Chapter 4

A Posteriori Error Estimation for Hyperbolic Problems

In this chapter, we construct efficient error estimators using superconvergence properties of the DG solutions of hyperbolic problems. We explicitly write the form of the leading asymptotic term of the local error for each type of elements and corresponding polynomial spaces which lead to new superconvergence properties of the DG method and derive optimal sets of orthogonal polynomials that span the leading term of the error. Moreover, we show that the local superconvergence results in chapter 3 extend to global DG solutions on general meshes when we use a corrected inflow boundary condition. We further present a numerical study of superconvergence properties for the DG method applied to time-dependent convection problems.

We observe that the DG solution does not exhibit any superconvergence on meshes consisting of elements of type I. Thus, elements of type I prevent superconvergence from propagating through the whole mesh. In order to recover the superconvergence results we use a modified DG method with corrected inflow boundary conditions and the augmented space $U_p$.

As described above, we note that our error estimation procedure is closely tied to the global superconvergence properties of the DG solutions. We further use these new results to discover the optimal finite element space for the error on each element and solve a local Galerkin problem that does not require numerical fluxes for the error and thus are much more efficient than the estimates presented in [45]. This property is ideal for parallel computation since no communication between processors are needed.

This chapter is organized as follows, in §4.1 we restate a model problem and recall the DG formulation. In §4.2 we present optimal finite elements for the error. We study transient convection problems in §4.3. In §4.4 we discuss an error estimation procedure and present several numerical examples confirming the $O(h^{p+2})$ pointwise superconvergence rates on all elements for three polynomial spaces in §4.5. Although the theory is not developed for
nonlinear problems and general meshes, computational results indicate they hold for a larger
class of problems and meshes.

4.1 Modified Discontinuous Galerkin Formulation

We consider the model problem (3.1) on general meshes with triangles of type I, II and III.
We approximate \(u(x, y)\) by a piecewise polynomial function \(U(x, y)\) whose restriction to \(\Delta\)
is in \(P_p, V_p\) or \(U_p\). The discrete modified DG method consists of finding \(U \in W_p\) such that

\[
\int_{\Gamma^-} a_n \hat{U} V ds + \int_{\Gamma^+} a_n U V ds + \iint_{\Delta} \nabla \hat{V} U dx dy = \iint_{\Delta} f V dx dy, \quad \forall \; V \in W_p.
\]

(4.1)

where \(W_p\) is either \(P_p, V_p\) or \(U_p\). In order to complete the definition of the DG method we
need to select the flux \(\hat{U}\) on \(\Gamma^-\). On meshes where all elements are of type III one may use
the standard flux \(\hat{U} = U^-\).

However, numerical evidence suggests that the DG solution does not exhibit any super-
convergence on meshes consisting of elements of type I. Thus, elements of type I prevent
superconvergence from propagating through the whole mesh. In order to recover the super-
convergence results we use a modified DG method with corrected inflow boundary conditions
and the augmented space \(U_p\). In the other word, to maintain these superconvergence rates
for the global solution on general meshes we use \(O(h^{p+2})\) approximations of the boundary
conditions at the inflow boundary of every element. This is possible on elements whose inflow
edges are on the inflow boundary of the domain while on the remaining elements we correct
the solution by adding an error estimate and use it as an inflow boundary condition.

Therefore, on meshes including elements of type I and II we will use the higher-order flux

\[
\hat{U} = U^- + E^-,
\]

(4.3)
on all interior inflow edges and \(\hat{U} = u\) on the physical boundary. Here \(E\) is an a posteriori
error estimate that will be discussed later.

The DG orthogonality condition for the local error \(\epsilon = u - U\) can be written on the canonical
triangle \(\Delta\) as (3.13).

In the next Lemma we introduce a set of orthogonal polynomials in \(L^2\) with respect to the
weight function \(w = 1 - \xi - \eta\) on the standard triangle.

Lemma 4.1. The \((i + j)\)-degree polynomials

\[
q^j_i = P^{1,0}_i \left( \frac{2\xi}{1 - \eta} - 1 \right)(1 - \eta)^j P^{2j+2,0}_i (2\eta - 1), \; i, j = 0, 1, \ldots
\]

(4.4a)
satisfy the orthogonality condition

\[ \int_0^1 \int_0^{1-\eta} (1 - \xi - \eta) q_i^j q_m^l \, d\xi \, d\eta = c_{ij}^{lm} \delta_{il} \delta_{jm}, \tag{4.4b} \]

where \( \delta_{ij} \) is the Kronecker symbol equal to 1 if \( i = j \) and 0 otherwise.

Thus, \( \{q_i^j, \, i, j = 0, 1, \ldots, p, \, i + j \leq p\} \) is a basis for \( P_p \).

Proof. Let us compute the inner product

\[ \int_0^1 \int_\Delta (1 - \xi - \eta) q_i^j q_m^l \, d\xi \, d\eta = \int_0^1 \left[ \int_0^{1-\eta} (1 - \xi - \eta) q_i^j q_m^l \, d\xi \right] \, d\eta = \int_0^1 P_{ij}^{2j+2,0}(2\eta - 1)P_i^{2m+2,0}(2\eta - 1)(1 - \eta)^{j+m}G(\eta) \, d\eta, \tag{4.5} \]

where

\[ G(\eta) = \int_0^{1-\eta} (1 - \xi - \eta)P_j^{1,0}(\frac{2\xi}{1-\eta} - 1)P_m^{1,0}(\frac{2\xi}{1-\eta} - 1) \, d\xi. \tag{4.6} \]

Applying the change of variables \( t = 2\eta - 1 \) and \( z = \frac{2\xi}{1-\eta} - 1 \), \( dt = 2d\eta \), \( dz = \frac{2d\xi}{1-\eta} \) and \( 1 - \xi - \eta = \frac{(1-\eta)(1-z)}{2} \) to (4.5) we obtain

\[ \int_\Delta (1 - \xi - \eta) q_i^j q_m^l \, d\xi \, d\eta = \frac{1}{2j+m+5} \int_{-1}^1 \left[ P_i^{2j+2,0}(t)P_i^{2m+2,0}(t)(1 - t)^{j+m+2} \int_{-1}^1 (1 - z)P_j^{1,0}(z)P_m^{1,0}(z) \, dz \right] \, dt = \frac{1}{2j+m+5} \left( \int_{-1}^1 (1 - t)^{j+m+2}P_i^{2j+2,0}(t)P_i^{2m+2,0}(t) \, dt \right) \left( \int_{-1}^1 (1 - z)P_j^{1,0}(z)P_m^{1,0}(z) \, dz \right). \tag{4.7} \]

Applying the orthogonality properties of Jacobi polynomials we prove that

\[ \left( \int_{-1}^1 (1 - t)^{j+m+2}P_i^{2j+2,0}(t)P_i^{2m+2,0}(t) \, dt \right) \left( \int_{-1}^1 (1 - z)P_j^{1,0}(z)P_m^{1,0}(z) \, dz \right) = c_{ij}^{lm} \delta_{il} \delta_{jm}. \]

Thus, \( q_i^j, \, 0 \leq i + j \leq p \) are orthogonal and form a basis for \( P_p \).

In our a posteriori error analysis we need the results from chapter 3. These theorems state the orthogonality conditions for the leading term of the local discontinuous Galerkin error.
4.2 Finite Element Spaces For the Error

In order to construct efficient \textit{a posteriori} error estimates for the DG method we will construct optimal finite spaces for the leading term, $Q_{p+1}$, of the error on each element. We apply several results from chapter 3 on the local error analysis and study each type of element separately for the three spaces $P_p$, $U_p$ and $V_p$. Finally, we present a technique to compute asymptotically correct \textit{a posteriori} estimates of the DG error $e = u - U$ on each element as well as on the whole domain.

4.2.1 Basis functions for element of type I

On triangles of type I there is no optimal set of functions that spans the leading term of the error for $P_p$ and $V_p$. Therefore, we will only consider $U_p$ on the reference triangle of type I defined by the vertices $(1, 0)$, $(1, 1)$ and $(0, 1)$ for $\alpha > 0$ and $\beta > 0$. If we assume the conditions of Theorem 3.6 are satisfied, we show that the leading term $Q_{p+1}$ of local DG finite element error can be written as

$$Q_{p+1}(\xi, \eta) = \sum_{i=1}^{p} c_{p+1-i}^i \chi_{p+1-i}^i(\xi, \eta),$$  \hspace{1cm} (4.8)

where $\chi_{p-i}^i$ are computed using Mathematica in terms of $\varphi_{j-i}^i$, $i, j = 0, ..., p$, and are given in Table 4.1.

Next, we outline a procedure to obtain the error basis functions $\chi_{p-i}^i$. Since Dubiner polynomials form a basis for $P_p$, the finite element error the standard triangle of type I can be approximated by its leading term in $P_p$ as

$$e(\xi, \eta) \approx E(\xi, \eta) = Q_{p+1} = \sum_{i=0}^{p+1} \sum_{j=i}^{p+1} c_{j-i}^i \varphi_{j-i}^i(\xi, \eta),$$  \hspace{1cm} (4.9)

which is defined by $\frac{(p+2)(p+3)}{2}$ parameters.

Applying the $\frac{p^2+3p+6}{2}$ error orthogonality conditions (3.107), (3.108), (3.110) and (3.113) from Theorem 3.6 yields

$$E(\xi, \eta) = \sum_{i=1}^{p} c_{p+1-i}^i \chi_{p+1-i}^i(\xi, \eta),$$  \hspace{1cm} (4.10)

where the functions $\chi_{p-i}^i$, shown in Table 4.1 are given in terms of $\varphi_{j-i}^i$, $i, j = 0, \cdots, p$, and $s = \alpha/\beta$. We display these basis functions in Figure 4.1 for $p = 1 - 4$ and $s = 1$.

We note that the leading term of the error in (4.10) is determined by $p$ parameters.
Table 4.1: Error basis functions for the spaces $\mathcal{U}_p$ for $p = 1$ to $3$ on elements of type I where $s = \alpha/\beta$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\chi^3_1 = (12(\varphi^0_1 + \varphi^1_1)s^2 + 2(10\varphi^0_0 - 2\varphi^1_0 - 3\varphi^0_1 - 2\varphi^2_0 + 12\varphi^1_1 + 5\varphi^0_0)s + 12\varphi^1_0 + 6\varphi^0_1 - 8\varphi^2_0 + 6\varphi^1_0 + 5\varphi^0_0)/10s + 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 2$</td>
<td>$\chi^3_1 = (75(2\varphi^1_0 + \varphi^2_0 + 2\varphi^3_0 + \varphi^1_1)s^4 + 5(28\varphi^0_0 + 28\varphi^1_0 - 42\varphi^2_0 + 12\varphi^3_0 + 54\varphi^1_1 + 51\varphi^0_0 - 12\varphi^1_0 + 96\varphi^2_0 + 51\varphi^0_0) + (140\varphi^0_0 + 532\varphi^0_0 + 126\varphi^0_0 - 68\varphi^0_0 + 36\varphi^1_0 + 140\varphi^0_0 - 180\varphi^0_0 + 540\varphi^2_0 + 315\varphi^2_0 + (140\varphi^0_0 + 280\varphi^1_0 + 100\varphi^2_0 - 12\varphi^1_0 + 10\varphi^2_0 - 180\varphi^0_0 + 240\varphi^2_0 + 165\varphi^2_0)s + 5(16\varphi^1_0 + 6\varphi^2_0 + 6\varphi^0_0 + 6\varphi^1_0 + 6\varphi^2_0 + 6\varphi^0_0))/15(s + 1)^4(5s + 2)$</td>
</tr>
<tr>
<td>$p = 3$</td>
<td>$\chi^3_1 = (420(2\varphi^1_0 + \varphi^2_0 + 2\varphi^3_0 + \varphi^1_1)s^5 + 4(84\varphi^0_0 + 228\varphi^1_0 - 80\varphi^2_0 + 870\varphi^0_0 + 455\varphi^0_0 - 54\varphi^2_0 + 180\varphi^0_0 - 324\varphi^1_0 - 116\varphi^0_0 + 600\varphi^0_0 + 455\varphi^0_0)s^4 + (2352\varphi^0_0 + 2976\varphi^1_0 - 1280\varphi^2_0 + 5520\varphi^1_0 + 3080\varphi^0_0 + 2736\varphi^0_0 - 2592\varphi^1_0 - 2520\varphi^0_0 - 1352\varphi^1_0 + 1596\varphi^2_0 + 2450\varphi^2_0 + 441\varphi^2_0)s^3 + 6(8\varphi^0_0 + 232\varphi^0_0 - 320\varphi^1_0 + 680\varphi^0_0 + 420\varphi^2_0 + 156\varphi^0_0 + 888\varphi^0_0 + 216\varphi^0_0 - 240\varphi^2_0 + 152\varphi^2_0 - 112\varphi^2_0 + 120\varphi^1_0 + 63\varphi^1_0)s^2 + (720\varphi^0_0 + 1440\varphi^0_0 - 1280\varphi^1_0 + 1320\varphi^1_0 + 980\varphi^2_0 - 864\varphi^2_0 + 2160\varphi^0_0 + 1296\varphi^1_0 + 520\varphi^2_0 - 372\varphi^2_0 - 190\varphi^2_0 + 63\varphi^0_0)s^2 - 20(16\varphi^0_0 - 6\varphi^2_0 - 7\varphi^2_0 - 20\varphi^2_0 - 0\varphi^2_0 + 2\varphi^2_0))/140(s + 1)^4(3s + 1)$</td>
</tr>
<tr>
<td>$p = 3$</td>
<td>$\chi^3_1 = (3360(\varphi^1_0 + \varphi^2_0 + \varphi^3_0)s^6 + 14(384\varphi^0_0 + 240\varphi^1_0 - 64\varphi^2_0 + 144\varphi^0_0 + 54\varphi^2_0 - 68\varphi^2_0 + 64\varphi^2_0 + 54\varphi^0_0 + 136\varphi^0_0 + 116\varphi^0_0 + 270\varphi^1_0 - 189\varphi^0_0) + (6720\varphi^0_0 + 29280\varphi^0_0 + 5216\varphi^0_0 + 48240\varphi^0_0 + 13968\varphi^0_0 + 4320\varphi^0_0 - 44712\varphi^1_0 + 3780\varphi^2_0 - 9680\varphi^3_0 + 22320\varphi^2_0 + 17640\varphi^2_0 - 4410\varphi^2_0 - 6(4984\varphi^0_0 + 6248\varphi^1_0 + 1984\varphi^2_0 + 976\varphi^3_0 + 924\varphi^4_0 - 1045\varphi^0_0 + 472\varphi^1_0 + 9108\varphi^0_0 + 3990\varphi^0_0 + 2944\varphi^0_0 + 196\varphi^0_0 - 4200\varphi^0_0 - 441\varphi^0_0)s^3 + (1440\varphi^0_0 - 11232\varphi^0_0 + 13776\varphi^1_0 + 3744\varphi^2_0 + 1536\varphi^3_0 + 1648\varphi^3_0 + 4420\varphi^0_0 - 5184\varphi^0_0 - 10856\varphi^0_0 - 18000\varphi^1_0 + 11610\varphi^3_0 - 5095\varphi^3_0 + (488\varphi^0_0 - 9696\varphi^0_0 - 7324\varphi^0_0 + 11520\varphi^0_0 + 17640\varphi^0_0 - 3600\varphi^0_0 + 14544\varphi^0_0 + 5400\varphi^0_0 + 1200\varphi^0_0 + 2008\varphi_0^0 + 9180\varphi^0_0 + 1530\varphi^0_0 + 2835\varphi^0_0 + 2375\varphi^0_0 + 3(-544\varphi^0_0 + 400\varphi^0_0 + 84\varphi^0_0 + 680\varphi^0_0) - 188\varphi^0_0 - 30\varphi^0_0 + 147\varphi^0_0))/252((s + 1)^4(3s + 1)$</td>
</tr>
</tbody>
</table>
4.2.2 Basis functions for elements of type II

