On the Theory of Optical Hilbert transform for incoherent objects

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Abstract: The Hilbert transform as been investigated abundantly in coherent imaging. To the best of our knowledge, it is for the first time investigated in the context of incoherent imaging. We present a two-pupil optical heterodyne scanning system and analyze mathematically the design of its two pupils such that the optical system can perform the Hilbert transform on incoherent objects. Computer simulations of the idea clarify the theoretical results.

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References and links

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1. Introduction

The Hilbert transform has drawn some attentions recently due to its ability for phase-retrieval as well as for image processing [1, 2]. The classical Hilbert transform can be implemented coherently by π - phase shifting in the Fourier plane of an optical system [3]. It is also well-known that by half-plane filtering in the Fourier plane, we can obtain the Hilbert transform of the original complex light field [4]. Unfortunately, the Hilbert transform of the information is superposed coherently with the original light field. Unless the two fields can be separated by some phase extraction methods, we cannot obtain the Hilbert transform of the original light field. For incoherent image processing, up till now, no optical systems have been able to extract the Hilbert transform information from incoherent objects, *i.e.*, *the quantity*, *such as intensity*, *to be Hilbert transformed is real and non-negative*. In this paper, we propose a *quarter-plane filtering* in the optical transfer function (OTF) domain to extract the Hilbert transform of the incoherent object. The notion of quarter-plane filtering is for 2-D transformation as half-plane filtering only leads to 1-D transformation. The proposed filtering can be performed with a two-pupil optical heterodyne scanning system originally developed

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by Poon and Korpel [5], and subsequently analyzed by Poon, using an optical transfer function approach [6]. In section 2, we first review and formulate the definition of an analytic signal of a function and from which we can obtain the Hilbert transform of the function. For brevity, one-dimension formalism is developed and extension to two dimensions is trivial. In section 3, we briefly discuss the two-pupil system and summarize the relevant results that are useful for our present development of the Hilbert transform. In section 4, we analyze the design of pupils so as to obtain the Hilbert transform and show some 1-D simulations to clarify the idea. Finally, in section 5, we make some concluding remarks.

2. Analytic signal and the Hilbert transform

Consider a real function $g(x)$ with its Fourier transform defined by [7]

$$F\{g(x)\} = G(k_x) = \int_{-\infty}^{\infty} g(x)\exp(jk_x x)dx, \tag{1}$$

where x and k_x are the Fourier transform variables. The integral

$$g_a^+(x) = \frac{1}{\pi}\int_0^{\infty} G(k_x)\exp(-jk_x x)dk_x \tag{2}$$

is known as the analytic signal associated with $g(x)$. $g_a^+(x)$ can be written in terms of the inverse Fourier transform as

$$g_a^+(x) = 2\frac{1}{2\pi}\int_{-\infty}^{\infty} G(k_x)U(k_x)\exp(-jk_x x)dk_x = 2F^{-1}\{G(k_x)U(k_x)\}\ , \tag{3}$$

where $U(k_x)$ is a unit step function:

$$U(k_x) = 1 \text{ for } k_x > 0 \text{ and } U(k_x) = 0 \text{ for } k_x < 0. \tag{4}$$

The analytic signal given by Eq. (2) is a complex function and can be written as follows:

$$g_a^+(x) = g(x) - j\hat{g}(x) \tag{5}$$

where $\hat{g}(x)$ is the Hilbert transform of $g(x)$:

$$\hat{g}(x) = H\{g(x)\} = \int_{-\infty}^{\infty} \frac{g(x')}{\pi(x-x')}dx'\ . \tag{6}$$

Hence we see that from the definition given by Eq. (5), we can extract the original function and its Hilbert transform by simply taking the real part of and the imaginary part of $g_a^+(x)$, respectively, i.e.,

$$g(x) = \mathrm{Re}\{g_a^+(x)\};\quad \hat{g}(x) = -\mathrm{Im}\{g_a^+(x)\}. \tag{7}$$

Using Eq. (3), we can re-write Eq. (7) as

$$g(x) = 2\,\mathrm{Re}[F^{-1}\{G(k_x)U(k_x)\}];\ \hat{g}(x) = -2\,\mathrm{Im}[F^{-1}\{G(k_x)U(k_x)\}]. \tag{8}$$

We can also create an analytic signal by including negative frequencies only. Similar to the definition given by Eq. (2), we then have

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$$g_a^-(x) = \frac{1}{\pi}\int_{-\infty}^{0} G(k_x)\exp(-jk_x x)dk_x = 2F^{-1}\{G(k_x)U(-k_x)\}$$
$$= g(x) + j\hat{g}(x), \tag{9}$$

and hence, for this case, we have

$$g(x) = 2\,\mathrm{Re}[F^{-1}\{G(k_x)U(-k_x)\}];\ \hat{g}(x) = 2\,\mathrm{Im}[F^{-1}\{G(k_x)U(-k_x)\}]. \tag{10}$$

We shall design the pupils in the two-pupil optical heterodyne scanning processor so that Eq. (10) can be realized.

