) to finitely many of its variables. We also know that the punchcards derived from the primary decomposition of $\mathsf{in}(\mathfrak{i}(Y_w))$ generate a symmetric polynomial, since the G-stability of Y_w implies the S_h-stability of $\mathfrak{i}(Y_w)$. Finally, the following lemma implies that Conjecture 5.4.11 is incompatible with these facts unless the chute-connected component doesn't bump up against the bottom of the matrix window.

Lemma 5.4.13. *Let D be a right-justified punchcard with connected component $\mathcal{C}(D)$ in $\mathbb{Z}_{>0} \times [h]$. Pick $m \in \mathbb{Z}$ so that $D \subset [m] \times [h]$, and let $\mathcal{C}'(D)$ be the connected component of D confined to $[m] \times [h]$. Let $f \in \mathbb{Z}[x_1, \ldots, x_h]$ be the generating function for $\mathcal{C}'(D)$. Then f is symmetric in $x_1, \ldots, x_h$ if and only if $\mathcal{C}'(D) = \mathcal{C}(D)$.*

Proof. If $\mathcal{C}'(D) = \mathcal{C}(D)$ then f is a Stanley polynomial. Assume instead that $\mathcal{C}'(D) \neq \mathcal{C}(D)$. Let $x^\lambda = {x_1}^{\lambda_1} \cdots {x_h}^{\lambda_h}$ be the monomial generator for D; D is right-justified, so x^λ is uniquely recessive within f under the revlex order on $x_1 > \cdots > x_h$. Thus x^λ has coefficient 1 in f, and if f is symmetric then so must be the uniquely dominant monomial ${x_1}^{\lambda_h} \cdots {x_h}^{\lambda_1}$. This monomial is the weight of the unique left-justified punchcard D_{left} in $\mathcal{C}(D)$. By assumption, there exists $D' \in \mathcal{C}(D)$ having coordinate (i, j) with $j > m$. Then any punchcard obtained from D' via chute moves—in particular, by Lemma 5.4.12, D_{left}—must also have some coordinate (i', j') with $j' < m$. Thus $D_{\text{left}} \notin \mathcal{C}(D)$ and f is not symmetric. $\square$

Remark 5.4.14. The reduced word–compatible sequence definition of affine Schur functions in [Lam06] implies a natural analog of the planar histories of finite permutations [BB93]. These consist of crosses and elbows on a $\mathbb{Z}_{>0} \times (\mathbb{Z}/n\mathbb{Z})$ grid, for which the affine Schur functions are then the generating functions. $\diamond$

For the remainder of this section we consider the effect on punchcards of increasing h and $l = m - h$. The discussion, and all performed examples, are consistent with the conjecture.

Definition 5.4.15. Let $m' > m$ with $h' \geq h$ and $l' \geq l$ and pick any punchcard $D \in (\mathcal{P}_w^{h,m})^*$. Construct

$$D' = \begin{matrix} \square & \square \\ D & D'' \\ \cdot & \cdot \end{matrix} \subset [m'] \times [h'], \tag{5.28}$$

where a punched (respectively, unpunched) block indicates that every entry in the block is punched (respectively, unpunched), and each row $i \in [m]$ of $D'' \subset [m] \times [h' - h]$ is either completely punched or unpunched as follows:

(a) D'' is punched through row i if $(i, h) \in D$.

(b) Supposing that D has $k \leq l$ punches in column h, D'' is punched through the topmost $l - k$ additional rows.

$\diamond$

Lemma 5.4.16 verifies the top third of (5.28). The helper Lemma 5.4.17 takes advantage of the natural action of $S_h < G$ on the set of subsets of $[m] \times [h]$ and on the ring $\mathbb{C}[M]$ that permutes column indices, while we relegate our expectation that punchcards of $Y_w^{h',m'}$ augment those of $Y_w^{h,m}$ as in (5.28) to an additional conjecture. These set the stage for Lemma 5.4.19, which justifies the definition for top punchcards punched in l rows.

Lemma 5.4.16. *Pick $D \in (\mathcal{P}_w^{h,m})^*$. Take $h' = h$ and $l' > l$ and construct D' from D as in (5.28). Then $D' \in (\mathcal{P}_w^{h,h+l'})^*$.*

Proof. It is enough to consider $l' = l + 1$. A Gröbner basis for the ideal $\mathfrak{i}_w^{h,m+1}$ may be obtained from one for $\mathfrak{i}_w^{h,m}$ by increasing all column indices by 1 and introducing the size-1, weight-0 shuffles (coordinates) $\mathrm{sh}_{1i}^0 = z_{1i}$ across $i = 1, \ldots, h$. Invoke the new leading terms z_{1i} under minimal poisoning to get D' from any D. □

Lemma 5.4.17. *Suppose that $E \in \mathcal{P}_w$ is punched in one of the columns $j, j+1$ but not in the other. Then $s_j(E) \in \mathcal{P}_w$.*

Proof. Take $S = \{q_1, \ldots, q_s\}$ to be the collection of generators of $\mathfrak{i}_w$ from Definition 4.3.14. Suppose that z^E is the leading term of $\sum_{t=1}^s f_t q_t$. Then $s_j(\sum_t f_t q_t) = \sum_t (s_j(f_t))(s_j(q_t))$ has leading term $x^{s_j(E)}$, which puts $s_j(E) \in \mathcal{P}_w$. □

Conjecture 5.4.18. *If $D \in (\mathcal{P}_w^{h,m})^*$ then there is $D' \in (\mathcal{P}_w^{h+1,m+1})^*$ with $D' \cap ([m] \times [h]) = D$. Moreover, D' can be selected to satisfy (b) in Definition 5.4.15.*

The conjecture encapsulates what more we need in order to justify Definition 5.4.15 and make strong combinatorial statements about $\mathcal{P}_w^*$. The first statement is confirmed in all examples for which computations under $h' > \gamma_1'$ (γ being the n-core of w) could be performed. The latter is less empirically supported, for lack of as many examples, but is made plausible by the antidiagonality of the underlying term order. A first step may be to show that the bottom row of D' is unpunched; the hyperplanes $V(z_{m+1,j})$ are transverse to $Y_w^{h+1,m+1}$, and the wider candidate ideals can be generated from the narrower ones via $\mathfrak{i}_w^{h+1,m+1} = \mathrm{GL}_{h'} \cdot \mathfrak{i}_w^{h,m}$, which suggest that no leading terms of $\mathsf{in}(\mathfrak{i}(Y_w^{h+1,m+1}))$, or at least of $\mathsf{in}(\mathfrak{i}_w^{h',m'})$, contain a coordinate $z_{m+1,j}$.

Lemma 5.4.19. *Pick a right-justified punchcard D of Y_w. Let $M' = M_{(m+1)\times(h+1)}$ and construct $D' \subset [m+1] \times [h+1]$ from D as in Definition 5.4.15. Then, up to Conjecture 5.4.18, $D' \in (\mathcal{P}_w^{h+1,m+1})^*$.*

Moreover, if $E' \in (\mathcal{P}_w')^*$ with $E' \cap D' = \{(i, h+1)\}$, where $(i, h) \in D$, then E' is punched in every column.

Proof. First we check that (a) is necessary for D' to poison $(\mathcal{P}_w^{h+1,m+1})^*$. Pick a row i with $(i, h) \in D$ (hence $(i, h+1) \in D'$). By minimality of poisonings, there is a dual punchcard $E \in \mathcal{P}_w$ such that $D \cap E = \{(i, h)\}$. Consider $E \subset [m+1] \times [h+1]$ and take $E' = s_h(E)$. Then $(i, h+1) \in E'$, so $(i, h+1) \in D' \cap E'$. Moreover, if $(i', h+1) \in E' \smallsetminus D'$ then $(i', h) \in E \smallsetminus D$. Since $(D' \cap E') \cap ([m+1] \times [h-1]) = (D \cap E) \cap ([m] \times [h-1])$, we may conclude that $D' \cap E' = \{(i, h+1)\}$, rendering $(i, h+1) \in D'$ necessary.

The conjecture requires that we may identify an appropriate subset of the $m + h + 1$ new positions in $[m+1] \times [h+1]$ for D' so that $D' \cap ([m] \times [h]) = D$. Since $\mathrm{mdeg}\, Y_w^{h,m}$ and mdeg $Y_w^{h+1,m+1}$ are polynomials of degree $hl-\ell(w)$ and $(h+1)l-\ell(w)$, respectively, these new positions must number l. Assuming that k are immediately rightward of positions $(i,h)\in D$, this leaves $l-k$ elsewhere, which the conjecture places in the topmost rows of D'' that are vacant in D.

For the second part, take $E'\in(\mathcal{P}'_w)^*$ with $E'\cap D'=\{(i,h+1)\}$. Suppose, contrary to claim, that E' has no punch in column j. By Lemma 5.4.17, $E:=s_hs_{h-1}\cdots s_jE'\in\mathcal{P}_w^{h+1,m+1}$ has no punches in column $h+1$. Let us view E as a subset of $[m]\times[h]$ and note that for every punch (i,j') in E either (i,j') or $(i,j'+1)$ is in E, depending on whether $j'<j$. Since D' and D are right-justified, therefore, if $(i,j')\in E\cap D$ then either (i,j') or $(i,j'+1)$ is in $E'\cap D'$. But this contradicts the choice of E'. $\square$

Example 5.4.20. In Example 4.3.11 we exhibited, for $w=[-2,5]\in\mathcal{W}^{\mathcal{P}}$ under $n=2$, the radical ideal

$$\mathfrak{i}(Y_w^{3,6})=\mathfrak{i}_w^{3,6}=(\Delta_{12},\mathrm{sh}_{34}^1,\mathrm{sh}_{56}^2).$$

The initial ideal $\mathfrak{j}_w^{3,6}=\mathsf{in}(\mathfrak{i}_w^{3,6})$ has a primary decomposition that realizes $V(\mathfrak{j}_w^{3,6})$ as a union

$$\bigcup_{L\in\mathcal{L}_w} L$$

of coordinate subspaces. The punchcards $D_L\in(\mathcal{P}_w^{3,6})^*)$ corresponding to the monomial generators of the L form chute-connected components with top punchcards

$$(\mathcal{P}_w^{3,6})^*_{\mathrm{top}}=\left\{D_{\mathrm{top},1}=\begin{array}{ccc}\cdot&\square&\square\\\cdot&\square&\square\\\cdot&\square&\square\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{array},\ D_{\mathrm{top},2}=\begin{array}{ccc}\cdot&\square&\square\\\cdot&\cdot&\square\\\square&\square&\square\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{array},\ D_{\mathrm{top},3}=\begin{array}{ccc}\cdot&\cdot&\square\\\square&\square&\square\\\cdot&\square&\square\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{array},\ D_{\mathrm{top},4}=\begin{array}{ccc}\cdot&\cdot&\cdot\\\square&\square&\square\\\cdot&\cdot&\cdot\\\square&\square&\square\\\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot\end{array}\right\}.$$

These spell out the Schur polynomial expansion

$$\mathrm{PD}_{3\times3}[Y_w^{3,6}]^T=(s_{(3)}+s_{(2,1)}+s_{(2,1)}+s_{(1,1,1)})(x_1,x_2,x_3)=\omega(s_{[-2,5]}^{(1)})(x_1,x_2,x_3).$$

Note in particular that $D_{\mathrm{top},4}$ is, in the rook sense (rather than the chute sense), right- but not top-justified. Taking only Conjecture 5.4.18 for granted, Lemma 5.4.19 requires that a corresponding right-justified punchcard $D'_{\mathrm{top},4}\in(\mathcal{P}_w^{4,7})^*$ have new punches at positions $(2,4)$ and $(4,4)$, leaving only position $(1,4)$ to be claimed by Definition 5.4.15. Indeed, $(\mathcal{P}_w^{4,7})^*$ consists of the chute-connected components generated by the top punchcards

$$(\mathcal{P}_w^{4,7})^*_{\mathrm{top}}=\left\{D'_{\mathrm{top},1}=\begin{array}{cccc}\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{array},\ D'_{\mathrm{top},2}=\begin{array}{cccc}\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{array},\ D'_{\mathrm{top},3}=\begin{array}{cccc}\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{array},\ D'_{\mathrm{top},4}=\begin{array}{cccc}\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{array}\right\}.$$

$\diamond$

5.5 GL-stable Gröbner geometry

We conclude this chapter by proposing a partial degeneration of Y_w suggested by several examples. The ideal $\mathfrak{i}_w$ is generated by shuffles, and the d–r revlex term order induces an order on the minors comprising any single shuffle, which agrees with the linear term order on $S_{h,m}$ induced from the order on $\mathcal{I}$ given in Definition 1.1.14.

Definition 5.5.1. Given a shuffle $\mathrm{sh}_{I,J}^k$, or any polynomial p in $\mathbb{C}[M]$ expressed as a linear combination of minors on a fixed column subset, designate the term including the minor containing the leading term of p in the d–r revlex order the *leading minor of* p. Given $Y_w \subset M_{m\times h}$, let the *initial determinantal ideal* $\mathfrak{k}_w = \mathfrak{k}_w^{h,m}$ be the ideal generated by the leading minors of the shuffles that generate $\mathfrak{i}_w$. $\diamond$

It is not clear whether $\mathsf{in}(\mathfrak{k}_w) = \mathsf{in}(\mathfrak{i}_w)$ in general. In all our calculations, the initial determinantal ideal has a decomposition into a unidimensional union of GL-stable subvarieties, many of which turn out to be matrix Schubert varieties. Let us illustrate with a couple of examples.

Example 5.5.2. Recall Example 5.4.7 again. From $\mathfrak{i}_w^{2,5} = \mathfrak{i}(Y_w^{2,5})$ we get

$$\mathfrak{k}_w^{2,5} = \langle \Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{24} \rangle.$$

We may retrieve the leading term $z_{21}z_{32}z_{41}$ via

$$\Delta_{23}z_{41} - \Delta_{24}z_{31} = -z_{21}\Delta_{34},$$

so in fact $\mathsf{in}(\mathfrak{i}_w^{2,5}) = \mathsf{in}(\mathfrak{k}_w^{2,5})$ in this case. The variety $V(\mathfrak{k}_w^{2,5})$ has two highest-dimensional components, following a decomposition

$$\mathfrak{k}_w^{2,5} = \langle \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34} \rangle \cap \langle \Delta_2, \Delta_{13} \rangle.$$

Thus $Y_w^{2,5}$ deforms, on its way to a union of coordinate subspaces, into the union of the matrix Schubert varieties $Y_{2,5}^{\ydiagram{3}} \cup Y_{2,5}^{\ydiagram{2,1}}$. Accordingly we find that

$$\operatorname{mdeg} Y_w^{2,5} = \omega(s_w^{(2)})(x_1,x_2) = s_{\ydiagram{3}}(x_1,x_2)+s_{\ydiagram{2,1}}(x_1,x_2) = \operatorname{mdeg} Y_{2,5}^{\ydiagram{3}}(x_1,x_2)+\operatorname{mdeg} Y_{2,5}^{\ydiagram{2,1}}(x_1,x_2),$$

with the latter matrix Schubert variety linearly transformed from its usual (opposite) form via $z_{1i} \leftrightarrow z_{2i}$, i.e. its first and second rows swapped. The reduced forms

$$\begin{pmatrix} 1 & \\ * & 0 \\ * & 0 \\ * & 0 \\ & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \\ 0 & 0 \\ * & 0 \\ & 1 \\ * & * \end{pmatrix}$$

make this clearer. $\diamond$

Example 5.5.3. Take $n = 2$ and $w = s_1 s_0 s_1 s_0 = [-3, 6]$. The initial determinantal ideal of $\mathfrak{i}_w^{4,8} = \mathfrak{i}(Y_w^{4,8})$ under an antidiagonal term order is

$$\mathfrak{k}_{[-3,6]}^{4,8} = \langle \Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{45}, \Delta_{156}, \Delta_{256}, \Delta_{356}, \Delta_{1467}, \Delta_{2467} \rangle,$$

where each Δ_I is a proxy for the collection of all Δ_I^J with $|J| = |I|$. One highest-dimensional component $V(\mathfrak{k}_{[-3,6]}^{4,8})$ is $V(\mathfrak{k}')$, with

$$\mathfrak{k}' = \mathfrak{k}_{[-3,6]}^{4,8} + \langle \Delta_3, \Delta_5, \Delta_{12} \rangle = \langle \Delta_3, \Delta_5, \Delta_{12}, \Delta_{1467}, \Delta_{2467} \rangle.$$

The row permutation $\sigma = [3, 4, 1, 5, 2, 6, 7, 8] \in S_8 < \mathrm{GL}_8$ on $M_{8\times 4}$ is a morphism of varieties taking $V(\mathfrak{k}')$ to the matrix Schubert variety

$$Y_{4,8}^{\ydiagram{3,1}} = V(\Delta_1, \Delta_2, \Delta_{34}, \Delta_{3567}, \Delta_{4567}).$$

Other components are identified in Appendix C, including $V(\mathfrak{k}'')$ where

$$\mathfrak{k}'' = \mathfrak{k}_{[-3,6]}^{4,8} + \langle \Delta_3, \Delta_{146}, \Delta_{246} \rangle = \langle \Delta_3, \Delta_{12}, \Delta_{45}, \Delta_{146}, \Delta_{156}, \Delta_{246}, \Delta_{256} \rangle.$$

The component admits the following column-reduced form:

$$\begin{pmatrix} 1 & & & \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & 1 & & \\ 0 & * & 0 & 0 \\ * & * & 0 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

This is not a matrix Schubert variety; in fact, its Gröbner degeneration yields the union

$$\bigcup_{D_L \in \mathcal{Q}^*}$$

of coordinate subspaces L whose punchcards comprise a single chute-connected component $\mathcal{Q}^*$. This collection has top punchcard

$$\mathcal{Q}^*_{\mathrm{top}} = \left\{ \begin{matrix} \cdot & \square & \square & \square \\ \cdot & \cdot & \square & \square \\ \square & \square & \square & \square \\ \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \right\}$$

and generating function (i.e. $V(\mathfrak{k}'')$ has multidegree) $(s_{\ydiagram{4,4,2,2}} + s_{\ydiagram{4,4,3,1}})(x_1, x_2, x_3, x_4)$. The contribution of this variety to $[Y_{[-3,6]}^{4,8}]^T$ is then

$$\mathrm{PD}_{4\times 4}(s_{\ydiagram{4,4,2,2}} + s_{\ydiagram{4,4,3,1}})(x_1, x_2, x_3, x_4) = (s_{\ydiagram{2,2}} + s_{\ydiagram{3,1}})(x_1, x_2, x_3, x_4).$$

$\diamond$

By virtue of corresponding to a connected component of punchcards, the multidegree of $V(\mathfrak{k}'')$ is a Stanley polynomial on $x_1, \ldots, x_4$. In this case we have the expansion

$$F_{[1,5,7,2,8,6,3,4]} = s_{(4,4,2,2)} + s_{(4,4,3,1)} \in \Lambda$$

of a Stanley symmetric function into the corresponding Schur functions, so

$$[V(\mathfrak{k}'')]^T = F_{[1,5,7,2,8,6,3,4]}|_{x_5=x_6=\cdots=0}.$$

Components like $V(\mathfrak{k}')$ and $V(\mathfrak{k}'')$ are similar (often isomorphic) to Schubert varieties, clearly project to cells in Gr parametrized by affine spaces, and admit rowwise linear transformations to better-organized forms as in Example 5.5.3. For the following result, for any subset $C \subset [h] \times [m-h]$ with m and h understood, write $C^\vee = ([h] \times [m-h]) \setminus C$.