First, we state prove a theorem for the space $\mathcal{P}_p$.

**Theorem 4.1.** Under the conditions of Theorem 3.1 the leading term $Q_{p+1}$ of local DG finite element error on an element of type II defined by the vertices $(0,0)$, $(1,0)$ and $(0,1)$ using the space $\mathcal{P}_p$ can be written as

$$Q_{p+1}(\xi, \eta) = cL_{p+1}(2\eta - 1) + \sum_{i,j \geq 0} c_i^j (1-\xi-\eta)P_i^{2j+2,0}(2\eta - 1)(1-\eta)^j P_j^{1,0}\left(\frac{2\xi}{1-\eta} - 1\right) . \quad (4.11)$$

**Proof.** Applying Theorem 3.2 we write

$$Q_{p+1}(\xi, \eta) = cL_{p+1}(2\eta - 1) + (1-\eta-\xi)\tilde{q}_p(\xi, \eta), \quad (4.12)$$

where $\tilde{q}_p \in \mathcal{P}_p$ and $L_{p+1}$ is the $(p+1)-$th degree Legendre polynomial on $[0,1]$.

Applying Lemma 3.1 we can write

$$\tilde{q}_p = \sum_{i,j \geq 0} c_i^j P_i^{2j+2,0}(2\eta - 1)(1-\eta)^j P_j^{1,0}\left(\frac{2\xi}{1-\eta} - 1\right) . \quad (4.13)$$

The orthogonality condition (3.33) yields

$$\int_0^1 \int_0^{1-\eta} (1-\xi-\eta)\tilde{q}_p V d\xi d\eta = 0, \forall V \in \mathcal{P}_{p-1}. \quad (4.14)$$
Combining (4.13) and (4.14) establishes (4.11) and completes the proof.

We observe that the error space belongs to space spanned by

\[\{L_{p+1}(2\eta - 1)\} + (1 - \eta - \xi)\{P^{2j+2,0}_i(2\eta - 1)(1 - \eta)^jP^{1,0}_j(\frac{2\xi}{1 - \eta} - 1), i, j = 0, ..., p, i + j = p\}.\]

We display these basis functions in Figure 4.2 for \(p = 0 - 4\).

Figure 4.2: Basis functions for the space \(P_p\) for \(p=1\) to 4 on elements of type II.

Another basis for the leading term of the error on a triangle of type II with the space \(P_p\) may be obtained by applying Theorem 3.1 where local finite error on an element \(\Delta\) of type II can be approximated by (3.36). On the outflow edge \(\eta = 1 - \xi\), \(Q_{p+1}\) is orthogonal to \(\varphi^i_{p-i}, i = 0, \cdots, p, i, e,\)

\[\int_0^1 Q_{p+1}(\xi, 1 - \xi)\varphi^i_{p-i}(\xi, 1 - \xi)d\xi = 0, \quad i = 0, ..., p.\]

We note that (3.36) involves \(2p + 3\) parameters. Using (4.16), the \(p + 1\) coefficients \(c^p_i, i = 0, \cdots, p\), can be expressed in terms of \(c^{p+1}_i, i = 0, \cdots, p + 1\), thus, \(Q_{p+1}\) can be written as

\[Q_{p+1} = \sum_{i=0}^{p+1} c^{p+1}_i \chi^{i}_{p+1-i}(\xi, \eta),\]
where the functions $\chi_{p+1,i}^j$, computed using *Mathematica*, are given in Table 4.2 in terms of $\varphi_{p-i}^i$ and $\varphi_{p+1-i}^i$.

Thus, $Q_{p+1}$ is determined by the $p + 2$ parameters, $d_i^{p+1}$, $i = 0, \ldots, p + 1$.

**Table 4.2:** Error basis functions for the spaces $P_p$ for $p = 0$ to 3 on elements of type II.

<table>
<thead>
<tr>
<th>$p = 0$</th>
<th>$\chi_0^0 = -\varphi_0^0 + \varphi_1^0$</th>
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<tbody>
<tr>
<td></td>
<td>$\chi_0^1 = -\frac{3}{2}\varphi_0^1 + \varphi_0^1$</td>
</tr>
<tr>
<td>$p = 1$</td>
<td>$\chi_0^0 = -\frac{3}{2}\varphi_0^0 + \varphi_1^0 + \varphi_0^0$</td>
</tr>
<tr>
<td></td>
<td>$\chi_0^1 = -\frac{3}{2}\varphi_0^1 - \frac{3}{2}\varphi_0^0 + \varphi_1^1$</td>
</tr>
<tr>
<td>$p = 2$</td>
<td>$\chi_0^0 = -10\varphi_0^0 + 6\varphi_1^1 - \frac{8}{3}\varphi_0^2 + \varphi_0^0$</td>
</tr>
<tr>
<td></td>
<td>$\chi_0^1 = -\frac{5}{2}\varphi_0^1 + \varphi_1^2$</td>
</tr>
<tr>
<td>$p = 3$</td>
<td>$\chi_0^0 = -\frac{5}{2}\varphi_0^0 + \varphi_1^1 - \frac{1}{2}\varphi_1^0 + \varphi_0^3 + \varphi_0^0$</td>
</tr>
<tr>
<td></td>
<td>$\chi_0^1 = -\frac{3}{8}\varphi_0^1 + \frac{18}{3}\varphi_0^2 - \frac{9}{3}\varphi_0^3 + \varphi_1^2$</td>
</tr>
<tr>
<td></td>
<td>$\chi_0^2 = -\frac{1}{2}\varphi_0^2 - \frac{3}{4}\varphi_0^0 + \varphi_2^1$</td>
</tr>
<tr>
<td></td>
<td>$\chi_0^3 = -\frac{3}{8}\varphi_0^3 + \frac{1}{4}\varphi_0^1 + \varphi_0^4$</td>
</tr>
</tbody>
</table>

In the following theorem we establish the asymptotic behavior of the DG error for the space $V_p$.

**Theorem 4.2.** Under the conditions of Theorem 3.1 the leading term $Q_{p+1}$ of local DG finite element error on an the reference element of type II defined by the vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ using the space $V_p$ can be written as (4.11) such that $C = [c_p^{p-1}, \ldots, c_p^0]^T$ is solution of the linear system $AC = B$ where

$$a_{ij} = \int_\Delta a.\nabla[(1 - \xi - \eta)q_j^i]c_i^j\eta^{p+1-i}d\eta d\xi, \ i, j = 1, \ldots, p,$$  \hspace{1cm} (4.18a)

$$b_i = (-c\beta \omega_i - c_0^p\gamma_i), \ i = 1, \ldots, p,$$  \hspace{1cm} (4.18b)

with

$$\gamma_i = \int_\Delta a.\nabla[(1 - \xi - \eta)q_i^0]\xi^i\eta^{p+1-i}d\eta d\xi, \ i = 1, \ldots, p,$$  \hspace{1cm} (4.18c)

$$\omega_i = \int_\Delta \frac{dL_{p+1}(2\eta - 1)}{d\eta}\xi^i\eta^{p+1-i}d\eta d\xi, \ i = 1, \ldots, p.$$  \hspace{1cm} (4.18d)
Proof. Since \( P_p \subset \mathcal{V}_p \), \( Q_{p+1} \) satisfies (4.11), which is combined with (3.83) to obtain the following algebraic system

\[
c\beta \omega_i + \sum_{j=0}^{p} c_j^{p-j} a_{ij} = 0, \quad i = 1, \ldots, p.
\]

Solving for \( c_1^{p-1}, \ldots, c_p^0 \), completes the proof.

Another set of basis functions can be found by writing the leading term \( Q_{p+1} \) as

\[
Q_{p+1}(\xi, \eta) = c_1^{p+1} \chi_{p+1}^0(\xi, \eta) + c_2^{p+1} \chi_{p+1}^0(\xi, \eta),
\]

where \( \chi_{p+1}^0 \) and \( \chi_{p+1}^0 \) are given in Table 4.3 in terms of \( \varphi_{p-i}^i \), \( \varphi_{p-i}^j \) and \( s = \alpha/\beta \). These basis functions are obtained by combining (4.17) and (3.83).

Table 4.3: Error basis functions for the spaces \( \mathcal{V}_p \) for \( p = 1 \) to \( 4 \) on elements of type II, \( s = \alpha/\beta \).

| \( p = 1 \) | \( \chi_2^0 = -\varphi_0^2 - \frac{7\varphi_1^0}{8} - \frac{3\varphi_2^0}{2} + \varphi_0^1 + \varphi_1^1 \\ \chi_0^2 = \frac{1}{8}(4s - 6)\varphi_0^2 - 6s\varphi_0^3 + 14\varphi_0^4 + 10\varphi_2^0) \\ \chi_3^0 = \frac{1}{8}(160\varphi_0^3 s^2 + 116\varphi_0^3 s + 150\varphi_1^3 s - 4(30s^2 + 48s + 19)\varphi_0^3 - 6(10s + 11)\varphi_1^3 - 5\varphi_0^2 + 64\varphi_0^3 + 60\varphi_1^3 + 30\varphi_2^0) \\ \chi_0^3 = \frac{1}{8}(160\varphi_0^3 s^2 - 2320\varphi_0^3 s^2 - 840\varphi_0^3 s^2 + 490\varphi_0^3 s - 1760\varphi_0^3 s - 1320\varphi_1^3 s - 140\varphi_3^0 + 1 + 40(35s^2 + 41x^2 + 37x + 11)\varphi_0^3 + 12(140s^2 + 115s + 43)\varphi_1^3 + 210\varphi_2^0 + 56\varphi_3^0 - 568\varphi_0^3 - 492\varphi_2^0 - 150\varphi_3^0 - 7\varphi_0^3) \\ \chi_0^4 = \frac{1}{8}(160\varphi_0^3 s^2 - 400\varphi_0^3 s^2 + 120\varphi_0^3 s^2 - 70\varphi_0^3 s + 992\varphi_0^3 s - 360\varphi_0^3 s + 20\varphi_3^0 - 8(25s^2 - 85s - 65s - 63)\varphi_0^3 - 60(4s^2 - 15s - 7)\varphi_1^3 + 290\varphi_2^0 + 36\varphi_3^0 - 504\varphi_0^3 - 588\varphi_1^3 - 70\varphi_2^0 - 63\varphi_3^0) |
Theorem 4.3. Under the conditions of Theorem 3.1 the leading term $Q_{p+1}$ of local DG finite element error on an the reference element of type II defined by the vertices (0,0), (1,0) and (0,1) using the space $\mathcal{U}_p$ can be written as (4.11) where

$$c_{\mathcal{P}}^p = \frac{\lambda_0(c\mu - (p + 1)s \sum_{i=1}^{p-1} c_i^{p-i} \kappa_i) - s\kappa_p(c(s + p + 2)\nu - (p + 1) \sum_{i=1}^{p-1} c_i^{p-i} \lambda_i)}{s(p + 1)(\kappa_0 \lambda_p - \kappa_p \lambda_0)}, \quad (4.21a)$$

$$c_{\mathcal{P}}^0 = \frac{-\lambda_0(c\mu - (p + 1)s \sum_{i=1}^{p-1} c_i^{p-i} \kappa_i) + s\kappa_0(c(s + p + 2)\nu - (p + 1) \sum_{i=1}^{p-1} c_i^{p-i} \lambda_i)}{s(p + 1)(\kappa_0 \lambda_p - \kappa_p \lambda_0)}, \quad (4.21b)$$

with

$$s = \frac{\alpha}{\beta}, \quad \mu = \int_0^1 L_{p+1}(1 - 2\xi)\xi^{p+1}d\xi, \quad \kappa_i = \int_{\Delta} (1 - \xi - \eta)q_i^{p-i}\xi d\eta d\xi$$

$$\nu = \int_0^1 L_{p+1}(2\eta - 1)\eta^{p+1}d\eta, \quad \lambda_i = \int_{\Delta} (1 - \xi - \eta)q_i^{p-i}\eta d\eta d\xi.$$ 

