Chapter 2

Governing Equations

As it can be seen in Figure 1.3, the diaphragm has a variable thickness, but small compared with the other dimensions. This suggests that shell theory might be suitable to formulate the problem. However, since the problem is complex, and we have made already some assumptions in Chapter 1, and because the thickness of the diaphragm is variable, we will use, as basis for analyzing the diaphragm deformation, Continuum Mechanics three-dimensional geometrically exact theory. In this chapter we will review some basic relations solid mechanics and we will define in terms of equations the diaphragm problem. A few constitutive models for rubber will also be discussed.

2.1 Deformation, strain and stress measures

Following [40] we consider a continuum body $\mathcal{S}$. The *initial* or *reference* configuration of the body is defined by the map

$$\Pi_R : \mathcal{S} \longrightarrow \ \Omega_R \subset \Re^3, \tag{2.1}$$

which associates a physical point in $\mathcal{S}$ with a point $\boldsymbol{X} = (X_1, X_2, X_3)$ in Ω_R. Similarly, the *deformed* or *present* configuration of the body is defined by the map

$$\Pi : \mathcal{S} \longrightarrow \ \Omega \subset \Re^3, \tag{2.2}$$

which associates to any material point in $\mathcal{S}$ a point in Ω. The domain Ω_R can be viewed as the domain occupied by the body in the reference configuration, and the domain Ω the domain occupied by the body in present configuration. Usually, in applications, by reference configuration it is understood Ω_R, and by present configuration Ω. We will use this notation. Also

$$\Sigma_R = \partial\Omega_R = \Pi_R(\partial\mathcal{S}), \qquad \Sigma = \partial\Omega = \Pi(\partial\mathcal{S}),$$

where $\partial\mathcal{S}$, $\partial\Omega_R$, $\partial\Omega$ define the boundary of the body, the boundary of Ω_R, and the boundary of Ω respectively. Using the definitions (2.1) and (2.2), the *deformation* of the body is defined by the one-to-one mapping

$$\boldsymbol{\chi} = \Pi \circ \Pi_R^{-1}, \quad \boldsymbol{\chi} : \Omega_R \ \longrightarrow \ \Omega, \quad \boldsymbol{\chi}\,(\boldsymbol{X}) = \boldsymbol{x}.$$

The inverse map $\boldsymbol{\chi}^{-1}$ is defined

$$\boldsymbol{\chi}^{-}1 = \Pi_R \circ \Pi^{-1}, \quad \boldsymbol{\chi}^{-1} : \Omega \ \longrightarrow \ \Omega_R, \quad \boldsymbol{\chi}^{-1}\,(\boldsymbol{x}) = \boldsymbol{X}.$$

The tangent of $\boldsymbol{\chi}$ is denoted by $\boldsymbol{F}$ and is called the *deformation gradient* [40]

$$F_{iA} = \frac{\partial \chi_i(\boldsymbol{X})}{\partial X_A} = \frac{\partial x_i}{\partial X_A}, \qquad J = \det \boldsymbol{F}, \tag{2.3}$$

and J, the determinant of $\boldsymbol{F}$ is called the jacobian of the transformation. Because the deformation $\boldsymbol{\chi}$ can be determined up to a rigid body motion, it can be written

$$\boldsymbol{\chi} = \boldsymbol{X} + \boldsymbol{U},$$

where $\boldsymbol{U}$ represents the displacement field. Associated with the deformation gradient, two important tensors are defined [40]:

- right Cauchy-Green tensor

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}, \qquad C_{AB} = F_{iA}F_{iB}, \tag{2.4}$$

- left Cauchy-Green tensor

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T, \qquad B_{ij} = F_{iA}F_{jA}. \tag{2.5}$$

For the right Cauchy-Green tensor, the principal invariants are defined

$$I_1 = \mathrm{tr}\boldsymbol{C}, \qquad I_2 = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{C})^2 - \mathrm{tr}\boldsymbol{C}^2\right], \qquad I_3 = \det[\boldsymbol{C}] = J^2, \tag{2.6}$$

$$I_1 = C_{AA}, \qquad I_2 = \frac{1}{2}\left(C_{AA}C_{BB} - C_{AB}C_{AB}\right), \qquad I_3 = \epsilon_{ABC}C_{A1}C_{B2}C_{C3},$$

where ϵ_{ABC} is the permutation tensor

$$\epsilon_{ABC} = \begin{cases} +1 & \text{for even permutation of } A, B, C \\ -1 & \text{for odd permutation of } A, B, C \\ 0 & \text{for} A = B, B = C, A = C \end{cases}$$

