Chapter 1

Introduction and Background

1.1 Introduction

Over the past several years the numerical approximation of partial differential equations (PDEs) has made important progress because of the rapid development of computer hardware technology and algorithm design. As a result, the numerical models can now give results faster than before. This subject is of particular interest because the analytical knowledge in this area (*e.g.* computational fluid dynamics) is relatively small. Solving some special set of PDEs (hyperbolic equations) accurately and efficiently is a challenge. Therefore, new numerical methods and algorithms need to be developed.

Systems of conservation laws are a special set of PDEs. The notion conservation laws refers to the fact, that they summarize some principal physical laws. Many physical phenomena are represented by such systems. They constitute very powerful and important models for physical phenomena arising from conservation principles. Important examples are namely the conservation of mass, Newton's Second Law, and the conservation of momentum and energy. They arise in important applications throughout science and engineering, including meteorology, oceanography, fluid mechanics (gas dynamics and compressible fluid mechanics), groundwater, acoustics, reactive flows, oil reservoir simulation, astrophysics, traffic modeling, modeling of aerodynamic or geophysical flows, biological and chemical transport, elasticity and plasticity, magnetohydrodynamics, electricity and magnetism, microelectronics, and several related areas. From the mathematical point of view, conservation laws are particularly interesting. They tend to develop discontinuous solutions even from smooth initial values because of the nonlinear nature of the equations. Designing and developing efficient, accurate, and robust numerical methods to address these problems have survived as a classical challenge since the earliest days of digital computation. The basic reasons for difficulty involve the development and tracking of discontinuities.

In this thesis a particular class of conservation laws is considered, namely those of hyper-

bolic type. The hyperbolic nature reflects the fact that these equations model transport or advection phenomena. In contrast to parabolic or elliptic PDEs which describe diffusion or equilibrium respectively, those of hyperbolic type model physical systems dominated by advection, like wave or flow phenomena. The mathematical models describing flow phenomena in the physical world are fully or partly hyperbolic. The Navier-Stokes equations, describing the movement of fluids under influence of body forces and friction, are partly hyperbolic due to the convective terms dominating these equations in general. However, they are also parabolic since friction and hence dissipation play an important role particularly in boundary layers. The Euler equations are the best investigated mathematical model for a hyperbolic system of conservation laws. They describe the flow of compressible gases or liquids at high pressure where viscous effects can be neglected. From the mathematical point of view they are interesting because of their nonlinear behaviour, which implies the development of discontinuous solution from smooth initial data in finite time. Standard numerical methods such as conforming finite element and finite volume methods become either very inefficient or impossible to use. Therefore, the goal is to find numerical schemes for conservation laws that serve different purposes like high accuracy when a shock forms and hence there is a strong need for nonlinear algorithms which reduce the oscillatory behavior of high order schemes which leads to important questions concerning the numerical approximation of this class of PDEs.

1.2 Literature Review

The finite element (FE) method was introduced by engineers in the late 1950′s to solve PDEs arising in structural mechanics. By now, the approximation method of finite elements is very classical. It approximates smooth functions by a linear combination of piecewise smooth basis functions. In order to find approximate solutions to PDEs discretized by finite elements or finite differences, discrete linear systems have to be solved. Several finite element formulations can exist for a given differential equation, but as Johnson [40] states, in general one must (i) transform the differential equation into a variational problem, (ii) discretize the infinite variable problem to a finite dimensional space, (iii) solve the discrete problem, and (iv) implement the method on a computer. The original Galerkin finite element methods were designed in such a way that the approximate solution was forced to be continuous across element boundaries and, as a result, are sometimes called conforming methods. Consult [69] and the references cited therein for more information on FE methods.

A Galerkin method is a FE method where the trial and test spaces are identical. If solutions are not required to be continuous across element interfaces of each element, we call it a discontinuous Galerkin (DG) method. DG method is a powerful approach for large gradient problems (including shocks) and is widely used in engineering applications. Standard continuous Galerkin FE methods have poor stability properties when applied to transport-dominated flow problems, so excessive numerical stabilization is needed. In contrast, the

DG method is known to have good stability properties when applied to first-order hyperbolic problems.

Scientists and engineers use hyperbolic balance and conservation laws to describe a broad range of physical phenomena. Hyperbolic systems are among the most difficult to simulate numerically, especially when shocks are present. DG finite element methods are a relatively new class of high-resolution simulation methods for hyperbolic problems.