Corollary 5.5.4. *Let $\lambda \subseteq ((m-h)^h)$, write $D_\lambda \subset [h] \times [m-h]$ for the set of positions of boxes of λ, and define the chart*

$$\begin{aligned} \varphi : \mathbb{C}^{|\lambda|} = \bigoplus_{(i,j)\in D_\lambda} \mathbb{C} z_{ij} &\to \Omega^\lambda_{h,m}, \\ a = (a_{ij})_{(i,j)\in D_\lambda} &\mapsto y_a G/G \end{aligned}$$

where y_a is the column-reduced matrix having entry a_{ij} in position $(m+1-h+i-j, i)$ for all $(i,j) \in D_\lambda$. (That is, each "free" parameter of $\Omega^\lambda_{h,m} \cong \mathbb{C}^{|\lambda|}$ corresponding to the box of λ at position (i,j) is filled with a_{ij}. Since all other entries are necessarily either 0 *or* 1 *and each column-reduced form is a canonical G-orbit representative, the map is bijective.) Pick $D \subseteq D_\lambda$. Denote the image*

$$\Omega^{D,\lambda}_{h,m} := \varphi\Big(\bigoplus_{(i,j)\in D} \mathbb{C} z_{ij}\Big) = \Omega^\lambda_{h,m} \cap V(z_{m+1-h+i-j,i} \mid (i,j) \in D_\lambda \setminus D).$$

For any $m' > m$ and $h' \geq h$ with $m' - h' \geq m - h$, when $\mu = \lambda^\vee$ and $E = D^\vee$ in the $h \times (m-h)$ rectangle then let $\lambda' = \mu^\vee$ and $D' = E^\vee$ in the $h' \times (m'-h')$ rectangle. Then

$$f^{(n-1)}_{D,\lambda} := \lim_{\infty \leftarrow h, m-h} \mathrm{PD}_{h\times(m-h)}[\Omega^{E^\vee,\mu^\vee}_{h,m}]^T \in \Lambda.$$

Definition 5.5.5. Assume the conditions of Corollary 5.5.4. If $f^{(n-1)}_{D,\lambda}$ is a Stanley symmetric function then call $\Omega^{D,\lambda}_{h,m}$ the *Stanley cell associated with D*. Also define the *associated Stanley variety* $X^{D,\lambda}_{h,m} := \overline{\Omega^{D,\lambda}_{h,m}}$ and *matrix Stanley variety* $Y^{D,\lambda}_{h,m} := \overline{\pi^{-1}(X^{D,\lambda}_{h,m})} \subset M$. $\diamond$

Example 5.5.6. In Example 5.5.3, $\mathfrak{k}''$ is the ideal of the matrix Stanley variety $Y^{D,(4^2)}_{4,8}$ with $D = \{(1,1),(1,4),(2,1),(2,2)\} \subset (4^2)$. (Remember again that the Ferrers diagram of λ is transposed from the appearance of λ in the matrix above.) $\diamond$

For a few more examples, see Appendix C.2, where tables of degenerations of Y_w into Stanley varieties are exhibited where computational power allowed. Beyond a more palatable description in terms of coordinates, two questions arise with respect to the motivation for this project.

Question 5.5.7. *(a) Is every $V(\mathfrak{k}_w)$ a union of matrix Stanley varieties?*

(b) Does every matrix Stanley variety admit a flat deformation into a chute-closed union of coordinate subspaces?

Affirmative answers would imply a natural, geometric expansion of the projective limits $\lim_{\infty \leftarrow h, m-h} \mathrm{PD}_{h\times(m-h)}(\operatorname{mdeg} Y_w^{h,m})$—under Conjectures 4.4.9 and 5.3.15, of involuted $(n-1)$-Schur functions—into rectangle-complements of Stanley symmetric functions. Whereas Stanley symmetric functions are known to expand positively into Schur functions, this would also provide a geometric construction of the Schur expansions of $(n-1)$-Schur functions.

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Appendix A

Root systems and Lie groups

Kac–Moody groups generalize simple Lie groups. They arise out of root systems subject to slightly weaker conditions than those underlying Lie algebras, which allows them to exhibit a wider range of behaviors; for instance, they may be infinite-dimensional. The structure of Kac–Moody algebras exhibits similar eigenspace decompositions, and one may also associate to a Kac–Moody algebra an associated Kac–Moody group that has the algebra as its tangent space. Any of several constructions may be employed, yielding different Kac–Moody groups; we identify our group of interest but do not consider its construction in detail.

Much of the following discussion, conventions, and propositions are borrowed from [Kac90, Rou98, Kum02]. Throughout, all Lie algebras and tensors will be taken over $\mathbb{C}$, while all root systems will exist in vector spaces over $\mathbb{R}$.

A.1 Lie algebras and root systems

The roots of a semisimple Lie algebra, which live in a vector space, analogize the eigenvalues of a linear transformation. Only a very few systems of roots—finite collections of vectors satisfying certain rigid relationships with each other—can be recovered as the roots of a semisimple Lie algebra.

Definition A.1.1. Let $\mathfrak{g}$ be a semisimple Lie algebra. For any $\alpha \in \mathfrak{h}^*$, define the *$\alpha$-root space*

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid \forall h \in \mathfrak{h},\ [h, x] = \alpha(h)x\},$$

of $\mathfrak{g}$. If $\alpha \neq 0$ and $\dim \mathfrak{g}_\alpha > 0$ then call α a *root of* $\mathfrak{g}$. Let $\Phi(\mathfrak{g})$ denote the collection of roots of $\mathfrak{g}$.

Call a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ a *Cartan subalgebra of* $\mathfrak{g}$ if $\mathfrak{h}$ is nilpotent and has normalizer $N(\mathfrak{h}) = \mathfrak{h}$. $\diamond$

Proposition A.1.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Then $\mathfrak{g}$ exhibits the root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha. \tag{A.1}$$

Example A.1.3. The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2 \subset M_{2\times 2}$ consists of the matrices of trace zero and has basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

under the commutator $[a,b] = ab - ba$. This obtains

$$[h,h] = [e,e] = [f,f] = 0, \quad [h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

These relations imply that $\mathfrak{h} = \mathbb{C}h \subset \mathfrak{g}$ is a Cartan subalgebra and that $\alpha_1 = 2 \in \mathbb{R} \subset \mathfrak{h}^* \cong \mathbb{C}$ and $\alpha_2 = -\alpha_1\mathfrak{h}^*$ are roots of $\mathfrak{g}$ with root spaces $\mathfrak{g}_1 = \mathfrak{g}_{\alpha_1} = \mathbb{C}e$ and $\mathfrak{g}_2 = \mathfrak{g}_{\alpha_2} = \mathbb{C}f$, respectively. Thus $\Phi(\mathfrak{sl}_2) = \{\pm 2\} \subset \mathbb{R}$ and $\mathfrak{gl}_2$ has the root space decomposition $\mathfrak{sl}_2 = \mathbb{C}e \oplus \mathfrak{h} \oplus \mathbb{C}f$. ◇

Example A.1.3 generalizes to any finite rank n:

Example A.1.4. Take the trace-zero matrices $\mathfrak{g} = \mathfrak{sl}_n \subset M_{n\times n}$, which have basis the off-diagonal coordinate vectors $E_{ij}, i \neq j$ and the trace-zero diagonal elements $H_i := E_{ii} - E_{i+1,i+1}$. The H_i generate a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We may view the dual space $\mathfrak{h}^* \subset (\mathbb{R}^n)^*$ as the subspace of $\mathbb{R}^n$ generated by the vectors $\alpha_i = E_{ij}^* - E_{i+1,i+1}^*$.

Meanwhile, each off-diagonal $(i,j)^{\text{th}}$ element E_{ij} satisfies

$$[H_k, E_{ij}] = \begin{cases} 2E_{ij} & (i,j) = (k,k+1) \\ E_{ij} & i = k \text{ and } j \neq k+1 \text{ or } j = k+1 \text{ and } i \neq k \\ 0 & \text{otherwise} \end{cases}$$

for $i < j$ and the negatives of these for $i > j$ and substituting $k-1$ for $k+1$. These adjoints may be recovered by the functionals $\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1} = E_{ii}^* - E_{jj}^* \in \mathfrak{h}^*$: for $i < j$,

$$\alpha_{ij}(H_k) = \begin{cases} 2 & (i,j) = (k,k+1) \\ 1 & i = k \text{ and } j \neq k+1 \text{ or } j = k+1 \text{ and } i \neq k \\ 0 & \text{otherwise,} \end{cases}$$

and similarly for $i > j$. This provides a base of size $\binom{n-1}{2}$ for the root system $\Phi = \Phi(\mathfrak{sl}_n) = \{\pm \sum_{i=j}^k \alpha_i\}_{1\leq j<k<n}$, the root spaces $\mathfrak{g}_{\alpha_{ij}} = \mathbb{C}E_{ij}$, and the root space decomposition

$$\mathfrak{sl}_n = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha = \mathbb{C}h \oplus \bigoplus_{i\neq j} \mathbb{C}E_{ij}.$$

◇

The roots of semisimple Lie algebras exhibit a rich combinatorial geometry, which we only touch upon here.

Definition A.1.5. Take any real vector space V with an inner product $\langle\ ,\ \rangle$. Given a nonzero vector $\alpha \in V$, let $\sigma_\alpha : V \to V$ be the reflection about the hyperspace H_α normal to α. A *root system* Φ in V is a finite, spanning collection of vectors satisfying

(i) if $c\alpha \in \Phi$ then $c = 1$ or $c = -1$,

(ii) $\sigma_\alpha \Phi = \Phi$ across $\alpha \in \Phi$, and

(iii) $\beta - \sigma_\alpha(\beta) \in \mathbb{Z}\alpha$ for any $\alpha, \beta \in \Phi$.

That is, the system is preserved under reflections about the H_α, contains no scalar multiples other than these reflections, and admits only certain rigid projections among its members. Should nonempty disjoint subsets $\Phi_1, \Phi_2 \subset \Phi$ exist such that $\langle \alpha_1, \alpha_2 \rangle = 0$ for every $\alpha_1 \in \Phi_1, \alpha_2 \in \Phi_2$, we say that Φ *reduces to* $\Phi_1 \times \Phi_2$; otherwise Φ is *irreducible.* Note that $\beta - \sigma_\alpha(\beta) = \frac{2\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle}$. ◇

Proposition A.1.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Then the roots $\Phi(\mathfrak{g})$ comprise a root system in $V = \mathfrak{h}^*$.*

Definition A.1.7. Take $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ to be a *base of* Φ if Δ is a basis for V and every $\beta \in \Phi$ can be written $\beta = \sum_{i=1}^{\ell} c_i\alpha_i$ with either all $c_i \in \mathbb{Z}_{\geq 0}$ or all $c_i \in \mathbb{Z}_{\leq 0}$. This definition distinguishes the "positive roots" Φ_+, expressible using only $c_i \geq 0$, from the similarly-defined "negative roots" $\Phi_- = -\Phi_+$ of Φ. The base Δ is said to be composed of *simple roots* α_i, which are not expressible as sums of two or more members of Φ_+. Write $\Delta_\pm = \Delta \cap \Phi_\pm$. Designate $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Delta_\pm} \mathfrak{g}_\alpha$ and $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$. The integral combinations of roots comprise the *root lattice*

$$Q = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i \subset V.$$

Identify the *coroots* $\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subset V^*$ given by $\alpha_i^\vee(v) = \frac{2\langle \alpha_i, v\rangle}{\langle \alpha_i,\alpha_i\rangle}$, which generate a *dual root system* $\Phi^\vee$ having cobase $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\}$ and coroot lattice $Q^\vee = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee \subset V^*$. ◇

Observe that (A.1) can be repackaged as

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

Example A.1.8. The root system in Example A.1.4 is called $\Phi(\mathfrak{sl}_n) = A_{n-1}$ (the subscript gives the dimension of the Cartan subalgebra). Observe that, under the usual inner product, $\frac{\langle \alpha_i,\alpha_{i+1}\rangle}{\langle \alpha_i,\alpha_i\rangle} = \frac{-1}{2}$ for each i and otherwise $\frac{\langle \alpha_i,\alpha_j\rangle}{\langle \alpha_i,\alpha_i\rangle} = 0$. The $\alpha_i = E_{ii} - E_{i+1,i+1} \in \mathfrak{h}^*$ thus comprise a base for A_{n-1} and generate the root lattice $Q \subset \mathfrak{h}^*$. ◇

Definition A.1.9. Given a root system Φ with base $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, construct the integer matrix matrix $C = (C_{ij})_{1\leq i,j\leq \ell}$ by $C_{ij} = \alpha_j(\alpha_i^\vee)$. Since $\sigma_{\alpha_i^\vee}(\alpha_i^\vee) = -\alpha_i^\vee$ we have $C_{ii} = 2$ across $1 \leq i \leq \ell$. The rigidity of the root system and the definition of base require that $\alpha_j(\alpha_i^\vee) \leq 0$ across $1 \leq i, j \leq \ell$. Thus C satisfies, across $1 \leq i, j \leq \ell$,

(i) $C_{ii} = 2$,

(ii) $C_{ij} \leq 0$ if $i \neq j$, and

(iii) $C_{ij} = 0$ if and only if $C_{ji} = 0$.

Any integer square matrix C satisfying (i)–(iii) is called a *generalized Cartan matrix*. If C is *positive definite*—that is, if, for every $I \subseteq [\ell]$, $\Delta_I^I(C) > 0$—then C is called a *Cartan matrix*. The matrix constructed from a root system Φ as above is a *Cartan matrix for* Φ. ◇

The Cartan matrices specifically serve as a classification scheme for finite-dimensional semisimple Lie algebras.

Proposition A.1.10. *Let C be a Cartan matrix. Then C is a Cartan matrix for some root system Φ, hence some Lie algebra $\mathfrak{g} = \mathfrak{g}(\Phi)$. Moreover, $\mathfrak{g}$ is simple if and only if C is indecomposable, and the correspondence is bijective up to simultaneous permutations of the rows and columns of C and isomorphisms of $\mathfrak{g}$.*

The rigidity (specifically the inner products mentioned in Definition A.1.9 together with the law of cosines) further requires that if $C_{ij} \leq C_{ji} < 0$ then $C_{ji} = -1$ and $C_{ij} \in \{-3, -2, -1\}$. Any distinct $\alpha_i, \alpha_j \in \Delta_+$ therefore form one of four possible angles: Taking C_{ij} to be -1, these angles are $\frac{2\pi}{3}$ ($C_{ji} = -1$), $\frac{3\pi}{4}$ ($C_{ji} = -2$), and $\frac{5\pi}{6}$ ($C_{ji} = -3$); the remaining case is $\frac{\pi}{2}$ ($C_{ji} = C_{ij} = 0$). In A_{n-1}, the adjacent roots α_i and α_{i+1} form the angle $\frac{2\pi}{3}$, while nonadjacent α_i and α_j are orthogonal.

Proposition A.1.11. *Given a root system Φ, there is a unique root $\theta = \sum_{\alpha\in\Delta_+} n_\alpha \alpha \in \Phi$ such that if $\beta = \sum_{\alpha\in\Delta_+} m_\alpha \alpha \in \Phi$ then every $m_\alpha \leq n_\alpha$. Call θ the highest root of Φ.*

Example A.1.12. The Lie algebra $\mathfrak{sl}_n$ and its root system A_{n-1} have Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & & 0 \\ & \vdots & & \ddots & \\ 0 & 0 & 0 & & 2 \end{pmatrix} \in M_{(n-1)\times(n-1)}.$$

Specifically, $\mathfrak{sl}_2$ has Cartan matrix (2). The highest root is $\theta = \alpha_{1,n-1} = \alpha_1 + \cdots + \alpha_{n-1}$. ◇

Definition A.1.13. The reflections $\sigma_{\alpha^\vee}$ across $\alpha^\vee \in \Phi^\vee$ generate the *Weyl group* $W <$ Aut$(\mathfrak{h}^*)$ *of* Φ. It is straightforward to check that W acts freely on $\mathfrak{h}^* \smallsetminus \bigcup_{\alpha\in\Phi} H_\alpha$, hence by continuity permutes connected components. Call these components the *chambers of* W and designate the *fundamental chamber* $C_\circ$ that contains the highest root θ. ◇

Definition A.1.14. A *Coxeter system* W, S or *Coxeter group* W *with generating set* S is the quotient of the free group on S by the relations

- $s^2 = 1$ across s (the generators are reflections) and
- $(ss')^{m_{s,s'}} = 1$ across $s \neq s'$, where $m_{s,s'} = m_{s',s} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$.

Define the *Bruhat–Chevalley partial order on* W as the transitive closure of the covering relation given by $w > s_i w$ when $\ell(w) > \ell(s_i w)$. Given $w \in W$, define the *length* $\ell(w)$ *of* w to be the minimum number of generators s_{i_j} required to write $w = s_{i_1} \cdots s_{i_\ell}$ (or 0 if w is the identity).

Given a subset $I \subset S$, denote $W_I \leq W$ the subgroup generated by I. Under the partial order, each coset of W/W_I has a unique minimum-length representative w satisfying $\ell(wv) = \ell(w) + \ell(v)$ for all $v \in W_I$. Let these representatives constitute the collection W^I. ◇

Proposition A.1.15. *Let $\tilde{S} < W$ be the subgroup generated by the reflections over the hyperplanes H_α ($\alpha \in \Delta_+$) containing the faces of $C_\circ$. Then $(W, \tilde{S})$ is a Coxeter system.*

Example A.1.16. Take $\mathfrak{h} \subset \mathfrak{sl}_n$ as in Example A.1.4. The hyperspaces $H_1, \ldots, H_{\binom{n-1}{2}}$ partition $\mathfrak{h}^* \setminus \bigcup_{\alpha \in A_{n-1}} H_\alpha$ into $n!$ chambers, each of which is obtained as $w(C_\circ)$ for some $w \in W$. For instance, in the case $n = 3$ we get roots $\alpha_1, \alpha_2, \alpha_{12} = \alpha_1 + \alpha_2$, and the three hyperspaces H_1, H_2, H_{12} partition $\mathbb{R}^2$ into six chambers. The Weyl group $W \cong S_3$ contains the reflections $s_1 = \sigma_1, s_2 = \sigma_2, s_1 s_2 s_1 = \sigma_{12}$ about these hyperspaces. ◇

A.2 Kac–Moody groups

The reflections about hyperspaces extend in this setting to reflections about (and translations along) a lattice of hyperplanes normal to the roots. The chambers of the original arrangement are partitioned into alcoves, which may be navigated by the affine Weyl group. This group is (finitely) generated by the simple reflections $\sigma_\alpha, \alpha \in \Delta_+$, with a translated reflection normal to the highest root.

Definition A.2.1. Let Φ be an irreducible root system with Weyl group $W < \mathrm{Aut}(\mathfrak{h}^*)$. Given any $k \in \mathbb{Z}$, denote the hyperplanes

$$H_{\alpha,k} = \{v \in \mathfrak{h}^* \mid v(\alpha^\vee) = k\}$$

parallel to H_α and their reflections $s_{\alpha,k}(v) = v - (v(\alpha^\vee) - k)\alpha$. Let $Q := \mathfrak{h}^*_{\mathbb{Z}} < \mathrm{Aut}(\mathfrak{h}^*)$ be the group of translations

$$\begin{aligned} t_{\alpha,k} : \mathfrak{h}^* &\to \mathfrak{h}^* \\ v &\mapsto v + k\alpha \end{aligned}$$

by members of Φ. The action of $W < \mathrm{Aut}(\mathfrak{h}^*)$ induces a multiplicative action of W on $Q < \mathrm{Aut}(\mathfrak{h}^*)$. The *affine Weyl group associated with* Φ is then the subgroup of $\mathrm{Aut}(\mathfrak{h}^*)$ generated by Q and W. $\diamond$

Lemma A.2.2. *$\mathcal{W}$ is the semidirect product $Q \rtimes W$.*

Proof. Given w, α, and k, we have $w(\alpha) = \Phi$ as well. So $wt_{\alpha,k}w^{-1} = t_{w(\alpha),k} \in Q$. $\square$

The connected components of $\mathfrak{h}^* \setminus \bigcup_{\substack{\alpha\in\Phi \\ k\in\mathbb{Z}}} H_{\alpha,k}$ are now of finite diameter. Call each component an *alcove*, and designate the *fundamental alcove* $A_\circ \subset C_\circ$ containing the origin in its closure.

Proposition A.2.3. *$A_\circ$ is uniquely determined, and $\mathcal{W}$ acts simply transitively on the set of alcoves.*

Take $\tilde{S} \subset \mathcal{W}$ to consist of the reflections in the faces of $A_\circ$. These faces are contained in the hyperspaces H_α for $\alpha \in \Delta_+$ and the hyperplane $H_{\theta,1}$. As in the finite case, $(\mathcal{W}, \tilde{S})$ is a Coxeter system.

Example A.2.4. Let $\Phi = A_{n-1}$, so that $\theta = \alpha_1 + \cdots + \alpha_{n-1}$. It can be checked directly that $s_\theta = s_1 s_2 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_2 s_1$. Meanwhile, the fundamental alcove $A_\circ$ is a simplex with faces given by the H_{α_i} through the origin and by the additional hyperplane $H_{\theta,1}$. The associated reflection $s_{\theta,1}$ factors (uniquely) into $t_{\theta,k}s_\theta \in Q \rtimes W = \mathcal{W}$ with $t_{\theta,k} \in Q$, where

$$ts_\theta(v) = s_{\theta,1}(v)$$
$$(v - (v(\theta^\vee)\theta) + k\theta = v - (v(\theta^\vee) - 1)\theta$$

shows that $k = 1$. Write $t_\theta := t_{\theta,1}$. $\diamond$

We will require several properties of this Weyl group in Section 2.2.

