Geometric Properties of Over-Determined Systems of Linear Partial Difference Equations

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We relate linear constant coefficient systems of partial difference equations (a discretization of a system of linear partial differential equations) satisfying some collection of scalar polynomial equations to systems defined over the coordinate ring of an algebraic variety. This motivates the extension of behavioral systems theory (a generalization of classical systems theory where inputs and outputs are lumped together) to the setting where the ring of operators is an affine domain and the signal space is restricted to signals which satisfy the same scalar polynomial equations. By recognizing the role of the kernel representation's Gröbner basis in the Cauchy problem, we extend notions of controllability from the classical behavioral setting to accommodate this generalization. We then address the question as to when an autonomous behavior admits a Liv̌sic-system state-space representation, where the state update equations are overdetermined leading to the requirement that the input and output signals satisfy their own compatibility difference equations. This leads to a frequency domain setting involving input and output holomorphic vector bundles and a transfer function given by a meromorphic bundle map. An analogue of the Hankel realization theorem developed by J. Ball and V. Vinnikov then leads to a Livšic-system state-space representation for an autonomous behavior satisfying some natural additional conditions.

Dedication

The memories of Israel Gohberg and Israel Gelfand.

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Chapter 1

Introduction

Before discussing the goals and results presented throughout this work, we must provide sufficient motivation for even considering these goals. We begin with a common "sales pitch" which can be found in the introductory sections of almost every paper/book on behavioral systems theory (e.g., [41, 51].) For what follows, we limit our discussion to *discrete-time* systems.

One of the most famous linear systems is the first-order input/state/output system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.1}$$

where x is the **state signal**, u is the **input signal**, y is the **output signal**, A, B, C, and D are constant matrices of appropriate sizes and each of these signals are defined over the **time domain** consisting of the non-negative integers $\mathbb{N}$. In the context of applications, one is concerned with certain properties exhibited by systems of the form (1.1). One of the most practical is whether or not the system is **controllable**, i.e., given that the system is at a current state, is there a finite amount of input one can feed into the system to reach a desired state? Another question is whether or not the output of the system gives rise to a unique state consistent with the observations; more precisely, for observed output y and input u, is there a unique initial state $x(0)$ for which y is given by (1.1)? In this situation we say that the system is **observable**. When systems of the form (1.1) are used to model phenomena over time, then there is a natural ordering of events which is consistent with the state evolution; this is usually referred to as **causality**.

We now analyze systems like (1.1) from a different point of view. Consider a difference equation of the form

$$y(t) = u(t) + u(t+1) + u(t+2) \tag{1.2}$$

To determine the value of y at time t, we must know what the input is for three different times; however, two of these measurements are in the future. To get around this, let us define the signals

$$\begin{aligned} x_1(t) &= u(t) \\ x_2(t) &= u(t+1) \qquad\qquad u'(t) = u(t+2), \end{aligned}$$

so that (1.2) becomes

$$y(t) = u'(t) + x_1(t) + x_2(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u'(t).$$

By the definition of x_1 and x_2, we have

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} u(t+1) \\ u(t+2) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ u'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u'(t).$$

This leads us to the input/state/output system

$$\begin{aligned} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u'(t) \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u'(t). \end{aligned}$$

We thus have a parametrization of the solutions of (1.2) by providing the **initial state** $x(0) := \left[\begin{smallmatrix} x_1(0) \\ x_2(0) \end{smallmatrix}\right]$ and the input signal $u'(t)$. By turning (1.2) into a **first order system** (i.e. a system of the form (1.1)), we produce a mechanism for solving the system iteratively: for every point in time, we feed an input into the system and the corresponding output is such that the trajectories satisfy the difference equation (1.2).

This leads us to what is known as a **behavior** or **behavioral system**. For signals of the form $w : \mathbb{N} \to \mathbb{R}$ and a polynomial matrix R we define the behavior as the set of solutions to the homogeneous system of difference equations

$$\mathcal{B} = \ker(R) = \{w = (w_1, \ldots, w_q) \in \{v : \mathbb{N} \to \mathbb{R}\}^q : Rw = 0\}.$$

Here the meaning of Rw is as follows. For $w \in \{v : \mathbb{N} \to \mathbb{R}\}$ we define $zw(t) \mapsto w(t+1)$ (i.e., $\mathbb{R}[z]$ acts on signal via the backward shift). For $R \in \mathbb{R}[z]^{p \times q}$ we then define Rw for $w \in \{v : \mathbb{N} \to \mathbb{R}\}$ by the standard matrix/vector operations. As an example, consider the case when $R = \begin{bmatrix} f(x) \end{bmatrix}$ and $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$; we then have

$$\mathcal{B} = \ker(R) = \{w \in \{v : \mathbb{N} \to \mathbb{R}\} : a_n w(t+n) + \cdots + a_1 w(t+1) + a_0 = 0 \text{ for all } t \in \mathbb{N}\}.$$

From a behavioral point of view one studies $\mathcal{B}$ as an $\mathbb{R}[z]$-module and its corresponding algebraic (and sometimes topological) structure. In this way, we have that $\mathcal{B}$ consists of all *solutions* to a homogeneous system of linear constant coefficient difference equations.

Going back to (1.2), one can see that all we really have is a difference equation. In the context of constructing solutions to this equation, there are no issues of causality, observability, or state. As a result, another route could be to just ignore the idea of state, input, and output. That is, let us just consider the set

$$\mathcal{B} = \{u, y \in \mathbb{R}^{\mathbb{N}} : y(t) = u(t) + u(t+1) + u(t+2) \text{ for all } t \in \mathbb{N}\}.$$

If one were to come up with some notion of input, output, state, or any other factor to reach a first order system, it would *always* bring the (observed) trajectories of the system back to $\mathcal{B}$. In other words, the notions of state, input and output are just convenient ways to parametrize $\mathcal{B}$ in terms of inputs and initial states; but if we ignore the state, changes of variables and those matters, the set of solutions will always be the same. As stated by Willems and repeated by many others, the set of solutions "is what it is."

Let us take another example to bring one other point to light. Consider the simple equation

$$F(t) = ma(t)$$

where we have F as force, a as acceleration, and m as a constant mass. In one application, the phenomena being modeled may have force as an adjustable value (i.e., an input) and in another application the acceleration could be the adjustable value. However, no matter what the application is, we have that it must always be the case that the derived signals are equal to

$$\mathcal{B} = \{F, a \in \mathbb{R}^{\mathbb{N}} : F(t) = ma(t) \text{ for all } t \in \mathbb{N}\}.$$

That is, no matter what parametrization is chosen, the space of trajectories satisfying the specified governing equations is equal to $\mathcal{B}$.

In both examples, the solution space of the system of difference equation is the common denominator. From a modeling point of view, the choice of inputs, outputs, and state each have physical ramifications. But from a mathematical point of view, none of these modeling decisions have any bearing on the solutions space since they are just a means of *parametrizing the solution space.* The purpose of behavioral systems theory is to keep the theoretical study of systems at the level of solutions to difference equations and provide a common language to relate systems under different modeling choices. In this way, we have that a **behavior** (the space of solutions) is the most fundamental form of a system. It may give rise to many different models, but each has the same set of trajectories.

At this point, we recognize that there is a fine line between being a practitioner of a theory and a developer of a theory. From an engineering point of view, systems like (1.1) are useful because they allow one to satisfy a set of governing equations like (1.2) while adjusting the input. The first-order structure allows a system like (1.1) to be used in tracking submarines, performing course corrections, or adjusting the response pattern of a hydrophone. A difference equation such as (1.2) has very little application in these iterative processes because it (at least appears) to require having the entire solution at once. To make the theory viable for modeling, one must find the structure sought out in an input/state/output model and present it in terms which do not depend on the choice of modeling parameters. For instance, a behavior which is controllable must yield a controllable input/state/output system and vice versa. To some extent, this has been accomplished for systems which exhibit suitable amount of "freeness." To put this last point into context, we now discuss the main problem.

The Main Problem

In the case of two-dimensional systems, there is a zoo of input/state/output models. Each model is an answer to one of the following questions.

1. (Givone-Roesser Systems) Should the state consist of "independent" components, each keeping information about evolution in independent directions?

2. (Fornasini-Marchesini Systems) Should we specify a direction of state-evolution and say that evolution of the state signal occurs in this direction?

3. (Livšic Systems) Can we specify state evolution equations for the state in each direction and ask that they be consistent?

The first two systems are quite well known since they provide one with a system which has free input, i.e., the choice of input at any given time does not place any limitations on the input at any other time. Livšic systems do not exhibit this property since the consistency condition, implicitly, is limiting the values that the input can take. The goal of this work is to put Livšic systems on the same footing as the other two system in behavioral systems theory. We now discuss what we mean by this last statement and why it is not a trivial task.

A Livšic system comes from[1] a two-dimensional input/state/output system of the form

$$\begin{aligned} x(t_1+1,t_2) &= A_1x(t_1,t_2) + B_1u(t_1,t_2) \\ x(t_1,t_2+1) &= A_2x(t_1,t_2) + B_2u(t_1,t_2) \\ y(t_1,t_2) &= Cx(t_1,t_2) + Du(t_1,t_2), \end{aligned} \tag{1.3}$$

where x is the state signal, y is the output signal, and u is the input signal. The solution space of a linear system must be linear; as such, it must allow for the zero signal to be one which satisfies the equations. We may choose to put zero input into the system and observe the state evolution to see

$$\begin{aligned} x(t_1+1,t_2+1) &= A_1x(t_1,t_2+1) = A_1A_2x(t_1,t_2) \\ x(t_1+1,t_2+1) &= A_2x(t_1+1,t_2) = A_2A_1x(t_1,t_2). \end{aligned}$$

Another modeling decision one usually makes is that every initial state of a system is admissible. As a consequence we must have commuting matrices $A_1A_2 = A_2A_1$ if this condition were to be allowed. For whatever input we have, consistent state evolution demands equality of the following

$$\begin{aligned} x(t_1+1,t_2+1) &= A_1(A_2x(t_1,t_2) + B_2u(t_1,t_2)) + B_1u(t_1+1,t_2) \\ x(t_1+1,t_2+1) &= A_2(A_1x(t_1,t_2) + B_1u(t_1,t_2)) + B_2u(t_1,t_2+1). \end{aligned}$$

Equality of the above and commutativity of A_1 and A_2 provides

$$A_1B_2u(t_1,t_2) + B_1u(t_1+1,t_2) = A_2B_1u(t_1,t_2) + B_2u(t_1,t_2+1).$$

That is, consistency of state-evolution requires that the input signal must solve a system. Due to the fact that the output is given by the state and input, we have that the input, state, and output must satisfy some conditions which limit the values they may take. In particular, the input signal *cannot be freely assigned* as in the case of Givonne-Roesser models and Fornasini-Marchesini models.

[1] A Livšic system is given by (1.3) with the addition of some various factorizations and compatibility conditions so that it admits enough structure to study in general.

One can easily write the set of solutions for a Livšic system as

$$\mathcal{B} = \{u, y : \text{ there exists } x \text{ such that } (u, x, y) \text{ satisfy } (1.3)\}.$$

By some basic matrix manipulations, one can even write down a somewhat explicit form so that it is given as the kernel of a polynomial matrix. However, is it possible to translate from an arbitrary behavior to a Livšic system? This, as we shall see, is not trivial since Livšic systems have many consistency equations which must be satisfied. There are other questions one might ask in terms of preservation of structural properties. For instance, since Livšic systems have their own definition for controllability, is there a way to observe this in the associated behavior? In order to address these questions, we must significantly rework the entire theory of behavioral systems.

For discrete-time behavioral systems, one usually works with the operator ring $\mathcal{D} = k[z_1, \ldots, z_d]$ (here, k is a field) and a signal space $\mathcal{A} = k^{\mathbb{N}^d}$. A monomial $\mathbf{z}^\alpha \in \mathcal{D}$ acts on a signal by the backward shift (i.e., $\mathbf{z}^\alpha w(t) = w(t + \alpha)$) and a scalar $a \in k$ acts on the value of the signal via point-wise multiplication. By extending these operations k-linearly, we have that $\mathcal{A}$ becomes a $\mathcal{D}$-module. Behaviors are, by definition, defined as the kernel of a polynomial matrix, i.e., for any $R \in \mathcal{D}^{p\times q}$ we have a behavior

$$\mathcal{B} = \ker_{\mathcal{A}}(R) = \{w \in \mathcal{A}^q : Rw = 0\}$$

and for every behavior $\mathcal{B}$ there exists a polynomial matrix R such that $\mathcal{B} = \ker_{\mathcal{A}}(R)$. In the study of behaviors, most of the literature involves the study of kernels of polynomial matrices as stated. Unfortunately, this special choice of $\mathcal{D}$ and $\mathcal{A}$ is not sufficient to study Livšic systems if we wish to relate certain structural properties.

In the study of Livšic systems, one notices that every trajectory $w = (x, u, y)$ (something which satisfies (1.3)) always satisfies some scalar polynomial equations. That is,

$$I := \{p \in \mathcal{D} : pw = 0 \text{ for all } w = (x, u, y) \text{ which satisfy } (1.3)\},$$

is a non-zero ideal in $\mathcal{D}$. Since I is non-zero, it turns out that most of the techniques employed in behavioral system theory become vacuous. In particular, if we lift to the quotient field to perform calculations we end up with the zero vector space.

One of the differences between behavioral systems and the study of systems with trajectories in ℓ^p is that one makes *no* assumptions on the signal space. It consists of every possible signal one could imagine. Naturally, we do not wish to diverge from this key aspect of the theory. However, since it must be the case that every system trajectory satisfies the scalar polynomial equations in I, we may at least restrict the signal space to the ones which already satisfy these equations. If we use the ring $\mathcal{D}_r = \mathcal{D}/I$ instead of $\mathcal{D}$ we have a well-defined $\mathcal{D}_r$ action on

$$\mathcal{A}_r := \{w \in k^{\mathbb{N}^d} : pw = 0 \text{ for all } p \in I\} \tag{1.4}$$

As explained above, this signal space consists of all of the solutions that we are looking for. It turns out that, after this slight change (and provided $\mathcal{D}_r$ is an integral domain), all of the algebraic theory opens up to us once again. In particular, techniques which rely on lifting to the quotient field to perform calculations are now no longer trivial since we have the solutions space now has a zero annihilator over the new ring $\mathcal{D}_r$; hence, there is *something* that is torsion-free.

This last point becomes more clear when we think of what linear algebra is accomplishing. If we have a vector space and a linear map on this space, then we can use relations in the kernel to "carve out" a subspace; in particular, we may *identify a vector space basis* for the kernel. In the classical behavioral setting, it turns out that the subspace we carve out corresponds to the controllable part of the system. However, when the system is all torsion (i.e., every component of the signal satisfies a non-zero scalar equation), the solution space is too complicated for there to be any usable basis. The reasoning is that scalar equations, unlike vector relations, really depend on the field of the vector space. As a result, if we change the field to something more suitable then we may once again identify subspaces. We then have that what is left is controllable, but in an entirely foreign sense.

One interesting aspect of behavioral systems is that there is a categorical duality between systems and finitely generated modules. In fact, after choosing a suitable topology, one has that this correspondence is almost akin to the one shared by algebraic sets and ideals in $\mathcal{D}$. (In particular, the topology is about just as weak as the Zariski topology.) However, since systems involve trajectories, we have that there is a correspondence between trajectories and how modules act on these trajectories. In the scalar case, there is a connection between a coordinate ring and the way this coordinate ring acts on discrete trajectories. Although not complete, we introduce some necessary ingredients which establish a connection between behaviors and their corresponding dual modules in this setting where we change the operator ring to a quotient ring. Unfortunately, this new approach is much more difficult to understand and thus has many subtle points which are still unknown. In particular, we have that the most natural correspondence is not between systems and modules, but systems and sheaves. In other words, one must consider locally what is occurring and not just globally.