Finite difference, finite volume, and finite element schemes have been developed to overcome the difficulties mentioned above to a certain extent. However, they each suffer limitations that can potentially be overcome by the DG method, which combines the best features of each of the more traditional approaches. In this thesis we shall concentrate on the DG method which enforces the PDE, the boundary conditions and continuity across element boundary in a weak sense [18].

The DG methods are a family of locally conservative, stable and high-order accurate methods that are easily coupled with other well-known methods and are well-suited to adaptive strategies. For these reasons, they have attracted the attention of many researchers working in computational mechanics, computational mathematics and computer science. They provide an appealing approach to address problems having discontinuities, such as those arising in hyperbolic conservation laws. The DG method does not require the approximate solutions to be continuous across element boundaries, it instead involve a flux term to account for the discontinuities. The first application of the DG method was introduced by Reed and Hill in 1973 [55] as a technique to solve neutron transport problems

$$\nabla.(\boldsymbol{a}u) + \sigma u = f, \tag{1.1}$$

where σ is a real number and $\boldsymbol{a}$ a constant vector. After this, the name DG appeared in a paper by Lesaint and Raviart [49] in 1974. A remarkable advantage of this method is that, because of the linear nature of the equation, the approximate solution can be computed element by element when elements are suitably ordered according to the characteristic direction. Since then, the DG method was studied for initial-value problems for ordinary differential equations [3, 17, 49, 58, 59]. Lesaint [48] presented the first numerical analysis of the method for a linear advection equation. Cockburn and Shu [26] extended the method to solve first-order hyperbolic PDEs of conservation laws. They also developed the Local Discontinuous Galerkin (LDG) method for convection-diffusion problems [27]. Consult [22] and the references cited therein for more information on DG methods. The DG methods have been used to solve hyperbolic [5, 15], parabolic [31, 32], and elliptic [13, 14] PDEs, to list just a few references. The DG method allows more general mesh configurations and discontinuous bases as illustrated in Figure 1.1, which simplify both *h-refinement* (mesh refinement and coarsening) and *p-refinement* (method order variation). However, for DG methods to be used in adaptive framework one needs *a posteriori* error estimates to guide adaptivity and stop the refinement process.

Moreover, the DG method may also be regarded as a way of extending finite volume methods

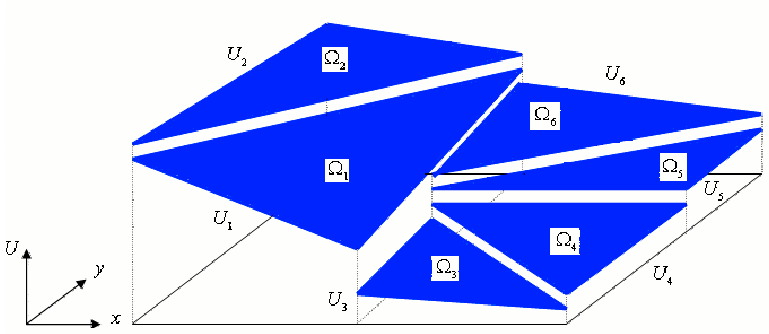

Figure 1.1: Nonconforming mesh with discontinuous approximations.

to arbitrarily high orders of accuracy. The DG method is somewhere between a finite element and a finite volume method and has many good features of both. The solution space consists of piecewise continuous functions (polynomial) relative to a structured or unstructured mesh and this allows solution discontinuities (e.g., shocks) to be captured sharply relative to element boundaries. It maintains local conservation on an elemental basis. The success of the DG method is due to the following properties: (*i*) The DG method simplifies adaptivity since inter-element continuity is neither required for *h*-refinement nor *p*-refinement so do not require continuity across element boundaries, (*ii*) is locally conservative (The method conserves the appropriate physical quantities, *e.g.*, mass, momentum, and energy on an elemental basis.), (*iii*) is well suited to solve problems on locally refined meshes with hanging nodes, (*iv*) is explicit, allowing the approximate solutions to be computed on triangle by triangle, or in parallel within layers of triangles, in accordance with domain of dependence requirements and (*v*) exhibit strong superconvergence of solutions for hyperbolic [3, 6], elliptic [19] and convection-diffusion problems [5] that can be used to estimate the discretization error.

The DG method has a simple communication pattern between elements with a common face that makes it useful for parallel computation (only neighboring elements are required to compute the solution on the current element). Furthermore, it can handle problems with complex geometries to high order.

