Weighted Optimality of Block Designs

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(ABSTRACT)

Design optimality for treatment comparison experiments has been intensively studied by numerous researchers, employing a variety of statistically sound criteria. Their general formulation is based on the idea that optimality functions of the treatment information matrix are invariant to treatment permutation. This implies equal interest in all treatments. In practice, however, there are many experiments where not all treatments are equally important. When selecting a design for such an experiment, it would be better to weight the information gathered on different treatments according to their relative importance and/or interest. This dissertation develops a general theory of weighted design optimality, with special attention to the block design problem.

Among others, this study develops and justifies *weighted* versions of the popular A, E and MV optimality criteria. These are based on the weighted information matrix, also introduced here. Sufficient conditions are derived for block designs to be weighted A, E and MV-optimal for situations where treatments fall into two groups according to two distinct levels of interest, these being important special cases of the "2-weight optimality" problem. Particularly, optimal designs are developed for experiments where one of the treatments is a control.

The concept of efficiency balance is also studied in this dissertation. One view of efficiency balance and its generalizations is that unequal treatment replications are chosen to reflect unequal treatment interest. It is revealed that efficiency balance is closely related to the weighted-E approach to design selection. Functions of the canonical efficiency factors may be interpreted as weighted optimality criteria for comparison of designs with the same replication numbers.

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Chapter 1

Introduction

1.1 Classical Optimality of Block Designs

A brief review of the classical optimality approach to block designs aims at helping highlight aspects and motivations of this research. Blocking, introduced by Fisher (1925), is one of the basic principles of experimental design. Blocking aims at making an experiment more efficient by grouping experimental units into "blocks", such that the units are as uniform within blocks as possible. The stratification of the experimental units may be performed in one direction, which is ordinary blocking, or in two, giving rise to row-column designs such as Latin squares and Youden designs. In this research, we investigate the optimality for ordinary block designs, for which the term "block design" is reserved throughout.

A block design is an assignment of v treatments to bk experimental units arranged in b blocks of size k. Particularly, if $k < v$, the design is called an *incomplete block design.* Any block design is connected if it allows unbiased estimation of all treatment contrasts. Only connected designs are considered here.

Let $\mathcal{D}(v, b, k)$ be the class of all connected block designs for given v, b and k. For a design $d \in \mathcal{D}(v,b,k)$, let $N_d = ((n_{dij}))$ denote the $v \times b$ incidence matrix whose entries are nonnegative integers indicating the number of units to which treatment i is assigned in block j. The matrix $N_d N_d^T$ is the treatment concurrence matrix of d, whose entries are $\lambda_{dii'} = \sum_{j=1}^{b} n_{dij} n_{di'j}$. The information matrix for treatment estimation, or C-matrix, obtained from the standard additive model (see (1.2)) for treatment and block effects is

$$C_d = D(r_d) - \frac{1}{k} N_d N_d^T, \tag{1.1}$$

where r_d is a $v \times 1$ vector whose i^{th} element $r_{di} = \sum_{j=1}^{b} n_{dij}$ is the replication of treatment i, and $D(r_d) = Diag(r_{di})$ (Chakrabarti, 1963). The above non-negative definite (nnd) information matrix depends on the design d and, provided that the design is connected, has rank $v-1$.

Useful measures of a design's "goodness" are based on increasing functions of contrast variances, and so are functions of the C-matrix through its Moore-Penrose inverse, denoted C^+. A good design will make variances small and so will minimize the appropriate such function. Different functions lead to different optimality criteria.

Definition 1.1. Let $0 < e_{d1} \leq e_{d2} \leq \ldots \leq e_{d,v-1}$ be the positive eigenvalues of C_d. Thus, $\frac{1}{e_{d1}} \geq \frac{1}{e_{d2}} \geq \ldots \geq \frac{1}{e_{d,v-1}}$ are the positive eigenvalues of C_d^+, called the *canonical variances.*

Definition 1.2. A design $\bar{d}$ is E-optimal in a design class $\mathcal{D}$ if it minimizes the maximal canonical variance, that is, if $\frac{1}{e_{\bar{d}1}} = \min_{d \in \mathcal{D}} \frac{1}{e_{d1}}$, or equivalently, $e_{\bar{d}1} = \max_{d \in \mathcal{D}} e_{d1}$.

Definition 1.3. A design $\bar{d}$ is A-optimal in a design class $\mathcal{D}$ if it minimizes the average canonical variance, that is, if $\sum_{i=1}^{v-1} \frac{1}{e_{\bar{d}i}} = \min_{d \in \mathcal{D}} \sum_{i=1}^{v-1} \frac{1}{e_{di}}$.

Definition 1.4. A design $\bar{d}$ is D-optimal in a design class $\mathcal{D}$ if it minimizes the product of the canonical variances, that is, if $\prod_{i=1}^{v-1} \frac{1}{e_{\bar{d}i}} = \min_{d \in \mathcal{D}} \prod_{i=1}^{v-1} \frac{1}{e_{di}}$, or equivalently, $\prod_{i=1}^{v-1} e_{\bar{d}i} = \max_{d \in \mathcal{D}} \prod_{i=1}^{v-1} e_{di}$.

Besides the E-, A-, and D-criteria, the MV-criterion, which is not a function of the canonical variances, is also investigated in this research.

Definition 1.5. Let $p_{d1} \geq p_{d2} \geq \ldots \geq p_{dv(v-1)/2}$ be the $\binom{v}{2}$ variances $var_d(\widehat{\tau_i - \tau_{i'}})$ for the pairwise treatment contrasts when using design d. A design $\bar{d}$ is MV-optimal over $\mathcal{D}(v, b, k)$ if it minimizes the maximum variance over all paired treatment contrasts, i.e. $p_{\bar{d}1} = \min_{d \in \mathcal{D}} p_{d1}$.

Kiefer (1975) introduced convex optimality functions Φ on the information matrices, and proved that balanced incomplete block designs ($BIBD$s) are universally optimal, i.e. minimize $\Phi(C_d)$ for every non-increasing, convex, permutation-invariant Φ. Following closely on the heels of Kiefer's work, John and Mitchell (1977), Cheng (1978, 1980), and Jacroux (1980a,b) used computer search and theoretical arguments to build optimal designs for the criteria defined above, all of which fall into Kiefer's framework. More recently, Majumdar and Notz (1983), Majumdar (1986), Jacroux and Majumdar (1989), Bagchi (1989, 1991, 1996, 2001) and Morgan (2000, 2003, 2005, 2007) have been working on design optimality for various classes of designs with blocking.

Let Y_{uj} be the observation on experimental unit u in block j. The commonly employed statistical model for any block design d, which in many cases is justifiable by randomization alone (see Hinkelmann and Kempthorne, 2008), is

$$Y_{uj} = \mu + \tau_{d[u,j]} + \beta_j + E_{uj}, \tag{1.2}$$

where

$\mu =$ mean response over all treatments and blocks,

$d[u, j] =$ the treatment assigned to unit u in block j by design d,

$\tau_{d[u,j]} =$ the effect of the treatment assigned to unit u in block j by design d,

$\beta_j =$ the effect of block j,

and the E_{uj}'s are uncorrelated, mean zero random variables with common variance σ_E^2. This model is employed in most of the papers cited above. It is the basis for the information matrix (1.1). We usually assume with no loss of generality that the unit variability σ_E^2 is $\sigma_E^2 = 1$. The symbol N is used for the total number of experimental units; $N = bk$.

1.2 Weighted Optimality of Block Designs

Kiefer's design optimality is based on functions of the information matrix that are invariant to treatment permutation, that is, $\Phi(PC_dP^T) = \Phi(C_d)$ for any permutation matrix P. This implies equal interest in all treatments. However, in practice there are many cases where not all treatments are equally important. For instance, we often encounter experimental situations where some test treatments are to be compared to a standard treatment (or control treatment). Sometimes the control is included specifically to verify the expectation of large treatment effects relative to control, after which the important comparisons among test treatments are performed. This indicates asymmetry of interest on test treatments and the control treatment, with (in this case) greater interest in test treatments than control.

Asymmetry of treatment interest implies that optimality based on the information matrix should not be invariant to all permutations. With the premise of asymmetric interest, the approach here is to group treatments into several subsets which are assigned distinct weights; larger weight reflects greater interest placed on estimating comparisons involving the corresponding treatments. In situations like that described above, this leads to a 2-weight design problem, that is, the weights take only two values, with one small weight and $v-1$ larger, equal weights.

The theme of this research is to remove the symmetry restriction in a meaningful way and develop new optimality theory accordingly. Asymmetry of treatment interest is incorporated into standard optimality criteria, then conditions for and constructions of designs respecting the weights are obtained. The 2-weight optimality problem is the starting point in tackling *weighted design optimality.*

Gupta(1999, 2002) has used the term "weighted A-optimal" when comparing a group of test treatments with a group of control treatments. Two different sets of contrasts, *treatment-control* and *treatment-treatment* were considered to be estimated with unequal precision. Compared to earlier work, this dissertation introduces a general idea of "weighted optimality" that can be applied to any experimental situation (not limited to *treatment-control*) and generalizing all standard (not just A) optimality criteria.

Design of experiments for which some of the treatments are controls has, from a special perspective, been extensively investigated in recent years. That perspective is formalized in the following definition.

Definition 1.6. A *test treatment versus control* (or TvC) experiment is an experiment in which

(i) one of the v treatments is a control, and

(ii) the sole purpose of the experiment is to compare the control to the other, *test* treatments.

Notable among the many papers seeking optimal designs for test treatment versus control experiments are Jacroux (1989), Jacroux and Majumdar (1989), Majumdar (1992, 1996a, b), Majumdar and Notz (1983), Stufken (1991a, b). Results in Section 2.1.1 will establish that optimality work for TvC experiments is a limiting case, as the weight on the control treatment goes to 1, of the weighted optimality framework constructed in this dissertation.

Notably absent from the design optimality literature is work for experiments of the type described in the first paragraph of this section. These we formalize with a second definition.

Definition 1.7. A *test treatment with control* (or TwC) experiment is an experiment in which

(i) one of the v treatments is a control, and

(ii) the control treatment is intended as a check on the expected efficacy of the other, *test* treatments, and accordingly less importance is placed on estimation of differences between the control and the test treatments.

The new approach advocated here allows for rigorous optimality investigation of TwC experiments. Weights can be chosen to incorporate the relative importance placed on test treatment versus control, and test treatment versus test treatment, comparisons. In the chapters that follow, all results for the 2-weight problem, where one of the weights is assigned to a single treatment, have applications to TwC experiments. From a larger perspective, an important contribution of the new approach is to bring TwC and TvC experiments under a common optimality umbrella.

1.3 Outline of Dissertation

This dissertation is organized as follows. In chapter 2, we give some preliminary results on weighted optimality and some basic theory on matrix convexity, which is critical to our research. In chapter 3, 4, and 5, theorems on sufficient conditions for weighted E, MV and A optimality are given, and some methods of construction are included as well. In chapter 6, future work on weighted optimality is proposed.

Chapter 2

Preliminary Results

The main purpose of this chapter is to introduce main concepts and definitions, and to develop some preliminary results on weighted optimality.

2.1 Definitions and Preliminary Results for Weighted Optimality

Just as many conventional criteria are functions of the eigenvalues of the information matrix C_d, many of the weighted criteria which will be used to evaluate design optimality are functions of the eigenvalues of the weighted information matrix C_{dw}, defined as follows.

Definition 2.1. Let positive weights w_1, w_2, ..., w_v be measures of interest on v treatments where without loss of generality $\sum_{i=1}^{v} w_i = 1$. Let W be a $v \times v$ diagonal matrix with w_i in the i^{th} diagonal position, that is, $W = Diag(w_i)$. Also, the square root matrix for W is denoted as $W^{1/2}$. Then the weighted information matrix C_{dw} for design d is defined as

$$C_{dw} = W^{-1/2} C_d W^{-1/2} = ((c_{dii'}/\sqrt{w_i w_{i'}})) \tag{2.1}$$

.

The use of C_{dw} will be justified in the following facts. The key is to see how applying weights to C_d induces weights on variances of treatment contrasts. Consider the spectral decomposition of the C_{dw}-matrix:

$$C_{dw} = W^{-1/2} C_d W^{-1/2} = \sum_{i=0}^{v-1} \theta_i f_i f_i' \tag{2.2}$$

where $\theta_0 < \theta_1 \leq \cdots \leq \theta_{v-1}$ are the eigenvalues of C_{dw} and the f_i are an orthonormal set of eigenvectors. For connected designs, both C_d and C_{dw} are of rank $v-1$, and $\theta_0 = 0$. Let $w = (w_1, w_2, \ldots, w_v)'$ be the vector of weights. The eigenvector corresponding to θ_0 is $f_0 = w^{1/2} = \left(\sqrt{w_1}, \sqrt{w_2}, \cdots, \sqrt{w_v}\right)'$.

In optimality studies, designs are sought to minimize functions of variances of normalized contrasts. A weighted variance will be a variance of a contrast but multiplied by an appropriate weight, and normalization will play into weighted variance as well. For notational convenience, the constant σ_E^2 will be taken to be $\sigma_E^2 = 1$ (also see (1.2) and following).

Lemma 2.1. *Let C_d^- and C_{dw}^- be arbitrary generalized inverses of C_d and C_{dw}, respectively. Then $W^{-1/2} C_{dw}^- W^{-1/2}$ is a generalized inverse of C_d, and $W^{1/2} C_d^- W^{1/2}$ is a generalized inverse of C_{dw}.*

Proof.

$$\begin{aligned} C_d W^{-1/2} C_{dw}^- W^{-1/2} C_d &= W^{1/2}(W^{-1/2} C_d W^{-1/2}) C_{dw}^- (W^{-1/2} C_d W^{-1/2}) W^{1/2} \\ &= W^{1/2} C_{dw} C_{dw}^- C_{dw} W^{1/2} = W^{1/2} C_{dw} W^{1/2} = C_d \end{aligned}$$

Similarly,

$$\begin{aligned} C_{dw} W^{1/2} C_d^- W^{1/2} C_{dw} &= W^{-1/2}(W^{1/2} C_{dw} W^{1/2}) C_d^- (W^{1/2} C_{dw} W^{1/2}) W^{-1/2} \\ &= W^{-1/2} C_d C_d^- C_d W^{-1/2} = W^{-1/2} C_d W^{-1/2} = C_{dw} \end{aligned}$$

$\square$

Define the $v \times v$ matrix $F = (f_0, f_1, f_2, \ldots, f_{v-1})$ and let $L = W^{1/2}F$ have columns $l_i = W^{1/2}f_i$. Then $LL^T = W$ and $L^T W^{-1} L = I$. Note $l_i'1 = 0$ for $i = 1, \ldots, v-1$ implies $l_1, l_2, \ldots, l_{v-1}$ are contrast vectors. Also, $l_0 = w$.

One of the generalized inverses of C_d is $W^{-1/2}C_{dw}^+ W^{-1/2}$, where $C_{dw}^+ = \sum_{i=1}^{v-1} \frac{1}{\theta_i} f_i f_i'$ is the Moore-Penrose inverse of C_{dw}. Now calculate the variance of the contrast $\widehat{l_i'\tau}$:

$$Var(\widehat{l_i'\tau}) = l_i' C_d^- l_i = l_i' W^{-1/2} C_{dw}^+ W^{-1/2} l_i = f_i' C_{dw}^+ f_i = \frac{1}{\theta_i}. \tag{2.3}$$

It can be seen that the variances of the $\widehat{l_i'\tau}$ are the inverses of the positive eigenvalues of the weighted information matrix C_{dw}. Note l_i are normalized relative to the weights: $l_i' W^{-1} l_i = 1$. That is, $l_1'\tau, \ldots, l_{v-1}'\tau$ are a set of $v-1$ contrasts of equal weight, and moreover,

$$\mathrm{Cov}(\widehat{l_i'\tau}, \widehat{l_j'\tau}) = l_i' C_d^- l_j = l_i' W^{-1/2} C_{dw}^+ W^{-1/2} l_j = f_i' C_{dw}^+ f_j = 0 = f_i' f_j = l_i' W^{-1} l_j. \tag{2.4}$$

The property $l_i' W^{-1} l_j = 0$ in (2.4) is termed *weighted orthogonality* of the contrast vectors $l_1, l_2, \ldots, l_{v-1}$.

Now consider an arbitrary contrast $c'\tau$. Be aware $l_1, l_2, \ldots, l_{v-1}$ are a basis for the space of all contrast vectors. Write $c = L_{(0)}q$ for some $q_{(v-1)\times 1}$, where $L_{(0)} = (l_1, l_2, \ldots, l_{v-1})$ is the $v \times (v-1)$ matrix with columns l_i excluding l_0. The variance of $\widehat{c'\tau}$ is

$$\begin{aligned} Var(\widehat{c'\tau}) &= q' L_{(0)}' W^{-1/2} C_{dw}^+ W^{-1/2} L_{(0)} q = q' F_{(0)}' C_{dw}^+ F_{(0)} q = q' = q' F_{(0)}' (\sum_{i=1}^{v-1} \frac{1}{\theta_i} f_i f_i') F_{(0)} q \\ &= q' Diag(\frac{1}{\theta_i}) q = \sum_{i=1}^{v-1} \frac{q_i^2}{\theta_i} \end{aligned} \tag{2.5}$$

The variance in (2.5) is a convex combination of the $1/\theta_i$ *provided* $q'q = 1$. This is the key to a proper definition of weighted variance. Now $L_{(0)}^T W^{-1} L_{(0)} = I_{v-1}$ so that $q'q = q' L_{(0)}' W^{-1} L_{(0)} q = c' W^{-1} c$.

Definition 2.2. The weighted variance for contrast $\widehat{c'\tau}$ is

$$Var_w(\widehat{c'\tau}) = [c' W^{-1} c]^{-1} Var(\widehat{c'\tau}) \tag{2.6}$$

In expression (2.6), the multiplier $[c'W^{-1}c]^{-1}$ is called the *weight of the contrast.* Observe that

- The contrasts $l_i'\tau$ have weight 1.
- In the unweighted case, in which all w_i are set to $1/v$, $Var_w(\widehat{c'\tau}) = \frac{1}{v}Var(\widehat{c'\tau})/c'c$, the usual variance of the normalized contrast.
- Optimality criteria that are functions of the θ_i through their inverses are evaluating weighted variances. This includes criteria of the form $\sum_i f(\theta_i)$, where f is convex and decreasing.

Again, the weights w_1, w_2, ..., w_v may be any positive numbers, subject only to the restriction $\sum_{i=1}^{v} w_i = 1$. However, it will often be the case that some weights are identical, in which case the weights define groups of equally weighted treatments. When there are g such groups, write the treatment set $V = \{1, 2, \ldots, v\}$ as

$$V = V_1 \cup V_2 \cup \ldots \cup V_g$$

where $|V_g| = v_g$. The 2-weight problem, mentioned in Chapter 1 and encountered frequently in this dissertation, is simply that of two distinct wrights w_1 and w_2, and $g = 2$ groups. Wherever appropriate and convenient, the set of weights will be referred to as the distinct numbers w_1, w_2, ..., w_g rather than the treatment-distinct numbers w_1, w_2, ..., w_v.

With this framework in place, specific criteria can be examined for the weighted case. As many conventional criteria are functions of the canonical variances, which are nothing but the inverted positive eigenvalues of the unweighted C-matrix, weighted criteria are defined as the same functions of the inverted positive eigenvalues of the C_w-matrix, which may now also be termed *canonical weighted variances.*

Definition 2.3. Let $0 < \theta_{d1} \leq \theta_{d2} \leq \ldots \leq \theta_{d,v-1}$ be the nonnegative eigenvalues of C_{dw}. Thus, $\frac{1}{\theta_{d1}} \geq \frac{1}{\theta_{d2}} \geq \ldots \geq \frac{1}{\theta_{d,v-1}}$ are the positive eigenvalues of C_d^+, called the *canonical weighted variances.*

2.1.1 Basics for the weighted E-criterion

Definition 2.4. The weighted E-value (written as E_w) for a design d is the largest canonical weighted variance for design d. That is,

$$E_w = \frac{1}{\theta_1}. \tag{2.7}$$

A design $\bar{d}$ is weighted E-optimal (or E_w-optimal) in a design class $\mathcal{D}$ if it minimizes the largest canonical weighted variance, that is, if

$$E_{\bar{d}w} = \min_{d \in \mathcal{D}} E_{dw}.$$

Result 2.1. *The weighted E-value is the largest weighted variance over all treatment contrasts.*

Proof. The largest weighted variance over all contrasts is

$$\max_{c'1=0} \left(\frac{Var(\widehat{c'\tau})}{c'W^{-1}c} \right) = \max_{c'1=0} \left(\frac{c'W^{-1/2}C_{dw}^+ W^{-1/2}c}{c'W^{-1}c} \right) = \max_{y'w^{1/2}=0} \left(\frac{y'C_{dw}^+ y}{y'y} \right).$$

$w^{1/2}$ is an eigenvector of C_{dw}^+ corresponding to eigenvalue 0, so this is the largest eigenvalue of C_{dw}^+, that is, $1/\theta_1$. □

Result 2.1 says that an E_w-optimal design factors importance of contrasts into design selection in minimizing impact of the worst case. It can be seen that for any design, increasing the weight placed on a treatment increases weighted variances (2.6) of contrasts in which it is involved. Minimizing summary functions of weighted variances (that is, minimizing functions of $1/\theta_i$), pushes variances of treatments with higher weight to be smaller, this being at the expense of variances of treatments with smaller weights.

Further insight into the E_w criterion is gained by examining the extremes for weights, done here for the 2-weight problem by letting one weight be arbitrarily close to 0. If the proposed approach is correct, then in the limit it should reduce to unweighted optimality for a subset of treatments.

Lemma 2.2. *Partition C_d as*

$$C_d = \begin{pmatrix} C_{d11} & C_{d12} \\ C_{d21} & C_{d22} \end{pmatrix}$$

where C_{d11} is $v_1 \times v_1$ and $2 \leq v_1 \leq v-1$. Write E_w = largest eigenvalue of C^+_{dw} = inverse of smallest eigenvalue of C_{dw}. Also let $C_{d11(2)} = C_{d11} - C_{d12}C^{-1}_{d22}C_{d21}$. Then for the 2-weight problem with v_1 treatments receiving weight w_1,

$$\lim_{\substack{w_2 \to 0 \\ w_1 \to 1/v_1}} \frac{w_1}{E_w} = \text{smallest positive eigenvalue of } C_{d11(2)}.$$

Proof. Write $\tilde{C}_{d11} = C_{d11} + Nw_1^2J$, $\tilde{C}_{d12} = C_{d12} + Nw_1w_2J = \tilde{C}^T_{d21}$, $\tilde{C}_{d22} = C_{d11} + Nw_2^2J$, and $P_w = W^{1/2}JW^{1/2}$, where $N = \sum_i r_{di}$ is the total number of experimental units of design d. E_w is the inverse of the smallest eigenvalue of

$$C_{dw} + NP_w = W^{-1/2} \begin{pmatrix} \tilde{C}_{d11} & \tilde{C}_{d12} \\ \tilde{C}_{d21} & \tilde{C}_{d22} \end{pmatrix} W^{-1/2}$$

which is positive definite. Thus E_w is the largest eigenvalue of $(C_{dw} + NP_w)^{-1}$, i.e.

$$\mathrm{E}_w = \max_x \frac{x'(C_{dw} + NP_w)^{-1}x}{x'x}$$

Partition $x' = (x_1', x_2')$ and

$$(C_{dw} + NP_w)^{-1} = W^{1/2} \begin{pmatrix} \tilde{C}_{d11(2)}^{-1} & -\tilde{C}_{d11(2)}^{-1}\tilde{C}_{d12}\tilde{C}_{d22}^{-1} \\ -\tilde{C}_{d22}^{-1}\tilde{C}_{d21}\tilde{C}_{d11(2)}^{-1} & \tilde{C}_{d22}^{-1} + \tilde{C}_{d22}^{-1}\tilde{C}_{d21}\tilde{C}_{d11(2)}^{-1}\tilde{C}_{d12} \end{pmatrix} W^{1/2}.$$

Then

$$\begin{aligned} \mathrm{E}_w &= \max_x \frac{1}{x'x}[w_1 x_1' \tilde{C}_{d11(2)}^{-1} x_1 - 2\sqrt{w_1 w_2} x_1' \tilde{C}_{d11(2)}^{-1}\tilde{C}_{d12}\tilde{C}_{d22}^{-1} x_2 \\ &\quad + w_2 x_2'(\tilde{C}_{d22}^{-1} + \tilde{C}_{d21}\tilde{C}_{d11(2)}^{-1}\tilde{C}_{d12}) x_2] \end{aligned} \tag{2.8}$$

Now

$$\begin{aligned} \lim_{w_2 \to 0} \tilde{C}_{d11} &= C_{d11} + \frac{N}{v_1} J, \\ \lim_{w_2 \to 0} \tilde{C}_{d12} &= C_{d12}, \\ \lim_{w_2 \to 0} \tilde{C}_{d22} = C_{d22} \quad &\Rightarrow \quad \lim_{w_2 \to 0} \tilde{C}_{d11(2)} = C_{d11(2)} + \frac{N}{v_1} J \end{aligned}$$

and so for any fixed x, the limit (ignoring for now the $\max_x$) of the expression in (2.8) is

$$\begin{aligned} \frac{1}{v_1} \frac{x_1'[C_{d11(2)} + \frac{N}{v_1}J]^{-1} x_1}{x'x} &\le \frac{1}{v_1} \frac{x_1'[C_{d11(2)} + \frac{N}{v_1}J]^{-1} x_1}{x_1' x_1} \\ &\le \frac{1}{v_1} \times \text{largest eigenvalue of } [C_{d11(2)} + \tfrac{N}{v_1}J]^{-1} \\ &= \frac{1}{v_1} \times [\text{ smallest eigenvalue of } C_{d11(2)} + \tfrac{N}{v_1}J]^{-1} \\ &= \frac{1}{v_1} \times [\text{ smallest positive eigenvalue of } C_{d11(2)}]^{-1} \end{aligned}$$

Since this is true for every fixed x, it follows that

$$\lim_{w_2 \to 0} \frac{w_1}{E_w} \ge \text{smallest positive eigenvalue of } C_{d11(2)}$$

and it is easy to see that equality is attained when x_1 is an eigenvector corresponding to the smallest positive eigenvalue. □

The matrix $C_{d11(2)}$ is the unweighted information matrix for estimation of the treatments V_1 receiving weight w_1. The smallest positive eigenvalue of $C_{d11(2)}$ is the worst variance over all normalized treatment contrasts of treatments in V_1. Lemma 2.2 says that if w_2 is very small, E_w essentially minimizes worst case variance for comparing treatments in V_1.

By symmetry, Lemma 2.2 includes the 2-weight problem for $v_1 = 1$ with $w_1 \to 0$ and $w_2 \to \frac{1}{v-1}$. The limiting result for the 2-weight problem with $v_1 = 1$ for $w_2 \to 0$ is substantially different, however. The following lemma is useful in sorting out the additional difficulties.

Lemma 2.3. *Let H be the $(v-1) \times v$ information matrix whose rows are the coefficients of the elementary contrast for estimating $\widehat{\tau_1 - \tau_i}$, i.e. $H = \left(1_{v-1} \vdots - I_{v-1}\right)$, and let C_{d22} be the $(v-1) \times (v-1)$ right lower submatrix of C_d. Then $(HC_d^- H^T)^{-1} = C_{d22}$.*

Proof. Partition C_d as

$$C_d = \begin{pmatrix} c_{d11} & C_{d12} \\ C_{d21} & C_{d22} \end{pmatrix}$$

where C_{d22} is $(v-1) \times (v-1)$, C_{d21} is $(v-1) \times 1$ and $C_{d21} = C_{d12}^T$. Write $\widetilde{c}_{d11} = c_{d11} + N$, $\widetilde{C}_{d12} = C_{d12} + N1'$, $\widetilde{C}_{d21} = C_{d21} + N1$, $\widetilde{C}_{d22} = C_{d22} + NJ_{v-1}$ and $a = \widetilde{c}_{d11} - \widetilde{C}_{d12}\widetilde{C}_{d22}^{-1}\widetilde{C}_{d21}$. Note that, since the rows of H are contrast vectors, $HC_d^- H^T$ is invariant to choice of generalized inverse of C_d. One choice is $C_d^- = (C_d + NJ)^{-1}$ so that for arbitrary $N \neq 0$,

$$
\begin{aligned}
(HC_d^- H^T)C_{d22} &= H(C_d+NJ)^{-1}\begin{pmatrix} -C_{d12} \\ -C_{d22} \end{pmatrix} \\
&= H(C_d+NJ)^{-1}\begin{pmatrix} -C_{d12} - N1' \\ -C_{d22} - NJ_{v-1} \end{pmatrix} + H(C_d+NJ)^{-1}(NJ_{v\times(v-1)}) \\
&= H\left[\frac{1}{a}\begin{pmatrix} 1 & -\widetilde{C}_{d12}\widetilde{C}_{d22}^{-1} \\ -\widetilde{C}_{d22}^{-1}\widetilde{C}_{d21} & a\widetilde{C}_{d22}^{-1} + \widetilde{C}_{d22}^{-1}\widetilde{C}_{d21}\widetilde{C}_{d12}\widetilde{C}_{d22}^{-1} \end{pmatrix}\right]\begin{pmatrix} -\widetilde{C}_{d12} \\ -\widetilde{C}_{d22} \end{pmatrix} \\
&\quad + H(C_d+NJ)^{-1}(NJ_{v\times(v-1)}) \\
&= I_{v-1} + H(C_d+NJ)^{-1}(NJ_{v\times(v-1)}) \qquad (2.9)
\end{aligned}
$$

The second term $H(C_d + NJ)^{-1}(NJ_{v\times(v-1)})$ in expression (2.9) equals zero, which follows from

$$
\begin{aligned}
&(C_d+NJ)^{-1}(C_d+NJ_v)J_{v\times(v-1)} = J_{v\times(v-1)} \\
\Rightarrow\ &(C_d+NJ)^{-1}(NvJ_{v\times(v-1)}) = J_{v\times(v-1)} \\
\Rightarrow\ &(C_d+NJ)^{-1}(NJ_{v\times(v-1)}) = \frac{1}{v}J_{v\times(v-1)} \\
\Rightarrow\ &H(C_d+NJ)^{-1}(NJ_{v\times(v-1)}) = \frac{1}{v}HJ_{v\times(v-1)} = 0.
\end{aligned}
$$

Therefore, $(HC_d^- H^T)C_{d22} = I$. $\square$

The information matrix in the test treatments versus control problem (TvC; see Section 1.2) is $(HC_dH^T)^{-1}$. Theorem 2.1 provides the link between the TvC problem and the weighted optimality approach.

Theorem 2.1. *Let f_i $(i = 0, 1, \ldots, v-1)$ be orthonormal eigenvectors of C_{dw} corresponding to eigenvalues θ_i, where $f_0 = w^{1/2}$ corresponds to the eigenvalue zero of C_{dw}. Let y_i $(i = 1, \ldots, v-1)$ be eigenvectors corresponding to eigenvalues e_i of $(HC_d^- H^T)^{-1}$, where $e_1 < e_2 < \cdots < e_{v-1}$. Then,* *(i)* $\lim\limits_{\substack{w_1\to 1\\ w_2\to 0}} w_2 C_{dw} = \begin{pmatrix} 0 & \underline{0}' \\ \underline{0} & C_{d22} \end{pmatrix} = \begin{pmatrix} 0 & \underline{0}' \\ \underline{0} & (HC_d^- H^T)^{-1} \end{pmatrix},$

(ii) $\lim\limits_{\substack{w_1\to 1\\ w_2\to 0}} w_2 \theta_i = e_i.$

Proof. (i) Write

$$w_2 C_{dw} = \begin{pmatrix} \frac{w_2}{w_1} c_{d11} & \frac{\sqrt{w_2}}{\sqrt{w_1}} C_{d12} \\ \frac{\sqrt{w_2}}{\sqrt{w_1}} C_{d21} & C_{d22} \end{pmatrix} \tag{2.10}$$

Letting $w_1 \to 1$ and thus $w_2 \to 0$, (i) follows easily from (2.10) and Lemma 2.3.

(ii) Partition $f_i' = (f_1, f_i^{*'})$. Observe that

$$\begin{aligned} & C_{dw} f_i = \theta_i f_i \\ \Rightarrow \quad & W^{-1/2} \begin{pmatrix} c_{d11} & C_{d12} \\ C_{d21} & C_{d22} \end{pmatrix} W^{-1/2} f_i = \theta_i f_i \\ \Rightarrow \quad & \begin{pmatrix} \frac{c_{d11}}{w_1} f_1 + \frac{1}{\sqrt{w_1 w_2}} C_{d12} f_i^* \\ \frac{1}{\sqrt{w_1 w_2}} C_{d21} f_1 + \frac{1}{w_2} C_{d22} f_i^* \end{pmatrix} = \begin{pmatrix} \theta_i f_1 \\ \theta_i f_i^* \end{pmatrix}. \end{aligned} \tag{2.11}$$

Now

$$\begin{aligned} f_i' w^{1/2} = 0 \quad \Rightarrow \quad & f_1 w_1^{1/2} + w_2^{1/2} (\sum_{i=2}^{v} f_i^*) = 0 \\ \Rightarrow \quad & f_1 w_1^{1/2} = -w_2^{1/2} (\sum_{i=2}^{v} f_i^*) \end{aligned} \tag{2.12}$$

and $|\sum_{i=2}^{v} f_i^*|$ is bounded. Thus as $w_1 \to 1$ and $w_2 \to 0$, the RHS of (2.12) goes to zero, so

$$\lim_{w_1 \to 1} f_1 = 0. \tag{2.13}$$

Combined with the fact that C_{dw} and $w_2 C_{dw}$ share the same eigenvectors, equation (2.13) and (i) together say that

$$\lim_{w_1 \to 1} f_i = \begin{pmatrix} 0 \\ y_i \end{pmatrix}$$

and thus

$$\begin{aligned}\lim_{w_1\to 1} w_2\theta_i f_i &= \lim_{w_1\to 1} w_2 C_{dw} f_i \\ &= \begin{pmatrix} 0 & \underline{0}' \\ \underline{0} & C_{d22} \end{pmatrix} \begin{pmatrix} 0 \\ y_i \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ e_i y_i \end{pmatrix}\end{aligned}$$

and (ii) follows. □

Theorem 2.1 says that as more weight is placed on the control treatment, the $v-1$ nonzero eigenvalues of C_{dw} become proportional to eigenvalues of $(HC_d^- H^T)^{-1}$, which as already noted is the information matrix for the test treatment versus control problem. Recall y_1 is the eigenvector corresponding to the smallest eigenvalue e_1 of $(HC_d^- H^T)^{-1}$, or equivalently, corresponding to the smallest eigenvalue of C_{d22}. We will show that the elements of y_1 have the same sign, i.e. $y_{1,i}$ are all non-positive or non-negative for $i = 1, \ldots, v-1$.

Since y_1 corresponds to the smallest eigenvalue of C_{d22}, it must satisfy $y_1' C_{d22} y_1 = \min_z z' C_{d22} z = e_1$, where z is an arbitrary normalized vector. Suppose the sign of some element in y_1 is different from that of some other element. Without loss of generality, we assume the first k elements, i.e. $y_{1,1}, \ldots, y_{1,k}$ have positive sign while the signs of $y_{1,k+1}, \ldots, y_{1,m}$ are all negative for some $k+1 < m \leq v-1$. Then

$$\begin{aligned}& y_1' C_{d22} y_1 \\ = & \sum \sum_{1\leq i,j\leq k} (C_{d22})_{ij} y_{1,i} y_{1,j} + \sum \sum_{k+1\leq i,j\leq m} (C_{d22})_{ij} y_{1,i} y_{1,j} + \sum \sum_{\substack{1\leq i\leq k \\ k+1\leq j\leq m}} (C_{d22})_{ij} y_{1,i} y_{1,j} \\ \geq & \sum \sum_{1\leq i,j\leq k} (C_{d22})_{ij} y_{1,i} y_{1,j} + \sum \sum_{k+1\leq i,j\leq m} (C_{d22})_{ij} y_{1,i} y_{1,j} + \sum \sum_{\substack{1\leq i\leq k \\ k+1\leq j\leq m}} (C_{d22})_{ij} (-y_{1,i}) y_{1,j}.\end{aligned}$$

The above inequality is because all off-diagonal elements of C_{d22} are non-positive. Observe that $(v-1)\times 1$ vector $y_1^* = (-y_{1,1}, \cdots, -y_{1,k}, y_{1,k+1}, \cdots, y_{1,m}, 0'\ )'$ is also a normalized vector and $y_1^{*'} C_{d22} y_1^* \leq y_1' C_{d22} y_1$. If the inequality is strict this is a contradiction and thus elements of y_1 are either all non-positive or all non-negative. Otherwise $y_1^* C_{d22} y_1^{*'} = y_1' C_{d22} y_1 = e_1$. This proves the following theorem.

Theorem 2.2. *The smallest eigenvalue e_1 of the information matrix $(HC_d^- H^T)^{-1}$ for comparing test treatments to a single control corresponds to an eigenvector for which all elements are non-negative.*

Note that if e_1 has multiplicity greater than 1, then there is at least one normalized eigenvector with the non-negative property.

The importance of Theorem 2.2 is that it tells us the precise meaning of the E-criterion in the test-treatment versus control problem, an issue previously unsettled in the literature (e.g. Majumdar and Notz, 1983). The inverse of e_1 is

$$\frac{1}{e_1} = y_1' H C_d^- H^T y_1 = \frac{1}{\sigma^2} Var(\widehat{h'\tau}) \tag{2.14}$$

where $h' = (\sum_{i=1}^{v-1} y_{1i}, -y_{11}, -y_{12}, \cdots, -y_{1,v-1})$ is a contrast vector comparing the control to an average of test treatments.

