# Chapter 3 -- Development of the State Vector Equation

## 3.1 Overview

The state vector equation which is to be solved is a two-point boundary value problem (BVP), but will appear when written in matrix-vector form as a first-order ordinary differential equation (ODE) with variable coefficients. The dependent variable vector contains only terms which are prescribable on meridional faces, with the independent variable measuring position along the meridian. The equation is derived for first-order transverse shear deformation theory. A Flügge-type approximation is used in the constitutive law, such that the equation is applicable to thin shells or shells of moderate thickness. The derivation results in a partial differential equation, with independent variables measuring position along the meridian and along a circle of latitude; this equation is converted into an ODE by assuming axisymmetric response. The equation will first be derived in the proper form for linear analysis by using the Hellinger-Reissner mixed variational principle, then in the geometrically nonlinear form by manipulation of the field equations derived in Chapter 2.

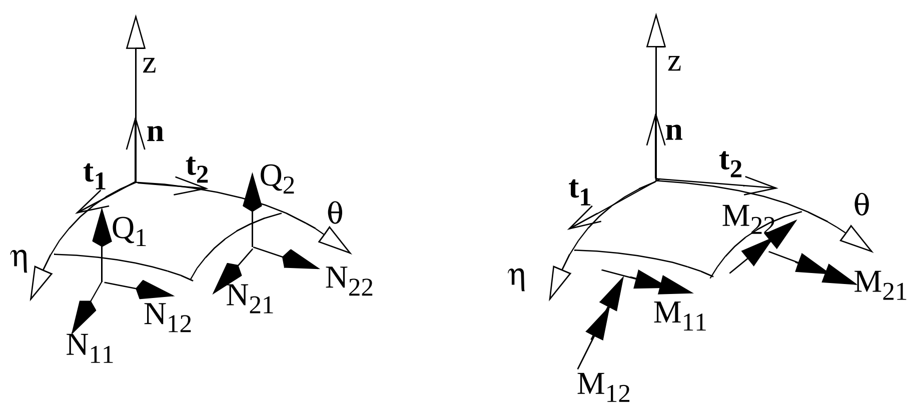


Fig. 3.1 Shell Stress Resultants and Stress Couples

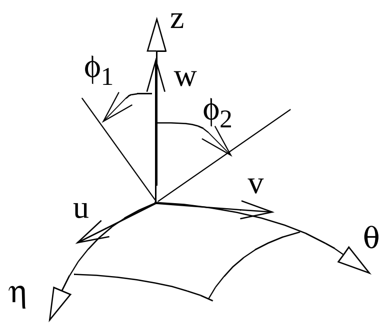


Fig. 3.2  Reference Surface Displacements and Rotations

### 3.2  The linear state vector equation

Reissner's potential function for a shell of revolution may be written as follows, for a circumferentially closed shell of revolution, with the meridional coordinate running from $a$ to $b$ [c.f. Steele and Kim (1992)]

$$J_R = \int_{Area} \{\varepsilon_1^T\sigma_1 + \varepsilon_2^T\sigma_2 - W_c\}dA - \int_{Area} D^T F dA - \int_{C_\sigma} D^T\tilde{\sigma}_1 A_2\Big|_a^b d\theta \; - \int_{C_D} (D-\tilde{D})^T\sigma_1 A_2\Big|_a^b d\theta \qquad \textbf{(3.1)}$$

where

$$\sigma_1 = [N_{11}, N_{12}, Q_1, M_{11}, M_{12}]^T$$
$$\sigma_2 = [N_{21}, N_{22}, Q_2, M_{21}, M_{22}]^T$$
$$\varepsilon_1 = [\varepsilon_{11}^o, \varepsilon_{12}^o, \gamma_{13}^o, \kappa_{11}, \kappa_{12}]^T$$
$$\varepsilon_2 = [\varepsilon_{21}^o, \varepsilon_{22}^o, \gamma_{23}^o, \kappa_{21}, \kappa_{22}]^T$$
$$D = [u, v, w, \varphi_1, \varphi_2]^T$$
$$F = [p_1, p_2, p_3, c_1, c_2]^T$$

The area integration is over the entire surface area, exclusive of the edges; the edges are included in $C_\sigma$ and $C_D$, over which stress resultants and displacements are prescribed, respectively.  Quantities with a tilde denote prescribed boundary values. The terms comprising the vectors $\sigma_1$ and $\sigma_2$ are the stress resultants and stress couples of shell theory (Fig. 3.1). The vectors $\varepsilon_1$ and $\varepsilon_2$ contain middle surface strains and curvatures. The vector $D$ contains middle surface displacements and rotations of the first-order transverse shear deformation theory (Fig. 3.2). See Chapter 2, "Derivation of Basic Equations," for further explanation of these terms. The vector $F$ is a surface load intensity vector, with $p_i$ representing forces per unit undeformed surface area in the meridional, circumferential and normal directions, respectively, and $c_1$ and $c_2$ representing applied moment intensities. The term $W_c$ represents the complementary strain energy density function as defined by Reissner so that

$$\begin{Bmatrix} \dfrac{\partial W_c}{\partial \sigma_1} \\ \dfrac{\partial W_c}{\partial \sigma_2} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1^T \\ \varepsilon_2^T \end{Bmatrix} \tag{3.2}$$