The discussion immediately following builds upon concepts from Section 2.2, stated in Kac–Moody generality.

Proposition A.2.5. *Let $\mathcal{G}$ be a Kac–Moody group, $\mathcal{W}$ its Weyl group, $\mathcal{B} < \mathcal{G}$ the standard Iwahori subgroup, and $\mathcal{P} < \mathcal{G}$ a standard parabolic subgroup (for instance the positive loop group when $\mathcal{G}$ is a loop group). Then $\mathcal{G}$ has the Bruhat decomposition*

$$\mathcal{G} = \bigsqcup_{w\in W^I} \mathcal{B}w\mathcal{P}, \tag{A.2}$$

which induces the (cellular) Schubert decomposition

$$\mathcal{G}/\mathcal{P} = \bigsqcup_{w\in \mathcal{W}^I} \mathcal{B}w\mathcal{P}/\mathcal{P} \tag{A.3}$$

on the quotient. The Schubert cells $\Omega_w := \mathcal{B}w\mathcal{P}/\mathcal{P}$ are parametrized by complex affine spaces and satisfy boundary conditions that recover the induced partial order on $\mathcal{W}^I$.

The closure $X_w = \overline{\Omega_w}$ of each Schubert cell is a *Schubert variety*; while the variety structure is common to Kac–Moody $\mathcal{G}/\mathcal{P}$, we will exhibit it only for the cases of concern to us in the next section. For now we note that each Schubert variety has an associated *Schubert class* $[X_w]_\bullet$ in homology, which is the image in $H_*(\mathcal{G}/\mathcal{P})$ of the fundamental class $[X_w]_\bullet \in H_*(X_w)$ under the map

$$H_*(X_w) \to H_*(\mathcal{G}/\mathcal{P})$$

induced by the inclusion $X_w \subseteq \mathcal{G}/\mathcal{P}$.

The decomposition (A.3) has important implications for homology and cohomology, by way of the following fact. Here and for the remainder of the document, unless otherwise noted, we use singular homology and cohomology over $\mathbb{Z}$.

Proposition A.2.6 ([Hat02])**.** *Let X be a (finite) CW-complex of cells in $\mathbb{C}$ with filtration*

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X,$$

where each X_i consists of the cells of dimension $\leq i$. Then

$$H_{2i}(X) \cong \mathbb{Z}^{p_i},$$

where p_i is the number of cells of dimension i, and all odd-degree homology groups are trivial.

Proposition A.2.7 ([Hat02]; Poincaré Duality)**.** *Let X be a smooth projective variety, and take $[X]_\bullet$ to be the fundamental class of X. Then there is an isomorphism*

$$\begin{aligned} D : H^i(X) &\to H_{n-i}(X) \\ [Y]^\bullet &\mapsto [Y]^\bullet \frown [X]_\bullet \end{aligned} \tag{A.4}$$

Corollary A.2.8. *Let $f : X \to Y$ be a closed embedding of smooth projective varieties of codimension d. Then there is a pushforward*

$$f_! : H^*(X) \to H^*(Y)$$

that takes the class $[Z]^\bullet \in H^*(X)$ of any subvariety $Z \subseteq X$ to $[Z]^\bullet \in H^*(Y)$.

Proof. Take $\dim Y = n$. Poincaré duality D_X on X and D_Y on Y (rightward arrows) yield the diagram

$$\begin{array}{ccc} H^{k-d}(X) & \xrightarrow[\sim]{D_X} & H_{n-k}(X) \\ {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} \\ H^k(Y) & \xrightarrow[D_Y]{\sim} & H_{n-k}(Y), \end{array}$$

through which the composition $f_! := {D_Y}^{-1} \circ f_* \circ D_X$ takes $[Z]^\bullet \in H^*(X)$ to $[Y \cap Z]^\bullet = [Z]^\bullet \in H^*(Y)$. $\square$

Elsewhere we just write $f_* = f_!$.

The former proposition implies that, for a parabolic $\mathcal{P} < \mathcal{G}$, the Schubert classes comprise a linear basis for the homology of $\mathcal{G}/\mathcal{P}$. When G/P, as in Sections B.1 and B.2 is a smooth projective variety, by the second proposition we have

$$H^{2(\ell(w_0^P)-\ell)}(G/P) \cong H_{2\ell}(G/P) \cong \mathbb{Z}^{p_\ell}, \tag{A.5}$$

where w_0^P is the longest element in W^I.

Borel exhibited an explicit description of $H^*(G/P)$:

Proposition A.2.9 ([Bor53]). *Let G be a semisimple Lie group of rank n and $P < G$ a parabolic subgroup. Then there is a ring homomorphism*

$$\mathbb{Z}[x_1, \ldots, x_n] \to H^*(G/P). \tag{A.6}$$

Much of the next appendix involves identifying canonical representatives on the left of (A.6) for the classes of Schubert varieties on the right, which facilitates Schubert calculus.

Appendix B

Schubert calculus

The four sections in this appendix present the algebraic combinatorics used to model the cohomology (and Borel–Moore homology) rings of the flag varieties of interest in the main text: the finite full flag variety Fl_n, finite Grassmannians $\mathrm{Gr}_{k,n}$, the infinite Grassmannian Gr_∞^0, and the affine Grassmannian $\mathrm{Gr}_{\mathrm{SL}_n}$. In each setting the central relation is a special case or a generalization of the Borel map (A.6), and the central combinatorial objects of study are the representatives of Schubert classes in the domain.

B.1 Flags and Schubert polynomials

This section complements the discussion of S_n in Section 1.1.

Definition B.1.1. Take S_n to act on the polynomial ring $R = \mathbb{Z}[x_1, \ldots, x_n]$ by $(wf)(x_1, \ldots, x_n) = f(x_{w(1)}, \ldots, x_{w(n)})$. Define the *divided difference operator* $\partial_w : R \to R$ as the composition $\partial_w = \partial_{i_1} \cdots \partial_{i_\ell}$ of operators

$$\partial_i(f) = \frac{1}{x_i - x_{i+1}}(f - s_i f),$$

where $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced word. Designate the "staircase partition" $\delta = (n-1, \ldots, 0)$ and write $x^\delta = {x_1}^{n-1}{x_2}^{n-2} \cdots x_{n-1}$. The *Schubert polynomial associated with* w is

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}(x^\delta), \tag{B.1}$$

so that $\mathfrak{S}_{w_0} = x^\delta$ and $\mathfrak{S}_{\mathrm{id}} = 1$. In general, $\deg \mathfrak{S}_w = \ell(w)$. ◇

While ∂_w does not depend on the choice of reduced word, the composition of ∂_i is zero if the word is not reduced.

Proposition B.1.2 ([LS82])**.** *In the full flag setting, (A.6) specializes to*

$$R := \mathbb{Z}[x_1, \dots, x_n] \to H^*(\mathrm{Fl}_n).$$
$$\mathfrak{S}_w \mapsto [X_w]^\bullet$$

Proposition B.1.3 ([Mac91])**.** *Given an ordered alphabet $x = \{x_1 > \cdots > x_n\}$ and a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$, write $x^\lambda = {x_1}^{\lambda_1} \cdots {x_n}^{\lambda_n}$. Fix a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$, let $\delta = (n-1 \geq \cdots \geq 0)$ be the "staircase partition", and let $w_0 \in S_n$ be the largest element. Then*

$$s_\lambda(x_1, \dots, x_n) = \partial_{w_0} x^{\delta+\lambda}.$$

The proposition is a special case of the general fact that $\partial_{w_0} x^\mu$ is the antisymmetrization $\sum_{w \in S_n} x^{w(\mu)}$ of x divided by the Vandermonde determinant (here $w(\mu)$ is the composition $(\mu_{w(1)}, \dots, \mu_{w(n)})$). In the case that..., this returns a definition of the Schur function on finitely many variables.

Fomin and Kirillov [FK96] introduced combinatorial diagrams, variably called "planar histories", "rc-graphs", and "pipe dreams", whose generating functions are the Schubert polynomials. Bergeron and Billey [BB93] used the combinatorial properties of rc-graphs to recover (and discover) several properties of Schubert polynomials. These objects are central to the Gröbner geometry of [KM05], to which the main text turns in Section 5.4.

Definition B.1.4. A permutation $w \in S_n$ has *one-line notation* $[w(1), \dots, w(n)]$. $\diamond$

It is easily seen that if $w = vs_i$ then $[w(1), \dots, w(i), w(i+1), w(n)] = [v(1), \dots, v(i+1), v(i), \dots, v(n)]$.

Example B.1.5. Let $w = [1, 5, 4, 2, 3]$. Then w factors as

$$[1, 5, 2, 4, 3]s_3 = [1, 5, 2, 3, 4]s_4 s_3 = [1, 2, 5, 3, 4]s_2 s_4 s_3 = [1, 2, 3, 5, 4]s_3 s_2 s_4 s_3 = s_4 s_3 s_2 s_4 s_3$$

into the word $w = s_4 s_3 s_2 s_4 s_3$, which is reduced because each reflection in the factorization removes an inversion. $\diamond$

Definition B.1.6. Consider $w = S_n$ and a reduced word $w = s_{i_1} \cdots s_{i_\ell}$ for $w \in S_n$. Call a composition (sequence of integers) $\alpha = (\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_\ell)$ *compatible with* I if

$$\alpha_j \leq i_j \text{ and } i_j < i_{j+1} \Rightarrow \alpha_j < \alpha_{j+1} \text{ for all } j. \tag{B.2}$$

Let $D = D(I, \alpha) \subset [n] \times [n]$, an *rc-graph for* w (for *r*educed word–*c*ompatible sequence), be the set of coordinates $(r, c) = (i_j - \alpha_j + 1, \alpha_j)$, and let

$$x^\alpha = \prod_{j=1}^{\ell} x_{\alpha_j}$$

be the *weight of* D. Denote the set of rc-graphs for w by $\mathcal{R}_w$. $\diamond$

Proposition B.1.7 ([BJS93, FK96]). *$\mathfrak{S}_w$ is the generating function for $\mathcal{R}_w$.*

Example B.1.8. The reduced word $s_4s_3s_2s_4s_3$ for w in Example B.1.5 has a compatible sequence $(1, 1, 1, 2, 2)$. This pair produces the rc-graph

$$\begin{array}{cc|ccccc} & & & & w(i) & & \\ & & 1 & 2 & 3 & 4 & 5 \\ & 1 & \cdot & + & + & + & \cdot \\ & 2 & \cdot & + & + & \cdot & \\ i & 3 & \cdot & \cdot & \cdot & & \\ & 4 & \cdot & \cdot & & & \\ & 5 & \cdot & & & & \end{array}$$

◇

Note that D is confined to the simplex $\Delta_n = \{(r, c) \mid r + c \leq n\}$. We depict D in matrix format with entries $+$ at coordinates in D and $\cdot$ at other coordinates of the simplex (see Example B.1.12). Our depictions are transposes of those in [BB93] and [KM05], consistent with our column-vector depiction of $\mathbb{C}^\infty$ and Gr_∞. This illustrates an injection from $\mathcal{R}_w$ to the set of planar histories for w.

Definition B.1.9. Let D be a subset of Δ_n and depict D by placing a "cross" $+$ at each coordinate in D and an "elbow" $\cdot$ at each coordinate in $\Delta_n \setminus D$. Let $w \in S_n$ be the permutation obtained by threading the depiction southwestward from columns $1, \ldots, n$ to rows $1, \ldots, n$ crossing at crosses and making orthogonal turns at elbows. Then D is a *planar history of* w. Call D *reduced* if $|D| = \mathrm{inv}\, w$, the number of inversions of w hence the minimal number of crosses required for w. ◇

Lemma B.1.10 ([BB93]). *If $D \in \mathcal{R}_w$ then D is a reduced planar history for w.*

Proof. We recover the specific reduced word for w by reading D bottom-to-top, left-to-right, and recording each coordinate (r, c) as s_{r+c-1} (spelling the word right-to-left). If we extend the planar history depiction of D by placing elbows at each (r, c) with $2 - n \leq r \leq 0$, $1 - r \leq c \leq n$, we obtain a northeast–southwest braid diagram; the order in which we read the crosses records the permutation of the braid (from southwest to northeast), which is therefore w. □

$\mathcal{R}_w$ is organized into components connected by chute moves. A *chute move* $C_{ij}(D)$ is allowed when, for some $k < j$, position (i, j) in D is marked by a cross; positions $(i + 1, j)$, (i, k), and $(i + 1, k)$, are marked by an elbow; and positions (i, p) and $(i + 1, p)$, $k < p < j$, are marked by a cross. The move C_{ij} adjusts position (i, j) to an elbow and position $(i + 1, k)$ to a cross. We write the *inverse chute move*, reversing the procedure, in this case as $C_{i+1,k}$.[1]

[1]The analogous *ladder* and *inverse ladder moves* $L_{ij}(D)$ are allowed when the chute move C_{ji} is allowed in the transpose diagram ${}^tD$ and adjusts positions so that $L_{ij}(D) = {}^t(C_{ji}({}^tD))$.

Proposition B.1.11 ([BB93]). *$\mathcal{R}_w$ may be recovered through inverse chute moves starting at a west-justified rc-graph $D_{\mathrm{bot}}(w)$, or through chute moves starting at a north-justified rc-graph $D_{\mathrm{top}}(w)$.*

Example B.1.12. Two chute moves take the rc-graph in Example B.1.8 to rc-graphs corresponding to the same reduced word (left) and the reduced word $s_3s_2s_4s_3s_2$, respectively:

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & \cdot & + & + & + & \cdot \\ 2 & \cdot & \cdot & + & \cdot & \\ 3 & + & \cdot & \cdot & & \\ 4 & \cdot & \cdot & & & \\ 5 & \cdot & & & & \end{array} \xleftarrow{C_{2,2}} \begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & \cdot & + & + & + & \cdot \\ 2 & \cdot & + & + & \cdot & \\ 3 & \cdot & \cdot & \cdot & & \\ 4 & \cdot & \cdot & & & \\ 5 & \cdot & & & & \end{array} \xrightarrow{C_{1,4}} \begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & \cdot & + & + & \cdot & \cdot \\ 2 & + & + & + & \cdot & \\ 3 & \cdot & \cdot & \cdot & & \\ 4 & \cdot & \cdot & & & \\ 5 & \cdot & & & & \end{array}$$

The three rc-graphs shown (correspond to rc-pairs that) have weights $x_1x_2{x_3}^2x_4$, ${x_2}^2{x_3}^2x_4$, and $x_1{x_2}^2{x_3}^2$, which are three terms in the monomial expansion of $\mathfrak{S}_w$. $\diamond$

B.2 Grassmannians and Schur polynomials

The Schubert polynomials for $\mathrm{Fl}_{I,n}$ specialize in the Grassmannian case $I = (k)$ to Schur polynomials, which are S_n-stable polynomials to be defined shortly. See Definition 1.1.16 and the surrounding discussion for basic definitions and observations involving partitions.

Example B.2.1. With $n = 5$ there is a 2-Grassmannian permutation $w = s_2s_1s_4s_3s_2 = [3,5,1,2,4]$. Then $w^{-1}w_0 = (s_2s_3s_4s_1s_2)(s_4s_3s_2s_1s_4s_3s_2s_4s_3s_4) = s_1s_4s_3s_2s_4$ and the associated Schubert polynomial is

$$\mathfrak{S}_w = \partial_{14324}({x_1}^4{x_2}^3{x_3}^2x_4) = {x_1}^3{x_2}^2 + {x_1}^2{x_2}^3.$$

$\diamond$

Definition B.2.2. Consider the S_n-fixed subring $\Lambda_n := \mathbb{Z}[x_1,\ldots,x_n]^{S_n}$ of *symmetric polynomials*. Extend the notation x^δ to any $a = (a_1,\ldots,a_n) \in \mathbb{Z}_{\geq 0}$, so that $x^a = {x_1}^{a_1}\cdots{x_n}^{a_n}$. Designate the following specific symmetric polynomials indexed by partitions, writing $f(x) = f(x_1,\ldots,x_n)$: the *monomial symmetric polynomials*

$$m_\lambda(x) = \sum_{\sigma\in S_n/\operatorname{Stab}_{S_n}\lambda} {x_1}^{\lambda_{\sigma(1)}}\cdots{x_\ell}^{\lambda_{\sigma(\ell)}},$$

where the sum is taken only over distinct monomials; the *elementary symmetric polynomials* $e_\lambda(x) = e_{\lambda_1}(x)\cdots e_{\lambda_\ell}(x)$, where

$$e_k(x) = \sum_{i_1<\cdots<i_k} x_{i_1}\cdots x_{i_k};$$

and the *complete homogeneous symmetric polynomials* $h_\lambda(x) = h_{\lambda_1}(x)\cdots h_{\lambda_\ell}(x)$, where

$$h_k(x) = \sum_{i_1\leq\cdots\leq i_k} x_{i_1}\cdots x_{i_k}.$$

$\diamond$

We will define the *Schur polynomials* $s_\lambda(x)$ in terms of tableaux, which also govern their multiplication and monomial expansions.

Definition B.2.3. A *filling of a partition* λ *of* N is an assignment of a unique integer to each box of λ, depicted as the Ferrers diagram of λ with assigned integers written inside their boxes. Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and pick $a = (a_1, \dots, a_m)$ so that $a_1 + \cdots + a_m = N$. Then a *tableau of shape* λ *and content* a is a filling of λ such that, for each $i = 1, \dots, m$, the boxes filled with $1, \dots, i$ constitute a partition and these filled with i constitute a horizontal a_i-strip. Denote by $\mathcal{T}_a(\lambda)$ the collection of tableaux of shape λ and content a, and set $K_{\lambda a} = |\mathcal{T}_a(\lambda)|$. $\diamond$

Remark B.2.4. We use the English convention for partitions throughout, though it prefers the matrix orientation transposed from ours, and we retain it for tableaux. In Section 3.2 this choice bumps up against the convention for n-cores, which prefers our matrix orientation; $(n-1)$-tableaux on these cores (see Appendix B.4) will be immediately distinguishable from finite tableaux for using a transposed French convention rather than the English. $\diamond$

Example B.2.5. The collection $\mathcal{T}_{(1,2,2,1)}(\ydiagram{3,2,1})$ consists of

$$\begin{ytableau} 1 & 2 & 2 \\ 3 & 3 \\ 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 3 \\ 2 & 3 \\ 4 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 3 \\ 2 & 4 \\ 3 \end{ytableau}\,, \quad \begin{ytableau} 1 & 2 & 4 \\ 2 & 3 \\ 3 \end{ytableau}\,,$$

so $K_{\ydiagram{3,2,1},(1,2,2,1)} = 4$. $\diamond$

The following several results can be found and contextualized in [Mac79].

Proposition B.2.6. *The Kostka numbers* $K_{\lambda\mu}$ *satisfy*

$$K_{\lambda\mu} = \begin{cases} 1 & \mu = \lambda \\ 0 & \mu \ntrianglelefteq \lambda. \end{cases}$$

The proposition means that the matrix $(K_{\lambda\mu})_{\lambda,\mu\vdash N}$ is invertible, which allows us to expand the Schur polynomials into complete homogeneous symmetric polynomials.

Definition B.2.7. Define the Schur polynomials as the solutions to the unitriangular system

$$h_\mu(x) = \sum_{\mu \trianglelefteq \lambda} K_{\lambda\mu} s_\lambda(x)$$

for $\lambda, \mu \vdash N$. It follows from the duality between the h_μ and the m_μ that the Schur polynomials also satisfy

$$s_\lambda(x) = \sum_{\mu \trianglelefteq \lambda} K_{\lambda\mu} m_\mu(x).$$

$\diamond$

Proposition B.2.8 (Pieri Rule and structure constants)**.** *The Schur polynomials satisfy the identity*

$$s_\lambda(x)s_{(k)}(x) = \sum_{\substack{\nu/\lambda \text{ is a} \\ \text{horizontal } k\text{-strip}}} s_\nu(x)$$

and, more generally,

$$s_\lambda(x)s_\mu(x) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(x).$$

The $c^\nu_{\lambda\mu}$ are called the Littlewood–Richardson coefficients.

Proposition B.2.9. *The Schur polynomial s_λ is the sum of the monomial weights of the tableaux of shape λ. That is,*

$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in \mathcal{T}_a(\lambda)} x^T = \sum_a K_{\lambda a} x^a,$$

where if $T \in \mathcal{T}_a(\lambda)$ then we define x^T to be x^a.