Regardless of the type of DG method, we need to know how well our computed solution approximates the exact solution to our problem. In practice, the exact solution of the problem is usually not available and a method to approximately evaluate the error needs to be devised. For these reasons, *a posteriori* error estimates have been developed for DG methods and provide some initial guidance for deciding on the degree of the approximation to use and the size of the mesh to use to maintain a prescribed level of accuracy. The preferred approach is to use an *hp*-adaptive method. As Adjerid *et al.* [2] state, these adaptive methods offer greater efficiency, reliability and robustness over the usual methods.

The computational error of a DG method has three sources:

• Galerkin discretization: The Galerkin discretization error arises because the solution is approximated by piecewise polynomials.

• Quadrature: The quadrature error comes from evaluating the integrals arising in the Galerkin formulation using numerical quadrature.

• Solution of the discrete problem: The discrete solution error results from solving the resulting discrete systems only approximately, using Newton's method for example.

Error estimates for Galerkin discretizations come in two forms: *a priori* error estimates and *a posteriori* error estimates.

An *a priori* error estimate relates the error between the exact and the approximate solution to the regularity properties of the exact (unknown) solution. *A priori* estimates are useful for predicting how the discretization error behaves with h-refinement or p-enrichment. Unfortunately, their usefulness in assessing the accuracy of a given numerical solution is limited since the estimate involves unknown constants and the exact solution we are approximating. The error estimates are *a priori* if the error bound already holds before calculating the approximate solution U and without detailed knowledge of U. It has two crucial disadvantages:

1. unless restrictive assumptions on the data are made, the regularity of the exact solution is unknown,

2. the constants in the estimates are either unknown or only given by very pessimistic upper bounds.

Therefore, *a priori* error estimates allow only to estimate the ratio accuracy vs. number of degrees of freedom asymptotically under some regularity assumptions on the (unknown) exact solution, but they are not suitable for a stopping criterion indicating the reach of a given accuracy. In practice, once the finite element solution U is computed, the following questions arise:

(i) Is it possible to use U for computing a bound for a suitable norm of the error $u - U$?

(ii) Is it possible to automatically refine the mesh (h-refinement) and increase the polynomial degree p (p-refinement) respectively, in an optimal fashion?

(iii) When should process stop? (stopping criterion)

These questions can be addressed by using, *a posteriori* error estimates which are computed in terms of the finite element solution U for computing bounds for the error $||u - U||$. The basic idea behind every *a posteriori* error estimate is to use the differential equation because the exact solution u is unknown. Therefore, an error estimate always quantifies how good the finite element solution U satisfies the differential equation.

Let E_Δ be the estimator associated with an element $\Delta \in \Omega_h$ of the triangulation of the given domain. It is called an *a posteriori* estimator if its evaluation depends on both the computed numerical solution and the given data of the problem. The quality of an *a posteriori* error

estimator is often measured by its efficiency index, *i.e.*, the ratio of the estimated error to the true error. An error estimator is said to be efficient if the efficiency index and its inverse remain bounded for all mesh-sizes. We say that an error estimator is asymptotically exact if its efficiency index approaches unity as the mesh-size converges to zero. This means that the error estimator will always converge to the exact error.

By adaptive mesh refinement, we mean that the mesh is refined automatically as the simulation goes on in regions where estimated errors are the largest. Given an *a posteriori* error estimator, an adaptive mesh-refinement process has the following general structure:

i) construct an initial coarse mesh Ω_0 that is a sufficiently good approximation of the geometry of the problem. Put $k := 0$.

ii) solve the discrete problem on Ω_k.

iii) compute an *a posteriori* error estimate for each element $\Delta \in \Omega_k$.

iv) if the estimated global error is sufficiently small, then stop. Otherwise decide which elements have to be refined and hence construct a mesh Ω_{k+1}. Replace k by $k+1$ and return to step (*ii*).

The inconvenience of an adaptive mesh-refinement process is that the number of elements increases rapidly during the simulation. Hence there is a risk of very high CPU time and we often face the problem of exceeding computer capability.