Corresponding to each tensor (2.4) and (2.5), the following measures for strain can be defined:

- Green-Lagrange tensor

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{C} - \boldsymbol{I}\right), \qquad E_{AB} = \frac{1}{2}\left(C_{AB} - \delta_{AB}\right), \tag{2.7}$$

- Almansi tensor

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{B}^{-1}\right), \qquad \epsilon_{ij} = \frac{1}{2}\left(\delta_{ij} - B^{-1}_{ij}\right). \tag{2.8}$$

Between the Green-Lagrange strains and the Almansi strains the following relation exists

$$\boldsymbol{E} = \boldsymbol{F}^T\boldsymbol{\epsilon}\boldsymbol{F}, \qquad E_{AB} = F_{iA}\epsilon_{ij}F_{jB}. \tag{2.9}$$

The concept of *stress* in the body is introduced through the Cauchy's postulate [40]. We denote by $\boldsymbol{\sigma}$ the Cauchy stress tensor, which is the *true stress* in the body. Additional stress measures can be defined.

- The first Piola-Kirchhoff stress tensor $\boldsymbol{P}$

$$\boldsymbol{P} = J\boldsymbol{\sigma}\left(\boldsymbol{F}^{-1}\right)^{T}, \qquad P_{iA} = J\sigma_{ij}F^{-1}_{Aj}. \tag{2.10}$$

- The second Piola-Kirchhoff stress tensor $\boldsymbol{S}$

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P}, \qquad S_{AB} = F^{-1}_{Ai}P_{iB}. \tag{2.11}$$

While, the first Piola-Kirchhoff stress tensor is unsymmetric, the Cauchy stress and the second Piola-Kirchhoff stress are symmetric

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{T}, \qquad \sigma_{ij} = \sigma_{ji}, \tag{2.12}$$

$$\boldsymbol{S} = \boldsymbol{S}^{T}, \qquad S_{AB} = S_{BA}. \tag{2.13}$$

The relation between the first Piola-Kirchhoff stress tensor and the second Piola-Kirchhoff stress tensor is given by

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{T}, \qquad \sigma_{ij} = \frac{1}{J}F_{iA}S_{AB}F_{jB}. \tag{2.14}$$

For a given problem different strains and stress measures can be used. However, it is natural to consider the stresses and the strains as conjugate quantities in the sense that their product gives mechanical work [5]. It follows that the Cauchy stress works with the Almansi strain, and the Green-Lagrange strain works with the second Piola-Kirchhoff stress tensor. Indeed, if we consider the deformation energy

$$\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\epsilon}\, d\Omega = \int_{\Omega_R} J\frac{1}{J}\boldsymbol{F}^{T}\boldsymbol{S}\boldsymbol{F} : \boldsymbol{\epsilon}\, d\Omega_R = \int_{\Omega_R} \boldsymbol{S} : \boldsymbol{F}^{T}\boldsymbol{\epsilon}\boldsymbol{F}\, d\Omega_R = \int_{\Omega_R} \boldsymbol{S} : \boldsymbol{E}\, d\Omega_R,$$

the above assertions become clear.

2.2 Hyperelastic materials

The relation between stress and strain is given by a *constitutive equation*. Several types of constitutive equations can be formulated [40]. Since the problem addressed by this thesis

involves a body made from rubber, we will focus on the category of *hyperelastic materials*, which are usually used to model rubber. A material is called *elastic* if the first Piola-Kirchhoff stress tensor (2.10) at a point in the body can be expressed as a function of *only* the deformation gradient at that point [40]. A material is called *hyperelastic* if there is a *stored energy function* W depending on $\boldsymbol{X}$,$\boldsymbol{F}$ with $J = \det[\boldsymbol{F}] > 0$ such that

$$\boldsymbol{P} = \frac{\partial W}{\partial \boldsymbol{F}}. \tag{2.15}$$

The function W must satisfy the *principle of frame indifference* or *objectivity*

$$\boldsymbol{Q}W\left(\boldsymbol{X},\boldsymbol{F}\right)\boldsymbol{Q}^{T} = W\left(\boldsymbol{X},\boldsymbol{QF}\right),$$

where $\boldsymbol{Q}$ is an orthogonal rotation matrix. Using the orthogonality condition $\boldsymbol{QQ}^{T} = \boldsymbol{I}$ where $\boldsymbol{I}$ is the 3×3 identity matrix

$$W\left(\boldsymbol{X},\boldsymbol{QF}\right) = W\left(\boldsymbol{X},\boldsymbol{F}\right). \tag{2.16}$$