Proof. First we note that the leading term $Q_{p+1}$ of the error satisfies the following orthogonality conditions

$$\int_{\Gamma^+} a \cdot n Q_{p+1} V ds - \int_{\Delta} a \cdot \nabla V Q_{p+1} d\xi d\eta = 0, \quad \forall V \in \mathcal{U}_p, \quad (4.22)$$

Testing against $V = \xi^{p+1}, \eta^{p+1}$ leads to

$$(\alpha + \beta) \int_0^1 Q_{p+1}(\xi, 1 - \xi)\xi^{p+1}d\xi - \alpha(p + 1) \int_0^1 \int_{1-\xi} Q_{p+1}(\xi, \eta)\xi d\eta d\xi = 0, \quad (4.23)$$

and

$$(\alpha + \beta) \int_0^1 Q_{p+1}(1 - \eta, \eta)\eta^{p+1}d\eta - \beta(p + 1) \int_0^1 \int_{1-\xi} Q_{p+1}(\xi, \eta)\eta d\eta d\xi = 0. \quad (4.24)$$

Combining these orthogonality conditions with (4.11) yields the algebraic system

$$c\beta \int_0^1 L_{p+1}(1 - 2\xi)\xi^{p+1}d\xi - \alpha(p + 1) \sum_{i=0}^p c_i^{p-i} \int_{\Delta} (1 - \xi - \eta)q_i^{p-i}\xi d\eta d\xi = 0, \quad (4.25)$$

and

$$c(\alpha + (p + 2)\beta) \int_0^1 L_{p+1}(2\eta - 1)\eta^{p+1}d\eta - \beta(p + 1) \sum_{i=0}^p c_i^{p-i} \int_{\Delta} (1 - \xi - \eta)q_i^{p-i}\eta d\eta d\xi = 0. \quad (4.26)$$

Solving for $c_{\mathcal{P}}^p$ and $c_{\mathcal{P}}^0$ completes the proof. \qed
We construct another set of shape functions for \( Q_{p+1} \) in terms of \( \varphi_{p-i}^i \) and \( \varphi_{p+1-i}^i \). Noting that (4.17) still holds and using (3.53) and (3.54), the leading term of local finite error on a reference element of Type II can be written as

\[
Q_{p+1} = \sum_{i=1}^{p} q_{p+1}^{i} \chi_{p+1-i}^i(\xi, \eta),
\]

(4.27)

where the \( \chi_{p+1-i}^i \), computed using Mathematica, are given in Table 4.4 in terms of \( s = \alpha/\beta \), \( \varphi_{p-i}^i \) and \( \varphi_{p+1-i}^i \).

We note that \( Q_{p+1} \) in (4.27) is determined by \( p \) parameters.

Table 4.4: Error basis functions for the spaces \( U_p \) for \( p = 1 \) to 4 on elements of type II, \( s = \alpha/\beta \).

<table>
<thead>
<tr>
<th>( p = 1 )</th>
<th>( \chi_1^i = ((-24 \varphi_0^i - 12 \varphi_0^i + 16 \varphi_0^i + 18 \varphi_0^i + 5 \varphi_0^i + 3 \varphi_0^i - 3(8 \varphi_0^i + 12 \varphi_0^i + 8 \varphi_0^i) - 18 \varphi_0^i - 5 \varphi_0^i)s - 24 \varphi_0^i + 6 \varphi_0^i + 15 \varphi_0^i)(s^2 + 3s + 3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p = 2 )</td>
<td>( \chi_2^i = ((-32 \varphi_0^i + 18 \varphi_0^i + \varphi_0^i - 24 \varphi_0^i - 24 \varphi_0^i - 6 \varphi_0^i)(s^2 + 4s + 6) )</td>
</tr>
<tr>
<td>( p = 3 )</td>
<td>( \chi_3^i = ((-32 \varphi_0^i + 30 \varphi_0^i + 7 \varphi_0^i - 40 \varphi_0^i - 24 \varphi_0^i - 2 \varphi_0^i)(s^2 - 5s + 10) )</td>
</tr>
<tr>
<td>( p = 4 )</td>
<td>( \chi_4^i = ((-48 \varphi_0^i + 30 \varphi_0^i - 3 \varphi_0^i + 4 \varphi_0^i + 36 \varphi_0^i + 8 \varphi_0^i)(s^2 - 6s + 15) )</td>
</tr>
</tbody>
</table>
4.2.3 Basis functions for elements of type III

In this section we state and prove results on the leading term of the local error by characterizing the optimal polynomial space and determining a basis for the leading term of the local error.

**Theorem 4.4.** Under the conditions of Theorem 3.1 let $Q_{p+1}$ be the leading term of local DG finite element error on an element of type III defined by the vertices $(0,0)$, $(1,0)$ and $(0,1)$ with $\alpha > 0$ and $\beta = 0$. Then,

For the space $\mathcal{P}_p$ we have

$$Q_{p+1} = cL_{p+1}(2\eta - 1) + \sum_{i,j \geq 0 \atop i+j=p} c_i^j (1 - \xi - \eta)P_i^{2j+2,0}(2\eta - 1)(1 - \eta)^i P_j^{1,0} \left(\frac{2\xi}{1 - \eta} - 1\right). \quad (4.28)$$

For the space $\mathcal{U}_p$ we have

$$Q_{p+1} = \sum_{i,j \geq 0 \atop i+j=p} c_i^j (1 - \xi - \eta)P_i^{2j+2,0}(2\eta - 1)(1 - \eta)^i P_j^{1,0} \left(\frac{2\xi}{1 - \eta} - 1\right), \quad (4.29a)$$

with

$$c_0^p = \frac{1}{p+1} \sum_{i=1}^{p} (-1)^{i+1}(p+1-i)c_i^{p-i}. \quad (4.29b)$$

For the space $\mathcal{V}_p$ we have

$$Q_{p+1} = c_1 L_{p+1}(2\eta - 1) + c_2 (1 - \eta)^{p+1} R_{p+1} \left(\frac{2\xi}{1 - \eta} - 1\right). \quad (4.30)$$

**Proof.** The proof of (4.28) is the same as (4.11). Using $Q(\xi, 1 - \xi) = 0$ we obtain (4.29a). We apply (3.57) for $k = p$ to prove (4.29b). We display these basis functions in Figure 4.3 for $p = 1 - 4$.

Next, we note that for the space $\mathcal{V}_p$ (4.11) holds which in turn by the orthogonality condition

$$\int_0^1 \int_0^{1-\eta} (1 - \xi - \eta)\tilde{q}_p(\xi, \eta) \frac{\partial V}{\partial \xi} = 0, \ \forall \ V \in \mathcal{V}_p.$$ 

leads to

$$Q_{p+1} = c_1 L_{p+1}(2\eta - 1) + \tilde{c}_2 (1 - \xi - \eta)(1 - \eta)^{p+1} P_p^{1,0} \left(\frac{2\xi}{1 - \eta} - 1\right). \quad (4.31)$$

We complete the proof of (4.30) by noting that $R_{p+1}(x) = (1 - x)P_p^{1,0}(x)$. We display these basis functions in Figure 4.4 for $p = 1 - 6$.

Finally, we note that the size of the error problem with an optimal basis is equal to the number of terms missing in the finite element space to have $\mathcal{P}_{p+1}$.
Figure 4.3: Basis functions for the space $U_p$ for $p=1$ to $4$ on elements of type III.

Figure 4.4: Basis functions for the space $V_p$ for $p=1$ to $6$ on elements of type III.
4.3 Time-Dependent Scalar Transport Equation

In this section we present a numerical study of superconvergence properties for the DG method applied to time-dependent scalar transport equation. In particular, we show that DG solutions are $O(h^{p+2})$ superconvergent at the Legendre points on the outflow edge for triangles having one outflow edge. For triangles having two outflow edges the finite element error is $O(h^{p+2})$ superconvergent at the end points of the inflow edge.

We first present the model problem and recall the DG formulation. The space discretization leads to a system of ordinary differential equations (ODE) with respect to the time variable. The fully discrete space-time spectral DG method is presented. Finally we present numerical results for linear time-dependent problems. The superconvergence results are confirmed numerically.

We consider the scalar transient convection problem

$$u_t + a \cdot \nabla u = f(x, y, t), \quad (x, y) \in \Omega = [0, 1]^2, \quad t > 0,$$

subject to the initial and boundary conditions

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega = [0, 1]^2,$$

$$u(x, 0, t) = g_0(x, t), \quad u(0, y, t) = g_1(y, t), \quad t > 0.$$

In our analysis we assume that $a = [\alpha, \beta]^T$ a non-zero constant velocity vector and select $f(x, y, t), g_0(x, t), g_1(y, t)$ such that the exact solution, $u$, is smooth.

In order to obtain the weak DG formulation, we partition the domain $\Omega$ into a regular mesh having $N$ triangular elements $\Delta_j, j = 1, ..., N$, of diameter $h > 0$.

Let us multiply (4.32a) by a test function $v$, integrate over an arbitrary element $\Delta$, and apply Green’s theorem to write

$$\iint_{\Delta} u_t v dx dy + \oint_{\Gamma^-} a \cdot n u v ds + \oint_{\Gamma^+} a \cdot n u v ds - \iint_{\Delta} a \cdot \nabla v u dx dy = \iint_{\Delta} f v dx dy, \quad (4.33)$$

where $\Gamma^+$ and $\Gamma^-$ denote the outflow boundary and inflow boundary, respectively, of $\Delta$.

Next we approximate $u$ by a piecewise polynomial function $U$ whose restriction to $\Delta$ is in $W^p$.

The discrete DG formulation consists of determining $U \in S_N^p$ such that

$$\iint_{\Delta} V U_t dx dy + \oint_{\Gamma^-} a \cdot n \hat{U} V ds + \oint_{\Gamma^+} a \cdot n U V ds -$$
\[
\int_{\Delta} \mathbf{a} \nabla V U \, dx dy = \int_{\Delta} f V \, dx dy, \quad \forall \, V \in \mathcal{W}_p. \quad (4.34)
\]

subject to the initial condition \( U(x, y, 0) \in \mathcal{W}_p \) obtained using either by the local \( L^2 \) projection on \( S^{N,p} \), i.e.,
\[
\int_{\Delta} U(x, y, 0)V \, dx dy = \int_{\Delta} u_0(x, y)V \, dx dy, \quad \forall \, V \in \mathcal{W}_p, \quad (4.35)
\]
or by solving the following (time-independent) problem at \( t = 0 \)
\[
\mathbf{a} \cdot \nabla u = \mathbf{a} \cdot \nabla u_0 = f_0, \quad (x, y) \in \Omega = [0,1]^2,
\]
\[
u(x, 0) = g_0(x, 0), \quad u(0, y) = g_1(y, 0). \quad (4.36)
\]

The discrete DG formulation for (4.36) consists of determining \( U(x, y, 0) \in \mathcal{W}_p \) such that
\[
\int_{\Gamma^-} \mathbf{a} \cdot \hat{n} U \phi_{-} \, ds + \int_{\Gamma^+} \mathbf{a} \cdot \hat{n} U \phi_{+} \, ds - \int_{\Delta} \mathbf{a} \nabla V U \, dx dy = \int_{\Delta} f_0 V \, dx dy, \quad \forall \, V \in \mathcal{W}_p. \quad (4.37)
\]

We map a physical triangle \( \Delta \) having vertices \((x_i, y_i), \ i = 1, 2, 3\), into the canonical triangle \((0,0), (1,0) \) and \((0,1)\) by the standard affine mapping equation.

Express \( U \) as a linear combination of Dubiner basis
\[
U(x, y, t) = \sum_{i=1}^{pp} \sum_{p=1}^{p+1} C_{i,p} (t) \varphi_{-i}^{p},
\]
and testing against functions \( V = \varphi_{-i}^{p} \), we obtain
\[
\left( \int_{\Delta} \varphi_{-k}^{p} \varphi_{-i}^{p} \, dx dy \right) \frac{dC_{i}(t)}{dt} + \int_{\Gamma^-} \mathbf{a} \cdot \hat{n} \varphi_{-i}^{p} \, ds + \int_{\Gamma^+} \mathbf{a} \cdot \hat{n} \varphi_{-i}^{p} \, ds - \int_{\Delta} \mathbf{a} \cdot \nabla \varphi_{-i}^{p} \, dx dy \right) C_{i}(t) = \int_{\Delta} f \varphi_{-i}^{p} \, dx dy, \quad i, k = 1, \cdots, pp. \quad (4.38)
\]

### 4.4 A Posteriori Error Estimation

Here we present weak finite element formulations to compute a posteriori error estimates.