The strain vectors $\varepsilon_1$ and $\varepsilon_2$ may be written in terms of the displacements and displacement gradients as

$$\varepsilon_1 = \frac{1}{A_\eta}\frac{\partial D}{\partial \eta} + G_1 D \qquad \varepsilon_2 = \frac{1}{A_\theta}\frac{\partial D}{\partial \theta} + G_2 D \tag{3.3}$$

with

$$G_1 = \begin{bmatrix} 0 & \Psi_\eta & 1/R_1 & 0 & 0 \\ \Psi_\eta & 0 & 0 & 0 & 0 \\ -1/R_1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \Psi_\eta \\ 0 & 0 & 0 & -\Psi_\eta & 0 \end{bmatrix} \qquad \Psi_\eta = \frac{A_{\eta,\theta}}{A_\eta A_\theta} \tag{3.4}$$

and

$$G_2 = \begin{bmatrix} 0 & -\Psi_\theta & 0 & 0 & 0 \\ \Psi_\theta & 0 & 1/R_2 & 0 & 0 \\ 0 & -1/R_2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\Psi_\theta \\ 0 & 0 & 0 & \Psi_\theta & 0 \end{bmatrix} \qquad \Psi_\theta = \frac{A_{\theta,\eta}}{A_\eta A_\theta} \qquad \textbf{(3.4), cont'd}$$

where the terms $A_\eta$ and $A_\theta$ represent the surface metrics in the meridional and circumferential directions, respectively. Note that with these definitions of $G_1$ and $G_2$, the equation (3.3) is merely a restatement of the equations (2.29). There is also a relationship between stress resultants and strain measures, given by the generalized Hooke's Law:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{Bmatrix} \qquad \text{or} \qquad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \end{Bmatrix} \qquad \textbf{(3.5)}$$

A semi-inversion of equation (3.5) (as shown by Steele and Kim) results in the expression

$$\begin{Bmatrix} \varepsilon_1 \\ \sigma_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{11}^{-1} & \alpha_{11}^{-1}\alpha_{12} \\ \alpha_{21}\alpha_{11}^{-1} & (\alpha_{22} - \alpha_{21}\alpha_{11}^{-1}\alpha_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \varepsilon_2 \end{Bmatrix} \qquad \textbf{(3.6)}$$

Reissner's variational principle states that the state of stress and displacement satisfying the conditions of prescribed tractions and displacements on the boundary of a structure, and also satisfying the equations of equilibrium and the stress-displacement relations interior to the structure is determined by the variational equation

$$\delta J_R = 0 \qquad \textbf{(3.7)}$$

The state vector equation is now found by using equations (3.2), (3.3) and the second line of equation (3.6) in equation (3.1), and taking the mixed variation with respect to $\sigma_1$ and $D$. Variation with respect to $\sigma_2$ is not performed, because only two of the three variables $\sigma_1$, $\sigma_2$, D are linearly independent, as may be seen from equations (3.3) and (3.5). This procedure yields the following equations, in which $I$ is a (5-by-5) identity matrix:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial s} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ D \end{Bmatrix} + \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \frac{\partial}{\partial t} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ D \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ D \end{Bmatrix} + \begin{Bmatrix} -F \\ 0 \\ 0 \end{Bmatrix} \qquad \textbf{(3.8)}$$

$$C_{11} = G_1^T - \frac{1}{A_\theta} \frac{\partial A_\theta}{\partial s} I \qquad C_{12} = G_2^T - \frac{1}{A_\eta} \frac{\partial A_\eta}{\partial t} I \qquad C_{13} = 0$$

$$C_{21} = S_{11} \qquad C_{22} = S_{12} \qquad C_{23} = -G_1$$

$$C_{31} = S_{21} \qquad C_{32} = S_{22} \qquad C_{33} = -G_2$$

and

$$ds = A_\eta d\eta$$

$$dt = A_\theta d\theta$$

with the boundary conditions

$$\delta D^T (\sigma_1 - \tilde{\sigma}_1) \Big|_a^b = 0 \qquad \textbf{(3.9)}$$

That is, at the ends $s = a$, $s = b$, each element of *D* must be prescribed, or its conjugate stress resultant must match the prescribed “tilde” value.

Derivation of equation (3.8) also requires the assumption of circumferential closedness of the shell. The equation (3.8) may be simplified under the assumptions of axisymmetric geometry and response: $\frac{\partial}{\partial t}(\ ) = 0$ where the parentheses may contain any dependent variable or geometric parameter. Then the third line of (3.8) yields an algebraic expression relating $\sigma_2$ to $\sigma_1$ and *D* as

$$\sigma_2 = S_{22}^{-1} [G_2 D - S_{21} \sigma_1] \qquad \textbf{(3.10)}$$

Use of equation (3.10) in equation (3.8) yields the linear state vector equation for axisymmetric geometry and response as

$$\frac{d}{ds}\grave{y}(s) = \begin{bmatrix} \beta_{11}(s) & \beta_{12}(s) \\ \beta_{21}(s) & \beta_{22}(s) \end{bmatrix} \grave{y}(s) + \begin{Bmatrix} -F(s) \\ 0 \end{Bmatrix}$$

$$\grave{y}(s) = [\sigma_1^T, D^T]^T \tag{3.11}$$

$$\beta_{11}(s) = G_1^T - G_2^T S_{22}^{-1} S_{21} - \frac{1}{A_\theta}\frac{dA_\theta}{ds} I \qquad \beta_{12}(s) = G_2^T S_{22}^{-1} G_2$$

$$\beta_{21}(s) = S_{11} - S_{12} S_{22}^{-1} S_{21} \qquad \beta_{22}(s) = S_{12} S_{22}^{-1} G_2 - G_1$$

subject to the boundary conditions of equation (3.9). Note that the state vector equation in the form of equation (3.11) contains all of the dependent variables which are prescribable on meridional faces, and nothing else (see Fig. 3.1 and Fig. 3.2) The stress resultants and stress couples on the circumferential faces may be found using equation (3.10), once the state vector equation is solved. Then, knowing the resultants, the middle surface strains may be found from the generalized Hooke's Law, then the ply stresses and strains may be found, using a procedure like that of classical laminated plate theory. Solution is thus completed, for the static analysis.