Theorem 2.3. *Consider all normalized averages $\sum_{i=1}^{v-1} a_i \tau_{i+1}$ of the test treatments ($\sum_{i=1}^{v-1} a_i^2 = 1$ and $a_i \geq 0$ for $i = 1, 2, \cdots, v-1$). An E-optimal design for the test treatment versus control problem minimizes the largest variance over all contrasts of the control with normalized test treatment averages:*

$$E_d = \max_{\substack{a_1,\cdots,a_{v-1} \\ \sum a_i^2 = 1, a_i \geq 0}} \frac{1}{\sigma^2} Var\Big(\widehat{(\sum_{i=1}^{v-1} a_i)\tau_1 - \sum_{i=1}^{v-1} a_i \tau_{i+1}} \Big).$$

Recall that for any contrast $c'\tau$,

$$Var_w(\widehat{c'\tau}) = (c'W^{-1}c)^{-1}Var(\widehat{c'\tau}).$$

Let $c = (\sum_i a_i, -a_1, -a_2, \cdots, -a_{v-1})'$ where $a_i \geq 0$ and $\sum_i a_i^2 = 1$. The weight for contrasts of this form is

$$(c'W^{-1}c)^{-1} = \left(\frac{(\sum_i a_i)^2}{w_1} + \frac{1}{w_2}\right)^{-1}.$$

As $w_1 \to 1$ this weight goes to 0. But as shown in Lemma 2.3, the correct way to evaluate weighted expressions asymptotically when $v_1 = 1$ is to multiply W^{-1} by w_2. Doing so gives

$$(w_2 c'W^{-1}c)^{-1} = \left(\frac{(\sum_i a_i)^2 w_2}{w_1} + 1\right)^{-1} \to 1 \qquad \text{as } w_1 \to 1.$$

This says that, for large w_1, all contrasts of this form have essentially the same weight. Thus minimizing the maximum of their unweighted variances is equivalent, asymptotically in the weights, to minimizing the maximum of their weighted variances.

In conclusion, *the E_w-criterion is asymptotically equivalent to the E-criterion for the test treatments versus control problem.* The latter can be thought of as minimizing maximal weighted variance where very large weight has been assigned to the control treatment. The class of contrasts evaluated by the E-criterion for the TvC problem is that of all comparisons of the control to normalized averages of the test treatments.

2.1.2 Basics for the weighted MV-criterion

From (2.6), the weighted variance of an elementary contrast $\widehat{\tau_i - \tau_j}$ is

$$Var_w(\widehat{\tau_i - \tau_j}) = \frac{w_i w_j}{w_i + w_j} Var(\widehat{\tau_i - \tau_j}). \tag{2.15}$$

This motivates the next definition.

Definition 2.5. The weighted MV-value (written as MV_w) is defined as the largest weighted variance over all pairwise comparisons:

$$MV_w = \max_{i \neq j} \frac{w_i w_j}{w_i + w_j} Var(\widehat{\tau_i - \tau_j}). \tag{2.16}$$

A design $\bar{d}$ is weighted MV-optimal (or MV_w-optimal) in a design class $\mathcal{D}$ if

$$MV_{\bar{d}w} = \min_{d \in \mathcal{D}} MV_{dw}.$$

Let $\mathcal{M}$ be the set of vectors $m'_{ij} = \frac{1}{\sqrt{w_i + w_j}}(0, \cdots, \sqrt{w_j}, \cdots, 0, \cdots, -\sqrt{w_i}, \cdots, 0)$ for all $i < j$ and $i, j \in \{1, \cdots v\}$, where $\sqrt{w_j}$ is in the i^{th} position and $\sqrt{w_i}$ is in the j^{th} position. The weighted variance for the pairwise comparison $\tau_i - \tau_j$ can be written

$$\frac{w_i w_j}{w_i + w_j} Var(\widehat{\tau_i - \tau_j}) = m'_{ij} W^{\frac{1}{2}} C_d^- W^{\frac{1}{2}} m_{ij} = m'_{ij} C_{dw}^- m_{ij} \tag{2.17}$$

showing that the MV_w is the largest value of $m'_{ij} C_{dw}^- m_{ij}$ over all $m_{ij} \in \mathcal{M}$. The MV_w-value is directly comparable to the unweighted MV-value: C_d is replaced by C_{dw}, and the pairwise contrast vector $(0 \cdots 1 \cdots 0 \cdots -1 \cdots 0)'$ is replaced by the normalized vector $\frac{1}{\sqrt{w_i + w_j}}(0 \cdots \sqrt{w_j} \cdots 0 \cdots - \sqrt{w_i} \cdots 0)'$. Furthermore, the weighted variance for a weighted contrast vector is invariant to the choice of generalized inverse C_{dw}^-.

The MV_w-criterion evaluates the weighted variances for the pairwise comparisons $\tau_i - \tau_j$. Further insight is gained by considering extreme weights for the 2-weight problem. To begin, let $v_1 = 1$ treatment have weight w_1 and $v_2 = v - 1$ treatments have weight $w_2 = \frac{1 - w_1}{v - 1}$ (this can be thought of as a control and several test treatments situation). Then

$$\begin{aligned} MV_w &= \max \left\{ \frac{w_1 w_2}{w_1 + w_2} \max_{j \geq 2} Var(\widehat{\tau_1 - \tau_j}), \frac{w_2^2}{2w_2} \max_{j, j' \geq 2} Var(\widehat{\tau_j - \tau_{j'}}) \right\} \\ &= \max \left\{ \frac{\big[1 - (v-1)w_2\big] w_2}{1 - (v-2)w_2} \max_{j \geq 2} Var(\widehat{\tau_1 - \tau_j}), \frac{w_2}{2} \max_{j, j' \geq 2} Var(\widehat{\tau_j - \tau_{j'}}) \right\} \end{aligned}$$

To normalize this MV_w-value, divide by a common denominator of

$$\sum_i \sum_{j>i} \frac{w_i w_j}{w_i + w_j} = (v-1)\frac{\big(1-(v-1)w_2\big)w_2}{1-(v-2)w_2} + \frac{(v-1)(v-2)}{2}\frac{w_2}{2}$$
$$= \frac{(v-1)(v+2)w_2 - [4(v-1)^2 + (v-1)(v-2)^2]w_2^2}{4[1-(v-2)w_2]}$$

Thus, the normalized MV_w-value MV_w^* is

$$MV_w^* = \max\Big\{\frac{4[1-(v-1)w_2]}{(v-1)(v+2) - [4(v-1)^2 + (v-1)(v-2)^2]w_2} \max_{j\geq 2} Var(\widehat{\tau_1 - \tau_j}),$$
$$\frac{2[1-(v-2)w_2]}{(v-1)(v+2) - [4(v-1)^2 + (v-1)(v-2)^2]w_2} \max_{j,j'\geq 2} Var(\widehat{\tau_j - \tau_{j'}})\Big\}$$

When $w_1 \to 1$ and $w_2 \to 0$,

$$\lim_{\substack{w_1\to 1\\ w_2\to 0}} MV_w^* = \max\Big\{\frac{4}{(v-1)(v+2)} \max_{j\geq 2} Var(\widehat{\tau_1 - \tau_j}), \frac{2}{(v-1)(v+2)} \max_{j,j'\geq 2} Var(\widehat{\tau_j - \tau_{j'}})\Big\}$$

When $w_1 \to 0$ and $w_2 \to \frac{1}{v-1}$,

$$\lim_{\substack{w_1\to 0\\ w_2\to 1}} MV_w^* = \max\Big\{0, \frac{2}{(v-1)(v-2)} \max_{j,j'\geq 2} Var(\widehat{\tau_j - \tau_{j'}})\Big\}$$

These results say that when large weight is assigned to the control, the MV_w-value places up to double weight on treatment/control comparisons, and so pushes $Var(\widehat{\tau_1 - \tau_{j'}})$ to be small. When large weight is evenly assigned to the $v-1$ treatments, then the control is essentially ignored and minimizing MV_w-value pushes the variances of pairwise contrasts within test treatments to be smaller.

Now consider the 2-weight problem for $v_1 > 1$ treatments having weight w_1 and $v_2 > 1$ treatments having weight w_2. Then

$$MV_w = \max\Big\{\frac{w_1}{2} \max_{i,i'\leq v_1} Var(\widehat{\tau_i - \tau_{i'}}), \frac{w_1 w_2}{w_1 + w_2} \max_{\substack{i\leq v_1\\ v_1+1\leq j\leq v}} Var(\widehat{\tau_i - \tau_j}), \frac{w_2}{2} \max_{v_1+1\leq j,j'\leq v} Var(\widehat{\tau_j - \tau_{j'}})\Big\}.$$

Divide by this common denominator to normalize the MV_w-value:

$$\frac{v_1(v_1-1)}{2}\frac{w_1}{2} + v_1 v_2\Big(\frac{w_1 w_2}{w_1 + w_2}\Big) + \frac{v_2(v_2-1)}{2}\frac{w_2}{2}$$

When $w_1 \to 1/v_1$ and $w_2 \to 0$,

$$\lim_{\substack{w_1\to 1/v_1\\ w_2\to 0}} MV_w^* = \max_{i,i'\leq v_1} \frac{2}{v_1(v_1-1)} Var(\widehat{\tau_i - \tau_{i'}}).$$

When $w_1 \to 0$ and $w_2 \to 1/v_2$,

$$\lim_{\substack{w_1 \to 0 \\ w_2 \to 1/v_2}} MV_w^* = \max_{v_1+1 \le j,j' \le v} \frac{2}{v_2(v_2-1)} Var(\widehat{\tau_j - \tau_{j'}}).$$

The above results say that MV_w values only depend on the pairwise contrast variance within one group if treatments in the other group are not of interest; this is exactly as expected. More generally, large values of one weight emphasizes small variance for pairwise comparisons of treatments within that weight group.

2.1.3 Basics for the weighted A-criterion

Definition 2.6. The weighted A-value (written as A_w) for a design d is the average of the canonical weighted variances for d:

$$A_w = \frac{1}{v-1} \sum_{i=1}^{v-1} \frac{1}{\theta_i} \tag{2.18}$$

A design $\bar{d}$ is weighted A-optimal (or A_w-optimal) in a design class $\mathcal{D}$ if

$$A_{\bar{d}w} = \min_{d \in \mathcal{D}} A_{dw}.$$

Result 2.2. *The weighted A-value is proportional to the average of the weighted variances for any $v-1$ weighted orthogonal contrasts.*

Proof. Let $\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_{v-1}$ be the coefficient vectors for any set of weighted orthogonal contrasts, and with no loss of generality assume the corresponding $v-1$ variance weights are 1. Then $\tilde{l}_i' W^{-1} \tilde{l}_i = 1$, $\tilde{l}_i' W^{-1} \tilde{l}_j = 0$ for $i \neq j$, and $\tilde{l}_i' W^{-1} l_0 = 0$. Writing $\tilde{L}_{(0)} \tilde{L}_{(0)}^T = \sum_{i=1}^{v-1} \tilde{l}_i \tilde{l}_i' =$

$W - l_0 l_0' = W - ww'$, then

$$\begin{aligned}
\sum_{i=1}^{v-1} Var(\widehat{\tilde{l}_i'\tau}) &= trace(\tilde{L}_{(v)}^T C_d^- \tilde{L}_{(v)}) = trace(\tilde{L}_{(v)}^T W^{-1/2} C_{dw}^+ W^{-1/2} \tilde{L}_{(v)}) \\
&= trace(C_{dw}^+ W^{-1/2} \tilde{L}_{(v)} \tilde{L}_{(v)}^T W^{-1/2}) = trace(C_{dw}^+ [I - f_0 f_0']) \\
&= trace(C_{dw}^+) = \sum_{i=1}^{v-1} \frac{1}{\theta_i} = (v-1) A_w \qquad (2.19)
\end{aligned}$$

□

Result 2.2 shows that A_w does not depend on a particular set of equal-weighted contrasts for a particular design; a design which minimizes A_w is minimizing the average weighted variance for *any* set of weight-orthogonal contrasts. The following Result shows that A_w can be expressed in terms of pairwise contrast variances.

Result 2.3. $A_w = \sum_i \sum_{j \neq i} \frac{w_i w_j}{v-1} Var(\widehat{\tau_i - \tau_j})$.

Proof. Let h_{ij} be the elementary contrast vector comparing treatments i and j, that is, $h_{ij}'\tau = \tau_i - \tau_j$. Let $b_{ij} = \sqrt{w_i w_j} h_{ij}$ be the columns of the $v \times v(v-1)$ matrix B. It can be checked that $W^{-1/2} B B^T W^{-1/2} = I - f_0 f_0'$. Then

$$\begin{aligned}
\sum_i \sum_{j \neq i} w_i w_j Var(\widehat{\tau_i - \tau_j}) &= \sum_i \sum_{j \neq i} Var(\widehat{b'_{ij}\tau}) = trace(B^T C_d^- B) \\
&= trace(C_{dw}^+ W^{-1/2} B B^T W^{-1/2}) = trace(C_{dw}^+) = (v-1) A_w
\end{aligned}$$

□

Result 2.3 indicates A_w is proportional to a simple weighted average of pairwise contrast variances. Minimization of A_w via choice of design implies that, on average, estimation of differences $\tau_i - \tau_j$ with larger $w_i w_j$ will be more precise than pairs with smaller weights.

Again, consider the extreme weight grouping having $v_1 = 1$ and $v_2 = v - 1$. Then

$$A_w = \sum_i \sum_{j \neq i} \frac{w_i w_j}{v-1} Var(\widehat{\tau_i - \tau_j})$$

$$= \frac{1}{v-1}\Big(\sum_{j\neq 1}\frac{w_1(1-w_1)}{v-1}Var(\widehat{\tau_1-\tau_j}) + \sum_{j>1}\sum_{\substack{j'>1\\ j'\neq j}}\frac{(1-w_1)^2}{(v-1)^2}Var(\widehat{\tau_j-\tau_{j'}})\Big).$$

Divide A_w by $\sum_i\sum_{j\neq i}\frac{w_iw_j}{v-1} = \frac{1}{v-1}\Big(\frac{w_1-w_1^2}{v-1}+\frac{(1-w_1)^2}{(v-1)^2}\Big)$ to give the normalized A_w as

$$A_w^* = \frac{w_1}{v(1+w_1)-2}\sum_{j\neq 1}Var(\widehat{\tau_1-\tau_j}) + \frac{1-w_1}{(v-1)[v(1+w_1)-2]}\sum\sum_{j\neq j'}Var(\widehat{\tau_j-\tau_{j'}})$$

As can be computed, the limiting values for A_w^* are

$$\begin{cases}\frac{1}{2(v-1)}\sum_{j\neq 1}Var(\widehat{\tau_1-\tau_j}) & \text{as } w_1\to 1 \text{ and } w_2\to 0\\ \frac{1}{(v-1)(v-2)}\sum\sum_{j\neq j'}Var(\widehat{\tau_j-\tau_{j'}}) & \text{as } w_1\to 0 \text{ and } w_2\to \frac{1}{v-1}\end{cases}$$

These results say that when full weight is assigned to the control only, the A_w-value depends only on the pairwise contrasts between control and treatments, while the pairwise contrasts within treatment group are not involved; if full weight is evenly assigned to the $v-1$ treatments and none to the control, then the A_w-value depends only on the pairwise contrasts within the treatment group.

Also, consider the 2-weight case with $v_1 > 1$ and $v_2 > 1$. Similar to the former case, the normalized A_w is

$$A_w^* = \frac{\sum_i\sum_{j\neq i}Var(\widehat{\tau_i-\tau_j})}{\frac{v_1(v_1-1)}{2}w_1^2 + v_1w_1(1-v_1w_1) + \frac{v_2-1}{2v_2}(1-v_1w_1)^2}$$

The limiting values for A_w^* are

$$\begin{cases}\frac{2v_1}{v_1-1}\sum\sum_{i,i'\in V_1}Var(\widehat{\tau_i-\tau_{i'}}) & \text{as } w_1\to \frac{1}{v_1} \text{ and } w_2\to 0\\ \frac{2v_2}{v_2-1}\sum\sum_{j,j'\in V_2}Var(\widehat{\tau_j-\tau_{j'}}) & \text{as } w_1\to 0 \text{ and } w_2\to \frac{1}{v_2}\end{cases}$$

These results show that when the full weight is evenly assigned to one group of treatments only, the A_w-value depends only on the pairwise contrasts within this treatment group, while treatments in the other group are not involved at all.

2.1.4 The weighted D-criterion

Another weighted criterion we introduce here is weighted D-optimality, although it turns out to be of little interest.

Definition 2.7. The weighted D-value (written as D_w) for design d is the product of the canonical weighted variances for d:

$$D_w = \prod_{i=1}^{v-1} \theta_i$$

A design $\bar{d}$ is weighted D-optimal (or D_w-optimal) in a design class $\mathcal{D}$ if $D_{\bar{d}w} = \min_{d \in \mathcal{D}} D_{dw}$.

Perhaps surprisingly, the D_w-optimal design is always identical to the D-optimal design for the unweighted case. This is a consequence of the following result.

Result 2.4. *If d_1 is D-better than d_2, then d_1 is D_w-better than d_2 for all choices of weights w.*

Proof. Since $w^{1/2}$ is the normalized eigenvector of C_{dw} having eigenvalue 0, the D_w value of any design d is

$$|C_{dw} + w^{1/2} w^{1/2'}|^{-1}.$$

The determinant in this expression can be simplified using relationships with adjugates:

$$\begin{aligned}
|C_{dw} + w^{1/2} w^{1/2'}| &= |C_{dw}| + w^{1/2'} adj(C_{dw}) w^{1/2} \quad \text{(Rao, 1973, pg. 32)} \\
&= 0 + w^{1/2'} adj(D(w^{-1/2})) adj(C_d) adj(D(w^{-1/2})) w^{1/2} \\
&= |D(w^{-1})| \times w^{1/2'} D(w^{1/2}) adj(C_d) D(w^{1/2}) w^{1/2} \\
&= |D(w^{-1})| \times D(w) adj(C_d) D(w).
\end{aligned}$$

The adjugate of C_d is $adj(C_d) = \kappa J$ for some κ depending on d (Chakrabarti, 1963). Thus the ratio of D_w-values for two designs d_1 and d_2 is

$$\begin{aligned}\frac{D_w(d_1)}{D_w(d_2)} &= \frac{|D(w^{-1})| \times D(w)adj(C_{d_2})D(w)}{|D(w^{-1})| \times D(w)adj(C_{d_1})D(w)} \\ &= \frac{D(w)(\kappa_2 J)D(w)}{D(w)(\kappa_1 J)D(w)} = \frac{\kappa_2}{\kappa_1}\end{aligned}$$

which is the same regardless of the weights w. □

That the D criterion cannot distinguish changes in relative treatment interest is a serious deficiency. There is no need to investigate the D_w criterion further, for it offers no information for a weighted evaluation of treatments.

2.1.5 A general class of weighted criteria

With the above criteria in place, we now define a general class of weighted criteria. An arbitrary criterion Φ is just a function $\Phi : C_{dw} \to \Re$ on the weighted information matrix C_{dw} of an arbitrary design d. A design is Φ_w-optimal if it minimize $\Phi(C_{dw})$ over $\mathcal{D}(v, b, k)$. Not any arbitrary Φ is of statistical interest. To be of interest, Φ must satisfy certain conditions.

As mentioned in Section 1.2, one property of conventional Φ-optimality is $\Phi(PC_dP^T) = \Phi(C_d)$ for every permutation matrix P, which means reordering treatments does not change the optimality measure. Obviously, this property cannot be applied to weighted optimality since the interest in the treatments is no longer uniform. However, for treatments with the same weights, order is irrelevant. The following properties on the optimality functions of the weighted information matrix are critical to this research.

Suppose the treatments are grouped into g sets with constant weight within each set. Let Q be any permutation matrix which preserves, in order, the weight groups. That is, Q may only perform within-group treatment permutation. Then a weighted optimality criterion Φ must satisfy

$$\Phi(C_{dw}) = \Phi(QC_{dw}Q^T) \tag{2.20}$$

for every such Q. It is apparent that the weighted criteria introduced above, E_w, MV_w, A_w, possess this property.

The second requirement of Φ is an obvious need that it be non-increasing in the following sense. If $C_{d_1w} - C_{d_2w}$ is *nnd*, then $\Phi(C_{d_1w}) \leq \Phi(C_{d_2w})$. Note that $C_{d_1w} - C_{d_2w}$ is *nnd* $\Leftrightarrow$ $C^+_{d_2w} - C^+_{d_1w}$ is *nnd* (Morgan, 2007), so this property says that if design d_1 dominates d_2 by estimating every contrast with no larger variance, then Φ cannot rank d_2 better than d_1.

The third requirement, while having no obvious statistical interpretation, is important for technical reasons. This is the property of matrix convexity:

$$\Phi(\alpha C_{d_1w} + (1-\alpha)C_{d_2w}) \leq \alpha\Phi(C_{d_1w}) + (1-\alpha)\Phi(C_{d_2w}) \tag{2.21}$$

Fortunately, the weighted versions of all the standard criteria introduced in the preceding subsections satisfy this property (see section 2.2).

Henceforth, all optimality criteria Φ considered are assumed to satisfy:

$$\begin{aligned} &(i) \quad \Phi \text{ is permutation invariant as in expression (2.20).} \\ &(ii) \quad \Phi \text{ is nonincreasing with respect to the } nnd \text{ ordering of matrices.} \\ &(iii) \quad \Phi \text{ is matrix convex.} \end{aligned} \tag{2.22}$$

Here is a class of criteria satisfying the above conditions:

Definition 2.8. The weighted Φ_p-optimality criteria are defined as

$$\Phi_p(C_{dw}) = \left(\frac{\sum_{i=1}^{v-1} \theta_{di}^{-p}}{v-1} \right)^{1/p} \tag{2.23}$$

for $0 < p < \infty$, where the θ_{di} are the nonzero eigenvalues of C_{dw}.

For $p = 1$, Φ_p is the A_w-value. As $p \to \infty$, Φ_p becomes the E_w-value. Letting $\mathcal{M}$ be the set of weighted pairwise contrast vectors described in definition 2.5, then

$$\Phi_{MV}(C_{dw}) = \max_{m \in \mathcal{M}} m' C_{dw}^{-} m \tag{2.24}$$

is the MV_w-value. Φ_{MV} is not in the Φ_p class, but it does satisfy (i)-(iii).

2.2 Convexity and Generalized Group Divisible Designs

This section provides some theorems related to convexity of the optimality criteria, which will be useful for proofs in chapters that follow. Convexity also leads to the concept of generalized group divisible designs.

Definition 2.9. Given a full rank symmetric matrix X, let its spectral decomposition be $X = UDU^T$, where $UU^T = I$ and $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ whose diagonal entries are the eigenvalues of X. For any real-valued function f, let $D_f = \text{diag}\,(f(\lambda_1), \ldots, f(\lambda_n))$, and define the correspondent matrix function $f(X)$ by $f(X) = UD_fU^T$. Then, f is said to be matrix convex if for symmetric matrices A, B and $\alpha \in [0, 1]$,

$$f(\alpha A + (1 - \alpha)B) \preceq \alpha f(A) + (1 - \alpha)f(B),$$

that is, $(\alpha f(A) + (1 - \alpha)f(B)) - f(\alpha A + (1 - \alpha)B)$ is nonnegtive definite.

We will use the next theorem to give some properties on the convexity of the information matrix.

Theorem 2.4. *(Bapat and Raghavan, 1997) Let $A_1, \ldots, A_n$ be $v \times v$ positive definite matrices. Then*

$$\frac{1}{n}(A_1 + \cdots + A_n) \succeq n\big(A_1^{-1} + \cdots + A_n^{-1}\big)^{-1}.$$

Theorem 2.5. *The function $f(t) = \dfrac{1}{t}$ is matrix convex on $t > 0$.*

Proof. We want to prove $[\alpha A + (1-\alpha)B]^{-1} \preceq \alpha A^{-1} + (1-\alpha)B^{-1}$ for $\alpha \in [0,1]$. Write $\alpha = m/n$, where m and n are two positive integers satisfying $1 \leq m < n$. Then,

$$\alpha A + (1-\alpha)B = m\left(\frac{A}{n}\right) + (n-m)\left(\frac{B}{n}\right). \tag{2.25}$$

Let $A_1 = \cdots = A_m = A$ and $A_{m+1} = \cdots = A_n = B$, then utilizing Theorem 2.4, (2.25) becomes

$$\begin{aligned} \alpha A + (1-\alpha)B &= \frac{1}{n}(A_1 + \cdots + A_n) \\ &\succeq n(A_1^{-1} + \cdots + A_n^{-1})^{-1} \\ &= (\frac{m}{n}A^{-1} + \frac{n-m}{n}B^{-1})^{-1} \\ &= [\alpha A^{-1} + (1-\alpha)B^{-1}]^{-1}. \end{aligned}$$

Equivalently, $[\alpha A + (1-\alpha)B]^{-1} \preceq \alpha A^{-1} + (1-\alpha)B^{-1}$ □

Corollary 2.1. $\sum_i \alpha_i A_i^{-1} - (\sum_i \alpha_i A_i)^{-1}$ *is positive definite for any* $\sum_i \alpha_i = 1$ *and* $\alpha_i \geq 0$ $\forall\ i$.

Theorem 2.6. *Let C_{d_1w} and C_{d_2w} be the $v \times v$ weighted information matrices of rank $v-1$ corresponding to any two connected designs d_1 and d_2 for weights w. Let $\bar{C}_{dw} = \alpha C_{d_1w} + (1-\alpha)C_{d_2w}$ for some $\alpha \in [0,1]$, and let l be an arbitrary vector satisfying $l'w^{1/2} = 0$. Then*

$$l'\bar{C}_{dw}^{-}l \leq \alpha l'C_{d_1w}^{-}l + (1-\alpha)l'C_{d_2w}^{-}l, \tag{2.26}$$

where $\bar{C}_{dw}^{-}$, $C_{d_1w}^{-}$ and $C_{d_2w}^{-}$ are generalized inverses for $\bar{C}_{dw}$, C_{d_1w} and C_{d_2w} respectively.

Proof. For any $v \times v$ weighted information matrix C_{dw} for any connected design d, $(C_{dw} + \varepsilon w^{1/2}w^{1/2'})$ is positive definite and $(C_{dw} + \varepsilon w^{1/2}w^{1/2'})^{-1}$ is a generalized inverse of C_{dw} for any real number $\varepsilon \neq 0$. Note that $\bar{C}_{dw}$ is also a weighted information matrix (i.e. it is the weighted version of the symmetric *nnd* matrix $\alpha C_{d_1} + (1-\alpha)C_{d_2}$ of rank $v-1$), though it need not correspond to any existing design. Moreover,

$$\bar{C}_{dw} + \varepsilon w^{1/2} w^{1/2'} = \alpha(C_{d_1 w} + \varepsilon w^{1/2} w^{1/2'}) + (1-\alpha)(C_{d_2 w} + \varepsilon w^{1/2} w^{1/2'}). \qquad (2.27)$$

It can be verified that $l' C_{dw}^{-} l$ is invariant to choice of generalized inverse of C_{dw}, hence, applying Theorem 2.5,

$$l'(\bar{C}_{dw} + \varepsilon w^{1/2} w^{1/2'})^{-1} l \leq \alpha l'(C_{d_1 w} + \varepsilon w^{1/2} w^{1/2'})^{-1} l + (1-\alpha) l'(C_{d_2 w} + \varepsilon w^{1/2} w^{1/2'})^{-1} l$$
$$\Rightarrow l' \bar{C}_{dw}^{-} l \leq \alpha l' C_{d_1 w}^{-} l + (1-\alpha) l' C_{d_2 w}^{-} l \qquad \square$$

Corollary 2.2. *The E_w and MV_w criteria are matrix convex.*

Proof. For the E_w criterion, let f_1, f_{11} and f_{12} be normalized eigenvectors corresponding to the smallest positive eigenvalues of $\bar{C}_{dw}$, $C_{d_1 w}$ and $C_{d_2 w}$, respectively. Then $\Phi_E(\bar{C}_{dw}) = f_1' \bar{C}_{dw}^{-} f_1$. Theorem 2.6 gives

$$\begin{aligned}\Phi_E(\bar{C}_{dw}) &= f_1' \bar{C}_{dw}^{-} f_1 \leq \alpha f_1' C_{d_1 w}^{-} f_1 + (1-\alpha) f_1' C_{d_2 w}^{-} f_1 \\ &\leq \alpha f_{11}' C_{d_1 w}^{-} f_{11} + (1-\alpha) f_{12}' C_{d_2 w}^{-} f_{12} = \alpha \Phi_E(C_{d_1 w}) + (1-\alpha)\Phi_E(C_{d_2 w})\end{aligned}$$

For the MV_w criterion, let $l_{ij} = m_{ij}$, where m_{ij} is defined following Definition 2.5, so that $\Phi_{MV}(C_{dw}) = \max_{ij} l_{ij}' C_{dw}^{-} l_{ij}$. Assume the maximal value is achieved by $l_{ij} = l_{i^* j^*}$, it follows that

$$\begin{aligned}\Phi_{MV}(C_{dw}) &= l_{i^* j^*}' C_{dw}^{-} l_{i^* j^*} \leq \alpha l_{i^* j^*}' C_{d_1 w}^{-} l_{i^* j^*} + (1-\alpha) l_{i^* j^*}' C_{d_2 w}^{-} l_{i^* j^*} \\ &\leq \alpha \max_{ij} l_{ij}' C_{d_1 w}^{-} l_{ij} + (1-\alpha) \max_{ij} l_{ij}' C_{d_2 w}^{-} l_{ij} \\ &= \alpha \Phi_{MV}(C_{d_1 w}) + (1-\alpha)\Phi_{MV}(C_{d_2 w}) \qquad \square\end{aligned}$$

Corollary 2.3. *The A_w criterion is matrix convex.*

Proof. Let $l_i = W^{1/2} f_i$ $(i = 1, \ldots, v-1)$ be the coefficient vectors for a set of weighted orthogonal contrasts corresponding to fixed w, where the f_i are as defined in (2.2) for C_{dw}.

Then by Result 2.2 and Theorem 2.6,

$$\Phi_A(C_{dw}) = \frac{1}{v-1}\sum_i l_i' C_d^- l \leq \frac{1}{v-1}\sum_i \left[\alpha l_i' C_{d_1}^- l + (1-\alpha) l_i' C_{d_2}^- l\right]$$
$$= \alpha \Phi_A(C_{d_1 w}) + (1-\alpha)\Phi_A(C_{d_2 w}).$$

The inequality uses the fact that Theorem 2.6 holds even if all weights are equal. □

The following results regarding various weighted optimal designs are based on the properties of matrix convexity.

Theorem 2.7. *Suppose treatments are grouped into g mutually disjoint sets, each of constant weight. Let C_{dw} be a weighted information matrix for design $d \in \mathcal{D}(v,b,k)$, let $\mathcal{Q}$ be a set of n permutation matrices Q corresponding to within-group permutations, and write $\bar{C}_{dw} = \sum_{Q\in\mathcal{Q}}(QC_{dw}Q^T)/n$. Then*

$$\Phi(\bar{C}_{dw}) \leq \Phi(C_{dw}),$$

where Φ is any optimality function satisfying (2.22).

Proof. Using conditions (i) & (iii) of (2.22) for optimality criterion Φ,

$$\Phi(\bar{C}_{dw}) = \Phi\big(\textstyle\sum_{Q\in\mathcal{Q}}(QC_{dw}Q^T)/n\big) \leq \sum_{Q\in\mathcal{Q}}\Phi(QC_{dw}Q^T)/n = \Phi(C_{dw}).$$ □

Corollary 2.4. *Let C_{dw} be the C_w-matrix for design d, and let $\bar{d}$ be a possibly hypothetical design corresponding to weighted information matrix $\bar{C}_{dw} = \sum_{Q\in\mathcal{Q}}(QC_{dw}Q^T)/n$, where Q is a within group permutation matrix and n is the number of such permutation matrices in $\mathcal{Q}$. Then $\bar{d}$ is E_w, A_w and MV_w superior to design d.*

We note that the matrix W in Definition 2.1 could conceivably be any positive definite matrix. That is, the theory of weighted optimality need not be restricted to diagonal W assigning individual weights to individual treatments. Any such alternative W would have to be very carefully selected, lest interpretation become highly problematic. Moreover, only very special, nondiagonal W will allow convexity properties of optimality criteria to be exploited via matrix averaging as in Theorem 2.7 and Corollary 2.4. For these reasons only diagonal W is pursued in this dissertation.

Corollary 2.4 says that within group permutation of C_{dw} will improve the weighted E, MV and A criteria. Moreover, the minimum $\Phi(\bar{C}_{dw})$ for given C_{dw} is achieved if $\mathcal{Q}$ in Theorem 2.4 contains *all* within-group permutation matrices, that is, the number of permutation matrices in $\mathcal{Q}$ is $n = \sum_{j=1}^{g} v_j!$, where v_j is the number of treatments in the j^{th} group. This implies that a generalized group divisible design ($GGDD(g)$ - see Definition 2.10 just below) with groups corresponding to the weight groups, if such design exists, has weighted information matrix that is invariant to all admissible permutations, so will be the best design among all designs with the same diagonal of the C-matrix. In fact, consider any design d, and let t_j be the sum of the diagonal elements in C_d for the treatments in weight group j. The trace t of C_d is $t = \sum_{j=1}^{g} t_j$. Then $\bar{C}_{dw}$ found by permuting C_{dw} over all $\sum_{j=1}^{g} v_j!$ admissible permutations is best over all designs with the same values of $t_1, \ldots, t_g$.

The $GGDD(g)$s are defined in terms of the elements of the information matrix as follows:

Definition 2.10. (Morgan and Srivastav, 1998) The design $d \in \mathcal{D}(v, b, k)$ is called a $GGDD(g)$ if the treatments in d can be divided into g mutually disjoint sets $V_1, V_2, \ldots, V_g$ of size $v_1, v_2, \ldots, v_g$ such that

(1) for $j = 1, \ldots, g$ and all $i \in V_j$, $c_{dii} = r_{di} - \lambda_{dii}/k = c_j$, where c_j depends on the set V_j but not otherwise on the treatment i,

(2) for $j, j' = 1, \ldots, g$ and all $i \in V_j$ and $i' \in V_{j'}$, with $i \neq i'$ if $j = j'$, $\lambda_{dii'} = -k\gamma_{jj'}$, where $\gamma_{jj'}$ depends on the sets V_j and $V_{j'}$ but not otherwise on the treatment i and i'.

While the general setup allows v arbitrary weights to be assigned to the v treatments (subject only to $\sum_{i=1}^{v} w_i = 1$), the situations of greatest interest employ g weight groups for some $g < v$. In this case the weights partition the treatments into subsets of sizes v_1, v_2, $\ldots$, v_g where $\sum_{i=1}^{g} v_i = v$, $\sum_{i=1}^{g} v_i w_i = 1$, and w_1, $\ldots$, w_g are all distinct. The $GGDD(g)$s are thus of primary interest in the search for weighted optimal designs, just as are the $BIBD$s for unweighted optimality. Much of the theory in this dissertation is for two distinct weights ($g = 2$), what we call the *two-weight problem*, or the *(v_1, v_2)-problem.* In general we work with the $(v_1, v_2, \ldots, v_g)$-problem, or g-weight problem.

For convenience, some properties of information matrices of $GGDD(2)$'s are stated here.
Case I. $v_1 = 1$ and $v_2 = v - 1$.
The information matrix can be written as:

$$C_d = \begin{pmatrix} \alpha_1 & \gamma 1'_{1\times(v-1)} \\ \gamma 1_{(v-1)\times 1} & \alpha_2 I_{v-1} + \beta_2 J_{v-1} \end{pmatrix}. \tag{2.28}$$

where in terms of Definition 2.10, $\beta_2 = \gamma_{22}$, $\alpha_2 = c_2 - \gamma_{22}$ and $\gamma = \gamma_{12}$.

Since $C_d 1 = 0$, it follows that

$$\alpha_1 + \gamma(v-1) = 0 \tag{2.29}$$

$$\alpha_2 + \beta_2(v-1) + \gamma = 0 \tag{2.30}$$

Let t be the trace for the C-matrix, so

$$\alpha_1 + \alpha_2(v-1) + \beta_2(v-1) = t \tag{2.31}$$

Of parameters α_1, α_2, β_2 and γ in C_d, only one parameter is free if the trace is fixed. If α_1 is taken as the free parameter, then

$$\alpha_2 = \frac{(v-1)t - v\alpha_1}{(v-1)(v-2)} \tag{2.32}$$

$$\beta_2 = \frac{2\alpha_1 - t}{(v-1)(v-2)} \tag{2.33}$$

$$\gamma = -\frac{\alpha_1}{v-1} \tag{2.34}$$

Since for any C-matrix, β_2 must be negative, it follows that $\alpha_1 < t/2$.

Case II. $v_1 \geq 2$ and $v_2 \geq 2$. The information matrix has the form:

$$C_d = \begin{pmatrix} \alpha_1 I_{v_1} + \beta_1 J_{v_1} & \gamma J_{v_1 \times v_2} \\ \gamma J_{v_2 \times v_1} & \alpha_2 I_{v_2} + \beta_2 J_{v_2} \end{pmatrix} \tag{2.35}$$

where in terms of Definition 2.10, $\beta_1 = \gamma_{11}$, $\alpha_1 = c_1 - \gamma_{11}$, $\beta_2 = \gamma_{22}$, $\alpha_2 = c_2 - \gamma_{22}$ and $\gamma = \gamma_{12}$.

$C_d 1 = 0$ gives

$$\alpha_1 + v_1\beta_1 + v_2\gamma = 0 \tag{2.36}$$

$$\alpha_2 + v_2\beta_2 + v_1\gamma = 0 \tag{2.37}$$

For fixed trace, we have

$$v_1(\alpha_1 + \beta_1) + v_2(\alpha_2 + \beta_2) = t \tag{2.38}$$

Or,

$$(v_1 - 1)\alpha_1 + (v_2 - 1)\alpha_2 - v\gamma = t \tag{2.39}$$

Note for fixed trace, there are two free parameters among α_1, β_1, α_2, β_2 and γ.

In the following chapters, certain types of $GGDD(2)$s will be shown to be optimal in terms of E_w, MV_w and A_w criteria for the 2-weight problem. Among other designs, the $GGDD(2)$s generalize the *reinforced* BIBDs originally introduced by Cox (1958) and Das (1958).

2.3 Efficiency Balance

In addition to the general theory of weighted optimality, efficiency balance is an area of application to which the notion of weighted information directly relates.