The condition (2.16) means that, if we rotate the reference configuration system, then the stress must transform by the same rotation. The frame indifference principle implies also the symmetry of the stress tensor. Using the polar decomposition theorem it can be shown that the principle of frame indifference principle also implies that the W depends implicit of $\boldsymbol{F}$ through $\boldsymbol{C}$ [40]. Hence the relation (2.15) can be written in the different form

$$\boldsymbol{S} = 2\frac{\partial W}{\partial \boldsymbol{C}} = \frac{\partial W}{\partial \boldsymbol{E}}, \qquad S_{AB} = 2\frac{\partial W}{\partial C_{AB}} = \frac{\partial W}{\partial E_{AB}}. \tag{2.17}$$

An elastic material is *isotropic* if

$$\boldsymbol{P}\left(\boldsymbol{X},\boldsymbol{FQ}\right) = \boldsymbol{P}\left(\boldsymbol{X},\boldsymbol{F}\right).$$

In the case of hyperelastic materials the isotropy condition is equivalent to

$$W\left(\boldsymbol{X},\boldsymbol{FQ}\right) = W\left(\boldsymbol{X},\boldsymbol{F}\right).$$

By combining the *frame indifference principle* with *isotropy* it can be shown that a hyperelastic material is frame indifferent, homogenous and isotropic if and only if [40]

$$W\left(\boldsymbol{X},\boldsymbol{F}\right) = \Phi\left(\lambda_1,\lambda_2,\lambda_3\right), \tag{2.18}$$

where Φ is a symmetric function of the principal stretches $\lambda_1, \lambda_2, \lambda_3$. The equivalent condition of (2.18) is that $W = W(I_1, I_2, I_3)$ where I_1, I_2, I_3 are the principal invariants of $\boldsymbol{C}$. The second Piola-Kirchhoff stress tensor then can be written

$$\boldsymbol{S} = 2\left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \boldsymbol{C}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \boldsymbol{C}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \boldsymbol{C}}\right),$$

and using the relations [56]

$$\frac{\partial I_1}{\partial \boldsymbol{C}} = \mathbf{1}, \quad \frac{\partial I_2}{\partial \boldsymbol{C}} = I_1\mathbf{1} - \boldsymbol{C}, \quad \frac{\partial I_3}{\partial \boldsymbol{C}} = I_2\mathbf{1} - I_1\boldsymbol{C} + \boldsymbol{C}^2 = I_3\boldsymbol{C}^{-1},$$

where $\mathbf{1}$ is the second order unit tensor, $(\mathbf{1})_{AB} = \delta_{AB}$, $\delta_{AB} = 1$ for $A = B$ and $\delta_{AB} = 0$ for $A \neq B$ we obtain

$$\boldsymbol{S} = 2\left[\left(\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2} + I_2\frac{\partial W}{\partial I_3}\right)\mathbf{1} - \left(\frac{\partial W}{\partial I_2} + I_1\frac{\partial W}{\partial I_3}\right)\boldsymbol{C} + \frac{\partial W}{\partial I_3}\boldsymbol{C}^2\right].$$

Having $\boldsymbol{S}$, the Cauchy stress can be determined using (2.14). In addition, another important tensor defined here is the forth order *material tensor* $\mathbf{C}$. By definition [40]

$$\mathbf{C} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}}, \qquad \mathrm{C}_{ABCD} = \frac{\partial S_{AB}}{\partial C_{CD}}, \tag{2.19}$$

and using the stored energy function

$$\mathbf{C} = 2\frac{\partial^2 W}{\partial \boldsymbol{C}\partial \boldsymbol{C}} = \frac{\partial^2 W}{\partial \boldsymbol{E}\partial \boldsymbol{E}}, \qquad \mathrm{C}_{ABCD} = 2\frac{\partial^2 W}{\partial C_{AB}\partial C_{CD}} = \frac{\partial^2 W}{\partial E_{AB}\partial E_{CD}}. \tag{2.20}$$

From the definition (2.19) and based on the symmetry of $\boldsymbol{S}$ and $\boldsymbol{C}$, it follows that

$$\mathrm{C}_{ABCD} = \mathrm{C}_{BACD} = \mathrm{C}_{ABDC} = \mathrm{C}_{CDAB}. \tag{2.21}$$