The introduction of this machinery allows us to study Livšic systems with tools that are mainstream in classical behavioral systems theory; furthermore, we may continue to use familiar concepts (such as transfer functions, inputs, outputs, controllability, etc.) in this new setting. Most importantly, it presents us with a way to approach this correspondence between Livšic systems and behaviors both rigorously and systematically.

Before concluding, we point out that, since most of this material is new, the search for intuition has to be done with some concern. In particular, we must be careful to notice assumptions which are implicitly satisfied in the classical setting but whose validity in the new setting is no longer so automatic. This leap is comparable to switching from modules to sheaves of modules: one wants global intuition to apply, but it is, in general, not always the case that it is true. One positive side to the generality, however, is that this new approach to behavioral systems opens a door to a much broader class of problems and also ties it to some of the more classical fields of mathematics.

Outline

Chapter 2 provides an overview of the ingredients from behavior systems that are used throughout this paper. Due to the necessary generality, we focus more on Oberst's approach to behavioral systems rather than the more conventional approach. (In particular, we make extensive use of signal flow systems.) We review some of the used topics from [39] and present them in a somewhat clearer form and with more modern terminology – especially the discussion involving Gröbner bases. Following this abstract treatment of systems, we present terminology which is common in the behavioral systems theory; this latter point is meant to provide context for the discussion which takes place in Chapter 5.

Chapter 3 discusses pure autonomy degree and the corresponding decomposition of the behavior by its annihilator. This is tied to trajectories in the behavior by means of a result obtained by Wood, Rogers and Owens in [53] which relates autonomy degree to the dimension (i.e., dimension of the quotient ring by the annihilator) of a behavior. In some sense, this result can be observed through the "initial condition set" involved in the canonical Cauchy problem by means of using the normal form for a Gröbner basis as a splitting of the vector space $k^{q\times\mathbb{N}^d}$. However, the autonomy degree is abstract (i.e., does not require a monomial ordering) and is based on the transcendence degree of field extension $k \subset Q(\mathcal{D}/\mathrm{Ann}(\mathcal{B}))$ as a means of defining the dimension of a ring. Following the discussion of autonomy degree, we use the primary decomposition of the annihilator and give explicit forms for the kernel representations of the behavior decomposed by its annihilator. We then group the decomposition by dimension so that the resulting behaviors have exponential trajectories with frequencies lying on algebraic varieties of the same dimension. In the absence of embedded primes, we have that this decomposition is unique. As a final result of the section, we provide necessary and sufficient conditions for the decomposition to be a direct sum.

Chapter 4 discusses exponential trajectories and their associated family of amplitude vector spaces. This sets the stage for developing a frequency domain theory of autonomous behaviors. Provided that the characteristic variety is a complex manifold, we can interpret the transfer function as a meromorphic bundle map between holomorphic vector bundles which correspond to the input and output amplitudes. Furthermore, this framework is also "behavioral" in that we can consider the system without an input/output structure; in this setting, we provide conditions for the behavior to give rise to a holomorphic vector bundle.

Chapter 5 discusses autonomous behaviors and a framework for putting them on equal footing with non-autonomous behaviors. Many of the results take advantage of the fact that Oberst developed a theory of multidimensional systems over affine domains. As a result, we present some of the mainstream terminology in this new setting. To facilitate the transition we introduce and use reduced behaviors, i.e., we reduce the signal space to signals which satisfy the annihilator conditions and switch to an appropriate ring of operators as is discussed around (1.4) above. The main observation in this setting is that controllability has natural algebraic and trajectory-based forms; however, the connection between the two becomes less direct since the algebraic form (image representation) necessarily requires that the ring be an integral domain while the trajectory-based form (j-controllability) only requires that the ring be equidimensional (hence, the behavior has pure autonomy degree). We present a special form of j-controllability, which is a generalization of controllability in the sense of patching trajectories, and demonstrate that it allows the behavior to have an image representation over its reduced ring. Due to the length of the material required to present j-controllability in an abstract form, we use a form which ties j-controllability to the canonical Cauchy problem so that it is more direct, but requires a monomial ordering. Following this discussion, we finish with two further developments. The first is that algebraic controllability allows one to directly relate the behavior to its frequency domain (which necessarily requires that the annihilator be prime). The final result is that algebraic controllability remains invariant under automorphisms of the operator ring. To establish this result we specialize the discussion of Zerz and Oberst in [59] by inducing an isomorphism on trajectories by means of the adjoint map of a k-algebra automorphism.

Chapter 6 concludes this dissertation with the main result – establishing a state space representation for two-dimensional autonomous behaviors with pure autonomy degree one. There are several necessary ingredients for this connection and each uses results developed throughout the dissertation. Firstly, we present a bird's-eye view of Livšic systems and their compatibility conditions. We then proceed to discuss Livšic systems as behaviors and construct an explicit kernel representation for both the input/state/output system and the input/output system. The main result of the section uses, in the context of meromorphic bundles maps, an analogue of the Hankel realization to produce a Livšic system from a transfer function and two vector bundles (the input and output bundles.) To apply the theorem, we define a generalization of the input/output structures discussed by Oberst in regard to signal flow systems. The input/output structure is constructive on identifying the input and transfer function but non-constructive for the output. We provide discussion on why this last unsavory point is unavoidable. Since the input and output bundles of a Livšic system have a very special form of being determinantal representations (a particular subclass of the so-called square systems discussed by Valcher in [46]), we provide necessary and sufficient conditions for a behavior to arise as a determinantal representation. If the characteristic variety is smooth, then one can computationally validate these conditions by computing the cohomology of the appropriate coherent sheaves (which can be performed by computer algebra systems such as Macaulay2.) The final ingredient is connecting Livšic controllability to algebraic controllability. After some work, this last step allows us to associate the behaviors at the signal flow system level. We then use the Hankel realization to conclude that a behavior which satisfies a sufficient hypotheses has a Livšic system as a state space representation. Furthermore, since this correspondence is computationally non-constructive, we emphasize that the equality of systems is given by a hypothesis placed only on the behavior.

Contributions

It is easy to show that, under some conditions, one can have a state-space representation for *some* two-dimensional subbehavior. The question is whether or not it is the behavior you started with. To address this latter point, we extend behavioral systems theory in several ways. We break these contributions into categories and state why they were necessary components in solving the main problem.

Decomposition of Systems

Problem. A behavior with autonomy degree one may have poles which are disjoint from its associated algebraic curves. For a controllable Livšic system, all poles are on an algebraic variety.

We provide a way of decomposing a behavior by its annihilator so as to drive the decomposition at the frequency domain level. We then state necessary and sufficient conditions for the decomposition to be unique and a direct sum decomposition. Our motivation comes from the observation that a pole corresponds to a lower-dimensional object living in the variety. In the case of two dimensions, this means that, for a prime ideal of dimension one, the poles are zero-dimensional ideals containing this ideal. If we come across poles which are not on the variety, then these are "disjoint" from the rest of the system and should be passed off to another system. This is precisely the pure autonomy degree decomposition of a behavior.

Connections to Algebraic Geometry.

Problem. Livšic systems give rise to various vector bundles and a bundle map (transfer function) between the two. Does the same hold in the behavioral setting?

We establish connections between behaviors and algebraic geometry via the frequency domain. In the classical setting this is not necessary since the variety one works over is $\mathbb{C}^d$. However, one can consider a behavior as giving rise to a family of vector spaces over an algebraic set. We then provide conditions for this family of vector spaces to be a holomorphic vector bundle. One can observe that the transfer function for the reduced behavior (under a specified i/o-structure) gives rise to a holomorphic bundle map between two trivial bundles.

Extensions of Systems Theory

Problem. When will the Livšic system given by the Hankel realization be *equal* to the given behavior.

The solution to this problem is quite lengthy and requires may additions to the existing theory. Some contributions include:

1. We develop a process by which a behavior can be **reduced** to a suitable operator ring and signal space so as to enjoy the use of linear algebra in a non-trivial way. This allows us to take advantage of the generality of results established by Oberst in [39].
2. We extend terminology and basic results to this new setting in a consistent way with the existing theory.
3. We develop a frequency domain theory over algebraic varieties and provide conditions for relating systems at the frequency domain level. Even from a classical point of view this is new outside of one-dimensional systems.
4. We define a trajectory-based version of controllability which implies a new version of algebraic controllability over affine domains.
5. We establish the invariance of various properties under isomorphisms and automorphisms of the operator ring.
6. We demonstrate the equivalence of Livšic controllability and the newly defined controllability for behaviors.

It is really this last point that requires all of the work. By extending Oberst's signal flow systems a little, we are able to use controllability to relate systems over the quotient field and then use minimality to yield equality. Demonstrating the Livšic controllability implies $\mathcal{D}_r$-controllability requires a series of reduction steps where we show results for a special type of system and then use a ring automorphism of the operator ring to bring any system to this form.

Livšic Systems

Problem. Livšic systems have a very distinct input/output structure. How can one identify this structure in behaviors? What is the typical structure of a Livšic system when the state is eliminated, i.e., what does its external behavior look like? Is there a relationship between the joint transfer function of a Livšic system and the transfer function of a behavior in the signal flow sense?

It turns out that there is no definitive answer to the first question. We provide conditions for a behavior to exhibit the same input/output structure as a Livšic system and demonstrate that every Livšic system satisfies these conditions. Two of three of the conditions are constructive and the last is a fiat hypothesis condition. However, after some explanation, it becomes clear why this is a necessary evil. The second question is addressed by identifying a standard structure for the minimal left annihilator of the Hautus pencil and doing some basic matrix calculations. The last question is addressed by augmenting Oberst's signal flow systems to handle cases where the input is not free, i.e., the signal flow system is given as the restricted graph of the Livšic joint transfer function.

There are several new results which are related to the discussion in this dissertation but are not necessary to address the main problem. Since the goal of this dissertation is to present a story which is both coherent and accessible while keeping distractions minimal, we withhold these results for later publication and future development.

Open Problems

The work contained in this dissertation, essentially, demonstrates that over-determined discrete-time linear systems theory is almost entirely driven by algebraic geometry. In particular, this work has lead to a wealth of open questions. Some problems which will keep the author busy for years to come are the following.

1. (Hautus Test-Like Results) One can show that for a two-dimensional system which is reduced to a one-dimensional affine domain, the singularities of the curve have systems theoretic implications. In particular, the rank of the kernel representation is constant off the singularities. In this case, let $\mathcal{D}_r$ be the affine domain and $\overline{\mathcal{D}_r}$ be its integral closure inside its quotient field. One interesting question is to find necessary and sufficient conditions for a $\mathcal{D}_r$-module $\mathcal{M}$ to be such that $\overline{\mathcal{D}_r} \otimes_{\mathcal{D}_r} \mathcal{M}$ is projective and for a projective $\overline{\mathcal{D}_r}$-module $\mathcal{M}'$ to be such that there exists an extension $\mathcal{M}$ where $\mathcal{M}' = \mathcal{M} \otimes_{\mathcal{D}_r} \overline{\mathcal{D}_r}$. Some reasonable questions are: (1) what does this mean for the time domain? (2) outside of this setting, are there general conditions for a system to give rise to a holomorphic vector bundle over the complex manifold one reaches by resolving the singularities of the associated algebraic variety? (3) what are the systems theoretic implications of being able to lift to a vector bundle?

2. (Resolution of Singularities) Algebraically, one can resolve some singularities by taking the integral closure of the coordinate ring. For any ideal (corresponding to singularities), we can blow the ring up along the ideal with the Rees algebra. How does this change the signals in the time domain? Establishing a strong understanding of the resolution of singularities

and its implications in the frequency/time domain appears to be a reasonable goal. Another question is to understand what, in terms of signals, does it mean to be defined over a smooth variety.

3. (Not-So-Nice Situations) Due to primary decomposition, we really only need to extend our understanding from the integral domain setting to the case where the ideal is primary. Developing an analogue of Oberst duality for schemes would probably be the cleanest way to understand higher order structure of the germs. Analogously, one could do this via jet bundles in the frequency domain. In this latter setting the work of Pommaret in [42] may be useful.

4. (Combinatorial Commutative Algebra) The Cauchy problem is what sums up any behavioral system. It turns the computation of solutions into a matter of a monomial ideal's staircase structure. Sturmfels and many others have dedicated countless hours to the study of monomial ideals (see for instance [36].) In the author's master's thesis [10], this matter was discussed in terms of defining a cohomology for the initial condition set associated to the Cauchy problem. Having a firm understanding of the connection between systems and their initial ideals (and even generic initial ideals) is crucial in really coming to grips with the time-domain side of overdetermined systems.

5. (Coordinate Free Behaviors) One shortcoming of the canonical Cauchy problem for systems of difference equations is that the initial condition set is based on the monomial ordering chosen. Generic initial ideals deal with the Gröbner basis under changes of coordinates. In a similar fashion, we should also define behaviors as modules over the action of $GL(n, k)$ or some other type of linear transformation like the Borel group or algebraic torus group. For instance, the structure of a system (in terms of the Cauchy problem) is significantly improved after a Noether normalization of the coordinate ring. However, under arbitrary transformations, one might benefit from having a better grasp of what is going on at the level of signals. This ties into system properties which are invariant under ring automorphisms.

6. (j-controllability) The author has conditions for signal spaces to be j-controllable. These deal with the structure of the initial ideal of the annihilator. However, the search for an analogue of trajectory patching for over-determined systems is far from complete; this is mainly a fact that j-controllability is a concept which is derived combining the classical setting with the Livšic setting. It may turn out that there is a definition which can better explain signal spaces and how we think of as them being free/controllable.

7. (Higher Dimensional State Space Representation) It is important to use Livšic systems as a springboard for the development of higher dimensional state space representations for behaviors with different degrees of autonomy. By tying the annihilator into the state space evolution equations, the input signal can satisfy various degrees of autonomy and, hence, each autonomy degree system could give rise to a hybrid of Givone-Roesser, Fornasini-Marchessini, Livšic, or some completely different state space representation.

Notation and Conventions

When appropriate, we provide references for results which are not new and especially when proof is not given. To keep the length manageable, several interesting proofs were not included due to the "ascending chain of lemmas and definitions" appearing to be non-Noetherian in nature. Terminology is indexed when introduced and each section or subsection states any assumptions in notation.

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Note. Significant portions of this dissertation have some overlap with [5].

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Chapter 2

Multidimensional Behavioral Systems

The purpose of this chapter is to define behavioral systems and present results which are either used or generalized in the following chapters. Since the amount of material covered in the literature is quite vast, we provide an overview of the material rather than a thorough investigation.

The focus of this section, as well as this entire work, is with *discrete-time systems*.

2.1 Algebraic Approach to Behaviors

In [39], Oberst presents a unifying approach to the study of multidimensional behaviors. While the methods are quite different from that of Willems in [51] and Rocha in [43], it allows a strong connection between behaviors and finitely generated modules. Over the past 20 years, the results introduced by Oberst are still employed in almost every proof.

Notes. (1) Due to the length of [39], this section provides only an abridged version of the ingredients used throughout this work. The proofs of some theorems and lemmas are provided if the exposition benefits from further elaboration. However, if proof is not provided then complete details can be found in [39]. We also note that this section contains no original work. In a few areas the exposition is adjusted slightly, however, there are no new contributions in this section besides these changes in exposition. In particular, *all* of the work from this section is derived from the work of Oberst in [39], Köthe in [31], and the other cited sources.

(2) Rather than defining behaviors as linear, shift-invariant, and closed subspaces of $\mathcal{A}^q$ with the topology of point-wise convergence as in [51, 43], we start with an algebraic/categorical approach. After suitable development, we then put a topology on $\mathcal{A}$ to link new terminology back to the original terminology of Willems. The motivation for this approach is that the topology on $\mathcal{A}$ is rigged so that the dual space is equal to $\mathcal{D}$. It also emphasizes that the choice of topology is not entirely necessary since the large injective cogenerator property (to be discussed) is able to, categorically, accomplish most of the results one is accustomed to by equipping the signal space with the weak-$*$ topology.