The idea of *efficiency balance* was introduced by Jones (1959). Efficiency in the sense of Jones measures the relative loss of information caused by the confounding of contrasts with blocks. As defined before, let $r_d = (r_{d1}, \ldots, r_{dv})'$ be the treatment replication numbers, and let $D(r_d) = Diag(r_{di})$. For these replication numbers, the C-matrix for a completely randomized (i.e. unblocked) design d is

$$C_d = D(r_d) - \frac{1}{N} r_d r_d' \tag{2.40}$$

where as usual $N = \sum_i r_{di}$ is the total number of experimental units. Now any connected information matrix C_d, whether or not d is a blocked design, can be written in spectral form as

$$C_d = \sum_{i=1}^{v-1} e_{di} h_{di} h_{di}' \tag{2.41}$$

where $0 < e_{d1} \le e_{d2} \le \cdots \le e_{d,v-1}$ are the positive eigenvalues of C_d and the h_{di} are a corresponding set of orthonormal eigenvectors each summing to zero. Consider the contrast $\widehat{h_{di}'\tau}$; its variance is

$$Var(\widehat{h_{di}'\tau}) = \frac{1}{e_{di}} = z_{di} \tag{2.42}$$

where z_{di}, the inverse of e_{di}, is called a *canonical variance* (see Definition 1.1). As pointed out in Section 2.1, many conventional optimality criteria, including A, D, and E, are functions of the canonical variances. One of the goals of this dissertation is to generalize conventional optimality work in a way that it includes the other common perspective on judging block designs, that based on *efficiency balance*, *partial efficiency balance*, and *canonical efficiency* *factors*. Here we state the basic ideas and define the basic quantities employed in any discussion of (partial) efficiency balance.

Efficiency balance works with a transformed version of the C-matrix:

$$C_{d,eff} = D(r_d)^{-1/2} C_d D(r_d)^{-1/2} \tag{2.43}$$

Let the spectral decomposition of $C_{d,eff}$ be as follows:

$$C_{d,eff} = \sum_{i=1}^{v-1} \tilde{\theta}_{di} f_{di} f'_{di} \tag{2.44}$$

Each f_{di} is orthogonal to $r_d^{1/2} = D(r_d)^{1/2}1$, since this is the eigenvector of $C_{d,eff}$ corresponding to the eigenvalue of zero. It is easy to show that a generalized inverse of $C_{d,eff}$ is $C^-_{d,eff} = D(r_d)^{1/2} C^+_d D(r_d)^{1/2}$ where C^+_d is the M-P inverse of C_d. Now define $s_{di} = D(r_d)^{-1/2} f_{di}$. Then $c_{di} = D(r_d) s_{di} = D(r_d)^{1/2} f_{di}$ is a contrast vector. The variance with which this contrast is estimated is

$$\begin{aligned}
\mathrm{VAR}_d(\widehat{c'_{di}\tau}) &= c'_{di} C^+_d c_{di} \\
&= f'_{di} D(r_d)^{1/2} C^+_d D(r_d)^{1/2} f_{di} \\
&= f'_{di} C^-_{d,eff} f_{di} \\
&= \frac{1}{\tilde{\theta}^2_{di}} f'_{di} C_{d,eff} C^-_{d,eff} C_{d,eff} f_{di} \\
&= \frac{1}{\tilde{\theta}^2_{di}} f'_{di} C_{d,eff} f_{di} \\
&= \frac{1}{\tilde{\theta}_{di}} \qquad (2.45)
\end{aligned}$$

Now suppose the design d we had started with was a completely randomized design (CRD) with information matrix C_d as shown in (2.40). Then $C_{d,eff}$ as given in (2.43) would be

$$\begin{aligned}
C_{\mathrm{CRD},eff} &= D(r_d)^{-1/2}[D(r_d) - \frac{1}{N} r_d r'_d] D(r_d)^{-1/2} \\
&= I - \frac{1}{N} D(r_d)^{1/2} J D(r_d)^{1/2} \qquad (2.46)
\end{aligned}$$

The nonzero eigenvalues for the matrix (2.46) are $\tilde{\theta}_{\text{CRD},1} = \cdots = \tilde{\theta}_{\text{CRD},v-1} = 1$. That is, when using a CRD, equation (2.45) says that the $v-1$ contrasts $c'_{di}\tau$ defined above are all estimated with the same variance 1. With blocked design d having information matrix C_d, the variances for the same contrasts, as shown in (2.45), are $1/\tilde{\theta}_{di}$ for $i = 1, \ldots, v-1$. Thus the efficiency when estimating $c'_{di}\tau$ with design d relative to a CRD is

$$\begin{aligned} \text{RE}_d(c'_{di}\tau) &= \frac{\text{VAR}_{\text{CRD}}}{\text{VAR}_d} \\ &= \frac{1}{1/\tilde{\theta}_{di}} \\ &= \tilde{\theta}_{di} \end{aligned}$$

The contrasts $c'_{di}\tau$ are called the *basic contrasts* for design d. The eigenvalues $\tilde{\theta}_{di}$ which evaluate their relative efficiencies also have a special name.

Definition 2.11. The eigenvalues $\tilde{\theta}_{di}$ of $C_{d,eff}$ are called the *canonical efficiency factors* for design d.

Definition 2.12. If $\tilde{\theta}_{d1} = \cdots = \tilde{\theta}_{d,v-1}$, then d is said to be *efficiency balanced* (EB). Otherwise d is *partially efficiency balanced* (PEB) with number of distinct efficiencies equal to the number of distinct $\tilde{\theta}_{di}$.

This terminology and an extended development may be found in Caliński and Kageyama (2000). Partial efficiency balance provides a method for judging the loss of information imposed by blocking. The perspective it offers is summarized in the following bulleted points:

- Begin by fixing the replication numbers $r_1, \ldots r_v$. These numbers are chosen to reflect the relative interest in the treatments. Specifically, if we were able to use a CRD, then the variance of any contrast $c'\tau$ depends only on the replication numbers selected: $\text{VAR}_{CRD}(\widehat{c'\tau}) = \sum c_i^2/r_i$. As shown by this expression, those treatments allocated more replication contribute less to variance than those allocated less replication. Since higher interest means contrasts involving a treatment should be estimated more precisely (less variance), replication numbers do indeed reflect interest, at least for a CRD.

- If we are not able to use a CRD, then some information will be lost to blocking. The canonical efficiency factors are a set of figures that summarize that loss and how it is distributed across the treatments.

- For efficiency balanced incomplete block designs all canonical efficiency factors are the same. Thus relative information on the treatments is the same as if we had used a CRD: the relative interest in the treatments as expressed in the replication numbers has been exactly preserved.

- Any incomplete block design that is not efficiency balanced is, by definition, partially efficiency balanced. Designs having only a few distinct efficiency factors that are fairly close to one another best preserve the relative interest expressed in the replication numbers.

If all replication numbers are the same, $r_{d1} = \cdots = r_{dv} = r$, then the canonical efficiency factors and the canonical variances have a very simple relationship:

$$r\tilde{\theta}_{di} = \frac{1}{z_{di}}.$$

In this case optimality arguments could be equivalently cast in terms of the canonical efficiency factors rather than in the conventional terms of canonical variances. But what if the replications are not the same?

As already explained, the EB perspective says that unequal replications are chosen to reflect unequal treatment interest. Conventional optimality theory is based on the presumption of equal treatment interest. Thus they appear to be two divergent theories for evaluating block designs. The general notion of weighted optimality introduced here unifies the two: it allows for unequal treatment interest, while still providing mathematical evaluation of designs that is aimed at maximizing the information *with respect to that interest* that can be extracted from an experiment.

There have been attempts in the literature to define optimality criteria based on canonical efficiency factors; see for example Das and Kageyama (1991) and Kozlowska (1996, 1999). These papers suffer from two severe shortcomings. The first is that they offer no statistical interpretation of the meaning of such criteria. Conventional optimality criteria are meaningful because they compare designs in terms of summary functions of variances. Correspondingly, the presentation here offers the first statistically meaningful interpretation of design selection via (appropriately chosen) summary functions of canonical efficiency factors: minimization of weighted variance.

The second shortcoming of the aforementioned papers is that they do not restrict design selection based on canonical efficiency factors to a fixed replication class. This is wrong in two fundamental ways:

- Since weights for the $\tilde{\theta}_{di}$ depend on the replication numbers, comparison of designs with different replications in terms of canonical efficiency factors is comparing weighted variances while imposing different weights for different designs. The destroys the meaning of the comparison as explained in the preceding paragraph.

- As explained earlier in this section, canonical efficiency factors are themselves relative measures, for they evaluate information in a blocked design relative to a CRD. *But the* *information in a CRD depends on the replication numbers*, invalidating comparisons of canonical efficiency factors for designs where these numbers differ. The dependence of CRD information on replication numbers is easily seen by calculating values of A, E, MV, etc. for two CRDs having the same N but only one of which is equally replicated.

One could, of course, choose weights $w_1, \ldots, w_v$ so that Nw_i is an integer for each i; design choice based on canonical efficiency factors is implicitly demanding this. Yet another advantage of the approach proposed here is that weight choice is *not* so constrained. Weights can be selected to reflect, as accurately as possible, experimenter interest in the various treatment comparisons.

In closing this discussion, it is worth mentioning that even if the weights $w_1, \ldots, w_v$ are such that Nw_i is an integer for each i, a weighted optimal design will not necessarily have replication numbers $r_i = Nw_i$. That is, weighted optimality does not necessarily imply replication proportional to weights. This statement will be justified by the weighted optimal designs presented as examples in the chapters to follow. As a general rule, choice of replications and choice of weights should be thought about as two distinct parts of the design selection process.

2.4 General balance, efficiency factors and weighted optimality

This dissertation is primarily concerned with the *intrablock information* in a block design. This is the information resulting from the *intrablock analysis*, that is, the analysis when the data follows the model given in (1.2). The intrablock information is summarized in the information matrix C_d given in (1.1). In model (1.2) the blocks exert fixed affects, and all treatment contrasts are based on within-block contrasts of observations (hence the names "intrablock information" and "intrablock analysis").

In some experiments it is reasonable to think of the blocks as exerting random effects. Before writing the corresponding model, let's introduce a few matrices that will make the expressions needed in this section easier to handle. If N is the total number of experimental units, partitioned into b sets of k units each by the blocks, define $L_{N\times b}$ to be the unit$\times$block incidence matrix:

$$((L))_{u,j} = \begin{cases} 1 & \text{if unit } u \text{ in block } j, \\ 0 & \text{otherwise.} \end{cases}$$

With suitable ordering of the units L can be expressed as $L = I_b \otimes 1_k$. Let A_d be the $N \times v$ unit$\times$treatment incidence matrix:

$$((A_d))_{u,i} = \begin{cases} 1 & \text{if unit } u \text{ receives treatment } i, \\ 0 & \text{otherwise.} \end{cases}$$

Then if $Y_{N\times 1}$ is the vector of observations, model (1.2) can be written in matrix form as

$$Y = \mu 1_N + A_d\tau + L\beta + E \tag{2.47}$$

where $E \sim (0, \sigma_E^2 I_N)$. In terms of A_d and L the information matrix (1.1) is

$$C_d = A_d'(I - P_L)A_d \tag{2.48}$$

where $P_L = L(L'L)^{-1}L'$ is the projector onto the column space of L.

The model with random block effects is a simple modification of (2.47):

$$Y = \mu 1_N + A_d\tau + LB + E \tag{2.49}$$

where $B \sim (0, \sigma_B^2 I_b)$ is uncorrelated with E. If model (2.49) is appropriate, then while the intrablock analysis is still applicable, if desired an additional analysis can be performed using the block totals as observations. The estimates of treatment contrasts coming from this *interblock analysis*, which are uncorrelated with the intrablock estimates, can then be combined with the intrablock estimates depending on σ_B^2 and σ_E^2. The goal here is to evaluate the quality of the interblock information relative to that of the intrablock information, and to understand how that is affected by weighting of treatments as proposed in this dissertation.

The information matrix for the interblock analysis is

$$C_{d,B} = A_d'(P_L - \frac{1}{N}J_N)A_d \tag{2.50}$$

Notice that $C_d + C_{d,B} = A_d'(I - \frac{1}{N}J_N)A_d = C_{\text{CRD}}$, the information matrix for a completely randomized design with replications equals to that of the incomplete block design d. The property of *general balance* for a block design can now be defined. For any positive definite matrix S with spectral decomposition $S = \sum e_i\xi_i\xi_i'$, use $S^{1/2}$ to denote the "square root matrix," $S^{1/2} = \sum \sqrt{e_i}\xi_i\xi_i'$.

Definition 2.13. Write $I_v = \sum_{t=1}^{s} T_t$ as the sum of orthogonal projectors. If

$$(A_d'A_d)^{-1/2}C_d(A_d'A_d)^{-1/2} = \sum_t \tilde{\theta}_{dt1}T_t$$

and

$$(A_d'A_d)^{-1/2}C_{d,B}(A_d'A_d)^{-1/2} = \sum_t \tilde{\theta}_{dt2}T_t$$

for nonnegative numbers $\tilde{\theta}_{dt1}$ and $\tilde{\theta}_{dt2}$, then design d is *generally balanced* with respect to the decomposition $T_1, T_2, \ldots, T_s$.

Equivalently, design d is generally balanced if $(A_d'A_d)^{-1/2}C_d(A_d'A_d)^{-1/2}$ and $(A_d'A_d)^{-1/2}C_{d,B}(A_d'A_d)^{-1/2}$ commute.

The import of general balance (GB) is that, corresponding to the T_t's, the intrablock and interblock analyses each admit the same partitioning of the treatment contrasts subspace, so that contrasts within the t^{th} partition are estimated with variance $\tilde{\theta}_{dt1}$ and $\tilde{\theta}_{dt2}$ in the intrablock and interblock analysis, respectively. For details see Houtman and Speed (1983) and Speed (1983). Houtman and Speed (1983) established that every block design for which all blocks have the same number k of experimental units is generally balanced.

Looking again at the definition of GB, the numbers $\tilde{\theta}_{dt1}$ are exactly the canonical efficiency factors previously encountered in Definition 2.11. They report the proportion of information on the corresponding contrasts that is accorded to the intrablock analysis, assuming that $\sigma_B^2 = 0$. Because σ_B^2 is typically much larger than σ_E^2, it is usually desired to have canonical efficiency factors as large as possible.

With GB defined and its basic ideas stated, consider the question of how these notions are impacted by a weighting of treatment interest via the weight matrix W. The weights can be incorporated into model (2.49) as follows:

$$\begin{aligned} Y &= \mu 1_N + A_d\tau + LB + E \\ &= \mu 1_N + A_d W^{-1/2} W^{1/2}\tau + LB + E \\ &= \mu 1_N + \tilde{A}_d\tilde{\tau} + LB + E \end{aligned} \tag{2.51}$$

where $\tilde{A}_d = A_d W^{-1/2}$ and $\tilde{\tau} = W^{1/2}\tau$. The intrablock information matrix for model (2.51) is

$$\tilde{A}_d'(I - P_L)\tilde{A}_d = W^{-1/2}A_d'(I - P_L)A_d W^{-1/2} = W^{-1/2}C_d W^{-1/2},$$

this being exactly C_{dw} as given in Definition 2.1. Likewise the interblock information matrix is

$$C_{d,Bw} = W^{-1/2}C_{dB}W^{-1/2}.$$

General balance with weighted treatment interest thus depends on the weighted information matrices C_{dw} and $C_{d,Bw}$.

Before proceeding, let's verify that model (2.51) does indeed produce weighted variances. Estimable functions of $\tilde{\tau}$ are represented by the column space of C_{dw}:

$$\mathcal{C}(C_{dw}) = \left\{c : c'w^{1/2} = 0\right\}.$$

The scaled variance of a normalized contrast is

$$\begin{aligned}
\frac{\mathrm{V}\textsc{ar}(\widehat{c'\tilde{\tau}})/\sigma_E^2}{c'c} &= \frac{c'C_{dw}^{-}c}{c'c} \\
&= \frac{c'W^{1/2}C_d^{-}W^{1/2}c}{c'c} \\
&= \frac{\tilde{c}'C_d^{-}\tilde{c}}{\tilde{c}'W^{-1}\tilde{c}} \qquad \text{where } \tilde{c} = W^{1/2}c \text{ is a contrast vector.}
\end{aligned}$$

This is (see Definition 2.6) the (scaled) weighted variance for the estimator of $\tilde{c}'\tau = c'\tilde{\tau}$, as desired. Clearly model (2.51) provides an equivalent route for pursuing the ideas of weighted treatment interest.

Working with model (2.51) the two matrices on which GB is based are

$$(\tilde{A}_d'\tilde{A}_d)^{-1/2}C_{dw}(\tilde{A}_d'\tilde{A}_d)^{-1/2} = (A_d'A_d)^{-1/2}C_d(A_d'A_d)^{-1/2}$$

and

$$(\tilde{A}_d'\tilde{A}_d)^{-1/2}C_{d,Bw}(\tilde{A}_d'\tilde{A}_d)^{-1/2} = (A_d'A_d)^{-1/2}C_{d,B}(A_d'A_d)^{-1/2}.$$

These are exactly the same matrices used to check GB in the unweighted case!

Theorem 2.8. *The block design d is generally balanced under the treatment-weighted model (2.51) if and only if it is generally balanced under the unweighted model (2.49).*

Theorem 2.8 has been derived from the definition of general balance, but can also be seen in (2.6). Because the weight $[c'W^{-1}c]^{-1}$ applied to $\mathrm{Var}_d(\widehat{c'\tau})$ does not depend on the design, the ratio of weighted variance using a CRD to that using the incomplete block design d is the same as the ratio of unweighted variances. That is, *the proportion of information in the intrablock analysis is the same, whether or not that information is weighted.*

In view of these results, Section 2.3, and other results in this chapter, the following observations concerning the canonical efficiency factors can be made:

- Canonical efficiency factors are unresponsive to changes in relative treatment interest.
- Popularly employed summary functions of canonical efficiency factors, such as their harmonic mean, summarize *weighted* variances for the special case of weights proportional to the replication numbers.
- Consequently, these summary functions are valid measures of design efficacy only if relative treatment interest is defined by this particular choice of weights.
- Likewise, these functions are valid for design comparison only if
    - (i) weights are proportional to replication numbers, *and*
    - (ii) all designs being compared have the same replication numbers.

These points are at considerable odds with routine use of the generic term "efficiency." Broadly taken, measures of efficiency in statistical inference allow competing inferential procedures to be compared on a relevant basis. Likewise, the common usage of efficiency in the design literature is as a quantity allowing comparison among competing designs. The proper use of canonical efficiency factors for comparing designs has been shown to be restricted to a very special case of weighted treatment evaluation. This special case is sufficiently narrow, and in any case is subsumed by a more general efficiency paradigm, that it would be better to rename these terms as *canonical information factors*. This name better describes what the $\tilde{\theta}_{di}$ of (2.43) and (2.44) do report: for fixed replication numbers, the part of the information on treatment contrasts found by the intrablock analysis.

Chapter 3

Weighted E-optimal Designs

We begin our investigation of conditions under which designs enjoy weighted optimality with the weighted E-criterion, in part because it will be shown to be closely related to efficiency balance. It is also a straightforward starting point. In this chapter, several sufficient conditions are derived for designs to be E_w-optimal.

3.1 Bounding Results and a Fundamental Theorem

Bounding arguments have been an important tool in tackling E-optimality problems. Here the standard bounds are generalized for seeking E_w-optimal designs. Lower bounds for E_{dw} are equivalently stated as upper bounds for θ_{d1}.

Lemma 3.1. *Let C_{dw} be the $v \times v$ weighted information matrix for an arbitrary design d. Let x be any $v \times 1$ vector not proportional to $f_0 = w^{1/2} = \left(\sqrt{w_1}, \sqrt{w_2}, \cdots, \sqrt{w_v}\right)'$, where f_0 is the normalized eigenvector of C_{dw} corresponding to eigenvalue zero. Denote the smallest nonzero eigenvalue of C_{dw} as θ_{d1}. Then*

$$\theta_{d1} \leq \frac{x' C_{dw} x}{x'(I - f_0 f_0')x} \tag{3.1}$$

Proof. Letting $x^\star = (I - f_0 f_0')x$, then $x^{\star\prime} f_0 = x'(I - f_0 f_0')f_0 = 0$. Let $B = x^{*\prime}x^* = x'(I - f_0 f_0')x$ and write $y = \frac{x^*}{\sqrt{B}}$. Then it is obvious that y is normalized and orthogonal to f_0. The fact that $\theta_{d1} = \min_{\substack{y'y=1 \\ y'f_0=0}} y'C_{dw}y$ thus gives

$$\theta_{d1} \le y'C_{dw}y = \frac{x^{*\prime}C_{dw}x^*}{B} = \frac{x'(I - f_0 f_0')C_{dw}(I - f_0 f_0')x}{B} = \frac{x'C_{dw}x}{B}. \qquad \square$$

Corollary 3.1. *For an arbitrary design d,*

$$\theta_{d1} \le \frac{c_{dii}}{w_i(1 - w_i)}, \tag{3.2}$$

for $i = 1, 2, \ldots, v$, where c_{dii} is the i^{th} diagonal element of C_d.

Proof. In Lemma 3.1, take $x' = (0, \ldots, 1, \ldots, 0)$, where 1 is the i^{th} entry in x. Then

$$\theta_{d1} \le \frac{c_{dwii}}{1 - w_i} = \frac{c_{dii}}{w_i(1 - w_i)}$$

. $\square$

Corollary 3.2. $\theta_{d1} \le \frac{c_{dwii} + c_{dwi'i'} - 2c_{dwii'}}{2 - w_i - w_{i'} + 2\sqrt{w_i w_{i'}}}$ *for $i \ne i' \in \{1, 2, \ldots, v\}$.*

Proof. In Lemma 3.1, take $x' = (0, \ldots, 1, \ldots, 0, \ldots, -1, \ldots, 0)$, where 1 and -1 are the i^{th} and i'^{th} entries in the vector, respectively. $\square$

Corollary 3.3. $\theta_{d1} \le \frac{c_{dwii} + c_{dwi'i'} + 2c_{dwii'}}{2 - w_i - w_{i'} - 2\sqrt{w_i w_{i'}}}$ *for $i \ne i' \in \{1, 2, \ldots, v\}$.*

Proof. In Lemma 3.1, take $x' = (0, \ldots, 1, \ldots, 0, \ldots, 1, \ldots, 0)$. $\square$

Corollary 3.4. *Let m be an integer satisfying $2 \le m \le v - 1$. Let $\mathbb{M}$ be a subset of $\{1, 2, \ldots, v\}$ of size m. Then*

$$\theta_{d1} \le \frac{\sum_{i \in \mathbb{M}} c_{dwii} + 2 \sum \sum_{\substack{i<i' \\ i \in \mathbb{M}}} c_{dwii'}}{m - (\sum_{i \in \mathbb{M}} \sqrt{w_i})^2}.$$

Proof. In Lemma 3.1, let the i^{th} entry in vector $x'_{v\times 1}$ be 1, where $i \in \mathbb{M}$, and the other entries be 0. $\square$

Optimality criterion bounds are generally used in one (or both) of two ways. One is to rule out classes of inferior designs defined by specific conditions without evaluating the individual members of that class. The other is to establish a best conceivable value of the criterion in question, so that any design achieving that value is thus optimal. This latter approach is followed in the first main result for E_w-optimality.

Theorem 3.1. *Consider $\mathcal{D}(v,b,k)$ for the general g-weight problem, having weight groups of size $v_1, v_2, \ldots, v_g$, which are assigned distinct weights $w_1, w_2, \ldots, w_g$ respectively for some $1 \le g \le v$. Let $\bar{d} \in \mathcal{D}(v,b,k)$ be a design satisfying*

(i) $C_{\bar{d}w} = \epsilon(I - f_0 f_0')$ *for some* ϵ, *and*

(ii) $\min_i \dfrac{c_{\bar{d}ii}}{w_i(1-w_i)} = \max_{d\in\mathcal{D}} \min_i \dfrac{c_{dii}}{w_i(1-w_i)}$,

Then, $\bar{d}$ is E_w-optimal among all $d \in \mathcal{D}$.

Proof. The eigenvalues of $C_{\bar{d}w}$ are:

$$\begin{cases} 0 & \text{with multiplicity } 1 \\ \epsilon & \text{with multiplicity } v-1 \end{cases}.$$

Equating diagonal elements of $C_{\bar{d}w}$ and $\epsilon(I - f_0 f_0')$ gives $\epsilon(1-w_i) = \frac{c_{\bar{d}ii}}{w_i}$. Thus, $\epsilon = \frac{c_{\bar{d}ii}}{w_i(1-w_i)}$ for all $i = 1, 2, \ldots, v$.

By Corollary 3.1, for an arbitrary design $d \in \mathcal{D}(v,b,k)$,

$$\theta_{d1} \le \min_i \frac{c_{dii}}{w_i(1-w_i)} \le \min_i \frac{c_{\bar{d}ii}}{w_i(1-w_i)} = \epsilon = \theta_{\bar{d}1}.$$

Since E_w-optimal designs are those designs that maximize θ_{d1} over all designs in $\mathcal{D}$, $\bar{d}$ is E_w-optimal over $\mathcal{D}(v,b,k)$. $\square$

Corollary 3.5. *Suppose a binary design* $\bar{d} \in \mathcal{D}(v, b, k)$ *satisfies* $C_{\bar{d}w} = \epsilon(I - f_0 f_0')$ *for some* ϵ. *Then,* $\bar{d}$ *is* E_w*-optimal among all* $d \in \mathcal{D}$. *Necessarily,* $\epsilon = \frac{bk-b}{\sum_{i=1}^{g} v_i w_i(1-w_i)}$.

Proof. We will show that among all designs in $\mathcal{D}$, $\max_{d \in \mathcal{D}} \min_i \frac{c_{dii}}{w_i(1-w_i)}$ is achieved by $\bar{d}$, that is, $\max_{d \in \mathcal{D}} \min_i \frac{c_{dii}}{w_i(1-w_i)} = \epsilon$.

Suppose there exists another design $\widehat{d}$ with $\min_i \frac{c_{\widehat{d}ii}}{w_i(1-w_i)} > \epsilon$. Then using the fact that $\epsilon = \frac{c_{\bar{d}ii}}{w_i(1-w_i)}$ for all i,

$$c_{\widehat{d}ii} > c_{\bar{d}ii} \Rightarrow \textstyle\sum_i c_{\widehat{d}ii} > \sum_i c_{\bar{d}ii} \Rightarrow trace(C_{\widehat{d}}) > trace(C_{\bar{d}}) \qquad \text{for all } i,$$

a contradiction.

Applying Theorem 3.1, $\bar{d}$ is E_w-optimal over $\mathcal{D}(v, b, k)$. Also,

$$c_{\bar{d}ii} = w_i(1 - w_i)\epsilon \qquad \text{for all } i$$

$$\Rightarrow trace(C_{\bar{d}}) = \epsilon \sum_{i=1}^{g} v_i w_i(1 - w_i)$$

$$\Rightarrow \epsilon = \frac{trace(C_{\bar{d}})}{\sum_{i=1}^{g} v_i w_i(1 - w_i)} = \frac{bk - b}{\sum_{i=1}^{g} v_i w_i(1 - w_i)},$$

as claimed. □

The condition in Theorem 3.1(i) extends the concept of efficiency balance, call it *efficiency balance with respect to weights w* or *weighted balance.* Comparing Theorem 3.1 to the definition of an efficiency balanced design in Section 2.3 reveals a very interesting fact: efficiency balance is implicitly concerned with E_w-optimality. Thus efficiency balance can be thought of as a special case of the weighted-E approach to design selection. Particularly, an efficiency balanced design is weighted E-optimal for weights proportional to selected replication numbers provided it meets condition (ii) of Theorem 3.1. Since the E_w-criterion minimizes the maximum of weighted contrast variances, it tends to keep these weighted variances close to one another, which is now revealed as the underlying goal of efficiency balance.

Theorem 3.1 will be employed frequently in the sections to follow. It is worth noting that although Theorem 3.1 has been stated for block designs, it holds for any class of designs for comparing v treatments where the $v \times v$ treatment information matrices have rank $v - 1$.

Weighted balance is also the generalization of the notion of variance balance to the weighted treatment information setup. The final result of this section clarifies the connection between these two concepts.

Lemma 3.2. *Design d is weight balanced if and only if $Var_{dw}(\widehat{c'\tau})$ is the same for every contrast $c'\tau$.*

Proof. For any d, from (2.5), $Var_d(\widehat{c'\tau}) = \sum_{i=1}^{v-1} \frac{q_i^2}{\theta_{di}}$ where $\sum_i q_i^2 = c'W^{-1}c$ and $\theta_{d1}, \ldots, \theta_{d,v-1}$ are the canonical weighted variances for d. If d is weight balanced, $\theta_{d1} = \cdots = \theta_{d,v-1} = \theta_d$ (say) $\Rightarrow Var_d(\widehat{c'\tau}) = [c'W^{-1}c]/\theta_d$. Putting this in (2.6) gives

$$Var_{dw}(\widehat{c'\tau}) = [c'W^{-1}c]^{-1} Var_d(\widehat{c'\tau}) = \frac{1}{\theta_d}$$

which does not depend on the contrast vector c.

Now suppose $Var_{dw}(\widehat{c'\tau}) = \varepsilon$, a constant not dependent on c. Let $l_i = W^{1/2} f_i$ where the spectral decomposition of C_{dw} is $C_{dw} = \sum_{i=1}^{v-1} \frac{1}{\theta_{di}} f_i f_i'$. Then again from (2.5) and (2.6), and the fact that $l_i' W^{-1} l_i = 1$,

$$\begin{aligned} \varepsilon = Var_{dw}(\widehat{l_i'\tau}) = Var_d(\widehat{l_i'\tau}) &= \frac{1}{\theta_{di}} \\ \Rightarrow \theta_{d1} = \cdots = \theta_{d,v-1} &= \frac{1}{\varepsilon} \\ \Rightarrow C_{dw} = \varepsilon \sum_{i=1}^{v-1} f_i f_i' &= \varepsilon(I - f_0 f_0') \end{aligned}$$

i.e. d is weight balanced. $\square$

3.2 E_w-optimal Designs for the 2-weight Problem with $v_1 = 1$

In this section, we explore E_w-optimality for the 2-weight problem with $v_1 = 1$. An important class of applications is experiments having a set of test treatments with a single control treatment. As indicated in Section 2.1.1, small weight on the control emphasizes comparisons among the test treatments, while large weight on the control places primary importance on estimation of comparisons of the test treatments with the control.

We show that many *GGDD*s are E_w-optimal for some particular weights and can be constructed via augmenting *BIBD*s. Without loss of generality, we assume treatment 1 is the control treatment throughout this section.

Theorem 3.2. *Let $\bar{d}$ be a binary $GGDD(2)$ design in $\mathcal{D}(v,b,k)$, with group sizes $v_1 = 1$ and $v_2 = v-1$. That is, $\bar{d}$ is a binary design with C-matrix of the form in (2.28). Write t for the trace of $C_{\bar{d}}$. Then $\bar{d}$ is E_w-optimal over $\mathcal{D}(v,b,k)$ for the 2-weight problem having $v_1 = 1$, $v_2 = v-1$, $w_1 = \frac{\alpha_{\bar{d}1}}{\alpha_{\bar{d}2}(v-1)} = \frac{\alpha_{\bar{d}1}(v-2)}{(v-1)t - v\alpha_{\bar{d}1}}$ and $w_2 = -\frac{\beta_{\bar{d}2}}{\alpha_{\bar{d}2}} = \frac{t-2\alpha_{\bar{d}1}}{(v-1)t-v\alpha_{\bar{d}1}}$.*

Proof. For $w_1 = \frac{\alpha_{\bar{d}1}}{\alpha_{\bar{d}2}(v-1)}$ and $w_2 = -\frac{\beta_{\bar{d}2}}{\alpha_{\bar{d}2}}$, it is claimed that the weighted information matrix for $\bar{d}$ is

$$C_{dw} = W^{-1/2} C_d W^{-1/2} = -\frac{\alpha_{\bar{d}2}^2}{\beta_{\bar{d}2}} \begin{pmatrix} 1 - w_1 & -\sqrt{w_1 w_2}1' \\ -\sqrt{w_1 w_2}1 & I_{v-1} - w_2 J_{v-1} \end{pmatrix} = -\frac{\alpha_{\bar{d}2}^2}{\beta_{\bar{d}2}}(I - f_0 f_0'). \quad (3.3)$$

Theorem 3.1 via Corollary 3.5 gives the result. To verify the claim, first note that by (2.32) and (2.33),

$$-\frac{\alpha_{\bar{d}2}^2}{\beta_{\bar{d}2}} = \frac{[(v-1)t - v\alpha_{\bar{d}1}]^2}{(v-1)(v-2)(t-2\alpha_{\bar{d}1})} \quad (3.4)$$

and

$$\alpha_{\bar{d}2} + \beta_{\bar{d}2} = \frac{t - \alpha_{\bar{d}1}}{v-1}. \tag{3.5}$$

Now check

$$\begin{aligned}
c_{\bar{d}w11} &= \frac{\alpha_{\bar{d}1}}{w_1} = \frac{(v-1)t - v\alpha_{\bar{d}1}}{v-2} \\
&= \frac{[(v-1)t - v\alpha_{\bar{d}1}]^2}{(v-1)(v-2)(t-2\alpha_{\bar{d}1})} \quad \frac{(v-1)(t-2\alpha_{\bar{d}1})}{(v-1)t - v\alpha_{\bar{d}1}} \\
&= -\frac{\alpha_{\bar{d}2}^2}{\beta_{\bar{d}2}}(1-w_1), \\
c_{\bar{d}wii} &= \frac{\alpha_{\bar{d}2} + \beta_{\bar{d}2}}{w_2} = \frac{[(v-1)t - v\alpha_{\bar{d}1}](t-\alpha_{\bar{d}1})}{(v-1)(t-2\alpha_{\bar{d}1})} \\
&= \frac{[(v-1)t - v\alpha_{\bar{d}1}]^2}{(v-1)(v-2)(t-2\alpha_{\bar{d}1})} \quad \frac{(v-2)(t-\alpha_{\bar{d}1})}{(v-1)t - v\alpha_{\bar{d}1}} \\
&= -\frac{\alpha_{\bar{d}2}^2}{\beta_{\bar{d}2}}(1-w_2), \qquad \text{for } i \in V_2 \\
c_{\bar{d}w1i} &= \frac{\gamma_{\bar{d}}}{\sqrt{w_1 w_2}} = -\frac{\alpha_{\bar{d}1}}{(v-1)w_1 w_2}\sqrt{w_1 w_2} \\
&= -\frac{\alpha_{\bar{d}1}}{(v-1)\frac{\alpha_{\bar{d}1}}{\alpha_{\bar{d}2}(v-1)}\frac{-\beta_{\bar{d}2}}{\alpha_{\bar{d}2}}}\sqrt{w_1 w_2} \\
&= \frac{\alpha_{\bar{d}2}^2}{\beta_{\bar{d}2}}\sqrt{w_1 w_2}, \\
c_{\bar{d}wii'} &= \frac{\beta_{\bar{d}2}}{w_2} = -\alpha_{\bar{d}2} = \left(\frac{\alpha_{\bar{d}2}^2}{\beta_{\bar{d}2}}\right)\left(-\frac{\beta_{\bar{d}2}}{\alpha_{\bar{d}2}}\right) = \frac{\alpha_{\bar{d}2}^2}{\beta_{\bar{d}2}} w_2 \qquad \text{for } i \neq i' \in V_2
\end{aligned}$$

□

$GGDD(2)$s as employed in Theorem 3.2 can be obtained by augmenting $BIBD$s, as shown next.

Theorem 3.3. *Suppose d is a $BIBD$ in $\mathcal{D}(v, b^*, k)$ with $k = v-1$. Let $GGDD(2)$ $\bar{d} \in \mathcal{D}(v, b, k)$ be constructed by appending $\hat{b}$ copies of one block in d, so that $b = b^* + \hat{b}$. Then, $\bar{d}$ is E_w-optimal for $v_1 = 1$, $v_2 = v-1$, $w_1 = \frac{b^*(k-1)}{[b^*(k-1)+\hat{b}(v-1)]v} = \frac{b^*(v-2)}{[b^*(v-2)+\hat{b}(v-1)]v}$ and $w_2 = \frac{1-w_1}{v_2} = \frac{b^*k(k-1)+\hat{b}v(v-1)}{[b^*(k-1)+\hat{b}(v-1)]kv} = \frac{b^*(v-2)+\hat{b}v}{[b^*(v-2)+\hat{b}(v-1)]v}$.*

Proof. The C-matrix of $\bar{d}$ is

$$C_{\bar{d}} = \frac{1}{k}\left(\frac{b^*k(k-1)}{v-1}I - \frac{b^*k(k-1)}{v(v-1)}J\right) + \frac{1}{k}\begin{pmatrix} 0 & 0 \\ 0 & \hat{b}kI_{v-1} - \hat{b}J_{v-1} \end{pmatrix}$$

and the correspondent parameters are

$$\alpha_{\bar{d}1} = \frac{1}{k}\Big[\frac{b^*k(k-1)}{v-1} - \frac{b^*k(k-1)}{v(v-1)}\Big] = \frac{b^*(k-1)}{v},$$
$$\alpha_{\bar{d}2} = \frac{1}{k}\Big[\frac{b^*k(k-1)}{v-1} + \hat{b}k\Big] = \frac{b^*(k-1)}{v-1} + \hat{b},$$
$$\beta_{\bar{d}2} = -\frac{1}{k}\Big[\frac{b^*k(k-1)}{v(v-1)} + \hat{b}\Big].$$

By Theorem 3.2, $\bar{d}$ is E_w-optimal for the following weights

$$w_1 = \frac{\alpha_{\bar{d}1}}{\alpha_{\bar{d}2}(v-1)} = \frac{b^*(k-1)}{[b^*(k-1) + \hat{b}(v-1)]v} = \frac{b^*(v-2)}{[b^*(v-2) + \hat{b}(v-1)]v}$$

and

$$w_2 = \frac{-\beta_{\bar{d}2}}{\alpha_{\bar{d}2}} = \frac{b^*k(k-1) + \hat{b}v(v-1)}{[b^*(k-1) + \hat{b}(v-1)]kv} = \frac{b^*(v-2) + \hat{b}v}{[b^*(v-2) + \hat{b}(v-1)]v}.$$

□

Example 3.1. *Consider the following GGDD(2) $\bar{d} \in \mathcal{D}(4,5,3)$ with $V_1 = \{1\}$ and $V_2 = \{2,3,4\}$*

1	*1*	*1*	*2*	*2*
2	*2*	*3*	*3*	*3*
3	*4*	*4*	*4*	*4*

$\bar{d}$ is constructed from a BIBD $d \in \mathcal{D}(4,4,3)$ by appending one block to d, so $b^* = 4$ and $\hat{b} = 1$. By Theorem 3.3, $\bar{d}$ is E_w-optimal for $w_1 = 2/11$ and $w_2 = 3/11$. Equivalently, the ratio of w_1 to w_2 is $w_1/w_2 = 2/3$.