LeSaint and Raviart [49] made the first analysis of the DG method and proved a rate of convergence of $O(h^p)$ in the $\mathcal{L}^2(\Omega)$ norm for general triangulations, where h is the mesh size and a rate of $O(h^{p+1})$ for tensor products of polynomials of degree p in one variable defined on Cartesian grids. Later, Johnson and Pitkarän̈ta [41] proved a rate of convergence of $O(h^{p+1/2})$ for general triangulations of size h and Peterson [54] numerically confirmed this rate to be optimal:

$$||u - u_h||_{\mathcal{L}^2(\Omega)} \leq ch^{p+1/2}||u||_{H^{p+1}(\Omega)}. \tag{1.2}$$

where c is a positive constant, independent of u and h. The estimate (1.2) predicts an $O(h^{1/2})$ gap from the optimal rate of convergence. In practice, however, the optimal $O(h^{p+1})$ rate is frequently seen. Richter [57] obtained the optimal rate of convergence of $O(h^{p+1})$ for some structured two-dimensional non-Cartesian grids (assuming some uniformity in the triangulation).

In all the above papers, the exact solution is assumed to be very smooth. The case in which the solution admits discontinuities was treated by Lin and Zhou [51] who proved the convergence of the method. The issue of the interrelation between the mesh and the order of convergence of the method was explored by Zhou and Lin [67] for $p = 1$, and later by Lin, Yan, and Zhou [50] for $p = 0$, and optimal error estimates were established under suitable assumptions on the mesh. Later, Falk and Richter [36] have obtained a rate of convergence of

$O(h^{p+1/2})$ for general triangulations for Friedrichs systems. Cockburn *et al.* [24] have shown how to postprocess the approximate solution to obtain a rate of convergence of $O(h^{2p+1})$ on Cartesian grids.

For nonlinear systems of conservation laws, variations of the original DG method such as the Runge-Kutta discontinuous Galerkin methods (RKDG) were introduced by Cockburn and Shu [26]. They give an explicit scheme that combines a discontinuous finite element formulation in space and a special total variation diminishing (TVD) Runge-Kutta time discretization, all in conjunction with a generalized slope limiter, where TVD means total variation diminishing. In 1998, Cockburn and Shu [27] introduced the local discontinuous Galerkin method, LDG, for time-dependent convection-diffusion systems and stated that the LDG method is an extension of the RKDG methods for purely hyperbolic systems. They further state that the basic idea for constructing the LDG method is to suitably rewrite the convection-diffusion system into a larger, degenerate, first-order system and then discretize it by the RKDG method.

There are various techniques for error estimation. The error estimator proposed by the authors, the Zienkiewicz-Zhu error estimator [70, 71], is the simplest and computationally most efficient. The essence of this error estimator is to use a recovered more accurate (even higher order) solution from the finite element approximation in place of the exact solution in the computation of the error.

Since we are interested in hyperbolic problems, we will give a brief summary of the development of the DG method along with some *a posteriori* error estimates and superconvergence results as they relate to the work presented here.

Superconvergence phenomena occur when computational solutions converge at certain points, lines, surfaces, or places at higher rates than elsewhere. Superconvergence is somehow related to convergence; that means we identify special places that converge at a faster rate than predicted by global *a priori* error estimates. In the other word the computational solution is more accurate at these places. It is well known that if h is mesh size, the local discretization errors converge at an $O(h^{p+1})$ rate. Superconvergence means that the local discretization errors converge at an $O(h^q)$ rate, where $q > p+1$. The rapid convergence of these points can be used in the design of post-processing techniques to enhance the accuracy of computations or in the design of error indicators for adaptive refinement. This superconvergence behavior of the classical FE method has been analyzed for several years. Superconvergence properties for DG methods have been studied in [3, 29, 42, 49] for ordinary differential equations, [3, 6, 7] for hyperbolic problems and [5, 19, 21] for diffusion and convection-diffusion problems. More generally, superconvergence may be used to (*i*) obtain quantities with greater accuracy, (*ii*) construct asymptotically exact *a posteriori* estimates of discretization errors for linear and nonlinear hyperbolic problems, and (*iii*) detect discontinuities, *e.g.*, to align meshes and/or apply stabilization (limiting).

Since their inception, *a posteriori* error estimates have been an integral part of every adaptive method and are used to guide the adaptive process by indicating regions where more or less

resolution is needed, to assess the quality of the solution and to stop the adaptive process. Flaherty [37] divides *a posteriori* error estimates into four broad classifications:

a) Residual error estimates are found by solving local finite element problems created on either an element or a subdomain.

b) Flux-projection error estimates are found by calculating a new flux via a post processing of the finite element solution and then taking the difference of the new smoother flux and the original flux.

c) Extrapolation error estimates are found by taking the difference of finite element solutions having different orders or different meshes.

d) Interpolation error estimates are used with estimates of the unknown constants.