#### 4.4.1 Error estimation for linear problems

We illustrate our procedure on meshes consisting of triangles of type II or III with \( \mathcal{U}_p \) where the finite element error on each element \( \Delta \) is approximated as
\[
e(x, y) \approx E(x, y) = \sum_{i=1}^{pp+1} \chi_{i}^{p} (\xi(x, y), \eta(x, y)), \quad (4.39)
\]
where \((\xi(x, y), \eta(x, y))\) is the linear transformation from the physical triangle to the standard triangle.

Replacing \(u\) in (3.1) by \(e + U\) and integrating on the physical element \(\Delta\) leads to

\[
\iint_{\Delta} (a \nabla (U + e) + c(U + e)) W \, dx \, dy = \iint_{\Delta} f W \, dx \, dy.
\]

Replacing \(e\) by \(E\) in (4.40) leads to the \(p \times p\) linear algebraic system for \(d_{i+1}^p, i = 1, \ldots, p\)

\[
\iint_{\Delta} (a \nabla (U + E) + c(U + E)) \chi_i^{p+1} \, dx \, dy = \iint_{\Delta} f \chi_i^{p+1} \, dx \, dy, \quad i = 1, 2, \ldots, p.
\]

This local weak formulation is applied to estimate the error for other polynomial spaces and types of elements using the appropriate error basis functions given in \(\S 4.2\).

An accepted efficiency measure of \textit{a posteriori} error estimates is the effectivity index. Various norms can be used to measure the error [71, 70, 68]. In this thesis we use the local effectivity indices in the \(L^2\) norm. However, the process of developing \textit{a posteriori} error estimators in other norms is similar. The local effectivity indices in the \(L^2\) norm is defined as

\[
\theta_i = \frac{||E||_{L^2(\Delta_i)}}{||e||_{L^2(\Delta_i)}},
\]

and the global effectivity index

\[
\theta = \frac{||E||_{L^2(\Omega)}}{||e||_{L^2(\Omega)}}.
\]

Ideally, the effectivity indices should approach unity under mesh refinement.

### 4.4.2 Error estimation for nonlinear problems

We also consider nonlinear hyperbolic problems of the form

\[
u_y + h(u)x = f(x, y),
\]

subject to appropriate boundary conditions.

The discrete DG method consists of finding \(U \in \mathcal{U}_p\) such that

\[
\int_{\Gamma^-} [h(\hat{U}), \hat{U}] \cdot n \, ds + \int_{\Gamma^+} [h(U), U] \cdot n \, ds - \iint_{\Delta} [h(U), U] \nabla V \, dx \, dy = \iint_{\Delta} f V \, dx \, dy, \quad \forall \, V \in \mathcal{U}_p.
\]

For smooth solutions the results of the previous chapter are still true. For instance, on an element of type III using \(\mathcal{V}_p\) we have the following theorem.
Theorem 4.5. Under the conditions of Theorem 3.1 the leading term $Q_{p+1}$ of local DG finite element error on an element of type III using the space $\mathcal{V}_p$ can be written as

$$Q_{p+1}(\xi, \eta) = c_1 L_{p+1}(2\eta - 1) + c_2 (1 - \eta)^{p+1} R_{p+1}(\frac{2\xi}{1 - \eta} - 1).$$  

(4.46)

**Proof.** First, we linearize (4.44) about the true solution $u$, find the DG orthogonality and expand the error into a power series. Then from there we follow the same line of reasoning as for linear problems. \hfill \square

The *a posteriori* error estimation problem on an element of type II and III and $\mathcal{P}_p$ consists determining

$$E = \sum_{k=0}^{p+1} c_{p+1-k}^{k} \chi_{p+1-k}^{k},$$

such that

$$\int_{\Delta} [h'(U), 1] \cdot \nabla (U + E) \chi_{p+1-k}^{k} dxdy = \int_{\Delta} f \chi_{p+1-k}^{k} dxdy, \quad k = 1, \ldots, p.$$  

(4.47)

We also use the solution of the linearized problem (4.47) as an initial guess for Newton’s iteration when solving the following nonlinear finite element problem for $E$

$$\int_{\Delta} [h'(U + E), 1] \cdot \nabla (U + E) \chi_{p+1-k}^{k} dxdy = \int_{\Delta} f \chi_{p+1-k}^{k} dxdy, \quad k = 1, \ldots, p.$$  

(4.48)

For instance, on elements of type III using the space $\mathcal{V}_p$, we determine

$$E = c_0^{p+1} \chi_0^{p+1} + c_{p+1}^{0} \chi_{p+1}^{0},$$

such that

$$\int_{\Delta} [h'(U), 1] \cdot \nabla (U + E) \chi_0^{p+1} dxdy = \int_{\Delta} f \chi_0^{p+1} dxdy, \quad (4.49a)$$

$$\int_{\Delta} [h'(U), 1] \cdot \nabla (U + E) \chi_{p+1}^{0} dxdy = \int_{\Delta} f \chi_{p+1}^{0} dxdy, \quad (4.49b)$$

where

$$\chi_0^{p+1}(x, y) = L_{p+1}(2\eta(x, y) - 1),$$  

(4.49c)

$$\chi_{p+1}^{0}(x, y) = (1 - \eta(x, y))^{p+1} R_{p+1}(\frac{2\xi(x, y)}{1 - \eta(x, y)} - 1).$$  

(4.49d)

We also use the solution of the linearized problem (4.49) as an initial guess for Newton’s iteration when solving the nonlinear finite element problem for $E$
\[
\iint_{\Delta} [h'(U + E), 1] \cdot \nabla (U + E) \chi^p_0 \, dx \, dy = \iint_{\Delta} f \chi^p_0 \, dx \, dy, \quad (4.50a)
\]
\[
\iint_{\Delta} [h'(U + E), 1] \cdot \nabla (U + E) \chi^0_{p+1} \, dx \, dy = \iint_{\Delta} f \chi^0_{p+1} \, dx \, dy. \quad (4.50b)
\]

### 4.4.3 Error estimation for time-dependent problems

We illustrate our procedure on meshes consisting of triangles of type I and II with \( \mathcal{U}_p \) where the finite element error on each element \( \Delta \) is approximated as

\[
e(x, y, t) \approx E(x, y, t) = \sum_{i=1}^{p} d_{p+1}^i(t) \chi^i_{p+1-i}(\xi(x, y), \eta(x, y)). \quad (4.51)
\]

where \((\xi(x, y), \eta(x, y))\) is the linear transformation from the physical triangle to the standard triangle.

Replacing \( u \) in (4.32) by \( e + U \) and integrating on the physical element \( \Delta \) leads to

\[
\iint_{\Delta} (U + e)_t W \, dx \, dy + \iint_{\Delta} a \cdot \nabla (U + e) W \, dx \, dy = \iint_{\Delta} f W \, dx \, dy. \quad (4.52)
\]

Neglect the higher-order term \( e_t \), replacing \( e \) by \( E \) in (4.52) leads to the \( p \times p \) linear algebraic system for \( d_{p+1}^i, \ i = 1, \ldots, p \)

\[
\iint_{\Delta} a \cdot \nabla E \chi^i_{p+1-i} \, dx \, dy = \iint_{\Delta} (f - U_t - a \cdot \nabla U) \chi^i_{p+1-i} \, dx \, dy, \ i = 1, 2, \ldots, p. \quad (4.53)
\]

We also apply the following error estimation procedure to compute error estimates

\[
\iint_{\Delta} (U + E)_t W \, dx \, dy + \iint_{\Delta} a \cdot \nabla (U + E) W \, dx \, dy = \iint_{\Delta} f W \, dx \, dy. \quad (4.54)
\]

For instance, on elements of type III using the space \( \mathcal{U}_p \), we solve an ordinary differential equation for \( d_{p+1}^i(t), \ i = 1, \ldots, p \)

\[
\iint_{\Delta} E_t \chi^i_{p+1-i} \, dx \, dy + \iint_{\Delta} a \cdot \nabla E \chi^i_{p+1-i} \, dx \, dy = \iint_{\Delta} (f - U_t - a \cdot \nabla U) \chi^i_{p+1-i} \, dx \, dy, \ i = 1, 2, \ldots, p. \quad (4.55)
\]

The initial values \( d_{p+1}^i(0), \ i = 1, 2, \ldots, N, \) are obtained using either by local \( L^2 \) projection or solving the problem at \( t = 0 \) using the DG method and then computing the error estimation.

This local weak formulation is applied to estimate the error for other polynomial spaces and types of elements using the appropriate error basis functions given in Chapter 4.
An accepted efficiency measure of \textit{a posteriori} error estimates is the effectivity index. In this paper we use the local effectivity indices in the $L^2$ norm

$$\theta_i(t) = \frac{||E(\cdot, \cdot, t)||_{L^2(\Delta_i)}}{||e(\cdot, \cdot, t)||_{L^2(\Delta_i)}}$$

and the global effectivity index

$$\theta(t) = \frac{||E(\cdot, \cdot, t)||_{L^2(\Omega)}}{||e(\cdot, \cdot, t)||_{L^2(\Omega)}}.$$ (4.57)

Ideally, the effectivity indices should stay close to one and should converge to one under mesh refinement.

For nonlinear transient hyperbolic problems of the form

$$u_t + g(u)_y + h(u)_x = f(x, y, t),$$ (4.58)

subject to appropriate boundary and initial conditions, the \textit{a posteriori} error estimation problem on an element of type I and II and $\mathcal{U}_p$ consists of determining

$$E(x, y, t) = \sum_{i=1}^{p} a_i^{p+1}(t) \chi_{p+1-i}(\xi(x, y), \eta(x, y)),$$

such that

$$\int_{\Delta} \left[ h'(U) \cdot \nabla (U + E) \chi_{p+1-k}^k \right] dxdy = \int_{\Delta} (f - U_t) \chi_{p+1-k}^k dxdy, \quad k = 1, \cdots, p. \quad (4.59)$$

We also use the solution of the linearized problem (4.59) as an initial guess for Newton’s iteration when solving the following nonlinear finite element problem for $E$

$$\int_{\Delta} \left[ h'(U+E) \cdot \nabla (U + E) \chi_{p+1-k}^k \right] dxdy = \int_{\Delta} (f - U_t) \chi_{p+1-k}^k dxdy, \quad k = 1, \cdots, p. \quad (4.60)$$

### 4.5 Computational Examples

In these examples we present new superconvergence results on triangular elements and show how to construct effective estimates of the finite element discretization error using superconvergence of DG solutions. We present new $O(h^{p+2})$ pointwise superconvergence results for first-order hyperbolic problems on triangular meshes using different finite element spaces. We present efficient techniques to compute asymptotically correct \textit{a posteriori} error estimates.
Example 4.1.

The purpose of this example is to show the efficiency of our \textit{a posteriori} error estimation technique on uniform meshes consisting of type III elements using $\mathcal{P}_p$, $\mathcal{V}_p$ and $\mathcal{U}_p$. Let us consider the linear problem with characteristics parallel to the $x$-axis.

\begin{equation}
 u_x + u = f(x, y), \quad (x, y) \in \Omega = [0, 1]^2,
\end{equation}

subject to the boundary conditions

\begin{equation}
 u(0, y) = g(y).
\end{equation}

We select $f(x, y)$ and $g$ such that the exact solution is

\begin{equation}
 u(x, y) = e^y + e^{-y}.
\end{equation}

We solve this problem on $\Omega$ using uniform triangular meshes obtained by partitioning the domain $\Omega$ into $n \times n$, $n = 4, 6, 8, 10$ squares and divide each square into two triangles connection the upper left and lower right vertices. Thus, each element of this mesh is of type III. We start by solving 4.61 using the space $\mathcal{P}_p$, $\mathcal{V}_p$ and $\mathcal{U}_p$ using the exact boundary condition $U^- = u$ on uniform meshes having mesh having $N = 32, 72, 128$ and 200 elements with $p = 0$ to 3. In Figure 4.5 we plot the zero-level curves of the true error on each element for $N = 32$ and $p = 0$, to 3. We present the maximum errors and orders of convergence at the superconvergence points in table 4.5. These results indicate that the DG solution is $O(h^{p+2})$ globally superconvergent at these points which is in full agreement with Theorem 3.11. We note that the error in the $L^2$ norm is only $O(h^{p+1})$. We obtain similar results when $U^-$ interpolates true boundary conditions at $p + 1$ Gauss-Legendre points.

We use the DG finite element (4.1), (4.2) with the spaces $\mathcal{P}_p$, $\mathcal{V}_p$ and $\mathcal{U}_p$, $p = 0, 1, 2, 3$. On each element we apply the error estimation procedure (4.41) to compute error estimates and present the local effectivity indices in Figures 4.6- 4.9. The global $L^2$ effectivity indices versus $N$ and $p$ shown in Tables 4.6 and 4.7 indicate that, for smooth solutions, our \textit{a posteriori} error estimate converges to the true error under both $h$- and $p$-refinements.

Example 4.2.

We consider the linear hyperbolic problem

\begin{equation}
 u_x + u_y + u = f(x, y), \quad (x, y) \in [0, 1]^2,
\end{equation}

subject to the boundary conditions

\begin{equation}
 u(x, 0) = g_0(x), \quad u(0, y) = g_1(y).
\end{equation}

We select $f(x, y)$, $g_0$ and $g_1$ such that the exact solution is

\begin{equation}
 u(x, y) = e^{x-y}.
\end{equation}
Figure 4.5: Zero-level curves of the true error for Example 4.1 on a uniform mesh having \(N = 32\) elements using the space \(V_p\) for \(p = 0\) to \(p = 3\) (upper left to lower right).