3. Two-pupil optical heterodyne scanning processor

Figure 1 shows a typical two-pupil optical heterodyne scanning image processor, which was originally developed and analyzed by Poon [6]. We shall briefly describe the principle of operation and then summarize some of the important results, which are relevant to our pursuit of the Hilbert transformation. Beamsplitters BS and BS1, and mirrors M and M1 form the Mach-Zehnder interferometer. The pupil, $p_1(x,y)$, is illuminated by a collimated laser at temporal frequency ω_0. The other pupil, $p_2(x,y)$, is illuminated by the laser of temporal frequency $\omega_0+\Omega$. The laser's temporal frequency offset by Ω is introduced by an acousto-optic frequency shifter (AOFS) as shown in the figure. The two pupils are located at the front focal planes of lens L1 and L2, both with a focal length of f . The two pupils are then combined by the beamsplitter, BS1, in order to focus the light onto the 2-D optical scanner, which is located on the back focal plane of lenses L1 and L2. The combined optical beams are then used to 2-D raster scan over an object of amplitude distribution $\Gamma_0(x,y;z)$, which is located at a distance of z away from the focal plane of the two lenses. Lens L3 is used to collect all the transmitted light (or scattered light if the object is diffusely reflecting) onto the photodetector (PD), which gives $i(x,y)$ as its current output. An electronic bandpass filter (BPF) tuned at the heterodyne frequency of Ω provides an output of a scanned and processed current $i_\Omega(x,y)$. Note that in the arguments of $i(x,y)$ and $i_\Omega(x,y)$, $x = x(t) = vt$ and $y = y(t) = vt$ if the scanning speed of the beam, v, is constant. Finally, $i_\Omega(x,y)$ is processed electronically as shown in Fig. 1 to give two outputs $i_c(x,y)$ and $i_s(x,y)$, which are given as follows [8]:

$$i_c(x,y) = \mathrm{Re}[F^{-1}\{F\{|\Gamma_0(x,y;z)|^2\}OTF_\Omega\}]\ ; \tag{11a}$$

$$i_s(x,y) = \mathrm{Im}[F^{-1}\{F\{|\Gamma_0(x,y;z)|^2\}OTF_\Omega\}], \tag{11b}$$

where $|\Gamma_0(x,y;z)|^2$ is the intensity distribution of the object $\Gamma_0(x,y;z)$ being scanned, and OTF_Ω is the optical transfer function of the optical system given by [6,8]

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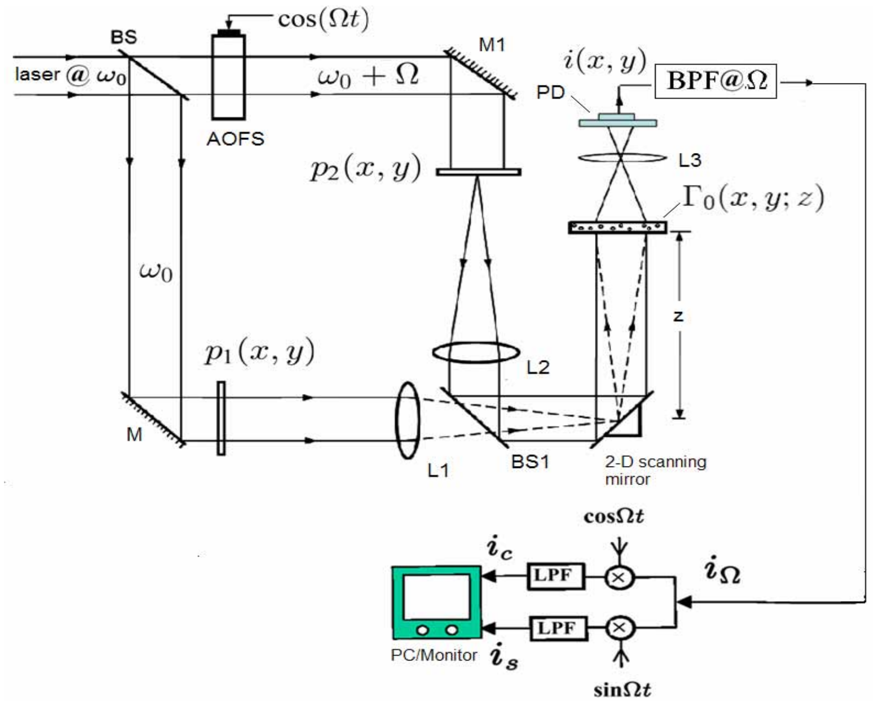