The expresion of the material tensor $\mathbf{C}$ in present configuration $\mathbf{c}$ is given by

$$\mathrm{c}_{ijkl} = \frac{1}{J}F_{iA}F_{jB}F_{kC}F_{lD}\mathrm{C}_{ABCD}. \tag{2.22}$$

2.3 Equations of motion

Using the balance of momentum, the equations of motion in the reference configuration are

$$\mathrm{DIV}\,\boldsymbol{P} = \rho_R\ddot{\boldsymbol{U}}, \qquad P_{iA,A} = \rho_R\ddot{U}_A, \qquad \text{in } \Omega_R, \tag{2.23}$$

where ρ_R is the material density in the reference configuration and $\ddot{\boldsymbol{U}}$ the acceleration. In the present configuration the equation of motions are given by [40]

$$\operatorname{div}\boldsymbol{\sigma} = \rho\ddot{\boldsymbol{U}}, \qquad \sigma_{ij,j} = \rho\ddot{U}_i, \qquad \text{in } \Omega, \tag{2.24}$$

where ρ is the material density in the present configuration. In (2.23), by U_A we understand the displacements expressed in reference configuration and in (2.24) by U_i the same displacements but expressed in the present configuration. The balance of moment of momentum yields to the symmetry of the stress tensor

$$\boldsymbol{F}\boldsymbol{P}^T = \boldsymbol{P}\boldsymbol{F}^T, \qquad F_{iA}P_{jA} = P_{iA}F_{jA},$$

in reference configuration, and

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \qquad \sigma_{ij} = \sigma_{ji}$$

The conservation of mass gives the relation between ρ_R and ρ

$$\rho_R = J\rho. \tag{2.25}$$

The boundary conditions in the reference configuration are

$$\boldsymbol{U} = \overline{\boldsymbol{U}}, \qquad \text{on } \Sigma_{RU}, \qquad \boldsymbol{P}\cdot\boldsymbol{N} = \overline{\boldsymbol{T}}, \qquad \text{on } \Sigma_{RT}, \tag{2.26}$$

and the same boundary conditions written in the present configurations are

$$\boldsymbol{U} = \overline{\boldsymbol{U}}, \qquad \text{on } \Sigma_{U}, \qquad \boldsymbol{\sigma}\cdot\boldsymbol{n} = \overline{\boldsymbol{T}}, \qquad \text{on } \Sigma_{T}, \tag{2.27}$$

where

$$\Sigma_R = \Sigma_{RU}\cup\Sigma_{RT}, \qquad \Sigma = \Sigma_U\cup\Sigma_T$$

define the surface of the body in reference and present configurations respectively, and $\boldsymbol{N}$, $\boldsymbol{n}$ are the normals to the boundary Σ_R, and respectively Σ. $\overline{\boldsymbol{U}}$ abd $\overline{\boldsymbol{T}}$, are the prescribed displacements and prescribed tractions respectively. The initial conditions for the equation (2.23) or (2.24) are

$$\boldsymbol{U}|_{t=0} = \boldsymbol{U}_0, \qquad \dot{\boldsymbol{U}}\Big|_{t=0} = \boldsymbol{V}_0,$$

where $\boldsymbol{U}_0$, $\boldsymbol{V}_0$ are the initial displacements and velocities. The equations of motion together with the boundary conditions and the initial conditions, at which we add a set of geometrical equations defining the strain, (2.7) or (2.8), and a constitutive equation (2.17), form a system of nonlinear equations. Solving this system directly is very difficult. In practice the procedure is to linearize the equations of motion and the boundary conditions, and to solve the linearized system. This leads to an *incremental* approach.

2.3.1 Linearization

Applying the linearization procedure as described in section 3.6 at the configuration $\overset{\circ}{\boldsymbol{\chi}}$ to the equation of motion (2.23) yields [40]

$$\mathrm{DIV}\left[\overset{\circ}{\boldsymbol{P}}+\left(\overset{\circ}{\boldsymbol{S}}\otimes\mathbf{1}+2\,\overset{\circ}{\boldsymbol{F}}\overset{\circ}{\mathbf{C}}\overset{\circ}{\boldsymbol{F}}^{T}\right)\cdot\mathrm{GRAD}\,\boldsymbol{u}\right]=\rho_R\left(\overset{\circ}{\boldsymbol{A}}+\ddot{\boldsymbol{u}}\right), \tag{2.28}$$

where $\overset{\circ}{\boldsymbol{A}}$ is the acceleration at the configuration $\overset{\circ}{\boldsymbol{\chi}}$, and $\boldsymbol{u}$ represents the increment of displacements which has to be determined and $\mathrm{GRAD}\,\boldsymbol{u}$ is the gradient of $\boldsymbol{u}$ in the reference configuration, see section 3.6.