Table 4.5: Maximum errors and orders of convergence at the superconvergence points for Example 4.1 on uniform meshes having \(N = 32, 72, 128, 200\) elements.
Figure 4.6: Local effectivity indices on a uniform mesh having \( N = 32 \) elements with \( P_p \), \( p = 0 \) to 3 (upper left to lower right) for Example 4.1.

Table 4.6: \( \|e\|_{L^2} \) and global effectivity indices for Example 4.1 versus \( N \) and \( p \) using \( P_p \)
Figure 4.7: Local effectivity indices on a uniform mesh having $N = 72$ elements with $P_p$, $p = 0$ to 3 (upper left to lower right) for Example 4.1.

Table 4.7: $|e|_{L^2}$ and global effectivity indices for Example 4.1 versus $N$ and $p$ using $V_p$
Figure 4.8: Local effectivity indices on a uniform mesh having $N = 32$ elements with $V_p$, $p = 0$ to 3 (upper left to lower right) for Example 4.1.
Figure 4.9: Local effectivity indices on a uniform mesh having $N = 72$ elements with $V_p$, $p = 0$ to 3 (upper left to lower right) for Example 4.1.
We solve (4.62) on a uniform meshes obtained by partitioning the domain into \( n \times n \), \( n = 4, 6, 8, 10 \), squares and dividing each square into two triangles by using the diagonals parallel to \( y = x \). Thus, the meshes have \( N = 32, 72, 128, 200 \) triangles of type III. We solve this problem using \( P_p, V_p \) and \( U_p \) with \( U^- = u \). In Figure 4.10 we plot the zero-level curves of the true error on each element for \( N = 32 \) and \( p = 0 \), to 3. We present the maximum errors and orders of convergence at the roots of \( (p + 1) \)-degree Legendre polynomial in Table 4.8. These results indicate that the DG solution is \( O(h^{p+2}) \) globally superconvergent at these points which is in full agreement with Theorem 3.11.

We use the DG finite element (4.1), (4.2) with the spaces \( P_p, V_p \) and \( U_p \), \( p = 0, 1, 2, 3 \). On each element we apply the error estimation procedure (4.41) to compute error estimates and present the local effectivity indices in Figures 4.11-4.12. We also present the maximum of the local effectivity index versus \( N \) and \( p \) in Figure 4.13. The global \( L^2 \) effectivity indices shown in Table 4.9 indicate that, for smooth solutions, our \textit{a posteriori} error estimate converges to the true error under both \( h \)- and \( p \)-refinements.

![Figure 4.10: Zero-level curves of the true error for Example 4.2 on uniform meshes having \( N = 32 \) elements using \( P_p \), \( p = 0 \) to 3 (upper left to lower right).](image-url)

Next we shall validate our theory on superconvergence for meshes having triangular elements of type I and II.

\textbf{Example 4.3.}
Table 4.8: Maximum errors and orders of convergence at the roots of \((p+1)\)-degree Legendre polynomial for Example 4.2 on uniform meshes having \(N = 32, 72, 128, 200\) elements using \(\mathcal{P}_p\).

\[
\begin{array}{cccccccc}
N & p = 0 & \|e\|_{\infty} & \text{order} & p = 1 & \|e\|_{\infty} & \text{order} & p = 2 & \|e\|_{\infty} & \text{order} & p = 3 & \|e\|_{\infty} & \text{order} \\
\end{array}
\]

Figure 4.11: Local effectivity indices on a uniform mesh having \(N = 32\) elements with \(\mathcal{P}_p\), \(p = 0\) to 3 (upper left to lower right) for Example 4.2.
Figure 4.12: Local effectivity indices on a uniform mesh having $N = 72$ elements with $\mathcal{P}_p$, $p = 0$ to 3 (upper left to lower right) for Example 4.2.

Figure 4.13: Global effectivity index versus $N$ (left) and versus $p$ (right) using the space $\mathcal{P}_p$ for Example 4.2.
Table 4.9: $L^2$ errors and global effectivity indices for Example 4.2 using uniform meshes having $N = 32, 72, 128, 200$, elements and the spaces $\mathcal{P}_p$, $\mathcal{V}_p$ and $\mathcal{U}_p$.

<table>
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<td></td>
<td>e</td>
<td></td>
<td>_{L^2}$</td>
<td>$\theta$</td>
<td>$</td>
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<tr>
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<td>3.2879e-9</td>
<td>1.0127</td>
<td>3.7335e-13</td>
<td>1.0138</td>
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</tr>
<tr>
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<td>3.1080e-3</td>
<td>1.0159</td>
<td>1.4556e-6</td>
<td>1.0076</td>
<td>1.4673e-14</td>
<td>1.0087</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>1.7607e-3</td>
<td>1.0116</td>
<td>4.6249e-7</td>
<td>1.0054</td>
<td>1.4744e-15</td>
<td>1.0064</td>
<td></td>
</tr>
<tr>
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<td>1.8991e-7</td>
<td>1.0043</td>
<td>2.4837e-16</td>
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<table>
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<tr>
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<td>_{L^2}$</td>
<td>$\theta$</td>
<td>$</td>
</tr>
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<td>32</td>
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<td>1.0117</td>
<td>4.0145e-13</td>
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<td>1.0159</td>
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<td>1.5778e-14</td>
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<tr>
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<td>4.6249e-7</td>
<td>1.0044</td>
<td>1.5854e-15</td>
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<td>200</td>
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<td>2.6706e-16</td>
<td>1.0021</td>
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</tbody>
</table>

We consider the linear hyperbolic problem.

\[ 2u_x + u_y = f(x, y), \quad (x, y) \in [0, 1]^2, \]

subject to the boundary conditions

\[ u(x, 0) = g_0(x), \quad u(0, y) = g_1(y). \]

We select $f(x, y)$, $g_0$ and $g_1$ such that the exact solution is

\[ u(x, y) = e^{x+y}. \]

We create uniform triangular meshes obtained by partitioning the domain into $n \times n$ squares with $n = 4, 6, 8, 10, 15, 20, 25$ squares and dividing each square into two triangles by connecting the upper-left and lower-right vertices. These meshes consist of $N = 32, 72, 128, 200, 450, 800, 1250$ triangles of type I and II.

First, we solve (4.63) using the DG method (4.1), (4.2) with the spaces $\mathcal{P}_p$, $\mathcal{U}_p$ and $\mathcal{V}_p$ and present the maximum errors at the superconvergence points over all elements in Table 4.10 which show $O(h^{p+1})$ convergence rates. In Figure 4.14 we plot the zero-level curves of the true error and mark the superconvergence points by $\times$ for $N = 32$ and $p = 1, 2, 3, 4$, where we note that the level curves do not pass close to the superconvergence points on most elements. These results indicate that the local superconvergence results of Chapter 3 do not hold globally on meshes having elements of type I and II since the convergence rates at the superconvergence points are the same as the global $L^2$ convergence rates.

In order to maintain the $O(h^{p+2})$ superconvergence globally on meshes consisting of elements of type I and II, we used the modified DG method (4.1), (4.3) where the inflow boundary
Figure 4.14: Zero-level curves of the true error for Example 4.3 on a mesh having \( N = 32 \) elements using \( P_1, V_1, U_1 \), (upper left to lower left) and \( P_2, V_2, U_2 \), (upper right to lower right) with DG method (4.1), (4.2).
Table 4.10: Maximum errors and orders of convergence at the superconvergence points for Example 4.3 on uniform meshes having $N = 32, 72, 128, 200$ elements.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$p = 1$</th>
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<th>$p = 4$</th>
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<td></td>
<td>$</td>
<td></td>
<td>e</td>
<td></td>
</tr>
<tr>
<td>32</td>
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<td>1.1198e-4</td>
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</tr>
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<td>72</td>
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<td>2.5927e-4</td>
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<td>2.8840</td>
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<tr>
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<td>1.1198e-4</td>
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</tr>
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<td>5.8186e-5</td>
<td>2.9339</td>
</tr>
</tbody>
</table>

conditions are corrected using our *a posteriori* error estimate. Now, we solve (4.63) using the same parameters as above with the space $U_p, p = 1, 2, 3, 4, N = 32, 72, 128, 200$ and present the maximum errors and orders of convergence in Table 4.11. We show the true error in each case in Figure 4.15. In Figure 4.16 we plot the zero-level curves of the true error and mark the superconvergence points by $\times$ for $N = 32$ and $p = 1, 2, 3, 4$, to show that the level curves pass close to the superconvergence points on most elements. These results show that the modified DG method (4.1), (4.3) on $U_p$ yields $O(h^{p+2})$ superconvergent solutions at the Legendre points on the *outflow* edges of elements of type II and at the endpoints of the *inflow* edges of elements of type I.

The local effectivity indices of Figure 4.17 for $N = 32$ and $U_p, p = 1, 2, 3, 4$, are close to unity on most elements. Moreover, the true $L^2$ errors and the corresponding global effectivity indices shown in Table 4.12 indicate that the *a posteriori* error estimates are asymptotically exact under mesh refinement.

Table 4.11: Maximum error and its Rate of convergence at the superconvergence points (marked by $\times$ in Figure 4.16) on meshes having $N = 32, 72, 128, 200$ elements with the spaces $U_p, p = 1 \ldots 4$, for Example 4.3.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$p = 1$</th>
<th>$p = 2$</th>
<th>$p = 3$</th>
<th>$p = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$</td>
<td></td>
<td>e</td>
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</tr>
<tr>
<td>32</td>
<td>3.3166e-3</td>
<td>8.3482e-4</td>
<td>1.1198e-4</td>
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<tr>
<td>72</td>
<td>1.5485e-2</td>
<td>2.5927e-4</td>
<td>1.8784</td>
<td>2.8840</td>
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<tr>
<td>128</td>
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<td>1.1939</td>
<td>1.1198e-4</td>
<td>2.9183</td>
</tr>
<tr>
<td>200</td>
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<td>1.9333</td>
<td>5.8186e-5</td>
<td>2.9339</td>
</tr>
</tbody>
</table>

Example 4.4.
Figure 4.15: Surface plot of the error for 4.3 on a 128 element uniform mesh with $p = 1$ (left) and $p = 2$ (right).

Figure 4.16: Zero-level curves of the true error for Example 4.3 on a mesh having $N = 32$ elements using the spaces $U_p$, $p = 1, 2, 3, 4$ (upper left to lower right).
Figure 4.17: Local effectivity indices for Example 4.3 on a mesh having $N = 32$ elements using the spaces $U_p$, $p = 1, 2, 3, 4$ (upper left to lower right).

Table 4.12: $L^2$ errors and global effectivity indices for Example 4.3 on uniform meshes having 32, 72, 128, 200, 450, 800 and 1250 elements using $U_p$, $p = 1, 2, 3, 4$.

<table>
<thead>
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<th>$|\varepsilon|_{L^2}$</th>
<th>$\theta$</th>
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<th>$\theta$</th>
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<th>$|\varepsilon|_{L^2}$</th>
</tr>
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<tbody>
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<td>3.2864e-4</td>
<td>0.9883</td>
<td>4.5666e-6</td>
<td>0.9711</td>
<td>5.0213e-8</td>
<td>0.9914</td>
<td></td>
</tr>
<tr>
<td>72</td>
<td>4.4195e-3</td>
<td>0.9843</td>
<td>4.1188e-5</td>
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<td>2.8433e-7</td>
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<td>1.5488e-9</td>
<td>0.9883</td>
<td></td>
</tr>
<tr>
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<td>0.9843</td>
<td>1.2216e-5</td>
<td>0.9845</td>
<td>5.6123e-8</td>
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<td>2.0345e-10</td>
<td>0.9932</td>
<td></td>
</tr>
<tr>
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<td>5.1564e-6</td>
<td>0.9848</td>
<td>1.7734e-8</td>
<td>0.9946</td>
<td>1.8243e-11</td>
<td>0.9933</td>
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</tr>
<tr>
<td>450</td>
<td>7.1128e-4</td>
<td>0.9850</td>
<td>2.6410e-6</td>
<td>0.9850</td>
<td>7.2711e-9</td>
<td>0.9950</td>
<td>1.5801e-11</td>
<td>0.9944</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>4.9426e-4</td>
<td>0.9850</td>
<td>1.5287e-6</td>
<td>0.9850</td>
<td>3.5064e-9</td>
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<tr>
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<td>0.9850</td>
<td>2.7369e-9</td>
<td>0.9952</td>
<td>4.9557e-12</td>
<td>0.9943</td>
<td></td>
</tr>
</tbody>
</table>
We solve problem (4.63) using the modified DG method (4.1), (4.3) on a uniform mesh having 72 elements and the spaces $U_p$ with the nonuniform degree distributions shown in Figure 4.18, where on the left the degree on each element is higher or equal to the degree on its downwind neighbors and on the right the degree on each element is lower or equal to the degree on its downwind neighbors. In both situations, we use exact boundary conditions. We show the zero-level curves of the true error in Figure 4.19 which pass close to the superconvergence points on most elements on the left while this not true for the mesh on the right. Consequently, the local effectivity indices shown in left of Figure 4.20 are close to unity on most elements while on the right they are smaller than unity on elements near the downwind end of the domain. The purpose of this example is to show that nonuniform polynomial degree distributions shown on the left of Figure 4.18 lead to accurate error estimates while degree distributions shown on the right lead to poor error estimates.

Figure 4.18: Two uniform meshes for Example 4.4 with nonuniform polynomial degree distributions.

Example 4.5.

In this example we show that the modified DG solution is $O(h^{p+2})$ superconvergent on triangular meshes having elements of type I, II and III.