Theorem 3.2 says *any* binary $GGDD(2)$ with group size $v_1 = 1$ and $v_2 = v-1$ is E_w-optimal, and weight balanced, for some specific w_1 and w_2. The next result extends the range of weights for which the optimality holds.

Theorem 3.4. *Consider the 2-weight problem with $v_1 = 1$ and $v_2 = v-1$. If a design $\bar{d}$ in $\mathcal{D}(v,b,k)$ has weighted information matrix of the form in (2.28), then $\bar{d}$ is E_w-optimal in $\mathcal{D}$*

for any weights w_1, w_2 *satisfying*

(i) $\max_{d\in\mathcal{D}} \min_i \dfrac{c_{dii}}{w_i(1-w_i)} = \dfrac{\alpha_{\bar{d}1}}{w_1(1-w_1)}$, *and*

(ii) $\dfrac{w_1}{w_2} \geq -\dfrac{\alpha_{\bar{d}1}}{(v-1)\beta_{\bar{d}2}}$.

Regardless of (i), if equality holds in (ii) then $\bar{d}$ *is weight balanced.*

Proof. The non-zero eigenvalues for $C_{\bar{d}w}$ are

$$\begin{cases} \dfrac{\alpha_{\bar{d}2}}{w_2} & \text{with multiplicity } v-2 \\ \dfrac{\alpha_{\bar{d}1}}{w_1} + \dfrac{\alpha_{\bar{d}2} + (v-1)\beta_{\bar{d}2}}{w_2} & \text{with multiplicity } 1. \end{cases}$$

By condition (ii),

$$-\frac{\alpha_{\bar{d}1}}{(v-1)\beta_{\bar{d}2}} \leq \frac{w_1}{w_2} \Rightarrow \frac{\alpha_{\bar{d}1}}{w_1} \leq -\frac{(v-1)\beta_{\bar{d}2}}{w_2} \Rightarrow \frac{\alpha_{\bar{d}1}}{w_1} + \frac{\alpha_{\bar{d}2} + (v-1)\beta_{\bar{d}2}}{w_2} \leq \frac{\alpha_{\bar{d}2}}{w_2}.$$

Thus, the smallest eigenvalue of $C_{\bar{d}w}$

$$\theta_{\bar{d}1} = \frac{\alpha_{\bar{d}1}}{w_1} + \frac{\alpha_{\bar{d}2} + (v-1)\beta_{\bar{d}2}}{w_2} = \frac{\alpha_{\bar{d}1}}{w_1} - \frac{\gamma_{\bar{d}}}{w_2} = \frac{\alpha_{\bar{d}1}}{w_1} + \frac{\alpha_{\bar{d}1}}{(v-1)w_2} = \frac{\alpha_{\bar{d}1}}{w_1(1-w_1)}.$$

For any other design $d \in \mathcal{D}(v,b,k)$,

$$\theta_{d1} \leq \min_i \frac{c_{dii}}{w_i(1-w_i)} \leq \frac{\alpha_{\bar{d}1}}{w_1(1-w_1)} = \theta_{\bar{d}1}. \qquad \square$$

Theorem 3.2 says that *any* binary $GGDD(2)$ $\bar{d}$ with $v_1 = 1$ is E_w-optimal for $\frac{w_1}{w_2} = -\frac{\alpha_{\bar{d}1}}{(v-1)\beta_{\bar{d}2}}$, since $\bar{d}$ is weight balanced and $\max_{d\in\mathcal{D}} \min_i \frac{c_{dii}}{w_i(1-w_i)} = \frac{c_{\bar{d}ii}}{w_i(1-w_i)}$ is always true for this specific choice of weights. Theorem 3.4 says the range of weights for which the optimality holds can be extended as shown in condition (ii), provided the maxmin condition holds for the *first* treatment, as shown in condition (i).

We would like to find ranges of weights for optimality of a design whenever possible. The next Lemma will prove useful in this regard.

Lemma 3.3. *Let* $C_{\bar{d}}$ *be the information matrix for* $\bar{d} \in \mathcal{D}(v,b,k)$ *with diagonal elements*

$c_{\bar{d}ii}$, $i \in 1, 2, \ldots, v$. *If the replication numbers* $n_{\bar{d}ij}$ *for the* i^{th} *treatment in block* j *satisfy*

(i) $n_{\bar{d}ij} \in \{a, a+1\}$ *if* $k = 2a+1$ *for some integer* $a > 0$, *or*

(ii) $n_{\bar{d}ij} = a$ *if* $k = 2a$ *for some integer* $a > 0$.

Then $\max_{d\in\mathcal{D}} \max_i c_{dii} = \max_i c_{\bar{d}ii}$.

Proof. The i^{th} diagonal element of $C_{\bar{d}}$ is $c_{\bar{d}ii} = \sum_{j=1}^{k} n_{\bar{d}ij} - \frac{1}{k}\sum_{j=1}^{k} n_{\bar{d}ij}^2$. For each j, the quadratic form $n_{dij} - \frac{1}{k}n_{dij}^2$ is maximized at the values given. □

Theorem 3.5. *Construct a* $GGDD(2)$ *design* $\bar{d}$ *from a* $BIBD$ $d^* \in \mathcal{D}(v^*, b, k^*)$ *having* v^* *treatments in* b *blocks of size* k^* *by adding the same new treatment in each block of* d^*, *and without loss of generality let it be the first treatment. Then* $\bar{d}$ *is* E_w*-optimal over* $\mathcal{D}(v, b, k)$ *for* $v = v^*+1$, $k = k^*+1$, $w_1 = (v-2)/[(v-1)(k-1)-1]$ *and* $w_2 = (k-2)/[(v-1)(k-1)-1]$.

Proof. It is easy to get $C_{\bar{d}}$ as

$$C_{\bar{d}} = \begin{pmatrix} \dfrac{b(k-1)}{k} & -\dfrac{b(k-1)}{k(v-1)}1' \\ -\dfrac{b(k-1)}{k(v-1)}1 & \Big[\dfrac{b(k-1)^2}{k(v-1)} - \gamma_{\bar{d}22}\Big]I + \gamma_{\bar{d}22}J \end{pmatrix} \tag{3.6}$$

where $\gamma_{22} = -\frac{b(k-1)(k-2)}{k(v-1)(v-2)}$.

Comparing the above $C_{\bar{d}}$ with the C-matrix of $GGDD(2)$ given in (2.28), then we have

$$\alpha_1 = \frac{b(k-1)}{k},$$

$$\alpha_2 = \frac{1}{k}\frac{b(k-1)^2(v-2) + b(k-1)(k-2)}{(v-1)(v-2)},$$

$$\beta_2 = \gamma_{\bar{d}22} = -\frac{b(k-1)(k-2)}{k(v-1)(v-2)},$$

$$\gamma = -\frac{b(k-1)}{k(v-1)}.$$

Applying Theorem 3.2, $\bar{d}$ is E_w-optimal for

$$w_1 = \frac{\alpha_1}{\alpha_2(v-1)} = \frac{v-2}{(v-1)(k-1)-1}, \text{ and}$$
$$w_2 = -\frac{\beta_2}{\alpha_2} = \frac{k-2}{(v-1)(k-1)-1}. \qquad \square$$

It will be shown that among designs in Theorem 3.5, those having block size $k = 3$ are E_w-optimal for a weight range instead of just a specific set of weights.

Corollary 3.6. *Suppose a design $\bar{d} \in \mathcal{D}(v, b, k)$ is obtained as described in Theorem 3.5 with block size $k = 3$. Then $\bar{d}$ is E_w-optimal over $\mathcal{D}$ provided that $w_1 \geq (v-2)/(2v-3)$, or equivalently, $\frac{w_1}{w_2} \geq v - 2$. In particular, $\bar{d}$ is E_w-optimal if $w_1 \geq 1/2$.*

Proof. This is an application of Theorem 3.4. From the C-matrix of $\bar{d}$ in (3.6), it can be obtained $\frac{\alpha_{\bar{d}1}}{(v-1)\beta_{\bar{d}2}} = -\frac{v-2}{k-2}$. It can also be checked that $\frac{w_1}{w_2} \geq -\frac{\alpha_{\bar{d}1}}{(v-1)\beta_{\bar{d}2}}$ iff $w_1 \geq \frac{v-2}{(v-1)(k-1)-1}$. Moreover, if $\frac{w_1}{w_2} \geq v - 2$, then $\frac{w_1(1-w_1)}{w_2(1-w_2)} = (v-1)[v - 1 - \frac{v-2}{1-w_2}] \geq \frac{v-1}{2} = \frac{c_{\bar{d}11}}{c_{\bar{d}22}}$. It follows that $\frac{c_{\bar{d}11}}{w_1(1-w_1)} \leq \frac{c_{\bar{d}22}}{w_2(1-w_2)}$. By Lemma 3.3, c_{d11} cannot be greater than $b(k-1)/k$ for $k = 3$, so it follows that $\max_{d \in \mathcal{D}} \min_i \frac{c_{dii}}{w_i(1-w_i)} = \frac{c_{\bar{d}11}}{w_1(1-w_1)}$. By Theorem 3.4, $\bar{d}$ is E_w-optimal over $\mathcal{D}(v, b, 3)$ for $w_1 \geq (v-2)/(2v-3)$. $\square$

Example 3.2. *Consider the design $\bar{d}$ with $v_1 = 1$, $v_2 = 4$, $b = 6$, and $k = 3$*

1	*1*	*1*	*1*	*1*	*1*
2	*2*	*2*	*3*	*3*	*4*
3	*4*	*5*	*4*	*5*	*5*

where $\bar{d}$ is built up by adding treatment 1 to every block in a $BIBD(4, 6, 2)$. By Corollary 3.6, $\bar{d}$ is E_w-optimal for $w_1/w_2 \geq v - 2 = 3 \Leftrightarrow w_1 \geq \frac{3}{7}$.

Theorem 3.6. *Given a BIBD $d^* \in \mathcal{D}(v^*, b, k^*)$, let $GGDD(2)$ design $\bar{d} \in \mathcal{D}(v, b, k)$ with $v_1 = 1$ and $v_2 = v-1$ be formed by adding the same number $\hat{k} \in \{k^*-1, k^*, k^*+1\}$ replications* *of treatment* 1 *to each block of* d^**. Then* $\bar{d}$ *is* E_w*-optimal over* $\mathcal{D}$ *for* $w_1 \geq \frac{(v-2)\hat{k}}{(v-1)(k-1)-\hat{k}}$ *and* $w_2 = \frac{1-w_1}{v-1}$*, where* $v = v^* + 1$ *and* $k = k^* + \hat{k}$*.*

Proof. Obviously, the parameters in $C_{\bar{d}}$ of form (2.28) are:

$$\alpha_{\bar{d}1} = c_{\bar{d}11} = \frac{b\hat{k}(k-\hat{k})}{k}$$

$$\alpha_{\bar{d}2} + \beta_{\bar{d}2} = c_{\bar{d}ii} = \frac{b(k-\hat{k})(k-1)}{k(v-1)} \qquad \text{for } i = 2, \cdots, v$$

$$\beta_{\bar{d}2} = -\frac{b(k-\hat{k})(k-\hat{k}-1)}{k(v-1)(v-2)}$$

It is simple to check $\frac{c_{\bar{d}11}}{w_1(1-w_1)} \leq \frac{c_{\bar{d}ii}}{w_i(1-w_i)}$ if and only if $w_1 \geq \frac{(v-2)\hat{k}}{(v-1)(k-1)-\hat{k}}$, that is, $\min_{i\in V} \frac{c_{\bar{d}ii}}{w_i(1-w_i)} = \frac{c_{\bar{d}11}}{w_1(1-w_1)}$. Moreover, by Lemma 3.3, $\max_{d\in\mathcal{D}} \frac{c_{d11}}{w_1(1-w_1)} = \frac{c_{\bar{d}11}}{w_1(1-w_1)}$ since $\hat{k} \in \{k^*-1, k^*, k^*+1\}$. Hence, $\max_{d\in\mathcal{D}} \min_{i\in V} \frac{c_{dii}}{w_i(1-w_i)} = \frac{c_{\bar{d}11}}{w_1(1-w_1)}$.

In addition, $w_1 \geq \frac{(v-2)\hat{k}}{(v-1)(k-1)-\hat{k}}$ gives

$$\frac{w_1}{w_2} \geq \frac{(v-2)\hat{k}}{k-\hat{k}-1} = -\frac{\alpha_{\bar{d}1}}{(v-1)\beta_{\bar{d}2}}.$$

By Theorem 3.4, d^* is E_w-optimal. □

Example 3.3. *Consider the design* $\bar{d}$ *constructed from BIBD* $d^* \in \mathcal{D}(6, 10, 3)$ *by adding two replications of treatment* 1 *in each block of* d^**,*

1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	3	3	3	4	4
3	3	4	5	6	4	5	6	5	5
4	5	6	7	7	7	6	7	6	7

Then $\bar{d}$ *is* E_w*-optimal in* $\mathcal{D}(7, 10, 5)$ *for the 2-weight problem with* $v_1 = 1$ *and* $\frac{w_1}{w_2} \geq 5$*.*

If adding 3 *replicates of the first treatment in each block of* d^**, then the new design is* E_w*-optimal over* $\mathcal{D}(7, 10, 6)$ *for* $\frac{w_1}{w_2} \geq 7.5$*.*

If adding 4 *replicates of the first treatment in each block of* d^**, then the new design is* E_w*-optimal over* $\mathcal{D}(7, 10, 7)$ *for* $\frac{w_1}{w_2} \geq 10$.

Theorem 3.2 says that *any* binary $GGDD(2)$ with $v_1 = 1$ is E_w-optimal with respect to specific weights for the 2-weight problem with $v_1 = 1$. The following theorem extends Corollary 3.5 from *binary* $GGDD(2)$s to *uniform* $GGDD(2)$s. The definition of uniform designs is given first.

Definition 3.1. (Morgan and Parvu, 2007) In a block design, the assignment of treatment i is said to be *uniform* if $|n_{ij} - n_{ij'}| \leq 1$ for every pair of blocks $j \neq j'$. A design is uniform if the assignments of all treatments are uniform.

A binary incomplete block design is uniform. Also, a uniform block design d maximizes the trace of C_d among all designs having the same replication numbers $r_1, r_2, \ldots, r_v$. Particularly, let n_{dij} (as usual) be the number of replications of treatment i in the j^{th} block of design d and write $\kappa_{di} = \max_j n_{dij}$. If $\kappa_{di} \leq int(\frac{k+1}{2})$, then uniform d maximizes the diagonal elements c_{ii} among all designs having $r_i \leq r_{di}$. We now generalize Corollary 3.5.

Theorem 3.7. *Let* $\bar{d}$ *be a uniform design in* $\mathcal{D}(v, b, k)$ *having* $\kappa_{\bar{d}i} \leq int(\frac{k+1}{2})$ *for all* i*. If* $\bar{d}$ *satisfies* $C_{\bar{d}w} = \epsilon(I - f_0 {f'}_0)$ *for some* ϵ *and* w*, that is, if* $\bar{d}$ *is weight balanced, then* $\bar{d}$ *is* E_w*-optimal over* $\mathcal{D}$*.*

Proof. Suppose there is a design d which is E_w-better than $\bar{d}$. $C_{\bar{d}w} = \epsilon(I - f_0 {f'}_0) \Rightarrow \frac{c_{\bar{d}ii}}{w_i(1-w_i)} = \epsilon = \theta_{\bar{d}1}\ \forall\ i$. By Corollary 3.1, if d is E_w-better than $\bar{d}$, then $\frac{c_{dii}}{w_i(1-w_i)} > \frac{c_{\bar{d}ii}}{w_i(1-w_i)} = \theta_{\bar{d}1}$ for all i. But if $\kappa_{\bar{d}i} = \frac{k+1}{2}$ for any i, then $c_{\bar{d}ii}$ cannot be exceeded. Otherwise, $\bar{d}$ has $\kappa_{\bar{d}i} \leq int(\frac{k}{2})$ for all i and is uniform, so

$$\begin{aligned} \frac{c_{dii}}{w_i(1-w_i)} &> \frac{c_{\bar{d}ii}}{w_i(1-w_i)} \qquad \forall\ i \\ \Rightarrow r_{di} &> r_{\bar{d}i} \qquad \forall\ i, \text{ a contradiction.} \end{aligned}$$

$\square$

Since there always exists a specific weight w_1 such that a $GGDD(2)$ with $v_1 = 1$ is weight balanced (see Theorem 3.4), Theorem 3.7 says a uniform $GGDD(2)$ $\bar{d}$ with $v_1 = 1$ having $\kappa_{\bar{d}i} \leq int(\frac{k+1}{2})$ is E_w-optimal for that w_1. Due to the discrete nature of these designs, it is expected that small changes in the weights will not change the design ordering with respect to the weighted criterion. The following theorem establishes a weight neighborhood of w for which *uniform GGDD*(2) designs are E_w-optimal.

Theorem 3.8. *For the 2-weight problem with $v_1 = 1$ and $v_2 = v - 1$, let $\bar{d}$ be a uniform $GGDD(2)$ design in $\mathcal{D}(v, b, k)$. Write $\kappa_{\bar{d}1} = \max_j n_{\bar{d}1j}$ and $\kappa_{\bar{d}2} = \max_{i\geq 2,j} n_{\bar{d}ij}$. If $\bar{d}$ satisfies $\kappa_{\bar{d}1} \leq int(\frac{k}{2})$ and $\kappa_{\bar{d}2} \leq int(\frac{k}{2})$, then $\bar{d}$ is E_w-optimal in $\mathcal{D}$ for $\frac{\alpha_{\bar{d}1} - \frac{k-(2\kappa_{\bar{d}1}-1)}{k}}{\alpha_{\bar{d}2}(v-1)} \leq w_1 \leq \frac{\alpha_{\bar{d}1}(v-2)}{(v-1)^2\left(\alpha_{\bar{d}2}+\beta_{\bar{d}2}-\frac{k-(2\kappa_{\bar{d}2}-1)}{k}\right)-\alpha_{\bar{d}1}}$, or equivalently, $\frac{1}{v-1} - \frac{\alpha_{\bar{d}1}(v-2)}{(v-1)^3\left(\alpha_{\bar{d}2}+\beta_{\bar{d}2}-\frac{k-(2\kappa_{\bar{d}2}-1)}{k}\right)-\alpha_{\bar{d}1}(v-1)} \leq w_2 \leq \frac{1}{v-1} - \frac{\alpha_{\bar{d}1} - \frac{k-(2\kappa_{\bar{d}1}-1)}{k}}{\alpha_{\bar{d}2}(v-1)^2}$.*

Proof. Denote the lower bound for w_2 by $w_{2L} \equiv \frac{1}{v-1} - \frac{\alpha_{\bar{d}1}(v-2)}{(v-1)^3(\alpha_{\bar{d}2}+\beta_{\bar{d}2}-\frac{k-(2\kappa_{\bar{d}2}-1)}{k})-\alpha_{\bar{d}1}(v-1)}$, and the upper bound by $w_{2U} \equiv \frac{1}{v-1} - \frac{\alpha_{\bar{d}1} - \frac{k-(2\kappa_{\bar{d}1}-1)}{k}}{\alpha_{\bar{d}2}(v-1)^2}$.

The minimum nonzero eigenvalue for $C_{\bar{d}w}$ is

$$\theta_{\bar{d}1} = \begin{cases} \frac{\alpha_{\bar{d}1}}{w_1(1-w_1)} & \text{if } w_2 \leq -\frac{\beta_{\bar{d}2}}{\alpha_{\bar{d}2}} \\ \frac{\alpha_{\bar{d}2}}{w_2} & \text{if } w_2 \geq -\frac{\beta_{\bar{d}2}}{\alpha_{\bar{d}2}} \end{cases}.$$

Let $t_{\bar{d}}$ be the trace of $C_{\bar{d}}$. Since $\bar{d}$ is $GGDD(2)$, equations (2.31)-(2.33) give

$$\begin{aligned}
-\frac{\beta_{\bar{d}2}}{\alpha_{\bar{d}2}} &= \frac{t_{\bar{d}} - 2\alpha_{\bar{d}1}}{(v-1)t_{\bar{d}} - v\alpha_{\bar{d}1}} = \frac{1}{v-1} - \frac{(v-2)\alpha_{\bar{d}1}}{(v-1)^2(t_{\bar{d}} - \frac{v}{v-1}\alpha_{\bar{d}1})} \\
&> \frac{1}{v-1} - \frac{(v-2)\alpha_{\bar{d}1}}{(v-1)^2(t_{\bar{d}} - \frac{v}{v-1}\alpha_{\bar{d}1}) - (v-1)^3\frac{k-(2\kappa_{\bar{d}2}-1)}{k}} \\
&= \frac{1}{v-1} - \frac{\alpha_{\bar{d}1}(v-2)}{(v-1)^3(\alpha_{\bar{d}2} + \beta_{\bar{d}2} - \frac{k-(2\kappa_{\bar{d}2}-1)}{k}) - \alpha_{\bar{d}1}(v-1)} \\
&= w_{2L}.
\end{aligned}$$

and

$$\begin{aligned}
-\frac{\beta_{\bar{d}2}}{\alpha_{\bar{d}2}} &= \frac{1}{v-1} - \frac{(v-2)\alpha_{\bar{d}1}}{(v-1)^2(t_{\bar{d}} - \frac{v}{v-1}\alpha_{\bar{d}1})} \\
&< \frac{1}{v-1} - \frac{(v-2)(\alpha_{\bar{d}1} - \frac{k-(2\kappa_{\bar{d}1}-1)}{k})}{(v-1)^2(t_{\bar{d}} - \frac{v}{v-1}\alpha_{\bar{d}1})} \\
&= \frac{1}{v-1} - \frac{\alpha_{\bar{d}1} - \frac{k-(2\kappa_{\bar{d}1}-1)}{k}}{\alpha_{\bar{d}2}(v-1)^2} \\
&= w_{2U}.
\end{aligned}$$

Since $\bar{d}$ is the average of all matrices which are obtained by permuting treatment 2, 3, ..., v over the information matrix of an arbitrary design in $\mathcal{D}$ having $c_{11} = c_{\bar{d}11}$ and $\sum_{i=2}^{v} c_{ii} = \sum_{i=2}^{v} c_{\bar{d}ii}$, no design having $c_{11} = c_{\bar{d}11}$ and $\sum_{i=2}^{v} c_{ii} = \sum_{i=2}^{v} c_{\bar{d}ii}$ can be E_w-better than $\bar{d}$ for any set of weights. Hence we only need to check designs with $c_{11} \neq c_{\bar{d}11}$ or $\sum_{i=2}^{v} c_{ii} \neq \sum_{i=2}^{v} c_{\bar{d}ii}$.

Case I. $w_{2L} \le w_2 \le -\frac{\beta_{\bar{d}2}}{\alpha_{\bar{d}2}} \Rightarrow \theta_{\bar{d}1} = \frac{\alpha_{\bar{d}1}}{w_1(1-w_1)} = \frac{c_{\bar{d}11}}{w_1(1-w_1)}$.

(i) Suppose a competitor $d^* \in \mathcal{D}$ has $r_{d^*1} > r_{\bar{d}1}$. So, $r_{d^*i} < r_{\bar{d}i}$ for at least one $i \in \{2, \ldots, v\}$. Without loss of generality, assume $r_{d^*2} < r_{\bar{d}2}$. Since the change in replication of any treatment must be an integer, it follows that $c_{d^*22} \le \alpha_{\bar{d}2} + \beta_{\bar{d}2} - \frac{k-(2\kappa_{\bar{d}2}-1)}{k}$ (which is attained iff $r_{d^*2} = r_{\bar{d}2} - 1$ and treatment 2 is uniform in d^*).

Now,

$$w_2 \geq w_{2L}$$

$$\Rightarrow w_2 \geq \frac{1}{v-1} - \frac{\alpha_{\bar{d}1}(v-2)}{(v-1)^3(\alpha_{\bar{d}2} + \beta_{\bar{d}2} - \frac{k-(2\kappa_{\bar{d}2}-1)}{k}) - \alpha_{\bar{d}1}(v-1)}$$

$$\Rightarrow w_2 \geq \frac{(v-1)(\alpha_{\bar{d}2} + \beta_{\bar{d}2} - \frac{k-(2\kappa_{\bar{d}2}-1)}{k}) - \alpha_{\bar{d}1}}{(v-1)^2(\alpha_{\bar{d}2} + \beta_{\bar{d}2} - \frac{k-(2\kappa_{\bar{d}2}-1)}{k}) - \alpha_{\bar{d}1}}$$

$$\Rightarrow \alpha_{\bar{d}1}(1-w_2) \geq (v-1)(\alpha_{\bar{d}2} + \beta_{\bar{d}2} - \frac{k-(2\kappa_{\bar{d}2}-1)}{k})[1-(v-1)w_2]$$

$$\Rightarrow \alpha_{\bar{d}1}(1-w_2) \geq (v-1)(\alpha_{\bar{d}2} + \beta_{\bar{d}2} - \frac{k-(2\kappa_{\bar{d}2}-1)}{k})w_1$$

$$\Rightarrow \frac{\alpha_{\bar{d}1}}{w_1(1-w_1)} \geq \frac{\alpha_{\bar{d}2} + \beta_{\bar{d}2} - \frac{k-(2\kappa_{\bar{d}2}-1)}{k}}{w_2(1-w_2)}$$

$$\Rightarrow \frac{c_{\bar{d}11}}{w_1(1-w_1)} \geq \frac{c_{d^*22}}{w_2(1-w_2)}$$

By Corollary 3.1, the E_w value of d^*, which is the minimum eigenvalue of C_{d^*w}, satisfies

$$\theta_{d^*1} \leq \min_i \frac{c_{d^*ii}}{w_i(1-w_i)} \leq \frac{c_{d^*22}}{w_2(1-w_2)} \leq \frac{c_{\bar{d}11}}{w_1(1-w_1)} = \theta_{\bar{d}1}.$$

(ii) Next consider a competitor $d' \in \mathcal{D}$ having $r_{d'1} \leq r_{\bar{d}1}$. Again, applying Corollary 3.1,

$$\theta_{d'1} \leq \frac{c_{d'11}}{w_1(1-w_1)} \leq \frac{c_{\bar{d}11}}{w_1(1-w_1)} = \theta_{\bar{d}1}.$$

It can be seen that (i) and (ii) cover all possible competitor designs, and so $\bar{d}$ is E_w-optimal for $-\frac{\beta_{\bar{d}2}}{\alpha_{\bar{d}2}} \geq w_2 \geq w_{2L}$.

Case II. $-\frac{\beta_{\bar{d}2}}{\alpha_{\bar{d}2}} \leq w_2 \leq w_{2U} \Rightarrow \theta_{\bar{d}1} = \frac{\alpha_{\bar{d}2}}{w_2}$.

Let d' and d^* be as defined in Case I. Then

$$r_{d^*1} > r_{\bar{d}1}$$

$$\Rightarrow \quad r_{d^*2} \leq r_{\bar{d}2} - 1 \qquad \text{WLOG assume } r_{d^*2} < r_{\bar{d}2}$$

$$\Rightarrow \quad \theta_{d^*1} \leq \frac{c_{d^*22}}{w_2(1-w_2)} \leq \frac{\alpha_{\bar{d}2} + \beta_{\bar{d}2} - \frac{k-(2\kappa_{\bar{d}2}-1)}{k}}{w_2(1-w_2)} < \frac{\alpha_{\bar{d}2}}{w_2} = \theta_{\bar{d}1},$$

the last inequality because

$$\begin{aligned}
&\frac{\alpha_{\bar{d}2}+\beta_{\bar{d}2}-\frac{k-(2\kappa_{\bar{d}2}-1)}{k}}{w_2(1-w_2)}-\frac{\alpha_{\bar{d}2}}{w_2}\\
&=\frac{1}{w_2(1-w_2)}\left(\beta_{\bar{d}2}-\frac{k-(2\kappa_{\bar{d}2}-1)}{k}+\alpha_{\bar{d}2}w_2\right)\\
&\leq\frac{1}{w_2(1-w_2)}\left(\beta_{\bar{d}2}-\frac{k-(2\kappa_{\bar{d}1}-1)}{k}+\frac{\alpha_{\bar{d}2}(v-1)-\alpha_{\bar{d}1}+\frac{k-(2\kappa_{\bar{d}1}-1)}{k}}{(v-1)^2}\right) \qquad \text{since } w_2<w_{2U}\\
&=\frac{-1}{w_2(1-w_2)}\left(1-\frac{1}{(v-1)^2}\right)\frac{k-(2\kappa_{\bar{d}1}-1)}{k} \qquad \text{by equation (2.32) and (2.33)}\\
&<0.
\end{aligned}$$

Next we will investigate d', as defined in Case I, which has $r_{d'1}\leq r_{\bar{d}1}$. We will need the following implication of $w_2\leq w_{2U}$:

$$\begin{aligned}
& w_2\leq\frac{1}{v-1}-\frac{\alpha_{\bar{d}1}-\frac{k-(2\kappa_{\bar{d}1}-1)}{k}}{\alpha_{\bar{d}2}(v-1)^2}\\
\Rightarrow\ & \alpha_{\bar{d}2}(v-1)w_1\geq\alpha_{\bar{d}1}-\frac{k-(2\kappa_{\bar{d}1}-1)}{k}\\
\Rightarrow\ & \frac{\alpha_{\bar{d}2}}{w_2}\geq\frac{\alpha_{\bar{d}1}-\frac{k-(2\kappa_{\bar{d}1}-1)}{k}}{w_1(1-w_1)}. \qquad (3.7)
\end{aligned}$$

There are two possible situations:

(i) $r_{d'1}<r_{\bar{d}1}$.

We have $r_{d'1}<r_{\bar{d}1}\Rightarrow\alpha_{d'1}\leq\alpha_{\bar{d}1}-\frac{k-(2\kappa_{\bar{d}1}-1)}{k}$. So, from Corollary 3.1 and (3.7),

$$\theta_{d'1}\leq\frac{\alpha_{d'1}}{w_1(1-w_1)}\leq\frac{\alpha_{\bar{d}1}-\frac{k-(2\kappa_{\bar{d}1}-1)}{k}}{w_1(1-w_1)}\leq\frac{\alpha_{\bar{d}2}}{w_2}=\theta_{\bar{d}1}.$$

(ii) $r_{d'1}=r_{\bar{d}1}$.

Those designs with $r_{d'1}=r_{\bar{d}1}$ but $r_{d'i}\neq r_{\bar{d}i}$ for at least one $i\in\{2,\cdots,v\}$ can be eliminated by case II(i), because $r_{d^*i}\leq r_{\bar{d}i}-1$ for at least one $i\in\{2,\cdots,v\}$.

It remains to evaluate those designs having $r_{d'i}=r_{\bar{d}i}$ for all $i\in\{1,2,\ldots,v\}$. Let d' be such a design. Since $\bar{d}$ is uniform, d' must have $c_{d'ii} \leq c_{\bar{d}ii}$ for all i. Let i and i' be any two treatments in $\{2, \ldots, v\}$. By Corollary 3.2, if d' is E_w-better than $\bar{d}$, then

$$\begin{aligned}
\theta_{\bar{d}1} = \frac{\alpha_{\bar{d}2}}{w_2} &= \frac{c_{\bar{d}ii} + c_{\bar{d}i'i'} + \frac{2\lambda_{\bar{d}ii'}}{k}}{2w_2} < \theta_{d'1} \leq \frac{c_{d'ii} + c_{d'i'i'} + \frac{2\lambda_{d'ii'}}{k}}{2w_2} \\
\Rightarrow \quad \lambda_{d'ii'} > \lambda_{\bar{d}ii'} &\qquad \text{since } c_{\bar{d}ii} + c_{\bar{d}i'i'} \geq c_{d'ii} + c_{d'i'i'} \qquad (3.8)
\end{aligned}$$

where $\lambda_{d'ii'}$ and $\lambda_{\bar{d}ii'}$ are the treatment concurrence counts for treatments i and i' for d' and $\bar{d}$, respectively. Since treatment concurrence are integers, $\lambda_{d'ii'} \geq \lambda_{\bar{d}ii'} + 1$. For any fixed $i \in \{2, \ldots, v\}$, $C_{d'}1 = 0$ and $C_{\bar{d}}1 = 0$ gives

$$\begin{aligned}
& c_{d'ii} - \frac{\lambda_{d'1i}}{k} - \sum_{i' \neq 1, i} \frac{\lambda_{d'ii'}}{k} = c_{\bar{d}ii} - \frac{\lambda_{\bar{d}1i}}{k} - \sum_{i' \neq 1, i} \frac{\lambda_{\bar{d}ii'}}{k} = 0 \\
\Rightarrow \quad & \lambda_{d'1i} \leq \lambda_{\bar{d}1i} - (v-2) \qquad \text{since } c_{d'ii} \leq c_{\bar{d}ii} \\
\Rightarrow \quad & c_{d'11} = \frac{\sum_{i=2}^{v} \lambda_{d'1i}}{k} \leq \frac{\sum_{i=2}^{v} \lambda_{\bar{d}1i}}{k} - \frac{(v-1)(v-2)}{k} \\
& = c_{\bar{d}11} - \frac{(v-1)(v-2)}{k} < \alpha_{\bar{d}1} - \frac{k - (2\kappa_{\bar{d}1} - 1)}{k}.
\end{aligned}$$

So, again invoking Corollary 3.1 and (3.7), $\theta_{d'1} \leq \frac{c_{d'11}}{w_1(1-w_1)} < \frac{\alpha_{\bar{d}1} - \frac{k-(2\kappa_{\bar{d}1}-1)}{k}}{w_1(1-w_1)} \leq \frac{\alpha_{\bar{d}2}}{w_2}$, a contradiction to the assumption that d' is E_w-better than $\bar{d}$. □

Example 3.4. *The following nonbinary design $\bar{d}$ in $\mathcal{D}(6, 15, 5)$ is weight balanced for $w_1 = 8/23$ and $w_2 = 3/23$:*

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	2	2	2	2	3
2	2	2	2	2	2	3	3	3	4	3	3	3	4	4
3	3	3	4	4	5	4	4	5	5	4	4	5	5	5
4	5	6	5	6	6	5	6	6	6	5	6	6	6	6

$\bar{d}$ is a uniform design and satisfies condition in Theorem 3.8, so it is E_w-optimal in $\mathcal{D}(6, 15, 5)$ for $\frac{39}{115} \leq w_1 \leq \frac{16}{41}$, or equivalently, $\frac{195}{76} \leq w_1/w_2 \leq \frac{16}{5}$.

Corollary 3.7. *For the 2-weight problem with $v_1 = 1$ and $v_2 = v-1$, a binary GGDD(2) $\bar{d} \in \mathcal{D}(v,b,k)$ with C-matrix given in (2.28) is E_w-optimal in $\mathcal{D}$ for $\frac{\alpha_{\bar{d}1}-\frac{k-1}{k}}{\alpha_{\bar{d}2}(v-1)} \leq w_1 \leq \frac{\alpha_{\bar{d}1}(v-2)}{(v-1)^2(\alpha_{\bar{d}2}+\beta_{\bar{d}2}-\frac{k-1}{k})-\alpha_{\bar{d}1}}$, equivalently, $\frac{1}{v-1} - \frac{\alpha_{\bar{d}1}(v-2)}{(v-1)^3(\alpha_{\bar{d}2}+\beta_{\bar{d}2}-\frac{k-1}{k})-\alpha_{\bar{d}1}(v-1)} \leq w_2 \leq \frac{1}{v-1} - \frac{\alpha_{\bar{d}1}-\frac{k-1}{k}}{\alpha_{\bar{d}2}(v-1)^2}$.*

Proof. A binary $\bar{d}$ has $\kappa_{\bar{d}1} = \kappa_{\bar{d}2} = 1$ in Theorem 3.8. □

Example 3.5. *Consider this binary GGDD $\bar{d}$ in $\mathcal{D}(v,b,k) = \mathcal{D}(4,5,3)$ with $v_1 = 1$ and $v_2 = 3$:*

$$\begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 & 3 \\ 3 & 4 & 4 & 4 & 4 \end{array}$$

By Theorem 3.1, $\bar{d}$ is weight balanced and hence E_w-optimal when $w_1 = 2/11$ and $w_2 = 3/11$, or equivalently, $w_1/w_2 = 2/3$. Corollary 3.7 says that $\bar{d}$ is E_w-optimal over $\mathcal{D}(4,5,3)$ for $0.4138 \leq w_1/w_2 \leq 1$.

Example 3.6. *The following $d \in \mathcal{D}(5,6,3)$ is a GGDD(2) with $v_1 = 1$ and $v_2 = 4$:*

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 & 3 & 4 \\ 3 & 4 & 5 & 4 & 5 & 5 \end{array}$$

It can be observed the replication of treatment 1 is twice that of the other treatments. By Theorem 3.1, d is weight balanced and thus E_w-optimal for $w_1 = 0.428$ and $w_2 = 0.143$, or $w_1/w_2 = 3$. By Corollary 3.7, the range of w_2 for which this design is E_w-optimal is $w_2 \in (0.077, 0.161)$, equivalently, $2.\overline{2} \leq w_1/w_2 \leq 9$. Recall Example 3.2 has shown d is E_w-optimal for $w_1/w_2 \geq 3$. Combining these two results, d is E_w-optimal for $w_1/w_2 \geq 2.\overline{2}$.

Example 3.7. *The following $d \in \mathcal{D}(6,10,4)$ is a $GGDD(2)$ with $v_1 = 1$ and $v_2 = 5$:*

$$\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\
3 & 3 & 3 & 4 & 4 & 5 & 4 & 4 & 5 & 5 \\
4 & 5 & 6 & 5 & 6 & 6 & 5 & 6 & 6 & 6
\end{array}$$

This design is weight balanced for $w_1 = 0.2857$ and $w_2 = 0.1429$, or $w_1/w_2 = 2$. Furthermore, by Corollary 3.7, this design is E_w-optimal in $\mathcal{D}(6,10,4)$ for $1.731 \leq w_1/w_2 \leq 2.\overline{6}$.

The continuous interval for w allow us to construct E_w-optimal designs in a more meaningful way. When designs are known to be optimal for a range of weights, experimenters are not required to be as precise in their specification of relative treatment interest.

3.3 E_w-optimal Designs for the 2-weight Problem with Group Sizes ≥ 2

The implementation of Theorem 3.1 and Corollary 3.5 to construct E_w-optimal designs for comparing two treatment sets will be presented in this section.