2.1.1 Behaviors as E-modules

Notation. D is a commutative Noetherian ring and A is a D-module. We also define $\mathbf{Mod}(D)$ as the category of D-modules with morphisms consisting of D-module homomorphisms; similarly, we define $\mathbf{Modf}(D)$ as the full subcategory of $\mathbf{Mod}(D)$ whose objects consist of finitely generated D-modules.

We start with the endomorphism ring

$$E = \mathrm{End}_{\mathcal{D}}(A) = \mathrm{Hom}_{\mathcal{D}}(A, A).$$

Because E consists of D-module homomorphisms, we may naturally impose that it also be a D-algebra. We define this action weakly through A as follows

$$(fR)(w) \mapsto R(fw) \qquad\qquad f \in D,\ R \in E,\ w \in A.$$

Using that E is the endomorphism ring of A, we have that it naturally induces an E-module structure on A as follows. For $e \in E$ and $a \in A$ define

$$E \times A \mapsto A \text{ by } (e, a) \mapsto e(a).$$

This allows A to be considered as a D-E-bimodule.

Remark. In the “original” development of behavioral systems as in [51], A is not usually considered as an D-E-bimodule but rather as just a D-module. We return to this matter in Section 2.1.4.

Definition 2.1.1. [50] Let $\mathcal{C}$ be a category. We define the **opposite category** of $\mathcal{C}$, denoted by $\mathcal{C}^{op}$, as the category where the source and target of the morphisms are interchanged and the composition of morphisms is performed in the reverse manner.

Similarly, for any D-module M we may impose a left E-module structure on $\mathrm{Hom}_D(M, A)$ as follows. For $e \in E$ and $f \in \mathrm{Hom}_D(M, A)$ define

$$E \times \mathrm{Hom}_D(M, A) \mapsto \mathrm{Hom}_D(M, A) \text{ by } (e, f) \mapsto e \circ f.$$

The prescribed actions are used to define the contravariant functor $\mathrm{Hom}_D(-, A)$,

$$\mathrm{Hom}_D(-, A) : \mathbf{Mod}(D)^{op} \to \mathbf{Mod}(E), \tag{2.1}$$

where the object map is $M \mapsto \mathrm{Hom}_D(M, A)$ and the morphism map is

$$(f : M \to N) \in \mathrm{Hom}_D(M, N) \mapsto (\mathrm{Hom}_D(f, A) : \mathrm{Hom}_D(N, A) \to \mathrm{Hom}_D(M, A)), \tag{2.2}$$

where $\mathrm{Hom}_D(f, A)(\phi) = \phi \circ f$ for $\phi \in \mathrm{Hom}_D(N, A)$.

Due to the assumption that D is a Noetherian ring, any finitely generated D-module M may be considered as a quotient module of the free module D^q with presentation matrix $R \in D^{p\times q}$, i.e.

$$M = D^q / R^T D^p \tag{2.3}$$

Recall that there is a canonical isomorphism

$$\mathrm{Hom}_D(D^q, A) \cong A^q,$$

where for $f \in \mathrm{Hom}_D(D^q, A)$ we define

$$f \mapsto (f(e_1), \ldots, f(e_q))^T \in A^q$$

where $e_1, \ldots, e_q$ form a basis of the free module D^q.

Returning to (2.3), we may use the residue classes of the basis to reach the map

$$f \in \mathrm{Hom}_D(D^q/R^T D^p, A) \mapsto (f([e_1]), \ldots, f([e_q]))^T \in A^q \tag{2.4}$$

where $[e_1], \ldots, [e_q]$ are residue classes of the basis vectors $e_1, \ldots, e_q$ for the free module D^q under the quotient map $\pi : D^q \to D^q/R^T D^p$. For $R \in D^{p\times q}$ with $R = \{r_{i,j}\}$ we may use $f(R^T D^p) = 0$ we reach

$$R(f([e_1]), \ldots, f([e_q]))^T = \sum_{j=1}^{q} r_{i_j} f\,([e_j]) = 0 \qquad\qquad \text{for all } i = 1, \ldots, p.$$

Conversely, for any $w(w_1, \ldots, w_q)^T \in A^q$ for which $Rw = 0$, we may define $f([e_i]) = w_i$ for all $i = 1, \ldots, q$ and span over D to reach $f \in \mathrm{Hom}_D(D^q/R^T D^p, A)$. This leads us to the main definition of this entire work.

Definition 2.1.2. We define a **behavior** as an E-D-bimodule of the form

$$\mathrm{Hom}_D(M, A) \cong \{w \in A^q : Rw = 0\}$$

where $M = D^q/R^T D^p$ for $R \in D^{p\times q}$. We call R a **kernel representation** of the behavior and M the **dual module** of the behavior. We define the category $\mathbf{Syst}(A)$ as the full subcategory of $\mathbf{Mod}(E)$ whose objects consist of behaviors.

Since $M = D^q/R^T D^p$ is a finitely generated D-module, we may define the functor induced by (2.1) as

$$\mathrm{Hom}_D(-, A) : \mathbf{Modf}(D)^{op} \to \mathbf{Syst}(A). \tag{2.5}$$

This leads to the diagram

$$\begin{tikzcd} \mathrm{Hom}_D(D^q, A) \arrow[r, "\mathrm{Hom}_D(R^T, A)"] \arrow[d, leftrightarrow, "\cong"] & \mathrm{Hom}_D(D^p, A) \arrow[d, leftrightarrow, "\cong"] \\ A^q \arrow[r, "R"] & A^p, \end{tikzcd}$$

where $\mathrm{Hom}_D(R^T, A)$ is defined by (2.2). The above formalism establishes a connection between behaviors and finitely generated D-modules in a purely algebraic manner. In the following section we show that there is a contravariant equivalence of the categories $\mathbf{Syst}(A)$ and $\mathbf{Modf}(D)$ provided A has a particularly nice structure with regard to finitely generated D-modules.

Before moving on, we define a pairing that is used extensively throughout this work. Define the D-bilinear non-degenerate pairing

$$\langle \cdot, \cdot \rangle : D^q \times A^q \to A \text{ as } \langle p, w \rangle = \sum_i p_i w_i, \tag{2.6}$$

where $p = (p_1, \ldots, p_q) \in D^q$ and $w = (w_1, \ldots, w_q) \in A^q$. Under the above pairing, we may identify A^q with $\mathrm{Hom}_D(D^q, A)$ by the map

$$w \mapsto \langle \cdot, w \rangle \qquad\qquad w \in A^q.$$

For a subset $U \subset D^q$, we define its orthogonal complement as the E-submodule

$$U^\perp = \{w \in A^q : \langle p, w \rangle = 0 \text{ for all } p \in U\}. \tag{2.7}$$

Similarly, for any subset $S \subset A^q$, we define the co-orthogonal complement as the D-submodule

$$S^\perp = \{p \in D^q : \langle p, w \rangle = 0 \text{ for all } w \in S\}. \tag{2.8}$$

2.1.2 Oberst Duality

Oberst duality is quite possibly the most important result systematically employed in behavioral systems theory. Indeed, almost every proof therein relies on the relationship shared between equations and their solutions. In some sense, this is analogous to the relationship shared by algebraic sets and their associated ideals. The applications can be quite deep as in the case of showing invariance of structural properties; on the other hand, the computational aspects (i.e. the use of Gröbner bases) hold only because of Oberst duality. In this section we start with properties of injective modules and proceed to the properties of injective cogenerators and large injective cogenerators. The main result demonstrates that there is a categorical duality between $\mathbf{Syst}(A)$ and $\mathbf{Modf}(D)$ provided A is a large injective cogenerator.

Injective modules have been extensively studied in various forms. From [32, Section 1.3] one can see that there is a wealth of equivalent definitions. For our purposes, we define injective modules as follows.

Definition 2.1.3. We say that the D-module A is **injective** if the functor $\mathrm{Hom}_D(-, A)$ is exact, i.e. the functor preserves exact sequences.

Recall that $\mathrm{Hom}_D(-, A)$ is always left exact, i.e., for an exact sequence of D-modules

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

we have

$$0 \longrightarrow \mathrm{Hom}_D(V_3, A) \longrightarrow \mathrm{Hom}_D(V_2, A) \longrightarrow \mathrm{Hom}_D(V_1, A)$$

is exact. If A is injective then we have, in addition, exactness of the following

$$0 \longrightarrow \mathrm{Hom}_D(V_3, A) \longrightarrow \mathrm{Hom}_D(V_2, A) \longrightarrow \mathrm{Hom}_D(V_1, A) \longrightarrow 0.$$

In light of the discussion of the previous section, one can reach the following equivalent form of injectivity.

Lemma 2.1.4 (Fundamental Principle of Ehrenpreis)**.** *The D-module A is injective if and only if for any exact sequence of the form*

$$D^p \xrightarrow{Q^T} D^q \xrightarrow{P^T} D^g \qquad\qquad Q \in D^{p\times q},\ P \in D^{q\times g}$$

we have that the sequence

$$\begin{tikzcd}
\mathrm{Hom}_D(D^g, A) \arrow[r, "\mathrm{Hom}_D(P^T,A)"] \arrow[d, equal] & \mathrm{Hom}_D(D^q, A) \arrow[r, "\mathrm{Hom}_D(Q^T,A)"] \arrow[d, equal] & \mathrm{Hom}_D(D^p, A) \arrow[d, equal] \\
A^g \arrow[r, "P"] & A^q \arrow[r, "Q"] & A^p
\end{tikzcd}$$

is exact. In particular, if $\ker_A(Q) = \mathrm{im}_A(P)$, then the equation $Pw = u$ for $u \in A^q$ has a solution if and only if $Qu = 0$.

The systems theoretic consequences of the above lemma are quite useful. If A is injective, then the image representation of a behavior (this is given by P in the above) may be determined through its dual module. In particular, the property can be posed in a way so that it is question of exactness between finitely generated D-modules. Computationally, this is quite favorable.

The following corollary provides further properties of the pairing (2.6) provided A is an injective D-module.

Corollary 2.1.5. *Let $U \subset D^q$ be D-submodule. The canonical quotient map $\pi : D^q \to D^q/U$ induces an E-module isomorphism*

$$\mathrm{Hom}_D(\pi, A) : \mathrm{Hom}_D(D^q/U, A) \cong U^\perp.$$

Provided that A is an injective D-module, the map

$$A^q/U^\perp \cong \mathrm{Hom}_D(U, A) \text{ given by } w \mapsto \langle -, w\rangle|_U$$

is an E-module isomorphism.

Proof. Define the inclusion map $\iota : U \hookrightarrow D^q$ so that we have the exact sequence

$$0 \longrightarrow U \xrightarrow{\iota} D^q \xrightarrow{\pi} D^q/U \longrightarrow 0.$$

After applying $\mathrm{Hom}_D(-, A)$ we reach the exact sequence

$$0 \longrightarrow \mathrm{Hom}_D(D^q/U, A) \xrightarrow{\mathrm{Hom}_D(\pi,A)} \mathrm{Hom}_D(D^q, A) \xrightarrow{\mathrm{Hom}_D(\iota,A)} \mathrm{Hom}_D(U, A).$$

Since $\ker(\mathrm{Hom}_D(\iota, A) : w \mapsto \langle -, w\rangle|_U) = U^\perp$, we may apply exactness to reach

$$\mathrm{im}(\mathrm{Hom}_D(\pi, A)) = \ker(\mathrm{Hom}_D(\iota, A) = U^\perp$$

since $\mathrm{Hom}_D(\pi A)$ is injective we have the first stated isomorphism.

If in addition A is injective, then $\mathrm{Hom}_D(-, A)$ is exact. From right exactness, $\mathrm{Hom}_D(\iota, A)$ is surjective and thus the first isomorphism theorem provides

$$\mathrm{Hom}_D(U, A) \cong \mathrm{Hom}_D(D^q, A)/\ker(\mathrm{Hom}_D(\iota, A)) \cong A^q/U^\perp.$$

□

We now move our discussion to modules that are cogenerators.

Definition 2.1.6. We say that the D-module A is a **cogenerator** if for every D-module M there exists a (possibly infinite) index set $\mathcal{I}$ and monomorphism $\phi : M \to A^{\mathcal{I}}$.

If A is an injective D-module, being a cogenerator significantly augments the structure of A.

Lemma 2.1.7. *An injective D-module A is a cogenerator if and only if for every D-module M, $\mathrm{Hom}_D(M, A)$ is zero if and only if M is zero.*

Proof. ($\Rightarrow$) Assume that A is an injective cogenerator and let M be an arbitrary D-module. First let us take $\mathrm{Hom}_D(M, A) = 0$. By assumption it follows that there exists $\{\phi_i : M \to A\}_{i\in\mathcal{I}}$ providing the injection $\prod \phi_i : M \to \mathcal{A}^{\mathcal{I}}$. Since each $\phi_i \in \mathrm{Hom}_D(M, A)$ we have each ϕ_i is zero and, by injectivity of the map $\prod \phi_i$, it must be the case that $M = 0$. If M is zero then clearly $\mathrm{Hom}_D(M, A)$.

($\Leftarrow$) Let A be an injective D-module, M be an arbitrary non-zero D-module and assume $\mathrm{Hom}_D(M, A) \neq 0$. Define the map $\phi : M \to A^{\mathrm{Hom}_D(M,A)}$ as

$$\phi(a) = \prod_{f\in\mathrm{Hom}_D(M,A)} f(a).$$

If $\phi(a) = 0$ for some non-zero $a \in M$, then we have $f(a) = 0$ for all $f \in \mathrm{Hom}_D(M, A)$. Define the submodule $M' = \ker(\phi) \subset M$. Since A is injective, the injection $\iota : M' \to M$ induces the surjective map $\iota' : \mathrm{Hom}(M, A) \to \mathrm{Hom}(M', A)$; in particular, for every $f' \in \mathrm{Hom}(M', A)$ there exists $f \in \mathrm{Hom}(M, A)$ such that $f \circ \iota = f'$. However, we then have $(f \circ \iota)(a') = 0$ for all $f \in \mathrm{Hom}_D(M, A)$. It follows that $\mathrm{Hom}(M', A) = 0$ and hence, by hypothesis, $M' = \ker(\phi) = 0$. We conclude that A is a cogenerator. □

Proposition 2.1.8. *Suppose that A is an injective cogenerator for the category of D-modules and M and N are D-modules.. Then $f : M \to N$ is injective if and only if $\mathrm{Hom}(f, A) : \mathrm{Hom}_D(N, A) \to \mathrm{Hom}_D(M, A)$ is surjective and $g : M \to N$ is surjective if and only if $\mathrm{Hom}(g, A) : \mathrm{Hom}_D(N, A) \to \mathrm{Hom}_D(M, A)$ is injective.*

Proof. Define $f^* = \mathrm{Hom}(f, A)$ and $g^* = \mathrm{Hom}(g, A)$. Since A is injective we have that $\mathrm{Hom}_D(-, A)$ is an exact contravariant functor; hence f being injective implies f^* is surjective and g being surjective implies g^* is injective. We now show the reverse implications.

Assume that $f^* : \mathrm{Hom}_D(N, A) \to \mathrm{Hom}_D(M, A)$ is surjective and define $V = \ker_D(f)$ and $\iota : V \hookrightarrow M$ as the inclusion map. Since A is injective, we have $\iota^* : \mathrm{Hom}_D(M, A) \to \mathrm{Hom}_D(V, A)$ is surjective;

hence for any $a \in \mathrm{Hom}_D(V, A)$ there exists $b \in \mathrm{Hom}_D(M, A)$ for which $b \circ \iota = a$. By hypothesis, f^* is surjective, so there also exists $c \in \mathrm{Hom}_D(N, A)$ for which $c \circ f = b$. In particular, $c \circ f \circ \iota = a$. However, $V = \ker_D(f)$ and so for any $w \in V$ we have $a(w) = c(f(\iota(w))) = c(f(w)) = c(0) = 0$. Because a was arbitrary, we have $\mathrm{Hom}_D(V, A) = 0$. By Lemma 2.1.7 and the assumption that A is an injective cogenerator we reach $V = 0$ and thus $\ker_D(f) = 0$.