Fig. 1. Typical two-pupil optical heterodyne image processor. [Figure 3.11 of T.-C. Poon, *Optical scanning holography with MATLAB* (Springer, 2007). With kind permission of Springer Science and Business Media.]

$$OTF_{\Omega} = \exp[j\frac{z}{2k_0}(k_x{}^2 + k_y{}^2)]$$

$$\times \iint p_1{}^*(x', y')p_2(x'+\frac{f}{k_0}k_x, y'+\frac{f}{k_0}k_y)\exp[j\frac{z}{f}(x'k_x + y'k_y)]dx'dy' \qquad (12)$$

with k_0 being the wave number of the light. Note that in Eq. (11), only the intensity of the object is processed and hence the system is an incoherent optical system.

4. Pupil function designs

The processing OTF, given by Eq. (12), can be altered by properly designing the pupil functions. It is interesting to point out that Eq. (11) has the same functional form as that of Eq. (10) by recognizing that the spectrum of $|\Gamma_0(x, y; z)|^2$ corresponds to $G(k_x)$, and OTF_{Ω} needs to be designed such that it is given by $U(-k_x)U(-k_y)$ in order to realize the Hilbert transform of $|\Gamma_0(x, y; z)|^2$ in two dimensions. We let $p_2(x, y) = \delta(x, y)$ and $p_1(x, y) = U(x)U(y)$. With these designed pupils, the OTF from Eq. (12) becomes

$$OTF_{\Omega} = \exp[-j\frac{z}{2k_0}(k_x{}^2 + k_y{}^2)]\, U(-k_x)U(-k_y)\,. \qquad (13)$$

Note the obtained OTF is masked by $U(-k_x)U(-k_y)$ and hence the notion of *quarter-plane filtering*. With this OTF, Eqs. (11a) and (11b) now can be written, respectively, as

$$H_{re}(x,y) = \mathrm{Re}[F^{-1}\{F\{|\Gamma_0|^2\}\exp[-j\frac{z}{2k_0}(k_x{}^2+k_y{}^2)]U(-k_x)U(-k_y)\}]; \tag{14a}$$

$$H_{im}(x,y) = \mathrm{Im}[F^{-1}\{F\{|\Gamma_0|^2\}\exp[-j\frac{z}{2k_0}(k_x{}^2+k_y{}^2)]U(-k_x)U(-k_y)\}]. \tag{14b}$$

It is clear that for $z=0$, the real-time scanned and processed outputs given by Eq. (14a) and (14b) represent the recovery of the original incoherent object, $|\Gamma_0(x,y;z)|^2$, and its Hilbert transform, respectively by comparing the equations with Eq. (10). In a general case, for $z \neq 0$, Eqs. (14a) and (14b), after some manipulations, can be worked out, in the spatial domain, to be

$$\begin{aligned} H_{re}(x,y) = &|\Gamma_0(x,y;z)|^2 * \frac{k_0}{2\pi z}\sin[\frac{k_0}{2z}(x^2+y^2)] \\ &+ H\{|\Gamma_0(x,y;z)|^2\} * \frac{k_0}{2\pi z}\cos[\frac{k_0}{2z}(x^2+y^2)]; \end{aligned} \tag{15a}$$

$$\begin{aligned} H_{im}(x,y) = &H\{|\Gamma_0(x,y;z)|^2\} * \frac{k_0}{2\pi z}\sin[\frac{k_0}{2z}(x^2+y^2)] \\ &- |\Gamma_0(x,y;z)|^2 * \frac{k_0}{2\pi z}\cos[\frac{k_0}{2z}(x^2+y^2)], \end{aligned} \tag{15b}$$

respectively, where $*$ represents 2-D convolution involving coordinates x and y. It is important to bring out some interpretations of Eq. (15). Note that for an arbitrary object $f(x,y)$, $f(x,y)*\sin[\frac{k_0}{2z}(x^2+y^2)]$ and $f(x,y)*\cos[\frac{k_0}{2z}(x^2+y^2)]$ represent the sine-FZP hologram and the cosine-FZP hologram of $f(x,y)$ in the context of optical scanning holography (OSH), respectively [8]. Hence, Eq. (15a) is the sum of the sine-FZP hologram of $|\Gamma_0(x,y;z)|^2$ and the cosine-FZP hologram of the Hilbert transform of $|\Gamma_0(x,y;z)|^2$, whereas Eq. (15b) is the difference between the sine-FZP hologram of the Hilbert transform of $|\Gamma_0(x,y;z)|^2$ and the cosine-FZP hologram of $|\Gamma_0(x,y;z)|^2$. In order to extract the Hilbert transform of $|\Gamma_0(x,y;z)|^2$, we need to reconstruct the holograms given by Eq. (15). For some chosen value of $k_0/2z$ and brevity, we simulate Eq. (15) in one dimension. We plot $H_{re}(x,y)$ and $H_{im}(x,y)$ in Figs. 2(a) and 2(b), respectively for an 1-D rectangular object. We see that $|\Gamma_0(x,y;z)|^2$ and its Hilbert transform are coded in a complicated way in the holograms of $H_{re}(x,y)$ and $H_{im}(x,y)$. However, if we reconstruct a complex hologram, $H_C(x,y)$, such that $H_C(x,y) = H_{re}(x,y) + jH_{im}(x,y)$, we can show that