## 3.3 The nonlinear equation

The equations for the linear response of shells of revolution were derived using the mixed variational formulation of Reissner, as implemented by Steele and Kim. The derivation required use of strain-displacement relations in the form of equations (3.3). A mixed variational principle for finite deformations was also given by Reissner but this principle may not be used here, for want of a nonlinear equivalent to (3.3). The equation for the geometrically nonlinear response must instead be found by manipulation of the nonlinear field equations for shells derived in chapter 2. The algebra involved in the derivation is extensive, but the manipulations are straight-forward. Only the major steps and the final results will be given here. The equations are derived in a form suitable for solution by an initial value method, such as multiple shooting.

For the geometrically nonlinear analysis, as in the linear analysis, the state vector will consist of those stress resultants and stress couples and those displacements which are defined on meridional edges of the shell, but the stress resultants and stress couples will be of the second

Piola-Kirchhoff type. We thus have

$$\bar{y} = [\overline{\sigma_1}^T, D^T]^T$$

$$\overline{\sigma_1} = [\overline{N}_{11}, \overline{N}_{12}, \overline{Q}_1, \overline{M}_{11}, \overline{M}_{12}]^T$$

$$D = [u, v, w, \phi_1, \phi_2]^T$$

In addition, we define

$$\overline{\sigma_2} = [\overline{N}_{22}, \overline{Q}_2, \overline{M}_{22}]^T$$

$$E_1 = [E_{11}^o, \Gamma_{12}^o, \Gamma_{13}^o, \chi_{11}, \chi_{12}]^T$$

$$E_2 = [E_{22}^o, \Gamma_{23}^o, \chi_{22}]^T$$

The equilibrium equations for geometrically nonlinear response of shells are given in equation (2.53). Modified for the special case of axially symmetric geometry and response, they become

$$\left[\frac{1}{A_1A_2}(A_2N_{11})_{,1} - \frac{A_{2,1}}{A_1A_2}N_{22} + \frac{1}{R_1}Q_1\right] = p\left(\frac{1}{A_1}w_{,1} - \frac{u}{R_1}\right)$$

$$\left[\frac{1}{A_1A_2}(A_2N_{12})_{,1} + \frac{A_{2,1}}{A_1A_2}N_{21} + \frac{1}{R_2}Q_2\right] = p\left(-\frac{v}{R_2}\right) \tag{3.12}$$

$$\left[\frac{1}{A_1A_2}(A_2Q_1)_{,1} - \frac{1}{R_1}N_{11} - \frac{1}{R_2}N_{22}\right] = p\left(1 + \frac{1}{A_1}u_{,1} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)w + \frac{A_{2,1}}{A_1A_2}u\right)$$

$$\left[\frac{1}{A_1A_2}(A_2M_{11})_{,1} - \frac{A_{2,1}}{A_1A_2}M_{22} - S_1\right] = 0$$

$$\left[\frac{1}{A_1A_2}(A_2M_{12})_{,1} + \frac{A_{2,1}}{A_1A_2}M_{21} - S_2\right] = 0$$

The correlation equations (2.50) are used to express the equilibrium equations (3.12) in terms of P-K-2 stress resultants and stress couples. A semi-inversion of the constitutive equations (2.56) yields a modified constitutive law in a form similar to equation (3.6), but in terms of P-K-2 resultants and nonlinear MS shell strains and changes of curvature.

$$\begin{Bmatrix} E_1 \\ \overline{\sigma_2} \end{Bmatrix} = B \begin{Bmatrix} \overline{\sigma_1} \\ E_2 \end{Bmatrix} \tag{3.13}$$

where $B$ is (8-by-8). We note that $E_2$ may be expressed in terms of displacements by use of equations (2.34) and (2.29). These displacements are all prescribable on meridional edges of the shell of revolution, as are all terms of $\overline{\sigma_1}$. We may thus express the dependent variables of $\overline{\sigma_2}$ in terms of dependent variables on the meridional faces only. We will bear this fact in mind, but will not make any substitutions for $\overline{\sigma_2}$ yet, because to do so would make the DE's which we are deriving unnecessarily long and difficult to write.

The first line of equation (3.13) gives us equations for $E_1$ in terms of $\overline{\sigma_1}$ and $E_2$. Other equations for $E_1$ are found by (2.36) and (2.29). We may simply compare expressions to get the ODE's for the displacements of the middle surface as