Theorem 3.9. *Let $d \in \mathcal{D}(v,b,k)$ be a $GGDD(2)$ with $v_g \geq 2$ for $g = 1,2$. Then $\gamma_{d11}\gamma_{d22} = \gamma_{d12}^2$ in (2.35) is a necessary and sufficient condition for d to have weighted information matrix $C_{dw} = \epsilon(I - w^{1/2}w^{1/2'})$ for some ϵ and some w_1 and w_2, and thus be a weight balanced design.*

Proof. If $\gamma_{d11}\gamma_{d22} = \gamma_{d12}^2$, then take $\frac{w_1}{w_2} = \frac{\gamma_{d11}}{\gamma_{d12}} = \frac{\gamma_{d12}}{\gamma_{d22}}$, or equivalently, $w_1 = \frac{\gamma_{d12}}{v_1\gamma_{d12}+v_2\gamma_{d22}}$ and $w_2 = \frac{\gamma_{d22}}{v_1\gamma_{d12}+v_2\gamma_{d22}}$, and $\epsilon = -\frac{(v_1\gamma_{d12}+v_2\gamma_{d22})^2}{\gamma_{d22}}$.

The elements of C_{dw} can be checked as following:

For $i, i' \in V_1$ and $i \neq i'$,

$$\begin{aligned}
c_{dwii} &= \frac{c_{dii}}{w_1} = -\frac{(v_1-1)\gamma_{d11} + v_2\gamma_{d12}}{\gamma_{d12}/(v_1\gamma_{d12} + v_2\gamma_{d22})} \\
&= -\frac{\big(v_1\gamma_{d12} + v_2\gamma_{d22}\big)^2}{\gamma_{d22}} \frac{[(v_1-1)\gamma_{d11} + v_2\gamma_{d12}]\gamma_{d22}}{(v_1\gamma_{d12} + v_2\gamma_{d22})\gamma_{d12}} \\
&= \epsilon \frac{(v_1-1)\gamma_{d12} + v_2\gamma_{d22}}{v_1\gamma_{d12} + v_2\gamma_{d22}} = \epsilon(1-w_1),
\end{aligned}$$

$$\begin{aligned}
c_{dwii'} &= \frac{c_{dii'}}{w_1} = \frac{\gamma_{d11}}{w_1} = \frac{\gamma_{d11}(v_1\gamma_{d12} + v_2\gamma_{d22})}{\gamma_{d12}} \\
&= \frac{\big(v_1\gamma_{d12} + v_2\gamma_{d22}\big)^2}{\gamma_{d22}} \frac{\gamma_{d12}}{v_1\gamma_{d12} + v_2\gamma_{d22}} = -\epsilon w_1.
\end{aligned}$$

For $i, i' \in V_2$ and $i \neq i'$,

$$\begin{aligned}
c_{dwii} &= \frac{c_{dii}}{w_2} = -\frac{(v_2-1)\gamma_{d22} + v_1\gamma_{d12}}{w_2} \\
&= -\frac{\big(v_1\gamma_{d12} + v_2\gamma_{d22}\big)^2}{\gamma_{d22}} \frac{(v_2-1)\gamma_{d22} + v_1\gamma_{d12}}{v_1\gamma_{d12} + v_2\gamma_{d22}} = \epsilon(1-w_2), \\
c_{dwii'} &= \frac{c_{dii'}}{w_2} = \frac{\gamma_{d22}}{w_2} \\
&= \frac{\big(v_1\gamma_{d12} + v_2\gamma_{d22}\big)^2}{\gamma_{d22}} \frac{\gamma_{d22}}{v_1\gamma_{d12} + v_2\gamma_{d22}} = -\epsilon w_2.
\end{aligned}$$

For $i \in V_1$ and $i' \in V_2$,

$$\begin{aligned}
c_{dwii'} &= \gamma_{d12}\sqrt{w_1 w_2} = (v_1\gamma_{d12} + v_2\gamma_{d22})\sqrt{\frac{\gamma_{d12}}{\gamma_{d22}}} \\
&= \frac{\big(v_1\gamma_{d12} + v_2\gamma_{d22}\big)^2}{\gamma_{d22}} \frac{\sqrt{\gamma_{d12}\gamma_{d22}}}{v_1\gamma_{d12} + v_2\gamma_{d22}} = -\epsilon\sqrt{w_1 w_2}
\end{aligned}$$

So, $C_{dw} = \epsilon(I - w^{1/2}w^{1/2'})$.

Also, if d has $C_{dw} = \epsilon(I - w^{1/2}w^{1/2'})$, it follows that $C_d = \epsilon(W - ww')$. So

$$\gamma_{d11} = -\epsilon w_1^2 \qquad \gamma_{d12} = -\epsilon w_1 w_2 \qquad \gamma_{d22} = -\epsilon w_2^2,$$

that is,

$$\frac{\gamma_{d11}}{\gamma_{d12}} = \frac{\gamma_{d12}}{\gamma_{d22}} = \frac{w_1}{w_2} \Leftrightarrow \gamma_{d11}\gamma_{d22} = \gamma_{d12}^2 \qquad \square$$

Corollary 3.8. *Let $d \in \mathcal{D}(v, b, k)$ be a design which is weight balanced for weights w_1 and w_2 as in Theorem 3.9. Let $d^\star$ be a design formed as $n \geq 1$ copies of d. Then $d^\star$ is weight balanced. If d and hence $d^\star$ is binary, then $d^\star$ is E_w-optimal in $\mathcal{D}(v, nb, k)$ for the same weights w_1 and w_2.*

Proof. $\frac{\gamma_{d^\star 11}}{\gamma_{d^\star 12}} = \frac{n\gamma_{d11}}{n\gamma_{d12}} = \frac{n\gamma_{d12}}{n\gamma_{d22}} = \frac{\gamma_{d^\star 12}}{\gamma_{d^\star 22}} = \frac{w_1}{w_2}$ indicates $d^\star$ is weight balanced. By Corollary 3.5 $d^\star$ is E_w-optimal. $\square$

Corollary 3.8 allows another weight balanced E_w-optimal design to be constructed by copying an existing balanced E_w-optimal design. But if an E_w-optimal design in $\mathcal{D}$ is not weight balanced, n copies may not be E_w-optimal over $\mathcal{D}(v, nb, k)$.

According to the sufficient and necessary condition $\gamma_{d12}^2 = \gamma_{d11}\gamma_{d22}$ for d to be weight balanced, settings of the parameters for possible binary weight balanced block designs can be enumerated. Existence of designs corresponding to these parameters can be determined using the GAP DESIGN software. The parameters for all binary weight balanced block designs having $v \leq 12$, $b \leq 30$, $k \leq v$ and $v_1, v_2 \geq 2$ are listed in Table 3.1. For each set of parameters, one design is given in Appendix A.

Theorem 3.10. *Let d_0 be a complete block design in $\mathcal{D}(v_0, b_0, k_0)$ having $b_0 = (v_0+1)^2$ blocks of size $k_0 = v_0$. Let $\bar{d}$ be built up from d_0 by adding one new treatment in blocks $1, 2, \ldots, v_0 + 1$, adding a second new treatment in blocks $(v_0 + 2), \ldots, 2(v_0 + 1)$, and so on. That is, add the j^{th} new treatment in blocks $(j - 1)(v_0 + 1) + 1, \ldots, j(v_0 + 1)$ for $j = 1, 2, \ldots, v_0 + 1$. Finally, append one block containing the $v_0 + 1$ new treatments so that $\bar{d}$ is a $GGDD(2)$ with two treatment sets of sizes $v_1 = v_0$ and $v_2 = v_0 + 1$. Then $\bar{d}$ is E_w-optimal in $\mathcal{D}(v, b, k)$ for $w_1 = \frac{1}{k}$ and $w_2 = \frac{1}{k^2}$, where $v = v_1 + v_2$, $b = b_0 + 1$ and $k = k_0 + 1$.*

Table 3.1: Parameters of weight balanced, binary block designs

v	(v_1, v_2)	k	b	r_1	r_2	λ_{11}	λ_{22}	λ_{12}	$w_1 : w_2$	Design#
5	(2,3)	3	10	9	4	9	1	3	3:1	1
			19	15	9	12	3	6	2:1	2
			20	18	8	18	2	6	3:1	3
			30	27	12	27	3	9	3:1	4
6	(2,4)	3	26	21	9	18	2	6	3:1	5
		4	13	12	7	12	3	6	2:1	6
			26	24	14	24	6	12	2:1	7
			27	22	16	18	8	12	3:2	8
	(3,3)	3	11	7	4	4	1	2	2:1	9
			22	14	8	8	2	4	2:1	10
			29	22	7	16	1	4	4:1	11
7	(2,5)	3	22	18	6	16	1	4	4:1	12
		4	17	14	8	12	3	6	2:1	13
	(3,4)	3	17	5	9	1	4	2	1:2	14
			23	15	6	9	1	3	3:1	15
		4	17	16	5	16	1	4	4:1	16
			21	16	9	12	3	6	2:1	17
			23	20	8	18	2	6	3:1	18
8	(2,6)	4	30	27	11	27	3	9	3:1	19
	(3,5)	4	26	18	10	12	3	6	2:1	20
	(4,4)	4	18	13	5	9	1	3	3:1	21
			29	17	12	9	4	6	3:2	22
9	(2,7)	3	24	15	6	9	1	3	3:1	23
		4	24	20	8	18	2	6	3:1	24
	(3,6)	6	21	20	11	20	5	10	2:1	25
	(4,5)	4	26	6	16	1	9	3	1:3	26
		5	26	25	6	25	1	5	5:1	27
10	(2,8)	4	18	16	5	16	1	4	4:1	28
	(3,7)	3	25	11	6	4	1	2	2:1	29
		5	15	3	7	8	2	4	2:1	30
	(5,5)	5	20	13	7	8	2	4	2:1	31
12	(2,10)	4	19	13	5	9	1	3	3:1	32

Proof. It is easy to get

$$C_{\bar{d}} = \frac{1}{k}\begin{pmatrix} k^3 I_{v_1} - k^2 J_{v_1} & -kJ_{v_1\times v_2} \\ -kJ_{v_2\times v_1} & k^2 I_{v_2} - J_{v_2} \end{pmatrix}.$$

So, $\gamma_{\bar{d}11}\gamma_{\bar{d}22} = (-k)(-\frac{1}{k}) = 1 = \gamma^2_{\bar{d}12}$. By Theorem 3.9, $\bar{d}$ must be a binary weight balanced design for some weights w_1 and w_2. Hence, by Corollary 3.5, $\bar{d}$ is E_w-optimal over $\mathcal{D}(v,b,k)$ for this w_1 and w_2. Necessarily, by Theorem 3.9, $w_1 = \frac{\gamma_{\bar{d}12}}{v_1\gamma_{\bar{d}12}+v_2\gamma_{\bar{d}22}} = \frac{1}{k}$ and $w_2 = \frac{\gamma_{\bar{d}22}}{v_1\gamma_{\bar{d}12}+v_2\gamma_{\bar{d}22}} = \frac{1}{k^2}$. $\square$

Example 3.8. *The following design in $\mathcal{D}(7,17,4)$ is E_w-optimal for $V_1 = \{1,2,3\}$, $V_2 = \{4,5,6,7\}$, $w_1 = 1/4$ and $w_2 = 1/16$*

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	4
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	5
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	6
4	4	4	4	5	5	5	5	6	6	6	6	7	7	7	7	7

Lemma 3.4. *Let $\bar{d}$ be a binary GGDD(2) in $\mathcal{D}(v,b,k)$. Let $c_{\bar{d}1}$ be the common diagonal element of $C_{\bar{d}}$ for V_1, and let $c_{\bar{d}2}$ be the common diagonal element of $C_{\bar{d}}$ for V_2. Without loss of generality, assume $\frac{c_{\bar{d}1}}{w_1(1-w_1)} \le \frac{c_{\bar{d}2}}{w_2(1-w_2)}$. Let a be the largest integer that satisfies $v_1/v_2 > a$. Then, a sufficient condition for $\max_{d\in\mathcal{D}}\min_i \frac{c_{dii}}{w_i(1-w_i)} = \frac{c_{\bar{d}1}}{w_1(1-w_1)}$ is:*

$$\frac{c_{\bar{d}2} - (a+1)\frac{k-1}{k}}{w_2(1-w_2)} \le \frac{c_{\bar{d}1}}{w_1(1-w_1)} \le \frac{c_{\bar{d}2}}{w_2(1-w_2)}.$$

Proof. Suppose there exists another design $d^* \in \mathcal{D}$ having $\min_i \frac{c_{d^*ii}}{w_i(1-w_i)} > \frac{c_{\bar{d}1}}{w_1(1-w_1)}$ for $i = 1,\ldots,v$. Write $r_{\bar{d}i} = r_{\bar{d}1}$ for all $i \in V_1$ and $r_{\bar{d}i'} = r_{\bar{d}2}$ for all $i' \in V_2$. Then

$$\frac{c_{d^*ii}}{w_1(1-w_1)} > \frac{c_{\bar{d}1}}{w_1(1-w_1)} \qquad \text{for all } i \in V_1$$

$$\Rightarrow r_{d^*i} \ge r_{\bar{d}1} + 1 \qquad \text{for all } i \in V_1, \text{ since } \bar{d} \text{ is binary}$$

$$\Rightarrow r_{d^*i'} \leq \frac{N - (r_{\bar{d}1} + 1)v_1}{v_2} = \frac{r_{\bar{d}2}v_2 - v_1}{v_2} \qquad \text{for some } i' \in V_2$$

$$\Rightarrow r_{d^*i'} \leq r_{\bar{d}2} - (a+1)$$

$$\Rightarrow c_{d^*i'i'} \leq c_{\bar{d}2} - (a+1)\frac{k-1}{k} \qquad \text{since } \bar{d} \text{ is binary}$$

$$\Rightarrow \frac{c_{d^*i'i'}}{w_2(1-w_2)} \leq \frac{c_{\bar{d}2} - (a+1)\frac{k-1}{k}}{w_2(1-w_2)} \leq \frac{c_{\bar{d}1}}{w_1(1-w_1)},$$

a contradiction. □

Corollary 3.9. *For the binary GGDD(2) $\bar{d}$ defined in Lemma 3.4, if $\bar{d}$ satisfies $\frac{c_{\bar{d}2} - \frac{k-1}{k}}{w_2(1-w_2)} \leq \frac{c_{\bar{d}1}}{w_1(1-w_1)} \leq \frac{c_{\bar{d}2}}{w_2(1-w_2)}$, then*

$$\max_{d \in \mathcal{D}} \min_i \frac{c_{dii}}{w_i(1-w_i)} = \frac{c_{\bar{d}1}}{w_1(1-w_1)}$$

Proof. This follows from the fact that $a \geq 0$ in Lemma 3.4. □

Theorem 3.11. *Let $\hat{d}$ be constructed from $\bar{d}$ described in Theorem 3.10 by adding one full block of the treatments in V_2. Then $\hat{d}$ is E_w-optimal for $v_1 = k-1$, $v_2 = k$, $w_1 = \frac{1}{k}$ and $w_2 = \frac{1}{k^2}$ in $\mathcal{D}(v, b, k)$, where $v = v_1 + v_2$ and $b = k^2 + 2$.*

Proof. The C-matrix of $\hat{d}$ is

$$C_{\hat{d}} = C_{\bar{d}} + \tfrac{1}{k}\begin{pmatrix} 0 & 0 \\ 0 & kI - J \end{pmatrix} \Rightarrow C_{\hat{d}w} = C_{\bar{d}w} + \begin{pmatrix} 0 & 0 \\ 0 & k^2 I - kJ \end{pmatrix}$$

It can be checked $\theta_{\hat{d}1} = k^3$.

Moreover, $\frac{c_{\hat{d}ii}}{w_i(1-w_i)} = k^3$ for $i \in V_1$, $\frac{c_{\hat{d}ii}}{w_i(1-w_i)} = k^3 + \frac{k^3}{k+1}$ and $\frac{c_{\hat{d}ii} - (k-1)/k}{w_i(1-w_i)} = k^3$ for $i \in V_2$. By Corollary 3.9, $\max_{d \in \mathcal{D}} \min_i \frac{c_{dii}}{w_i(1-w_i)} = k^3 = \theta_{\hat{d}1}$. The result follows immediately upon application of Corollary 3.1. □

Example 3.9. *The following design $\hat{d} \in \mathcal{D}(7, 18, 4)$, which is constructed from the design* *described in Example 3.8, is E_w-optimal for $w_1 = 1/4$ and $w_2 = 1/16$.*

1	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*4*	*4*
2	*2*	*2*	*2*	*2*	*2*	*2*	*2*	*2*	*2*	*2*	*2*	*2*	*2*	*2*	*2*	*5*	*5*
3	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*3*	*6*	*6*
4	*4*	*4*	*4*	*5*	*5*	*5*	*5*	*6*	*6*	*6*	*6*	*7*	*7*	*7*	*7*	*7*	*7*

Theorem 3.12. *Suppose design $d^* \in \mathcal{D}(v^*, b^*, k^*)$ is a BIBD. Let the GGDD(2) design $\tilde{d} \in \mathcal{D}(v, b, k)$ be built up from d^* via the following steps:*

(1) To each block of d^*, add new treatments v^*+1, v^*+2, $\ldots$, $v^*+(v^*-k^*)$ so that the new block size is $k = v^*$ and the total number of treatments is $v = v^* + (v^* - k^*)$.

(2) Append $\hat{b} = int\left(\dfrac{b^*k^*(\frac{1}{k} - \frac{k^*}{k^2})}{k-1}\right) + 1$ blocks containing the treatments 1, 2, $\ldots$, v^* so that the total number of blocks is $b = b^* + \hat{b}$.

Now, $\tilde{d}$ consists of two sets of treatments, V_1 and V_2, where V_1 contains the treatments $1, 2, \ldots, v^*$ from d^* and V_2 contains the treatments v^*+1, v^*+2, $\ldots$, v. Then $\tilde{d}$ is E_w-optimal in $\mathcal{D}(v, b, k)$ for $w_1 = k^*/k^2$ and $w_2 = (1 - v_1 w_1)/v_2 = 1/k$.

Proof. Write $\gamma_{12} = \frac{-\lambda_{12}}{k}$ and $\gamma^*_{11} = \frac{-\lambda^*_{11}}{k}$, where λ_{12} is the concurrence between treatment sets V_1 and V_2, and λ^*_{11} is the concurrence within V_1 in d^*. The information matrix of $\tilde{d}$ can be obtained as:

$$C_{\tilde{d}} = \frac{1}{k}\begin{pmatrix} \hat{b}kI - \hat{b}J & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{k}\begin{pmatrix} \left[\dfrac{b^*k^*}{v^*}(k-1) - k\gamma^*_{11}\right]I + k\gamma^*_{11}J & k\gamma_{12}J \\ k\gamma_{12}J & b^*kI - b^*J \end{pmatrix}$$

in which

$$\gamma_{12} = \frac{-b^*k^*}{kv^*} = \frac{-b^*k^*}{k^2}$$

and

$$\gamma^*_{11} = \frac{-b^*k^*}{kv^*}\left(\frac{k^*-1}{v^*-1}\right) = \frac{-b^*k^*(k^*-1)}{k^2(k-1)}.$$

Then, utilizing the fact $v_1 = v^* = k$ and $v_2 = v^* - k^* = k - k^*$, the nonzero eigenvalues for $C_{\tilde{d}w}$ are

$$\begin{aligned}
e_{\tilde{d}w1} &= \frac{1}{k}\big[\hat{b}k + \frac{b^*k^*}{v^*}(k-1) + \frac{b^*k^*(k^*-1)}{k(k-1)}\big]/w_1 \\
&= \frac{\hat{b}k}{k^*} + \frac{b^*k(k-1)}{v^*} + \frac{b^*(k^*-1)}{k-1} \\
&= b^*k + \frac{\hat{b}k^2}{k^*} - \frac{b^*(k-k^*)}{k-1} \qquad \text{with multiplicity } v_1 - 1 \\
e_{\tilde{d}w2} &= \frac{\frac{1}{k}b^*k}{w_2} = b^*k \qquad \text{with multiplicity } v_2 - 1 \\
e_{\tilde{d}w3} &= trace(C_{\tilde{d}w}) - e_{\tilde{d}w1}(v_1 - 1) - e_{\tilde{d}w2}(v_2 - 1) \\
&= \frac{1}{k}v_1\big[\hat{b}k - \hat{b} + \frac{b^*k^*(k-1)}{v^*}\big]/w_1 + \frac{1}{k}v_2(b^*k - b^*)/w_2 \\
&\quad -\big[b^*k + \frac{\hat{b}k^2}{k^*} - \frac{b^*(k-k^*)}{k-1}\big](v_1 - 1) - b^*k(v_2 - 1) \\
&= \frac{kv_1(\hat{b}k - \hat{b})}{k^*} + \frac{kv_1}{k^*}\frac{b^*k^*(k-1)}{v^*} + (b^*k - b^*)v_2 \\
&\quad -\big[b^*k + \frac{\hat{b}k^2}{k^*} - \frac{b^*(k-k^*)}{k-1}\big](v_1 - 1) - b^*k(v_2 - 1) \\
&= b^*k \qquad \text{with multiplicity } 1
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\hat{b} = int\left(\frac{b^*k^*(\frac{1}{k} - \frac{k^*}{k^2})}{k-1}\right) + 1 \\
&\Rightarrow \hat{b} > \frac{b^*k^*(\frac{1}{k} - \frac{k^*}{k^2})}{k-1} \\
&\Rightarrow \frac{\hat{b}k^2}{k^*} - \frac{b^*(k-k^*)}{k-1} > 0
\end{aligned}$$

Thus, $\theta_{\tilde{d}1} = e_{\tilde{d}w2} = e_{\tilde{d}w3} = b^*k$. Also for $i \in V_2$, $\frac{c_{\tilde{d}ii}}{w_i(1-w_i)} = \frac{c_{\tilde{d}2}}{w_2(1-w_2)} = \frac{b^*(k-1)/k}{\frac{1}{k}(1-\frac{1}{k})} = b^*k$ cannot be bigger than $\frac{c_{\tilde{d}1}}{w_1(1-w_1)}$ by Corollary 3.1. So, $\min_i \frac{c_{\tilde{d}ii}}{w_i(1-w_i)}$ occurs for any treatment in V_2. Moreover, for $i \in V_1$

$$\frac{c_{\tilde{d}ii} - (k-1)/k}{w_1(1-w_1)}$$

$$= \frac{k^3}{k^*(k^2-k^*)}\big[\hat{b}(k-1) + \frac{b^*k^*}{v^*}(k-1)\big] - \frac{k^3(k-1)}{k^*(k^2-k^*)}$$
$$< \frac{k^3}{k^*(k^2-k^*)}\big[\big(\frac{b^*k^*(\frac{1}{k}-\frac{k^*}{k^2})}{k-1}+1\big)(k-1) + \frac{b^*k^*}{v^*}(k-1) - (k-1)\big] = b^*k$$

By Corollary 3.9, $\max_{d\in\mathcal{D}}\min_i \frac{c_{dii}}{w_i(1-w_i)} = b^*k = \theta_{\tilde{d}1}$. The proof is completed by applying Corollary 3.1. □

Example 3.10. *For $v_1 = 6$, $v_2 = 3$, $w_1 = 1/12$ and $w_2 = 1/6$, the following design is E_w-optimal over $\mathcal{D}(v, b, k) = (9, 11, 6)$:*

1	1	1	1	1	2	2	2	3	3	1
2	2	3	4	5	3	4	5	4	4	2
3	4	5	6	6	6	5	6	5	6	3
7	7	7	7	7	7	7	7	7	7	4
8	8	8	8	8	8	8	8	8	8	5
9	9	9	9	9	9	9	9	9	9	6

It can be observed that the above design is built up by adding the treatments 7, 8 and 9 to every block in the $BIBD(6, 10, 3)$, then appending one block of treatment symbols $\{1, \dots, 6\}$.

Theorem 3.13. *Suppose d^* is a $BIBD$ in $\mathcal{D}(v, b^*, k)$. Let $\tilde{d}$ be obtained from d^* by appending $\hat{b}$ copies of one block in d^*, so that $\tilde{d}$ is a $GGDD(2)$ in $\mathcal{D}(v, b, k)$, with group sizes $v_1 = k$, $v_2 = v - k$, and $b = b^* + \hat{b}$. Let a be the largest integer that satisfies $k/v > a/(a+1)$, and let λ denote the common treatment concurrence in d^*. Then $\tilde{d}$ is E_w-optimal in $\mathcal{D}(v, b, k)$ for $w_1 = \frac{\lambda+\hat{b}}{\lambda v+\hat{b}k}$ and $w_2 = \frac{\lambda}{\lambda v+\hat{b}k}$, provided*

(i) $\hat{b} > 0$ if $\lambda(v-k-1) \le (a+1)(k-1)$,

(ii) $\hat{b} \in \big[1, int(\frac{\lambda(a+1)(k-1)}{\lambda(v-k-1)-(a+1)(k-1)})\big]$ if $\lambda(v-k-1) > (a+1)(k-1)$.

Proof. The C-matrix of $\tilde{d}$ is

$$C_{\tilde{d}} = \frac{1}{k}\left(\frac{b^*k(k-1)}{v-1}I - \lambda J\right) + \frac{1}{k}\begin{pmatrix} \hat{b}kI - \hat{b}J & 0 \\ 0 & 0 \end{pmatrix}$$

So, the corresponding eigenvalues for $C_{\tilde{d}w}$ are

$$\begin{cases} e_{w1} = \dfrac{\frac{b^*k(k-1)}{v-1} + \hat{b}k}{kw_1} & \text{with multiplicity } v_1 - 1 \\ e_{w2} = \dfrac{b^*k(k-1)}{k(v-1)w_2} & \text{with multiplicity } v_2 - 1 \\ e_{w3} = \dfrac{b^*k(k-1)}{kv(v-1)w_1w_2} & \text{with multiplicity } 1 \end{cases}$$

For $w_1 = \frac{\lambda+\hat{b}}{\lambda v+\hat{b}k}$ and $w_2 = \frac{\lambda}{\lambda v+\hat{b}k}$, it is simple to see $e_{w1} = e_{w3} < e_{w2}$, thus $\theta_{\tilde{d}1} = e_{w1} = \frac{(\lambda v+\hat{b}k)^2}{k(\lambda+\hat{b})}$.

Meanwhile,

$$\frac{c_{\tilde{d}ii}}{w_i(1-w_i)} = \theta_{\tilde{d}1} \qquad \text{for } i \in V_1 \ ,$$

implying by Corollary 3.1 that $\frac{c_{\tilde{d}1}}{w_1(1-w_1)} = \min_i \frac{c_{\tilde{d}ii}}{w_i(1-w_i)}$ for $w_1 = \frac{\lambda+\hat{b}}{\lambda v+\hat{b}k}$ and $w_2 = \frac{\lambda}{\lambda v+\hat{b}k}$.

Also, since $k/v > a/(a+1)$ gives $v_1/v_2 > a$, by Lemma 3.9 if

$$\frac{c_{\tilde{d}1}}{w_1(1-w_1)} \geq \frac{c_{\tilde{d}2} - (a+1)(k-1)/k}{w_2(1-w_2)}, \tag{3.8}$$

then $\frac{c_{\tilde{d}1}}{w_1(1-w_1)} = \max_{d\in\mathcal{D}} \min_i \frac{c_{dii}}{w_i(1-w_i)}$. The two solutions to the above inequality are

(i) $\lambda(v-k-1) \leq (a+1)(k-1)$ and $\hat{b} > 0$,

(ii) $\lambda(v-k-1) > (a+1)(k-1)$ and $0 < \hat{b} \leq int\left(\frac{\lambda(a+1)(k-1)}{\lambda(v-k-1)-(a+1)(k-1)}\right)$.

By Corollary 3.1, $\tilde{d}$ is E_w-optimal over $\mathcal{D}$. □

Example 3.11. *The following two designs are built up from $BIBD(7,7,3)$. The first design adds $\hat{b} = 1$ copy of the first block in $BIBD(7,7,3)$ to obtain a $GGDD(2)$. The second one adds $\hat{b} = 2$ copies of the first block in $BIBD(7,7,3)$.*

For $w_1 = .2$, $w_2 = .1$, $v_1 = 3$ and $v_2 = 4$, this design is E_w-optimal in $\mathcal{D}(v,b,k) = (7,8,3)$:

1	1	1	2	2	3	3	1
2	4	6	4	5	4	5	2
3	5	7	6	7	7	6	3

For $w_1 = 3/13$, $w_2 = 1/13$, $v_1 = 3$ and $v_2 = 4$, this design is E_w-optimal in $\mathcal{D}(v, b, k) = (7, 9, 3)$:

1	1	1	2	2	3	3	1	1
2	4	6	4	5	4	5	2	2
3	5	7	6	7	7	6	3	3

Example 3.12. *Consider the following $BIBD(5, 10, 3)$:*

1	1	1	1	1	1	2	2	2	3
2	2	2	3	3	4	3	3	4	4
3	4	5	4	5	5	4	5	5	5

$k/v = 3/5$. Observe that $2/3 > k/v > 1/2$, so $a = 1$ in Theorem 3.13. It can be checked that $\lambda(v - k - 1) < 2(k - 1)$, so $\hat{d}$ can be any positive number. Then designs constructed by appending $\hat{b}$ copies of one block in the above BIBD are E_w-optimal in $\mathcal{D}(v, b, k) = (5, 10 + \hat{b}, 3)$ for $v_1 = 3$, $v_2 = 2$ and $w_1/w_2 = (\lambda + \hat{b})/\lambda = \frac{\hat{b}}{3} + 1$.

Lemma 3.5. *Let $\bar{d}$ be a binary GGDD(2) in $\mathcal{D}(v, b, k)$ with $r_1 > r_2$, where r_1 and r_2 are treatment replication numbers for the two sets of treatments V_1 and V_2 respectively, and $v_1, v_2 \geq 2$. Then there exist w_1 and $w_2 = \frac{1 - v_1 w_1}{v_2}$ such that $\max_{d \in \mathcal{D}} \min_i \frac{c_{dii}}{w_i(1-w_i)} = \frac{c_{\bar{d}1}}{w_1(1-w_1)}$.*

Proof. Again, write $c_{\bar{d}1}$ as the common diagonal element of $C_{\bar{d}}$ for V_1, and let $c_{\bar{d}2}$ be the common diagonal element of $C_{\bar{d}}$ for V_2. Corollary 3.9 shows a sufficient condition for $\max_{d \in \mathcal{D}} \min_i \frac{c_{dii}}{w_i(1-w_i)} = \frac{c_{\bar{d}1}}{w_1(1-w_1)}$ is:

$$\frac{c_{\bar{d}2} - \frac{k-1}{k}}{w_2(1 - w_2)} \leq \frac{c_{\bar{d}1}}{w_1(1 - w_1)} \leq \frac{c_{\bar{d}2}}{w_2(1 - w_2)}, \tag{3.9}$$

or equivalently,

$$\frac{(r_2 - 1)(k - 1)}{k w_2(1 - w_2)} \leq \frac{r_1(k - 1)}{k w_1(1 - w_1)} \leq \frac{r_2(k - 1)}{k w_2(1 - w_2)}. \tag{3.10}$$

Denote $\frac{w_2(1-w_2)}{w_1(1-w_1)} = T$. The above inequalities are equivalent to $\frac{r_2-1}{r_1} \leq T \leq \frac{r_2}{r_1}$. The proof is completed by solving for w_1 and w_2.

i) $\quad \dfrac{r_2-1}{r_1} \leq T$

$$\Leftrightarrow (r_2-1)w_1(1-w_1) \leq \frac{1-v_1w_1}{v_2}\left(1-\frac{1-v_1w_1}{v_2}\right)r_1$$

$$\Leftrightarrow S_1(w_1) \equiv [r_1v_1^2-(r_2-1)v_2^2]w_1^2+[(r_2-1)v_2^2+r_1v_1v_2-2r_1v_1]w_1-r_1v_2+r_1 \leq 0$$

Now $S_1(w_1=0) = -r_1(v_2-1) < 0$ and $S_1(w_1=1) = r_1(v_1-1)(v-1) > 0$. Note that S_1 is a quadratic form of w_1, thus by the intermediate value theorem, there exists some $w_1^* \in (0,1)$ such that $S_1 < 0$ for all $0 < w_1 < w_1^*$ and $S_1(w_1^*) = 0$.

ii) $\quad T \leq \dfrac{r_2}{r_1}$

$$\Leftrightarrow S_2(w_2) \equiv [r_1v_1^2-r_2v_2^2]w_1^2+[r_2v_2^2+r_1v_1v_2-2r_1v_1]w_1-r_1v_2+r_1 \geq 0$$

Now $S_2(w_1=0) = -r_1(v_2-1) < 0$ and $S_2(w_1=1) = r_1(v_1-1)(v-1) > 0$. So, there exists $w_1^{**} \in (0,1)$ such that $S_2 > 0$ for all $w_1^{**} < w_1 < 1$ and $S_2(w_1^{**}) = 0$.

Moreover, $S_2(w_1) - S_1(w_1) = v_2^2w_1(1-w_1) > 0)$, indicating $w_1^{**} < w_1^*$. So, $w_1^{**} \leq w_1 \leq w_1^*$ solves $\frac{r_2-1}{r_1} \leq \frac{w_2(1-w_2)}{w_1(1-w_1)} \leq \frac{r_2}{r_1}$. $\square$

Theorem 3.14. *Let $\bar{d}$ be a binary GGDD(2) in $\mathcal{D}(v,b,k)$, and let r_1, r_2 be the replication numbers for treatments in sets V_1 and V_2, respectively, with $r_1 > r_2$. As usual, let λ_{12} be the concurrence between treatment sets V_1 and V_2, λ_{11} be the concurrence within V_1 and λ_{22} be the concurrence within V_2. Then the weighted information matrix for $\bar{d}$ is*

$$C_{\bar{d}w} = \frac{1}{k}\begin{pmatrix} \frac{[r_1(k-1)+\lambda_{11}]I-\lambda_{11}J}{w_1} & -\frac{\lambda_{12}}{\sqrt{w_1w_2}}J \\ -\frac{\lambda_{12}}{\sqrt{w_1w_2}}J & \frac{[r_2(k-1)+\lambda_{22}]I-\lambda_{22}J}{w_2} \end{pmatrix}.$$

Let $w_1 = [r_1(k-1)-v_2\lambda_{12}]/[v_1r_1(k-1)-v_2\lambda_{12}]$. Let $T1 = (r_2-1)(k-1)[r_1(k-1)-v_2\lambda_{12}]$, $T2 = r_2(k-1)[r_1(k-1)-v_2\lambda_{12}]$ and $T = \lambda_{12}[v_1r_1(k-1)-v_2\lambda_{12}-\lambda_{12}(v_1-1)]$. Then if $T1 \leq T \leq T2$, $\bar{d}$ is E_w-optimal over $\mathcal{D}(v,b,k)$.

Proof. Note if $w_1 = [r_1(k-1) - v_2\lambda_{12}]/[v_1 r_1(k-1) - v_2\lambda_{12}]$, then $w_2 = [\lambda_{12}(v_1-1)]/[v_1 r_1(k-1) - v_2\lambda_{12}]$. It is simple to check

$$T1 \leq T \leq T2 \Leftrightarrow \frac{r_2 - 1}{r_1} \leq \frac{w_2(1-w_2)}{w_1(1-w_1)} \leq \frac{r_2}{r_1}.$$

Since $C_{\bar{d}}1 = 0$, the parameters of $\bar{d}$ satisfy

$$r_1(k-1) = \lambda_{11}(v_1 - 1) + v_2\lambda_{12}$$

and

$$r_2(k-1) = \lambda_{22}(v_2 - 1) + \lambda_{12}v_1$$

Thus, the three non-zero eigenvalues of $C_{\bar{d}w}$ are:

$$\begin{cases} e_{\bar{d}w1} = \dfrac{r_1(k-1) + \lambda_{11}}{kw_1} = \dfrac{v_1 r_1(k-1) - v_2\lambda_{12}}{kw_1(v_1 - 1)} & \text{with multiplicity } v_1 - 1 \\ e_{\bar{d}w2} = \dfrac{r_2(k-1) + \lambda_{22}}{kw_2} = \dfrac{v_2 r_2(k-1) - v_1\lambda_{12}}{kw_2(v_2 - 1)} & \text{with multiplicity } v_2 - 1 \\ e_{\bar{d}w3} = \dfrac{\lambda_{12}}{kw_1 w_2} & \text{with multiplicity } 1 \end{cases}$$

Moreover, $w_1 = \frac{r_1(k-1) - v_2\lambda_{12}}{v_1 r_1(k-1) - v_2\lambda_{12}}$ is equivalent to $\lambda_{12} = \frac{r_1 w_2(k-1)}{1-w_1}$. So, $\theta_{\bar{d}1} = e_{\bar{d}w1} = e_{\bar{d}w3}$.

Now compare $e_{\bar{d}w2}$ with $e_{\bar{d}w3}$,

$$\begin{aligned} e_{\bar{d}w2} - e_{\bar{d}w3} &= \frac{v_2 r_2 w_1(k-1) - v_1 w_1 \lambda_{12} + \lambda_{12} - v_2\lambda_{12}}{kw_1 w_2(v_2 - 1)} \\ &= \frac{v_2}{kw_1 w_2(v_2 - 1)}\big[r_2 w_1(k-1) - \lambda_{12}(1 - w_2)\big] > 0 \end{aligned}$$

since the condition $\frac{w_2(1-w_2)}{w_1(1-w_1)} < \frac{r_2}{r_1}$ gives $\frac{r_2 w_1(k-1)}{1-w_2} > \frac{r_1 w_2(k-1)}{1-w_1} = \lambda_{12}$.

Again, denote $c_{\bar{d}1}$ as the common diagonal element of $C_{\bar{d}}$ for V_1, and denote $c_{\bar{d}2}$ as the common diagonal element of $C_{\bar{d}}$ for V_2. So now,

$$\theta_{\bar{d}1} = e_{\bar{d}w1} = e_{\bar{d}w3} = \frac{\lambda_{12}}{kw_1 w_2} = \frac{r_1(k-1)}{kw_1(1-w_1)} = \frac{c_{\bar{d}1}}{w_1(1-w_1)}.$$

Furthermore,

$$\frac{r_2 - 1}{r_1} < \frac{w_2(1-w_2)}{w_1(1-w_1)} < \frac{r_2}{r_1} \Rightarrow \max_{d \in \mathcal{D}} \min_i \frac{c_{dii}}{w_i(1-w_i)} = \frac{c_{\bar{d}1}}{w_1(1-w_1)} = \theta_{\bar{d}1},$$

which follows immediately from Lemma 3.5. The proof will be completed by applying Corollary 3.1. $\square$

If let $V_1 = \{7, 8, 9\}$ and $V_2 = \{1, 2, 3, 4, 5, 6\}$, the design in Example 3.10 is E_w-optimal for $w_1 = 1/6$ and $w_2 = 1/12$, since it can be checked $T1 = 500$, $T2 = 600$ and $T = 550$, and thus satisfies $T1 \leq T \leq T2$.