$$\mathrm{GRAD}\,\boldsymbol{u}=(\bigtriangledown\boldsymbol{u}\circ\boldsymbol{\chi})\boldsymbol{F}=\bigtriangledown\boldsymbol{u}\cdot\boldsymbol{F},\qquad(\bigtriangledown\boldsymbol{u})_{ij}=\frac{\partial u_i}{\partial x_j}.$$

The corresponding form of (2.28) in the present configuration

$$\mathrm{div}\left[\overset{\circ}{\boldsymbol{\sigma}}+\left(\overset{\circ}{\boldsymbol{\sigma}}\otimes\mathbf{1}+\overset{\circ}{\mathbf{c}}\right)\cdot\bigtriangledown\boldsymbol{u}\right]=\rho\left(\overset{\circ}{\boldsymbol{A}}+\ddot{\boldsymbol{u}}\right). \tag{2.29}$$

In (2.28) and (2.29) $\mathbf{1}$ defines the second order unit tensor, and $\otimes$ is the symbol for the direct product, i.e. $(\boldsymbol{a}\otimes\boldsymbol{b})_{ijkl}=a_{ij}b_{kl}$. $\mathbf{C}$ and $\mathbf{c}$ represents the material tensor written in the reference and present configuration respectively related by (2.22). The application of the linearization process to the boundary conditions (2.26) respectively (2.27) yields

$$\boldsymbol{u}=\overline{\boldsymbol{U}}-\overset{\circ}{\boldsymbol{U}},\qquad\text{on}\quad\Sigma_{RU},$$

$$\left[\left(\overset{\circ}{\boldsymbol{S}}\otimes\mathbf{1}+2\,\overset{\circ}{\boldsymbol{F}}\overset{\circ}{\mathbf{C}}\overset{\circ}{\boldsymbol{F}}^{T}\right)\cdot\mathrm{GRAD}\,\boldsymbol{u}\right]\cdot\boldsymbol{N}=\overline{\boldsymbol{T}}-\overset{\circ}{\boldsymbol{P}}\cdot\boldsymbol{N},\qquad\text{on}\ \Sigma_{RT},$$

in the reference configuration, and

$$\boldsymbol{u} = \overline{\boldsymbol{U}} - \overset{\circ}{\boldsymbol{U}}, \qquad \text{on} \quad \Sigma_U,$$

$$\left[\left(\overset{\circ}{\boldsymbol{\sigma}} \otimes \mathbf{1} + \overset{\circ}{\mathbf{c}}\right) \cdot \bigtriangledown \boldsymbol{u}\right] \cdot \boldsymbol{n} = \overline{\boldsymbol{T}} - \overset{\circ}{\boldsymbol{\sigma}} \cdot \boldsymbol{n}, \qquad \text{on } \Sigma_T,$$

in the present configuration. If the configuration $\overset{\circ}{\boldsymbol{\chi}}$ is a natural (stress free) state, then the first linearization gives the equations of linear elasticity. The linearized equations (2.28) or (2.29) can be solved using the methods of linear elasticity. The total applied load can be divided in a number of steps, and the solution can be obtained step-by-step, the new solution being determined from the previous one. In Chapter 3 we will review the linearization process in details but for a weak formulation of the problem.

2.4 Additive and multiplicative deformation decomposition

An essential part in the finite element model, which will be presented in Chapter 3, is related to decomposition of the deformation energy into volumetric deformation energy and deviatoric deformation energy. To clarify this idea, let us consider the case of linear elasticity. In this case it is known that the strain tensor can be written as a sum between its spherical and deviatoric part

$$\boldsymbol{\varepsilon} = \tilde{\boldsymbol{\varepsilon}} + \frac{1}{3}\Theta\mathbf{1}, \quad \tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} - \frac{1}{3}\Theta\mathbf{1}, \quad \Theta = \operatorname{tr}\boldsymbol{\varepsilon}, \quad \operatorname{tr}\tilde{\boldsymbol{\varepsilon}} = 0, \tag{2.30}$$

where

$$\boldsymbol{\varepsilon} = \bigtriangledown^s \boldsymbol{u}, \qquad \bigtriangledown^s \boldsymbol{u} = \frac{1}{2}\left[\bigtriangledown \boldsymbol{u} + (\bigtriangledown \boldsymbol{u})^T\right], \quad \operatorname{tr}\boldsymbol{\varepsilon} = \varepsilon_{kk}. \tag{2.31}$$