$$\begin{aligned}
\frac{du}{ds} &= b_{11}\overline{N}_{11} + b_{12}\overline{N}_{12} + b_{14}\overline{M}_{11} + b_{15}\overline{M}_{12} - \frac{w}{R_1} + \left( b_{16}E^o_{22} + b_{18}\chi_{22} - \frac{(\varepsilon^o_{13})^2}{2} \right) \\
\frac{dv}{ds} &= b_{21}\overline{N}_{11} + b_{22}\overline{N}_{12} + b_{24}\overline{M}_{11} + b_{25}\overline{M}_{12} + \Psi_2 v + (b_{26}E^o_{22} + b_{28}\chi_{22} - \varepsilon^o_{13}\varepsilon^o_{23}) \\
\frac{dw}{ds} &= b_{33}\overline{Q}_1 + \frac{u}{R_1} - \phi_1 + (b_{37}\Gamma^o_{23}) \\
\frac{d\phi_1}{ds} &= b_{41}\overline{N}_{11} + b_{42}\overline{N}_{12} + b_{44}\overline{M}_{11} + b_{45}\overline{M}_{12} + \left( \frac{\varepsilon^o_{13}}{R_1}\phi_1 + b_{46}E^o_{22} + b_{48}\chi_{22} + \frac{(\varepsilon^o_{13})^2}{2R_1} \right) \\
\frac{d\phi_2}{ds} &= \left[ b_{51} - \frac{b_{21}}{R_2} \right]\overline{N}_{11} + \left[ b_{52} - \frac{b_{22}}{R_2} \right]\overline{N}_{12} + \left[ b_{54} - \frac{b_{24}}{R_2} \right]\overline{M}_{11} + \left[ b_{55} - \frac{b_{25}}{R_2} \right]\overline{M}_{12} \\
&\quad + \Psi_2 v\left[ \frac{1}{R_1} - \frac{1}{R_2} \right] + \Psi_2\phi_2 + \left( \left[ b_{56} - \frac{b_{26}}{R_2} \right]E^o_{22} + \left[ b_{58} - \frac{b_{28}}{R_2} \right]\chi_{22} + \frac{\varepsilon^o_{13}\phi_2}{R_2} + \frac{\varepsilon^o_{13}\varepsilon^o_{23}}{R_2} + \frac{\varepsilon^o_{23}\phi_1}{R_1} \right)
\end{aligned} \tag{3.14}$$

where $b_{ij}$ represents the ($ij^{\text{th}}$) element of the coefficient matrix $B$ of equation (3.13), and $\Psi_2 = A_{2,1}/(A_1 A_2)$, as in equation (3.4). The final expressions in the parentheses of equations

(3.14) contain nonlinear terms and terms which may be expressed in terms of the dependent variables. For brevity, we omit the expansion.

Equations (3.14) form one half of the state vector equation which is sought. The other half is found by simply rearranging the equations (3.12), after expressing them in terms of P-K-2 stress resultants by use of equations (2.50). We get

$$\frac{d\bar{N}_{11}}{ds} = -\left[\Psi_2 + \frac{\varepsilon_{13}^o}{R_1}\right]\bar{N}_{11} - \frac{\varepsilon_{23}^o}{R_1}\bar{N}_{12} - \frac{1}{R_1}\bar{Q}_1 + \frac{\gamma_{13}^o}{R_1^2}\bar{M}_{11} + \left(\frac{\bar{M}_{12}\phi_2}{R_1R_2} + \bar{N}_{22}\Psi_2\right) \qquad \textbf{(3.15)}$$

$$+ (b_{33}\bar{Q}_1 - \phi_1 + b_{37}\gamma_{23}^o)p$$

$$\frac{d\bar{N}_{12}}{ds} = -\left[\frac{\varepsilon_{13}^o}{R_2} + 2\Psi_2\right]\bar{N}_{12} + \left[\frac{\varepsilon_{13}^o}{R_2^2} + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\Psi_2 + \frac{1}{R_2^2}\frac{dR_2}{ds}\right]\bar{M}_{12}$$

$$+ \left(\frac{(\varepsilon_{23}^o + \gamma_{23}^o)}{R_2^2}\bar{M}_{22} + \frac{\bar{M}_{12}\phi_1}{R_1R_2} - \frac{\varepsilon_{23}^o}{R_2}\bar{N}_{22} - \frac{2}{R_2}\bar{Q}_2\right) - \frac{\nu}{R_2}p$$

$$\frac{d\bar{M}_{11}}{ds} = \bar{Q}_1 - \left[\frac{\varepsilon_{13}^o}{R_1} + \Psi_2\right]\bar{M}_{11} - \frac{\varepsilon_{23}^o}{R_1}\bar{M}_{12} + (\Psi_2\bar{M}_{22})$$

$$\frac{d\bar{M}_{12}}{ds} = -\left[\frac{\varepsilon_{13}^o}{R_2} - 2\Psi_2\right]\bar{M}_{12} + \left(\bar{Q}_2 - \frac{\varepsilon_{23}^o}{R_2}\bar{M}_{22}\right)$$

and

$$\frac{d\bar{Q}_1}{ds} = \frac{-1}{(1+b_{33}\tilde{N})}[q_1\bar{N}'_{11} + q_2\bar{N}'_{12} + q_4\bar{M}'_{11} + q_5\bar{M}'_{12} + q_7 v' + q_9\phi_1{}' + q_{10}\phi_2{}'] \quad \textbf{(3.15), cont'd}$$

$$+\frac{-1}{(1+b_{33}\tilde{N})}\left\{\left[b'_{37}\gamma^o_{23} + \varepsilon^o_{23}\Psi_2 - \frac{1}{R_1}\right]\bar{N}_{11}\right.$$

$$\left. + \varepsilon^o_{23}\Psi_2\bar{N}_{12} + \Psi_2\bar{Q}_1 + \left[\gamma^o_{13}\frac{R'_1}{R_1^2} - \frac{b'_{37}}{R_1}\gamma^o_{23} - \frac{\gamma^o_{13}\Psi_2}{R_1}\right]\bar{M}_{11}\right\}$$