Example B.2.10. The three tableaux on the partition $\lambda = (2, 1, 1)$ with entries in $[3]$ are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}\,.$$

The Schur polynomial in $\mathbb{Z}[x_1, x_2, x_3]^{S_3}$ associated with $\lambda = (2, 1, 1)$ is $s_{(2,1,1)} = {x_1}^2 x_2 x_3 + x_1 {x_2}^2 x_3 + x_1 x_2 {x_3}^2 = m_{(2,1,1)}(x_1, x_2, x_3)$. This agrees with Definition B.2.7 since the only $\mu \vdash 4$ dominated by $(2, 1, 1)$ are $(2, 1, 1)$ itself and (1^4), and $m_{(1^4)}(x_1, x_2, x_3) = 0$. Accordingly, when $n \geq 4$ we have

$$s_{(3,1)}(x_1, x_2, x_3) = m_{(3,1)}(x_1, x_2, x_3) + m_{(2,2)}(x_1, x_2, x_3) + 2m_{(2,1,1)}(x_1, x_2, x_3) + 3m_{(1^4)}(x_1, x_2, x_3).$$

◇

Definition B.2.11. Call a box (i, j) of λ *removable* if $\lambda / \{(i, j)\}$ is a partition; call a position (box) (i, j) not in λ *addable* if $\lambda \cup \{(i, j)\}$ is a partition. Assign each box $(i, j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ the *residue* $i - j + k$. ◇

Corollary B.2.12 ([Win96])**.** *Let $P = P_k$ and $w \in W^P$, and choose $\lambda \in$ Par so that $\Omega_\lambda = \Omega_w$. Then λ has an addable box of residue r if $s_r w \in W^P$ with $\ell(s_r w) < \ell(w)$, and a removable box of residue r if $s_r w \in W^P$ and $\ell(s_r w) > \ell(w)$.*

Proposition B.2.13 ([Ful97])**.** *The Borel map A.6 in the Grassmannian setting is given by*

$$\mathbb{Z}[x_1, \dots, x_n] \to H^*(\mathrm{Gr}_{k,n}) \cong \mathbb{Z}[x_1, \dots, x_k]^{S_k} / (s_\lambda(x_1, \dots, x_k) \mid \lambda \nsubseteq ((n-k)^k)) \tag{B.3}$$

$$s_\lambda \mapsto \begin{cases} [X_w]^\bullet & \lambda \subseteq ((n-k)^k) \\ 0 & \textit{otherwise.} \end{cases}$$

where w is obtained from λ as in Corollary B.2.12. In particular,

$$[X_\lambda]^\bullet \smile [X_\mu]^\bullet = \sum_\nu c_{\lambda\mu}^\nu [X_\nu]^\bullet. \tag{B.4}$$

This trick is a special case of Lemma 3.2.3, using $n > \max(\lambda_1, \lambda_1')$.

Example B.2.14. Recall $w = [3, 5, 1, 2, 4]$ from Example B.2.1. Obtain the partition $\lambda \subseteq ((n-k)^k) = (3^2)$ associated with w from the boxless array of residues having $k = 2$ in the northwest corner as follows:

$$s_2 s_1 s_4 s_3 s_2 \cdot \begin{array}{ccc} 2 & 3 & 4 \\ 1 & 2 & 3 \end{array} \rightsquigarrow s_2 s_1 s_4 s_3 \cdot \begin{array}{ccc} \boxed{2} & 3 & 4 \\ 1 & 2 & 3 \end{array} \rightsquigarrow s_2 s_1 s_4 \cdot \begin{array}{ccc} \boxed{2} & \boxed{3} & 4 \\ 1 & 2 & 3 \end{array} \rightsquigarrow s_2 s_1 \cdot \begin{array}{ccc} \boxed{2} & \boxed{3} & \boxed{4} \\ 1 & 2 & 3 \end{array} \rightsquigarrow s_2 \cdot \begin{array}{ccc} \boxed{2} & \boxed{3} & \boxed{4} \\ \boxed{1} & 2 & 3 \end{array} \rightsquigarrow \begin{array}{ccc} \boxed{2} & \boxed{3} & \boxed{4} \\ \boxed{1} & \boxed{2} & 3 \end{array}$$

Thus $\lambda = (3, 2)$. Accordingly, the Schubert polynomial is $\mathfrak{S}_w = s_{\ydiagram{3,2}}(x_1, x_2)$. ◇

B.3 The infinite Grassmannian and symmetric functions

The homology and cohomology of the ind-Grassmannians are modeled using symmetric functions, which we review here. For a thorough discussion, see [Mac79].

While we do not examine it in detail, Borel–Moore homology underlies the homology of the ind-spaces Gr_∞ and $\mathrm{Gr}_{\mathrm{GL}_n}$ and is invoked in the service of equivariant homology (until approximation spaces obviate the need). The upcoming proposition is of especial importance to the equivariant (co)homology maps that comprise (5.12).

Definition B.3.1. Let X be a locally compact topological space satisfying certain additional conditions laid out in [CG10] Section 2.6. The *one-point compactification $\hat{X}$ of X* is given by $\hat{X} = X$ if X is compact and otherwise by $\hat{X} = X \cup \{\infty\}$, where $\infty \notin X$ and for any sequence $\{x_i\}_{i=1}^\infty \subset X$, if $\lim_{i\to\infty} x_i$ does not exist in X then $\lim_{i\to\infty} x_i = \infty$ in $\hat{X}$. The *Borel–Moore homology groups $H_*^{BM}(X)$ of X* are then given by $H_i^{BM}(X) = H_i(\hat{X}, \infty)$, where H_* refers to (relative) singular homology with complex coefficients. ◇

Proposition B.3.2 ([CG10] Section 2.6 (4))**.** *Let M be a smooth, orientable real m-manifold with $X \subseteq M$ a closed subset, and suppose that X is a proper deformation retract of a closed neighborhood $U \supseteq X$. Then there is a canonical isomorphism*

$$H_i^{BM}(X) \cong H^{m-i}(M, M \setminus X). \tag{B.5}$$

In particular, if $X = M$ then (B.5) becomes

$$H_i^{BM}(M) \cong H^{m-i}(M). \tag{B.6}$$

The unitriangularity of Schur polynomials extends to the ring of symmetric polynomials, by which we may define the Schur functions $s_\lambda \in \Lambda$ analogously to the finite case. We begin with another class of symmetric functions, the Stanley symmetric functions, which actually include the Schur functions. Their definition requires a fact about the projective limits of Schubert polynomials, and they are naturally tied to planar histories.

Definition B.3.3. Given $n' < n$, there are canonical surjective algebra homomorphisms

$$\mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}[x_1, \ldots, x_{n'}]$$
$$x_i \mapsto \begin{cases} x_i & i \le n' \\ 0 & i > n' \end{cases}$$

that make the polynomial rings into a projective system. The symmetric subrings $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ form a subsystem, whose inverse limit

$$\Lambda = \lim_{\infty \leftarrow n} \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$$

is called the *ring of symmetric functions*. Whereas each evaluation takes $e_\lambda(x_1, \ldots, x_n)$ to $e_\lambda(x_1, \ldots, x_{n'})$ and similarly preserves the other bases of the finite rings of symmetric polynomials, we can define the *elementary*, *complete homogeneous*, *monomial*, and *Schur functions*, which we shall denote $e_\lambda, h_\lambda, m_\lambda, s_\lambda \in \Lambda$, as the projective limits of these bases. They therefore expand into each other exactly as in the finite case. ◇

An important automorphism of Λ will be essential to interpreting our data and drawing the appropriate conjecture in Section 5.3.

Proposition B.3.4. *The map*

$$\omega : \Lambda \to \Lambda, \quad e_i \mapsto h_i \tag{B.7}$$

induces an involutive ring automorphism that sends s_λ to $s_{\lambda'}$.

Definition B.3.5. Define the *Hall inner product* $\langle \cdot, \cdot \rangle$ on Λ to be the symmetric bilinear form determined by $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$. ◇

Lemma B.3.6. *The Schur functions are self-dual: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.*

Proof. If we write $s_\lambda = \sum_\mu S_{\lambda\mu} s_\mu$ then we can use Definition B.2.7 to factor S as $S = (K_{\lambda\mu})(K_{\lambda\mu})^{-1}$. □

Lemma B.3.7. *Consider $S_n \times S_{n'} \subset S_{n+n'}$ and $w \in S_{n'}$. Then $\mathfrak{S}_w = \mathfrak{S}_{1_{S_n} \times w}(0, \ldots, 0, x_1, \ldots, x_{n'})$. Moreover, $F_w := \lim_{\infty \leftarrow n} \mathfrak{S}_{1_{S_n} \times w}$ is a symmetric function, called the Stanley symmetric function associated with w. If $w \in S_n$ is Grassmannian then $F_w = s_\lambda$, with λ as in Corollary B.2.12.*

Proof. The finite statement is easily proved using the properties of planar histories, in particular via an injection $\mathcal{R}_w \hookrightarrow \mathcal{R}_{1_{S_m}\times w}$ given by sending each coordinate (i,j) to $(i,j+m)$. That the projective limit is symmetric follows from lemmata in Section 5.4. □

Whereas the Schur functions comprise an orthonormal basis of Λ, any symmetric function $f \in \Lambda$ is uniquely determined as $f = \sum_{s_\lambda} \langle f, s_\lambda \rangle s_\lambda$ by its Hall inner products with the Schur basis.

Definition B.3.8. Define the *Littlewood–Richardson coefficients* $c^\nu_{\lambda\mu}$ via

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu.$$

Define the *skew Schur functions* $s_{\nu/\mu} \in \Lambda$ by

$$s_{\nu/\mu} = \sum_\mu c^\nu_{\lambda\mu} s_\mu.$$

Define a *skew tableau of (skew) shape* ν/μ *and content* $a = (a_1, \ldots, a_m)$ to be a filling of ν/μ by $1, \ldots, m$ so that, for each $i = 1, \ldots, m$, the boxes filled with $1, \ldots, i$, together with μ, constitute a partition, while those filled with i constitute a horizontal a_i-strip. Denote by $\mathcal{T}_a(\nu/\mu)$ the collection of skew tableaux of shape ν/μ and content a. ◇

Proposition B.3.9 ([Ful97])**.** *The skew Schur function $s_{\nu/\mu}$ is the generating function for the skew tableaux of shape ν/μ.*

To conduct algebraic topology on the infinite or affine Grassmannian, we must understand how Borel–Moore homology interfaces with singular cohomology, which may be defined in the usual way. The following results arrive at a skew Schur function description of the cap product

$$\frown\colon H_j(\mathrm{Gr}^0_\infty) \times H^i(\mathrm{Gr}^0_\infty) \to H_{j-i}(\mathrm{Gr}^0_\infty).$$

Lemma B.3.10. *The cap product $\frown\colon H_*(\mathrm{Gr}_{h,m}) \times H^*(\mathrm{Gr}_{h,m}) \to H_*(\mathrm{Gr}_{h,m})$ is governed by its evaluation*

$$[X^\lambda]^\bullet \frown [X^{\mu^\vee}]_\bullet = \sum_{\nu \subseteq ((m-h)^h)} c^\mu_{\lambda\nu} [X^{\nu^\vee}]_\bullet \tag{B.8}$$

at the basis of Schubert classes.

Proof. By Poincaré duality (A.4) and the structure maps (B.4) we have

$$[X^\lambda]^\bullet \frown [X^{\mu^\vee}]_\bullet = ([X^\lambda]^\bullet \smile [X^{\mu^\vee}]^\bullet) \frown [\mathrm{Gr}_{h,m}]_\bullet = \left(\sum_{\nu^\vee} c^{\nu^\vee}_{\lambda\mu^\vee} [X^{\nu^\vee}]^\bullet \right) \frown [\mathrm{Gr}_{h,m}]_\bullet$$

It is then enough to show that $c^{\nu^\vee}_{\lambda\mu^\vee} = c^\mu_{\lambda\nu}$ for any triplet of partitions $\lambda, \mu, \nu \subseteq ((m-h)^h)$.

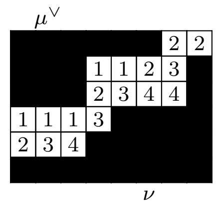


Figure B.1: A skew tableau of shape $\nu^\vee/\mu^\vee$ and content λ, where $\lambda = (5,4,4,3)$, $\mu = (8,8,8,5,5,2)$, and $\nu = (8,5,4,1,1)$ lie inside the box $((m-h)^h) = (8^6)$.

It is immediate from definitions that $c^{\nu^\vee}_{\lambda\mu^\vee} = 0 \Leftrightarrow c^{\mu}_{\lambda\nu} = 0$. Let us instead assume that $c^{\nu^\vee}_{\lambda\mu^\vee} > 0$. Then $c^{\nu^\vee}_{\lambda\mu^\vee}$ counts the ways of obtaining the partition $\nu^\vee$ by pasting horizontal λ_i-strips to $\mu^\vee$ over $i = 1, \dots, h$, i.e. the skew tableaux $\mathcal{T}_\lambda(\nu^\vee/\mu^\vee)$. In Fig. B.1, the boxes comprising each horizontal λ_i-strip are labeled by i.

We may rotate Fig. B.1 by π to see that, similarly, $c^{\mu}_{\lambda\nu} = |\mathcal{T}_\lambda(\mu/\nu)|$. From the original diagram we can also see that $c^{\mu}_{\lambda\nu}$ counts the number of skew tableaux of shape $\nu^\vee/\mu^\vee$ and content $(\lambda_\ell, \dots, \lambda_1)$. The symmetry of the skew Schur functions and Proposition B.3.9 then imply that $c^{\mu}_{\lambda\nu} = c^{\nu^\vee}_{\lambda\mu^\vee}$. □

Corollary B.3.11. *The cap product on* Gr^0_∞ *is given by*

$$[X^\lambda]^\bullet \frown [X^\mu]_\bullet = \sum_\nu c^{\mu}_{\lambda\nu}[X^\nu]_\bullet. \tag{B.9}$$

Therefore, under the ring and graded Λ*-module isomorphisms*

$$\begin{aligned} H_*(\mathrm{Gr}^0_\infty) &\to \Lambda \\ [X^\lambda]_\bullet &\mapsto s_\lambda \end{aligned} \quad \text{and} \quad \begin{aligned} \Lambda &\to H^*(\mathrm{Gr}^0_\infty) \\ s_\lambda &\mapsto [X^\lambda]^\bullet \end{aligned},$$

respectively, the cap product is realized by the action

$$s_\lambda \cdot s_\mu = s_{\mu/\lambda}.$$

Proof. Retrieve the inclusion $\mathrm{Gr}_{h,m} \hookrightarrow \mathrm{Gr}^0_\infty$ from Definition 2.1.8, using suitable h and m for the λ and μ under consideration. In our notation, $X^{\lambda^\vee} \subset \mathrm{Gr}_{h,m}$ is the preimage of $X^\lambda \subset \mathrm{Gr}^0_\infty$, so that $\dim(X^{\lambda^\vee}) = |\lambda|$. The result then follows from Lemma B.3.10. □

B.4 The affine Grassmannian and affine Schur functions

We now introduce the symmetric functions specific to affine Type A. While approaching the MacDonald positivity conjecture, Lapointe, Lascoux, and Morse [LLM03] identified the "k-Schur functions", which behave analogously to the Schur functions in many ways. While several conjecturally equivalent definitions have appeared, we define the k-Schur functions as in [LM05] so that we may take their duality with the affine Schur functions for granted.

See Definition 3.2.1 and the surrounding discussion for basic definitions and observations involving hooks, cores, and boundaries. Recall in particular that the bijection between $(k+1)$-cores γ and k-bounded partitions λ, made explicit in Remark 3.2.10, is given by left-justifying the rows of $\partial\gamma$; several examples follow.

Definition B.4.1. Taking $|\gamma| = N$ and $a = (a_1, \ldots, a_m)$ a composition of N, a *(weak) k-tableau of shape γ and k-weight a* [LM08, LLMS10] is a filling T of γ such that, for each $i = 1, \ldots, m$,

- the boxes $\gamma^{(i)}$ filled with $1, \ldots, i$ constitute a $(k+1)$-core; and
- if $\lambda^{(i)}$ is the k-bounded partition corresponding to $\gamma^{(i)}$ and $(\mu^{(i)})'$ that corresponding to $(\gamma^{(i)})'$, then $\lambda^{(i)}/\lambda^{(i-1)}$ is a vertical a_i-strip and $\mu^{(i)}/\mu^{(i-1)}$ is a horizontal a_i-strip.

(Call μ the *k-conjugate of* λ.) Denote by $\mathcal{T}_a^{(k)}(\gamma)$ the collection of k-tableaux of shape γ and k-weight a. Define the *k-Kostka number* $K_{\lambda a}^{(k)} = |\mathcal{T}_a^{(k)}(\gamma)|$, where the k-bounded partition λ corresponds to the $(k+1)$-core γ. ◇

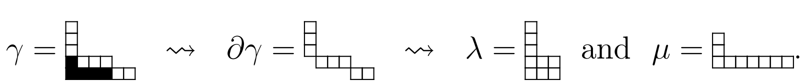
Example B.4.2. Take $\gamma = (7,4,1,1,1)$ and $a = (1,2,1,2,1,2)$. Then γ is a 4-core with residues

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 0 \\ \cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-4} 3 & 0 & 1 & 2 \\ \hline 0 & 1 & 2 & 3 & 0 & 1 & 2 \\ \hline\end{array}$$

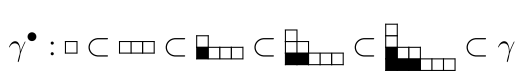
and 3-tableaux

$$\mathcal{T}_a^{(3)}(\gamma) = \left\{ \begin{array}{|c|c|c|c|c|c|c|}\hline 6 \\ \cline{1-1} 5 \\ \cline{1-1} 4 \\ \cline{1-4} 3 & 4 & 5 & 6 \\ \hline 1 & 2 & 2 & 3 & 4 & 5 & 6 \\ \hline\end{array}\ ,\ \begin{array}{|c|c|c|c|c|c|c|}\hline 6 \\ \cline{1-1} 5 \\ \cline{1-1} 3 \\ \cline{1-4} 2 & 4 & 5 & 6 \\ \hline 1 & 2 & 3 & 4 & 4 & 5 & 6 \\ \hline\end{array} \right\}.$$

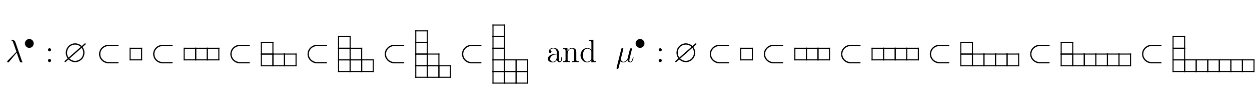
Under the bijection, γ corresponds to the 3-bounded partition $\lambda = (3,3,1,1,1)$ and k-conjugate $\mu = (7,1,1)$ as



The left 3-tableau above corresponds to the sequence



of 4-cores, with boxes of hook-length ≥ 4 filled, and to the sequences



of 3-bounded partitions. ◇

Proposition B.4.3 ([LM07]). *The k-Kostka numbers satisfy*

$$K^{(k)}_{\lambda\mu} = \begin{cases} 1 & \mu = \lambda \\ 0 & \mathfrak{c}(\mu) \lhd \mathfrak{c}(\lambda), \end{cases}$$

where $\mathfrak{c}$ takes a k-bounded partition to its associated $(k+1)$-core [LM07].

Definition B.4.4. Fix $k \geq 1$. Define the *k-Schur functions* $s^{(k)}_\lambda \in \Lambda$ by inverting the unitriangular system

$$h_\mu = \sum K^{(k)}_{\lambda\mu} s^{(k)}_\lambda \qquad \text{for } \lambda_1,\ \mu_1 \leq k.$$

$\diamond$

We may also index the k-Schur functions by Grassmannian affine permutations. Fix $n \geq 2$. Recall the action of $\tilde{S}_n$ on $\mathbb{Z}$ leading up to Theorem 3.1.10. $w \in \tilde{S}_n$ has one-line notation $[w(1), w(2), \cdots, w(n)]$, from which any $w(i+cn) = w(i) + cn$ may be recovered from $w(i)$ when $i \in [n]$. If $w \in \mathcal{W}^{\mathcal{P}}$ then the almost natural set $I = w(\mathbb{Z}_{>0})$ corresponds to a partition γ, ignoring the infinite contiguous absent indices before I and present indices after $\mathbb{Z} \smallsetminus I$. This γ is shown to be an n-core in Section 3.2. If λ is its associated $(n-1)$-bounded partition then we write $s^{(n-1)}_w = s^{(n-1)}_\lambda$. Henceforth, as in the main text, we substitute $n-1$ for k.