Assume that $g^* : \mathrm{Hom}_D(N, A) \to \mathrm{Hom}_D(M, A)$ is injective and define $U = N/\mathrm{im}_D(g)$ and $\pi : N \to N/\mathrm{im}_D(g)$ as the surjective quotient map. From the assumption that A is injective we have $\pi^* : \mathrm{Hom}_D(U, A) \to \mathrm{Hom}_D(N, A)$ is injective; hence, for any $a \in \mathrm{Hom}_D(U, A)$ we have $(a \circ \pi)(w) = 0$ for all $w \in U$ implies $a = 0$. Similarly, g^* being injective implies that for any $b \in \mathrm{Hom}_D(N, A)$ that $(b \circ g)(w) = 0$ for all $w \in M$ implies $b = 0$. Using that both π^* and g^* are injective, we have $g^* \circ \pi^*$ is injective and hence for any $a \in \mathrm{Hom}_D(U, A)$ we have $(a \circ \pi \circ g)(w) = 0$ for all $w \in U$ implies $a = 0$. However, $\pi \circ g = 0$ since $\ker_D(\pi) = \mathrm{im}_D(g)$, which this leads us to $\mathrm{Hom}_D(U, A) = 0$. By Lemma 2.1.7 and the assumption that A is an injective cogenerator we may conclude that $U = 0$. In particular, $N = \mathrm{im}_D(g)$ and hence g is surjective. □

Corollary 2.1.9. *Suppose that A is an injective cogenerator for the category of D-modules and M, N and V are D-modules. Let $f : M \to N$ and $g : V \to M$ be D-module homomorphisms. We have* $\ker_D(f) = \mathrm{im}_D(g)$ *if and only if* $\mathrm{im}(\mathrm{Hom}_D(f, A)) = \ker(\mathrm{Hom}_D(g, A))$.

Proof. We assume M, N, and V are all non-zero modules. In the case that $V = 0$ or $N = 0$ we have the result holds by Proposition 2.1.8. In the case that $M = 0$ then the result follows from the assumption that A is an injective cogenerator.

Define $f^* = \mathrm{Hom}(f, A)$ and $g^* = \mathrm{Hom}(g, A)$. Since A is injective we have $\mathrm{Hom}_D(-, A)$ is an exact contravariant functor; hence it preserves exactness and so $\ker_D(f) = \mathrm{im}_D(g)$ implies $\mathrm{im}_D(f^*) = \ker_D(g^*)$. We now show the reverse implication.

Assume $\mathrm{im}_D(f^*) = \ker_D(g^*)$. If $a \in \ker(g^*)$ then $(a \circ g)(w) = 0$ for all $w \in V$. It follows that $\ker(g^*) \cong \mathrm{Hom}_D(M/\mathrm{im}_D(g), A)$. Similarly for $c \in \mathrm{im}(f^*)$ there exists $b \in \mathrm{Hom}_D(N, A)$ for which $c(w) = (b \circ f)(w)$ for all $w \in M$. For such a c we have $c(w) = 0$ for all $w \in \ker_D(f)$. Hence $\mathrm{im}(f^*) \cong \mathrm{Hom}_D(M/\ker_D(f), A)$. Due to the assumption $\mathrm{im}_D(f^*) = \ker_D(g^*)$ and that A is an injective cogenerator, we may appeal to Proposition 2.1.8 to conclude that $\ker_D(f) = \mathrm{im}_D(g)$. □

Provided A is an injective cogenerator, this leads to favorable properties of the pairing (2.6).

Corollary 2.1.10. *If A is an injective cogenerator and $U \subset D^q$ is a D-submodule, then $U = U^{\perp\perp}$.*

Proof. By Corollary 2.1.5 we may observe $A^q/U^\perp \cong \mathrm{Hom}_D(U, A)$. Since $U \subset U^{\perp\perp}$ we have the inclusion $U \hookrightarrow U^{\perp\perp}$ which induces the surjection

$$(A^q/U^{\perp\perp\perp} \cong \mathrm{Hom}_D(U^{\perp\perp}, A)) \to (A^q/U^\perp \cong \mathrm{Hom}_D(U, A))$$

However, since $U^\perp = U^{\perp\perp\perp}$ it must be the case that $\mathrm{Hom}_D(U^{\perp\perp}, A) = \mathrm{Hom}_D(U, A)$ By Proposition 2.1.8 we may conclude that $U = U^{\perp\perp}$. □

It is important to note that, even though being an injective cogenerator allows us to relate exactness of diagrams, it does *not* allow us to make the claim that we can inject D-modules into *finite* products of A. D-module with this special property are given by the following.

Definition 2.1.11. An injective D-module A is called a **large injective cogenerator** if for every finitely generated D-module M, there is a D-linear monomorphism $\phi : M \to A^n$ for some $n \in \mathbb{N}$.

Before proving an equivalent property of large injective cogenerators, we now review some basic results and terminology from module theory.

Lemma 2.1.12. *[32] If D is a Noetherian ring, then any direct sum of injective D-modules is injective.*

Definition 2.1.13. A D-module is **indecomposable** if it is non-zero and cannot be written as the direct sum of two non-zero D-modules.

Definition 2.1.14. [32] For any D-modules $M \subset N$ we say that N is an **essential extension** if for every non-zero submodule $M' \subset M$, $M' \cap N \neq 0$.

Definition 2.1.15. [32] For any D-module M we call an essential extension which is also an injective D-module an **injective envelope** of M. One can verify that for every D-module M, M has an injective envelope and that such a module is unique up to isomorphism; as a consequence, we denote the injective envelope of M by $\widehat{M}$.

Lemma 2.1.16. *[32] If D is a Noetherian ring, then any injective module M is a direct sum of indecomposable injective submodules.*

Definition 2.1.17. Let M be a D-module. We call a prime ideal $I \subset D$ an **associated prime** of M if there exists some non-zero $f \in M$ for which $pf = 0$ for all $p \in I$.

Lemma 2.1.18. *[32] If D is a Noetherian ring and M is an indecomposable injective D-module, then M has only one associated prime.*

Definition 2.1.19. For a ring D we define $\mathrm{Spec}(D)$ as the set of prime ideals of D.

Define $\mathcal{I}(D)$ as the set of equivalence classes (where the equivalence is defined via D-module isomorphism) of indecomposable injective D-modules. For each $[M] \in \mathcal{I}(D)$, because D is Noetherian, M has one associated prime. As a result, we may define the surjective map $\alpha : \mathcal{I}(R) \to \mathrm{Spec}(R)$ where $\alpha([M]) = \mathrm{Ass}(M)$. The following well known theorem states that, since D is commutative, we have α is bijective.

Theorem 2.1.20 (Matlis' Theorem)**.** *Let D be a commutative Noetherian ring. Then α is a bijective map. Furthermore, the indecomposable injective D-modules are, up to isomorphism,*

$$\left\{ \widehat{R/p} : p \in \mathrm{Spec}(R) \right\}$$

where $\widehat{R/p}$ is the injective envelope of R/p.

We may use Matlis' Theorem to show that it is only necessary to check the property of large injective cogenerator over the indecomposable injective D-modules.

Theorem 2.1.21. *Let A be an injective D-module. Then A is a large injective cogenerator if and only if for every prime ideal $I \subset D$ there exists a D-linear monomorphism $\phi : D/I \to A^n$ for some $n \in \mathbb{N}$.*

Proof. ($\Rightarrow$). This is clearly satisfied since D/I is a finitely generated D-module.

($\Leftarrow$). Let M be a finitely generated D-module and denote its injective envelope as $\widehat{M}$, while noting that $M \subset \widehat{M}$. By Lemma 2.1.16, there exists indecomposable injective modules M_i such that $\widehat{M} = \oplus_{i\in\mathcal{I}} M_i$ for a possibly infinite index set $\mathcal{I}$. Because $\widehat{M}$ is an essential extension of M, for each M_i we have $M \cap M_i \neq 0$. In particular, $M = \oplus_{i\in\mathcal{I}} M_i \cap M$ where each $M_i \cap M$ is non-zero. From the assumption that D is a Noetherian ring, the ascending chain condition assures us that $\ell := |\mathcal{I}| < \infty$.

By Matlis' Theorem, there exists a finite set of prime ideals $p_1, \ldots, p_\ell \in \operatorname{Spec}(D)$ for which $M_i \cong \widehat{D/p_i}$. By hypothesis, there exists monomorphisms $\phi_i : D/p_i \to A^{n(i)}$ for each prime ideal. Because each $\widehat{D/p_i}$ is an injective D-module, by the inclusion $D/p_i \hookrightarrow \widehat{D/p_i}$, we may extend ϕ_i to $\widetilde{\phi}_i : \widehat{D/p_i} \to A^{n(i)}$ where $\widetilde{\phi}_i|_{D/p_i} = \phi_i$. Since ϕ_i is injective, if $\widetilde{\phi}_i$ were to have a kernel, it would have to be contained in the elements used to construct the injective envelope. As a result,

$$\ker(\widetilde{\phi}_i) \cap (D/p_i) = \ker(\widetilde{\phi}_i|_{D/p_i}) = \ker(\phi_i) = 0.$$

Note that $\ker(\widetilde{\phi}_i)$ is a D-submodule of $\widehat{D/p_i}$. Furthermore, since $\widehat{D/p_i}$ is an essential extension of D/p_i, every non-zero submodule of $\widehat{D/p_i}$ must intersect D/p_i non-trivially; since $\ker(\widetilde{\phi}_i) \cap D/p_i = 0$, it must be the case that $\ker(\widetilde{\phi}_i) = 0$. We thus reach that

$$\widetilde{\phi}_i : \widehat{D/p_i} \to A^{n(i)} \qquad\qquad i \in \mathcal{I},$$

is a monomorphism. Define $n = \sum_{i\in\mathcal{I}} n(i) \in \mathbb{N}$ so that

$$\phi := \prod_{i\in\mathcal{I}} \widetilde{\phi}_i : \widehat{M} \cong \bigoplus \widehat{D/p_i} \to \prod_{i\in\mathcal{I}} A^{n(i)} = A^n$$

is injective. Because $M \subset \widehat{M}$, $\phi|_M$ is injective. We thus conclude that A satisfies the "large" condition.

We now demonstrate that A is an injective cogenerator. Let N be an arbitrary non-zero D-module. Choose a finitely generated non-zero D-submodule $M \subset N$. We demonstrated that there is a D-linear monomorphism $\phi : M \to A^n$. Because M is non-zero, for $\phi = (\phi_1, \ldots, \phi_k)$, there must exists ϕ_i which is non-zero. From the assumption that A is an injective module, we have that there exists an extension $\widetilde{\phi}_i : N \to A$ where $\widetilde{\phi}_i|_M = \phi_i$. From the observation that ϕ_i is non-zero, we have $\widetilde{\phi}_i$ is non-zero and hence $\operatorname{Hom}_D(N, A)$ is non-zero. By Lemma 2.1.7, this demonstrates that A is an injective cogenerator. $\square$

Corollary 2.1.22. *Let A be an injective D-module. Then A is a large injective cogenerator if and only for every prime ideal $I \subset D$, there are finitely many elements $w_1, \ldots, w_n \in A$ such that $I = \{p \in D : pw_1 = \cdots = pw_n = 0\}$.*

Proof. ($\Rightarrow$). Let $I \subset D$ be a prime ideal. Since D is Noetherian, I is a finitely generated D-module and, by the assumption A is a large injective cogenerator, there exists an injective map $\phi = (\phi_1, \dots, \phi_n) : D/I \to A^n$ for some $n \in \mathbb{N}$. Let $\pi : D \to D/I$ be the canonical quotient map. Define $f = (f_1, \dots, f_k) := (\phi_1 \circ \pi, \dots, \phi_n \circ \pi)$ as the composed map so that, since ϕ is injective, $\ker(f) = I$. Because $f_i(p) = pf_i(1)$, define $w_i := f_i(1)$ so that

$$\ker(f_i) = \{p \in D : pw_i = 0\} = \mathrm{Ann}(w_i) \qquad\qquad i = 1, \dots, n$$

and

$$I = \ker(f) = \bigcap_{i=1}^{n} \ker(f_i) = \bigcap_{i=1}^{n} \mathrm{Ann}(w_i) = \{p \in D : pw_1 = \cdots = pw_n = 0\}.$$

($\Leftarrow$). By Theorem 2.1.21 it suffices to demonstrate the necessary injection on quotients of prime ideals. Let $I \subset D$ be a given prime ideal. By assumption, there exists $w_1, \dots, w_n$ such that $I = \{p \in D : pw_1 = \cdots = pw_n = 0\}$. Because $I \subset \mathrm{Ann}(w_i)$, we may define the maps $\phi_i : D/I \to A$ by $\phi_i(p) = p\phi_i(1) = pw_i$ for $i = 1, \dots, n$ to directly see $\ker(\phi_i) = \mathrm{Ann}(w_i)$. However, for $\phi := (\phi_1, \dots, \phi_n) : D/I \to A^n$,

$$\ker(\phi) = \bigcap_{i=1}^{n} \ker(\phi_i) = \bigcap_{i=1}^{n} \mathrm{Ann}(w_i) = I \equiv 0 \text{ in } D/I.$$

It follows that ϕ is injective and thus A is a large injective cogenerator. □

After this long journey, we reach the main theorem of this section.

Theorem 2.1.23 (Oberst Duality Theorem)**.** *Let the D-module A be a large injective cogenerator for the category* $\mathbf{Modf}(D)$.

1. *A left E-module S is a system, i.e. S is E-isomorphic to some E-module of the form*

$$\mathrm{Hom}_D(M, A) \qquad\qquad M \in \mathbf{Modf}(D)$$

if and only if S is a finitely generated E-submodule of some A^q for $q \in \mathbb{N}$. In particular, an E-finitely generated submodule of a system S is also a system.

2. *The functor* $\mathrm{Hom}_D(-, A)$ *is exact and induces the categorical duality*

$$S : \mathrm{Hom}_D(-, A) : \mathbf{Modf}(D)^{op} \to \mathbf{Syst}(A).$$

In other words, for any arbitrary finitely generated D-modules M and N, the map

$$\mathrm{Hom}_D(-, A) : \mathrm{Hom}_D(M, N) \to \mathrm{Hom}_E(\mathrm{Hom}_D(N, A), \mathrm{Hom}_D(M, A))$$

is bijective and a D-module isomorphism.

3. *The full subcategory* $\mathbf{Syst}(A)$ *of* $\mathbf{Mod}(E)$ *whose objects are systems and morphisms are* E*-homomorphisms is closed under taking kernels, images, and finite direct sums. In particular,* $\mathbf{Syst}(A)$ *is an abelian category and* $\mathrm{Hom}_D(-, A)$ *is an exact functor from* $\mathbf{Modf}(D)$ *to* $\mathbf{Syst}(A)$*. Furthermore, for* $q \geq 0$*, the set of all* E*-finitely generated submodules of* A^q *is closed under finite sums and intersections; in particular, this set forms a lattice.*

What is remarkable about the Oberst duality theorem is that it is more of a framework than a result. The only assumptions placed on D are that it be commutative and Noetherian. Provided one can create a D-module A which is a large injective cogenerator, then the duality theorem applies. In particular, the duality theorem can be used in both discrete and continuous linear systems over different operator rings and signal spaces. In this work, quite possibly the most important observation is that for the ideal $I \subset D$, $\mathrm{Hom}_D(-, A)$ also induces a duality between D/I and $A_I = \{a \in A : Ia = 0\}$.

Theorem 2.1.24 (Induced Duality)**.** *[39, page 46-47] Let* $I \subset D$ *be an ideal and the* D*-module* A *be a large injective cogenerator for the category* $\mathbf{Modf}(D)$*. Then the functor* $\mathrm{Hom}_D(-, A)$ *used in Theorem 2.1.23 induces the duality*

$$\mathrm{Hom}_D(-, A) : \mathbf{Modf}(D/I)^{op} \cong \mathbf{Syst}(A_I) = \mathbf{Syst}(\{w \in A : Ia = 0\}).$$

We make extensive use of this in the following chapters when we discuss the reduction of a behavior by its annihilator.