$$-\frac{1}{(1+b_{33}\tilde{N})}\left(b'_{33}\tilde{N}\bar{Q}_1 - \frac{\bar{N}_{22}}{R_2} - \frac{\bar{M}_{12}\phi_2}{R_2}\Psi_2 + \frac{\bar{M}_{12}\phi_2}{R_2^2}R'_2 + \frac{b_{37}R'_2}{R_2^2}\tilde{N}v + \frac{R'_2}{R_2^2}\bar{N}_{12}v\right)$$

$$-\frac{1}{(1+b_{33}\tilde{N})}\left(1 + \Psi_2 u + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)w + u'\right)p$$

where

$$\tilde{N} = \bar{N}_{11} - \frac{\bar{M}_{11}}{R_1}$$

$$q_1 = \varepsilon^o_{13} \qquad q_2 = \varepsilon^o_{23} \qquad q_4 = -\frac{\gamma^o_{13}}{R_1} \qquad q_5 = -\frac{\phi_2}{R_2}$$

$$q_7 = -\left[\frac{b_{37}}{R_2}\tilde{N} + \frac{\bar{N}_{12}}{R_2}\right] \qquad q_9 = -\bar{N}_{11} \qquad q_{10} = b_{37}\tilde{N} - \frac{\bar{M}_{12}}{R_2}$$

in which primes are used on the right-hand side as a shorthand notation: $(\ )' = d(\ )/ds$. Appropriate boundary conditions are found by translating equation (2.54) into P-K-2 resultants by use of equations (2.50). These boundary conditions are that one member must be prescribed from each of the following pairs, on each meridional face of the shell:

$$(\bar{N}_{11}, u), \left(\left[\bar{N}_{12} + \frac{\bar{M}_{12}}{R_2}\right], v\right)$$

$$\left(\left[\bar{Q}_1 + \varepsilon^o_{13}\bar{N}_{11} + \varepsilon^o_{23}\bar{N}_{12} - \frac{\gamma^o_{13}}{R_1}\bar{M}_{11} - \frac{\phi_2}{R_2}\bar{M}_{12}\right], w\right) \qquad \textbf{(3.16)}$$

$$(\bar{M}_{11}, \phi_1), (\bar{M}_{12}, \phi_2)$$

Equations (3.14), (3.15) and (3.16) together with equations (2.29), (2.34) and the second line of

(3.13) may be manipulated to form the nonlinear state vector equation:

$$\begin{aligned} \bar{y}' &= F[s, \bar{y}(s)] \qquad a < s < b \\ G_L(\bar{y}(a)) &= \beta_a \qquad G_R(\bar{y}(b)) = \beta_b \end{aligned} \tag{3.17}$$

but, as has been previously noted, the manipulations result in a set of equations which is quite unwieldy.

We note from the last of equations (3.15) that the system is coupled, and is thus poorly suited for exact solution. We also note, however, that all of the coupling occurs in the equation for $Q'_1$, so that it is easy to incorporate the equations into a numerical procedure by simply calculating $Q'_1$ last. We further note that the equations require foreknowledge of the dependent variables in order to calculate the derivatives, which are used to find the dependent variables. The available boundary conditions do not provide sufficient information to begin the process of exact solution. This difficulty is circumvented by use of an initial value method, in which initial values of the dependent variables are *assumed*. We will solve using the method of multiple shooting.

## 3.4 Equations for Newton's method

A complete description of the multiple shooting method will be given in Chapter 4, but it should be noted that the method is only appropriate for *linear* ODE's. For this work, we will use Newton's method to find a solution to the nonlinear problem by solving a succession of linear problems, upon each of which the multiple shooting method may be used.

### 3.4.1 Newton's method

The central idea of Newton's method is that the solution to a nonlinear problem may be found by making successive approximations of the solution, iterating and updating until the final approximation is sufficiently close to the "true" solution. The procedure described here is discussed also in Ascher, et. al. (1988), pp. 48-49.

The true solution to (3.17) is denoted by $\overline{y}^*$, so that $\overline{y}^{*\prime} - F[s, \overline{y}^*] = 0$. For other values of $\bar{y}$, we have

$$\bar{y}' - F[s, \bar{y}] = R \quad \textbf{(3.18)}$$

where $R$ is known as a *residual* function, and $\overline{y^*}$ satisfies $R = 0$.

For the purpose of iteration, we begin with an initial guess $\overline{y_0}$, and take

$$\overline{y_{i+1}} = \overline{y_i} + \Delta\overline{y_{i+1}} \quad \textbf{(3.19)}$$

which gives

$$(\overline{y_{i+1}})' = (\overline{y_i})' + (\Delta\overline{y_{i+1}})' \quad \textbf{(3.20)}$$

We also have from (3.18),

$$(\overline{y_{i+1}})' = F[s, \overline{y_{i+1}}] + R_{i+1} \quad \textbf{(3.21)}$$

Then, assuming the "correction" term $\Delta\overline{y_{i+1}}$ to be small, we may take a single-term Taylor series expansion for $F[s, \overline{y_{i+1}}]$ to get

$$F[s, \overline{y_{i+1}}] = F[s, \overline{y_i}] + \left.\frac{\partial F}{\partial \bar{y}}\right|_{\bar{y}_i} \Delta\overline{y_{i+1}} + R_{i+1} = \overline{y_i}' - R_i + \left.\frac{\partial F}{\partial \bar{y}}\right|_{\bar{y}_i} + R_{i+1} \quad \textbf{(3.22)}$$

with the last equality following from (3.18). By (3.20)-(3.22) we get

$$(\Delta\overline{y_{i+1}})' = \left.\frac{\partial F}{\partial \bar{y}}\right|_{\bar{y}_i} \Delta\overline{y_{i+1}} + (R_{i+1} - R_i)$$

Finally, we assume $\overline{y_{i+1}} = \overline{y^*}$, so that $R_{i+1} = 0$, yielding

$$(\Delta\overline{y_{i+1}})' = \left.\frac{\partial F}{\partial \bar{y}}\right|_{\bar{y}_i} \Delta\overline{y_{i+1}} - R_i \quad \textbf{(3.23)}$$

Appropriate boundary conditions are found by a similar method, beginning with the BC's of equations (3.17). The set of equations (3.23) with the boundary conditions thus form a linear two-point boundary value problem, suitable for solution by shooting. The Jacobian matrix $\left.\frac{\partial F}{\partial \bar{y}}\right|_{\bar{y}_i}$ is found by collecting the coefficients of linear $\Delta\overline{y_{i+1}}$ terms; terms of second and higher orders are discarded. The residuals $R_i$ are found by (3.18).