Proposition B.4.5 ([LM05]). *For any $(n-1)$-bounded partition λ,*

$$s^{(n-1)}_\lambda = s_\lambda + \sum_{\mu \lhd \lambda} d^{(n-1)}_{\lambda\mu} s_\mu, \qquad d^{(n-1)}_{\lambda\mu} \in \mathbb{Z}.$$

This fact—that the $(n-1)$-Schur functions expand integrally, positively, and triangularly into Schur functions, will inform Conjecture 5.3.15 in the main text.

Definition B.4.6. In a separate generalization of the Schur functions, define the *affine Schur functions* $F^{(n-1)}_w$ as the generating functions for k-tableaux,

$$F^{(n-1)}_w = F^{(n-1)}_\lambda = \sum_T x^T,$$

using the $(n-1)$-bounded partition λ associated with w and the k-weight $x^T = {x_1}^{a_1} \cdots {x_N}^{a_N}$ of Definition B.4.1. $\diamond$

Example B.4.7. Examples 3.1.9, 3.2.4, and 3.2.7 trace the combinatorial geometry of an (embedded) affine Schubert variety. The 2-bounded partition of $w = [-2, 2, 6]$ is $(2, 1, 1)$ and the 3-core of w is $(3, 1, 1)$, which has the 2-tableaux

$$\mathcal{T}^{(2)}\left(\begin{array}{|c|cc} \cline{1-1} \ \\ \cline{1-1} \ \\ \hline \ & \multicolumn{1}{c|}{\ } & \multicolumn{1}{c|}{\ } \\ \hline \end{array}\right) = \left\{ \begin{array}{|c|cc} \cline{1-1} l \\ \cline{1-1} k \\ \hline i & \multicolumn{1}{c|}{j} & \multicolumn{1}{c|}{k} \\ \hline \end{array}\, , \ \begin{array}{|c|cc} \cline{1-1} k \\ \cline{1-1} j \\ \hline i & \multicolumn{1}{c|}{k} & \multicolumn{1}{c|}{l} \\ \hline \end{array} \right\}$$

across positive integers $i < j < k < l$. Accordingly, w is associated with the affine Schur function $F^{(2)}_{\begin{array}{|c|c} \cline{1-1} \ \\ \cline{1-1} \ \\ \hline \ & \multicolumn{1}{c|}{\ } \\ \hline \end{array}} = 2m_{\begin{array}{|c|} \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array}}$. $\diamond$

The affine Schur functions exhibit a unitriangular expansion into monomial symmetric functions, which implies that they dualize the k-Schur functions [Lam06].

Definition B.4.8. Designate the subring $\Lambda_{(n)} = \mathbb{Z}[h_1, \dots, h_{n-1}] \subset \Lambda$. By Definition B.4.4, the $(n-1)$-Schur functions comprise a basis for $\Lambda_{(n)}$. Take $\Lambda^{(n)} = \Lambda/\mathfrak{i}$ with $\mathfrak{i}$ the ideal generated by the m_λ for λ $(n-1)$-bounded. Note that the $m_\lambda + \mathfrak{i}$ with $\lambda_1 < n$ comprise a basis for $\Lambda^{(n)}$. By Proposition B.4.9, then, so do the $\tilde{F}_\lambda^{(n-1)} = F_\lambda^{(n-1)} + \mathfrak{i}$ with $\lambda_1 < n$. $\diamond$

Proposition B.4.9 ([LM08]). *The affine Schur functions satisfy*

$$F_\lambda^{(n-1)} = \sum_{\lambda \trianglelefteq \mu} K_{\lambda\mu}^{(n-1)} m_\mu.$$

Furthermore, under a nondegenerate bilinear form $\Lambda_{(n)} \times \Lambda^{(n)} \to \mathbb{Z}$ induced by the Hall inner product, and with $\lambda_1, \mu_1 < n$,

$$\langle s_\lambda^{(n-1)}, F_\mu^{(n-1)} \rangle = \delta_{\lambda\mu}.$$

The combinatorial machinery established, let us turn to the topology of the affine Grassmannian $\mathrm{Gr}_{\mathrm{SL}_n}$ of Definition 2.2.1, which is the loop group of $\mathrm{SL}_n(\mathbb{C})$.

The loop group topology of G lends insight into the topology of Gr_G. The *loop space* LX of a topological space X is the collection of continuous maps $S^1 \to X$, called *loops*, under the *compact open topology* given by the base of open sets $V(K, U) = \{f : S^1 \to X \mid f(K) \in U\}$ where $X \subset S^1$ is compact and $U \subset X$ is open. Given a base point $x_0 \in X$, we denote the subspace of *based loops*

$$\begin{aligned} f : S^1 &\to X \\ f(1) &= x_0, \end{aligned}$$

by $\Omega X \subset LG$. When $X = G$ is an algebraic group, we take ΩG to consist of loops based at the identity e, and LG and ΩG admit a group structure governed by the product $\mu\nu(t) = \mu(t)\nu(t)$. When, as in the case $G = \mathrm{SL}_n$, G is a simply connected simple complex Lie group, G has a maximal compact subgroup K to which it deformation retracts, and it is furthermore known that ΩK is homotopy equivalent to Gr_G.

The product induces a graded product on homology, imbuing $H_*(\mathrm{Gr}_G)$ with a ring structure. Bott [Bot58] showed that the homology and cohomology rings, with the cap product

$$\frown : H_j(\mathrm{Gr}_G) \times H^i(\mathrm{Gr}_G) \to H_{j-i}(\mathrm{Gr}_G),$$

form dual Hopf algebras [Bot58]. (When $G = \mathrm{SL}_n$ we have $K = \mathrm{SU}_n$.) Lam [Lam08] described the explicit ring isomorphisms

$$\begin{array}{ccc} \Lambda_{(n)} & \to & H_*(\mathrm{Gr}_{\mathrm{SL}_n}) \\ s_w^{(n-1)} & \mapsto & [X_w]_\bullet \end{array} \qquad \text{and} \qquad \begin{array}{ccc} H^*(\mathrm{Gr}_{\mathrm{SL}_n}) & \to & \Lambda^{(n)} \\ [X_w]^\bullet & \mapsto & \tilde{F}_w^{(n-1)} \end{array}, \tag{B.10}$$

providing the Schubert class representatives conjectured by Shimozono and Morse.

It is the goal of this project to retrieve explicit polynomial representatives for affine Schubert classes in homology via a natural geometric construction, as was done for cohomology in the finite case by Knutson and Miller [KM05]. We conjecture in Section 5.3 that these are the k-Schur functions.

Appendix C

Example calculations

C.1 Slide thresholds

This appendix computes the shuffle threshold $k_P(I)$ for several illustrative choices of P and I. Some are very specific and others more general, and along the way we state (somewhat) general results when they capture an insight that complements or motivates the main text. In each case, let $x \in \pi^{-1}(V)$.

Example C.1.1 (One heralding pivot, one row index)**.** Recall $w_{rn} = [-rn + (1+r), 2 + r, \ldots, n+r] \in \mathcal{W}^{\mathcal{P}}$ from Proposition 3.1.4. This permutation has n-core $\gamma(w_{rn}) = (r^{n-1}, (r-1)^{n-1}, \ldots, 1^{n-1})$. Any $V \in \Omega^{r,rn}_{w_{rn}} \subset \mathrm{Gr}_{r,rn}$ has a basis heralded by a single vector. Equivalently, if $x \in \pi^{-1}(V)$ then x has u-reduced form $(\, v \;\, t\,v \;\, \cdots \;\, t^{r-1}v \,)$. It follows that $\hat{P} = (1)$ and $P = (1, 1+n, \ldots, 1+(r-1)n)$.

Pick any $i \in [rn]$. Write $x = (\, x_1 \;\, \cdots \;\, x_r \,)$ (not necessarily reduced) and find $x_j \in E_1 \setminus E_2$. Set $k = \left\lfloor \frac{i-1}{n} \right\rfloor$, so that $t^c x_j$ vanishes if and only if $c > k$, and in particular $\mathrm{sh}^{k+1}_{(i),(j)}(x) = \Delta_{(i)}(t^{k+1}x_j) = 0$. This also implies that $\mathrm{sh}^{k}_{(i),(j)}(x) = \Delta_{(i)}(t^{k}x_j) = x_{i-kn,j}$ is the $(i-kn)^{\text{th}}$ coordinate of x_j, which in general will be nonzero. Thus $k_{w_{rn}}((i)) = \left\lfloor \frac{i-1}{n} \right\rfloor$.

More generally, if $w_{(r-1)n} < w \leq w_{rn}$ then again $V \in \Omega^{r,rn}_{w}$ has a single heralding vector. If $\hat{P} = (\hat{p})$ then $P = (\hat{p}, \hat{p}+n, \ldots, \hat{p}+(r-1)n)$, so given $i \in [rn]$ we get $k_w((i)) = \left\lfloor \frac{i-\hat{p}}{n} \right\rfloor$, similarly to before. ◇

Next we'll extend this "floor function" solution to the case of a larger row index set I. This will require us to consider the various ways that indices in I may be matched with indices in P so as to violate rank conditions for as large a value of k as possible. We will also have to take into account cancellation due to different shifts of the $t^c v$ aligning with each other.

Example C.1.2 (One heralding pivot, multiple row indices)**.** Take $w_{(r-1)n} < w \leq w_{rn}$ and other parameters as in Example C.1.1, but consider $I = (i_1, \ldots, i_\ell)$ with $\ell > 1$. First consider

$x = (\, x_1 \;\cdots\; x_r \,) = (\, v \; t\,v \;\cdots\; t^{r-1}v \,)$. Given J and $k > 0$, there are three ways that the determinantal summands $\Delta_I^J(t^{r_\nu}x_{j_\nu})$ that comprise $\mathrm{sh}_{I,J}^k(x)$ either vanish or cancel in pairs:

1. If some $r_{\nu'} > \left\lfloor \frac{i_{\nu'}-\hat{p}}{n} \right\rfloor - j_{\nu'} + 1$—which includes the straightforward case $r_{\nu'} > h - j_{\nu'}$—then $\Delta_I^J(t^{r_\nu}x_{j_\nu}) = 0$ since $t^{r_{\nu'}}x_{j_{\nu'}} = 0$.
2. If $\nu' \neq \nu''$ but $j_{\nu'} + r_{\nu'} = j_{\nu''} + r_{\nu''}$ then $\Delta_I^J(t^{r_\nu}x_{j_\nu}) = 0$ since $t^{r_{\nu'}}x_{j_{\nu'}} = t^{r_{\nu''}}x_{j_{\nu''}}$.
3. If $\nu' < \nu''$ but $j_{\nu'} + r_{\nu'} > j_{\nu''} + r_{\nu''}$ then take $R' \in \mathbb{N}_k^\ell$ to agree with R except by selecting $r'_{\nu'}$ and $r'_{\nu''}$ so that $j_{\nu'} + r'_{\nu'} = j_{\nu''} + r_{\nu''}$ and $j_{\nu''} + r'_{\nu''} = j_{\nu'} + r_{\nu'}$. Then $\Delta_I^J(t^{r'_\nu}x_{j_\nu}) = -\Delta_I^J(t^{r_\nu}x_{j_\nu})$ since the submatrices are related by the transposition of column ν' with ν''.

Let us take a moment to characterize the cancellation of Case 3.

Definition C.1.3. Fix P, and pick $R \in \mathbb{N}_k^\ell$. If $p_{\nu'} + r_{\nu'}n \geq p_{\nu'+1}$ for some $\nu' \in [\ell - 1]$ then say that R *straddles* P *(at* ν'*)*. Note that R straddles P at ν' whenever the inequality $p_{\nu'} + r_{\nu'}n \geq p_{\nu''}$ is satisfied for any $\nu'' > \nu'$. In any case that R straddles P, define $\tilde{R} \in \mathbb{N}_k^\ell$ as follows: Take ν' to be minimal so that R straddles P at ν' and set

$$\begin{aligned} \tilde{r}_{\nu'} &= (p_{\nu'+1} + r_{\nu'+1}n - p_{\nu'})/n \\ \tilde{r}_{\nu'+1} &= (p_{\nu'} + r_{\nu'}n - p_{\nu'+1})/n \\ \tilde{r}_\nu &= r_\nu \quad \text{for } \nu \notin \{\nu', \nu'+1\}. \end{aligned}$$

This proto-definition prefaces Definition 4.3.2. ◇

Lemma C.1.4. *Take* ν' *to be minimal so that* R *straddles* P *at* ν'*. Then* $\tilde{R}$ *straddles* P *at* ν' *but at no* $\nu'' < \nu'$*. Consequently, the map* $R \mapsto \tilde{R}$ *is an involution on* $\mathrm{strad}(P) := \{R \in \mathbb{N}_k^\ell \mid R$ *straddles* $P\}$.

Proof. By construction, we need only show that (a) $\tilde{R}$ straddles P at ν' and that (b) $\tilde{R}$ does not straddle P at any $\nu'' < \nu'$.

(a) $p_{\nu'} + \tilde{r}_{\nu'}n = p_{\nu'} + (p_{\nu'+1} + r_{\nu'+1}n - p_{\nu'}) = p_{\nu'+1} + r_{\nu'+1}n \geq p_{\nu'+1}$.

(b) Assume $\tilde{R}$ straddles P at ν'' with $\nu'' < \nu'$. Then $p_{\nu''} + r_{\nu''}n = p_{\nu''} + \tilde{r}_{\nu''}n \geq p_{\nu''+1}$, so R straddles P at ν'', contrary to hypothesis.

□

Each involutive pair $(R,\tilde{R})$ satisfies $\Delta_I^J(t^{\tilde{r}_\nu}x_{j_\nu})=-\Delta_I^J(t^{r_\nu}x_{j_\nu})$, so any value of k for which all R straddle $P=(\hat{p}<\hat{p}+n<\cdots<\hat{p}+(r-1)n)$ produces a shuffle that evaluates to zero at x. We leave to the reader the verification that, provided each $i_\nu\geq p_\nu=\hat{p}+(\nu-1)n$,

$$k_w(I)=\left\lfloor\frac{i_\ell-\hat{p}}{n}\right\rfloor-\ell+1$$

is the maximum value of k for which some determinantal summand of $\mathrm{sh}_{I,J}^k(x)$ does not cancel due to straddling. This also turns out to be the maximum k for which, at some J, $\mathrm{sh}_{I,J}^k(x)\neq 0$. (If any $i_\nu<p_\nu$ then $\mathrm{sh}_{I,J}^k$ violates rank conditions and vanishes anyway.) $\diamond$

Remark C.1.5. The gist of the main result, for full-rank matrices in Y_w, is already laid out: The maximum k for which $\mathrm{sh}_{I,J}^k\mid_{\Omega_w}\not\equiv 0$ (the shuffle threshold) is that which permits exactly one non-straddling $R\in\mathbb{N}_k^\ell$ (the maximum slide weight). $\diamond$

In the next case we take P to contain multiple heralding pivots. For simplicity we also suppose that I contains one index in the same congruence class (modulo n) as each heralding pivot (and greater than that pivot) in order to make the point that the highest-weight shuffles at I are obtained by most closely approximating this assignment. (In general it will be enough to consider only I with indices pairwise distinct modulo n.) We then invoke straddling to confine the combinatorial problem to one of matching I to $\hat{P}$.

Example C.1.6 (Multiple heralding pivots, congruent to their associated row indices)**.** Take $x\in\pi^{-1}(\Omega_w^{h,m})$ to be u-reduced with heralding pivots $\hat{P}=(\hat{p}_1<\cdots<\hat{p}_\ell)$ and heralding columns $x_{\hat{j}_\nu}\in E_{\hat{p}_\nu}\smallsetminus E_{\hat{p}_\nu+1}$. Set $c_\nu=\left\lfloor\frac{m-\hat{p}_\nu}{n}\right\rfloor$. Note that, for each ν, c_ν is the maximum power c for which $t^c x_{\hat{j}_\nu}\neq 0$. Let $k=\sum_\nu c_\nu$ and $i_\nu=\hat{p}_\nu+c_\nu n\in[m-n+1,m]$ across $1\leq\nu\leq\ell$, then take $I=(i_1,\ldots,i_\ell)$ (unsorted). We will show that $k=k_w(I)$.

Let $\hat{J}=(\hat{j}_1,\ldots,\hat{j}_\ell)$ and $R\in\mathbb{N}_k^\ell$. If $R\neq(c_1,\ldots,c_\ell)$ then there is a μ such that $r_\mu>c_\mu$, so that $t^{r_\mu}x_{\hat{j}_\mu}=0$ and $\Delta_I(t^{r_\mu}x_{\hat{j}_\mu})=0$. But $\Delta_I(t^{c_\mu}x_{\hat{j}_\mu})\neq 0$, which means that $\mathrm{sh}_{I,\hat{J}}^k(x)\neq 0$.

However, $\mathrm{sh}_{I,\hat{J}}^{k+1}(x)=0$ since every summand $\Delta_I(t^\nu x_{\hat{j}_\nu})$ vanishes. If instead we take $J=(j_1<\cdots<j_\ell)$ and suppose that $j_\nu\geq\hat{j}_\nu$ for all ν, then again $\mathrm{sh}_{I,J}^{k+1}(x)=0$ for the same reason. Suppose instead that $j_\nu<\hat{j}_\nu$ for some ν. Then there exist ν',ν'' such that $x_{j_{\nu''}}=t^c x_{j_{\nu'}}$.

Given J, for $1\leq\nu\leq\ell$ we define c'_ν as follows. If there are $c\geq 1$ and μ such that $x_{j_\mu}=t^c x_{j_\nu}$, let $c'_\nu=c-1$ for the maximum such c. Otherwise take $c'_\nu=\left\lfloor\frac{m-p_{j_\nu}}{n}\right\rfloor$. Set $k'=\sum_{\nu=1}^\ell c'_\nu$. Then for all $R\in\mathbb{N}_{k'}^\ell$ except $(c'_1,\ldots,c'_\ell)$, R straddles some $\hat{P}_\xi=(p_{j_\nu}\mid p_{j_\nu}\equiv\hat{p}_\xi)$, or else some $t^{r_\nu}x_{j_\nu}=0$. Meanwhile, if $R=(c'_1,\ldots,c'_\ell)$ then $\Delta_I^J(t^{r_\nu}x_{j_\nu})\neq 0$. Since $k'\leq k$, we may conclude that $k_w(I)=k$.

More generally, if $\Omega_w^{h,m}$ has heralding pivots $\hat{P}$ and we take a row subset $I=(i_1,\ldots,i_\ell)$ with each $i_\nu\equiv\hat{p}_\nu$ and $\hat{p}_\nu\leq i_\nu$, then

$$k_P(I)=\frac{1}{n}\sum_{\nu=1}^\ell(i_\nu-\hat{p}_\nu).$$

◇

The process in this case of maximizing k specifically to each column subset, then taking the maximum of these, motivates the eponymous Definition 4.3.14 of Section 4.3, which captures the combinatorial interplay between I and the various J—equivalently, when x is u-reduced, between I and P.

C.2 Gröbner bases, multidegrees, and punchcards

As discussed in Section 5.3, we used `Macaulay 2`, in particular the `primaryDecomposition` function, to compute punchcards for $Y_w^{h,m}$ across all choices of n, w, m, and h (writing $l = m - h$) our computing power could handle. The function `MASVdegxT` receives `n` $= n$, `gamma` $= \gamma$ the n-core of w, `a` $= l$, and `b` $= h$ (plus the variable `kk` explained below), then performs the following tasks:

1. Construct the ring `R` $= R = \mathbb{Z}/32003\mathbb{Z}[z_{ij} \mid i = 0, \ldots, m-1; j = 0, \ldots, h-1]$. `Macaulay 2` prefers to begin indexing at zero. The construction imbues R with the d–r revlex term order (see Section 5.4). (Matrix coordinate rings in `Macaulay 2` are automatically imbued with r–d revlex; we actually constructed $h \times m$ matrices and performed calculations sideways. The order happens to be graded, but this is irrelevant when working with homogeneous polynomials.) Following tradition (for instance [EGSS02]), we use the field `k` $= \mathbb{Z}/32003\mathbb{Z}$ to avoid filling the extra memory required to perform computations in $\mathbb{Q}$, with the prime 32003 large enough for the substitution to have no effect in our small examples.

2. Construct the ideal `I` $= \mathfrak{i}_w^{h,m} \subset R$ from the generators of Definition 4.3.14. We generate the vanishing minors (`ladderMinors`) of the containing matrix Schubert variety $Y_{h,m}^{\gamma^\vee}$ and the other vanishing shuffles (`windowShuffles`) separately, in each case including a generator for each suitably-sized column subset. So far we have only required one shuffle—the one having weight $k_w(I) + 1$—at each row subset I.

