

Chapter 6

Conclusion and Future Work

6.1 Contributions

Discontinuous Galerkin (DG) methods have gained in popularity during the last 25 years because of their ability to address problems having discontinuities, such as those that arise in hyperbolic conservation laws. DG method use a discontinuous finite element basis which simplifies hp -adaptivity and leads to a simple communication pattern across faces that makes them useful for parallel computation. In order for the DG method to be useful in an adaptive setting, techniques for estimating the discretization errors should be available both to guide adaptive enrichment and to provide a stopping criteria for the adaptive solution process.

In this work we proved new superconvergence results for DG solutions. We have performed a numerical and theoretical study of the existence of superconvergent points for DG methods applied to a two-dimensional hyperbolic problems. Several superconvergence phenomena have been discovered. The local error analysis is performed on three types of elements: (i) elements with one *inflow* and two *outflow* edges (type I), (ii) elements with two *inflow* and one *outflow* edges (type II), and (iii) elements with one *inflow* edge, one *outflow* edge and one edge parallel to the characteristics (type III). We have studied the effect of finite element spaces on the superconvergence properties of DG solutions on all three types of triangular elements. In particular, we showed that the DG solution on elements of type I is $O(h^{p+2})$ superconvergent at the two vertices of the *inflow* edge using the space \mathcal{U}_p . On elements of type II, the DG solution is $O(h^{p+2})$ superconvergent at Legendre points on the *outflow* edges for \mathcal{P}_p , \mathcal{V}_p and \mathcal{U}_p as well as at interior mesh-orientation dependent points for the polynomial spaces \mathcal{V}_p and \mathcal{U}_p . On elements of type III, the DG solution is $O(h^{p+2})$ superconvergent at the Legendre points on the *outflow* edge using \mathcal{P}_p and \mathcal{V}_p while it is $O(h^{p+2})$ superconvergent at every point of the *outflow* edge using \mathcal{U}_p . Furthermore, we discovered additional mesh-orientation dependent superconvergence points in the interior of triangles. The dependence of these points on mesh orientation is explicitly given. We also showed that, locally, the

flux is $O(h^{2p+2})$ superconvergent on average on the *outflow* boundary of the first *inflow* element.

We proved a global superconvergence result on meshes consisting of triangles of type III having one *inflow* and one *outflow* edges. In particular, we show that the global error on the *outflow* boundaries is superconvergent at the roots of Legendre polynomial of degree $p + 1$ and that the flux is strongly superconvergent on average on the *outflow* boundaries while the above superconvergence results hold on meshes consisting of elements of type III only, DG solutions do not exhibit superconvergence on meshes consisting of elements of type I which prevent superconvergence from propagating through the whole mesh. In order to recover the superconvergence for the global solution on general meshes we used a modified DG method with corrected *inflow* boundary conditions and the augmented space \mathcal{U}_p . This correction is possible both on elements whose *inflow* edges are on the *inflow* boundary of the domain and on interior elements where we correct the solution by adding an error estimate and use it as an *inflow* boundary condition to obtain $O(h^{p+2})$ approximations of inflow boundary conditions.

We have applied our local superconvergence results to construct simple, efficient and accurate *a posteriori* estimates of the two-dimensional DG finite element error for hyperbolic problems on triangular meshes. The error estimation procedure presented in this work yields error estimates in several norms and does not require communication across neighboring elements which makes it useful for parallel computations. *A posteriori* error estimates are also needed to guide adaptive enrichment and to provide a measure of solution accuracy for any numerical method. We developed an inexpensive *a posteriori* error estimation technique for DG solutions of conservation laws based on superconvergence. We explicitly write the basis functions for the error spaces corresponding to several finite element spaces. The leading term of the discretization error on each triangle was estimated by solving a local problem where no boundary conditions are needed which leads to a much more efficient estimates than those presented in [45]. The computed error estimates are shown to converge to the true error under mesh refinement in smooth solution regions. We further presented a numerical study of superconvergence properties for the DG method applied to time-dependent convection problems. We also constructed asymptotically correct *a posteriori* error estimates by solving local hyperbolic problems with no boundary conditions on general unstructured meshes. Global superconvergence results are numerically confirmed for transient problems. Finally, the *a posteriori* error estimates were tested on several linear and nonlinear problems to show their efficiency and accuracy under mesh refinement.

There are several limitations to the current theory which should be overcome. Some of these are relatively simple and were omitted to maintain a clarity of the presentation. They include system of hyperbolic problems in several space dimensions of the form (1.3). Others, such as problems where coefficients change rapidly or nonlinear problems with discontinuities offer significant challenges and may, indeed, not be amenable to our approach.

General domains can be most easily partitioned into unstructured triangle meshes. In order

to obtain superconvergence results for all types of elements and construct asymptotically correct *a posteriori* error estimates on general unstructured meshes, it is recommended to use a DG method with corrected *inflow* boundary conditions i.e., higher-order flux, and the augmented space \mathcal{U}_p . If not, the DG solution does not exhibit any superconvergence.

6.2 Future Work

There are several long term goals in this dissertation. We plan to develop a *hp*-adaptive code of the modified DG method applied to hyperbolic systems on general unstructured triangular meshes. We plan to study the superconvergence properties of the DG method applied to hyperbolic problems on tetrahedral meshes including the rates of convergence and hope to derive error estimates. We intend to extend the *a posteriori* error estimates and superconvergence results of the DG method to locally refined meshes with hanging nodes and unstructured tetrahedral meshes. We note that our error analysis does not apply near discontinuities and, so far, we are not able to construct asymptotically correct error estimates near discontinuities. We predict that the *a posteriori* error estimates developed in this dissertation can be extended to three-dimensional tetrahedral elements. This constitutes a future focal point.

High-order numerical schemes produce spurious oscillations near discontinuities, which may, indeed, lead to nonlinear (numerical) instabilities and unbounded computational solutions. Less severe oscillations may produce nonphysical solutions, such as negative pressures or temperatures, which would generally lead to physical instabilities. We plan to describe a strategy for detecting discontinuities and for limiting spurious oscillations near such discontinuities when solving hyperbolic systems of conservation laws by high-order DG methods. The approach will be based on the strong superconvergence results in smooth regions.

The theoretical analysis in higher dimensions is extremely difficult and is a subject of ongoing research. This preliminary analysis illustrates new differences among DG methods. The existence of superconvergent points for three-dimensional hyperbolic problems on tetrahedral meshes remains an open problem and will be investigated in the future. We expect that some types of elements will have simple properties while others will not. Additional work will be necessary to establish rigorous proofs of the global *a posteriori* error analysis for elements of type I and II. The proof of a global superconvergence result will also be investigated.