### 3.4.2 The linearized equations for Newton's method

The Jacobian matrix $\left.\frac{\partial F}{\partial \bar{y}}\right|_{\bar{y}_i}$ may be difficult to obtain exactly, and its calculation is numerically expensive. On the other hand, we have in equations (3.14) and (3.15) exact equations for the geometrically nonlinear response, and may apply equation (3.19) directly and linearize appropriately in the $\Delta\bar{y}$ terms to get an expression for $\left.\frac{\partial F}{\partial \bar{y}}\right|_{\bar{y}_i}\Delta\overline{y_{i+1}}$ suitable for use in equation (3.23). In so doing, we get (with all terms evaluated at the $i$+1$^{\text{th}}$ iteration step, unless subscripted otherwise)

$$\Delta u' = b_{11}\Delta\overline{N}_{11} + b_{12}\Delta\overline{N}_{12} + b_{14}\Delta\overline{M}_{11} + b_{15}\Delta\overline{M}_{12} - \frac{1}{R_1}\Delta w \tag{3.24}$$

$$+ (b_{16}\Delta E^{o}_{22} + b_{18}\Delta\chi_{22} - (\varepsilon^{o}_{13})_i\Delta\varepsilon^{o}_{13})$$

$$\Delta v' = b_{21}\Delta\overline{N}_{11} + b_{22}\Delta\overline{N}_{12} + b_{24}\Delta\overline{M}_{11} + b_{25}\Delta\overline{M}_{12} + \Psi_2\Delta v$$
$$+ (b_{26}\Delta E^{o}_{22} + b_{28}\Delta\chi_{22} - (\varepsilon^{o}_{13})_i\Delta\varepsilon^{o}_{23} - \Delta\varepsilon^{o}_{13}(\varepsilon^{o}_{23})_i)$$

$$\Delta w' = b_{33}\Delta Q_1 + \frac{1}{R_1}\Delta u - \Delta\phi_1 + (b_{37}\Delta\gamma^{o}_{23})$$

$$\Delta\phi_1' = b_{41}\Delta\overline{N}_{11} + b_{42}\Delta\overline{N}_{12} + b_{44}\Delta\overline{M}_{11} + b_{45}\Delta\overline{M}_{12} + \frac{(\varepsilon^{o}_{13})_i}{R_1}\Delta\phi_1$$

$$+ \left(b_{46}\Delta E^{o}_{22} + b_{48}\Delta\chi_{22} + \left(\frac{(\varepsilon^{o}_{13})_i}{R_1} + \frac{(\phi_1)_i}{R_1}\right)\Delta\varepsilon^{o}_{13}\right)$$

$$\Delta\phi_2{}' = \left[b_{51} - \frac{b_{21}}{R_2}\right]\Delta\bar{N}_{11} + \left[b_{52} - \frac{b_{22}}{R_2}\right]\Delta\bar{N}_{12} + \left[b_{54} - \frac{b_{24}}{R_2}\right]\Delta\bar{M}_{11} + \left[b_{55} - \frac{b_{25}}{R_2}\right]\Delta\bar{M}_{12}$$

$$+ \left[\frac{1}{R_1} - \frac{1}{R_2}\right]\Psi_2\Delta v + \frac{(\varepsilon_{23}^o)_i}{R_1}\Delta\phi_1 + \left[\Psi_2 + \frac{(\varepsilon_{13}^o)_i}{R_2}\right]\Delta\phi_2$$

$$+ \left(\left[b_{56} - \frac{b_{26}}{R_2}\right]\Delta E_{22}^o + \left[b_{58} - \frac{b_{28}}{R_2}\right]\Delta\chi_{22} + \left[\frac{(\phi_1)_i}{R_1} + \frac{(\varepsilon_{13}^o)_i}{R_2}\right]\Delta\varepsilon_{23}^o + \left[\frac{(\varepsilon_{23}^o)_i}{R_2} + \frac{(\phi_2)_i}{R_2}\right]\Delta\varepsilon_{13}^o\right)$$

$$\Delta(\bar{N}_{11})' = \left[-\Psi_2 - \frac{(\varepsilon_{13}^o)_i}{R_1}\right]\Delta\bar{N}_{11} - \frac{(\varepsilon_{23}^o)_i}{R_1}\Delta\bar{N}_{12} - \frac{1}{R_1}\Delta\bar{Q}_1 + \frac{(\gamma_{13}^o)_i}{R_1^2}\Delta\bar{M}_{11} + \frac{(\phi_2)_i}{R_1R_2}\Delta\bar{M}_{12}$$