3. Using `monomialIdeal`, compute the initial ideal `J` $= J$ of I.

4. Using `primaryDecomposition`, compute the primary decomposition `P` of J.

5. Construct the symmetric algebra `T` $= \mathbb{Z}/32003\mathbb{Z}[x_0, \ldots, x_{h-1}]$.

6. With the custom function `multidegree`, compute the multidegree `xmdeg` $= f_w^{h,m}$ of `P` in `T`.

7. Construct an equivalent symmetric algebra `S` $\cong$ `T`, define the $h \times l$ rectangle complement map from `T` to `S`, and compute the rectangle complement $\mathrm{PD}_{h\times l}(f_w^{h,m})$. (Here we use Demazure character functions written by Mark Shimozono.)

We also computed $\sqrt{\mathfrak{i}_w^{h,m}}$ from $\mathfrak{i}_w^{h,m}$ whenever possible. All code is available upon request.

```
-- irredundant vanishing minors by size and a list of their leading terms
ladderMinors = (lambda,a,b,m,rows,pivs,R) -> (
  j := 0;
  F := {};
  G := {};
  rowsubs := {};
  usedsubs := {};
  keepsubs := {};
  while j < b do (
    j = j+1;
    prevsubs = flatten(usedsubs);
    keepsubs = {};
    subs := subsets(a+b,j);
    Ftemp := {};
    for i from 0 to #(subs) - 1 do (
      if (
        (subs_i)_(j-1) < pivs_(j-1) and not member(true,
        for k from 0 to #prevsubs-1 list isSubset(prevsubs_k,subs_i))
      )
      then (
        Ftemp = append(Ftemp,minors(j,submatrix(m,rows,subs_i)));
        keepsubs = append(keepsubs,subs_i)
      )
    );
    if sum(Ftemp) == 0 then F = append(F,ideal(0_R)) else (
      F = append(F,sum(Ftemp));
      leads := {};
      for k from 0 to #keepsubs-1 do (
        rowsubs = subsets(b,j);
        for l from 0 to #rowsubs-1 do (
          leads = append(leads,sqantidiag(m,rowsubs_l,keepsubs_k))
        )
      );
      if leads == {} then G = append(G,0_R) else G = append(G,ideal(leads))
    );
    usedsubs = append(usedsubs,keepsubs);
  );
  {F,G}
);
```

```
-- (max-k + 1)-shuffles on subsets confined to length-n windows
windowShuffles = (n, m, rows, cols, pivs, R) ->
( shufinds = {};
  Shtemp := {};
  Sh = {};
  for l from 2 to min(n, #rows) do
  ( syms := permutations(l);
    pivset := subsets(pivs, l);
    colset := subsets(cols, l);
    rowset := subsets(rows, l);
    for c from 0 to #colset-1 do if max(colset_c)-min(colset_c) < n then
    ( maxk := -1;
      for p from 0 to #pivset-1 do
      ( for w from 0 to #syms-1 do if
        all(colset_c, (pivset_p)_(syms_w), (x, y) -> x >= y) then
        ( buffers := for i from 0 to l-1 list
          ( b := ((pivset_p)_(syms_w))_i;
            while b+n < #cols and not member(b+n, pivset_p) do b = b+n;
            b
          );
          beads := for i from 0 to l-1 list
          floor((min((colset_c)_i, buffers_i)-((pivset_p)_(syms_w))_i)/n);
          if any(beads, x -> x < 0) then print {l, c, p, w};
          maxk = max(maxk, sum(beads))
        )
      );
      if maxk >= 0 then
      ( k := maxk+1;
        shufinds = append(shufinds, {colset_c, l, {k, k}});
        cv := subsetCV(l, k, k);
        for r from 0 to #rowset-1 do
        ( subdets := {};
          for v from 0 to #cv-1 do
          ( if isSubset(colset_c-n*cv_v, cols) then
            subdets = append(subdets, det(submatrix(m, rowset_r, colset_c-n*cv_v)))
          );
          if sum(subdets) != 0 then Shtemp = append(Shtemp, ideal(sum(subdets)))
        )
      )
    );
    if sum(Shtemp) == 0 then
    Sh = append(Sh, ideal(0_R)) else Sh = append(Sh, sum(Shtemp))
```

```
  );
  {Sh,shufinds}
);

-- calculate multidegree from primary decomposition and rings
multidegree = (P,R,T,wts) -> (
  wt := map(T,R,wts);
  M := 0_T;
  for i from 0 to #P-1 do (
    genP := generators(P_i);
    dimn := rank source genP;
    term := 1_T;
    for j from 0 to dimn-1 do ( term = term*wt(genP_(0,j)) );
    M = M + term;
  );
  M
);

MASVMdegxT = (n,gamma,a,b,kk) -> (
  gammaT = transposePartition(gamma);
  if a < gammaT_0 then a = gammaT_0;
  if b <= #gammaT then b = #gammaT else
  gammaT = join(gammaT,for i from #(gammaT) to b - 1 list 0);
  lambda = apply(1..#gammaT, i -> a - gammaT_(#gammaT-i));
  rows = inds(b);
  cols = inds(a+b);
  pivs = append( for i from 0 to #(lambda) - 1 list
  i + lambda_(#(lambda) - 1 - i) , a+b);
  pivsn := join(pivs,last(pivs)+1..last(pivs)+n);
  esslpivs = {};
  for i from 0 to #pivsn-1 do (if not(member(0,apply(esslpivs,j->(pivsn_i-j)%n)))
  then esslpivs=append(esslpivs,pivsn_i));
  -- changes: switch variable indices in ring & table --
  R = kk[z_(0,0)..z_(a+b-1,b-1)];
  m = matrix table(b,a+b,(i,j)->z_(j,i));
  stuff = ladderMinors(lambda,a,b,m,rows,pivs,R);
  F = stuff_0;
  shuff = windowShuffles(n,m,rows,cols,pivs,esslpivs,R);
  Sh = shuff_0;
  I = idealSum(F,Sh,R);
  J = monomialIdeal(I);
  if J == 0 then P = {J} else P = primaryDecomposition(J);
```

```
  T = kk[x_1..x_b];
  xwts = flatten(for i from 1 to a+b list (for j from 1 to b list (x_j)_T));
  xmdeg = multidegree(P,R,T,xwts);
  S = kk[x_0..x_(b-1)];
  xndeg = (isom(S,T))(xmdeg);
  xExpn = demCharExpansion(xndeg,S);
  PDxExpn = PDCharsIg(xExpn,b,a);
  PDxndeg = demCharPoly(PDxExpn,b,S);
  PDxmdeg = (isom(T,S))(PDxndeg)
);
```

To check the Schur expansions of multidegrees against ω-involuted $(n-1)$-Schur functions, we used the `*-combinat` patches to the open-source system `Sage`.

All calculations were consistent with the following:

- $\sqrt{\mathfrak{i}_w^{h,m}} = \mathfrak{i}_w^{h,m}$ (Conjecture 4.4.9);
- $V(\mathsf{in}(\mathfrak{i}_w^{h,m}))$ is equidimensional, i.e. every $D \in (\mathcal{P}_w^{h,m})^*$ has the same number of punches (consistent with Conjecture 5.4.9);
- $\mathrm{PD}_{h\times l}(\mathrm{mdeg}_M\, Y_w^{h,m}) = \omega(s_w^{(n-1)})(x_1,\ldots,x_h)$ (Conjecture 5.3.15);
- $(\mathcal{P}_w^{h,m})^*$ is chute-closed (Conjecture 5.4.11);
- Given $m' > h' > 0$ with $h' \geq h$ and $l' = m' - h' \geq l$, the $D' \in (\mathcal{P}_w^{h',m'})^*_{\mathrm{top}}$ biject with the $D \in (\mathcal{P}_w^{h,m})^*_{\mathrm{top}}$ as in Definition 5.4.15 (Conjecture 5.4.18).

Moreover, if we performed calculations in the rectangle $h' \times l'$ then we performed them in every rectangle contained in $h' \times l'$ and containing $\gamma_1 \times \gamma_1'$. We therefore summarize the results by presenting, for each $w \in \mathcal{W}^{\mathcal{P}}$,

1. a note of the largest dimension h' of a rectangle in which calculations were performed (extending $m-h$ is computationally trivial);
2. the generators of $i_w^{h,m}$, with a note as to whether we verified that $\sqrt{\mathfrak{i}_w^{h,m}} = \mathfrak{i}_w^{h,m}$;
3. the top punchcards $(\mathcal{P}_w^{h,m})^*_{\mathrm{top}}$, using $h = \gamma_1$ and $l = \gamma_1'$;
4. the Schur expansion of $f_w^{(n-1)}$, grouped by which terms in the polynomial localization arise from which chute-connected components of $(\mathcal{P}_w^{h,m})^*$ (recall that the correspondence passes through a transpose), and whether it equals the involuted $(n-1)$-Schur function $s_\lambda^{(n-1)}$; and

5. where performed, the T-stable partial Gröbner degeneration of $Y_w^{h,m}$ into matrix Stanley varieties.

The data for $w = [-1, 4]$ is complete and fully labeled as a guide.

$n = 2$

$$w = [0, 3], \quad \gamma = \ydiagram{1}, \quad \lambda = \ydiagram{1}, \quad h' = 5$$

$$(\mathcal{P}_w^{1,2})^*_{\text{top}} = \left\{ \begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix} \right\}$$

$$\text{PD}_{1\times 1}[Y_w^{1,2}]^T \;=\; s_{\ydiagram{1}} \;=\; \omega(s^{(1)}_{\ydiagram{1}})$$

Permutation, n-core, $(n-1)$-bounded partition, and largest dimension h':

$$w = [-1, 4], \quad \gamma = \ydiagram{1,2}, \quad \lambda = \ydiagram{1,1}, \quad h' = 5$$

Ideal (and whether radical):

$$\mathfrak{i}_w^{2,4} = (\Delta_{12}, \text{sh}^1_{34}) = \sqrt{\mathfrak{i}_w^{2,4}}$$

T-stable Gröbner degeneration and (additional) leading minors:

$$\begin{array}{ccccc} \begin{pmatrix} 1 & \\ a & \\ & 1 \\ b & a \end{pmatrix} & \rightsquigarrow & \begin{pmatrix} 1 & \\ * & 0 \\ * & 0 \\ & 1 \end{pmatrix} & & \begin{pmatrix} 1 & \\ 0 & 0 \\ & 1 \\ * & * \end{pmatrix} \\ \Delta_{12}, \Delta_{23} & \rightsquigarrow & \Delta_{13} & & \Delta_2 \end{array}$$

Top punchcards:

$$(\mathcal{P}_w^{2,4})^*_{\text{top}} = \left\{ \begin{smallmatrix} \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}, \; \begin{smallmatrix} \cdot & \cdot \\ \square & \square \\ \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix} \right\}$$

Rectangle-complemented multidegree:

$$\text{PD}_{2\times 2}[Y_w^{2,4}]^T \;=\; s_{\ydiagram{2}} \;+\; s_{\ydiagram{1,1}} \;=\; \omega(s^{(1)}_{\ydiagram{1,1}})$$

$$w = [-2, 5], \quad \gamma = \ydiagram{1,2,3}, \quad \lambda = \ydiagram{1,1}, \quad h' = 5$$

$$\mathfrak{i}_w^{3,6} = (\Delta_{12}, \Delta_{134}, \Delta_{234}, \mathrm{sh}_{34}^1, \mathrm{sh}_{56}^2) = \sqrt{\mathfrak{i}_w^{3,6}}$$

$$\begin{array}{ccccc}
\begin{pmatrix} 1 & & \\ a & & \\ & 1 & \\ b & a & \\ & & 1 \\ c & b & a \end{pmatrix} & \rightsquigarrow &
\begin{pmatrix} 1 & & \\ * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \\ & 1 & \\ & & 1 \end{pmatrix} &
\begin{pmatrix} 1 & & \\ * & 0 & 0 \\ 0 & 0 & 0 \\ & 1 & \\ * & * & 0 \\ & & 1 \end{pmatrix} &
\begin{pmatrix} 1 & & \\ 0 & 0 & 0 \\ & 1 & \\ 0 & * & 0 \\ * & * & 0 \\ & & 1 \end{pmatrix} \quad
\begin{pmatrix} 1 & & \\ 0 & 0 & 0 \\ & 1 & \\ 0 & 0 & 0 \\ & & 1 \\ * & * & * \end{pmatrix} \\
\begin{array}{c}\Delta_{12}, \Delta_{23}, \Delta_{34}, \\ \Delta_{145}, \Delta_{245}\end{array} & \rightsquigarrow &
\begin{array}{c}\Delta_{13}, \Delta_{14}, \\ \Delta_{24}\end{array} & \Delta_3 & \Delta_2, \Delta_{135} \quad \Delta_2, \Delta_4
\end{array}$$

$$(\mathcal{P}_w^{3,6})^*_{\mathrm{top}} = \left\{
\begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array},
\begin{array}{ccc} \cdot & \square & \square \\ \cdot & \cdot & \square \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array},
\begin{array}{ccc} \cdot & \cdot & \square \\ \square & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array},
\begin{array}{ccc} \cdot & \cdot & \cdot \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}
\right\}$$

$$\mathrm{PD}_{3\times 3}[Y_w^{3,6}]^T \;=\; s_{\ydiagram{3}} \;+\; s_{\ydiagram{2,1}} \;+\; s_{\ydiagram{2,1}} \;+\; s_{\ydiagram{1,1,1}} \;=\; \omega(s_{\ydiagram{1,1,1}}^{(1)})$$

$$w = [-3,6], \quad \gamma = \yng(1,2,4), \quad \lambda = \yng(1,1), \quad h' = 5$$

$$\mathfrak{i}_w^{4,8} = (\Delta_{12}, \Delta_{134}, \Delta_{234}, \Delta_{1356}, \Delta_{2356}, \Delta_{1456}, \Delta_{2456}, \Delta_{3456}, \mathrm{sh}_{34}^1, \mathrm{sh}_{56}^2, \mathrm{sh}_{78}^3)$$

$$\begin{pmatrix} 1 & & & \\ a & & & \\ & 1 & & \\ b & a & & \\ & & 1 & \\ c & b & a & \\ & & & 1 \\ d & c & b & a \end{pmatrix} \quad \rightsquigarrow$$

$$\begin{gathered} \Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{45}, \\ \Delta_{156}, \Delta_{256}, \Delta_{356}, \\ \Delta_{1467}, \Delta_{2467} \end{gathered} \quad \rightsquigarrow$$

$$\underset{\substack{\Delta_{13},\Delta_{14},\Delta_{15},\\ \Delta_{24},\Delta_{25},\Delta_{35}}}{\begin{pmatrix} 1 \\ *000 \\ *000 \\ *000 \\ *000 \\ \ \ 1 \\ \ \ \ \ 1 \\ \ \ \ \ \ \ 1 \end{pmatrix}} \quad \underset{\Delta_4,\Delta_{13}}{\begin{pmatrix} 1 \\ *000 \\ *000 \\ 0000 \\ \ \ 1 \\ **00 \\ \ \ \ \ 1 \\ \ \ \ \ \ \ 1 \end{pmatrix}} \quad \underset{\substack{\Delta_3,\\ \Delta_{146},\Delta_{246}}}{\begin{pmatrix} 1 \\ *000 \\ 0000 \\ \ \ 1 \\ 0*00 \\ **00 \\ \ \ \ \ 1 \\ \ \ \ \ \ \ 1 \end{pmatrix}} \quad \underset{\substack{\Delta_3,\Delta_5,\\ \Delta_{12}}}{\begin{pmatrix} 1 \\ *000 \\ 0000 \\ \ \ 1 \\ 0000 \\ \ \ \ \ 1 \\ ***0 \\ \ \ \ \ \ \ 1 \end{pmatrix}} \quad \underset{\substack{\Delta_2,\Delta_{35},\\ \Delta_{136},\Delta_{146}}}{\begin{pmatrix} 1 \\ 0000 \\ \ \ 1 \\ 0*00 \\ 0*00 \\ **00 \\ \ \ \ \ 1 \\ \ \ \ \ \ \ 1 \end{pmatrix}} \quad \underset{\substack{\Delta_2,\Delta_4,\\ \Delta_{135},\Delta_{136}}}{\begin{pmatrix} 1 \\ 0000 \\ \ \ 1 \\ 0000 \\ **00 \\ **00 \\ \ \ \ \ 1 \\ \ \ \ \ \ \ 1 \end{pmatrix}} \quad \underset{\substack{\Delta_2,\Delta_5,\\ \Delta_{1367}}}{\begin{pmatrix} 1 \\ 0000 \\ \ \ 1 \\ 0*00 \\ 0000 \\ \ \ \ \ 1 \\ ***0 \\ \ \ \ \ \ \ 1 \end{pmatrix}} \quad \underset{\substack{\Delta_2,\Delta_4,\Delta_{56},\\ \Delta_{1357},\Delta_{1367}}}{\begin{pmatrix} 1 \\ 0000 \\ \ \ 1 \\ 0000 \\ \ \ \ \ 1 \\ 00*0 \\ ***0 \\ \ \ \ \ \ \ 1 \end{pmatrix}} \quad \underset{\Delta_2,\Delta_4,\Delta_6}{\begin{pmatrix} 1 \\ 0000 \\ \ \ 1 \\ 0000 \\ \ \ \ \ 1 \\ 0000 \\ \ \ \ \ \ \ 1 \\ **** \end{pmatrix}}$$

$$(\mathcal{P}_w^{4,8})^*_{\mathrm{top}} = \left\{ \begin{matrix} \cdot\,\square\,\square\,\square \\ \cdot\,\square\,\square\,\square \\ \cdot\,\square\,\square\,\square \\ \cdot\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \end{matrix}, \begin{matrix} \cdot\,\square\,\square\,\square \\ \cdot\,\square\,\square\,\square \\ \cdot\;\cdot\,\square\,\square \\ \square\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \end{matrix}, \begin{matrix} \cdot\,\square\,\square\,\square \\ \cdot\;\cdot\,\square\,\square \\ \square\,\square\,\square\,\square \\ \cdot\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \end{matrix}, \begin{matrix} \cdot\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\,\square \\ \square\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \square\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \end{matrix}, \begin{matrix} \cdot\;\cdot\,\square\,\square \\ \square\,\square\,\square\,\square \\ \cdot\,\square\,\square\,\square \\ \cdot\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \end{matrix}, \begin{matrix} \cdot\;\cdot\,\square\,\square \\ \square\,\square\,\square\,\square \\ \cdot\;\cdot\,\square\,\square \\ \square\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \end{matrix}, \begin{matrix} \cdot\;\cdot\;\cdot\,\square \\ \square\,\square\,\square\,\square \\ \cdot\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \square\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \end{matrix}, \begin{matrix} \cdot\;\cdot\;\cdot\,\square \\ \square\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \square\,\square\,\square\,\square \\ \cdot\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \end{matrix}, \begin{matrix} \cdot\;\cdot\;\cdot\;\cdot \\ \square\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \square\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \square\,\square\,\square\,\square \\ \cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{4\times 4}[Y_w^{4,8}]^T \;=\; s_{\yng(4)} + s_{\yng(3,1)} + \left(\, s_{\yng(2,2)} + s_{\yng(3,1)} \,\right) + s_{\yng(2,1,1)} + s_{\yng(3,1)} + s_{\yng(2,2)} + s_{\yng(2,1,1)} + s_{\yng(2,1,1)} + s_{\yng(1,1,1)} \;=\; \omega(s^{(1)}_{\yng(1,1,1,1)})$$

$$w = [-4, 7], \quad \gamma = \ydiagram{1,2,3,4,5}, \quad \lambda = \ydiagram{1,1,1,1,1}, \quad h' = 5$$

$$(\mathcal{P}_w^{5,10})^*_{\mathrm{top}} = \left\{ \begin{array}{ccccccc}
\begin{matrix} \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\square\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\cdot\cdot\square\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\cdot\cdot\cdot\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\cdot\cdot\cdot\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\cdot\cdot\square\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix}, \\[1em]
\begin{matrix} \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\cdot\cdot\cdot\square \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\cdot\cdot\square\square \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\cdot\cdot\square\square \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\square\square\square\square \\ \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix}, \\[1em]
\begin{matrix} \cdot\square\square\square\square \\ \cdot\cdot\cdot\square\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\square\square\square\square \\ \cdot\cdot\cdot\square\square \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\square\square\square\square \\ \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\square\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\square\square\square \\ \square\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix},
& \begin{matrix} \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\square\square\square\square \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \\ \cdot\cdot\cdot\cdot\cdot \end{matrix}
\end{array} \right\}$$

$$\begin{aligned}
\mathrm{PD}_{5\times 5}[Y_w^{5,10}]^T &= s_{\ydiagram{2,2,1}} + s_{\ydiagram{3,2}} + s_{\ydiagram{2,2,1}} + s_{\ydiagram{1,1,1,1,1}} + s_{\ydiagram{2,1,1,1}} + s_{\ydiagram{2,1,1,1}} + s_{\ydiagram{3,1,1}} \\
&\quad + s_{\ydiagram{3,2}} + s_{\ydiagram{2,1,1,1}} + \left(s_{\ydiagram{2,2,1}} + s_{\ydiagram{3,1,1}} \right) + s_{\ydiagram{3,1,1}} + s_{\ydiagram{4,1}} + s_{\ydiagram{3,2}} + s_{\ydiagram{2,1,1,1}} \\
&\quad + \left(s_{\ydiagram{2,2,1}} + s_{\ydiagram{3,1,1}} \right) + \left(s_{\ydiagram{2,2,1}} + s_{\ydiagram{3,1,1}} \right) + \left(s_{\ydiagram{3,2}} + s_{\ydiagram{4,1}} \right) + s_{\ydiagram{3,1,1}} + \left(s_{\ydiagram{3,2}} + s_{\ydiagram{4,1}} \right) + s_{\ydiagram{4,1}} + s_{\ydiagram{5}} \\
&= \omega\left(s^{(1)}_{\ydiagram{1,1,1,1,1}}\right)
\end{aligned}$$