Let us consider the problem

$$u_y = f(x, y), \quad (x, y) \in [0, 1]^2,$$  \hspace{1cm} (4.64a)

subject to the boundary conditions

$$u(x, 0) = g_0(x), \quad u(0, y) = g_1(y).$$  \hspace{1cm} (4.64b)

We select $f(x, y)$, $g_0$ and $g_1$ such that the exact solution is

$$u(x, y) = e^{3x-y}. \hspace{1cm} (4.64c)$$
Figure 4.19: Zero-level curves of the error for Example 4.4 with the $p$-distributions of Figure 4.18.

Figure 4.20: Local effectivity indices for Example 4.4 and the $p$-distributions of Figure 4.18.
Following [54] we construct meshes consisting of elements of type I, II and III, obtained by partitioning the domain $\Omega = [0, 1]^2$ into $N = 2n \times (2n+1)$ triangles where $h = 1/n$ as shown in Figure 4.21.

![Figure 4.21: Triangulation of $\Omega$ for $h = 1/6$.](image)

We solve (4.64) using the modified DG method (4.1), (4.3) on meshes having $N = 42, 72, 110, 156$ elements with the spaces $U_p$, $p = 1, 2, 3, 4$. We plot the zero-level curves of the true error on each element in Figure 4.22 for $N = 72$ and $p = 1, 2, 3, 4$. We present the maximum errors and the orders of convergence at the shifted roots of $(p+1)$-degree Legendre polynomial on the outflow edge of type II and III in Table 4.13. We also present the maximum errors and orders of convergence at the endpoints of the inflow edges of elements of type I in Table 4.14. The results of Tables 4.13 and 4.14 indicate that the $p$-degree modified DG solution is $O(h^{p+2})$ superconvergent at Legendre points on the outflow edges for each element of type II and at the endpoints of inflow edges of elements of type I. In Figure 4.23 and Table 4.15 we present the global $L^2$ effectivity indices versus $N$ and $p$ which show that the error estimates converge to the true error under $h$-refinement.

Table 4.13: Maximum errors and orders of convergence at the roots of $(p+1)$-degree Legendre polynomial on the outflow edges over all elements of type II for Example 4.5 on meshes having $N = 42, 72, 110, 156$ elements using $U_p$ with $p = 1, 2, 3, 4$.

<table>
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<td></td>
<td>$</td>
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<td>e</td>
<td></td>
</tr>
<tr>
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<td>3.8835</td>
<td>4.8350</td>
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<tr>
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<tr>
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<td>4.8350</td>
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</table>

Example 4.6.
Figure 4.22: Zero-level curves of the true error for Example 4.5 on a mesh having $N = 72$ elements using the spaces $U_p$, $p = 1, 2, 3, 4$ (upper left to lower right).

Table 4.14: Maximum errors and rates of convergence at the endpoints of the inflow edge on all elements of type I for Example 4.5 on uniform meshes having $N = 42, 72, 110, 156$ elements using $U_p$ with $p = 1$ to 4.

<table>
<thead>
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<td></td>
<td>$</td>
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</tr>
<tr>
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<td>8.1613e-4</td>
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</tr>
<tr>
<td>72</td>
<td>1.6601e-2</td>
<td>2.9775</td>
<td>2.6449e-4</td>
<td>3.9167</td>
</tr>
<tr>
<td>110</td>
<td>8.7853e-3</td>
<td>2.8519</td>
<td>1.1182e-4</td>
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</tr>
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<td>5.1994e-3</td>
<td>2.8770</td>
<td>5.5081e-5</td>
<td>3.8855</td>
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</table>

Table 4.15: $||e||_{L^2}$ and global effectivity indices for Example 4.5 versus $h$ and $p$ using $U_p$.

<table>
<thead>
<tr>
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<th>$p = 1$</th>
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<tr>
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<td>1.066</td>
<td>2.252e-9</td>
<td>1.061</td>
</tr>
</tbody>
</table>
We consider the following problem with a contact discontinuity

\[ u_x + u_y = 0, \quad (x, y) \in [0, 1]^2, \]  

subject to the boundary conditions

\[ u(x, 0) = e^{-x}, \quad 0 \leq x \leq 1, \]  
\[ u(0, y) = e^y + 0.25, \quad 0 < y \leq 1. \]

The exact solution is

\[ u(x, y) = \begin{cases} 
  e^{-x+y} + 0.25 & \text{if } x < y, \\
  e^{-x+y} & \text{if } x \geq y.
\end{cases} \]  

The true solution has a contact discontinuity along \( y = x \). Therefore, the smoothness assumption of Theorem 3.1 is violated and as a result we expect the \textit{a posteriori} error estimate to perform poorly near the discontinuity.

We solve (4.65) on meshes having 450, 1250, 5000 and 20000 elements with \( U_p, p = 1, 2 \) and present the local effectivity indices in Figure 4.24. These computational results indicate that the local effectivity indices on elements away from the discontinuity converge to unity under mesh refinement while they perform poorly on elements near the discontinuity. Since we are not using limiting to suppress spurious oscillations near the discontinuity, the region around the discontinuity where the error is underestimated gets wider as \( p \) increases. Krivodonova \textit{et al.} [46] described a strategy for detecting discontinuities and for limiting spurious oscillations near such discontinuities when solving hyperbolic systems of conservation laws by high-order discontinuous Galerkin methods. The approach is based on a strong superconvergence at the outflow boundary of each element in smooth regions of the flow. By detecting discontinuities...
Figure 4.24: Local effectivity indices for 4.6 with $(N, p) = (450,1), (450,2), (1250,1), (1250,2), (5000,1), (5000,2), (20000,1),$ and $(20000,2)$ (upper left to lower right).
in such variables as density or entropy, limiting may be applied only in these regions; thereby, preserving a high order of accuracy in regions where solutions are smooth.

**Example 4.7.**

We consider the non-constant coefficients hyperbolic problem

\[(x + 2) \, u_x + (y + 2) \, u_y = f(x, y), \quad (x, y) \in [0, 1]^2 \quad (4.66a)\]

subject to the boundary conditions

\[u(x, 0) = g_0(x), \quad u(0, y) = g_1(y). \quad (4.66b)\]

We select \(f(x, y)\), \(g_0\) and \(g_1\) such that the exact solution is

\[u(x, y) = x \, e^y + y \, e^x. \quad (4.66c)\]

We solve problem (4.66) on a uniform mesh obtained by partitioning the domain into \(n \times n\), \(n = 4, 6, 8, 10\) squares and dividing each square into two triangles by connecting the upper-left and lower-right vertices. Thus, we obtain meshes having \(N = 32, 72, 128, 200\) triangles of type I and II.

In order to obtain the \(O(h^{p+2})\) superconvergence on global meshes having elements of type I and II, we used the modified DG method (4.1), (4.3) where the inflow boundary condition is corrected using the *a posteriori* error estimate. We solve (4.66) with the space \(U_p\), \(p = 1, 2, 3, 4\), \(N = 32, 72, 128, 200\) and present the maximum errors and order of convergence at the superconvergence points over all elements in Table 4.16 which show \(O(h^{p+2})\) convergence rates. In Figure 4.25 we plot the zero-level curves of the true error and mark the superconvergence points by \(\times\) for \(N = 32\) and \(p = 1, 2, 3, 4\), where we note that the level curves pass close to the superconvergence points on most elements. These results show the modified DG method (4.1), (4.3) on \(U_p\) yields \(O(h^{p+2})\) superconvergent solutions at the Legendre points on the *outflow* edges of elements of type II and at the endpoints of the *inflow* edges of elements of type I.

The local effectivity indices in Figure 4.26 for \(N = 32\) and \(U_p\), \(p = 1, 2, 3, 4\), are close to unity on most elements. The true \(L^2\) errors and the corresponding global effectivity indices shown in Table 4.17 indicate that the *a posteriori* error estimates are asymptotically exact under mesh refinement.

**Example 4.8.**

We consider the divergence-free linear hyperbolic problem

\[(y + 2) \, u_x + (x + 2) \, u_y = f(x, y), \quad (x, y) \in [0, 1]^2, \quad (4.67a)\]
Table 4.16: Maximum error and its order of convergence at the superconvergence points for Example 4.7 on meshes having $N = 32, 72, 128, 200$ elements with the spaces $U_p, p = 1, 2, 3, 4$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$p = 1$</th>
<th></th>
<th>$p = 2$</th>
<th></th>
<th>$p = 3$</th>
<th></th>
<th>$p = 4$</th>
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</tr>
</thead>
<tbody>
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<td></td>
<td>$|e|_{\infty}^*$ order</td>
<td>$|e|_{\infty}^*$ order</td>
<td>$|e|_{\infty}^*$ order</td>
<td>$|e|_{\infty}^*$ order</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>3.0961e-4</td>
<td>3.6919e-5</td>
<td>1.3101e-6</td>
<td>6.9567e-9</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td></td>
</tr>
<tr>
<td>128</td>
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<td>2.8655</td>
<td>2.5706e-6</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>2.2842e-5</td>
<td>2.8980</td>
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<td>3.9022</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.25: Zero-level curves of the true error for Example 4.7 on a mesh having $N = 32$ elements using the spaces $U_p, p = 1, 2, 3, 4$ (upper left to lower right).
Figure 4.26: Local effectivity indices for Example 4.7 on a mesh having $N = 32$ elements using the spaces $U_p$, $p = 1, 2, 3, 4$ (upper left to lower right).

Table 4.17: $L^2$ errors and global effectivity indices for Example 4.7 on uniform meshes having 32, 72, 128 and 200 elements using $U_p$, $p = 1 − 4$.

<table>
<thead>
<tr>
<th>$N$</th>
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<th>$p = 3$</th>
<th>$p = 4$</th>
</tr>
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<tr>
<td></td>
<td>$</td>
<td></td>
<td>e</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>6.4299e-3</td>
<td>1.0218</td>
<td>8.8980e-5</td>
<td>0.9809</td>
</tr>
<tr>
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<td>1.2603e-3</td>
<td>1.0148</td>
<td>7.8142e-6</td>
<td>0.9868</td>
</tr>
<tr>
<td>128</td>
<td>3.9728e-4</td>
<td>1.0112</td>
<td>1.3910e-6</td>
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</tr>
<tr>
<td>200</td>
<td>1.6237e-4</td>
<td>1.0090</td>
<td>3.8930e-7</td>
<td>0.9583</td>
</tr>
</tbody>
</table>
subject to the boundary conditions

\[ u(x, 0) = g_0(x), \quad u(0, y) = g_1(y). \]  

(4.67b)

We select \( f(x, y) \), \( g_0 \) and \( g_1 \) such that the exact solution is

\[ u(x, y) = x e^y + y e^x. \]  

(4.67c)

We solve problem (4.67) using the same parameters and meshes as for Example 4.7 and present the maximum errors and order of convergence at the superconvergence points over all elements in Table 4.18 which show \( O(h^{p+2}) \) convergence rates. In Figure 4.27 we plot the zero-level curves of the true error and mark the superconvergence points by \( \times \) for \( N = 32 \) and \( p = 1, 2, 3, 4 \), where we note that the level curves pass close to the superconvergence points on most elements. These results show the modified DG method (4.1), (4.3) on \( U_p \) yields \( O(h^{p+2}) \) superconvergent solutions at the Legendre points on the outflow edges of elements of type II and at the endpoints of the inflow edges of elements of type I. The true \( L^2 \) errors and the corresponding global effectivity indices shown in Table 4.19 indicate that the \textit{a posteriori} error estimates are asymptotically exact under mesh refinement.

Table 4.18: Maximum error and its order of convergence at the superconvergence points for Example 4.8 on meshes having \( N = 32, 72, 128, 200 \) elements with the spaces \( U_p, p = 1, 2, 3, 4 \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( p = 1 )</th>
<th>( |e|_{L^\infty} )</th>
<th>( \text{order} )</th>
<th>( |e|_{L^\infty} )</th>
<th>( \text{order} )</th>
<th>( |e|_{L^\infty} )</th>
<th>( \text{order} )</th>
<th>( |e|_{L^\infty} )</th>
<th>( \text{order} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>5.6742e-7</td>
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</tr>
<tr>
<td>72</td>
<td>9.1255e-5</td>
<td>2.8911</td>
<td>2.0255e-6</td>
<td>3.9599</td>
<td>7.4230e-8</td>
<td>4.9724</td>
<td>7.9212e-10</td>
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</tr>
<tr>
<td>128</td>
<td>3.9789e-5</td>
<td>2.8852</td>
<td>6.4990e-7</td>
<td>3.9514</td>
<td>1.8086e-8</td>
<td>4.9084</td>
<td>1.4629e-10</td>
<td>5.8714</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>2.0779e-5</td>
<td>2.9114</td>
<td>2.6967e-7</td>
<td>3.9419</td>
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<td>4.9256</td>
<td>3.9632e-11</td>
<td>5.8526</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.19: \( L^2 \) errors and global effectivity indices for Example 4.8 on uniform meshes having 32, 72, 128 and 200 elements using \( U_p, p = 1 - 4 \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( p = 1 )</th>
<th>( |e|_{L^2} )</th>
<th>( \theta )</th>
<th>( |e|_{L^2} )</th>
<th>( \theta )</th>
<th>( |e|_{L^2} )</th>
<th>( \theta )</th>
<th>( |e|_{L^2} )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>6.3109e-3</td>
<td>1.0786</td>
<td>8.9636e-5</td>
<td>0.9757</td>
<td>2.2588e-8</td>
<td>0.9888</td>
<td>1.5701e-11</td>
<td>0.9937</td>
<td></td>
</tr>
<tr>
<td>72</td>
<td>1.2406e-3</td>
<td>1.0878</td>
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<td>0.9785</td>
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<td>0.9934</td>
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</tr>
<tr>
<td>128</td>
<td>3.9409e-4</td>
<td>1.0737</td>
<td>1.4058e-6</td>
<td>0.9811</td>
<td>8.2738e-11</td>
<td>0.9909</td>
<td>1.5396e-14</td>
<td>0.9936</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>1.6135e-4</td>
<td>1.0295</td>
<td>3.6834e-7</td>
<td>0.9832</td>
<td>1.3705e-11</td>
<td>0.9910</td>
<td>1.6601e-15</td>
<td>0.9931</td>
<td></td>
</tr>
</tbody>
</table>

Example 4.9.