$$+ \frac{(\bar{M}_{12})_i}{R_1R_2}\Delta\phi_2 + \left(\Psi_2\Delta\bar{N}_{22} - \frac{(\bar{N}_{11})_i}{R_1}\Delta\varepsilon_{13}^o - \frac{(\bar{N}_{12})_i}{R_1}\Delta\varepsilon_{23}^o + \frac{(\bar{M}_{11})_i}{R_1^2}\Delta\gamma_{13}^o\right)$$

$$+ [b_{33}\Delta\bar{Q}_1 - \Delta\phi_1 + b_{37}\Delta\gamma_{23}^o]p$$

$$\Delta(\bar{N}_{12})' = \left[-2\Psi_2 - \frac{(\varepsilon_{13}^o)_i}{R_2}\right]\Delta\bar{N}_{12} + \left[\Psi_2\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \frac{R_2{}'}{R_2^2} + \frac{(\varepsilon_{13}^o)_i}{R_2^2} + \frac{(\phi_2)_i}{R_1R_2}\right]\Delta\bar{M}_{12}$$

$$+ \frac{(\bar{M}_{12})_i}{R_1R_2}\Delta\phi_1 + \left(\left[\frac{(\varepsilon_{23}^o)_i}{R_2^2} + \frac{(\gamma_{23}^o)_i}{R_2^2}\right]\Delta\bar{M}_{22} + \left[-\frac{2}{R_2}\right]\Delta\bar{Q}_2 + \left[\frac{(\bar{M}_{12})_i}{R_2^2} - \frac{(\bar{N}_{12})_i}{R_2}\right]\Delta\varepsilon_{13}^o\right.$$

$$\left. + \left[\frac{(\bar{M}_{22})_i}{R_2^2} - \frac{(\bar{N}_{22})_i}{R_2}\right]\Delta\varepsilon_{23}^o + \frac{(M_{22})_i}{R_2^2}\Delta\gamma_{23}^o - \frac{(\varepsilon_{23}^o)_i}{R_2}\Delta\bar{N}_{22}\right) - \frac{\Delta v}{R_2}p$$

$$\Delta(\bar{M}_{11})' = \Delta\bar{Q}_1 + \left[-\Psi_2 - \frac{(\varepsilon_{13}^o)_i}{R_1}\right]\Delta\bar{M}_{11} - \frac{(\varepsilon_{23}^o)_i}{R_1}\Delta\bar{M}_{12}$$

$$+ \left(-\frac{(\bar{M}_{11})_i}{R_1}\Delta\varepsilon_{13}^o - \frac{(\bar{M}_{12})_i}{R_1}\Delta\varepsilon_{23}^o + \Psi_2\Delta\bar{M}_{22}\right)$$

$$\Delta(\overline{M}_{12})'$$

$$= \left[-2\Psi_2 - \frac{(\varepsilon_{13}^o)_i}{R_2}\right]\Delta\overline{M}_{12} + \left(-\frac{(\varepsilon_{23}^o)_i}{R_2}\Delta\overline{M}_{22} + \Delta\overline{Q}_2 - \frac{(\overline{M}_{12})_i}{R_2}\Delta\varepsilon_{13}^o - \frac{(\overline{M}_{22})_i}{R_2}\Delta\varepsilon_{23}^o\right)$$

$$\Delta(\overline{Q}_1)' = -\frac{1}{(1 + b_{33}\tilde{N}_i)}[(q_1)_i\Delta(\overline{N}_{11})' + (q_2)_i\Delta(\overline{N}_{12})' + (q_4)_i\Delta(\overline{M}_{11})'$$

$$+ (q_5)_i\Delta(\overline{M}_{12})' + (q_7)_i\Delta v' + (q_9)_i\Delta\phi_1' + (q_{10})_i\Delta\phi_2']$$

$$-\frac{1}{(1 + b_{33}\tilde{N}_i)}$$

$$[r_1\Delta\overline{N}_{11} + r_2\Delta\overline{N}_{12} + r_3\Delta\overline{Q}_1 + r_4\Delta\overline{M}_{11} + r_5\Delta\overline{M}_{12} + r_7\Delta v + r_{10}\Delta\phi_2]$$

$$-\frac{1}{(1 + b_{33}\tilde{N}_i)}\left[-\frac{1}{R_2}\Delta\overline{N}_{22} + [(\overline{N}_{11})'_i + (\overline{N}_{11})_i\Psi_2]\Delta\varepsilon_{13}^o\right.$$

$$+ [(\overline{N}_{12})'_i + (\overline{N}_{12})_i\Psi_2]\Delta\varepsilon_{23}^o + \left[\frac{(\overline{M}_{11})_i(R_1)'}{R_1^2} - \frac{(\overline{M}_{11})'_i}{R_1} - \frac{(\overline{M}_{11})_i\Psi_2}{R_1}\right]\Delta\gamma_{13}^o$$

$$\left. + b'_{37}\tilde{N}\Delta\gamma_{23}^o\right] - \frac{1}{(1 + b_{33}\tilde{N}_i)}\left[\Delta u' + \Psi_2\Delta u + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\Delta w\right]p$$

with $(q_k)_i$, $N_i$ defined as for equation (3.15), evaluated following the $i^{\text{th}}$ iteration step, and

$$r_1 = \left[-(\phi_1)'_i + b_{37}(\phi_2)'_i - \frac{b_{37}(v')_i}{R_2} - \frac{1}{R_1} + \left(b'_{33}\overline{Q}_1 + \frac{b_{37}(R_2)'}{R_2^2}v + \Psi_2\varepsilon_{13}^o + b'_{37}\gamma_{23}^o\right)_i\right]$$