3.4 E_w-optimal Designs with Blocks of Size $k = v$

The most popular of all block designs are the randomized complete block designs, or $RCBD$s. These are designs with blocks of size $k = v$ for which each treatment is assigned to one experimental unit in each block. $RCBD$s are known to be universally optimal over $\mathcal{D}(v, b, k = v)$ in the unweighted case (Kiefer 1975). This section explores the E_w-behavior of $RCBD$s and their competitors for the 2-weight problem.

Lemma 3.6. *For the 2-weight problem with $v_1 = 1$, the two eigenvalues of the C_w-matrix for an RCBD are $\frac{b}{w_2}$ with multiplicity $v - 2$ and $\frac{b(v-1)}{vw_1(1-w_1)}$ with multiplicity 1. Furthermore, the minimum of the eigenvalues of C_w for an RCBD is*

$$\theta_{RCBD,1} = \begin{cases} \dfrac{b}{w_2} & \text{for } w_1 \leq \dfrac{1}{v} \\ \dfrac{b(v-1)}{vw_1(1-w_1)} & \text{for } w_1 \geq \dfrac{1}{v}. \end{cases}$$

Let $r_1, r_2, \ldots, r_v$ be the replication numbers for treatments 1, 2, ..., v, respectively. For the 2-weight problem with $v_1 = 1$, without loss of generality, assume $r_2 \leq r_3 \leq \cdots \leq r_v$ throughout this section.

Lemma 3.7. *Consider the 2-weight problem with $v_1 = 1$, $k = v$ and $b \leq v - 2$. Let d be an arbitrary design in $\mathcal{D}(v, b, k)$ having $r_{d1} \leq b$. Then, d cannot be E_w-better than an RCBD.*

Proof. It is simple to observe $(r_{d2} + r_{d3}) \leq 2b$. By Corollary 3.2, the smallest eigenvalue θ_{d1} of C_{dw} satisfies

$$\begin{aligned}
\theta_{d1} &\leq \frac{1}{2w_2}(c_{d22} + c_{d33} - 2c_{d23}) \\
&= \frac{1}{2w_2}\left(\sum_{j=1}^{b} n_{d2j} - \frac{\sum_{j=1}^{b} n^2{}_{d2j}}{v} + \sum_{j=1}^{b} n_{d3j} - \frac{\sum_{j=1}^{b} n^2{}_{d3j}}{v} + \frac{2\sum_{j=1}^{b} n_{d2j}n_{d3j}}{v}\right) \\
&= \frac{1}{2w_2}\left[\left(\sum_{j=1}^{b} n_{d2j} + \sum_{j=1}^{b} n_{d3j}\right) - \left(\frac{\sum_{j=1}^{b} n^2{}_{d2j}}{v} + \frac{\sum_{j=1}^{b} n^2{}_{d3j}}{v} - \frac{2\sum_{j=1}^{b} n_{d2j}n_{d3j}}{v}\right)\right] \\
&= \frac{1}{2w_2}\left(r_{d2} + r_{d3} - \frac{1}{v}\sum_{j=1}^{b}(n_{d2j} - n_{d3j})^2\right) \\
&\leq \frac{1}{2w_2}(r_{d2} + r_{d3}) \\
&\leq \frac{b}{w_2}
\end{aligned}$$

If $w_1 \leq \frac{1}{v}$, by Lemma 3.6, we have

$$\theta_{d1} \leq \frac{b}{w_2} = \theta_{RCBD,1}.$$

If $w_1 \geq \frac{1}{v}$, by Corollary 3.1,

$$\theta_{d1} \leq \frac{c_{d11}}{w_1(1-w_1)} \leq \frac{b(v-1)}{vw_1(1-w_1)} = \theta_{RCBD,1}.$$

□

Theorem 3.15. *Consider the 2-weight problem with $v_1 = 1$, $k = v$ and $b \leq v - 2$. Then, an RCBD is E_w-optimal over $\mathcal{D}(v, b, k)$ for $w_1 \leq \frac{1}{v}$.*

Proof. All competitors having $r_1 \leq b$ have been ruled out by Lemma 3.7, so it is sufficient to show that $d \in \mathcal{D}$ having $r_{d1} \geq b+1$ is E_w-inferior to an $RCBD$ for $w_1 \leq \frac{1}{v}$. Since $r_{d1} \geq b+1$,

$r_{d2} \leq b-1$. For $w_1 \leq \frac{1}{v}$, by Corollary 3.1,

$$\theta_{d1} \leq \frac{c_{d22}}{w_2(1-w_2)} \leq \frac{(b-1)(v-1)}{vw_2(1-w_2)} = \frac{b(1-\frac{1}{b})(v-1)}{vw_2(1-w_2)} \leq \Big(\frac{b}{w_2}\Big)\Big(\frac{(1-\frac{1}{b})(v-1)}{v(1-\frac{1}{v-1})}\Big)$$
$$\leq \Big(\frac{b}{w_2}\Big)\Big(\frac{(1-\frac{1}{v-2})(v-1)}{v(1-\frac{1}{v-1})}\Big) < \frac{b}{w_2} = \theta_{RCBD,1}.$$

□

Lemma 3.8. *Consider the 2-weight problem with $v_1 = 1$, $k = v$ and $b = v-1$. Let $d \in \mathcal{D}(v,b,k)$ be an arbitrary design having $2 \leq r_{d1} \leq v-1$. Then, d cannot be E_w-better than an RCBD.*

Proof. Since $r_{d1} \geq 2$, it is easy to see $r_{d2} + r_{d3} \leq 2b$. Using reasoning similar to that in Lemma 3.7, for $w_1 \leq \frac{1}{v}$,

$$\theta_{d1} \leq \frac{1}{2w_2}(c_{d22} + c_{d33} - 2c_{d23}) \leq \frac{1}{2w_2}(r_{d2} + r_{d3}) \leq \frac{b}{w_2} = \theta_{RCBD,1}.$$

If $w_1 \geq \frac{1}{v}$, by Corollary 3.1,

$$\theta_{d1} \leq \frac{c_{d11}}{w_1(1-w_1)} \leq \frac{r_{d1}(k-1)}{kw_1(1-w_1)} \leq \frac{(v-1)^2}{vw_1(1-w_1)} = \theta_{RCBD,1}.$$

□

Lemma 3.9. *Consider the 2-weight problem with $v_1 = 1$, $k = v$ and $b = v-1$. Let $\hat{d} \in \mathcal{D}(v,b,k)$ be a nonbinary GGDD(3) constructed from an RCBD by replacing treatment 1 with treatment $i+1$ in the i^{th} block for $i = 2,3,\ldots,v-1$. Let $\mathcal{D}_1 \subset \mathcal{D}(v,b,k)$ be the subclass in which all designs have $r_1 = 1$. Then $\hat{d}$ is E_w-optimal over $\mathcal{D}_1$ for any weight.*

Proof. Note all designs in $\mathcal{D}$ must have $r_2 \leq v-1$. Let $\mathcal{D}_2 \subset \mathcal{D}(v,b,k)$ be the subclass having $r_1 = 1$, $r_2 = b = v-1$ and $n_{2j} = 1$ for $j = 1,2,\ldots,b$, and thus $c_{22} = (v-1)^2/v$.

Obviously, $\hat{d} \in \mathcal{D}_2 \subset \mathcal{D}_1$. Also, $\hat{d}$ maximizes the trace of C_d over $d \in \mathcal{D}_1$. The C-matrix of $\hat{d}$ is

$$C_{\hat{d}} = \begin{pmatrix} \frac{v-1}{v} & -\frac{1}{v} & -\frac{1}{v}1'_{v-2} \\ -\frac{1}{v} & \frac{(v-1)^2}{v} & -1'_{v-2} \\ -\frac{1}{v}1_{v-2} & -1_{v-2} & \frac{(v^2-1)}{v}I_{v-2} - \frac{(v+1)}{v}J_{v-2} \end{pmatrix}.$$

It can be seen $C_{\hat{d}}$ is the average of all permuted C-matrices obtained by permuting over treatments $3, 4, \ldots, v$ for an arbitrary design in $\mathcal{D}_2$ with the same trace as $C_{\hat{d}}$. By convexity, $\hat{d}$ is E_w-best among designs having the same trace in $\mathcal{D}_2$. Now consider competitors $d \in \mathcal{D}_2$ having smaller trace. Averaging over treatments $3, 4, \ldots, v$ produces the $GGDD(3)$ information matrix $\bar{C}_d$ having smaller trace than $C_{\hat{d}}$:

$$\bar{C}_d = \begin{pmatrix} \frac{v-1}{v} & -\frac{1}{v} & -\frac{1}{v}1'_{v-2} \\ -\frac{1}{v} & \frac{(v-1)^2}{v} & -1'_{v-2} \\ -\frac{1}{v}1_{v-2} & -1_{v-2} & \frac{c_{\bar{d}33}(v-2)-\frac{v+1}{v}}{v-3}I_{v-2} - \frac{c_{\bar{d}33}-\frac{v+1}{v}}{v-3}J_{v-2} \end{pmatrix}. \tag{3.11}$$

where by assumption, $c_{\bar{d}33} < c_{\hat{d}33}$. It follows that $trace(\bar{C}_{dw}) < trace(C_{\hat{d}w})$.

By Morgan and Parvu (2007), two nonzero eigenvalues of both $\bar{C}_{dw}$ and $C_{\hat{d}w}$ are the two eigenvalues of the following 3×3 reduced matrix,

$$\begin{pmatrix} \frac{v-1}{vw_1} & \frac{-1}{v\sqrt{w_1w_2}} & -\frac{v-2}{v\sqrt{w_1w_2}} \\ \frac{-1}{\sqrt{vw_1w_2}} & \frac{(v-1)^2}{vw_2} & -\frac{v-2}{w_2} \\ -\frac{1}{v\sqrt{w_1w_2}} & -\frac{1}{w_2} & \frac{v+1}{vw_2} \end{pmatrix}. \tag{3.12}$$

The remaining positive eigenvalue of $\bar{C}_{dw}$ has multiplicity $v-3$, as does that of $C_{\hat{d}w}$. Since $trace(\bar{C}_{dw}) < trace(C_{\hat{d}w})$, that eigenvalue for $C_{\hat{d}w}$ is larger than that of $\bar{C}_{dw}$. Consequently, d cannot be E_w-better than $\hat{d}$.

It remains to compare $\hat{d}$ with those designs $d \in \mathcal{D}_1/\mathcal{D}_2$ having $c_{d22} < c_{\hat{d}22} = (v-1)^2/v$. Specifically, $c_{d22} \leq \frac{v^2-2v-1}{v}$.

The positive eigenvalues of $C_{\hat{d}w}$ are

$$\begin{cases} \dfrac{v^2-1}{vw_2} & \text{with multiplicity } v-3 \\ \dfrac{1}{vw_1w_2} & \text{with multiplicity } 1 \\ \dfrac{v^2-v+1}{vw_2} & \text{with multiplicity } 1. \end{cases}$$

The minimum eigenvalue of $C_{\hat{d}w}$ follows as

$$\theta_{\hat{d}1} = \begin{cases} \dfrac{v^2-v+1}{vw_2} & \text{for } w_1 \leq \dfrac{1}{v^2-v+1} \\ \dfrac{1}{vw_1w_2} & \text{for } w_1 \geq \dfrac{1}{v^2-v+1}. \end{cases}$$

If $w_1 \leq \frac{1}{v^2-v+1}$, by Corollary 3.1,

$$\begin{aligned} \theta_{\hat{d}1} - \theta_{d1} &\geq \frac{v^2-v+1}{vw_2} - \frac{c_{d22}}{w_2(1-w_2)} \geq \frac{(v^2-v+1)}{vw_2} - \frac{(v^2-2v-1)}{vw_2(1-w_2)} \\ &= \frac{(v^2-v+1)(1-w_2)-(v^2-2v-1)}{vw_2(1-w_2)} \\ &\geq \frac{(v^2-v+1)(1-\frac{1}{v-1})-(v^2-2v-1)}{vw_2(1-w_2)} \qquad \text{since } w_2 \leq \frac{1}{v-1} \\ &= \frac{2-\frac{1}{v-1}}{vw_2(1-w_2)} \\ &> 0. \end{aligned}$$

If $w_1 \geq \frac{1}{v^2-v+1}$, again by Corollary 3.1,

$$\theta_{\hat{d}1} = \frac{1}{vw_1w_2} = \frac{v-1}{vw_1(1-w_1)} = \frac{c_{d11}}{w_1(1-w_1)} \geq \theta_{d1}.$$

□

Theorem 3.16. *For the 2-weight problem with $v_1 = 1$, $k = v$ and $b = v-1$, the GGDD(3) $\hat{d} \in \mathcal{D}(v,b,k)$ defined in Lemma 3.9 is E_w-optimal over $\mathcal{D}$ for $w_1 \leq \frac{1}{v^2-v}$.*

Proof. Lemma 3.9 has shown $\hat{d}$ is E_w-optimal over all designs in $\mathcal{D}$ having $r_1 = 1$. Lemma 3.8 has shown no design having $2 \leq r_1 \leq v-1$ is E_w-better than an $RCBD$. Moreover, the proof of Lemma 3.8 shows no design having $r_1 \geq 2$ is E_w-better than an $RCBD$ for $w_1 \leq \frac{1}{v}$ and thus for $w_1 \leq \frac{1}{v^2-v}$. So, it is sufficient to show $\hat{d}$ is E_w-better than $RCBD$ for $w_1 \leq \frac{1}{v^2-v}$. Using results in the proof of Lemma 3.9, when $w_1 \leq \frac{1}{v^2-v+1}$,

$$\theta_{\hat{d}1} = \frac{v^2-v+1}{vw_2} > \frac{v-1}{w_2} = \theta_{RCBD,1}.$$

If $\frac{1}{v^2-v+1} \leq w_1 \leq \frac{1}{v^2-v}$,

$$\theta_{\hat{d}1} = \frac{1}{vw_1w_2} \geq \frac{1}{v(\frac{1}{v^2-v})w_2} = \frac{v-1}{w_2} = \theta_{RCBD,1}.$$

□

Example 3.13. *Consider this design in $\mathcal{D}(v,b,k)$ with $k=v=5$ and $b=4$:*

1	2	2	2
2	3	3	3
3	3	4	4
4	4	4	5
5	5	5	5

This design is E_w-optimal in $\mathcal{D}(5,4,5)$ for $w_1 \leq \frac{1}{v^2-v} = \frac{1}{20}$, or equivalently, $\frac{w_1}{w_2} \leq \frac{v-1}{v^2-v-1} = \frac{4}{19}$. For $\frac{1}{20} \leq w_1 \leq \frac{1}{5}$, or equivalently, $\frac{4}{19} \leq \frac{w_1}{w_2} \leq 1$, an RCBD is E_w-optimal as shown next.

Theorem 3.17. *For the 2-weight problem with $v_1 = 1$, $k=v$ and $b=v-1$, an RCBD is E_w-optimal over $\mathcal{D}(v,b,k)$ for $\frac{1}{v^2-v} \leq w_1 \leq \frac{1}{v}$.*

Proof. Lemma 3.8 has eliminated all competitors in $\mathcal{D}$ having $2 \leq r_1 \leq v-1$. Lemma 3.9 has shown $GGDD(3)$ $\hat{d}$ having $r_1 = 1$ is E_w-best among all designs having $r_1 = 1$. So first compare an $RCBD$ with this $\hat{d}$. Again, we use results in the proof of Lemma 3.9 along with Corollary 3.1:

$$\theta_{\hat{d}1} = \frac{1}{vw_1w_2} \leq \frac{1}{v(\frac{1}{v^2-v})w_2} = \frac{v-1}{w_2} = \theta_{RCBD,1}.$$

It remains to compare an $RCBD$ with an arbitrary competitor $d \in \mathcal{D}$ having $r_{d1} \geq v = b+1$. Since $r_{d1} \geq b+1$, $r_{d2} \leq b-1$ and

$$\begin{aligned} \theta_{d1} \leq \frac{c_{d22}}{w_2(1-w_2)} &\leq \frac{(b-1)(v-1)}{vw_2(1-w_2)} \leq \Big(\frac{b}{w_2}\Big)\Big(\frac{(1-\frac{1}{b})(v-1)}{v(1-\frac{1}{v-1})}\Big) \\ &= \Big(\frac{b}{w_2}\Big)\Big(\frac{(1-\frac{1}{v-1})(v-1)}{v(1-\frac{1}{v-1})}\Big) < \frac{b}{w_2} = \theta_{RCBD,1} \end{aligned}$$

□

Lemma 3.10. *Consider the 2-weight problem with $v_1 = 1$, $k = v$ and $b = v$. let d be an arbitrary design in $\mathcal{D}(v, b, k)$ having $3 \leq r_{d1} \leq v$. Then, d cannot be E_w-better than an RCBD.*

Proof. Since $r_{d1} \geq 3$, it follows $r_{d2} + r_{d3} + \cdots + r_{dv} \leq v^2 - 3$. So, it is simple to observe $r_{d2} + r_{d3} \leq 2v$. Corollary 3.2 gives

$$\theta_{d1} \leq \frac{1}{2w_2}(c_{d22} + c_{d33} - 2c_{d23}).$$

If $w_1 \leq \frac{1}{v}$, using the proof in Lemma 3.7, we get

$$\theta_{d1} \leq \frac{1}{2w_2}(r_{d2} + r_{d3}) \leq \frac{v}{w_2} = \theta_{RCBD,1}. \tag{3.13}$$

If $w_1 \geq \frac{1}{v}$, by Corollary 3.1,

$$\theta_{d1} \leq \frac{c_{d11}}{w_1(1-w_1)} \leq \frac{r_{d1}(k-1)}{kw_1(1-w_1)} \leq \frac{v-1}{w_1(1-w_1)} = \theta_{RCBD,1}.$$

□

Theorem 3.18. *Consider the 2-weight problem with $v_1 = 1$, $k = v$ and $b = v$. Let the nonbinary GGDD(3) $\hat{d} \in \mathcal{D}(v, b, k)$ be constructed from an RCBD by replacing treatment 1 with treatment i in the i^{th} block for $i = 3, 4, \ldots, v$. Then, $\hat{d}$ is E_w-optimal in $\mathcal{D}(v, b, k)$ for $\frac{1}{v^2+1} \leq w_1 \leq \frac{2}{v^2}$.*

Proof. Since $r_{\hat{d}1} = 2$, $r_{\hat{d}2} = v$, $r_{\hat{d}3} = \cdots = r_{\hat{d}v} = v + 1$, the C-matrix for $\hat{d}$ is

$$C_{\hat{d}} = \frac{1}{v} \begin{pmatrix} 2(v-1) & -2 & -21'_{v-2} \\ -2 & v^2 - v & -(v+1)1'_{v-2} \\ -21_{v-2} & -(v+1)1_{v-2} & (v^2 + v - 1)I_{v-2} - (v+2)J_{v-2} \end{pmatrix}. \tag{3.14}$$

The $v - 1$ non-zero eigenvalues of $C_{\hat{d}w}$ are

$$\begin{cases} \dfrac{(v-1)(v^2+v-1)}{v(1-w_1)} & \text{with multiplicity } v - 3 \\ \dfrac{(v-1)(v^2+1)}{v(1-w_1)} & \text{with multiplicity } 1 \\ \dfrac{2(v-1)}{vw_1(1-w_1)} & \text{with multiplicity } 1. \end{cases} \tag{3.15}$$

So, the minimum eigenvalue of $C_{\hat{d}w}$ is

$$\theta_{\hat{d}1} = \begin{cases} \dfrac{(v-1)(v^2+1)}{v(1-w_1)} & \text{for } 0 < w_1 \leq \dfrac{2}{v^2+1} \\ \dfrac{2(v-1)}{vw_1(1-w_1)} & \text{for } w_1 \geq \dfrac{2}{v^2+1}. \end{cases} \tag{3.16}$$

For any design in $\mathcal{D}$ having $r_1 = 2$, it is simple to see $r_2 \leq v$. Optimality of $\hat{d}$ will be established, by comparing $\hat{d}$ to competitors in these six disjoint subclasses of $\mathcal{D}$ for $\frac{1}{v^2+1} \leq w_1 \leq \frac{2}{v^2}$:

(i) designs having $r_1 = 1$

(ii) designs having $3 \leq r_1 \leq v$

(iii) designs having $r_1 \geq v+1$

(iv) designs having $r_1 = 2$, $r_2 \leq v-1$

(v) designs having $r_1 = 2$, $r_2 = v$ and $c_{22} < v-1$

(vi) designs other than $\hat{d}$ having $r_1 = 2$, $r_2 = v$ and $c_{22} = v-1$

Case (i). Compare $\hat{d}$ with designs having $r_1 = 1$.

Let d be an arbitrary design in (i). By Corollary 3.1, $\theta_{d1} \leq \frac{c_{d11}}{w_1(1-w_1)} = \frac{v-1}{vw_1(1-w_1)}$. So, if $\frac{1}{v^2+1} \leq w_1 \leq \frac{2}{v^2+1}$,

$$\theta_{\hat{d}1} = \frac{(v-1)(v^2+1)}{v(1-w_1)} = \frac{v-1}{v(\frac{1}{v^2+1})(1-w_1)} \geq \frac{v-1}{vw_1(1-w_1)} \geq \theta_{d1}.$$

If $\frac{2}{v^2+1} \leq w_1 \leq \frac{2}{v^2}$,

$$\theta_{\hat{d}1} = \frac{2(v-1)}{vw_1(1-w_1)} > \frac{v-1}{vw_1(1-w_1)} \geq \theta_{d1}.$$

Case (ii). Compare $\hat{d}$ with designs having $3 \leq r_1 \leq v$.

Let d be an arbitrary design in (ii). Recall Lemma 3.10 has shown no design with $3 \leq r_1 \leq v$ can be E_w-better than an $RCBD$. Thus $\hat{d}$ need only to be compared to an $RCBD$.

If $\frac{1}{v^2+1} \leq w_1 \leq \frac{2}{v^2+1}$,

$$\theta_{\hat{d}1} = \frac{(v-1)(v^2+1)}{v(1-w_1)} = \frac{(v-1)(v+\frac{1}{v})}{1-w_1} > \frac{v(v-1)}{1-w_1} = \frac{v}{w_2} = \theta_{RCBD,1}.$$

If $\frac{2}{v^2+1} \leq w_1 \leq \frac{2}{v^2}$,

$$\theta_{\hat{d}1} = \frac{2(v-1)}{vw_1(1-w_1)} = \frac{2}{vw_1w_2} \geq \frac{2}{v(\frac{2}{v^2})w_2} = \frac{v}{w_2} = \theta_{RCBD,1}.$$

Case (iii). Compare $\hat{d}$ with designs having $r_1 \geq v+1$.

Let d be an arbitrary design in (iii). It can be seen that $r_{d2} \leq v-1$, so $c_{d22} \leq \frac{r_{d2}(k-1)}{k} = \frac{(v-1)^2}{v}$.

If $\frac{1}{v^2+1} \leq w_1 \leq \frac{2}{v^2+1}$,

$$\begin{aligned}
\theta_{\hat{d}1} &= \frac{(v-1)(v^2+1)}{v(1-w_1)} = \frac{(v^2+1)(1-w_2)}{vw_2(1-w_2)} = \frac{(v^2+1)\left(1-\frac{1-w_1}{v-1}\right)}{vw_2(1-w_2)} \\
&\geq \frac{(v^2+1)\left(\frac{v-2+\frac{1}{v^2+1}}{v-1}\right)}{vw_2(1-w_2)} = \frac{v^2+1-\left(v+1+\frac{1}{v-1}\right)}{vw_2(1-w_2)} \qquad (3.17) \\
&> \frac{v^2+1-2v}{vw_2(1-w_2)} \geq \theta_{d1} \qquad \text{by Corollary 3.1}
\end{aligned}$$

If $\frac{2}{v^2+1} \leq w_1 \leq \frac{2}{v^2}$,

$$\begin{aligned}
\theta_{\hat{d}1} &= \frac{2(v-1)}{vw_1(1-w_1)} = \frac{2(1-w_2)}{vw_1w_2(1-w_2)} = \frac{\frac{2}{w_1}\left(\frac{v-2+w_1}{v-1}\right)}{vw_2(1-w_2)} \\
&= \frac{\frac{2(v-2)}{w_1(v-1)}+\frac{2}{v-1}}{vw_2(1-w_2)} \geq \frac{\frac{2(v-2)}{(v-1)\left(\frac{2}{v^2}\right)}+\frac{2}{v-1}}{vw_2(1-w_2)} = \frac{v^2\left(\frac{v-2}{v-1}\right)+\frac{2}{v-1}}{vw_2(1-w_2)} = \frac{v^2-\frac{v^2-2}{v-1}}{vw_2(1-w_2)} \qquad (3.18) \\
&= \frac{(v-1)^2+\frac{v^2-3v+3}{v-1}}{vw_2(1-w_2)} = \frac{(v-1)^2+\frac{\left(v-\frac{3}{2}\right)^2+\frac{3}{4}}{v-1}}{vw_2(1-w_2)} > \frac{(v-1)^2}{vw_2(1-w_2)} \\
&\geq \theta_{d1} \qquad \text{by Corollary 3.1}
\end{aligned}$$

Case (iv). Compare $\hat{d}$ with designs having $r_1 = 2$ and $r_2 \leq v-1$.

Let d be an arbitrary design in (iv). The proof is the same as in (iii), as $c_{d22} \leq \frac{(v-1)^2}{v}$.

Case (v). Compare $\hat{d}$ with designs having $r_1 = 2$, $r_2 = v$ and $c_{22} < v-1$.

Let d be an arbitrary design in (v). Observe that $r_{d2} = v$ and $c_{d22} < v-1$ implies treatment 2 appears more than once in at least one block, so $c_{d22} \leq \frac{v^2-v-2}{v}$.

If $\frac{1}{v^2+1} \leq w_1 \leq \frac{2}{v^2+1}$, using (3.17) and Corollary 3.1,

$$\theta_{\hat{d}1} = \frac{(v-1)(v^2+1)}{v(1-w_1)} \geq \frac{v^2+1-\left(v+1+\frac{1}{v-1}\right)}{vw_2(1-w_2)} = \frac{v^2-v-\frac{1}{v-1}}{vw_2(1-w_2)} \geq \frac{v^2-v-2}{vw_2(1-w_2)} \geq \theta_{d1}.$$

If $\frac{2}{v^2+1} \leq w_1 \leq \frac{2}{v^2}$, using (3.18) and Corollary 3.1,

$$\begin{aligned}\theta_{\hat{d}1} &= \frac{2(v-1)}{vw_1(1-w_1)} \geq \frac{v^2 - \frac{v^2-2}{v-1}}{vw_2(1-w_2)} = \frac{(v^2-v-2) + \left(v+2-\frac{v^2-2}{v-1}\right)}{vw_2(1-w_2)} \\ &= \frac{(v^2-v-2)+\frac{v}{v-1}}{vw_2(1-w_2)} > \frac{v^2-v-2}{vw_2(1-w_2)} \geq \theta_{d1}.\end{aligned}$$

Case (vi). Compare $\hat{d}$ with designs having $r_1 = 2$, $r_2 = v$ and $c_{22} = v - 1$

a). Let $\mathcal{D}_3 \subset \mathcal{D}$ be the subclass of designs in Case (vi) having $c_{11} = 2(v-1)/v$, that is, the two replicates of treatment 1 of any design in $\mathcal{D}_3$ appear in two different blocks. Let d be an arbitrary design other than $\hat{d}$ in $\mathcal{D}_3$. It can be observed that $trace(C_d) \leq trace(C_{\hat{d}}) \Rightarrow trace(C_{dw}) \leq trace(C_{\hat{d}w})$ for any fixed w_1 and w_2. Also notice that if $trace(C_{dw}) = trace(C_{\hat{d}w})$, then $C_{\hat{d}w} = \sum_{P\in\mathcal{P}} PC_{dw}P^T/(v-2)!$, where $\mathcal{P}$ is a class of all $v \times v$ permutation matrices that permute treatments $3, \ldots, v$. So, $\hat{d}$ is E_w-equal or better than any d in $\mathcal{D}_3$ having $trace(C_d) = trace(C_{\hat{d}})$.

To compare $\hat{d}$ with $d \in \mathcal{D}_3$ having $trace(C_d) < trace(C_{\hat{d}})$, write $\bar{C}_d \equiv C_{\bar{d}} = \sum_{P\in\mathcal{P}} PC_dP^T/(v-2)!$, where $\mathcal{P}$ is defined above. Since $c_{\hat{d}11} = c_{\bar{d}11}$, $c_{\hat{d}22} = c_{\bar{d}22}$ and $\gamma_{\hat{d}12} = \gamma_{\bar{d}12}$, by the same argument surrounding (3.12) it is known two eigenvalues of $C_{\hat{d}w}$ and $C_{\bar{d}w}$ are the same, but the third eigenvalue of $C_{\bar{d}w}$ (with multiplicity $v-3$) is less than that of $C_{\hat{d}w}$. So, $\theta_{\hat{d}1} \geq \theta_{\bar{d}1} \geq \theta_{d1}$.

b). Let $\mathcal{D}_4 \subset \mathcal{D}$ be the subclass of designs in Case (vi) for which the two replicates of treatment 1 both appear in the same block, so that $c_{11} = 2(v-2)/v$. Let d be an arbitrary design in $\mathcal{D}_4$. It can be seen that if $r_{dv} > v+1$, then $r_{d2} = r_{d3} = v$. Following the proof of Lemma 3.10, we get $\theta_{d1} \leq \frac{r_{d2}+r_{d3}}{2w_2} = \frac{v}{w_2} \leq \theta_{RCBD,1}$.

Now we only need to consider those d having $r_{d3} = \cdots = r_{dv} = v+1$. For any such d define the $GGDD(3)$ $\bar{d}$ by $C_{\bar{d}} \equiv \bar{C}_d = \sum_{P\in\mathcal{P}} PC_dP^T/(v-2)!$ where $\mathcal{P}$ is defined in (a) above. Then $C_{\bar{d}}$ is

$$C_{\bar{d}} = \begin{pmatrix} c_{\bar{d}11} & -\frac{2}{v} & -\frac{c_{\bar{d}11}-\frac{2}{v}}{v-2}1'_{v-2} \\ -\frac{2}{v} & v-1 & -\frac{v+1}{v}1'_{v-2} \\ -\frac{c_{\bar{d}11}-\frac{2}{v}}{v-2}1_{v-2} & -\frac{v+1}{v}1_{v-2} & \frac{c_{\bar{d}33}(v-2)-\frac{c_{\bar{d}11}+v-1-\frac{4}{v}}{v-2}}{v-3}I_{v-2} - \frac{c_{\bar{d}33}-\frac{c_{\bar{d}11}+v-1-\frac{4}{v}}{v-2}}{v-3}J_{v-2} \end{pmatrix}. \quad (3.19)$$

Since d is E_w-inferior to $\bar{d}$, it is sufficient to show $\theta_{\bar{d}1} \le \theta_{\hat{d}1}$. Recall that $\hat{d}$ is also a $GGDD(3)$, with C-matrix similar to (3.19) except $c_{\hat{d}11} = 2(v-1)/v$, while $c_{\bar{d}11} = 2(v-2)/v$, and $c_{\hat{d}33} > c_{\bar{d}33}$.

Both $\hat{d}$ and $\bar{d}$ have three distinct nonzero eigenvalues. Among these two positive eigenvalues for $\hat{d}$ and $\bar{d}$ are the two nonzero eigenvalues for the reduced matrix shown here:

$$\begin{pmatrix} \frac{c_{11}}{w_1} & -\frac{2}{v\sqrt{w_1w_2}} & -\frac{c_{11}-\frac{2}{v}}{\sqrt{w_1w_2}} \\ -\frac{2}{v\sqrt{w_1w_2}} & \frac{v-1}{w_2} & -\frac{(v+1)(v-2)}{vw_2} \\ -\frac{c_{11}-\frac{2}{v}}{(v-2)\sqrt{w_1w_2}} & -\frac{v+1}{vw_2} & \frac{c_{11}+v-1-\frac{4}{v}}{(v-2)w_2} \end{pmatrix}. \quad (3.20)$$

These eigenvalues are two solutions to the quadratic equation:

$$Q(x) = x^2 + q_1x + q_0 = 0, \quad (3.21)$$

where

$$q_1 = -\left(\frac{c_{11}}{w_1} + \frac{c_{22}}{w_2} + \frac{c_{11}+c_{22}+2\gamma_{12}}{(v-2)w_2}\right),$$

$$q_0 = \frac{c_{11}c_{22}}{w_1w_2} + \frac{c_{11}c_{22}}{(v-2)w_1w_2} + \frac{c_{11}c_{22}}{(v-2)w_2^2} - \frac{\gamma_{12}^2}{w_1w_2} - \frac{\gamma_{12}^2}{(v-2)w_1w_2} - \frac{\gamma_{12}^2}{(v-2)w_2^2}.$$

The two solutions for (3.21) are $x_1 = -q_1 - \sqrt{q_1^2 - 4q_0}$ and $x_2 = -q_1 + \sqrt{q_1^2 - 4q_0}$. Specifically, q_1 and q_0 for $C_{\hat{d}}$ are:

$$q_{\hat{d}1} = -\frac{1}{v}\left(\frac{2(v-1)}{w_1} + \frac{v(v-1)}{w_2} + \frac{(v+2)(v-1)-4}{(v-2)w_2}\right),$$

$$q_{\hat{d}0} = \frac{1}{v^2}\left(\frac{2v(v-1)^2}{w_1w_2} + \frac{2v(v-1)^2}{(v-2)w_1w_2} + \frac{2v(v-1)^2}{(v-2)w_2^2} - \frac{4}{w_1w_2} - \frac{4}{(v-2)w_1w_2} - \frac{4}{(v-2)w_2^2}\right),$$

and q_1 and q_0 for $C_{\bar{d}}$ are:

$$q_{\bar{d}1} = -\frac{1}{v}\left(\frac{2(v-2)}{w_1} + \frac{v(v-1)}{w_2} + \frac{v(v+1)-8}{(v-2)w_2}\right),$$

$$q_{\bar{d}0} = \frac{1}{v^2}\left(\frac{2v(v-1)(v-2)}{w_1w_2} + \frac{2v(v-1)(v-2)}{(v-2)w_1w_2} + \frac{2v(v-1)(v-2)}{(v-2)w_2^2} - \frac{4}{w_1w_2} - \frac{4}{(v-2)w_1w_2} - \frac{4}{(v-2)w_2^2}\right).$$

It is simple to see from (3.15) and (3.16) that the minimum positive eigenvalue of $C_{\hat{d}w}$, $\theta_{\hat{d}1}$, is the smaller solution to $Q_{\hat{d}}(x) = 0$. So now it is sufficient to show the smaller solution to $Q_{\hat{d}}(x) = 0$ is greater than or equal to the smaller solution to $Q_{\bar{d}}(x) = 0$, i.e.

$$-q_{\hat{d}1} - \sqrt{q_{\hat{d}1}^2 - 4q_{\hat{d}0}} \geq -q_{\bar{d}1} - \sqrt{q_{\bar{d}1}^2 - 4q_{\bar{d}0}}. \tag{3.22}$$

Rearranging inequality (3.22), it can be checked that a sufficient condition for (3.22) to hold

$$(q_{\bar{d}1} - q_{\hat{d}1})(q_{\bar{d}0}q_{\hat{d}1} - q_{\hat{d}0}q_{\bar{d}1}) - (q_{\hat{d}0} - q_{\bar{d}0})^2 \geq 0. \tag{3.23}$$

Using the expressions above for $q_{\hat{d}1}$, $q_{\hat{d}0}$, $q_{\bar{d}1}$ and $q_{\bar{d}0}$, the LHS of (3.23) simplifies to

$$\frac{4(v-1)^2[w_1(v^2+1)-2]^2}{(v-2)v^4(1-w_1)^4w_1^3}$$

which is clearly nonnegative ($= 0$ at $w_1 = \frac{2}{v^2+1} \Leftrightarrow \frac{w_1}{w_2} = \frac{2}{v+1}$).

□

Lemma 3.11. *Consider the 2-weight problem with $v_1 = 1$, $k = v$ and $b = v$. Let $\mathcal{D}_1 \subset \mathcal{D}(v, b, k)$ be the subclass in which all designs have $r_1 = 1$. Let $\hat{d} \in \mathcal{D}_1$ be the nonbinary GGDD(2) constructed from an RCBD by replacing treatment 1 with treatment i in the i^{th} block for $i = 2, 3, \ldots, v$. Then, $\hat{d}$ is E_w-best over $\mathcal{D}_1$ for any weight.*

Proof. The C-matrix for $\hat{d}$ is

$$C_{\hat{d}} = \frac{1}{v}\begin{pmatrix} v-1 & -1'_{v-1} \\ -1_{v-1} & (v^2+v-1)I_{v-1} - (v+2)J_{v-1}. \end{pmatrix} \tag{3.24}$$

The two nonzero eigenvalues of $C_{\hat{d}w}$ are

$$\begin{cases} \dfrac{v^2+v-1}{vw_2} & \text{with multiplicity } v-2 \\ \dfrac{v-1}{vw_1(1-w_1)} & \text{with multiplicity } 1. \end{cases}$$

Accordingly,

$$\theta_{\hat{d}1} = \begin{cases} \dfrac{v^2+v-1}{vw_2} & \text{for } w_1 \leq \dfrac{1}{v^2+v-1} \\ \dfrac{v-1}{vw_1(1-w_1)} & \text{for } w_1 \geq \dfrac{1}{v^2+v-1}. \end{cases}$$

Moreover, it can be seen $C_{\hat{d}}$ is the average of all permuted information matrices obtained by permuting over treatments 2, 3, ..., v for an arbitrary design in $\mathcal{D}_1$ having the same trace as $C_{\hat{d}}$. By convexity, no design in $\mathcal{D}_1$ with the same trace as $C_{\hat{d}}$ can be E_w-better than $\hat{d}$.