$n = 3$

$$w = [0, 2, 4], \quad \gamma = \ydiagram{1}, \quad \lambda = \ydiagram{1}, \quad h' = 3$$

$$(\mathcal{P}_w^{1,2})^*_{\mathrm{top}} = \left\{ \begin{matrix} \cdot \\ \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{1\times 1}[Y_w^{1,2}]^T = s_{\ydiagram{1}} = \omega\left(s^{(2)}_{\ydiagram{1}}\right)$$

$$w = [-1,3,4], \quad \gamma = \ydiagram{1,1}, \quad \lambda = \ydiagram{1,1}, \quad h' = 3$$

$$(\mathcal{P}_w^{1,3})^*_{\text{top}} = \left\{ \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{1\times 2}[Y_w^{1,3}]^T \;=\; s_{\ydiagram{2}} \;=\; \omega(s^{(2)}_{\ydiagram{1,1}})$$

$$w = [0,1,5], \quad \gamma = \ydiagram{2}, \quad \lambda = \ydiagram{2}, \quad h' = 4$$

$$(\mathcal{P}_w^{2,3})^*_{\text{top}} = \left\{ \begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{2\times 1}[Y_w^{2,3}]^T \;=\; s_{\ydiagram{1,1}} \;=\; \omega(s^{(2)}_{\ydiagram{2}})$$

$$w = [-2,3,5], \quad \gamma = \ydiagram{1,1,2}, \quad \lambda = \ydiagram{1,1,1}, \quad h' = 4$$

$$\mathfrak{i}_w^{2,5} = (\Delta_{12}, \Delta_{13}, \Delta_{23}, \mathrm{sh}^1_{45}) = \sqrt{\mathfrak{i}_w^{2,5}}$$

$$\begin{array}{ccccc}
\begin{pmatrix} 1 & \\ a & \\ b & \\ & 1 \\ c & a \end{pmatrix} & \rightsquigarrow & \begin{pmatrix} 1 & \\ 0 & 0 \\ * & 0 \\ & 1 \\ * & * \end{pmatrix} & \begin{pmatrix} 1 & \\ * & 0 \\ * & 0 \\ * & 0 \\ & 1 \end{pmatrix} \\
\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{24} & \rightsquigarrow & \Delta_2 & \Delta_{14}, \Delta_{34}
\end{array}$$

$$(\mathcal{P}_w^{2,5})^*_{\text{top}} = \left\{ \begin{matrix} \cdot & \square \\ \square & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}, \quad \begin{matrix} \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{2\times 3}[Y_w^{2,5}]^T \;=\; s_{\ydiagram{2,1}} + s_{\ydiagram{3}} \;=\; \omega(s^{(2)}_{\ydiagram{1,1,1}})$$

$$w = [-1, 1, 6], \quad \gamma = \ydiagram{1,3}, \quad \lambda = \ydiagram{1,2}, \quad h' = 5$$

$$\mathfrak{i}_w^{3,5} = (\Delta_{12}, \mathrm{sh}^1_{345}) = \sqrt{\mathfrak{i}_w^{3,5}}$$

$$\begin{pmatrix} 1 & & \\ a & & \\ & 1 & \\ & & 1 \\ b & \alpha & a \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & & \\ * & 0 & 0 \\ & 1 & \\ * & * & 0 \\ & & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & & \\ 0 & 0 & 0 \\ & 1 & \\ & & 1 \\ * & * & * \end{pmatrix}$$

$$\Delta_{12}, \Delta_{234} \quad \rightsquigarrow \quad \Delta_{134} \qquad \Delta_2$$

$$(\mathcal{P}_w^{3,5})^*_{\mathrm{top}} = \left\{ \begin{matrix} \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}, \quad \begin{matrix} \cdot & \cdot & \cdot \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{3\times 2}[Y_w^{3,5}]^T \;=\; s_{\ydiagram{2,1}} + s_{\ydiagram{1,1,1}} \;=\; \omega(s^{(2)}_{\ydiagram{1,2}})$$

$$w = [-3, 4, 5], \quad \gamma = \ydiagram{1,1,2,2}, \quad \lambda = \ydiagram{1,1,1,1}, \quad h' = 4$$

$$\mathfrak{i}_w^{2,6} = (\Delta_{12}, \Delta_{13}, \Delta_{23}, \mathrm{sh}^1_{45}, \mathrm{sh}^1_{46}, \mathrm{sh}^1_{56}) = \sqrt{\mathfrak{i}_w^{2,6}}$$

$$\begin{pmatrix} 1 & \\ a & \\ b & \\ & 1 \\ c & a \\ d & b \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & \\ * & 0 \\ 0 & 0 \\ * & 0 \\ & 1 \\ * & * \end{pmatrix} \quad \begin{pmatrix} 1 & \\ * & 0 \\ * & 0 \\ * & 0 \\ * & 0 \\ & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \\ 0 & 0 \\ 0 & 0 \\ & 1 \\ * & * \\ * & * \end{pmatrix}$$

$$\begin{matrix} \Delta_{12}, \Delta_{13}, \Delta_{23}, \\ \Delta_{24}, \Delta_{34}, \Delta_{35} \end{matrix} \quad \rightsquigarrow \quad \Delta_3, \Delta_{14} \qquad \begin{matrix} \Delta_{14}, \Delta_{15}, \\ \Delta_{25}, \Delta_{45} \end{matrix} \qquad \Delta_2, \Delta_3$$

$$(\mathcal{P}_w^{2,6})^*_{\mathrm{top}} = \left\{ \begin{matrix} \cdot & \square \\ \cdot & \square \\ \square & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}, \quad \begin{matrix} \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}, \quad \begin{matrix} \cdot & \cdot \\ \square & \square \\ \square & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{2\times 4}[Y_w^{2,6}]^T \;=\; s_{\ydiagram{3,1}} \;+\; s_{\ydiagram{4}} \;+\; s_{\ydiagram{2,2}} \;=\; \omega(s^{(2)}_{\ydiagram{1,1,1,1}})$$

$$w = [-2,2,6], \quad \gamma = \ydiagram{1,3}, \quad \lambda = \ydiagram{1,2}, \quad h' = 4$$

$$\mathfrak{i}_w^{3,6} = (\Delta_{12}, \Delta_{13}, \Delta_{23}, \mathrm{sh}^1_{456}) = \sqrt{\mathfrak{i}_w^{3,6}}$$

$$\begin{array}{ccccc}
\begin{pmatrix} 1 & & \\ a & & \\ b & & \\ & 1 & \\ & a & 1 \\ c & b & \alpha \end{pmatrix} & \rightsquigarrow & \begin{pmatrix} 1 & & \\ * & 0 & 0 \\ * & 0 & 0 \\ & 1 & \\ * & * & 0 \\ & & 1 \end{pmatrix} & & \begin{pmatrix} 1 & & \\ * & 0 & 0 \\ 0 & 0 & 0 \\ & 1 & \\ & & 1 \\ * & * & * \end{pmatrix} \\
\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{345} & \rightsquigarrow & \Delta_{145}, \Delta_{245} & & \Delta_3
\end{array}$$

$$(\mathcal{P}_w^{3,6})^*_{\mathrm{top}} = \left\{ \begin{matrix} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}, \quad \begin{matrix} \cdot & \square & \square \\ \cdot & \cdot & \cdot \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{3\times 3}[Y_w^{3,6}]^T \;=\; s_{\ydiagram{3,1}} \;+\; s_{\ydiagram{2,1,1}} \;=\; \omega(s^{(2)}_{\ydiagram{1,1,2}})$$

$$w = [-1,0,7], \quad \gamma = \ydiagram{2,4}, \quad \lambda = \ydiagram{2,2}, \quad h' = 4$$

$$\mathfrak{i}_w^{4,6} = (\Delta_{123}, \mathrm{sh}^1_{456}) = \sqrt{\mathfrak{i}_w^{4,6}}$$

$$\begin{array}{cccccc}
\begin{pmatrix} 1 & & & \\ & 1 & & \\ a & \alpha & & \\ & & 1 & \\ & & & 1 \\ b & \beta & a & \alpha \end{pmatrix} & \rightsquigarrow & \begin{pmatrix} 1 & & & \\ & 1 & & \\ * & * & 0 & 0 \\ * & * & 0 & 0 \\ & & 1 & \\ & & & 1 \end{pmatrix} & \begin{pmatrix} 1 & & & \\ & 1 & & \\ 0 & * & 0 & 0 \\ & & 1 & \\ * & * & * & 0 \\ & & & 1 \end{pmatrix} & \begin{pmatrix} 1 & & & \\ & 1 & & \\ 0 & 0 & 0 & 0 \\ & & 1 & \\ & & & 1 \\ * & * & * & * \end{pmatrix} \\
\Delta_{123}, \Delta_{234}, \Delta_{1345} & \rightsquigarrow & \Delta_{124}, \Delta_{134} & \Delta_{23}, \Delta_{1245} & \Delta_3
\end{array}$$

$$(\mathcal{P}_w^{4,6})^*_{\mathrm{top}} = \left\{ \begin{matrix} \cdot & \cdot & \square & \square \\ \cdot & \cdot & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}, \quad \begin{matrix} \cdot & \cdot & \cdot & \square \\ \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}, \quad \begin{matrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{4\times 2}[Y_w^{4,6}]^T \;=\; s_{\ydiagram{2,2}} \;+\; s_{\ydiagram{2,1,1}} \;+\; s_{\ydiagram{1,1,1,1}} \;=\; \omega(s^{(2)}_{\ydiagram{2,2}})$$

$$w = [-4,4,6], \quad \gamma = \ydiagram{1,1,2,2,3}, \quad \lambda = \ydiagram{1,1,1,1,1}, \quad h' = 4$$

$$\mathfrak{i}_w^{3,8} = (\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{145}, \Delta_{146}, \Delta_{156}, \Delta_{245}, \Delta_{246}, \Delta_{256}, \Delta_{345}, \Delta_{346}, \Delta_{356}, \Delta_{456},$$
$$\mathrm{sh}^1_{45}, \mathrm{sh}^1_{46}, \mathrm{sh}^1_{56}, \mathrm{sh}^2_{78}, \mathrm{sh}^2_{79}, \mathrm{sh}^2_{89}, \mathrm{sh}^1_{178}, \mathrm{sh}^1_{179}, \mathrm{sh}^1_{189}, \mathrm{sh}^1_{278}, \mathrm{sh}^1_{279}, \mathrm{sh}^1_{289}, \mathrm{sh}^1_{378}, \mathrm{sh}^1_{379}, \mathrm{sh}^1_{389}) = \sqrt{\mathfrak{i}_w^{3,8}}$$

$$(\mathcal{P}_w^{3,8})^*_{\mathrm{top}} = \left\{ \begin{array}{ccc}\cdot&\square&\square\\ \square&\square&\square\\ \cdot&\square&\square\\ \cdot&\square&\square\\ \cdot&\cdot&\square\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{array}, \begin{array}{ccc}\cdot&\square&\square\\ \square&\square&\square\\ \cdot&\square&\square\\ \cdot&\cdot&\cdot\\ \square&\square&\square\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{array}, \begin{array}{ccc}\cdot&\square&\square\\ \cdot&\square&\square\\ \cdot&\square&\square\\ \square&\square&\square\\ \cdot&\cdot&\square\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{array}, \begin{array}{ccc}\cdot&\square&\square\\ \cdot&\square&\square\\ \cdot&\square&\square\\ \cdot&\square&\square\\ \cdot&\square&\square\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{array}, \begin{array}{ccc}\cdot&\square&\square\\ \cdot&\cdot&\square\\ \square&\square&\square\\ \square&\square&\square\\ \cdot&\cdot&\square\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{array}, \begin{array}{ccc}\cdot&\cdot&\square\\ \square&\square&\square\\ \square&\square&\square\\ \cdot&\square&\square\\ \cdot&\cdot&\square\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{array}, \begin{array}{ccc}\cdot&\cdot&\square\\ \square&\square&\square\\ \square&\square&\square\\ \cdot&\cdot&\cdot\\ \square&\square&\square\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{array} \right\}$$

$$\mathrm{PD}_{3\times 5}[Y_w^{3,8}]^T \;=\; s_{\ydiagram{4,1}} + s_{\ydiagram{3,1,1}} + s_{\ydiagram{4,1}} + s_{\ydiagram{5}} + s_{\ydiagram{3,2}} + s_{\ydiagram{3,2}} + s_{\ydiagram{2,2,1}} = \omega(s^{(2)}_{\ydiagram{1,1,1,1,1}})$$

$$w = [-3,2,7], \quad \gamma = \ydiagram{1,1,2,4}, \quad \lambda = \ydiagram{1,1,1,2}, \quad h' = 4$$

$$\mathfrak{i}_w^{4,8} = (\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{145}, \Delta_{245}, \Delta_{345}, \mathrm{sh}^1_{45}, \mathrm{sh}^1_{146}, \mathrm{sh}^1_{156}, \mathrm{sh}^1_{246}, \mathrm{sh}^1_{256}, \mathrm{sh}^1_{346}, \mathrm{sh}^1_{356}, \mathrm{sh}^1_{456})$$

$$\begin{pmatrix}1&&&\\ a&&&\\ b&&&\\ &1&&\\ c&a&&\\ &b&1&\\ &&&1\\ d&c&\alpha&a\end{pmatrix} \rightsquigarrow \begin{pmatrix}1&&&\\ 0&0&0&0\\ *&0&0&0\\ &1&&\\ *&*&0&0\\ *&*&0&0\\ &&1&\\ &&&1\end{pmatrix} \quad \begin{pmatrix}1&&&\\ 0&0&0&0\\ *&0&0&0\\ &1&&\\ 0&*&0&0\\ &&1&\\ *&*&*&0\\ &&&1\end{pmatrix} \quad \begin{pmatrix}1&&&\\ 0&0&0&0\\ *&0&0&0\\ &1&&\\ 0&0&0&0\\ &&1&\\ &&&1\\ *&*&*&*\end{pmatrix} \quad \begin{pmatrix}1&&&\\ *&0&0&0\\ *&0&0&0\\ *&0&0&0\\ &1&&\\ *&*&0&0\\ &&1&\\ &&&1\end{pmatrix} \quad \begin{pmatrix}1&&&\\ *&0&0&0\\ *&0&0&0\\ 0&0&0&0\\ &1&&\\ &&1&\\ *&*&*&0\\ &&&1\end{pmatrix}$$

$$\begin{array}{c}\Delta_{12}, \Delta_{13}, \Delta_{23},\\ \Delta_{145}, \Delta_{345}, \Delta_{24},\\ \Delta_{1567}, \Delta_{2567}, \Delta_{3567},\\ \Delta_{4567}, \Delta_{4568}\end{array} \rightsquigarrow \begin{array}{c}\Delta_2, \Delta_{146},\\ \Delta_{156}, \Delta_{346},\\ \Delta_{356}, \Delta_{456}\end{array} \quad \Delta_2, \Delta_{45} \quad \Delta_2, \Delta_5 \quad \begin{array}{c}\Delta_{14}, \Delta_{34},\\ \Delta_{156}, \Delta_{256},\\ \Delta_{356}, \Delta_{456}\end{array} \quad \Delta_4$$

$$(\mathcal{P}_w^{4,8})^*_{\mathrm{top}} = \left\{ \begin{array}{cccc}\cdot&\square&\square&\square\\ \square&\square&\square&\square\\ \cdot&\cdot&\square&\square\\ \cdot&\cdot&\square&\square\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{array}, \begin{array}{cccc}\cdot&\square&\square&\square\\ \square&\square&\square&\square\\ \cdot&\cdot&\cdot&\square\\ \cdot&\square&\square&\square\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{array}, \begin{array}{cccc}\cdot&\square&\square&\square\\ \square&\square&\square&\square\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \square&\square&\square&\square\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{array}, \begin{array}{cccc}\cdot&\square&\square&\square\\ \cdot&\square&\square&\square\\ \cdot&\square&\square&\square\\ \cdot&\cdot&\square&\square\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{array}, \begin{array}{cccc}\cdot&\square&\square&\square\\ \cdot&\square&\square&\square\\ \cdot&\cdot&\cdot&\square\\ \square&\square&\square&\square\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\cdot\end{array} \right\}$$

$$\mathrm{PD}_{4\times 4}[Y_w^{4,8}]^T \;=\; s_{\ydiagram{3,2}} + \left(s_{\ydiagram{3,1,1}} + s_{\ydiagram{2,2,1}} \right) + s_{\ydiagram{2,1,1,1}} + s_{\ydiagram{4,1}} + s_{\ydiagram{3,1,1}} = \omega(s^{(2)}_{\ydiagram{1,1,1,2}})$$

$$w = [-5,5,6], \quad \gamma = (3,3,2,2,1,1), \quad \lambda = (1,1,1,1,1,1), \quad h' = 3$$

$$(\mathcal{P}_w^{3,9})^*_{\mathrm{top}} = \left\{ \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \cdot \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \square & \square & \square \\ \square & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \cdot & \square \\ \square & \square & \square \\ \square & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \cdot & \square \\ \square & \square & \square \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \cdot & \square \\ \square & \square & \square \\ \square & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \cdot & \square \\ \square & \square & \square \\ \square & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \cdot \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \cdot & \cdot \\ \square & \square & \square \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \square & \square & \square \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{3\times 6}[Y_w^{3,9}]^T = s_{(5,1)} + s_{(4,1,1)} + s_{(5,1)} + s_{(6)} + s_{(4,2)} + \left(s_{(4,2)} + s_{(2,2,2)} \right) + s_{(3,2,1)} + s_{(4,2)} + s_{(3,2,1)} + s_{(2,2,2)}$$

$$= \omega\left(s^{(2)}_{(1,1,1,1,1,1)} + s_{(2,2,2)}\right) \neq \omega\left(s^{(2)}_{(1,1,1,1,1,1)}\right)^1$$

---

[1]The multidegree, generated in `Macaulay 2` from the appropriate shuffles, does not yield, under duality and transposition, the k-Schur polynomial associated with w in this case; it includes the unexpected summand $s_{(3,3,0)}$. We are investigating this case in detail.