Remark. In this work we focus on the special choice $A = \mathrm{Hom}_k(D, k)$; we emphasize that such a special choice is not necessary as there are other large injective cogenerators.

2.1.3 $\mathrm{Hom}_k(D, k)$ is a Large Injective Cogenerator

Notation. We assume that k is a field, D is an affine k-algebra, and $A = \mathrm{Hom}_k(D, k)$ is the (algebraic) dual space of k.

In this section we show that the signal space $\mathcal{A} := k^{\mathbb{N}^d}$ is a large injective cogenerator for the category of finitely generated $k[z_1, \ldots, z_d]$-modules. This observation allows one to apply the Oberst Duality theorem to discrete time behaviors. In fact, we even show a more general result for affine domains.

Motivation. First let us consider the ring $D = k[z_1, \ldots, z_d]$. We may make the identification

$$p(\mathbf{z}) = \sum_{i \in \mathbb{N}^d} a_i \mathbf{z}^i \mapsto \{(a_i, i)\}_{i \in \mathbb{N}^d} \in \bigoplus_{i \in \mathbb{N}^d} k := k^{(\mathbb{N}^d)}, \tag{2.9}$$

where only a finite number the coefficients a_i are non-zero. We employ the notation $(\mathbb{N}^d)$ is to emphasize that sums are finite and hence the integer lattice $\mathbb{N}^d$ serves as a basis. In this way, we may identify $k[z_1, \ldots, z_d]$ as a k-vector space. For any $w \in \mathrm{Hom}_k(D, k)$ and $p \in D$ we have

$$w(p) = w\left(\sum_{i \in \mathbb{N}^d} a_i \mathbf{z}^i\right) = \sum_{i \in \mathbb{N}^d} a_i w(\mathbf{z}^i).$$

We may make the identification

$$w \mapsto \{w(\mathbf{z}^i)\}_{i\in\mathbb{N}^d} \in k^{\mathbb{N}^d} := \prod_{i\in\mathbb{N}^d} k \tag{2.10}$$

where we have an infinite product rather than direct sum; in particular, $\mathrm{Hom}_k(D,k)$ is a k-vector space. Note that the D-module structure on $k^{\mathbb{N}^d}$ is induced by the D-module structure on $\mathrm{Hom}_k(D,K)$, i.e., for any $a_j\mathbf{z}^j \in D$ we have

$$a_j\mathbf{z}^j\langle \mathbf{z}^i, w\rangle = \langle a_j\mathbf{z}^{i+j}, w\rangle = a_j w(\mathbf{z}^{i+j}) = \{a_j w(\mathbf{z}^{i+j})\}_{i\in\mathbb{N}^d}.$$

By linearity and that polynomials are finite sums of forms we have that, component-wise, the D-module structure induced on $k^{\mathbb{N}^d}$ is well-defined. By the choice of basis on D, we have that monomials in D act on $\mathrm{Hom}_k(D,k)$ via the **backward shift**:

$$\mathbf{z}^j\{w(\mathbf{z}^i)\}_{i\in\mathbb{N}^d} \mapsto \{w(\mathbf{z}^{i+j})\}_{i\in\mathbb{N}^d}.$$

Considering any $w \in k^{\mathbb{N}^d}$ as a function $w(t) : \mathbb{N}^d \to k$, where $w(t) = w(\mathbf{z}^t)$, we have $w'(t) := (\mathbf{z}^i w)(t) = w(t+i)$. *For what follows, we consider k-vector spaces rather than the specific choices of $D = \mathcal{D}$ and $A = \mathcal{A}$.*

Let V be a k-vector space and $V^* = \mathrm{Hom}_k(V,k)$ be its dual space. There is a natural k-bilinear pairing

$$\langle -,-\rangle : V \times V^* \to k$$

that induces the k-linear maps

$$\begin{aligned} V^* &\to \mathrm{Hom}_k(V,k) & \qquad v^* &\mapsto \langle -, v^*\rangle \\ V &\to \mathrm{Hom}_k(V^*,k) & \qquad v &\mapsto \langle v, -\rangle. \end{aligned}$$

Recall that a paring is called **non-degenerate** if the map $V^* \to \mathrm{Hom}_k(V,k)$ is an isomorphism. Note that, without the addition of some topology, the mapping $V \to \mathrm{Hom}_k(V^*,k)$ is an k-vector space isomorphism if and only if V is finite-dimensional.

We may define a D-module structure on A as follows. For $p,q \in D$ and $w \in A$ we define

$$(pw)(q) = \langle q, pw\rangle = \langle pq, w\rangle = w(pq).$$

Let M be a D-module and V be a k-vector space. We may impose a D-module structure on $\mathrm{Hom}_k(M,V)$ as follows. For $p \in D$, $f \in \mathrm{Hom}_k(M,V)$, and $m \in M$ we define the mapping $D \times \mathrm{Hom}_k(M,V) \to \mathrm{Hom}_k(M,V)$ as $(p,f) \mapsto pf$ where the action of pf on M is given weakly by

$$(pf)(m) = f(pm). \tag{2.11}$$

If we consider the special choice $M = D$, then $\mathrm{Hom}_k(D,-)$ is the functor

$$\mathrm{Hom}_k(D,-) : \mathbf{Mod}(k) \mapsto \mathbf{Mod}(D) \qquad\qquad V \mapsto \mathrm{Hom}_k(D,V).$$

Because D is k-algebra, $\mathrm{Hom}_k(D, V)$ may naturally be regarded as a k-vector space and hence we may also consider the forgetful functor

$$F : \mathbf{Mod}(D) \to \mathbf{Mod}(k) \qquad\qquad F : M \mapsto M.$$

Through the observation that $\mathrm{Hom}_k(D, -)$ is the right adjoint (e.g., [11, Proposition 11, Section 1.8]) of F we have the following set bijection

$$\mathrm{Hom}_k(F(M), V) \cong \mathrm{Hom}_D(M, \mathrm{Hom}_k(D, V)). \tag{2.12}$$

where for $f \in \mathrm{Hom}_k(M, V)$, $g \in \mathrm{Hom}_D(M, \mathrm{Hom}_k(D, V))$ and $p \in D$ we define

$$f(m) = g(m)(1) \qquad\qquad g(m)(p) = f(pm).$$

Note. One usually foregoes writing $F(M)$ and just writes M since $\mathrm{Hom}_k(M, k)$ implies that we are just using the k-vector space structure of M and hence forgetting the D-module structure.

Recall the following well known result.

Lemma 2.1.25. *The k-D-bimodule $A := \mathrm{Hom}_k(D, k)$ is an injective D-module.*

Proof. Let $g : A \to B$ and $h : A \to k$ be linear maps where g is injective. We may take A as a subspace of B since g is injective. Because A is a vector space, it admits a basis. Since A is a subspace of B, the basis of A can be extended to a basis of B. We may define f as the map on B by the the basis elements of A and arbitrarily on the additional basis elements and extend linearly; in this manner we reach $f \circ g = h : A \to k$. We thus reach that k is an injective k-module.

Since D is a k-D-bimodule, for any D-module M,

$$\mathrm{Hom}_D(M, \mathrm{Hom}_k(D, k)) = \mathrm{Hom}_k(M \otimes_D k, k) = \mathrm{Hom}_k(M, k).$$

Since $\mathrm{Hom}_k(-, k)$ is exact, we have $\mathrm{Hom}_D(-, \mathrm{Hom}(D, k))$ is exact; we conclude that $\mathrm{Hom}_k(D, k)$ is an injective D-module. $\square$

Remark. An alternative (but practically identical) approach to demonstrating any vector space is injective is as follows. First note that any field is injective since we may take any linearly independent element and, as in Lemma 2.1.25, extend the basis to any vector space. We then use that the product of injective modules is injective.

Lemma 2.1.26. *The k-D-bimodule $\mathrm{Hom}_k(D, k)$ is an injective cogenerator over D-modules.*

Proof. By Lemma 2.1.25 we know that $\mathrm{Hom}_k(D, k)$ is injective. One may observe that for a D-module M, any $a \in M$ for which $w(a) = 0$ for all $w \in \mathrm{Hom}_D(M, \mathrm{Hom}_k(D, k)) = \mathrm{Hom}_k(M, k)$ must necessarily have (since $\mathrm{Hom}_k(D, k)$ is the algebraic dual space) the equality $a = 0$. Clearly if $M = 0$ then $\mathrm{Hom}_k(M, k) = 0$. From Lemma 2.1.7 we may conclude that $\mathrm{Hom}_k(D, k)$ is an injective cogenerator. $\square$

Unfortunately, it is much harder to show that $\mathrm{Hom}_k(D,k)$ is a large injective cogenerator. The "large" property is essential for keeping things finitely generated when one starts with a finitely generated D-module. The approach we take to show that A is an injective cogenerator by a series of reduction steps. We now proceed to show that we can reduce the problem to checking on a polynomial ring in one indeterminate.

Lemma 2.1.27. *If every affine integral domain can be D-linearly embedded into some $A^q = \mathrm{Hom}_k(D,k)^q$, then $\mathrm{Hom}_k(D,k)$ is a large injective cogenerator for D-modules.*

Proof. Let D be an affine k-algebra with $A = \mathrm{Hom}_k(D,k)$. By Theorem 2.1.21, it suffices to demonstrate that for every prime ideal $\mathfrak{p} \subset D$, that $D/\mathfrak{p}$ D-linearly embeds into A^q with finite q.

Assume that for any prime ideals $\mathfrak{p}$ there exists $D/\mathfrak{p}$-linear injection $\phi : D/\mathfrak{p} \hookrightarrow \mathrm{Hom}_k(D/\mathfrak{p},k)^q$ for some $q \in \mathbb{N}$. Then the (surjective) canonical quotient map $\pi : D \to D/\mathfrak{p}$ induces the D-linear injection

$$\mathrm{Hom}(\pi,k) : \mathrm{Hom}_k(D/\mathfrak{p},k) \to \mathrm{Hom}_k(D,k) = A.$$

However, by assumption, we also have the D-$D/\mathfrak{p}$-linear injection

$$\phi : D/\mathfrak{p} \hookrightarrow \mathrm{Hom}_k(D/\mathfrak{p},k)^q.$$

By composition, $\mathrm{Hom}(\pi,k) \circ \phi$ is the desired D-linear injection into A^q. □

Lemma 2.1.28. *If every polynomial algebra can be D-linearly embedded into some $A^q = \mathrm{Hom}_k(D,k)^q$, then every affine integral domain can be D-linearly embedded into some $A^q = \mathrm{Hom}_k(D,k)^q$*

Proof. By the Noether Normalization Theorem (see [21, pg. 231]), D contains a polynomial algebra $\widetilde{D} = k[z_1,\ldots,z_d]$ such that D is a finite extension of $\widetilde{D}$. Without loss of generality, assume that the inclusion map $\iota : \widetilde{D} \hookrightarrow D$ is a finite extension (otherwise, we change the following restrictions to preimages.) Since k is a field, as seen in the proof of Lemma 2.1.25, it is injective and hence $\mathrm{Hom}_k(-,k)$ is exact. As a result, ι induces the $\widetilde{D}$-linear surjection

$$\mathrm{Hom}(\iota,k) : A = \mathrm{Hom}_k(D,k) \to \widetilde{A} = \mathrm{Hom}_k(\widetilde{D},k) \qquad a \mapsto a|_{\widetilde{D}}$$

by restriction of scalars. By hypothesis, there exists a monomorphism $\gamma : \widetilde{D} \to \widetilde{A}^n$ where

$$\widetilde{p} \in \widetilde{D} \mapsto \{\widetilde{p}\widetilde{a_i}\}_{i=1}^n \in \widetilde{A}^n \qquad \widetilde{a_1},\ldots,\widetilde{a_n} \in \widetilde{A}.$$

Since $\mathrm{Hom}(\iota,k)$ is surjective, each $\widetilde{a_i} \in \widetilde{A}$, which are homomorphisms on $\widetilde{D} \subset D$, can be extended to homomorphisms $a_i : D \to k$ such that $a_i|_{\widetilde{D}} = \widetilde{a_i}$. In particular, $a_i \in \mathrm{Hom}_k(D,k) \cong A$ so they induce the D-homomorphism

$$(a_1,\ldots,a_n) : D \to A^n \qquad p \mapsto \{pa_i\}_{i=1}^n.$$

Let the ideal $I = \ker(a_1,\ldots,a_n) \subset D$ be the kernel. Since $(\widetilde{p})|_{\widetilde{D}}\widetilde{a_i} = \widetilde{p}\widetilde{a_i}$ for $i = 1,\ldots,n$ and γ is injective, we have $I \cap \widetilde{D} = \ker(\gamma) = 0$.

We have $\langle 0\rangle_{\widetilde{D}}$ is a prime ideal in $\widetilde{D}$ (since it is a polynomial ring) and I lies over $\langle 0\rangle_{\widetilde{D}}$. We also have $\langle 0\rangle_D$ lies over $\langle 0\rangle_{\widetilde{D}}$ so both $\langle 0\rangle_D \subset I$ lie over $\langle 0\rangle_{\widetilde{D}}$. Note, however, by assumption we have D is an integral domain and hence $\langle 0\rangle_D$ is a prime ideal. By the lying-over theorem (or more specifically, [12, Corollary 1, page 32]) we have the equality $\langle 0\rangle_D = I$. Since $\ker\{(a_1,\ldots,a_n) : D \to A^n\} = I = \langle 0\rangle_D$, we conclude that the map is injective. $\square$

Remark. Lemma 2.1.28 can be seen as an extension of [12, Corollary 2, page 33].

For the final step, we construct a countable D-linearly independent set that we can map into. Consider the case when $D = k[z]$ (the polynomial ring in one determinate) and $A = \mathrm{Hom}_k(D,k) \cong k^{\mathbb{N}}$. By associating monomials to points in $\mathbb{N}$, there is a k-linear injection $k[z] \to A$ defined as follows. For $p \in D$, write

$$p(z) = \alpha_\ell z^\ell + \alpha_{\ell-1} z^{\ell-1} + \cdots + \alpha_0 \qquad\qquad \alpha_\ell,\ldots,\alpha_0 \in k, \deg(p) = \ell.$$

and define the map $\phi : D \to A$ as

$$\phi(p)(t) = \begin{cases} 0 & t > \ell \\ \alpha_t & t \le \ell \end{cases}$$

However, we wish to construct a D-linear injection rather than a k-linear injection. This requires a slightly different approach.

We may act on A by the backward shift by defining for $a \in A$ and $p = \sum_{i=0}^{\ell=\deg(p)<\infty} \alpha_i z^i$ the D-module action given by

$$(pa)(t) = \alpha_\ell a(t+\ell) + \alpha_{\ell-1} a(t+\ell-1) + \cdots + \alpha_0 a(t) \qquad\qquad t \in \mathbb{N}.$$

We now ask if there exists $a \in A$ such that for all $p \in D$, $p \mapsto pa$ is injective, i.e., a is a D-linearly independent element of A.