It is obvious that $\hat{d}$ maximizes trace of the C-matrix over $\mathcal{D}_1$. Now, let d be any design in $\mathcal{D}_1$ which has smaller trace than $\hat{d}$, and let $\bar{C}_d$ be the average of all permutations of C_d over treatments 2, 3, ..., v. If write $\theta_{\bar{d}1}$ as the smallest positive eigenvalue of $\bar{C}_d$, then the convexity property says $\theta_{d1} \leq \theta_{\bar{d}1}$, so it is sufficient to show $\bar{d}$ is E_w-inferior to $\hat{d}$. It can be seen that both $C_{\hat{d}}$ and $\bar{C}_d \equiv C_{\bar{d}}$ have the form as (3.24), in which $\alpha_{\bar{d}1} = \alpha_{\hat{d}1} = \frac{v-1}{v}$. From (2.28), we can see

$$\alpha_{\bar{d}2} = \frac{(v-1)trace(C_{\bar{d}}) - v\alpha_{\bar{d}1}}{(v-1)(v-2)} < \frac{(v-1)trace(C_{\hat{d}}) - v\alpha_{\hat{d}1}}{(v-1)(v-2)} = \alpha_{\hat{d}2} = \frac{v^2+v-1}{v}$$

since $trace(C_{\bar{d}}) < trace(C_{\hat{d}})$.

If $w_1 \leq \frac{1}{v^2+v-1}$, one eigenvalue of $C_{\bar{d}w}$ is $\frac{\alpha_{\bar{d}2}}{w_2}$. So, for $w_1 \leq \frac{1}{v^2+v-1}$,

$$\theta_{d1} \leq \theta_{\bar{d}1} \leq \frac{\alpha_{\bar{d}2}}{w_2} < \frac{\alpha_{\hat{d}2}}{w_2} = \theta_{\hat{d}1}.$$

If $w_1 \geq \frac{1}{v^2+v-1}$, by Corollary 3.1, $\theta_{d1} \leq \theta_{\bar{d}1} \leq \frac{v-1}{vw_1(1-w_1)} = \theta_{\hat{d}1}$. □

Theorem 3.19. *Consider the 2-weight problem with $v_1 = 1$, $k = v$ and $b = v$. Let $\bar{d}$ be the GGDD(2) having $r_1 = 1$ defined in Lemma 3.11. Then $\bar{d}$ is E_w-optimal in $\mathcal{D}(v, b, k)$ for $w_1 \leq \frac{1}{v^2+1}$.*

Proof. By Lemma 3.11, only competitors having $r_1 \geq 2$ need to be considered. We will compare $\bar{d}$ with other designs belonging to the following disjoint design classes:

(i) Designs having $r_1 \geq 3$

(ii) Designs having $r_1 = 2$ and $c_{22} \leq \frac{v^2-v-2}{v}$

(iii) Designs having $r_1 = 2$, $r_2 = v$ and $c_{22} = v - 1$

Case (i). Designs having $r_1 \geq 3$.

Let d be an arbitrary design in (i). By Lemma 3.10, d cannot be E_w-better than $RCBD$. So, it is sufficient to show $\bar{d}$ is E_w-better than $RCBD$ for $w_1 \leq \frac{1}{v^2+1}$.

If $w_1 \leq \frac{1}{v^2+v-1}$,

$$\theta_{\bar{d}1} = \frac{v^2+v-1}{vw_2} > \frac{v}{w_2} = \theta_{RCBD,1}.$$

If $\frac{1}{v^2+v-1} \leq w_1 \leq \frac{1}{v^2+1}$,

$$\theta_{\bar{d}1} = \frac{v-1}{vw_1(1-w_1)} = \frac{1}{vw_1w_2} \geq \frac{1}{v(\frac{1}{v^2+1})w_2} = \frac{v^2+1}{vw_2} > \frac{v}{w_2} = \theta_{RCBD,1}.$$

Case (ii). Designs having $r_1 = 2$ and $c_{22} \leq \frac{v^2-v-2}{v}$.

Let d be an arbitrary design in (ii).

If $w_1 \leq \frac{1}{v^2+v-1}$,

$$\begin{aligned}\theta_{\bar{d}1} &= \frac{v^2+v-1}{vw_2} = \frac{(v^2+v-1)(1-w_2)}{vw_2(1-w_2)} \geq \frac{(v^2+v-1)(1-\frac{1}{v-1})}{vw_2(1-w_2)} \qquad \text{since } w_2 < \frac{1}{v-1} \\ &= \frac{v^2-2-\frac{v}{v-1}}{vw_2(1-w_2)} > \frac{v^2-v-2}{vw_2(1-w_2)} \geq \theta_{d1}.\end{aligned}$$

If $\frac{1}{v^2+v-1} \le w_1 \le \frac{1}{v^2+1}$,

$$\begin{aligned}
\theta_{\bar{d}1} &= \frac{v-1}{vw_1(1-w_1)} = \frac{1-w_2}{vw_1w_2(1-w_2)} > \frac{(1-w_2)}{v(\frac{1}{v^2+1})w_2(1-w_2)} = \frac{(v^2+1)(1-w_2)}{vw_2(1-w_2)} \\
&\ge \frac{(v^2+1)(1-\frac{v^2+v-2}{(v^2+v-1)(v-1)})}{vw_2(1-w_2)} = \frac{v^2-v-\frac{v^2+2v-3}{(v^2+v-1)(v-1)}}{vw_2(1-w_2)} > \frac{v^2-v-2}{vw_2(1-w_2)} \\
&\ge \theta_{d1}.
\end{aligned}$$

Case (iii). Designs having $r_1 = 2$, $r_2 = v$ and $c_{22} = v-1$.

Let d be an arbitrary design in (iii). Then d is either the competitor in Case (vi) or $\hat{d}$ defined in Theorem 3.18. It was shown that $\hat{d}$ was E_w-better than those designs in Case (vi) of Theorem 3.18 for any weight. So, we only need to show $\bar{d}$ is E_w-better than $\hat{d}$ for $w_1 \le \frac{1}{v^2+1}$. If $w_1 \le \frac{1}{v^2+v-1}$,

$$\theta_{\bar{d}1} = \frac{v^2+v-1}{vw_2} = \frac{(v-1)(v^2+v-1)}{v(1-w_1)} > \frac{(v-1)(v^2+1)}{v(1-w_1)} = \theta_{\hat{d}1}.$$

If $\frac{1}{v^2+v-1} \le w_1 \le \frac{1}{v^2+1}$,

$$\theta_{\bar{d}1} = \frac{v-1}{vw_1(1-w_1)} \ge \frac{v-1}{v(\frac{1}{v^2+1})(1-w_1)} = \frac{(v-1)(v^2+1)}{v(1-w_1)} = \theta_{\hat{d}1}.$$

$\square$

Example 3.14. *By Theorem 3.19, the following GGDD(2) design in $\mathcal{D}(v,b,k)$ with $v = b = k = 5$ is E_w-optimal in $\mathcal{D}(5,5,5)$ for $w_1/w_2 \le 4/25$.*

1	*2*	*2*	*2*	*2*
2	*2*	*3*	*3*	*3*
3	*3*	*3*	*4*	*4*
4	*4*	*4*	*4*	*5*
5	*5*	*5*	*5*	*5*

By Theorem 3.18, this design is E_w-optimal in $\mathcal{D}(5,5,5)$ for $4/25 \le w_1/w_2 \le 8/23$.

1	*1*	*2*	*2*	*2*
2	*2*	*3*	*3*	*3*
3	*3*	*3*	*4*	*4*
4	*4*	*4*	*4*	*5*
5	*5*	*5*	*5*	*5*

Finally, an RCBD is E_w-optimal in $\mathcal{D}(5,5,5)$ for $8/23 \le w_1/w_2 \le 1$, as shown next.

Theorem 3.20. *Consider the 2-weight problem with $v_1 = 1$, $k = v$ and $b = v$. Then an RCBD is E_w-optimal in $\mathcal{D}(v,b,k)$ for $\frac{2}{v^2} \le w_1 \le \frac{1}{v}$.*

Proof. Let d be an arbitrary competitor in $\mathcal{D}$.

If $r_{d1} \le 2$, for $\frac{2}{v^2} \le w_1 \le \frac{1}{v}$, by Corollary 3.1,

$$\theta_{d1} \le \frac{c_{d11}}{w_1(1-w_1)} \le \frac{2(v-1)}{vw_1(1-w_1)} = \frac{2}{vw_1w_2} = \Big(\frac{v}{w_2}\Big)\Big(\frac{2}{v^2w_1}\Big) \le \frac{v}{w_2} = \theta_{RCBD,1}$$

If $3 \le r_{d1} \le v$, by Lemma 3.10, d cannot be E_w-better than an $RCBD$ for any weight.

So now it is sufficient to show an $RCBD$ is E_w-better than d having $r_{d1} \ge v+1 \Rightarrow r_{d2} \le v-1$. So, for $\frac{2}{v^2} \le w_1 \le \frac{1}{v}$, by Corollary 3.1,

$$\begin{aligned}\theta_{d1} &\le \frac{c_{d22}}{w_2(1-w_2)} \le \frac{(v-1)^2}{vw_2(1-w_2)} = \Big(\frac{v}{w_2}\Big)\Big(\frac{(v-1)^2}{v^2(1-w_2)}\Big) \\ &\le \Big(\frac{v}{w_2}\Big)\Big(\frac{(v-1)^2}{v^2(1-\frac{1-\frac{2}{v^2}}{v-1})}\Big) = \Big(\frac{v}{w_2}\Big)\Big(\frac{(v-1)^3}{v^2(v-2)+2}\Big) < \frac{v}{w_2} = \theta_{RCBD,1}\end{aligned}$$

□

In this Section we have found the E_w-optimal designs in the design classes $\mathcal{D}(v,b,k)$ for $k = v$ and $b \le v$ for the 2-weight problem with $v_1 = 1$ and $w_1 \le 1/v$ (or equivalently,

$w_1/w_2 \leq 1$). This is a TwC experimental situation (compare Definition 1.7 and the discussion in Section 1.2) where comparisons with the control are of less interest. Surprisingly, if the number of blocks is no greater than $v-2$, the E_w-best choice is always an $RCBD$. For $b = v-1$ or $b = v$ the E_w-best design depends on $w_1 \leq \frac{1}{v}$, with the $RCBD$ being best only if w_1 is sufficiently close to $1/v$, the equal weight case.

Chapter 4

Weighted A-optimal Designs

In this chapter, we generalize some sufficient conditions for designs to be A_w-optimal.

Theorem 4.1. *For the 2-weight problem with $v_1 = 1$ and $v_2 = v - 1$, let $\tilde{d}$ be a binary $GGDD(2)$ in $\mathcal{D}(v, b, k)$ with weighted information matrix*

$$C_{\tilde{d}w} = \begin{pmatrix} \frac{c_1}{w_1} & \frac{-c_1}{(v-1)\sqrt{w_1 w_2}} & \cdots & \frac{-c_1}{(v-1)\sqrt{w_1 w_2}} \\ \frac{-c_1}{(v-1)\sqrt{w_1 w_2}} & \frac{c_2}{w_2} & \frac{c_1 - c_2(v-1)}{(v-1)(v-2)w_2} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-c_1}{(v-1)\sqrt{w_1 w_2}} & \frac{c_1 - c_2(v-1)}{(v-1)(v-2)w_2} & \cdots & \frac{c_2}{w_2} \end{pmatrix},$$

Let $d^* \in \mathcal{D}$ be a possibly hypothetical design having completely symmetric C-matrix with $trace(C_{d^*}) = trace(C_{\tilde{d}})$. Note d^* is a $BIBD$ if a $BIBD$ exists.

(i) If $c_1 > c_2$, then $\tilde{d}$ is A_w-better than d^* if and only if $w_1 > \frac{(v-2)c_1}{(v-1)^2 c_2 - c_1}$.

(ii) If $c_1 < c_2$, then $\tilde{d}$ is A_w-better than d^* if and only if $w_1 < \frac{(v-2)c_1}{(v-1)^2 c_2 - c_1}$.

Proof. (i) Denote the non-zero eigenvalues for $C_{\tilde{d}w}$ as $e_{\tilde{d}w1}$ with multiplicity 1 and $e_{\tilde{d}w2}$ with multiplicity $(v - 2)$, which can be computed by solving these equations:

$$\begin{cases} e_{\tilde{d}w1} + (v-2)e_{\tilde{d}w2} = \dfrac{c_1}{w_1} + (v-1)\dfrac{c_2}{w_2} \\ e_{\tilde{d}w2} = \dfrac{c_2}{w_2} - \dfrac{c_1 - c_2(v-1)}{(v-1)(v-2)w_2} \end{cases}$$

Thus, the two nonzero eigenvalues of $C_{\tilde{d}w}$ are

$$e_{\tilde{d}w1} = \frac{c_1}{w_1} + \frac{c_1}{(v-1)w_2} \text{ and } e_{\tilde{d}w2} = \frac{c_2(v-1)^2 - c_1}{(v-1)(v-2)w_2}.$$

So the A_w-value for $\tilde{d}$ is

$$\Phi_A(C_{\tilde{d}w}) = \frac{1}{e_{\tilde{d}w1}} + (v-2)\frac{1}{e_{\tilde{d}w2}} = \frac{(v-1)w_1w_2}{c_1} + \frac{(v-1)(v-2)^2w_2}{c_2(v-1)^2 - c_1}$$

Denote $c_{d^*ii} = c$ for all $i \in \{1, 2, \ldots, v\}$, thus $c = \frac{c_1+(v-1)c_2}{v}$. Then,

$$\begin{aligned} &\Phi_A(C_{\tilde{d}w}) - \Phi_A(C_{d^*w}) \\ &= \left(\frac{w_1}{c_1} - \frac{w_1}{c} + \frac{(v-2)^2}{c_2(v-1)^2 - c_1} - \frac{(v-2)}{cv}\right)(v-1)w_2. \end{aligned}$$

$\tilde{d}$ is A_w-superior to d^* iff

$$\begin{aligned} &\Phi_A(C_{\tilde{d}w}) - \Phi_A(C_{d^*w}) < 0 \\ \Leftrightarrow\ & w_1 > \frac{\dfrac{(v-2)^2}{c_2(v-1)^2 - c_1} - \dfrac{v-2}{cv}}{\dfrac{1}{c} - \dfrac{1}{c_1}} = \frac{(v-2)c_1}{(v-1)^2c_2 - c_1} \end{aligned}$$

The proof for (ii) follows similar steps. □

Here we compare $\frac{(v-2)c_1}{(v-1)^2c_2-c_1}$ with $1/v$. Denote trace for a binary design by t. Applying the fact $c_1 + (v-1)c_2 = t$, it follows that

$$\frac{(v-2)c_1}{(v-1)^2c_2 - c_1} = \frac{(v-2)c_1}{(v-1)t - vc_1}.$$

Obviously, the above function is increasing in c_1 and is bigger than $1/v$ when $c_1 = t/v$.

So, $c_1 > c_2 \Rightarrow \frac{(v-2)c_1}{(v-1)^2c_2-c_1} > 1/v$. Thus $w_1 < w_2$ is a sufficient condition for d^* to be A_w-better than d if $c_1 > c_2$.

Theorem 4.2. *For the 2-weight problem with $v_1 = 1$, let $\tilde{d}$ be a binary GGDD(2) design in $\mathcal{D}(v, b, k)$, for which $C_{\tilde{d}}$ has information matrix as shown in (2.28). Let $t = b(k-1)$ be the trace for information matrices of binary designs in $\mathcal{D}$. Then, $\tilde{d}$ is A_w-optimal over $\mathcal{D}$*

for $w_1 = \frac{v(v-2)^2\alpha_{\tilde{d}1}^2}{[(v-1)t-v\alpha_{\tilde{d}1}]^2}$ *and* $w_2 = \frac{[(v-1)t-v\alpha_{\tilde{d}1}]^2-v[(v-2)\alpha_{\tilde{d}1}]^2}{(v-1)[(v-1)t-v\alpha_{\tilde{d}1}]^2}$ *if either of the following conditions is met:*

(i) $k > \sqrt{v} + \frac{\sqrt{v}}{\sqrt{v}+1}$,

(ii) $k \leq \sqrt{v} + \frac{\sqrt{v}}{\sqrt{v}+1}$ *and* $\alpha_{\tilde{d}1} < \frac{(v-1)t}{(v-2+\sqrt{v})\sqrt{v}}$.

Proof. For an arbitrary design $d \in \mathcal{D}$, let $\bar{C}_d$ be the averaged information matrix obtained by permuting C_d within the treatment set $\{2, 3, \ldots, v\}$. Then $\Phi_A(\bar{C}_{dw}) \leq \Phi_A(C_{dw})$. Thus, it is sufficient to show $\Phi_A(C_{\tilde{d}w}) \leq \Phi_A(\bar{C}_{dw})$ for any d with $trace(\bar{C}_d) \leq t$.

It is known that $\bar{C}_d$ has form given in (2.28). The eigenvalues of its correspondent $\bar{C}_{dw} = W^{-1/2}\bar{C}_d W^{-1/2}$ are

$$\begin{cases} \theta_1 = \dfrac{\alpha_1}{w_1} + \dfrac{\alpha_2 + (v-1)\beta_2}{w_2} & \text{with multiplicity } 1 \\ \theta_2 = \dfrac{\alpha_2}{w_2} & \text{with multiplicity } v-2. \end{cases}$$

Then the A_w-best design for a fixed trace minimizes

$$\Phi_A(\bar{C}_{dw}) = \frac{1}{\theta_1} + (v-2)\frac{1}{\theta_2} = \frac{1}{\frac{\alpha_1}{w_1} + \frac{\alpha_2+(v-1)\beta_2}{w_2}} + \frac{w_2(v-2)}{\alpha_2}.$$

Utilizing facts that $\alpha_1 = -(v-1)\gamma$ and $\alpha_2 + (v-1)\beta_2 = -\gamma$ given in (2.29) and (2.30),

$$\Phi_A(\bar{C}_{dw}) = \frac{w_2(v-2)}{\alpha_2} - \frac{w_1 w_2}{\gamma}.$$

The trace of the information matrix for $\bar{C}_{dw}$ is

$$trace(\bar{C}_d) = trace(C_d) = \alpha_1 + (v-1)(\alpha_2+\beta_2) = \alpha_1 + (v-2)\alpha_2 - \gamma = (v-2)\alpha_2 + \frac{v}{v-1}\alpha_1.$$

So,

$$\begin{aligned} \Phi_A(\bar{C}_{dw}) &= \frac{w_2(v-2)}{\alpha_2} + \frac{w_1 w_2}{trace(C_d) - \alpha_1 - (v-2)\alpha_2} \\ &= \frac{w_2(v-2)}{\alpha_2} + \frac{w_1 w_2 v}{trace(C_d) - (v-2)\alpha_2} \qquad \text{by (2.32).} \end{aligned} \tag{4.1}$$

Equivalently, $\Phi_A(\bar{C}_{dw})$ can be expressed in terms of α_1:

$$\Phi_A(\bar{C}_{dw}) = (1-w_1)\Big[\frac{(v-2)^2}{(v-1)trace(C_d) - v\alpha_1} + \frac{w_1}{\alpha_1}\Big] \tag{4.2}$$

To investigate the minimum of this expression, assume the parameters in $\bar{C}_d$ can change continuously. Setting $trace(C_d) = t$ and then taking the derivative of expression (4.1) of $\Phi_A(\bar{C}_{dw})$ with respect to α_2 and setting it to zero, we can get the value of α_2, call it $\alpha_{\tilde{d}2}$, such that design $\bar{d}$ is A_w-optimal over the binary subclass of $\mathcal{D}$ for fixed weight w_1 and w_2.

$$\begin{aligned} \frac{\partial \Phi_A(\bar{C}_{dw})}{\partial \alpha_2} &= -\frac{w_2(v-2)}{\alpha_2^2} + \frac{w_1 w_2 v(v-2)}{\left[t-(v-2)\alpha_2\right]^2} \overset{set}{=} 0 \\ \Rightarrow \frac{1}{\alpha_{\tilde{d}2}^2} &= \frac{w_1 v}{\left[t-(v-2)\alpha_{\tilde{d}2}\right]^2} \end{aligned} \tag{4.3}$$

(4.3) can be rewritten as:

$$w_1 = \frac{\left[\frac{t}{\alpha_{\tilde{d}2}} - (v-2)\right]^2}{v}.$$

Correspondingly,

$$w_2 = \frac{v - \left[\frac{t}{\alpha_{\tilde{d}2}} - (v-2)\right]^2}{v(v-1)}.$$

Equivalently, we can use $\alpha_{\tilde{d}1}$ instead of $\alpha_{\tilde{d}2}$ in the expressions of w_1 and w_2, giving

$$w_1 = \frac{v(v-2)^2\alpha_{\tilde{d}1}^2}{[(v-1)t - v\alpha_{\tilde{d}1}]^2}$$

and

$$w_2 = \frac{[(v-1)t - v\alpha_{\tilde{d}1}]^2 - v[(v-2)\alpha_{\tilde{d}1}]^2}{(v-1)[(v-1)t - v\alpha_{\tilde{d}1}]^2}.$$

It can also be checked $\frac{\partial \Phi_A^2(C_w)}{\partial \alpha_2^2} > 0$, so $\Phi_A(C_{\tilde{d}w}) \le \Phi_A(C_{dw})$ for any binary d.

We also need to make sure $0 < w_1 < 1$. It is apparent $w_1 > 0$ is satisfied, but to ensure $w_1 < 1$, the following restriction must be added:

$$w_1 = \frac{v\alpha_{\tilde{d}1}^2(v-2)^2}{[(v-1)t - v\alpha_{\tilde{d}1}]^2} < 1 \Rightarrow \alpha_{\tilde{d}1} < \frac{(v-1)t}{(v-2+\sqrt{v})\sqrt{v}}. \tag{4.4}$$

Since any binary design has $\alpha_{\tilde{d}1} \le \frac{b(k-1)}{k} = \frac{t}{k}$, the inequality (4.4) is always true if

$$t/k < \frac{(v-1)t}{(v-2+\sqrt{v})\sqrt{v}} \quad \Leftrightarrow \quad k > \frac{(v-2+\sqrt{v})\sqrt{v}}{(v-1)} = \sqrt{v} + \frac{\sqrt{v}}{\sqrt{v}+1}. \tag{4.5}$$

So, if $k > \sqrt{v}+\frac{\sqrt{v}}{\sqrt{v}+1}$, $\tilde{d}$ is A_w-better for the specified weights than any other binary $GGDD(2)$ having $v_1 = 1$. If $k \leq \sqrt{v} + \frac{\sqrt{v}}{\sqrt{v}+1}$, then $\alpha_{\tilde{d}1} < \frac{(v-1)t}{(v-2+\sqrt{v})\sqrt{v}}$ need to be satisfied for $\tilde{d}$ to be A_w-better.

Next, we will show $\tilde{d}$ is A_w-superior to all nonbinary designs in $\mathcal{D}$ for the above w_1 and w_2. Let $\widehat{d}$ be an arbitrary nonbinary design in $\mathcal{D}$. Let $\bar{C}_{\widehat{d}w}$ be the averaged information matrix obtained by permuting treatments 2, 3, ..., v of $C_{\widehat{d}w}$. By Theorem 2.7, $\Phi_A(\bar{C}_{\widehat{d}w}) \leq \Phi_A(C_{\widehat{d}w})$. If $\widehat{d}$ is A_w-better than $\tilde{d}$, then for fixed w_1, w_2 and v, using (4.2),

$$
\begin{aligned}
& \Phi_A(\bar{C}_{\widehat{d}w}) < \Phi_A(C_{\tilde{d}w}) \\
\Rightarrow\ & \Phi_A(\bar{C}_{\widehat{d}w})\frac{trace(C_{\widehat{d}})}{t} < \Phi_A(C_{\tilde{d}w}) \qquad \text{since } trace(C_{\widehat{d}}) < t \\
\Rightarrow\ & (1-w_1)\Big[\frac{(v-2)^2}{(v-1)trace(C_{\widehat{d}}) - v\alpha_{\widehat{d}1}} + \frac{w_1}{\alpha_{\widehat{d}1}}\Big]\frac{trace(C_{\widehat{d}})}{t} < (1-w_1)\Big[\frac{(v-2)^2}{(v-1)t - v\alpha_{\tilde{d}1}} + \frac{w_1}{\alpha_{\tilde{d}1}}\Big] \\
\Rightarrow\ & \Big[\frac{(v-2)^2}{(v-1)t - v\alpha_{\widehat{d}1}\big(\frac{t}{trace(C_{\widehat{d}})}\big)} + \frac{w_1}{\alpha_{\widehat{d}1}\big(\frac{t}{trace(C_{\widehat{d}})}\big)}\Big] < \Big[\frac{(v-2)^2}{(v-1)t - v\alpha_{\tilde{d}1}} + \frac{w_1}{\alpha_{\tilde{d}1}}\Big]
\end{aligned}
$$

The above inequality says a binary $GGDD$ with $\alpha_1 = \frac{\alpha_{\widehat{d}1}t}{trace(C_{\widehat{d}})}$ is A_w-better than $\tilde{d}$, which is a contradiction to the former proof that no binary design can be A_w-better than $\tilde{d}$. $\square$

Here is one of the possible applications of Theorem 4.2. Similar results can be written for other constructions given earlier in this dissertation.

Corollary 4.1. *Let $\bar{d}$ be constructed as in Theorem 3.3. Then $\bar{d}$ is A_w-optimal over $\mathcal{D}$ for* $w_1 = \big[\frac{(b^*+\hat{b})(k-1)(v-1)}{b^*(k-1)+\hat{b}(v-1)} - (v-2)\big]^2/v$ *and* $w_2 = \big[v - \big(\frac{(b^*+\hat{b})(k-1)(v-1)}{b^*(k-1)+\hat{b}(v-1)} - (v-2)\big)^2\big]/[v(v-1)]$.

Proof. Recall in the proof for Theorem 3.3, we showed that $\bar{d}$ is $GGDD(2)$ for which $v_1 = 1$ and $v_2 = v-1$ along with

$$
\begin{aligned}
\alpha_{\bar{d}1} &= \frac{b^*(k-1)}{v}, \\
\alpha_{\bar{d}2} &= \frac{1}{k}\Big[\frac{b^*k(k-1)}{v-1} + \hat{b}k\Big],
\end{aligned}
$$

$$\beta_{\bar{d}2} = -\frac{1}{k}\left[\frac{b^* k(k-1)}{v(v-1)} + \hat{b}\right].$$

Now apply Theorem 4.2 to get w_1 and w_2. □

Example 4.1. *Again, consider the following design $\tilde{d}$ in $\mathcal{D}(4,5,3)$,*

$$\begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 & 3 \\ 3 & 4 & 4 & 4 & 4 \end{array}$$

The C-matrix of $\tilde{d}$

$$C_{\tilde{d}} = \begin{pmatrix} 2 & -\frac{2}{3}1_{1\times 3} \\ -\frac{2}{3}1_{3\times 1} & \frac{11}{3}I_3 + (-1)J_3 \end{pmatrix}.$$

Thus, $\alpha_{\tilde{d}1} = 2$ and $\alpha_{\tilde{d}2} = 11/3$. By Theorem 4.2, $\tilde{d}$ is A_w-optimal design among $\mathcal{D}(4,5,3)$ for $w_1 = 16/121$ and $w_2 = 35/121$. Recall from Example 3.1 that this design is weight balanced and E_w-optimal for $w_1 = 2/11$ and $w_2 = 3/11$.

Chapter 5

Weighted MV-optimal Designs

In this chapter, we derive sufficient conditions for a $GGDD$ design $\bar{d}$ to be MV_w-optimal for the 2-weight problem. The first result provides conditions for which a weight balanced design is MV_w-optimal, so long as each weight group contains at least two treatments.

Theorem 5.1. *Suppose v treatments are divided into two treatment sets with size v_1, $v_2 \geq 2$, that are assigned distinct weight w_1 and w_2. Let $\bar{d}$ be a binary $GGDD(2)$ in $\mathcal{D}(v,b,k)$ that has information matrix shown in (2.35) and satisfying $C_{\bar{d}w} = \epsilon(I - f_0 f_0')$ for some ϵ. Then $\bar{d}$ is MV_w-optimal over $\mathcal{D}$ if $w_1 \leq \dfrac{1}{\sqrt{vv_1}}$ and $w_2 \leq \dfrac{1}{\sqrt{vv_2}}$.*

Proof. Let d be an arbitrary design in $\mathcal{D}$. By Theorem 2.7, $\Phi_{MV}(\bar{C}_{dw}) \leq \Phi_{MV}(C_{dw})$ where $\bar{C}_{dw}$ is the averaged version of C_{dw} obtained by within-group permutation. Let $\hat{d}$ be a design having $C_{\hat{d}w} = \hat{\epsilon}(I - f_0 f_0')$ for some $\hat{\epsilon}$, and satisfying $trace(C_{\hat{d}}) = trace(\bar{C}_d) = trace(C_d) = t$. Obviously, both $C_{\hat{d}}$ and $\bar{C}_d$ have $GGDD(2)$ form. So, we start the proof by showing that, for $w_1 \leq \frac{1}{\sqrt{vv_1}}$ and $w_2 \leq \frac{1}{\sqrt{vv_2}}$, $\Phi_{MV}(C_{\hat{d}w}) \leq \Phi_{MV}(\bar{C}_{dw})$.

The Moore-Penrose inverse of C-matrix having $GGDD(2)$ form in (2.35) is (Majumdar, 1986)

$$C^+ = \begin{pmatrix} \frac{1}{\alpha_1} I_{v_1} - \frac{1}{\alpha_1 v_1} J_{v_1} - \frac{v_2}{\gamma v_1 v^2} J_{v_1} & \frac{1}{\gamma v^2} 1_{v_1 \times v_2} \\ \frac{1}{\gamma v^2} 1_{v_2 \times v_1} & \frac{1}{\alpha_2} I_{v_2} - \frac{1}{\alpha_2 v_2} J_{v_2} - \frac{v_1}{\gamma v_2 v^2} J_{v_2} \end{pmatrix}. \tag{5.1}$$

Let l be the contrast vector for comparing treatment i and i'. Using (2.15) with (5.1), the weighted variances of pairwise contrasts are

$$Var_w(\widehat{\tau_i - \tau_{i'}}) = \frac{w_i w_{i'}}{w_i + w_{i'}} l C^+ l' = \begin{cases} \frac{w_1}{\alpha_1} & \text{for } i, i' \in V_1 \\ \frac{w_2}{\alpha_2} & \text{for } i, i' \in V_2 \\ \frac{w_1 w_2}{w_1 + w_2} \left[\frac{1}{\alpha_1} - \frac{1}{\alpha_1 v_1} + \frac{1}{\alpha_2} - \frac{1}{\alpha_2 v_2} - \frac{1}{\gamma v^2} (\frac{v_2}{v_1} + \frac{v_1}{v_2} + 2) \right] & \\ \quad \text{for } i \in V_1 \text{ and } i' \in V_2 & \end{cases}$$

Note from Lemma 3.2 that $C_{\hat{d}w} = \epsilon(I - f_0 f_0')$ implies $Var_{\hat{d}w}(\widehat{\tau_i - \tau_{i'}})$ is constant for $i, i' \in \{1, \ldots, v\}$ and $i \neq i'$. Writing $Var_{\hat{d}w}(\widehat{\tau_i - \tau_{i'}}) \equiv m$ for $i, i' \in \{1, 2, \ldots, v\}$, then m can be calculated by setting

$$\frac{w_1}{\alpha_{\hat{d}1}} = \frac{w_2}{\alpha_{\hat{d}2}} = \frac{w_1 w_2}{w_1 + w_2} \left(\frac{1}{\alpha_{\hat{d}1}} - \frac{1}{\alpha_{\hat{d}1} v_1} + \frac{1}{\alpha_{\hat{d}2}} - \frac{1}{\alpha_{\hat{d}2} v_2} - \frac{1}{\gamma_{\hat{d}} v^2} (\frac{v_2}{v_1} + \frac{v_1}{v_2} + 2) \right).$$

Solving these equations with (2.39) gives

$$\alpha_{\hat{d}1} = -\frac{\gamma_{\hat{d}}}{w_2}, \qquad \alpha_{\hat{d}2} = -\frac{\gamma_{\hat{d}}}{w_1}, \qquad \text{and} \qquad \gamma_{\hat{d}} = -\frac{w_1 w_2 t}{v_1 w_1 (1 - w_1) + v_2 w_2 (1 - w_2)}.$$

Thus,

$$m = \frac{w_1}{\alpha_{\hat{d}1}} = \frac{w_2}{\alpha_{\hat{d}2}} = -\frac{w_1 w_2}{\gamma_{\hat{d}}} = \frac{v_1 w_1 (1 - w_1) + v_2 w_2 (1 - w_2)}{t}.$$

Let the possibly hypothetical design d^* have information matrix $C_{d^*} = \bar{C}_d$. Then C_{d^*} has $GGDD(2)$ form in (2.35) and trace t. To improve on $\hat{d}$, d^* must satisfy $Var_{d^*w}(\widehat{\tau_i - \tau_{i'}}) < m$ for all $i \neq i'$, and so

$$\frac{w_1}{\alpha_{d^*1}} < \frac{w_1}{\alpha_{\hat{d}1}} \Rightarrow \alpha_{d^*1} > \alpha_{\hat{d}1}, \tag{5.2}$$

$$\frac{w_2}{\alpha_{d^*2}} < \frac{w_2}{\alpha_{\hat{d}2}} \Rightarrow \alpha_{d^*2} > \alpha_{\hat{d}2}. \tag{5.3}$$

Combining the inequalities (5.2) and (5.3) with the fact from (2.39) that $\gamma_{d^*} = \frac{1}{v}[(v_1 - 1)\alpha_{d^*1} + (v_2 - 1)\alpha_{d^*2} - t]$, it is simple to see $\gamma_{d^*} > \gamma_{\hat{d}}$ must also be satisfied, since t is fixed.

Now we investigate the implications for the pairwise variance

$$Var_{d^*w}(\widehat{\tau_i - \tau_{i'}}) = \frac{w_1 w_2}{w_1 + w_2}\left[\frac{1}{\alpha_1} - \frac{1}{\alpha_1 v_1} + \frac{1}{\alpha_2} - \frac{1}{\alpha_2 v_2} - \frac{1}{\gamma v^2}\left(\frac{v_2}{v_1} + \frac{v_1}{v_2} + 2\right)\right] \tag{5.4}$$

where for simplicity the d^* subscript has been omitted from the design parameters. Note that for fixed t, there are two free parameters, which are taken as α_1 and α_2. Treating the weighted variance for this pairwise comparison as a continuous function of the parameters α_1 and α_2 for $i \in V_1$ and $i' \in V_2$, and noting that $\frac{1}{\gamma v^2}(\frac{v_2}{v_1} + \frac{v_1}{v_2} + 2) = \frac{1}{\gamma v_1 v_2}$,

$$\begin{aligned}
\frac{\partial}{\partial \alpha_1} Var_w(\widehat{\tau_i - \tau_{i'}}) &= \frac{w_1 w_2}{w_1 + w_2}\frac{\partial}{\partial \alpha_1}\left(\frac{1}{\alpha_1} - \frac{1}{\alpha_1 v_1} + \frac{1}{\alpha_2} - \frac{1}{\alpha_2 v_2} - \frac{1}{\gamma v_1 v_2}\right)\\
&= \frac{w_1 w_2}{w_1 + w_2}\frac{\partial}{\partial \alpha_1}\left[\frac{1}{\alpha_1} - \frac{1}{\alpha_1 v_1} + \frac{1}{\alpha_2} - \frac{1}{\alpha_2 v_2} - \frac{v}{[(v_1-1)\alpha_1 + (v_2-1)\alpha_2 - t]v_1 v_2}\right]\\
&= \frac{w_1 w_2}{w_1 + w_2}\left[-\frac{1}{\alpha_1^2}\left(1 - \frac{1}{v_1}\right) + \frac{v}{v_1 v_2}\frac{v_1 - 1}{\big[(v_1-1)\alpha_1 + (v_2-1)\alpha_2 - t\big]^2}\right]\\
&= \frac{w_1 w_2}{w_1 + w_2}\frac{v_1 - 1}{v_1}\left[\frac{v}{v_2\big[(v_1-1)\alpha_1 + (v_2-1)\alpha_2 - t\big]^2} - \frac{1}{\alpha_1^2}\right].
\end{aligned}$$

Evaluate the above derivative at any point (α_1^*, α_2^*) for which $\alpha_1^* > \alpha_{\hat{d}1}$ and $\alpha_2^* > \alpha_{\hat{d}2}$,

$$\left.\frac{\partial}{\partial \alpha_1} Var_w(\widehat{\tau_i - \tau_{i'}})\right|_{(\alpha_1^*, \alpha_2^*)} = \frac{w_1 w_2}{w_1 + w_2}\frac{v_1 - 1}{v_1}\left[\frac{v}{v_2\big[(v_1-1)\alpha_1^* + (v_2-1)\alpha_2^* - t\big]^2} - \frac{1}{\alpha_1^{*2}}\right].$$

For convenience, we change back to γ^* and use the facts $w_1 \leq \frac{1}{\sqrt{vv_1}}$ and $w_2 \leq \frac{1}{\sqrt{vv_2}}$. Then

$$\begin{aligned}
\left.\frac{\partial}{\partial \alpha_1} Var_w(\widehat{\tau_i - \tau_{i'}})\right|_{(\alpha_1^*, \alpha_2^*)} &= \frac{w_1 w_2}{w_1 + w_2}\frac{v_1 - 1}{v_1}\left(\frac{1}{vv_2\gamma^{*2}} - \frac{1}{\alpha_1^{*2}}\right)\\
&> \frac{w_1 w_2}{w_1 + w_2}\frac{v_1 - 1}{v_1}\left(\frac{w_2^2}{\gamma^{*2}} - \frac{1}{\alpha_1^{*2}}\right)\\
&> \frac{w_1 w_2}{w_1 + w_2}\frac{v_1 - 1}{v_1}\left(\frac{w_2^2}{\gamma_{\hat{d}}^2} - \frac{1}{\alpha_{\hat{d}1}^2}\right) = 0,
\end{aligned}$$

the last inequality because γ must be negative and thus $\gamma^{*2} < \gamma_{\hat{d}}^2$.