These four techniques are not necessarily independent of one another. The *a posteriori* error estimate developed in this dissertation is a residual error estimate.

A good *a posteriori* error estimate should:

(*i*) be asymptotically correct and provide an accurate measure of the error in the sense that they converge to the true error under h-and p-refinement.

(*ii*) be computationally inexpensive to compute relative to the cost of obtaining the finite element solution.

(*iii*) robust in the sense that they provide an estimate that is computable in several norms.

The reliability of the *a posteriori* error estimates is measured by the effectivity index which is the ratio of the estimated error to the exact error. Thus a reliable estimate is one for which the effectivity index is close to one. Several *a posteriori* estimation methods produce very good estimates of the actual error, with effectivity indices near unity in many cases, and with remarkably good local error indicators in most of the cases considered. This approach offers several advantages over other methods: (*i*) it is easy to compute; (*ii*) it is asymptotically exact; (*iii*) it provides asymptotically exact estimates one order higher than the current order. *A posteriori* estimators rely on the derivation of computable bounds on the error and may be used to signify where refinement in spatial quantities or polynomial degree may be adaptively modified. They can be particularly useful in applications where solution gradients vary in orders of magnitude across spatial domains, such as those arising in convection dominated transport. Most of the *a posteriori* error estimates for conforming FE methods were focused on global error bounds in the energy norm. *A posteriori* estimates of discretization errors are known for elliptic [8, 9, 10, 11, 12, 72, 71], parabolic [4, 31, 33, 47, 60, 65] and hyperbolic problems [6, 64]. Local contributions to global error estimates furnish error indicators that are typically used to control adaptive enrichment through mesh refinement/coarsening (h-refinement) and/or method order variation (p-refinement). Thus, meshes are refined and/or method orders increased where error indicators are large and an opposite course is taken where error indicators are small.

In 1994 Biswas *et al.* [16] developed a parallel FE method for hyperbolic conservation laws in one and two-dimensions using the DG method to discretize in space with a basis of piecewise Legendre polynomials. In this work they also discovered evidence of superconvergence of the DG method approximate solution at the Radau points and used this result along with p-refinement to construct efficient *a posteriori* error estimates in space. Lowrie [52] showed numerical evidence of an order $O(h^{2p+1})$ accurate component of the DG solution. Cockburn *et al.* [25] used a post-processing technique, involving a convolution with a kernel, applied only once to the approximate solution, to obtain a rate of convergence of $O(h^{2p+1})$ on Cartesian grids when the exact solution is globally smooth.

Recently, Adjerid *et al.* [3] proved that DG solutions of one-dimensional linear and nonlinear hyperbolic problems using $p-$degree polynomial approximations exhibit an $O(h^{p+2})$ superconvergence rate at the roots of Radau polynomial of degree $p+1$ on each element. Radau points are the roots of the sum or difference of successive Legendre polynomials of degree p and $p+1$. The choice of sign fixes one root at the left or right end of each element. They used this result to construct asymptotically correct *a posteriori* error estimates. They further established a strong $O(h^{2p+1})$ superconvergence at the downwind end of every element.

Krivodonova and Flaherty [45] constructed *a posteriori* error estimates that converge to the true error under mesh refinement on unstructured triangular meshes. They showed that the leading term of the local discretization error on triangles having one *outflow* edge is spanned by a suboptimal set of orthogonal polynomials of degree p and $p+1$ and computed DG error estimates by solving local problems involving numerical fluxes, thus requiring information from neighboring *inflow* elements.

Adjerid and Massey [6, 7] proved superconvergence results for multi-dimensional problems on rectangular meshes and presented an error analysis for linear and nonlinear problems. They showed that the leading term in the true local error is spanned by two $(p+1)$-degree Radau polynomials in the x and y directions, respectively. They further showed that a p-degree discontinuous finite element solution exhibits an $O(h^{p+2})$ superconvergence at Radau points obtained as a tensor product of the roots of Radau polynomial of degree $p+1$. For a linear model problem they established that, locally, the solution flux is $O(h^{2p+2})$ superconvergent on average on the $outflow$ element boundary and the global solution flux converges at an $O(h^{2p+1})$ rate on average at the $outflow$ boundary of the domain. They used these superconvergence results to construct asymptotically exact *a posteriori* error estimates for linear and nonlinear hyperbolic problems.