We consider the following linear hyperbolic problem

\[ u_t + 3u_x + 2u_y = f(x, y, t), \quad (x, y) \in [0, 1]^2, \quad 0 \leq t \leq 1, \]  

(4.68a)
Figure 4.27: Zero-level curves of the true error for Example 4.8 on a mesh having $N = 32$ elements using the spaces $U_p$, $p = 1, 2, 3, 4$ (upper left to lower right).
subject to the initial and boundary conditions

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega = [0, 1]^2,$$

$$u(x, 0, t) = g_0(x, t), \quad u(0, y, t) = g_1(y, t), \quad t > 0.$$  (4.68c)

We select $f(x, y)$, $u_0(x, y)$, $g_0(x, t)$ and $g_1(y, t)$ such that the exact solution is

$$u(x, y, t) = e^{x^2-y^2+t^2+1}.$$  (4.68d)

We solve this problem using the DG method (4.1), (4.2) on triangular meshes having $N = 48, 108, 192, 300$ elements obtained by dividing $\Omega$ into $n \times 3n/2$, $n = 4, 6, 8, 10$ rectangles and using diagonals parallel to $y = 2x/3$ to split each rectangle into two triangles. We compute the finite element solutions using the spaces $P_p$, $V_p$ and $U_p$ with $p = 0$ to 3 and plot the zero-level curves of the true error $(u - U)(x, y, 1)$ on each element in Figures 4.28, 4.29 and 4.30. We observe that the zero-level curves pass through the superconvergence points marked by $\times$.

The maximum errors at the superconvergence points described in Chapter 3 as well as their order of convergence at $t = 1$ are shown in Table 4.20. Time integration is performed using a fourth-order classical explicit Runge-Kutta method. On the space $P_p$ we compute the maximum error at the shifted roots of the $(p+1)$-degree Legendre polynomial over all outflow edges. If $U$ is in $V_p$, we compute the maximum error at the shifted Legendre roots and at the interior superconvergence points. If $U$ is in $U_p$, the maximum error is computed at 15 uniformly distributed points on each outflow edge and the interior superconvergence points described in Chapter 3. We observe $O(h^{p+2})$ convergence rates. This table indicate that the DG solution is $O(h^{p+2})$ superconvergent at the superconvergence points while globally it is only $O(h^{p+1})$ convergent. This is in full agreement with the theory. We note that $P_1 = U_0$ and $P_0 = V_0$ yield the same errors.

The global effectivity indices at $t = 1$ shown in Table 4.21 are close to unity and indicate that the a posteriori error estimates are asymptotically exact under both $h$- and $p$-refinement.

We perform a second experiment using $L^2$ projection of $u_0(x, y)$ into a $p^{th}$ degree piecewise polynomial on $\Delta$ for the initial conditions with all other parameters having the same values as before

$$\int_\Delta U(x, y, 0)V dxdy = \int_\Delta u_0(x, y)V dxdy, \quad \forall V \in W_p.$$  (4.69)

The maximum errors at the superconvergence points as well as their order of convergence at $t = 1$ are shown in Table 4.22. Again time integration is performed using a fourth-order classical explicit Runge-Kutta method. We observe $O(h^{p+2})$ convergence rates. This table indicate that the DG solution is $O(h^{p+2})$ superconvergent at the superconvergence points.

We also show the global effectivity indices in Table 4.23. We observe that while error estimates in both experiments converge to the true error under mesh refinement, results
obtained using the initial values by solving the problem at \( t = 0 \) are slightly better than those obtained using the \( L^2 \) projection. Since both methods produce similar results, for the remaining computational examples we will present numerical results using initial values obtained by solving the problem at \( t = 0 \) using the DG method.

As a final test, we consider the same problem and on each element we apply the error estimation procedure (4.54) to compute error estimates. The initial values are obtained by solving the problem at \( t = 0 \) using the DG method and then computing the error estimation. The initial values obtained using the \( L^2 \) projection gives similar results.

We present the global effectivity indices at \( t = 1 \) in Table 4.24 which indicate that the error estimates converge to the true errors under both \( h \)- and \( p \)-refinements. We conclude that the assumption \( E_t = 0 \) does not affect the quality of the \( a \) posteriori error estimate for \( u \). For the remaining computational examples we will present numerical results and we will assume the temporal discretization error to be negligible and compute estimates of the spatial discretization error.

We note that for all experiments the effectivity indices stay close to unity for a wide range of mesh sizes and polynomial degrees. Numerical results further indicate that the error estimates converge to the true error with decreasing mesh size and increasing polynomial degree \( p \).

Table 4.20: Maximum error at \( t = 1 \) and its order of convergence at superconvergence points for Example 4.9 on meshes having \( N = 48, 108, 192, 300 \) elements.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( p = 0 )</th>
<th>( p = 1 )</th>
<th>( p = 2 )</th>
<th>( p = 3 )</th>
</tr>
</thead>
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<tr>
<td></td>
<td>(</td>
<td></td>
<td>e</td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>1.4951e-2</td>
<td>2.8426e-4</td>
<td>3.9599e-6</td>
<td>4.6555e-8</td>
</tr>
<tr>
<td>108</td>
<td>6.8771e-3</td>
<td>1.9153</td>
<td>8.2899e-5</td>
<td>3.8829e-7</td>
</tr>
<tr>
<td>192</td>
<td>3.9115e-3</td>
<td>1.9614</td>
<td>3.8144e-5</td>
<td>2.9177e-7</td>
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<td>1.9997</td>
<td>1.9810e-5</td>
<td>2.9362e-7</td>
</tr>
</tbody>
</table>

In the next example we consider meshes consisting of elements of types I and II. We show that the results of Chapter 4 holds.

Example 4.10.
Figure 4.28: Zero-level curves of the true error at \( t = 1 \) for Example 4.9 on a mesh having \( N = 48 \) elements using the spaces \( P_p, \ p = 0, 1, 2, 3 \) (upper left to lower right).

Table 4.21: \( L^2 \) errors and global effectivity indices at \( t = 1 \) for Example 4.9 on uniform meshes having \( N = 48, 108, 192, 300 \) elements using and \( P_p, V_p, U_p, \ p = 1, 2, 3, 4 \).
Table 4.22: Maximum error at $t = 1$ and its order of convergence at superconvergence points for Example 4.9 on meshes having $N = 48, 108, 192, 300$ elements using $L^2$ projections.

<table>
<thead>
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<th>$|e|_\infty$</th>
<th>order</th>
<th>$|e|_\infty$</th>
<th>order</th>
<th>$|e|_\infty$</th>
<th>order</th>
<th>$|e|_\infty$</th>
<th>order</th>
</tr>
</thead>
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<tr>
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<tr>
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<td>1.1604e-7</td>
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</tbody>
</table>

$V_p^n$
Figure 4.30: Zero-level curves of the true error at $t = 1$ for Example 4.9 on a mesh having $N = 48$ elements using the spaces $U_p$, $p = 0, 1, 2, 3$ (upper left to lower right).

Table 4.23: $L^2$ errors and global effectivity indices at $t = 1$ for Example 4.9 on uniform meshes having $N = 48, 108, 192, 300$ elements using $L^2$ projections and $\mathcal{P}_p, \mathcal{V}_p, U_p$, $p = 1, 2, 3, 4$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$p = 0$</th>
<th>$\theta$</th>
<th>$p = 1$</th>
<th>$\theta$</th>
<th>$p = 2$</th>
<th>$\theta$</th>
<th>$p = 3$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4.0762e-7</td>
<td>0.99054</td>
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<tr>
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<tr>
<td>108</td>
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<td>2.1357e-5</td>
<td>0.98919</td>
<td>1.9829e-6</td>
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<td>5.6750e-8</td>
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</tr>
<tr>
<td>192</td>
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<td>0.98074</td>
<td>1.1288e-5</td>
<td>0.98985</td>
<td>8.2446e-7</td>
<td>0.99044</td>
<td>1.8646e-8</td>
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</tr>
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<td>6.6101e-6</td>
<td>0.98996</td>
<td>4.0762e-7</td>
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<td>7.6618e-9</td>
<td>0.99206</td>
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$\mathcal{V}_p$

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<td>1.9829e-6</td>
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</tr>
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<td>8.2446e-7</td>
<td>0.99044</td>
<td>1.8646e-8</td>
<td>0.99258</td>
</tr>
<tr>
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<td>5.5978e-4</td>
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<td>6.6101e-6</td>
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<td>0.99054</td>
<td>7.6618e-9</td>
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</tr>
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<td>5.9712e-6</td>
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<tr>
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<td>0.99234</td>
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<tr>
<td>192</td>
<td>7.9667e-4</td>
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<td>0.98985</td>
<td>8.2446e-7</td>
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</tr>
<tr>
<td>300</td>
<td>5.5978e-4</td>
<td>0.98102</td>
<td>6.6101e-6</td>
<td>0.98996</td>
<td>4.0762e-7</td>
<td>0.99054</td>
<td>7.6618e-9</td>
<td>0.99206</td>
</tr>
</tbody>
</table>

$U_p$
We consider the linear time-dependent problem
\[ u_t + 2u_x + u_y = f(x, y, t), \quad (x, y) \in [0, 1]^2, \quad 0 \leq t \leq 1, \quad (4.70a) \]
subject to the initial and boundary conditions
\[ u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega = [0, 1]^2, \quad (4.70b) \]
\[ u(x, 0, t) = g_0(x, t), \quad u(0, y, t) = g_1(y, t), \quad t > 0. \quad (4.70c) \]
We select \( f(x, y), u_0(x, y), \ g_0(x, t) \) and \( g_1(y, t) \) such that the exact solution is
\[ u(x, y, t) = e^{x+y+t}. \quad (4.70d) \]

We solve problem (4.70) on a uniform mesh obtained by partitioning the domain into \( n \times n \), \( n = 4, 6, 8, 10 \) squares and dividing each square into two triangles by connecting the upper-left and lower-right vertices. Thus, we obtain meshes having \( N = 32, 72, 128, 200 \) triangles of type I and II.

In order to obtain the \( O(h^{p+2}) \) superconvergence on global meshes having elements of type I and II, we used the modified DG method (4.1), (4.3) where the inflow boundary condition is corrected using the \textit{a posteriori} error estimate. We solve (4.70) using the space \( U_p, \ p = 1, 2, 3, 4, \ N = 32, 72, 128, 200 \) and in Table 4.25 we present the maximum error and order of convergence at the superconvergence points described in Chapter 3 and \( t = 1 \). Time integration is performed using a fourth-order classical explicit Runge-Kutta method. In Figure 4.31 we plot the zero-level curves of the true error \( (u - U)(x, y, 1) \) and mark the superconvergence points by \( \times \) for \( N = 32 \) and \( p = 1, 2, 3, 4 \), where we note that the level
curves pass close to the superconvergence points on most elements. These results show the modified DG method (4.34), (4.3) on $U_p$ yields $O(h^{p+2})$ superconvergent solutions at the Legendre points on the outflow edges of elements of type II and at the endpoints of the inflow edges of elements of type I.

The effectivity indices stay close to unity for all times and converge under $h$- and $p$-refinements. We present the global effectivity indices in Figure 4.32 versus $p$ and $N$ and in Table 4.26 at $t = 1$ which are close to unity and indicate that the $a$ posteriori error estimates are asymptotically exact under mesh refinement.

![Figure 4.31: Zero-level curves of the true error at $t = 1$ for Example 4.10 on a mesh having $N = 32$ elements using the spaces $U_p$, $p = 1, 2, 3, 4$ (upper left to lower right).](image)

We will consider a nonlinear examples to validate our superconvergence results for nonlinear problems with both smooth solutions and solutions with a shock.

**Example 4.11.**
Table 4.25: Maximum error at \( t = 1 \) and its order of convergence at the superconvergence points marked by \( \times \) in Figure 4.31 for Example 4.10 on meshes having \( N = 32, 72, 128, 200 \) elements with the spaces \( U_p, \ p = 1, 2, 3, 4 \).