Choose an increasing sequence of natural numbers

$$\mu(1) < \mu(2) < \mu(3) < \cdots \qquad\qquad \mu(i) \in \mathbb{N}$$

so that $\lim_{j\to\infty}(\mu(j+1) - \mu(j)) = \infty$ (for instance, $\mu(j) = j!$ works.) Define $S = \{\mu(j)\}_{j=1}^{\infty}$ and also define $a \in k^{\mathbb{N}}$ as the characteristic function of S, i.e.

$$a(t) = \delta_S(t) = \begin{cases} 1 & t \in S \\ 0 & t \notin S \end{cases}$$

Let $p = \sum_{i=0}^{\ell} \alpha_i z^i \in D$ be any given polynomial and define $\ell = \deg(p)$ (i.e., $\alpha_\ell \neq 0$ and $a_s = 0$ for all $s > \ell$.) By construction there exists $N \in \mathbb{N}$ such that $\mu(j) - \mu(j-1) > \ell$ for all $j \ge N$. As a consequence, for any $j \ge N$ and $t = \mu(j) - \ell$,

$$a(t+\ell) = a(\mu(j) - \ell + \ell) = a(\mu(j)) = 1 \quad \text{and} \quad a(t+\ell-1) = \cdots = a(t) = 0$$

and hence

$$(pa)(t) = \alpha_\ell a(t+\ell) + \alpha_{\ell-1} a(t+\ell-1) + \cdots + \alpha_0 a(t) = \alpha_\ell a(t+\ell) = \alpha_\ell \neq 0.$$

It follows that $pa \neq 0$. Since $p \in D$ was arbitrary, we have that a is D-linearly independent. This leads us to the following simple observation.

Lemma 2.1.29. *For the polynomial ring $k[z]$ in one indeterminate,* $\mathrm{Hom}_k(k[z], k) = k^{\mathbb{N}}$ *is a large injective cogenerator.*

Proof. The element a which is $k[z]$-linearly independent provides the map $\phi : k[z] \to \mathrm{Hom}_k(k[z], k)$ via $p \in k[z] \mapsto pa \in \mathrm{Hom}_k(k[z], k)$. We have $pa = p'a$ for $p, p' \in k[z]$ implies $(p - p')a = 0$ and, since a is $k[z]$-linearly independent, $(p - p')a = 0$ implies $p = p'$; in particular, $\ker(\phi) = 0$. □

The following lemma streamlines the above process to produce a countable family of such elements. The reasoning here is that we wish to have an inductive process for adding more indeterminates. For this to work, there has to be ample linearly independent elements to compensate for the addition of more indeterminates.

Lemma 2.1.30. *For polynomial ring $D = k[z]$ in one indeterminate, the D-module $A = \mathrm{Hom}_k(D, k)$ admits a countable family of D-linearly independent elements.*

Proof. The proof is contained in [39, pg. 55-57]. The approach centers on constructing a countable family of characteristic functions by constructing a sequence of D-linearly independent elements as above. □

Recall that for a family of non-degenerate k-bilinear forms

$$\langle \cdot, \cdot \rangle_i : V(i) \times \widehat{V}(i) \to k \qquad i \in \mathcal{I},$$

induce the k-bilinear form

$$\langle \cdot, \cdot \rangle : \bigoplus_{i \in \mathcal{I}} V(i) \times \prod_{i \in \mathcal{I}} \widehat{V}(i) \to k \tag{2.13}$$

is also non-degenerate. As an application, let D be an affine integral domain and $A = \mathrm{Hom}_k(D, k)$ be its dual space with the non-degnerate k-bilinear form

$$D \times A \to k \qquad (p, a) \mapsto \langle p, a \rangle := a(p).$$

We may use (2.13) to reach the non-degenerate k-bilinear form

$$\langle \cdot, \cdot \rangle : D^{(\mathbb{N})} \times A^{\mathbb{N}} \to k$$

where

$$\langle p, a \rangle := \langle \{p(j) : j \in \mathbb{N}\}, \{a(j) : j \in \mathbb{N}\} \rangle = \sum_j \langle p(j), a(j) \rangle.$$

Note that in the above $D^{(\mathbb{N})}$ means that components are almost all zero so that the sum is finite and thus well-defined.

We may identify $D^{(\mathbb{N})} = D[z]$ via

$$p = \sum_{j \in \mathbb{N}} p(j) s^j \qquad p \in D^{(\mathbb{N})}.$$

Similarly, we may identify $A^{\mathbb{N}} = \mathrm{Hom}_k(D[z], k)$ where for $a \in A^{\mathbb{N}}$,

$$a = \langle \cdot, a \rangle : \sum_j p(j)s^j \mapsto \sum \langle p(j), a(j) \rangle.$$

In particular, we have that the $D[z]$-module structure on $A^{\mathbb{N}}$ is defined as

$$(z^i a)(j) = a(i+j) \qquad\qquad i, j \in \mathbb{N}.$$

This trick allows us to write $k[z_1, z_2] = k[z_1][z_2]$ and use that $\mathrm{Hom}_k(k[z_1], k)$ is a large injective cogenerator and then use the countable basis to demonstrate that $\mathrm{Hom}_k(k[z_1][z_2], k)$ is a large injective cogenerator. In the following lemma we make this statement more precise.

Lemma 2.1.31. *Let D be an affine domain and assume that the D-module $A = \mathrm{Hom}_k(D, k)$ admits a countable family of D-linearly independent elements. Then for the polynomial algebra in one indeterminate $D[z]$, the $D[z]$-module $A = \mathrm{Hom}_k(D[z], k)$ also admits a countable family of $D[z]$-linearly independent elements.*

Proof. By following the above construction, we may identify $D[z] \leftrightarrow D^{(\mathbb{N})}$ and $\mathrm{Hom}_k(D[z], k) \leftrightarrow A^{\mathbb{N}}$. By hypothesis, there exists a countable family of D-linearly independent elements in A. Let ι be the "diagonal" bijection $\iota : \mathbb{N} \to \mathbb{N}^2$. We may choose a doubly indexed family $\{a_i(j) \in A\}_{i,j\in\mathbb{N}}$ of D-linearly independent elements in A by using ι. We now show that the family of sequences $a_i = \{a_i(j)\}_{j\in\mathbb{N}}$ are $D[z]$-linearly independent. Say that there exists $p = \{p_i\} \in D^{(\mathbb{N})}$ where $p_i = \sum_i p_i(j)z^j \in D[z]$, almost all $p_i(j)$ are zero and $\sum p_i a_i = 0$. It follows that

$$0 = \left(\sum p_i a_i\right)(0) = \sum_{i,j} p_i(j) a_i(j),$$

and thus $p_i = 0$ since, by hypothesis, each $a_i(j)$ is D-linearly independent. We conclude that $A = \mathrm{Hom}_k(D[z], k)$ admits a countable family of $D[z]$-linearly independent elements. □

We may extend Lemma 2.1.31 to a polynomial algebra in any number of indeterminates by starting with the observation $\mathrm{Hom}_k(k[z], k)$ admits a countable family of $k[z]$-linearly independent elements and adding indeterminates until the desired number is reached. That is

$$D_1 = k[z_1] \quad \Rightarrow \quad D_2 = D_1[z_2] = k[z_1, z_2] \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad D_d = D_{d-1}[z_d] = k[z_1, \dots, z_d].$$

Theorem 2.1.32. *For an affine k-algebra D, $\mathrm{Hom}_k(D, k)$ is a large injective cogenerator.*

Proof. First note that since for any polynomial algebra D, we have $\mathrm{Hom}_k(D, k)$ has a D-linearly independent element a. As a result, the map $p \in D \mapsto pa \in \mathrm{Hom}_k(D, k)$ is injective. By Lemmas 2.1.28 and 2.1.27 this means that $\mathrm{Hom}_k(D, k)$ is a large injective cogenerator when D is an affine k-algebra. □

From (2.9) and (2.10) the identifications

$$D = k[z_1, \dots, z_d] \qquad\qquad A = k^{\mathbb{N}^d}$$

allow us to state that $\mathcal{A} := k^{\mathbb{N}^d}$ is a large injective cogenerator for the category of finitely generated $\mathcal{D} := k[z_1, \dots, z_d]$-modules. We also have that for any prime ideal $I \subset \mathcal{D}$ that $\mathrm{Hom}_k(\mathcal{D}/I, k)$ is a large injective cogenerator over the category of $\mathcal{D}/I$ modules.

2.1.4 Topological Characterization of Behaviors

Notation. Let k be a field equipped with the discrete topology, D be an affine k-algebra, and $A = \mathrm{Hom}_k(D, A)$.

For the case $d = 1$, $D = \mathbb{R}[z]$, and $A = \mathbb{C}^{\mathbb{Z}}$, Willems in [51] characterizes all behaviors as linear, shift-invariant and closed subspaces of $(\mathbb{R}^{\mathbb{Z}})^q$ equipped with the topology of point-wise convergence. By using $D = \mathbb{R}[z_1, \ldots, z_d]$ is a Noetherian ring, the proof Willems provides (which employs that the ring is a PID, but really only requires that it be Noetherian) naturally extends to the multidimensional setting; this case has been addressed by Rocha in [43]. One application is that we may consider the closure of a shift-invariant subspace of trajectories as a behavior. When Willems develops a frequency domain theory, this topological correspondence is used to link a behavior to the closed subspace containing its exponential trajectories. Outside of this example, typical applications involve constructing a behavior by means other than a kernel representation.

One shortcoming of the mentioned setting is that it requires working with a scalar field which is $\mathbb{C}$, $\mathbb{R}$, or a field with a Euclidean topology. Furthermore, the proof requires that the ring is a polynomial ring rather than an affine k-algebra. Oberst provides a clean and general proof that places a topology on A which is complementary to the algebraic structure; as a consequence, the topological characterization is both sharper and allows a deeper understanding of the algebraic structure placed on A.

Before moving onto the main contents of the section, we present some topological preliminaries. All of the stated results can be found in [30], however, the proofs have been adjusted to keep the prerequisites minimal.

The topology we are concerned with is the **weak-$*$ topology** on A. In our setting, we have the k-vector space D and its algebraic dual $A = \mathrm{Hom}_k(V, k)$. The pairing

$$\langle v^*, v\rangle \in A \times D \mapsto v^*(v) \in k$$

can be used to induce the weak-$*$ topology on A by defining the base of neighborhoods of zero for A as

$$\{v^* \in A : v^*(v_1) = \cdots = v^*(v_r) = 0\}$$

for all finite dimensional subspaces $\mathrm{span}_k\{v_1, \ldots, v_r\} \subset D$[1] This is an example of a **linear topology** on a vector space (giving rise to a linear topological vector space) where a base for the neighborhoods of zero consists of linear subspaces and neighborhoods of other points are translates of neighborhoods of zero.

When $D = k[z_1, \ldots, z_d]$, we may identify the polynomial $p \in D$

$$p(\mathbf{z}) = \sum_{i\in\mathbb{N}^d} a_i \mathbf{z}^i \mapsto \{a_i\}_{i\in\mathbb{N}^d} \in k^{(\mathbb{N}^d)} := \bigoplus_{i\in\mathbb{N}^d} k.$$

This follows because every polynomial is a *finite* sum of forms and so in $\{a_i\}_{i\in\mathbb{N}^d}$ all but a finite number of the a_i values are zero. In this way, we have a basis for D given by $\{e_i\}_{i\in\mathbb{N}^d}$ so that every

[1] This is the standard weak-$*$ topology used in basic functional analysis (e.g., [44]) for the case where the scalar field is given the discrete topology.

element in D has a unique representation as a finite sum of the basis elements with coefficients in k. For the dual space $A = \mathrm{Hom}_k(D, k)$, we have a similar identification for $v^* \in A$,

$$v^* \mapsto \{v^*(\mathbf{z}_i)\}_{i\in\mathbb{N}^d} \in k^{\mathbb{N}^d} := \prod_{i\in\mathbb{N}^d} k.$$

For each e_i, we may take the subspace $\mathrm{span}_k\{e_i\} \subset k^{(\mathbb{N}^d)}$ and construct a linear functional e_i^* such that

$$e_i^*(e_i) = 1 \qquad\qquad e_i^*(e_j) = 0 \text{ for all } j \neq i.$$

Existence follows by the fact $\mathrm{span}_k\{e_i\}$ is a one-dimensional vector space and, as such, there is a dual element; we may extend e_i^* to a a linear functional on $k^{(\mathbb{N}^d)}$ by specifying that it take zero values on the complementary basis provided. Even though the elements $\{e_i^*\}_{i\in\mathbb{N}^d}$ are not a basis of A, they serve as a "dual system" under evaluation on D; in particular, for any vector $\{a_i\} \in k^{\mathbb{N}^d}$ we have that the functional $\ell = \sum_{i\in\mathbb{N}^d} a_i e_i^*$ can be evaluated on D since

$$\ell(e_j) = \sum_{i\in\mathbb{N}^d} a_i e_i^*(e_j) = a_j e_j^*(e_j) = a_j.$$

In this way, we have every linear functional $\ell \in A$ can be represented by a vector $\{a_i\} \in k^{\mathbb{N}^d}$. Under such a representation, we have (recall the notation $-^{\perp}$ from (2.7))

$$e_i^{\perp} = \left\{ \{a_j\} \in k^{\mathbb{N}^d} : \sum_{j\in\mathbb{N}^d} a_j e_j^*(e_i) = a_i e_i^*(e_i) = a_i = 0 \right\} = \Big\{ \{a_j\} \in k^{\mathbb{N}^d} : a_i = 0 \Big\}.$$

Similarly, for any finite set $\mathcal{I}$ we have

$$(\{e_i\}_{i\in\mathcal{I}})^{\perp} = \Big\{ \{a_j\} \in k^{\mathbb{N}^d} : a_i = 0 \text{ for all } i \in \mathcal{I} \Big\}.$$

This leads us to the well-known proposition.

Proposition 2.1.33. *When A is equipped with the weak-$*$ topology, A is topologically isomorphic to the topological product $k^{\mathbb{N}^d} = \prod_{\mathbb{N}^d} k$ where k is equipped with the discrete topology.*

Proof. Let $U \subset k^{\mathbb{N}^d}$ be an open neighborhood of zero. It follows that $U = \prod_{i\in\mathbb{N}^d} U_i$ where $U_i = k$ for all but finitely many indices $i \in \mathcal{I}$. Since k is discrete, we have every $U' \subset A$ where $U \subset U'$ is also an open neighborhood of zero. In particular, all neighborhoods of U are open. We may thus take a base of neighborhoods of zero to be $(\{e_i\}_{i\in\mathcal{I}})^{\perp}$. Because this topology has the same base of neighborhoods of zero as the weak-$*$ topology, they induce the same topology. $\square$

We now establish some basic implications of the topology placed on A. First let us recall the following definitions.

Definition 2.1.34. Let X be a set. We define a **filter** as a non-empty subclass $\mathcal{F} = \{F_i\}$ of subsets of X with the following properties:

1. Ever subset of X containing $F_i \in \mathcal{F}$ belongs to $\mathcal{F}$.
2. The intersection of finitely many elements of $\mathcal{F}$ also belongs to $\mathcal{F}$.
3. The empty set does not belong to $\mathcal{F}$.

A non-empty subclass $\mathcal{B}$ of a filter $\mathcal{F}$ is called **filter base** of $\mathcal{F}$ if it satisfies the following conditions.

1. The intersection of two elements $\mathcal{B}$ *contains* a set of $\mathcal{B}$.
2. The empty set does not belong to $\mathcal{B}$.

A point $x \in X$ is said to be a **cluster point**[2] of the filter $\mathcal{F}$ if x is a closure point of every element $F_i \in \mathcal{F}$.

One can easily verify that if $\mathcal{B}$ is a filter base of the filter $\mathcal{F}$, then when validating whether or not a point $x \in X$ is a cluster point of $\mathcal{F}$ one need only show that x is a cluster point of the filter base $\mathcal{B}$.

Definition 2.1.35. We say that a topological vector space X is **linearly compact** if every filter with a base of linear affine subspaces (i.e. of the form $x + H$ where $x \in X$ is a point and $H \subset X$ is a linear subspace) has a cluster point in X. Another way of defining this property is as follows: for any collection of closed linear affine subspaces such that a finite sub-collection has non-empty intersection it follows that the whole collection has non-empty intersection.

Note that one version of the standard notion of compactness is exactly this second formulation of linear compactness, but with the collection of closed linear affine sets taken to be any collection of closed sets.