We can similarly get $\left.\frac{\partial}{\partial \alpha_2} Var_{dw}(\widehat{\tau_i - \tau_{i'}})\right|_{(\alpha_1^*, \alpha_2^*)} > 0$. Consequently, the variance of $\widehat{\tau_i - \tau_{i'}}$

for $i \in V_1$ and $i' \in V_2$ is larger for d^* thus for $\hat{d}$. Since all pairwise comparisons with $\hat{d}$ have the same variance,

$$\max_{i \neq i'} Var_{d^*w}(\widehat{\tau_i - \tau_{i'}}) > \max_{i \neq i'} Var_{\hat{d}w}(\widehat{\tau_i - \tau_{i'}}),$$

a contradiction. So, such $C_{d^*} = \bar{C}_d$ does not exist, that is, no d can improve on a weight balanced design of the same trace.

We now know $\hat{d}$ is MV_w-best among designs in $\mathcal{D}$ with fixed trace of C-matrix. It remains to show $\bar{d}$ with maximum trace of C-matrix is MV_w-better than $\hat{d}$. Since both designs are weight balanced, it is sufficient to show that any elementary contrast has lower weighted variance with $\bar{d}$. Now since $t < trace(C_{\bar{d}})$ implies $\hat{\epsilon} < \epsilon$, it follows that

$$\alpha_{\hat{d}1} = \frac{\hat{\epsilon}}{\epsilon}\alpha_{\bar{d}1} < \alpha_{\bar{d}1},$$

$$\alpha_{\hat{d}2} = \frac{\hat{\epsilon}}{\epsilon}\alpha_{\bar{d}2} < \alpha_{\bar{d}2},$$

so $Var_{\bar{d}w}(\widehat{\tau_i - \tau_{i'}}) < Var_{\hat{d}w}(\widehat{\tau_i - \tau_{i'}})$ for any $i \neq i'$ in the same weight group. $\square$

Corollary 5.1. *For the 2-weight problem where $v_1, v_2 \geq 2$, without loss of generality let $w_1 < w_2$. Then, a binary design $\bar{d}$ having $C_{\bar{d}w} = \epsilon(I - f_0 f_0')$ is MV_w-optimal over $\mathcal{D}(v, b, k)$ provided that $\frac{v_1}{v_2} > \frac{w_2}{w_1}(\frac{w_2}{w_1} - 2)$.*

Proof. Write $\frac{w_2}{w_1} = \rho$ for some $\rho > 1$, then

$$\frac{w_1}{w_2} = \frac{1}{\rho} \Rightarrow w_1 < \frac{1}{v} < \frac{1}{\sqrt{vv_1}}.$$

Furthermore,

$$\begin{aligned}
\frac{v_1}{v_2} > \frac{w_2}{w_1}(\frac{w_2}{w_1} - 2) &\Rightarrow v > (\rho - 1)^2 v_2 \\
&\Rightarrow v > (\frac{w_2}{w_1} - 1)^2 v_2 \\
&\Rightarrow \frac{w_2}{(1 - v_2 w_2)/v_1} - 1 < \sqrt{\frac{v}{v_2}} \\
&\Rightarrow w_2 < \frac{1}{\sqrt{vv_2}}
\end{aligned}$$

By Theorem 5.1, $\bar{d}$ is MV_w-optimal since both $w_1 < \frac{1}{\sqrt{vv_1}}$ and $w_2 < \frac{1}{\sqrt{vv_2}}$ hold. $\square$

Note that if $\frac{w_2}{2} \le w_1 < w_2$ holds, then $\frac{v_1}{v_2} > \frac{w_2}{w_1}(\frac{w_2}{w_1} - 2)$ holds. So weighted-balanced designs are MV_w-optimal so long as one weight is not more than twice the other.

Example 5.1. *The following binary* $GGDD(2)$ *design in* $\mathcal{D}(6, 11, 3)$

$$\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\
2 & 4 & 4 & 5 & 4 & 4 & 5 & 4 & 4 & 5 & 5 \\
3 & 5 & 6 & 6 & 5 & 6 & 6 & 5 & 6 & 6 & 6
\end{array}$$

has $V_1 = \{1, 2, 3\}$*,* $V_2 = \{4, 5, 6\}$*,* $\lambda_{11} = 1$*,* $\lambda_{12} = 2$*,* $\lambda_{22} = 4$*, and satisfies* $\lambda_{11}\lambda_{22} = \lambda_{12}^2$*. Thus, for* $w_1/w_2 = \lambda_{12}/\lambda_{22} = 1/2$*, it is weight balanced and* E_w*-optimal over* $\mathcal{D}(6, 3, 11)$ *for* $v_1 = v_2 = 3$*,* $w_1 = 1/9$ *and* $w_2 = 2/9$*. Moreover,* $w_1 < 1/\sqrt{vv_1} = 0.2357$ *and* $w_2 < 1/\sqrt{vv_2} = 0.2357$*, so it is also* MV_w*-optimal over* $\mathcal{D}(6, 3, 11)$*. It can also be checked that* $\frac{v_1}{v_2} > \frac{w_2}{w_1}(\frac{w_2}{w_1} - 2) = 0$ *for* $w_2/w_1 = 2$*.*

In practice, it is not difficult to find (see Chapter 3) E_w-optimal designs that satisfy $C_{dw} = \epsilon(I - f_0 f_0')$. However, of these E_w-optimal designs, only a few satisfy the extra conditions stated in Theorem 5.1 for MV_w-optimality. That is, $w_1 < \frac{1}{\sqrt{vv_1}}$ and $w_2 < \frac{1}{\sqrt{vv_2}}$ also need to be met.

Those MV_w-optimal designs above are optimal with respect to a certain weight setting. However, can we show a class of designs are MV_w-superior to another class of designs in $\mathcal{D}(v, b, k)$ for some weights? The following Theorem will be needed regarding this question.

Theorem 5.2. *For the* 2*-weight problem with* $v_1 = 1$*,* $v_2 = v - 1$ *and* $w_1 + (v-1)w_2 = 1$*, let* $\bar{d} \in \mathcal{D}$ *be a binary* $GGDD(2)$ *with information matrix of the form shown in (2.28). Let* $d^* \in \mathcal{D}$ *be a possibly hypothetical design having completely symmetric* C*-matrix with*

$trace(C_{d^*}) = trace(C_{\bar{d}}) = t$. Then

(i) If $\alpha_{\bar{d}1} < \frac{t}{v}$, d^* is MV_w-better than $\bar{d}$ for $w_1 > \frac{1}{v}$, while $\bar{d}$ is MV_w-better than d^* for $w_1 < \frac{1}{v}$.

(ii) If $\frac{t}{v} < \alpha_{\bar{d}1} < \frac{t}{2}$, $\bar{d}$ is MV_w-better than d^* for $w_1 > \frac{(v-2)t}{(v^2-2)t-2v(v-1)\alpha_{\bar{d}1}}$, while d^* is MV_w-better than $\bar{d}$ for $w_1 < \frac{(v-2)t}{(v^2-2)t-2v(v-1)\alpha_{\bar{d}1}}$.

(Note that $\alpha_{\bar{d}1} = \frac{t}{v}$ implies $C_{d^*} = C_{\bar{d}}$.)

Proof. A g-inverse of $C_{\bar{d}}$ is

$$\begin{aligned} C_{\bar{d}}^- &= \left(C_d + \frac{\alpha_{\bar{d}1}}{v-1}J\right)^{-1} \\ &= \begin{pmatrix} \dfrac{v-1}{\alpha_{\bar{d}1}v} & 0 \\ 0 & \dfrac{(v-1)(v-2)}{(\alpha_{\bar{d}2}+\beta_{\bar{d}2})(v-1)^2-\alpha_{\bar{d}1}}\left[I_{v-1} - \dfrac{(\alpha_{\bar{d}1}-\alpha_{\bar{d}2}-\beta_{\bar{d}2})(v-1)}{\alpha_{\bar{d}1}v(v-2)}J_{v-1}\right] \end{pmatrix}. \end{aligned}$$

Using the properties (2.32) - (2.34) stated in Section 2.2, it follows that

$$Var_{\bar{d}w}(\widehat{\tau_1-\tau_j}) = \frac{w_1w_2}{w_1+w_2}\left[\frac{(v-2)^2}{(v-1)t-v\alpha_{\bar{d}1}} + \frac{1}{\alpha_{\bar{d}1}}\right] \text{ for } j \neq 1,$$

$$Var_{\bar{d}w}(\widehat{\tau_j-\tau_{j'}}) = \frac{w_2(v-1)(v-2)}{(v-1)t-v\alpha_{\bar{d}1}} \text{ for } j \text{ and } j' \in \{2,3,\ldots,v\} \text{ and } j \neq j'.$$

So, the MV_w value for $\bar{d}$ is

$$MV_{\bar{d}w} = \begin{cases} \dfrac{w_1w_2}{w_1+w_2}\left[\dfrac{(v-2)^2}{(v-1)t-v\alpha_{\bar{d}1}} + \dfrac{1}{\alpha_{\bar{d}1}}\right] & \text{if } w_1 > \dfrac{(v-2)\alpha_{\bar{d}1}}{(v-1)t-v\alpha_{\bar{d}1}} \\ \dfrac{w_2(v-1)(v-2)}{(v-1)t-v\alpha_{\bar{d}1}} & \text{otherwise.} \end{cases}$$

The MV_w value for d^* is

$$MV_{d^*w} = \begin{cases} \dfrac{w_1w_2}{w_1+w_2}\dfrac{2(v-1)}{t} & \text{if } w_1 > \dfrac{1}{v}, \\ \dfrac{(v-1)w_2}{t} & \text{otherwise.} \end{cases}$$

Case I. $\alpha_{\bar{d}1} < \frac{t}{v}$.

It is easy to see that $\frac{(v-2)\alpha_{\bar{d}1}}{(v-1)t-v\alpha_{\bar{d}1}} < \frac{1}{v}$. So for $w_1 < \frac{(v-2)\alpha_{\bar{d}1}}{(v-1)t-v\alpha_{\bar{d}1}}$, it follows that

$$MV_{\bar{d}w} = \frac{w_2(v-1)(v-2)}{(v-1)t - v\alpha_{\bar{d}1}} < \frac{w_2(v-1)}{t} = MV_{d^*w},$$

For $\frac{(v-2)\alpha_{\bar{d}1}}{(v-1)t-v\alpha_{\bar{d}1}} < w_1 < \frac{1}{v}$,

$$\frac{w_1}{w_1+w_2} < \frac{1}{2} < \frac{\left[(v-1) - \frac{v\alpha_{\bar{d}1}}{t}\right]\alpha_{\bar{d}1}}{(v-4)\alpha_{\bar{d}1} + t} = \frac{(v-1)/t}{\frac{(v-2)^2}{(v-1)t-v\alpha_{\bar{d}1}} + \frac{1}{\alpha_{\bar{d}1}}}.$$

So,

$$MV_{\bar{d}w} = \frac{w_1 w_2}{w_1+w_2}\left[\frac{(v-2)^2}{(v-1)t-v\alpha_{\bar{d}1}} + \frac{1}{\alpha_{\bar{d}1}}\right] < \frac{w_2(v-1)}{t} = MV_{d^*w}.$$

For $w_1 > \frac{1}{v}$,

$$MV_{\bar{d}w} = \frac{w_1 w_2}{w_1+w_2}\left[\frac{(v-2)^2}{(v-1)t-v\alpha_{\bar{d}1}} + \frac{1}{\alpha_{\bar{d}1}}\right] > \frac{w_1 w_2}{w_1+w_2}\frac{2(v-1)}{t} = MV_{d^*w},$$

since $\left[\frac{(v-2)^2}{(v-1)t-v\alpha_{\bar{d}1}} + \frac{1}{\alpha_{\bar{d}1}}\right] - \frac{2(v-1)}{t} = \frac{(v-1)(t-v\alpha_{\bar{d}1})(t-2\alpha_{\bar{d}1})}{[(v-1)t-v\alpha_{\bar{d}1}]\alpha_{\bar{d}1}t} > 0$.

Case II. $\frac{t}{v} < \alpha_{\bar{d}1} < \frac{t}{2}$. Then

$$\frac{t}{v} < \alpha_{\bar{d}1} < \frac{t}{2}$$

$$\Leftrightarrow (t-2\alpha_{\bar{d}1})(t-v\alpha_{\bar{d}1}) < 0$$

$$\Leftrightarrow (v-1)t^2 - (v-1)(v+2)\alpha_{\bar{d}1}t + 2v(v-1)\alpha_{\bar{d}1}^2 < 0$$

$$\Leftrightarrow (v-1)t^2 - v\alpha_{\bar{d}1}t < (v^2-2)\alpha_{\bar{d}1}t - 2v(v-1)\alpha_{\bar{d}1}^2$$

$$\Leftrightarrow \frac{(v-2)\alpha_{\bar{d}1}}{(v-1)t-v\alpha_{\bar{d}1}} > \frac{(v-2)t}{(v^2-2)t-2v(v-1)\alpha_{\bar{d}1}}.$$

Moreover, $\frac{(v-2)t}{(v^2-2)t-2v(v-1)\alpha_{\bar{d}1}} > \frac{1}{v}$ since the former expression is increasing in $\alpha_{\bar{d}1}$.

For $w_1 < \frac{1}{v}$,

$$MV_{\bar{d}w} = \frac{w_2(v-1)(v-2)}{(v-1)t - v\alpha_{\bar{d}1}} > \frac{w_2(v-1)}{t} = MV_{d^*w}.$$

For $\frac{1}{v} < w_1 < \frac{(v-2)t}{(v^2-2)t-2v(v-1)\alpha_{\bar{d}1}}$,

$$w_1 < \frac{(v-2)t}{(v^2-2)t-2v(v-1)\alpha_{\bar{d}1}}$$

$$\Rightarrow w_2 = \frac{1-w_1}{v-1} > \frac{vt-2v\alpha_{\bar{d}1}}{(v^2-2)t-2v(v-1)\alpha_{\bar{d}1}}.$$

Thus,

$$\frac{w_1}{w_2} < \frac{(v-2)t}{vt-2v\alpha_{\bar{d}1}} \Rightarrow \frac{w_1}{w_1+w_2} < \frac{(v-2)t}{2[(v-1)t-v\alpha_{\bar{d}1}]}.$$

So,

$$MV_{\bar{d}w} = \frac{w_2(v-1)(v-2)}{(v-1)t-v\alpha_{\bar{d}1}} > \frac{w_1w_2}{w_1+w_2}\frac{2(v-1)}{t} = MV_{d^*w}.$$

For $\frac{(v-2)t}{(v^2-2)t-2v(v-1)\alpha_{\bar{d}1}} < w_1 < \frac{(v-2)\alpha_{\bar{d}1}}{(v-1)t-v\alpha_{\bar{d}1}}$,

$$w_1 > \frac{(v-2)t}{(v^2-2)t-2v(v-1)\alpha_{\bar{d}1}} \Rightarrow \frac{w_1}{w_2} > \frac{(v-2)t}{vt-2v\alpha_{\bar{d}1}} \Rightarrow \frac{w_1}{w_1+w_2} > \frac{(v-2)t}{2[(v-1)t-v\alpha_{\bar{d}1}]}.$$

So,

$$MV_{\bar{d}w} = \frac{w_2(v-1)(v-2)}{(v-1)t-v\alpha_{\bar{d}1}} < \frac{w_1w_2}{w_1+w_2}\frac{2(v-1)}{t} = MV_{d^*w}.$$

It can also be checked that

$$\frac{t}{v} < \alpha_{\bar{d}1} < \frac{t}{2}$$

$$\Leftrightarrow (v-1)(t-2\alpha_{\bar{d}1})(t-v\alpha_{\bar{d}1}) < 0$$

$$\Leftrightarrow (v-1)t^2 + [(v-2)^2 - v - 2(v-1)^2]\alpha_{\bar{d}1}t + 2v(v-1)\alpha_{\bar{d}1}^2 < 0$$

$$\Leftrightarrow [(v-2)^2\alpha_{\bar{d}1} - v\alpha_{\bar{d}1} + (v-1)t]t < 2(v-1)[(v-1)t-v\alpha_{\bar{d}1}]\alpha_{\bar{d}1}$$

$$\Leftrightarrow \frac{(v-2)^2}{(v-1)t-v\alpha_{\bar{d}1}} + \frac{1}{\alpha_{\bar{d}1}} < \frac{2(v-1)}{t}.$$

So for $w_1 > \frac{(v-2)\alpha_{\bar{d}1}}{(v-1)t-v\alpha_{\bar{d}1}}$,

$$MV_{\bar{d}w} = \frac{w_1w_2}{w_1+w_2}\left[\frac{(v-2)^2}{(v-1)t-v\alpha_{\bar{d}1}} + \frac{1}{\alpha_{\bar{d}1}}\right] < \frac{w_1w_2}{w_1+w_2}\frac{2(v-1)}{t} = MV_{d^*w}.$$

Note that in Section 2.2, we have stated $\alpha_{\bar{d}1} > \frac{t}{2}$ is impossible. □

We can now give conditions under which weight balance implies MV_w-optimality when $v_1 = 1$.

Theorem 5.3. *Let $\bar{d} \in \mathcal{D}$ be a binary $GGDD(2)$ with information matrix of the form shown in (2.28) and satisfying $\alpha_{\bar{d}1} < \frac{(bk-b)(v-1)}{\sqrt{v}(\sqrt{v}+v-2)}$. Then for the 2-weight problem with $v_1 = 1$, $\bar{d}$ is MV_w-optimal over $\mathcal{D}$ for $w_1 = \frac{(v-2)\alpha_{\bar{d}1}}{(v-1)t-v\alpha_{\bar{d}1}}$, where t is the maximal trace for designs in $\mathcal{D}$.*

Proof. The weighted pairwise variances are given for any binary $GGDD(2)$ in the proof of Theorem 5.2. As shown there, $Var_{\bar{d}w}(\widehat{\tau_1 - \tau_j}) = Var_{\bar{d}w}(\widehat{\tau_j - \tau_{j'}}) = \frac{w_2(v-1)(v-2)}{(v-1)t - v\alpha_{\bar{d}1}}$ if $w_1 = \frac{(v-2)\alpha_{\bar{d}1}}{(v-1)t - v\alpha_{\bar{d}1}}$. Then any other binary $GGDD(2)$ $d^* \in \mathcal{D}(v, b, k)$ can beat $\bar{d}$ only if

$$\frac{w_2(v-1)(v-2)}{(v-1)t - v\alpha_{d^*1}} < \frac{w_2(v-1)(v-2)}{(v-1)t - v\alpha_{\bar{d}1}} \Rightarrow \alpha_{d^*1} < \alpha_{\bar{d}1}$$

Differentiate $Var_w(\widehat{\tau_1 - \tau_j})$ for an arbitrary binary $GGDD(2)$ with respect to α_1:

$$\frac{\partial}{\partial \alpha_1} Var_w(\widehat{\tau_1 - \tau_j}) = \frac{w_1 w_2}{(w_1 + w_2)} \left(\frac{v(v-2)^2}{[(v-1)t - v\alpha_1]^2} - \frac{1}{\alpha_1^2} \right).$$

It follows that $\frac{\partial}{\partial \alpha_1} Var_{dw}(\widehat{\tau_1 - \tau_j}) < 0$ when $\alpha_1 < \frac{t(v-1)}{\sqrt{v}(\sqrt{v}+v-2)} = \frac{(bk-b)(v-1)}{\sqrt{v}(\sqrt{v}+v-2)}$, which implies $Var_{d^*w}(\widehat{\tau_1 - \tau_j}) > Var_{\bar{d}w}(\widehat{\tau_1 - \tau_j})$ as $\alpha_{d^*1} < \alpha_{\bar{d}1}$. So d^* does not exist.

So far we have shown that $\bar{d}$ is MV_w-better than other binary $GGDD(2)$ designs. Once again, applying Theorem 2.7 enables us to rule out all binary non-$GGDD(2)$ designs. It remains to eliminate all nonbinary designs. Let d' be an arbitrary nonbinary design with trace $t' < t$. By Theorem 2.7, $\Phi_{MV}(\bar{C}_{d'w}) \leq \Phi_{MV}(C_{d'w})$, where $\bar{C}_{d'}$ is the C-matrix obtained by permuting $C_{d'}$ within the V_2 group. Moreover, $\frac{t}{t'}\bar{C}_{d'}$ is an information matrix with $GGDD(2)$ form and the same trace as $C_{\bar{d}}$, so $\Phi_{MV}(C_{\bar{d}w}) \leq \Phi_{MV}(\frac{t}{t'}\bar{C}_{d'w}) \leq \Phi_{MV}(\bar{C}_{d'w}) \leq \Phi_{MV}(C_{d'w})$. □

Example 5.2. *This design $\bar{d}$ in $\mathcal{D}(4, 5, 3)$ has been previously seen in Examples 3.1 and 4.1:*

1	1	1	2	2
2	2	3	3	3
3	4	4	4	4

Obviously, $\bar{d}$ is a $GGDD(2)$ with $V_1 = \{1\}$, $V_2 = \{2, 3, 4\}$, $\alpha_{\bar{d}1} = 2$. Furthermore, $\alpha_{\bar{d}1} < \frac{(bk-b)(v-1)}{\sqrt{v}(\sqrt{v}+v-2)} = 15/4$, so $\bar{d}$ is an MV_w- optimal design over $\mathcal{D}(4, 5, 3)$ for $w_1 = 2/11$ and $w_2 = 3/11$. Note $\bar{d}$ is also E_w-optimal over $\mathcal{D}$ for $w_1 = 2/11$ and $w_2 = 3/11$ as shown in Example 3.1.

Theorem 5.4. *Suppose a binary GGDD(2) $\bar{d} \in \mathcal{D}(v,b,k)$ has $\alpha_{\bar{d}1} = \frac{(v-1)t}{\sqrt{v}(\sqrt{v}+v-2)}$. Then $\bar{d}$ is MV_w-optimal over $\mathcal{D}$ for weight $w_1 \in [\frac{1}{\sqrt{v}}, 1)$.*

Proof. Recall in the proof of Theorem 5.3 we get $\frac{\partial}{\partial \alpha_1} Var_w(\widehat{\tau_1 - \tau_j}) = \frac{w_1 w_2}{(w_1+w_2)}\left[\frac{v(v-2)^2}{[(v-1)t - v\alpha_1]^2} - \frac{1}{\alpha_1^2}\right]$. Thus $\frac{\partial}{\partial \alpha_1} Var_w(\widehat{\tau_1 - \tau_j}) = 0$ when $\alpha_1 = \frac{(v-1)t}{\sqrt{v}(\sqrt{v}+v-2)}$. Furthermore, the second order derivative of $Var_w(\widehat{\tau_1 - \tau_j})$ with respect to α_1 is

$$\frac{\partial^2}{\partial \alpha_1^2} Var_w(\widehat{\tau_1 - \tau_j}) = \frac{w_1 w_2}{(w_1 + w_2)}\left(\frac{2v^2(v-2)^2}{[(v-1)t - v\alpha_1]^3} + \frac{2}{\alpha_1^3}\right) > 0.$$

As a result, $Var_w(\widehat{\tau_1 - \tau_j})$ is minimized at $\alpha_{\bar{d}1} = \frac{(v-1)t}{\sqrt{v}(\sqrt{v}+v-2)}$ for any fixed weights. Moreover, it can be checked that $\bar{d}$ is weighted-balanced for $w_1 = 1/\sqrt{v}$, and so when $w_1 \geq \frac{1}{\sqrt{v}}$, $Var_{\bar{d}w}(\widehat{\tau_1 - \tau_j}) \geq Var_{\bar{d}w}(\widehat{\tau_j - \tau_{j'}})$. □

Design $\bar{d}$ in Theorem 5.4 may not be easy to find. The appealing aspect of this result lies in that once such a design is feasible, it will be MV_w-optimal for a continuous interval of weights.

Chapter 6

Proposed Future Work

The purpose of this research is to explore weighted optimality of incomplete block designs. Sufficient conditions for designs being weighted optimal with respect to certain weights have been derived. Promising avenues for future research along these lines include the following directions:

1. This dissertation establishes weight intervals for E_w-optimal designs covering all possible TwC situations with $k = v$ and having smaller weight on the contrast. Future work could attempt to establish weight intervals for larger w_1 (i.e. $w_1 \geq 1/v_1$), including the asymptotic weights of TvC situations. Also, A_w and MV_w-optimal designs having $k = v$ under both TvC and TwC situations can be investigated.

2. A neighborhood of weights for which $GGDD(2)$ designs maintain E_w-optimality in $\mathcal{D}(v, b, k)$ would be highly desirable for experimental situations in which $v_1, v_2 \geq 2$. Also, neighborhoods of weights based on known weighted optimal $GGDD(2)$s could be investigated for weighted A and MV optimality.

3. The intervals of weights shown in Theorem 3.8 should be broadened, if possible. For

example, it is not difficult to verify the following design $d \in \mathcal{D}(6, 10, 3)$

$$\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\
3 & 3 & 3 & 4 & 4 & 5 & 4 & 4 & 5 & 5 \\
4 & 5 & 6 & 5 & 6 & 6 & 5 & 6 & 6 & 6
\end{array}$$

is E_w-optimal for $1 \leq w_1/w_2 \leq 2.\overline{6}$, while the interval of the ratio between w_1 and w_2 given by Theorem 3.8 is $1.731 \leq w_1/w_2 \leq 2.\overline{6}$. The former is more appealing because it indicates d maintains E-optimality even in the unweighted case.

4. The MV_w-optimality criterion is important for TvC situations. However, results on MV_w-optimality are limited in this dissertation. Clearly more work is needed for this important situation. Furthermore, sufficient conditions for designs being MV_w-optimal for specific weights, or for weight intervals, should be explored.

5. Computational tools can be employed to construct weighted optimal designs, not just weight balanced designs as done here. It would be useful to compile a catalog of optimal designs covering a practical range of (v, b, k), and a large weight range, for comparing a control with several treatments.

Appendix A

Table of Designs

The table below contains all weight balanced, binary block designs for the 2-weight problem for v_1, $v_2 \geq 2$, $v \leq 12$, $b \leq 30$, and $k \leq v-1$. The notation $(1,2,4) \times 3$, for example, means 3 copies of a block containing treatments 1, 2 and 4.

design#	block
1	$(1,2,3) \times 3$, $(1,2,4) \times 3$, $(1,2,5) \times 3$, $(3,4,5)$
2	$(1,2,3) \times 4$, $(1,2,4) \times 4$, $(1,2,5) \times 4$, $(1,3,4)$, $(1,3,5)$, $(1,4,5)$, $(2,3,4)$ $(2,3,5)$, $(2,4,5)$, $(3,4,5)$
3	two copies of #1
4	three copies of #1
5	$(1,2,3) \times 5$, $(1,2,4) \times 3$, $(1,2,5) \times 5$, $(1,2,6) \times 5$, $(1,3,4)$, $(1,4,5)$ $(1,4,6)$, $(2,3,4)$, $(2,4,5)$, $(2,4,6)$, $(3,5,6)$, $(3,5,6)$
6	$(1,2,3,4) \times 2$, $(1,2,3,5) \times 2$, $(1,2,3,6) \times 2$, $(1,2,4,5) \times 2$, $(1,2,4,6) \times 2$ $(1,2,5,6) \times 2$, $(3,4,5,6)$
7	two copies of #6
8	$(1,2,3,4) \times 3$, $(1,2,3,5) \times 3$, $(1,2,3,6) \times 3$, $(1,2,4,5) \times 3$, $(1,2,4,6) \times 3$ $(1,2,5,6) \times 3$, $(1,3,4,5)$, $(1,3,4,6)$, $(1,3,5,6)$, $(1,4,5,6)$ $(2,3,4,5)$, $(2,3,4,6)$, $(2,3,5,6)$, $(2,4,5,6)$, $(3,4,5,6)$
9	$(1,2,3)$, $(1,2,4)$, $(1,2,5)$, $(1,2,6)$, $(1,3,4)$, $(1,3,5)$, $(1,3,6)$, $(2,3,4)$ $(2,3,5)$, $(2,3,6)$, $(4,5,6)$
10	two copies of #9

design#	block
11	$(1,2,3)\times 10$, $(1,2,4)\times 2$, $(1,2,5)\times 2$, $(1,2,6)\times 2$, $(1,3,4)\times 2$, $(1,3,5)\times 2$ $(1,3,6)\times 2$, $(2,3,4)\times 2$, $(2,3,5)\times 2$, $(2,3,6)\times 2$, $(4,5,6)$
12	$(1,2,3)\times 4$, $(1,2,4)\times 3$, $(1,2,5)\times 3$, $(1,2,6)\times 3$, $(1,2,7)\times 3$, $(1,4,6)$, $(1,5,7)$ $(2,4,7)$, $(2,5,6)$, $(3,4,5)$, $(3,6,7)$
13	$(1,2,3,4)\times 2$, $(1,2,3,5)\times 2$, $(1,2,3,7)$, $(1,2,4,6)\times 2$, $(1,2,4,7)$, $(1,2,5,6)\times 2$ $(1,2,5,7)$, $(1,2,6,7)$, $(1,3,6,7)$, $(1,4,5,7)$, $(2,3,6,7)$, $(2,4,5,7)$, $(3,4,5,6)$
14	$(1,2,3)$, $(1,4,5)\times 2$, $(1,6,7)\times 2$, $(2,4,6)\times 2$, $(2,5,7)\times 2$, $(3,4,7)\times 2$ $(3,5,6)\times 2$, $(4,5,6)$, $(4,5,7)\times 2$, $(5,6,7)$
15	$(1,2,3)\times 4$, $(1,2,4)\times 2$, $(1,2,5)$, $(1,2,6)$, $(1,2,7)$, $(1,3,4)$, $(1,3,5)\times 2$, $(1,3,6)$ $(1,3,7)$, $(1,6,7)$, $(2,3,4)$, $(2,3,5)$ $(2,3,6)\times 2$, $(2,3,7)$, $(2,5,7)$, $(3,4,7)$, $(4,5,6)$
16	$(1,2,3,4)\times 4$, $(1,2,3,5)\times 4$, $(1,2,3,6)\times 4$, $(1,2,3,7)\times 4$, $(4,5,6,7)$
17	$(1,2,3,4\times 2$, $(1,2,3,5)\times 2$, $(1,2,3,6)\times 3$, $(1,2,3,7)\times 2$, $(1,2,4,5)\times 2$, $(1,2,6,7)$ $(1,3,4,7)\times 2$, $(1,3,5,6)$, $(1,5,6,7)$, $(2,3,4,6)$ $(2,3,5,7)\times 2$, $(2,4,6,7)$, $(3,4,5,6)$
18	$(1,2,3,4)\times 5$, $(1,2,3,5)\times 4$, $(1,2,3,6)\times 4$, $(1,2,3,7)\times 4$, $(1,2,5,6)$ $(1,3,6,7)$, $(1,4,5,7)$, $(2,3,5,7)$, $(2,4,6,7)$, $(3,4,5,6)$
19	$(1,2,3,4)\times 2$, $(1,2,3,5)$, $(1,2,3,6)\times 2$, $(1,2,3,7)\times 2$, $(1,2,3,8)\times 2$, $(1,2,4,5)\times 2$ $(1,2,4,6)\times 2$, $(1,2,4,7)\times 2$, $(1,2,4,8)$, $(1,2,5,6)\times 2$, $(1,2,5,7)\times 2$, $(1,2,5,8)\times 2$ $(1,2,6,7)$, $(1,2,6,8)\times 2$, $(1,2,7,8)\times 2$, $(3,4,5,8)$, $(3,5,6,7)$, $(4,6,7,8)$
20	$(1,2,3,6)\times 2$, $(1,2,3,7)\times 2$, $(1,2,3,8)\times 2$, $(1,2,4,5)\times 3$, $(1,2,6,7)$ $(1,2,6,8)$, $(1,2,7,8)$, $(1,3,4,7)\times 2$, $(1,3,4,8)$, $(1,3,5,6)\times 2$, $(1,3,5,8)$ $(2,3,4,6)\times 2$, $(2,3,4,8)$, $(2,3,5,7)\times 2$, $(2,3,5,8)$, $(4,6,7,8)$, $(5,6,7,8)$
21	$(1,2,3,4)$, $(1,2,3,5)$, $(1,2,3,6)$, $(1,2,3,7)$, $(1,2,3,8)$, $(1,2,4,5)$ $(1,2,4,6)$, $(1,2,4,7)$, $(1,2,4,8)$, $(1,3,4,5)$, $(1,3,4,6)$, $(1,3,4,7)$ $(1,3,4,8)$, $(2,3,4,5)$, $(2,3,4,6)$, $(2,3,4,7)$, $(2,3,4,8)$, $(5,6,7,8)$
22	$(1,2,3,4)$, $(1,2,3,6)\times 2$, $(1,2,3,8)\times 2$, $(1,2,4,6)\times 2$, $(1,2,4,7)\times 2$ $(1,3,4,5)\times 2$, $(1,3,7,8)\times 2$, $(1,4,5,8)\times 2$, $(1,5,6,7)\times 2$, $(2,3,4,5)\times 2$ $(2,3,5,7)\times 2$, $(2,4,7,8)\times 2$, $(2,5,6,8)\times 2$, $(3,4,6,7)\times 2$
23	$(1,2,3)\times 3$, $(1,2,4)$, $(1,2,5)$, $(1,2,6)$, $(1,2,7)$, $(1,2,8)$, $(1,2,9)$, $(1,4,6)$ $(1,4,9)$, $(1,5,6)$, $(1,5,8)$, $(1,7,8)$, $(1,7,9)$, $(2,4,5)$, $(2,4,8)$, $(2,5,7)$ $(2,6,7)$, $(2,6,9)$, $(2,8,9)$, $(3,4,7)$, $(3,5,9)$, $(3,6,8)$
24	$(1,2,3,4)\times 2$, $(1,2,3,6)\times 2$, $(1,2,4,5)$, $(1,2,4,7)$, $(1,2,4,8)$, $(1,2,4,9)$ $(1,2,5,6)$, $(1,2,5,7)$, $(1,2,5,8)\times 2$, $(1,2,6,7)$, $(1,2,6,8)$, $(1,2,6,9)$, $(1,2,7,9)\times 2$ $(1,2,8,9)$, $(1,3,5,9)$, $(1,3,7,8)$, $(2,3,5,9)$, $(2,3,7,8)$, $(4,5,6,7)$, $(4,6,8,9)$

design#	block
25	$(1,2,3,4,5,6)\times 2$, $(1,2,3,4,5,7)\times 2$, $(1,2,3,4,6,8)\times 2$, $(1,2,3,4,7,9)\times 2$ $(1,2,3,4,8,9)\times 2$, $(1,2,3,5,6,9)\times 2$, $(1,2,3,5,7,8)\times 2$, $(1,2,3,5,8,9)\times 2$ $(1,2,3,6,7,8)\times 2$, $(1,2,3,6,7,9)\times 2$, $(4,5,6,7,8,9)$
26	$(1,2,3,4)$, $(1,5,6,7)\times 2$, $(1,5,8,9)$, $(1,6,8,9)$, $(1,7,8,9)$, $(2,5,6,8)\times 2$ $(2,5,7,9)$, $(2,6,7,9)$, $(2,7,8,9)$, $(3,5,6,9)$, $(3,5,7,8)\times 2$, $(3,6,7,9)$ $(3,6,8,9)$, $(4,5,6,9)$, $(4,5,7,9)$, $(4,5,8,9)$, $(4,6,7,8)\times 2$, $(5,6,7,8)$ $(5,6,7,9)$, $(5,6,8,9)$, $(5,7,8,9)$, $(6,7,8,9)$
27	$(1,2,3,4,5)\times 5$, $(1,2,3,4,6)\times 5$, $(1,2,3,4,7)\times 5$, $(1,2,3,4,8)\times 5$ $(1,2,3,4,9)\times 5$, $(5,6,7,8,9)$
28	$(1,2,3,4)$, $(1,2,3,8)$, $(1,2,3,9)$, $(1,2,3,10)$, $(1,2,4,5)$, $(1,2,4,6)$, $(1,2,4,7)$ $(1,2,5,8)$, $(1,2,5,9)$, $(1,2,5,10)$, $(1,2,6,8)$, $(1,2,6,9)$, $(1,2,6,10)$ $(1,2,7,8)$, $(1,2,7,9)$, $(1,2,7,10)$, $(3,5,6,7)$, $(4,8,9,10)$
29	$(1,2,3)\times 4$, $(1,4,5)$, $(1,4,7)$, $(1,5,7)$, $(1,6,9)$, $(1,6,10)$, $(1,8,9)$, $(1,8,10)$ $(2,4,6)$, $(2,4,10)$, $(2,5,6)$, $(2,5,8)$, $(2,7,8)$, $(2,7,9)$, $(2,9,10)$, $(3,4,8)$ $(3,4,9)$, $(3,5,9)$, $(3,5,10)$, $(3,6,7)$, $(3,6,8)$, $(3,7,10)$
30	$(1,2,3,4,7)$, $(1,2,3,5,7)$, $(1,2,3,6,8)$, $(1,2,3,6,9)$, $(1,2,3,7,10)$, $(1,2,3,8,9)$ $(1,2,4,5,6)$, $(1,2,4,9,10)$, $(1,3,4,5,8)$, $(1,3,5,9,10)$, $(1,6,7,8,10)$, $(2,3,4,8,10)$ $(2,3,5,6,10)$, $(2,5,7,8,9)$, $(3,4,6,7,9)$
31	$(1,2,3,4,6)$, $(1,2,3,4,7)$, $(1,2,3,5,7)$, $(1,2,3,5,10)$, $(1,2,3,9,10)$, $(1,2,4,5,6)$ $(1,2,4,5,9)$, $(1,2,4,8,10)$, $(1,3,4,5,8)\times 2$, $(1,3,5,6,9)$, $(1,4,5,7,10)$, $(1,6,7,8,9)$ $(2,3,4,5,6)$, $(2,3,4,8,9)$, $(2,3,5,7,8)$, $(2,4,5,7,9)$, $(2,5,6,8,10)$, $(3,4,5,9,10)$ $(3,4,6,7,10)$
32	$(1,2,3,4)$, $(1,2,3,5)$, $(1,2,4,9)$, $(1,2,5,9)$, $(1,2,6,8)$, $(1,2,6,10)$, $(1,2,7,8)$ $(1,2,7,10)$, $(1,2,11,12)$, $(1,3,10,12)$, $(1,4,7,11)$, $(1,5,6,11)$, $(1,8,9,12)$ $(2,3,8,11)$, $(2,4,6,12)$, $(2,5,7,12)$, $(2,9,10,11)$, $(3,6,7,9)$, $(4,5,8,10)$

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