Castillo [19] investigated the superconvergence behavior of the LDG method applied to a two-point elliptic boundary-value problem using the numerical flux proposed in [20]. He showed that on each element the $p-$degree LDG solution gradient is $O(h^{p+1})$ superconvergent at the shifted roots of the $p-$degree Legendre polynomial.

Adjerid and Klauser [5] presented a study of the LDG method for transient convection-diffusion problems in one-dimension. They showed that $p-$degree piecewise polynomial discontinuous finite element solutions of convection-dominated problems are $O(h^{p+2})$ super-

convergent at Radau points. For diffusion-dominated problems, the solutions derivative is $O(h^{p+2})$ superconvergent at the roots of the derivative of Radau polynomial of degree $p+1$. They used these results to construct several asymptotically exact *a posteriori* finite element error estimates.

Celiker and Cockburn [21] uncovered a new superconvergence property of a large class of FE methods for one-dimensional convection-diffusion problems. This class includes DG methods defined in terms of numerical traces, discontinuous Petrov-Galerkin methods and hybridized mixed methods. They proved that the so-called numerical traces of both variables superconverge at all the nodes of the mesh, provided that the traces are conservative, that is, provided they are single-valued. In particular, for a LDG method, they showed that the superconvergence is $O(h^{2p+1})$ when polynomials of degree at most p are used.

Since *a posteriori* error estimates lie in the heart of every adaptive finite element algorithm for differential equations, the subject has attained a certain level of maturity for conforming FE methods applied to elliptic problems [66]. However, *a posteriori* error estimation is much less developed for DG methods applied to hyperbolic problems. The first estimates were developed by Adjerid *et al.* [3] who constructed the first implicit *a posteriori* DG error estimates for one-dimensional linear and nonlinear hyperbolic problems by assuming that the error on each element behaved like a Radau polynomial of degree $p+1$. They further showed that estimators based on Radau polynomials are asymptotically exact for smooth solutions. Later, Adjerid and Massey [6, 7] showed how to construct accurate error estimates for two-dimensional problems on rectangular meshes where they showed that the leading term of error is spanned by two $(p+1)$-degree Radau polynomials in the x and y directions, respectively. They further showed that similar results hold for nonlinear hyperbolic problems. Krivodonova and Flaherty [45] showed that the leading term of the local discretization error on triangles having one *outflow* edge is spanned by a suboptimal set of orthogonal polynomials of degree p and $p+1$ and constructed implicit *a posteriori* error estimates that converge to the true error under mesh refinement on unstructured triangular meshes by solving local problems with boundary conditions and a large polynomial space for the error. They computed the DG error estimates by solving local problems involving numerical fluxes, thus requiring information from neighboring *inflow* elements. These estimates can be used to estimate the total discretization error of space-time solution methods or the spatial error when using a method-of-lines approach. Recently, Adjerid and Klauser [5] used superconvergence properties of LDG solution to construct asymptotically exact *a posteriori* error estimates for one-dimensional convection-diffusion problems. They solved local time-dependent problems for the error estimate using Radau polynomials to compute asymptotically exact *a posteriori* error estimates in several norms. These estimates are also asymptotically correct for nonlinear problems such as the transient viscous Burgers equation. The error estimates that we propose are based on superconvergence. They offer the possibility of providing inexpensive accuracy estimates with properties such as those previously listed.

1.3 Statement of the Problem and Research Objectives

Let us consider a system of hyperbolic conservation laws in several space dimensions of the form [53]

$$\boldsymbol{u}_t + \nabla.\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{u}_t + \sum_{i=1}^{n} \frac{\partial \boldsymbol{F}_i(\boldsymbol{u})}{\partial x_i} = \boldsymbol{r}(\boldsymbol{u}), \quad \boldsymbol{x} \in \mathbb{R}^n, \quad t > 0, \tag{1.3a}$$

$$\boldsymbol{u} = \boldsymbol{g}, \quad at \quad t = 0, \tag{1.3b}$$

with well posed boundary data prescribed on $\partial\Omega$. The subscript t refers to partial differentiation with respect to time, and ∇ is the spacial gradient operator. The components of the solution $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) = [u^1(\boldsymbol{x}, t), ..., u^m(\boldsymbol{x}, t)]^T$ are the densities of various conserved quantities and $\nabla.\boldsymbol{F}(\boldsymbol{u})$ is the divergence of the flux function $\boldsymbol{F}(\boldsymbol{u})$. The rate of change of these densities within a bounded region Ω is given by the flux function $\boldsymbol{F(u)} = [\boldsymbol{F}_1(\boldsymbol{u}), ..., \boldsymbol{F}_m(\boldsymbol{u})]$ which controls the rate of change of $\boldsymbol{u}$ through the boundary of Ω where $\boldsymbol{F}_i(\boldsymbol{u}) : \mathbb{R}^m \to \mathbb{R}^n$ is the i^{th} component of the flux $\boldsymbol{F}$. The function $\boldsymbol{g}$ describes the initial condition of $\boldsymbol{u}$.