<table>
<thead>
<tr>
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<th>( p = 3 )</th>
<th>( p = 4 )</th>
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<tbody>
<tr>
<td></td>
<td>(</td>
<td></td>
<td>e</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>3.0872e-3</td>
<td></td>
<td>1.2887e-4</td>
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</tr>
<tr>
<td>72</td>
<td>9.5836e-4</td>
<td></td>
<td>2.7178e-5</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>4.0928e-4</td>
<td></td>
<td>2.9575</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>2.1078e-4</td>
<td></td>
<td>2.9737</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.26: \( L^2 \) errors and global effectivity indices at \( t = 1 \) for Example 4.10 on uniform meshes having \( N = 32, 72, 128, 200 \) elements using \( U_p, \ p = 1, 2, 3, 4 \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( p = 1 )</th>
<th>( p = 2 )</th>
<th>( p = 3 )</th>
<th>( p = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(</td>
<td></td>
<td>e</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>4.0839e-2</td>
<td>0.97328</td>
<td>9.2538e-4</td>
<td>0.98968</td>
</tr>
<tr>
<td>72</td>
<td>2.3149e-2</td>
<td>0.97934</td>
<td>4.0341e-4</td>
<td>0.99139</td>
</tr>
<tr>
<td>128</td>
<td>1.5048e-2</td>
<td>0.98291</td>
<td>2.1321e-4</td>
<td>0.99205</td>
</tr>
<tr>
<td>200</td>
<td>1.0574e-2</td>
<td>0.98320</td>
<td>1.2486e-4</td>
<td>0.99216</td>
</tr>
</tbody>
</table>

Figure 4.32: Effectivity indices \( \theta_{L^2}(t) \) versus time for Example 4.10 using \( U_p \) with \( p = 1 \) and \( N = 32, 72, 128, 200 \) (left), \( N = 32 \) and \( p = 1, 2, 3, 4 \) (right).
We consider the inviscid Burger’s equation
\[ u_y + uu_x = f(x, y), \quad (x, y) \in [0, 1]^2, \quad (4.71a) \]
subject to the boundary conditions
\[ u(x, 0) = g_0(x), \quad \text{and} \quad u(y, 0) = g_1(x). \quad (4.71b) \]
We select \( f, g_0 \) and \( g_1 \) such that the exact solution is
\[ u(x, y) = \sqrt{1 + x^2 + 5y^2}. \quad (4.71c) \]

We solve (4.71) using the modified DG method (4.45), (4.3) on uniform meshes having \( N = 32, 72, 128, 200, 450, 800, 1250 \) triangular elements with the space \( \mathcal{U}_p \), \( p = 1, 2, 3, 4 \). Since the true solution is positive over the whole domain, our meshes will consist of elements of type I and II. We plot the 0-level curves of the DG error in Figure 4.33 with Legendre points (type II) and endpoints (type I) marked by \( \times \). In Table 4.27 we present the maximum errors and their order of convergence at the shifted roots of \((p + 1)\)-degree Legendre polynomial on the outflow edges on elements of type II and at the endpoints of the inflow edges for elements of type I. These results indicate that the \( p \)-degree DG solution is \( O(h^{p+2}) \) superconvergent at Legendre points on the outflow edges of elements of type II and at the endpoints of the inflow edges of elements of type II. Thus, our local superconvergence results in Chapter 3 on elements of type I and II extend to the whole mesh.

The \textit{a posteriori} error estimates are computed by solving the linear problem (4.47) and show the local effectivity indices in Figure 4.34. We also show the global effectivity indices in Table 4.28. Here, we applied Newton’s iteration to solve the nonlinear problem (4.48) where we used linear error estimate (4.47) as an initial guess and present the effectivity indices in Table 4.29. While the effectivity indices for both estimators are within 2\% from unity for most meshes and polynomial degrees, the error estimates obtained by solving the linear problem (4.47) are more efficient. Thus, for the remaining nonlinear computational examples we will present numerical results for the linear error estimator (4.47) only.

Table 4.27: Maximum error and its order of convergence at the superconvergence points for problem (4.71) on meshes having \( N = 32, 72, 128, 200 \) elements with the spaces \( \mathcal{U}_p \), \( p = 1, 2, 3, 4 \).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{N} & \text{p = 1} & \text{p = 2} & \text{p = 3} & \text{p = 4} \\
\hline
\text{||e||}_\infty & \text{order} & \text{||e||}_\infty & \text{order} & \text{||e||}_\infty & \text{order} & \text{||e||}_\infty & \text{order} \\
\hline
32 & 8.4554e-4 & 1.5292e-4 & 6.7141e-8 & 4.8521e-8 \\
128 & 1.1330e-4 & 2.8959 & 4.1362e-6 & 3.9477 & 7.3728e-10 & 4.9050 & 2.1447e-10 & 5.9095 \\
200 & 5.9373e-5 & 2.8959 & 4.1362e-6 & 3.9275 & 7.3728e-10 & 4.9050 & 2.1447e-10 & 5.9095 \\
\hline
\end{array}
\]

**Example 4.12.**
Table 4.28: Effectivity indices Problem (4.71) using the linearized error estimator (4.47) on meshes having 32, 72, 128, 200, 400, 800 and 1250 elements and $U_p$, $p = 1, 2, 3, 4$.  

<table>
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<tr>
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<th>$p = 3$</th>
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<td>0.9982</td>
<td>0.9808</td>
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<tr>
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<td>0.9941</td>
<td>0.9932</td>
<td>1.0031</td>
<td>0.9982</td>
</tr>
<tr>
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<td>0.9943</td>
<td>1.0039</td>
<td>1.0034</td>
</tr>
<tr>
<td>200</td>
<td>0.9948</td>
<td>0.9946</td>
<td>1.0045</td>
<td>1.0039</td>
</tr>
<tr>
<td>400</td>
<td>0.9949</td>
<td>0.9948</td>
<td>1.0049</td>
<td>1.0040</td>
</tr>
<tr>
<td>800</td>
<td>0.9949</td>
<td>0.9948</td>
<td>1.0050</td>
<td>1.0041</td>
</tr>
<tr>
<td>1250</td>
<td>0.9949</td>
<td>0.9949</td>
<td>1.0052</td>
<td>1.0042</td>
</tr>
</tbody>
</table>

Table 4.29: Global effectivity indices for problem (4.71) using the nonlinear error estimator (4.48) on meshes having 32, 72, 128, 200, 450, 800 and 1250 elements and $U_p$, $p = 1, 2, 3, 4$.  

<table>
<thead>
<tr>
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<td>0.9832</td>
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<td>0.9945</td>
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<td>1.0032</td>
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<tr>
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<td>0.9949</td>
<td>1.0052</td>
<td>1.0043</td>
</tr>
</tbody>
</table>

Figure 4.33: Zero level curves for problem (4.71) on a uniform mesh having 32 elements with $U_p$, $p = 1, 2, 3, 4$, (upper left to lower right).
Figure 4.34: Local effectivity indices for problem (4.71) on a uniform mesh having 32 elements with $\mathcal{U}_p$, $p = 1, 2, 3, 4$, (upper left to lower right).
Here we consider the homogeneous inviscid Burger’s equation

\[ u_y + uu_x = 0, \quad (x, y) \in [0, 1] \times [0, b], \quad b > 0, \]

subject to the initial conditions

\[ g_0(x, 0) = 1 + \frac{1}{2} \sin(2\pi x). \]

and select \( g_1(0, y) \) such that the true solution is periodic and forms a shock discontinuity at the point \( (\frac{1}{2}, \frac{1}{2}) \), which propagates along \( y = x \).

First, we apply the modified DG method (4.45), (4.3) to solve this problem on \([0, 1] \times [0, 0.3]\) with a smooth solution on uniform meshes having 32, 72, 128, 200, 450, 800, 1250 elements of type I and II with \( U_p, \quad p = 1, 2, 3, 4 \). We compute an error estimate by solving (4.47) and present global effectivity indices in Table 4.30. The computational results show that the effectivity indices converge to unity under \( h \)-refinement.

Table 4.30: Effectivity indices for problem (4.72) on \([0, 1] \times [0, 0.3]\) with the error estimate (4.47) on meshes with \( N = 32, 72, 128, 200, 450, 800, 1250 \) elements and \( U_p, \quad p = 1, 2, 3, 4 \).

<table>
<thead>
<tr>
<th>( N )</th>
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<th>( p = 2 )</th>
<th>( p = 3 )</th>
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<tbody>
<tr>
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<td>1.0423</td>
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<tr>
<td>128</td>
<td>1.0157</td>
<td>0.9636</td>
<td>0.9561</td>
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</tr>
<tr>
<td>200</td>
<td>1.0214</td>
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<td>0.9315</td>
<td>0.9681</td>
</tr>
<tr>
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<td>0.9458</td>
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<tr>
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<td>1.0230</td>
<td>1.0136</td>
<td>0.9878</td>
<td>0.9958</td>
</tr>
<tr>
<td>1250</td>
<td>1.0197</td>
<td>1.0067</td>
<td>0.9867</td>
<td>0.9969</td>
</tr>
</tbody>
</table>

Next, we solve (4.72) on \([0, 1] \times [0, 0.999]\) using uniform meshes with \( N = 32, 72, 128, 200, 450, 800, 1250 \) and plot the zero-level curves of the DG error for \( N = 32 \) and \( p = 1, 2 \) in Figure 4.35 with Legendre points for every element of type II and endpoints of \( inflow \) edges of elements of type I marked by \( \times \). We observe that the level curves pass close to the superconvergence points on most triangles away from the shock discontinuity.

These computational results indicate that the maximum errors at the shifted roots of \((p+1)\)-degree Legendre polynomial on the \( outflow \) edge (for type II) and at the endpoints (for type I) on elements away from the discontinuity converge as \( O(h^{p+2}) \) under mesh refinement as shown in Table 4.31. The results shown in Table 4.32 indicate that superconvergence is lost near the shock discontinuity.

Our superconvergence analysis perform poorly on elements near the discontinuity. Since we are not using limiting to suppress spurious oscillations near the discontinuity, the region around the discontinuity where the error is underestimated gets wider as \( p \) increases.
Table 4.31: Maximum errors and orders of convergence at the superconvergence points in the strip $[0,1] \times [0,0.3]$ for Example 4.12 on meshes having $N = 32, 72, 128, 200$ elements with the spaces $\mathcal{U}_p$, $p = 1, 2, 3, 4$.

<table>
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<tbody>
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<td>$5.8855e-4$</td>
<td></td>
<td>$9.8014e-6$</td>
<td></td>
<td>$5.9773e-7$</td>
<td></td>
</tr>
<tr>
<td>72</td>
<td>$3.0301e-2$</td>
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<td>$2.9381$</td>
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<td>$3.9240$</td>
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<tr>
<td>128</td>
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<td>$2.9039$</td>
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<td>$3.9004e-5$</td>
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<td>$2.9108$</td>
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<td>$1.6256e-5$</td>
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<td>$4.8670$</td>
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</tr>
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</table>

Table 4.32: Maximum errors and orders of convergence at the superconvergence points in $[0,1] \times [0,0.999]$ for Example 4.12 on meshes having $N = 32, 72, 128, 200$ elements with the spaces $\mathcal{U}_p$, $p = 1, 2, 3, 4$.

<table>
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<th>order</th>
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<tbody>
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</tr>
<tr>
<td>72</td>
<td>$6.5140e-3$</td>
<td></td>
<td>$2.2770$</td>
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<td>$2.4891e-4$</td>
<td></td>
<td>$3.0311$</td>
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</tr>
<tr>
<td>128</td>
<td>$3.3835e-3$</td>
<td></td>
<td>$2.2770$</td>
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<td>$2.4891e-4$</td>
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<td>$3.0311$</td>
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<tr>
<td>200</td>
<td>$2.2358e-3$</td>
<td></td>
<td>$1.8567$</td>
<td></td>
<td>$1.2656e-4$</td>
<td></td>
<td>$3.2016$</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.35: Zero level curves for of the error using $p = 1, 2$ approximations with 32 elements (left to right) for Example 4.12 on $[0,1] \times [0,0.999]$. 
Thus, our superconvergence results hold only on elements with a smooth solution and away from the discontinuity.

We conclude by solving the previous problem on $[0, 1] \times [0, 0.999]$ on two uniform meshes having $N = 1800, 20000$ elements and $U_p, \ p = 1, 2$. We plot the local effectivity indices in Figure 4.36. These computational results indicate that our theoretical results are valid for nonlinear problems in regions where the solution is smooth and away from discontinuities. Thus, the local effectivity indices converge to unity under mesh refinement in regions where the solution is smooth.

![Figure 4.36: Local effectivity indices for the homogeneous Burger’s equation 4.12 with initial condition (4.72b) on $[0, 1] \times [0, 0.999]$ using meshes having $(N, p) = (1800, 1), (1800, 2), (20000, 1), (20000, 2)$ (upper left to lower right).](image)

### 4.6 Implementation

The nonlinear systems are solved with a Newton’s method. These codes were running using Matlab version 7 release 14, on a Dell laptop PC Pentium M with processor 1.86 GHz,
1.00 GB of RAM. Computer code for obtaining the basis function for the \textit{a posteriori} error estimation were written in Mathematica.

4.7 Conclusion

In this chapter we have extended the one-dimensional DGM \textit{a posteriori} and superconvergence results of Adjerid \textit{et al.} \cite{adjerid} to two-dimensional hyperbolic problems on triangular meshes. To maintain the superconvergence results for the global solution one needs to use a $O(h^{p+2})$ approximation of boundary conditions at the \textit{inflow} boundary of every element. Since we know the \textit{inflow} boundary conditions this is not a problem for all elements whose \textit{inflow} boundary is on the \textit{inflow} boundary of the problem. For the remaining elements we correct the solution by adding an error estimate and use it for \textit{inflow} boundary condition. We also determined explicitly the form of the leading asymptotic term of the local error which is used to discover new superconvergence properties of the DG method. Finally, we note that our error estimates are not accurate near discontinuities.

Triangles meshes are very useful in applications especially when we have complicated domains. These domains can be most easily partitioned into unstructured triangle meshes. In order to obtain superconvergence results for all types of elements and construct asymptotically correct \textit{a posteriori} error estimates on general unstructured meshes, it is recommended to use a DG method with corrected \textit{inflow} boundary conditions and the augmented space $U_p$. In the next chapter, extend these results to unstructured meshes in two dimension for hyperbolic problems.