Lemma 2.1.36. *If k has the discrete topology, then k is linearly compact.*

Proof. Let $\mathcal{F} = \{F_i\}$ be a filter with a base $B = \{b_j\}$ of linear affine subspaces. Any affine subspace of k is necessarily the entire space k or zero. If $b_j = \{0\}$ then 0 is a cluster point since every set containing zero intersects either k or $\{0\}$ non-trivially. Since every $F_i \in \mathcal{F}$ contains an element of the base, we conclude that k is linearly compact. $\square$

An analogue of Tychonoff's Theorem for the linearly compact setting is given by the following.

Lemma 2.1.37 (Tychonoff's Theorem)**.** *The topological product of arbitrarily many linearly compact spaces is linearly compact.*

Proof. Let $V = \prod_{i \in \mathcal{I}} V_i$ where each V_i is linearly compact. For any filter $\mathcal{F}$ with a base of linear affine subspaces B, we have that the continuous projection map $\pi_i : V \to V_i$ produces the filter $\mathcal{F}_i = \pi_i(\mathcal{F})$ with a base of linear affine subspaces $B_i = \pi_i(B)$. By hypothesis, each $\mathcal{F}_i$ has a cluster

[2] In [31, page 12], Köthe uses the term **adherent points** of the filter when speaking of cluster points of the filter.

point $x_i \in V_i$. We argue that $x = \{x_i\}_{i \in \mathcal{I}}$ is a cluster point of $\mathcal{F}$. For any open neighborhood U of x, we have $U = \prod H_i$ where $H_i = V_i$ for all but a finite number of indices $\mathcal{J}$. For each $i \in \mathcal{J}$ one observe that $\pi_i(H_i)$ intersects every element of the filter $\mathcal{F}_i$ non-trivially and so H_i intersects every element of the filter $\mathcal{F}$ non-trivially; in particular, x is a cluster point of the filter $\mathcal{F}$. We conclude that V is linearly compact. □

Corollary 2.1.38. *When* $\mathrm{Hom}_k(D, k)$ *is given the weak-$*$ topology,* $\mathrm{Hom}_k(D, k)$ *is linearly compact.*

Proof. This follows from the identification $\mathrm{Hom}_k(D, k) \leftrightarrow k^{\mathbb{N}^d}$ and Lemmas 2.1.36 and 2.1.37. □

Before discussing properties of maps on linearly compact spaces, we need the following lemma.

Lemma 2.1.39. *Let* $v_1, \dots, v_r, v \in D$ *be given. Then there exists* $a_1, \dots, a_r \in k$ *such that* $v = a_1 v_1 + \cdots + a_r v_r$ *if and only if* $v^*(v) = 0$ *for all*

$$v^* \in X := \{v^* \in A : v^*(v_1) = \cdots = v^*(v_r) = 0\}$$

Proof. $(\Rightarrow)$ For any $v^* \in X$,

$$v^*(v) = a_1 v^*(v_1) + \cdots + a_r v^*(v_r) = 0.$$

$(\Leftarrow)$ Define the map $S : A \to k^r$ by $S(v^*) = (v^*(v_1), \dots, v^*(v_r))$ for $v^* \in A$. If $S(v^*) = S(w^*)$ then

$$S(v^*) - S(w^*) = ((v^* - w^*)(v_1), \dots, (v^* - w^*)(v_r)) = 0,$$

which implies that $v^* - w^* \in X$. By hypothesis, this implies that $(v^* - w^*)(v) = 0$ and thus $v^*(v) = w^*(v)$. As a result, $T : S(A) \to k$ defined as $T(S(v^*)) = v^*(v)$ is a well defined linear functional on the vector subspace $S(A) \subset k^r$. Let $\widehat{T}$ be an extension of T to a linear functional on k^r. Since k^r is a finite dimensional vector space, we have that $\widehat{T}$ is given by

$$\widehat{T}(x_1, \dots, x_r) = a_1 x_1 + \cdots a_r x_r$$

for some $a_1, \dots, a_r \in k$. Since $\widehat{T}|_{S(A)} = T$,

$$v^*(v) = T(S(v^*)) = \widehat{T}(S(v^*)) = a_1 v^*(v_1) + \cdots + a_r v^*(v_r).$$

In particular,

$$v^*(v - a_1 v_1 - \cdots - a_r v_r) = 0$$

for all $v^* \in A$. Since $A = \mathrm{Hom}_k(D, k)$ we may conclude that $v = a_1 v_1 + \cdots + a_r v_r$. □

Lemma 2.1.40. *If* A *is equipped with the weak-$*$ topology, then* D *is equal to the space of continuous linear functionals on* A.

Proof. We have that the base of weak neighborhoods of zero are given by sets of the form

$$U(u_1, \dots, u_n) := \{v^* \in A : v^*(u_1) = \dots = v^*(u_n) = 0\}$$

where $u_1, \dots, u_n \in D$. For any $u \in D$ we have $v^*(u) = 0$ for each $v^* \in U(u)$ and so u is continuous at zero since $U(u)$ is open in A. Conversely, for any continuous linear functional $u \in D$ on A, there exists a weak neighborhood of zero, $\{u_1, \dots, u_n\}^\perp \subset A$, on which u vanishes. By Lemma 2.1.39, u is a linear combination of $u_1, \dots, u_n$ and so $u \in D$. $\square$

We also have that the following properties hold for spaces and bounded linear maps on spaces with a linearly compact topology; hence they also hold for the weak-$*$ topology.

Lemma 2.1.41. *For a linearly compact space V the following hold.*

1. *A closed linear subspace $V' \subset V$ is linearly compact.*
2. *Every continuous linear map R from a linearly compact space to a topological vector space has a linearly compact null space.*
3. *Every continuous linear map R from a linearly compact space to a topological vector space has a linearly compact image.*
4. *If $V' \subset V$ is a closed linear subspace then V/V' is linearly compact.*

Proof. (1) Since V' is closed it contains its cluster points. Any filter $\mathcal{F}$ on V' can be considered as a filter on V. If $\mathcal{F}$ is a filter with a base of affine subspaces which has a cluster point in V, we have that V' contains this cluster point since it is closed.

(2) Let $R : V \to V'$ be continuous. Then the pre-image $R^{-1}(0)$ is closed in V; hence, the null-space is closed in V. By (1) we reach that the null-space is linearly compact.

(3) Let $R : V \to V'$ be continuous. Let $\mathcal{F} = \{F_i\}$ be a filter on $R(V_1)$ with a base consisting of affine subspaces. Since R is linear, we have that the preimage $R^{-1}(F_i)$ generates a filter $\mathcal{F}'$ on V which also has a base consisting of affine subspaces. Since V is linearly compact, $\mathcal{F}'$ has a cluster point $x \in V$ and, by continuity, we have $R(x)$ is cluster point of the filter $\mathcal{F}$. Since $\mathcal{F}$ was arbitrary, we have $R(V)$ is linearly compact.

(3) Let $R : V \to V/V'$ be the quotient map. Since V' is closed, the quotient topology induced on V/V' makes it a linear topological vector space. Because R is continuous and surjective, by (2) we may conclude that V/V' is linearly compact. $\square$

There is also a relationship between the maps between vector spaces and the continuous maps between their dual spaces.

Lemma 2.1.42. *Let D and D' be k-vector spaces and $A = \operatorname{Hom}_k(D, k)$ and $A' = \operatorname{Hom}_k(D', k)$ be their respective dual spaces equipped with the weak-$*$ topology. If $R : D \to D'$ is a k-linear map, then the induced adjoint map $R^T : A' \to A$ is continuous; conversely, every continuous k-linear map $R^T : A' \to A$ is the adjoint of some k-linear map $R : D \to D'$.*

Proof. Let $R : D \to D'$ be a k-linear map. We establish continuity on the base neighborhoods of zero. Let $\{v_1, \dots, v_r\}^\perp \subset A$ be a neighborhood of zero. For any $v^* \in \{Rv_1, \dots, Rv_r\}^\perp \subset A'$ we have

$$0 = \langle v^*, a_1 Rv_1 + \cdots + a_r Rv_r \rangle = \langle R^T v^*, a_1 v_1 + \cdots + a_r v_r \rangle = R^T v^*(a_1 v_1 + \cdots + a_r v_r).$$

Since $R^T \{Rv_1, \dots, Rv_r\}^\perp \subset \{v_1, \dots, v_r\}^\perp$, R^T is continuous.

Conversely, let R^T be a continuous k-linear map $R^T : A' \to A$. We thus have for any $w \in D$ that $v^* \mapsto (R^T v^*)(w) \in k$ is a continuous linear functional on A'. Because A' is equipped with the weak-$*$ topology, the continuous dual of A' is equal to D'; as a consequence, there exists a *unique* $w' \in D'$ such that $v^*(w') = (R^T v^*)(w)$ for all $v^* \in A'$. We may define the linear map $\widehat{R} : D \to D'$ via $\widehat{R}w = w'$ by choosing w to be basis elements of D for any chosen basis and extending by linearity. Through the pairing we have

$$\langle R^T v^*, w \rangle = \langle v^*, w' \rangle = \langle v^*, \widehat{R}w \rangle.$$

In particular, $\widehat{R}$ is the adjoint of R^T. □

For later use, we also present a version of the Hahn-Banach theorem for this setting.

Theorem 2.1.43 (Hahn-Banach Theorem)**.** *Let D be a k-vector space and $A = \mathrm{Hom}_k(D, k)$ have the weak-$*$ topology. If w^* is an element of A not lying in the closed linear subspace $B \subset A$, there exists a $\widehat{u} \in D$ with $w^*(\widehat{u}) = 1$ and $v^*(\widehat{u}) = 0$ for all $v^* \in B$.*

Proof. Since B is closed, we have that there exists an open neighborhood of zero, U, for which $(w^* + U) \cap B = \emptyset$. In particular, we have $w^* \cap (B + U) = \emptyset$ and $B + U$ is a linear subspace (recall that U was an open neighborhood of zero). Let $V = \mathrm{span}_k(w^*)$ be a subspace of A and note that, since $B + U$ is a linear subspace, we have $V \cap (B + U) = \emptyset$. By construction, V is spanned by one element and so it is finite-dimensional; in particular, w^* has a dual element u such that $u(w^*) = 1$. Let $\{x_i\}_{i \in \mathcal{I}}$ be an algebraic complementary basis of w^* in A and define the extension of u by $\{x_i\}_{i \in \mathcal{I}}$ as $\widehat{u}(x_i) = 0$ for all $i \in \mathcal{I}$ and extend k-linearly. While this demonstrates $\widehat{u} \in \mathrm{Hom}_k(A, k)$ (the algebraic dual, not the continuous dual) we now argue that it is also the case $\widehat{u} \in D$, i.e., $\widehat{u}$ is a continuous linear functional on A.

Since $\widehat{u}|_{B+U} = 0$ and $0 \in B$, we have $\widehat{u}_U = 0$ and thus $\widehat{u}$ vanishes on an open neighborhood of zero; as a consequence, we have that $\widehat{u}$ is continuous. Because A has the weak-$*$ topology, by Lemma 2.1.40 this implies that $\widehat{u} \in D$. By construction, $\widehat{u}$ satisfies the stated properties. □

After this introduction to the weak-$*$ for vector spaces over a field with the discrete topology, we return to behaviors. Denote by $\widehat{\mathbf{Mod}(k)}$ the abelian category of linearly compact k-vector spaces with continuous k-linear maps as morphisms. For $X, Y \in \widehat{\mathbf{Mod}(k)}$, denote the morphisms (i.e. continuous k-linear maps) between X and Y as $\widehat{\mathrm{Hom}_k}(X, Y)$. For $f \in \widehat{\mathrm{Hom}_k}(X, Y)$, by Lemma 2.1.41 we have that $\ker(f)$, $\mathrm{im}(f)$, and $\mathrm{coker}(f)$ are linearly compact with their induced topologies; furthermore, if f is a bijection, then it is a topological isomorphism.

Let V be a k-vector space with the discrete topology and $\{v_i\}_{i\in\mathcal{I}}$ be a k-basis of V. One may observe that the identification

$$V^* = \mathrm{Hom}_k(V,k) \to k^{\mathcal{I}} \qquad\qquad v^* \mapsto \{v^*(v_i)\}_{i\in\mathcal{I}}$$

is a topological isomorphism. Recall the non-degenerate k-bilinear pairing

$$\langle -,-,\rangle : V \times V^* \to k \qquad\qquad \langle v, v^*\rangle = v^*(v)$$

with the induced k-linear maps

$$\begin{aligned} V^* &\to \mathrm{Hom}_k(V,k) &\qquad v^* &\mapsto \langle -, v^*\rangle \\ V &\to \mathrm{Hom}_k(V^*,k) &\qquad v &\mapsto \langle v, -\rangle. \end{aligned}$$

If we equip V^* with the weak-$*$ topology then $v \mapsto \langle v, -\rangle$ is continuous by Lemma 2.1.40 and we also have $V \cong \widehat{\mathrm{Hom}_k}(V^*,k)$; note that the introduction of the topology on V^* reduces $\mathrm{Hom}_k(V^*,k)$ to $\widehat{\mathrm{Hom}_k}(V^*,k)$ so that the injective mapping $v \mapsto \langle v, -\rangle$ is now bijective. In particular, if $\{v_i\}_{i\in\mathcal{I}}$ is a k-basis of V and $V^* = \mathrm{Hom}_k(V,k)$, then we have

$$V = k^{(\mathcal{I})} \cong \widehat{\mathrm{Hom}_k}(V^*,k).$$

In this way we have that the functor $\mathrm{Hom}_k(-,k)$ provides the duality

$$(-)^* := \mathrm{Hom}_k(-,k) : \mathbf{Mod}(k)^{op} \cong \widehat{\mathbf{Mod}(k)}. \tag{2.14}$$

Let M be a D-module; since D is an affine k-algebra and k has the discrete topology, the dual space $M^* = \mathrm{Hom}_k(M,k)$ is linearly compact and is also a D-module as demonstrated in (2.11); in particular, we have the D-module structure as follow

$$(fm^*)(m) = \langle m, fm^*\rangle = \langle fm, m^*\rangle = m^*(fm) \qquad\qquad m \in M,\ m^* \in M^*,\ f \in D.$$

Denote by $\widehat{\mathbf{Mod}(D)}$ the category of linearly compact D-modules with continuous D-module actions, i.e., for $f \in D$, we have that $x \mapsto fx$ is continuous. Naturally, we define the morphisms of $\widehat{\mathbf{Mod}(D)}$ to be continuous D-module homomorphisms. Since D is an affine k-algebra, from (2.14) we reach

$$(-)^* = \mathrm{Hom}_k(-,k) : \mathbf{Mod}(D)^{op} \cong \widehat{\mathbf{Mod}(D)}. \tag{2.15}$$

For finitely generated modules we have the identification from (2.12)

$$M^* = \mathrm{Hom}_k(M,k) = \mathrm{Hom}_D(M, \mathrm{Hom}_k(D,k)) = \mathrm{Hom}_D(M,A)$$

given by

$$m^*(m)(f) = m^*(fm) = \langle fm, m^*\rangle = \langle m, fm^*\rangle \qquad\qquad m \in M,\ m^* \in M^*,\ f \in D.$$

In particular, the functors $\mathrm{Hom}_D(-,A)$ and $\mathrm{Hom}_k(-,k)$ coincide on the category of finitely generated D-modules. As a consequence,

$$(D^q)^* = \mathrm{Hom}_D(D^q,A) = \mathrm{Hom}_k(D^q,k) = A^q \qquad\qquad q \in \mathbb{N};$$

hence both can be considered as an E-module or as a linearly compact D-module. This is rather important since, as we now show, it connects D-module homomorphisms to E-module homomorphisms.