For instance, the one-dimensional *Euler equations* for compressible gas flow provides an example of such a system. This system has the following form

$$\rho_t + (\rho v)_x = 0, \quad \text{(conservation of mass)} \tag{1.4a}$$

$$(\rho v)_t + (\rho v^2 + p)_x = 0, \quad \text{(conservation of momentum)} \tag{1.4b}$$

$$(\rho E)_t + (\rho E v + p v)_x = 0. \quad \text{(conservation of energy)} \tag{1.4c}$$

in $\mathbb{R} \times (0, \infty)$, where ρ is the fluid mass density, v the velocity, E is the energy density per unit mass. Evans [35] assumes $E = e + \frac{v^2}{2}$ where e is the internal energy per unit mass and also that the pressure $p = p(\rho, e)$ is a known function of ρ and e.

Although the formulation applies to arbitrary conservation laws, our discussion is restricted to two-dimensional scalar hyperbolic problems. We will consider three scalar model problems

(i) A convection reaction problem

$$\boldsymbol{a}.\nabla u + \phi(u) = \alpha u_x + \beta u_y + \phi(u) = f(x, y), \quad (x, y) \in \Omega = [0, 1]^2, \tag{1.5}$$

subject to the boundary conditions $u(x, 0) = g_0(x)$, $u(0, y) = g_1(y)$, where c, $\alpha \geq 0$, $\beta \geq 0$, $\alpha^2 + \beta^2 > 0$ are real constants and $\phi(u)$ is a smooth function.

(ii) A linear time-dependent problem

$$u_t + \boldsymbol{a}.\nabla u = f(x, y, t), \qquad (x, y) \ \in [0, 1]^2, \ t > 0, \tag{1.6}$$

subject to the initial conditions $u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega = [0, 1]^2$ and boundary conditions $u(x, 0, t) = g_0(x, t), \quad u(0, y, t) = g_1(y, t), \quad t > 0$.

(*iii*) A nonlinear hyperbolic problem of the form

$$h(u)_x + g(u)_y = f(x,y), \quad (x,y) \in \Omega = [0,1]^2, \tag{1.7}$$

subject to the boundary conditions $u(x,0) = h_0(x), \;\; u(0,y) = h_1(y)$, where we assume that $h'(u) > 0$ and $g'(u) > 0$.

Superconvergence and *a posteriori* DG error analysis on triangular meshes are more challenging and less trivial. The first attempt to estimate the DG finite element error on triangular meshes was done by Krivodonova and Flaherty [45]. They showed that the leading term of the local discretization error on each triangle is spanned by $p-$ and $p+1-$degree Dubiner polynomials and estimated the error by solving local problems involving numerical fluxes, thus requiring information from $inflow$ elements. However, no pointwise superconvergence has been observed. Therefore, the idea of finding superconvergence results for hyperbolic problems on triangular meshes is entirely new.

In this thesis we extend the one-dimensional results of Adjerid *et al.* [3] to two-dimensional hyperbolic problems on triangular meshes and the work of Flaherty and krivodonova [45] to prove new superconvergence results for DG solutions. We investigate the superconvergence properties of the DG method applied to scalar first-order hyperbolic PDEs. The analysis is performed for three types of elements. We show how to select the finite element space such that the leading term in the true local error is $O(h^{p+2})$. In particular, we present several new $O(h^{p+2})$ pointwise superconvergence results for hyperbolic problems on triangular meshes consisting of one $outflow$ edge elements as well as on meshes having both one and two $outflow$ edge elements and we show how to construct simple, efficient and asymptotically correct *a posteriori* finite element error estimates of the finite element discretization error using superconvergence of DG solutions.