Further let suppose the isotropic case. The deformation energy can be written in this case

$$W = G\tilde{\boldsymbol{\varepsilon}} : \tilde{\boldsymbol{\varepsilon}} + K\Theta^2, \qquad G = \frac{E}{2(1+\nu)}, \qquad K = \frac{E}{3(1-2\nu)}, \tag{2.32}$$

where E is the Young's modulus, G the shear modulus, ν the Poisson's coefficient, and K the bulk modulus. It can be seen that the energy can be split in two parts: a part which is due to the change in shape, and a part which is due to the change in the volume of element of the body. In other words the energy can be decomposed, in a *deviatoric* and *volumetric* part

$$W(\boldsymbol{\varepsilon}) = U(\Theta) + \tilde{W}(\tilde{\boldsymbol{\varepsilon}}), \qquad U(\Theta) = K\Theta^2, \qquad \tilde{W}(\tilde{\boldsymbol{\varepsilon}}) = G\tilde{\boldsymbol{\varepsilon}} : \tilde{\boldsymbol{\varepsilon}}. \tag{2.33}$$

Generalizing this concept of strain splitting and energy decoupling in nonlinear elasticity, Simo, Taylor and Pister [49] and Simo and Taylor [50] introduce the following decomposition of the deformation gradient

$$\boldsymbol{F} = \boldsymbol{F}_{\text{vol}}\tilde{\boldsymbol{F}}, \quad \boldsymbol{F}_{\text{vol}} = J^{1/3}\mathbf{1}, \quad \tilde{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F}, \quad \boldsymbol{C} = J^{2/3}\tilde{\boldsymbol{C}}, \quad \tilde{\boldsymbol{C}} = \tilde{\boldsymbol{F}}^T\tilde{\boldsymbol{F}}. \tag{2.34}$$

From the construction (2.34), it is clear that

$$\det[\boldsymbol{F}_{\text{vol}}] = J, \qquad \det[\tilde{\boldsymbol{F}}] = 1.$$

We note here the difference between (2.30), where there is an additive decomposition, and (2.34), where there is a multiplicative split. The corresponding expression (2.33) can be written in this case

$$W(\boldsymbol{X}, \boldsymbol{C}) = U(J) + \tilde{W}(\boldsymbol{X}, \tilde{\boldsymbol{C}}). \tag{2.35}$$

Equation (2.35) will be used in the finite element formulation in Chapter 3.

2.5 Rubber models

In section 2.3 we have presented the equation of motion assuming that the material is hyperelastic. A class of materials widely associated with the hyperelastic behavior is the class of rubber materials. Based on thermodynamic considerations it can be shown [55] that for rubbers there is a *stored energy function* such that the equations (2.17) hold. Therefore the rubber materials are considered belonging to the class of hyperelastic materials. In this

thesis we will consider only the case of homogenous, isotropic hyperelastic materials and we will use these models to describe the material response of the diaphragm. The behaviour of rubber materials is characterized by the fact that the volume deformation is negligible compared to shear deformation. This leads to the third invariant of the right Cauchy Green tensor (2.6) $I_3 = \det[\boldsymbol{C}] \to 1$ and Poisson's ratio $\nu \to 0.5$. The case $\det[\boldsymbol{C}] = 1, \nu = 0.5$ corresponds to fully incompressible material. There are many forms of the stored energy function proposed in the literature [55]. We can divide these functions into two categories.

- Fully incompressible models

$$I_3 = \det[\boldsymbol{C}] = 1, \qquad \text{and} \qquad \nu = 0.5. \tag{2.36}$$

- Quasi incompressible models

$$I_3 = \det[\boldsymbol{C}] \to 1, \qquad \text{and} \qquad \nu \to 0.5. \tag{2.37}$$

2.5.1 Fully incompressible models

In this case $I_3 = 1$ and consequently the stored energy function $W = W(I_1, I_2)$. A widely used expression for W is the polynomial form [55]

$$W = \sum_{i=0,j=0}^{\infty} C_{ij}(I_1 - 3)^i (I_2 - 3)^j,$$

where C_{ij} are constants determined from experiments. For different values of i, j different models can be obtained [55]. We mention here two models

Neo-Hookeean model In this case $j = 0, i = 1$.

$$W = C_{10}(I_1 - 3).$$

Mooney-Rivlin model In this case $j = 1, i = 1$.

$$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3). \tag{2.38}$$

An important point in establishing the form of W has been made by Ogden [43], who proposed for W a series expansion in the principal stretches

Ogden model

$$W = \sum_n \frac{\mu_n}{\alpha_n} \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3\right),$$

in which λ_i are the principal stretches, α_n may have any values, positive or negative not necessary integers and μ_n are constants.