$$w = [\,], \quad \gamma = (4,3,3,2,2,1,1), \quad \lambda = (1,1,1,1,1,1,1), \quad h' = 4$$

$$(\mathcal{P}_w^{4,11})^*_{\mathrm{top}} = \left\{
\begin{matrix}\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
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\begin{matrix}\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},\right.$$

$$\begin{matrix}\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},$$

$$\left.\begin{matrix}\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\cdot&\square&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix},
\begin{matrix}\cdot&\cdot&\cdot&\square\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\square&\square&\square&\square\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\square&\square&\square&\square\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{matrix}\right\}$$

$$\mathrm{PD}_{4\times 7}[Y_w^{4,11}]^T = s_{(5,2)} + (s_{(4,2,1)} + s_{(5,1,1)}) + s_{(4,1,1,1)} + s_{(6,1)} + s_{(5,1,1)} + s_{(4,3)} + (s_{(3,3,1)} + s_{(4,2,1)}) + s_{(3,2,1,1)}$$

$$+ s_{(6,1)} + s_{(5,1,1)} + s_{(6,1)} + s_{(7)} + s_{(5,2)} + (s_{(4,3)} + s_{(5,2)}) + s_{(4,2,1)} + (s_{(4,3)} + s_{(5,2)}) + (s_{(3,3,1)} + s_{(4,2,1)})$$

$$+ s_{(3,2,2)} + s_{(4,3)} + (s_{(3,3,1)} + s_{(4,2,1)}) + s_{(3,2,1,1)} + s_{(5,2)} + s_{(4,2,1)} + s_{(3,2,2)} + s_{(3,2,2)} + s_{(2,2,2,1)}$$

$$= \omega(s^{(2)}_{(1,1,1,1,1)})$$

$$w = [\,],\quad \gamma = \ydiagram{1,1,2,2,3,3,4,4},\quad \lambda = \ydiagram{1,1,1,1,1,1,1,1},\quad h' = 4$$

$$\begin{aligned}
\mathrm{PD}_{4\times 8}[Y_w^{4,12}]^T \;=\;& s_{\ydiagram{8}} + 3s_{\ydiagram{7,1}} + 6s_{\ydiagram{6,2}} + 6s_{\ydiagram{5,3}} + 3s_{\ydiagram{4,4}} + 3s_{\ydiagram{6,1,1}} + 8s_{\ydiagram{5,2,1}} \\
&+ 7s_{\ydiagram{4,3,1}} + 6s_{\ydiagram{4,2,2}} + 3s_{\ydiagram{3,3,2}} + s_{\ydiagram{5,1,1,1}} + 3s_{\ydiagram{4,2,1,1}} + 3s_{\ydiagram{3,2,2,1}} + s_{\ydiagram{2,2,2,2}} \\
=\;& \omega(s^{(2)}_{\ydiagram{1,1,1,1,1,1,1,1}} - 2s_{\ydiagram{4,2,2}}) \neq \omega(s^{(2)}_{\ydiagram{1,1,1,1,1,1,1,1}})
\end{aligned}$$

$n = 4$

$$w = [0,1,2,7],\quad \gamma = \ydiagram{3},\quad \lambda = \ydiagram{3},\quad h' = 4$$

$$(\mathcal{P}_w^{3,4})^*_{\mathrm{top}} = \left\{ \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{1\times 3}[Y_w^{3,4}]^T \;=\; s_{\ydiagram{1,1,1}} \;=\; \omega(s^{(3)}_{\ydiagram{3}})$$

$$w = [-1,1,4,6],\quad \gamma = \ydiagram{1,2},\quad \lambda = \ydiagram{1,2},\quad h' = 3$$

$$(\mathcal{P}_w^{2,4})^*_{\mathrm{top}} = \left\{ \begin{array}{cc} \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{2\times 2}[Y_w^{2,4}]^T \;=\; s_{\ydiagram{2,1}} \;=\; \omega(s^{(3)}_{\ydiagram{1,2}})$$

$$w = [-2,3,4,5],\quad \gamma = \ydiagram{1,1,1},\quad \lambda = \ydiagram{1,1,1},\quad h' = 3$$

$$(\mathcal{P}_w^{1,4})^*_{\mathrm{top}} = \left\{ \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \right\}$$

$$\mathrm{PD}_{3\times 1}[Y_w^{3,4}]^T \;=\; s_{\ydiagram{3}} \;=\; \omega(s^{(3)}_{\ydiagram{1,1,1}})$$

$$w = [-1, 1, 2, 8], \quad \gamma = \ydiagram{1,4}, \quad \lambda = \ydiagram{1,3}, \quad h' = 5$$

$$(\mathcal{P}_w^{4,6})^*_{\text{top}} = \left\{ \begin{array}{cccc} \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}, \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{4\times 2}[Y_w^{4,6}]^T \;=\; s_{\ydiagram{2,1,1}} \;+\; s_{\ydiagram{1,1,1,1}} \;=\; \omega(s^{(3)}_{\ydiagram{1,3}})$$

$$w = [-2, 1, 4, 7], \quad \gamma = \ydiagram{1,1,3}, \quad \lambda = \ydiagram{1,1,2}, \quad h' = 4$$

$$(\mathcal{P}_w^{3,6})^*_{\text{top}} = \left\{ \begin{array}{ccc} \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{3\times 6}[Y_w^{3,6}]^T \;=\; s_{\ydiagram{2,1,1}} \;+\; s_{\ydiagram{3,1}} \;=\; \omega(s^{(3)}_{\ydiagram{1,1,2}})$$

$$w = [-1, 0, 5, 6], \quad \gamma = \ydiagram{2,2}, \quad \lambda = \ydiagram{2,2}, \quad h' = 3$$

$$(\mathcal{P}_w^{2,4})^*_{\text{top}} = \left\{ \begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{2\times 2}[Y_w^{2,4}]^T \;=\; s_{\ydiagram{2,2}} \;=\; \omega(s^{(3)}_{\ydiagram{2,2}})$$

$$w = [-3, 3, 4, 6], \quad \gamma = \ydiagram{1,1,1,2}, \quad \lambda = \ydiagram{1,1,1,1}, \quad h' = 4$$

$$(\mathcal{P}_w^{2,6})^*_{\text{top}} = \left\{ \begin{array}{cc} \cdot & \square \\ \square & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}, \begin{array}{cc} \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{2\times 4}[Y_w^{2,6}]^T \;=\; s_{\ydiagram{3,1}} \;+\; s_{\ydiagram{4}} \;=\; \omega(s^{(3)}_{\ydiagram{1,1,1,1}})$$

$$w = [-1, 0, 2, 9], \quad \gamma = \ydiagram{2,5}, \quad \lambda = \ydiagram{2,3}, \quad h' = 5$$

$$(\mathcal{P}_w^{5,7})^*_{\mathrm{top}} = \left\{ \begin{array}{ccccc} \cdot & \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \square & \square \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}, \begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \square \\ \cdot & \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}, \begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \square & \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{5\times 2}[Y_w^{5,7}]^T \;=\; s_{\ydiagram{2,2,1}} \;+\; s_{\ydiagram{2,1,1,1}} \;+\; s_{\ydiagram{1,1,1,1,1}} \;=\; \omega(s^{(3)}_{\ydiagram{2,3}})$$

$$w = [-2, 1, 3, 8], \quad \gamma = \ydiagram{1,1,4}, \quad \lambda = \ydiagram{1,1,3}, \quad h' = 4$$

$$(\mathcal{P}_w^{4,7})^*_{\mathrm{top}} = \left\{ \begin{array}{cccc} \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}, \begin{array}{cccc} \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{4\times 3}[Y_w^{4,7}]^T \;=\; s_{\ydiagram{3,1,1}} \;+\; s_{\ydiagram{2,1,1,1}} \;=\; \omega(s^{(3)}_{\ydiagram{1,1,3}})$$

$$w = [-2, 0, 5, 7], \quad \gamma = \ydiagram{1,2,3}, \quad \lambda = \ydiagram{1,2,2}, \quad h' = 4$$

$$(\mathcal{P}_w^{3,6})^*_{\mathrm{top}} = \left\{ \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \cdot & \square \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{3\times 3}[Y_w^{3,6}]^T \;=\; s_{\ydiagram{3,2}} \;+\; s_{\ydiagram{2,2,1}} \;=\; \omega(s^{(3)}_{\ydiagram{1,2,2}})$$

$$w = [-3, 2, 4, 7], \quad \gamma = \ydiagram{1,1,1,3}, \quad \lambda = \ydiagram{1,1,1,2}, \quad h' = 4$$

$$(\mathcal{P}_w^{3,7})^*_{\mathrm{top}} = \left\{ \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{3\times 4}[Y_w^{3,7}]^T \;=\; s_{\ydiagram{3,1,1}} \;+\; s_{\ydiagram{4,1}} \;=\; \omega(s^{(3)}_{\ydiagram{1,1,1,2}})$$

$$w = [-4, 3, 5, 6], \quad \gamma = \yng(1,1,1,2,2), \quad \lambda = \yng(1,1,1,1,1), \quad h' = 3$$

$$(\mathcal{P}_w^{2,7})^*_{\mathrm{top}} = \left\{ \begin{array}{cc} \cdot & \square \\ \square & \square \\ \square & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}, \; \begin{array}{cc} \cdot & \square \\ \cdot & \square \\ \square & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}, \; \begin{array}{cc} \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{2\times 5}[Y_w^{2,7}]^T \; = \; s_{\yng(3,2)} \; + \; s_{\yng(4,1)} \; + \; s_{\yng(5)} \; = \; \omega(s^{(3)}_{\yng(1,1,1,1,1)})$$

$$w = [-3, 2, 3, 8], \quad \gamma = \yng(1,1,1,4), \quad \lambda = \yng(1,1,1,3), \quad h' = 4$$

$$(\mathcal{P}_w^{4,8})^*_{\mathrm{top}} = \left\{ \begin{array}{cccc} \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}, \; \begin{array}{cccc} \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{4\times 4}[Y_w^{4,8}]^T \; = \; s_{\yng(4,1,1)} \; + \; s_{\yng(3,1,1,1)} \; = \; \omega(s^{(3)}_{\yng(1,1,1,3)})$$

$$w = [-4, 2, 5, 7], \quad \gamma = \yng(1,1,1,2,3), \quad \lambda = \yng(1,1,1,1,2), \quad h' = 3$$

$$(\mathcal{P}_w^{3,8})^*_{\mathrm{top}} = \left\{ \begin{array}{ccc} \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \; \begin{array}{ccc} \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \cdot & \square \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \; \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \; \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{3\times 5}[Y_w^{3,8}]^T \; = \; s_{\yng(4,2)} \; + \; s_{\yng(3,2,1)} \; + \; s_{\yng(4,1,1)} \; + \; s_{\yng(5,1)} \; = \; \omega(s^{(3)}_{\yng(1,1,1,1,2)})$$

$$w = [-5, 4, 5, 6], \quad \gamma = \yng(1,1,1,2,2,2), \quad \lambda = \yng(1,1,1,1,1,1), \quad h' = 3$$

$$(\mathcal{P}_w^{2,8})^*_{\text{top}} = \left\{ \begin{smallmatrix} \cdot & \square \\ \cdot & \square \\ \square & \square \\ \square & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}, \begin{smallmatrix} \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \square & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}, \begin{smallmatrix} \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}, \begin{smallmatrix} \cdot & \cdot \\ \square & \square \\ \square & \square \\ \square & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix} \right\}$$

$$\mathrm{PD}_{2\times 6}[Y_w^{2,8}]^T \;=\; s_{\yng(4,2)} \;+\; s_{\yng(5,1)} \;+\; s_{\yng(6)} \;+\; s_{\yng(3,3)} \;=\; \omega(s^{(3)}_{\yng(1,1,1,1,1,1)})$$

$n = 5$

$$w = [-3, 3, 4, 5, 6], \quad \gamma = \yng(4), \quad \lambda = \yng(4), \quad h' = 5$$

$$(\mathcal{P}_w^{4,5})^*_{\text{top}} = \left\{ \begin{smallmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{smallmatrix} \right\}$$

$$\mathrm{PD}_{4\times 1}[Y_w^{4,5}]^T \;=\; s_{\yng(1,1,1,1)} \;=\; \omega(s^{(4)}_{\yng(4)})$$

$$w = [-2, 1, 4, 5, 7], \quad \gamma = \yng(1,3), \quad \lambda = \yng(1,3), \quad h' = 4$$

$$(\mathcal{P}_w^{3,5})^*_{\text{top}} = \left\{ \begin{smallmatrix} \cdot & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{smallmatrix} \right\}$$

$$\mathrm{PD}_{3\times 2}[Y_w^{3,5}]^T \;=\; s_{\yng(2,1,1)} \;=\; \omega(s^{(4)}_{\yng(1,3)})$$

$$w = [-1, 1, 2, 5, 8], \quad \gamma = \yng(1,1,2), \quad \lambda = \yng(1,1,2), \quad h' = 3$$

$$(\mathcal{P}_w^{2,5})^*_{\text{top}} = \left\{ \begin{smallmatrix} \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix} \right\}$$

$$\mathrm{PD}_{2\times 3}[Y_w^{2,5}]^T \;=\; s_{\yng(3,1)} \;=\; \omega(s^{(4)}_{\yng(1,1,2)})$$

$$w = [0,1,2,3,9], \quad \gamma = \ydiagram{1,1}, \quad \lambda = \ydiagram{1,1}, \quad h' = 2$$

$$(\mathcal{P}_w^{1,5})^*_{\text{top}} = \left\{ \begin{matrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{1\times 4}[Y_w^{1,5}]^T \;=\; s_{\ydiagram{4}} \;=\; \omega(s^{(4)}_{\ydiagram{1,1,1,1}})$$

$$w = [-4,3,4,5,7], \quad \gamma = \ydiagram{1,5}, \quad \lambda = \ydiagram{1,4}, \quad h' = 5$$

$$(\mathcal{P}_w^{5,7})^*_{\text{top}} = \left\{ \begin{matrix} \cdot & \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}, \quad \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \square & \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{5\times 2}[Y_w^{5,7}]^T \;=\; s_{\ydiagram{2,1,1,1}} \;+\; s_{\ydiagram{1,1,1,1,1}} \;=\; \omega(s^{(4)}_{\ydiagram{1,4}})$$

$$w = [-3,1,4,5,8], \quad \gamma = \ydiagram{1,1,4}, \quad \lambda = \ydiagram{1,1,3}, \quad h' = 4$$

$$(\mathcal{P}_w^{4,7})^*_{\text{top}} = \left\{ \begin{matrix} \cdot & \square & \square & \square \\ \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}, \quad \begin{matrix} \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{4\times 3}[Y_w^{4,7}]^T \;=\; s_{\ydiagram{2,1,1,1}} \;+\; s_{\ydiagram{3,1,1}} \;=\; \omega(s^{(4)}_{\ydiagram{1,1,3}})$$

$$w = [-2,0,4,6,7], \quad \gamma = \ydiagram{2,3}, \quad \lambda = \ydiagram{2,3}, \quad h' = 3$$

$$(\mathcal{P}_w^{3,5})^*_{\text{top}} = \left\{ \begin{matrix} \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \right\}$$

$$\mathrm{PD}_{3\times 2}[Y_w^{3,5}]^T \;=\; s_{\ydiagram{2,2,1}} \;=\; \omega(s^{(4)}_{\ydiagram{2,3}})$$

$$w = [-1, 0, 2, 6, 8], \quad \gamma = \ydiagram{1,2,2}, \quad \lambda = \ydiagram{1,2,2}, \quad h' = 2$$

$$(\mathcal{P}_w^{2,5})^*_{\text{top}} = \left\{ \begin{array}{cc} \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{2\times 3}[Y_w^{2,5}]^T \;=\; s_{\ydiagram{3,2}} \;=\; \omega(s^{(4)}_{\ydiagram{1,2,2}})$$

$$w = [-2, 1, 2, 5, 9], \quad \gamma = \ydiagram{1,1,1,3}, \quad \lambda = \ydiagram{1,1,1,2}, \quad h' = 4$$

$$(\mathcal{P}_w^{3,7})^*_{\text{top}} = \left\{ \begin{array}{ccc} \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \quad \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{3\times 4}[Y_w^{3,7}]^T \;=\; s_{\ydiagram{4,1}} \;+\; s_{\ydiagram{3,1,1}} \;=\; \omega(s^{(4)}_{\ydiagram{1,1,1,2}})$$

$$w = [-1, 1, 2, 3, 10], \quad \gamma = \ydiagram{1,1,1,1,2}, \quad \lambda = \ydiagram{1,1,1,1,1}, \quad h' = 3$$

$$(\mathcal{P}_w^{2,7})^*_{\text{top}} = \left\{ \begin{array}{cc} \cdot & \square \\ \square & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}, \quad \begin{array}{cc} \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{2\times 5}[Y_w^{2,7}]^T \;=\; s_{\ydiagram{4,1}} \;+\; s_{\ydiagram{5}} \;=\; \omega(s^{(4)}_{\ydiagram{1,1,1,1,1}})$$

$$w = [-3, 0, 4, 6, 8], \quad \gamma = \ydiagram{1,2,4}, \quad \lambda = \ydiagram{1,2,3}, \quad h' = 4$$

$$(\mathcal{P}_w^{4,7})^*_{\text{top}} = \left\{ \begin{array}{cccc} \cdot & \square & \square & \square \\ \cdot & \cdot & \square & \square \\ \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}, \quad \begin{array}{cccc} \cdot & \cdot & \square & \square \\ \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{4\times 3}[Y_w^{4,7}]^T \;=\; s_{\ydiagram{3,2,1}} \;+\; s_{\ydiagram{2,2,1,1}} \;=\; \omega(s^{(4)}_{\ydiagram{1,2,3}})$$

$$w = [-2,-1,5,6,7], \quad \gamma = \ydiagram{3,3}, \quad \lambda = \ydiagram{3,3}, \quad h' = 4$$

$$(\mathcal{P}_w^{3,5})^*_{\mathrm{top}} = \left\{ \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{3\times 2}[Y_w^{3,5}]^T \;=\; s_{\ydiagram{2,2,2}} \;=\; \omega(s^{(4)}_{\ydiagram{3,3}})$$

$$w = [-3,1,3,5,9], \quad \gamma = \ydiagram{1,1,1,4}, \quad \lambda = \ydiagram{1,1,1,3}, \quad h' = 4$$

$$(\mathcal{P}_w^{4,8})^*_{\mathrm{top}} = \left\{ \begin{array}{cccc} \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \\ \square & \square & \square & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}, \quad \begin{array}{cccc} \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \\ \cdot & \square & \square & \square \\ \cdot & \cdot & \cdot & \square \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{4\times 4}[Y_w^{4,8}]^T \;=\; s_{\ydiagram{3,1,1,1}} \;+\; s_{\ydiagram{4,1,1}} \;=\; \omega(s^{(4)}_{\ydiagram{1,1,1,3}})$$

$$w = [-1,0,1,7,8], \quad \gamma = \ydiagram{2,2,2}, \quad \lambda = \ydiagram{2,2,2}, \quad h' = 3$$

$$(\mathcal{P}_w^{2,5})^*_{\mathrm{top}} = \left\{ \begin{array}{cc} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{2\times 3}[Y_w^{2,5}]^T \;=\; s_{\ydiagram{3,3}} \;=\; \omega(s^{(4)}_{\ydiagram{2,2,2}})$$

$$w = [-2,0,2,6,9], \quad \gamma = \ydiagram{1,1,2,3}, \quad \lambda = \ydiagram{1,1,2,2}, \quad h' = 4$$

$$(\mathcal{P}_w^{3,7})^*_{\mathrm{top}} = \left\{ \begin{array}{ccc} \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \quad \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{3\times 4}[Y_w^{3,7}]^T \;=\; s_{\ydiagram{3,2,1}} \;+\; s_{\ydiagram{4,2}} \;=\; \omega(s^{(4)}_{\ydiagram{1,1,2,2}})$$

$$w = [-2, 1, 2, 4, 10], \quad \gamma = \ydiagram{1,1,1,1,3}, \quad \lambda = \ydiagram{1,1,1,1,2}, \quad h' = 3$$

$$(\mathcal{P}_w^{3,8})^*_{\text{top}} = \left\{ \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \square & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}, \quad \begin{array}{ccc} \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \square & \square \\ \cdot & \cdot & \square \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{3\times 5}[Y_w^{3,8}]^T \ = \ s_{\ydiagram{4,1,1}} \ + \ s_{\ydiagram{5,1}} \ = \ \omega(s^{(4)}_{\ydiagram{1,1,1,1,2}})$$

$$w = [-1, 0, 2, 3, 11], \quad \gamma = \ydiagram{1,1,1,1,2,2}, \quad \lambda = \ydiagram{1,1,1,1,1,1}, \quad h' = 3$$

$$(\mathcal{P}_w^{2,8})^*_{\text{top}} = \left\{ \begin{array}{cc} \cdot & \square \\ \square & \square \\ \square & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}, \quad \begin{array}{cc} \cdot & \square \\ \cdot & \square \\ \square & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}, \quad \begin{array}{cc} \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \square \\ \cdot & \cdot \\ \cdot & \cdot \end{array} \right\}$$

$$\mathrm{PD}_{2\times 6}[Y_w^{2,8}]^T \ = \ s_{\ydiagram{4,2}} \ + \ s_{\ydiagram{5,1}} \ + \ s_{\ydiagram{6}} \ = \ \omega(s^{(4)}_{\ydiagram{1,1,1,1,1,1}})$$