$$H_C(x,y) = [-j|\Gamma_0(x,y;z)|^2 + H\{|\Gamma_0(x,y;z)|^2\}] * \exp[j\frac{k_0}{2z}(x^2+y^2)]. \tag{16}$$

Equation (16) can be interpreted as a complex-FZP hologram of the complex field $-j|\Gamma_0(x,y;z)|^2 + H\{|\Gamma_0(x,y;z)|^2$. This complex hologram can now be reconstructed digitally by performing the following convolution operation:

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$$H_C(x,y) * h(x,y;z), \tag{17}$$

where $h(x,y;z) = \frac{jk_0}{2\pi z}\exp[\frac{-jk_0}{2z}(x^2+y^2)]$, which is used to simulate Fresnel diffraction of the complex hologram upon plane wave illumination for digital reconstruction. To extract the information from the hologram, we take the real part and the imaginary part of the reconstruction, i.e, $\mathrm{Re}[H_C(x,y) * h(x,y;z)]$ and $\mathrm{Im}[H_C(x,y) * h(x,y;z)]$ and they are shown in Figs. 2(c) and 2(d), respectively. We clearly observe the reconstruction of the original rectangular object and its Hilbert transform.

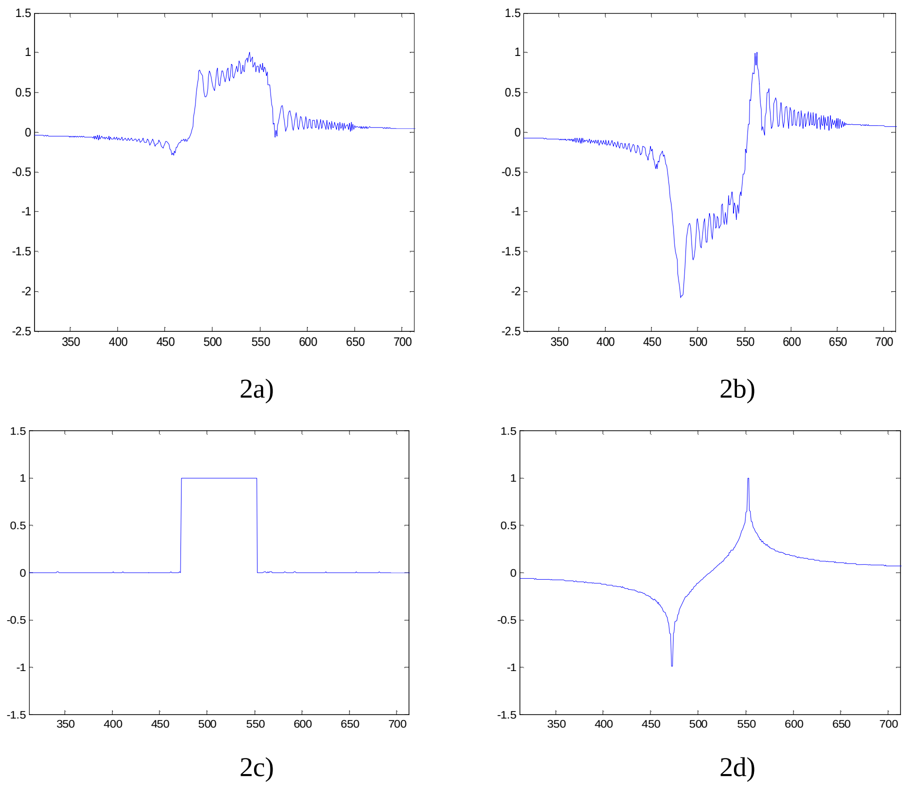

Fig. 2. (a). Hologram $H_{re}(x,y)$ and (b). hologram $H_{im}(x,y)$, (c). real part of and (d). imaginary part of reconstruction from $H_C(x,y) = H_{re}(x,y) + jH_{im}(x,y)$.

5. Concluding remarks

We have analyzed mathematically a two-pupil optical heterodyne scanning image processor by designing the pupils for the extraction of the Hilbert transform of incoherent objects. We believe this is the first time the Hilbert transform of incoherent objects has been addressed and the result of it could be useful for image processing applications as well as real-time holographic recording of the Hilbert transform of a 3-D object. Experimental investigations are currently underway.

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