2.5.2 Quasi incompressible models

These models are obtained from the fully incompressible models by relaxing the conditions (2.36) to the conditions (2.37). This leads to slight compressibility but makes the model more suitable for numerical calculations. The condition (2.37) becomes an internal constraint and can be introduced by the Lagrange multiplier, penalty method or augmented lagrangean method [12]. In Chapter 3 we will describe the introduction of (2.37) in the finite element analysis, and therefore we present below some models implemented in the finite element programs used for our analysis.

Mooney-Rivlin (DYNA3D) In DYNA3D the model is based on the work of Whirley, Engelman and Maker cited from [2]. The strain energy density function in DYNA3D is defined as

$$W = A(I_1 - 3) + B(I_2 - 3) + C\left(\frac{1}{I_3^2} - 3\right) + D(I_3 - 1)^2,$$

where A and B are empirical coefficients determined from experiments,

$$C = \frac{1}{2}A + B, \qquad D = \frac{A(5\nu - 2) + B(11\nu - 5)}{2(1 - 2\nu)},$$

and ν is Poisson's ratio. I_1 and I_2 are the invariants of the right Cauchy-Green tensor $\boldsymbol{C}$ (2.6). For small strains the shear modulus of linear elasticity is given by

$$G = 2(A + B).$$

The recommendation for this material model is to be used for problems involving moderate to large strains keeping the Poisson's ratio $0.48 \leq \nu < 0.5$.

Mooney-Rivlin (NIKE3D) This model is based on the work of Simo and Taylor and uses the decomposition (2.35) for the stored energy function

$$W = A(\tilde{I}_1 - 3) + B(\tilde{I}_2 - 3) + \frac{1}{2}K\left[\ln(\Theta)\right]^2, \quad K = \frac{4(A+B)(1+\nu)}{3(1-2\nu)}, \quad \text{(NIKE3D)}, \quad (2.39)$$

where $\tilde{I}_1, \tilde{I}_2$ are the invariants of $\tilde{\boldsymbol{C}}$ (2.34), Θ the relative volume or the volumetric deformation and K is the bulk modulus. By choosing $\nu \to 0.5$ the bulk modulus becomes a penalty parameter, see Chapter 3. As $\Theta \to 1$ the standard Moony-Rivlin is recovered.

Blatz-Ko model The stored energy function is defined [14]

$$W = \frac{Gf}{2}\left\{J_1 - 3 + \frac{1-2\nu}{\nu}\left[J_3^{-2\nu/(1-2\nu)} - 1\right]\right\} +$$

$$+\frac{G(1-f)}{2}\left\{J_2 - 1 - 3 + \frac{1-2\nu}{\nu}\left[J_3^{2\nu/(1-2\nu)} - 1\right]\right\}, \quad (2.40)$$

where f is an experimental parameter, and

$$J_1 = I_1, \qquad J_2 = I_2/I_3, \qquad J_3 = \sqrt{I_3}.$$

Experimental data indicate $f = 1, \nu = 0.463$ as a good approximation for continuum rubbers [14]. It can be shown that for $f = 1$, the second Piola-Kirchhoff stress tensor is

$$\boldsymbol{S} = G\left(\frac{1}{\Theta}\boldsymbol{C} - \Theta^{-1/(1-2\nu)}\mathbf{1}\right).$$

The Cauchy stress tensor can be calculated with (2.14).