Suppose a domain Ω has been divided into triangles. We first classify triangular elements into three types: (*i*) elements with one $inflow$ and two $outflow$ edges are of type I, (*ii*) elements with two $inflow$ and one $outflow$ edges are of type II and (*iii*) elements with one $inflow$ edge, one $outflow$ edge, one edge parallel to the characteristics are of type III. This classification will be defined more precisely later. We show that the solution on elements of type I is $O(h^{p+2})$ superconvergent at the two vertices of the $inflow$ edge using an appropriate polynomial space. Moreover, for some spaces bigger than the space of polynomials of degree p and smaller than the polynomial space of degree $p+1$ we discovered additional problem-dependent superconvergence points in the interior of each triangle. On elements of type II, the DG solution is $O(h^{p+2})$ superconvergent at the Legendre points on the $outflow$ edge as well as at interior problem-dependent points using the appropriate polynomial space. On elements of type III , the DG solution is $O(h^{p+2})$ superconvergent at the Legendre points on the $outflow$ edge and for some polynomial spaces the DG solution is $O(h^{p+2})$ at every point of the $outflow$ edge. We also proved the existence of several interior superconvergence points.

We determine explicitly the form of the leading asymptotic term of the local error which

is used to discover new superconvergence properties of the DG method. To maintain the superconvergence results for the global solution one needs to use a $O(h^{p+2})$ approximation of boundary conditions at the $inflow$ boundary of every element. Since we know the $inflow$ boundary conditions this is not a problem for all elements whose $inflow$ boundary is on the $inflow$ boundary of the problem. For the remaining elements we correct the solution by adding an error estimate and use it for $inflow$ boundary condition. We perform several studies with several finite element spaces and investigate their performance. These results are used to construct simple, efficient and asymptotically correct *a posteriori* finite element error estimates for two-dimensional hyperbolic problems in regions where solutions are smooth. Numerical simulations are examined and detailed for several case studies.

As described above, our error estimation procedure is closely tied to the global superconvergence properties. We further use these new results to discover the optimal finite element space for the error on each element and solve a "fully" local Petrov-Galerkin problem that does not require numerical fluxes for the error and thus are much more efficient than the estimates presented in [45]. We further present a numerical study of superconvergence properties for the DG method applied to time-dependent convection problems. Similar results hold for nonlinear hyperbolic problems. We also construct asymptotically correct *a posteriori* error estimates by solving local hyperbolic problems with no boundary conditions on general unstructured meshes. The global superconvergence results are numerically confirmed. Finally, several numerical simulations for both linear and nonlinear problems are performed to show the effectiveness of the error estimate and to validate the theory.

1.4 Organization of the Thesis

In this work we address the numerical approximation of hyperbolic equations using the DG method. Chapter two starts with a presentation of the theoretical framework of conservation laws and a survey on the fundamental problems, ideas and solution approaches. We also recall some theory on the method of characteristics. We specify the types of problems; we introduce most of our notations and present the DG method for multi-dimensional conservation laws. This chapter also includes a condensed overview of orthogonal nodal basis, mappings and numerical quadratures. Finally, we present several approaches concerning the construction of finite element spaces.

Chapter three is the heart of this thesis, since it is dealing with the local and global error analysis. We present the model problem and recall the DG method formulation and state few preliminary results. Notions and theoretical results as well as proofs for the local and global error analysis are introduced. We present the error analysis and we prove the existence of new superconvergence results for all three types of elements using three spaces. We show new $O(h^{p+2})$ pointwise superconvergence results for first-order hyperbolic problems on triangular meshes consisting of one $outflow$ edge elements as well as on meshes having both one and two $outflow$ edge elements. After a summary of our superconvergence results, we discuss

global superconvergence on special triangular meshes and present numerical results. Finally, we provide examples illustrating the error patterns and rates of convergence under mesh refinement.

In chapter four we show how to construct several simple, efficient and asymptotically correct *a posteriori* finite element error estimates for all types of elements using superconvergence of DG solutions. We present efficient techniques to compute asymptotically correct *a posteriori* error estimates. The global superconvergence results are numerically confirmed. Computational results are presented for several linear and nonlinear stationary and transient problems.

In chapter five, the DG is studied for hyperbolic problems on general unstructured meshes. The presentation of the method is followed by a convergence analysis of the spatial discretization error. A fully discrete DG method is presented. The global superconvergence results are numerically confirmed. Finally, the *a posteriori* error estimates are tested on several linear and nonlinear problems to show their efficiency and accuracy under mesh refinement.

Finally, we conclude and discuss our results in chapter six. A short summary and an outline of future perspectives and ongoing research concerning the topic of superconvergence and error estimation are given.