$$r_2 = \left[-\frac{1}{R_2}(v')_i + \left(\Psi_2\varepsilon_{23}^o + \frac{(R_2)'}{R_2^2}v\right)_i\right] \qquad r_3 = b'_{33}\tilde{N}_i + \Psi_2$$

$$r_4 = \left[\left(-\frac{b_{33}(\bar{Q}_1)'}{R_1} + \frac{b_{37}v'}{R_1R_2} - \frac{b_{37}\phi_2'}{R_1}\right)_i + \left(-\frac{b'_{33}\bar{Q}_1}{R_1} - \frac{b_{37}R_2'v}{R_1R_2^2} + \left[\frac{R_1'}{R_1^2} - \frac{\Psi_2}{R_1}\right]\gamma_{13}^o - \frac{b'_{37}\gamma_{23}^o}{R_1}\right)_i\right]$$

$$r_5 = \left[-\frac{(\phi_2)'_i}{R_2} + \left(\frac{R_2'}{R_2^2} - \frac{\Psi_2}{R_2}\right)(\phi_2)_i\right] \qquad r_7 = \left[(b_{37}\tilde{N}_i + \bar{N}_{12})_i\frac{R_2'}{R_2^2}\right]$$

$$r_{10} = \left[\frac{-(\bar{M}_{12})'_i}{R_2} - \left(\frac{\Psi_2}{R_2} - \frac{R_2'}{R_2^2}\right)(\bar{M}_{12})_i\right]$$

We thus, finally, have a set of linear ODE's in the correction terms, which is solvable by shooting. As in the general nonlinear equations, the linearized equations for Newton's method are coupled, and require that the coefficients of $\Delta(Q_1)'$ be calculated last. Again, the coupling may be undone, but the equations obtained in so doing are quite lengthy. The form chosen for presentation here is thought also to be more easily incorporated into a computational code.

## 3.5 Incorporation of elastic boundary conditions

Appropriate boundary conditions for the state vector equation take on the form

$$G[y(a), y(b)] = \beta,$$

or, for separated boundary conditions,

$$G_a[y(a)] = \beta_a \qquad \text{and} \qquad G_b[y(b)] = \beta_b \tag{3.25}$$

Allowing for elastic supports, equation (3.25) takes on the linear form

$$B_a y(a) = \beta_a \qquad \text{and} \qquad B_b y(b) = \beta_b \tag{3.26}$$

For this special case, the matrices $B_a$, $B_b$ and the vectors $\beta_a$, $\beta_b$ are found by consideration of equilibrium at the location of the support, as well as the constitutive law. Incorporation of elastic supports is a significant step in the analysis, as it allows the dome to provide closure to any arbitrary pressure vessel, with the following restriction: the vessel must display axisymmetry of geometry, loading and response in the immediate vicinity of the vessel/dome interface. Likewise, the dome may itself be capped by an arbitrary, but locally axisymmetric, structure.

The elastic response of the support structure is defined by

$$\sigma_e = \sigma_e(p, \delta)$$

in which $\sigma_e$ is the vector of meridional-face stress resultants acting upon the support at the joint location, $p$ is the applied load vector, and $\delta$ is the displacement vector at the joint. In general, $\sigma_e$ is a nonlinear function of ($p$, $\delta$). The function may be linearized about a chosen reference state $(p, \delta) = (p^*, \delta^*)$ by use of a Taylor series expansion:

$$\sigma_e(p, \delta) = \sigma_e(p^*, \delta^*) + \left.\frac{\partial \sigma_e}{\partial p}\right|_{p^*,\, \delta^*} \Delta p + \left.\frac{\partial \sigma_e}{\partial \delta}\right|_{p^*,\, \delta^*} \Delta \delta$$

If we now take $\Delta p = p - p^*$ $\quad \Delta\delta = \delta - \delta^*$,
we get

$$\sigma_e(p, \delta) = \sigma_e^* + K\delta + Vp\,, \tag{3.27}$$

$$K = \left.\frac{\partial \sigma_e}{\partial \delta}\right|_{p^*,\, \delta^*} \qquad V = \left.\frac{\partial \sigma_e}{\partial p}\right|_{p^*,\, \delta^*}$$

$$\sigma_e^* = \sigma_e(p^*, \delta^*) - K\delta^* - Vp^*$$

Thus, assuming the terms $\sigma_e(p^*, \delta^*), K, V$ may be found, elastic BC's may be implemented which are valid in the neighborhood of $(p^*, \delta^*)$.

For the special case of linear elastic supports, we may take the reference state to be given by $(p^*, \delta^*) = (0, 0)$, which is chosen for convenience, and assume that this reference state corresponds to a zero-stress state $\sigma_e(0, 0) = 0$. Then by equation , $\sigma_e^* = 0$, and (3.27) becomes

$$\sigma_e(p, \delta) = K\delta + Vp \tag{3.28}$$

The relation (3.28) must hold for any $(p, \delta)$, so that $K$, $V$ may be found by evaluation of $\sigma_e$ for chosen $(p, \delta)$. For example, letting $\delta$ be the unit direction vector $\hat{e}_i$, $i = 1, 2, \ldots$ and taking $p$ to be a zero vector, we see that the $i^{\text{th}}$ column of $K$ must be equal to $\sigma_e$. Similarly, letting $p$ be given by $\hat{e}_i$ with $\delta = 0$, we get the $i^{\text{th}}$ column of $V$. Philosophically, this corresponds to taking the elastic response to be a superposition of that “force” necessary to achieve a zero-displacement state at unit loading, multiplied by the load (i.e., $Vp$) and the “force” correspondent to the actual displacements, if there were no other loading (that is, $K\delta$).