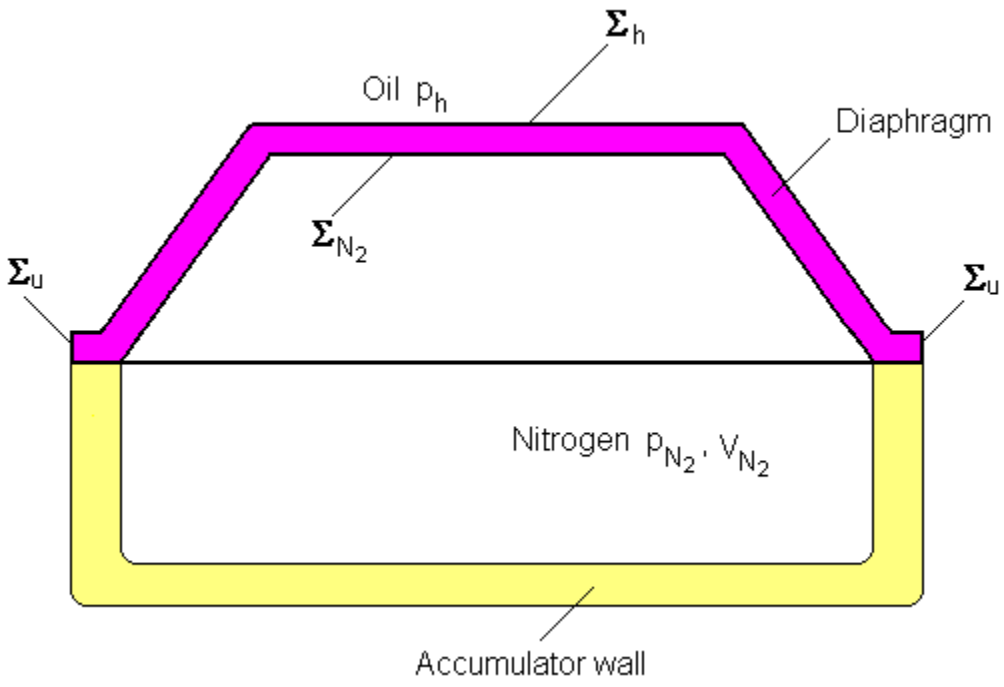

Figure 2.1: The diaphragm domain.

2.6 The diaphragm problem

Based on the physical observations from Chapter 1 and using the continuum equations described in this chapter, we can formulate the diaphragm problem. The diaphragm is made from polyurethane, which belongs to the rubber materials class, and consequently we expect an elastic response which may be in the range of large strains. Therefore we will use the full nonlinear theory as described in the section 2.1.

With reference to Figure 2.1, the equations (2.23) for the diaphragm problem in the reference configuration are

$$\text{DIV}\,\boldsymbol{P} = \rho_R \ddot{\boldsymbol{U}}, \qquad P_{iA,A} = \rho_R \ddot{U}_A, \qquad \text{in } \Omega_R, \tag{2.41}$$

with the following boundary conditions

$$\boldsymbol{U} = 0, \qquad \text{on } \Sigma_U, \tag{2.42}$$

$$\boldsymbol{P} \cdot \boldsymbol{N} = \Delta p \boldsymbol{N}, \qquad \Delta p = p_{N_2} - p_h \qquad \text{on } \Sigma_h. \tag{2.43}$$

Admitting that the diaphragm start from zero, the initial conditions are

$$\boldsymbol{U}|_{t=0} = 0, \qquad \dot{\boldsymbol{U}}\Big|_{t=0} = 0. \tag{2.44}$$

To the equations (2.41), we will add the geometric equation (2.7), a constitutive law given by (2.17) with the stored energy function of the type described in section 2.5 and an evolution law for the nitrogen gas. We assume that the diaphragm is moving *sufficiently slowly* such that at each state, the gas is able to establish its thermal equilibrium, corresponding to every instantaneous position of the diaphragm i.e. a *reversible* evolution. This is true in our case because the speed of the diaphragm is of order of $mm/msec$ and the sound speed is of order of $mm/\mu sec$. Also we consider that the movement of the diaphragm is *sufficiently fast* such that the nitrogen is not able to exchange heat with the external medium (the diaphragm and the walls of the accumulator) i.e. an *adiabatic* evolution. Therefore we can say that the nitrogen has an *isentropic* evolution [36].

$$p V_{N_2}^{\kappa} = \text{constant}, \qquad \kappa = \frac{c_p}{c_V}, \tag{2.45}$$

where V_{N_2} is the enclosed nitrogen volume and c_p, c_V are the specific heat at constant pressure and volume respectively [36]. The link between (2.41) and (2.45) is given by the enclosed volume of nitrogen inside the accumulator, under the diaphragm

$$V_{N_2} = \int_{V_{N_2}(\boldsymbol{u})} dV.$$

In conclusion the diaphragm problem is to find the solution of (2.41) with the boundary conditions (2.42), (2.43), the initial conditions (2.44) and the evolution law (2.45). Due to complexity of the geometry of the diaphragm and also to the equations involved we will seek a numerical solution for this problem using the Finite Element Method.