Let $M, M' \in \mathbf{Modf}(D)$ be two finitely generated D-modules and let $\mathcal{B} = \mathrm{Hom}_D(M, A)$ and $\mathcal{B}' = \mathrm{Hom}_D(M', A)$ be their associated behaviors. By Lemma 2.1.42 and the established dualities we reach the following correspondences.

$$\mathrm{Hom}_D(M, M') \cong \mathrm{Hom}_E(\mathcal{B}', \mathcal{B}) \qquad \mathrm{Hom}_D(M, M') \cong \widehat{\mathrm{Hom}_D}(M'^*, M^*).$$

As a consequence, for a D-modules homomorphism $R : M \to M'$, we may identify

$$R^* = \mathrm{Hom}_k(R, k) = \mathrm{Hom}_D(R, A) = R^T.$$

By combining the above identifications, we reach

$$\mathrm{Hom}_E(\mathrm{Hom}_D(M', A), \mathrm{Hom}_D(M, A)) = \widehat{\mathrm{Hom}_D}(M'^*, M^*),$$

which leads to the following theorem.

Theorem 2.1.44. *Let $A = \mathrm{Hom}_k(D, k)$ have the linearly compact topology. For finitely generated D-modules M and M', and associated behaviors $\mathcal{B} = \mathrm{Hom}_D(M, A)$, $\mathcal{B}' = \mathrm{Hom}_D(M', A)$, we have the following relationships*

$$\mathrm{Hom}_D(M, M') \cong \mathrm{Hom}_E(\mathcal{B}', \mathcal{B}) = \widehat{\mathrm{Hom}_D}(M'^*, M^*) \subset \mathrm{Hom}_k(\mathcal{B}', \mathcal{B}),$$

where for the matrix $R \in \mathrm{Hom}_D(M, M')$,

$$R \leftrightarrow \mathrm{Hom}_D(R, A) = \mathrm{Hom}_E(R, A) = \mathrm{Hom}_k(R, k) = R^* \leftrightarrow R^T.$$

As a consequence, if $\widehat{R} : \mathcal{B}' \to \mathcal{B}$ is a k-linear map, then $\widehat{R}$ is also an E-linear map if and only if $\widehat{R} = \mathrm{Hom}_D(R, A) = R^T$ for some $R \in \mathrm{Hom}_D(M, M')$, or if and only if $\widehat{R}$ is a continuous D-linear map.

A further consequence is the following.

Theorem 2.1.45. *Let $A = \mathrm{Hom}_k(D, k)$ have the linearly compact topology. For a k-subspace $\mathcal{B} \subset A^q$, $q \in \mathbb{N}$ the following are equivalent*

1. *$\mathcal{B}$ is a behavior, i.e., there exists $R \in D^{p\times q}$ such that*

$$\mathcal{B} = \ker_A(R) = \{w \in A^q : Rw = 0\} \subset A^q.$$

2. *$\mathcal{B}$ is a finitely generated E-submodule of A^q.*

3. *$\mathcal{B}$ is a closed D-submodule of A^q.*

Proof. (1) $\iff$ (2) follow from (1) of Theorem 2.1.23 (the Oberst duality theorem.) In particular, behaviors are precisely the finitely generated E-submodules of A^q.

(1) $\Rightarrow$ (3). Assume that $\mathcal{B} \subset A^q$ is a behavior. We have the identifications

$$\mathcal{B} = \ker_A(R) = \mathrm{Hom}_D(\mathrm{coker}_D(R^T), A) = (\mathrm{coker}_D(R^T))^* \subset (D^q)^* = A^q.$$

By Theorem 2.1.44 the D-module map $\iota : D^q \to \mathrm{coker}_D(R^T)$ provides the continuous D-module inclusion $\iota^* : (\mathrm{coker}_D(R^T))^* \hookrightarrow (D^q)^*$. We thus have that the behavior $\mathcal{B}$ is a D-submodule of A^q and is the closed image in A^q of the map ι^*.

(3) $\Rightarrow$ (1). Let $\mathcal{B} \subset A^q = (D^q)^*$ be a closed D-submodule of A^q; as a consequence, $\mathcal{B}$ is a sub-object of A^q in $\widehat{\mathbf{Mod}(D)}$. Due to the duality given by (2.15), there exists a D-module M and matrix $R \in D^{p\times q}$ such that $M = \mathrm{coker}_D(R^T)$ and

$$\mathcal{B} = M^* = (\mathrm{coker}_D(R^T))^* = \mathrm{Hom}_D(\mathrm{coker}_D(R^T), A) = \{w \in A^q : Rw = 0\} \subset A^q.$$

We conclude that $\mathcal{B}$ is a behavior. □

Before concluding the section, we would like to point out that the above topological/categorical correspondence between behaviors and their dual modules can be established purely via topological means as is done in [30]; in particular, one can forgo Oberst duality if the development is performed in a completely topological manner. This argument is made (in words more precise than the preceding sentence) by Yekutieli in [56]. This correspondence, although somewhat less demanding in terms of mathematical content, does not extend to the continuous time setting and lacks the overall flexibility that Oberst duality offers. The reason that this usually fails in the continuous time setting is that, unlike in the discrete time setting, one usually does not take A as the algebraic dual space of the ring of operators. Furthermore one usually already prescribes some topology on the space of signals when defining the systems – we cannot adjust the topology once it is fixed!

2.1.5 Applications of Oberst Duality

Notation. Let D be a commutative Noetherian ring and A be a large injective cogenerator for the category of finitely generated D-modules.

The following applications of Oberst Duality are heavily used throughout this work.

Lemma 2.1.46. *The behavior A is injective in the category* $\mathbf{Syst}(A)$*. In particular, if $F : \mathcal{B} \to \mathcal{B}'$ is an E-monomorphism between behaviors $\mathcal{B}$ and $\mathcal{B}'$, any E-linear map $G_1 : \mathcal{B} \to A$ can be extended to an E-linear map $G_2 : \mathcal{B}' \to A$ such that $G_2F = G_1$.*

Proof. Since $\mathcal{B}$ and $\mathcal{B}'$ are behaviors, we have $\mathcal{B} = \ker_A(R)$ and $\mathcal{B}' = \ker_A(R')$ for some matrices $R \in D^{p\times q}$ and $R' \in D^{p'\times q'}$. Define $\mathcal{M} = D^q/\mathrm{im}_D(R^T)$ and $\mathcal{M}' = D^{q'}/\mathrm{im}_D(R'^T)$ so that $\mathcal{B} = \mathrm{Hom}_D(\mathcal{M}, A)$ and $\mathcal{B}' = \mathrm{Hom}_D(\mathcal{M}', A)$. We have that F induces the surjective map $F^T : \mathcal{M}' \to \mathcal{M}$

and G_1 induces the map $G_1^T : D \to \mathcal{M}$. In particular, we have the diagram

$$\begin{tikzcd}
& D \arrow[dl, dotted, "G_2^T"'] \arrow[d, "G_1^T"] & \\
\mathcal{M}' \arrow[r, "F^T"] & \mathcal{M} \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is exact and G_2^T is a D-module homomorphism such that $F^T G_2^T = G_1^T$. Note that G_2^T since D is a free D-module it is also a projective module. By Oberst duality (Theorem 2.1.23), we may apply $\mathrm{Hom}_D(-, A)$ to the diagram to reach

$$\begin{tikzcd}
& & A \arrow[d, dotted, "G_2"] \\
0 \arrow[r] & \mathcal{B} \arrow[ur, "G_1"] \arrow[r, "F"] & \mathcal{B}'
\end{tikzcd}$$

where the bottom row is exact and $G_2 F = G_1$. By definition[3], we have that A is injective in the category $\mathbf{Syst}(A)$. $\square$

Before continuing, we need a simple lemma which is, essentially, the third isomorphism theorem.

Lemma 2.1.47. *Let $\mathcal{B}_1 \subset \mathcal{B}_2$ be two sub-behaviors of A^q and $\iota : \mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ be the inclusion map. Then the induced E-linear canonical map $\tau : A^q/\mathcal{B}_1 \to A^q/\mathcal{B}_2$ is both well-defined and surjective.*

Proof. Consider the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{B}_1 \arrow[r, "a_1"] \arrow[d, "\iota"] & A^q \arrow[r, "b_1"] \arrow[d, equal] & A^q/\mathcal{B}_1 \arrow[r] \arrow[d, dotted, "\tau"] & 0 \\
0 \arrow[r] & \mathcal{B}_2 \arrow[r, "a_2"] & A^q \arrow[r, "b_2"] & A^q/\mathcal{B}_2 \arrow[r] & 0.
\end{tikzcd}$$

Since b_1 is surjective, for any $t \in A^q/\mathcal{B}_1$, there exists $s \in A^q$ such that s represents t, or equivalently $b_1(s) = t$. Define $\tau(t) = b_2(s)$; to reach that τ is E-linear one need only observe that b_1, b_2, and ι are all E-linear. We now verify that τ is well-defined.

Let $t \in A^q/\mathcal{B}_1$ be given and let $s_1, s_2 \in A^q$ be two distinct representatives, i.e. $s_1 \neq s_2$ and $b_1(s_1 - s_2) = 0$. Say that $b_2(s_1) \neq b_2(s_2)$ or, equivalently, that $b_2(s_1 - s_2) \neq 0$. Define $h = s_1 - s_2$ so that $h \notin \ker(b_2) = \mathcal{B}_2$ and, by the inclusion $\mathcal{B}_1 \subset \mathcal{B}_2$, $h \notin \mathcal{B}_1$. It follows that $h \notin \ker(b_1)$ which implies $b_1(h) = b_1(s_1 - s_2) \neq 0$ thus contradicting the assumption that $b_1(s_1 - s_2) = 0$. We conclude that τ is well-defined and surjective.

$\square$

The following theorem is a remarkably simple result that is used to make many inferences about behaviors.

[3] Although this is not Definition 2.1.3, it is an equivalent definition of an injective module. See [32, page 60].

Theorem 2.1.48. *Let $\mathcal{B}_1 = \ker_A(R_1)$ and $\mathcal{B}_2 = \ker_A(R_2)$, where $R_i \in D^{p(i)\times q}$, be two sub-behaviors of A^q. Then $\mathcal{B}_1 \subset \mathcal{B}_2$ if and only if there is a matrix $X \in D^{p(2)\times p(1)}$ such that $R_2 = XR_1$.*

Proof. ($\Rightarrow$). Let $\mathcal{B}_1 \subset \mathcal{B}_2$. For $i = 1, 2$ define the E-module homomorphism $\pi_i : A^q \to A^q/\mathcal{B}_i$ as the canonical quotient maps and $F_i : A^q/\mathcal{B}_i \to A^{p(i)}$ as the maps induced by R_i; in this situation both F_1 and F_2 are E-monomorphisms. We may factor each R_i through its kernel as

$$A^q \xrightarrow{\pi_i} A^q/\mathcal{B}_i \xrightarrow{F_i} A^{p(i)}$$

so that $R_i = F_i \circ \pi_i : A^q \to A^{p(i)}$. By Lemma 2.1.47 the map E-linear $\iota : A^q/\mathcal{B}_1 \to A^q/\mathcal{B}_2$ is both well-defined surjective. By Lemma 2.1.46, there exists $G : A^{p(2)} \to A^{p(1)}$ such that $GF_1 = F_2 \circ \iota$. The maps defined thus far lead us to the following commutative diagram

$$\begin{tikzcd} A^q/\mathcal{B}_1 \arrow[r, "F_1"] \arrow[d, "\iota"] & A^{p(1)} \arrow[d, "G"] \\ A^q/\mathcal{B}_2 \arrow[r, "F_2"] & A^{p(2)}. \end{tikzcd}$$

Note that, since both ι and π_1 are surjective, $\iota\circ\pi_1$ is surjective. However, since π_2 is also surjective, both $\iota\circ\pi_1$ and π_2 have the same domain and codomain and the diagram is commutative, $\iota\circ\pi_1 = \pi_2$. This leads us to the following observation

$$GR_1 = GF_1 \circ \pi_1 = F_2 \circ \iota \circ \pi_1 = F_2 \circ \pi_2 = R_2.$$

Since G is an E-linear map, by Theorem 2.1.23 there exists a matrix $X \in D^{p(2)\times p(1)}$ representing G. We thus have $XR_1 = R_2$.

($\Leftarrow$). Let X be such that $R_2 = XR_1$. Then $R_1 w = 0$ implies that $R_2 w = XR_1 w = 0$. It follows that $\mathcal{B}_1 \subset \mathcal{B}_2$. $\square$

A simple application of the above theorem provides us with the following result.

Corollary 2.1.49. *Let $\mathcal{B} \subset A^q$ be a behavior such that $\mathcal{B} = \ker_A(R_1) = \ker_A(R_2)$. Then there exists matrices X_1 and X_2 of appropriate size so that $R_2 = X_1R_1$ and $R_1 = X_2R_2$.*

2.1.6 I/O Structure and Transfer Classes

Notation. Let k be a field, D be an affine domain and A be a large injective cogenerator. Also define $K = Q(D)$ as the quotient field of D.

The idea of an input/output structure for behaviors dates back to Willems in [51]. To define an input/output structure we present a kernel representation, which is an AR system, as an ARMA system. Let $\mathcal{B}$ be a behavior with kernel representation $R \in \mathcal{D}^{p\times q}$. Up to elementary column operations, we may write

$$R = [-Q \quad P] \qquad \text{rank}(R) \geq \text{rank}(P) \qquad P \text{ has full column rank.}$$

The behavior then becomes

$$\mathcal{B} = \{w = (u, y) \in \mathcal{A}^m \times \mathcal{A}^p : Qu = Py\}.$$

Implicitly, we have selected a decomposition of the trajectories $w = (u, y)$, where u is the **input** and y is the **output**. In this setting, one is concerned with the existence of a rational matrix H such that $PH = Q$, i.e., a transfer function. However, instead of confining ourselves to specific choices of D and A we discuss results in a more general setting.

Consider a behavior $\mathcal{B} = \ker_A(R) \subset A^q$ where $R \in D^{p\times q}$. Let the **output index set**, $\mathcal{O} \subset \{1, \ldots, q\}$, be a subset of indices and the **input index set**, $\mathcal{I} = \{i \in \{1, \ldots, q\} : i \notin \mathcal{O}\}$ be its complement; note that, as consequence of the decomposition, we have $\mathcal{I} \cup \mathcal{O} = \{1, \ldots, q\}$ and $\mathcal{I} \cap \mathcal{O} = \emptyset$. We call the sets $(\mathcal{I}, \mathcal{O})$ an **i/o-structure** of a behavior.

For a given i/o-structure of a behavior, we may decompose the kernel representation into an i/o form as follows. Let $R = [r_1, \ldots, r_q]$ be the columns of R; for an i/o-structure $(\mathcal{I}, \mathcal{O})$ we write

$$P = [r_i]_{i\in\mathcal{O}} \qquad\qquad Q = [-r_i]_{i\in\mathcal{I}}.$$

This can also be achieved through linear transformations. Let $T_\mathcal{I} \in A^{q\times|\mathcal{I}|}$ and $T_\mathcal{O} \in A^{q\times|\mathcal{O}|}$ be the matrices defined as

$$T_\mathcal{I} = [e_i]_{i\in\mathcal{I}} \qquad\qquad T_\mathcal{O} = [e_i]_{i\in\mathcal{O}}.$$

where e_* are the standard basis vectors for A^q. It is then clear that the following holds

$$P = RT_\mathcal{O} \qquad\qquad -Q = RT_\mathcal{I}.$$

In a corresponding way, we define the **input signals** and **output signals** as, respectively, $A^\mathcal{I}$ and $A^\mathcal{O}$. For $w = (w_1, \ldots, w_q) \in A^q$ we have

$$u = T_\mathcal{I} w = (w_i)_{i\in\mathcal{I}} \qquad\qquad y = T_\mathcal{O} w = (w_i)_{i\in\mathcal{O}}.$$

In this way, we have the equivalent ways of defining $\mathcal{B}$.

$$\mathcal{B} = \{w \in A^q : Rw = 0\} = \{w \in A^q : PT_\mathcal{O} w = QT_\mathcal{I} w\} = \{(u, y) \in A^\mathcal{I} \times A^\mathcal{O} : Py = Qu\}.$$

Note that for *any* i/o-structure, the behavior is identical since the trajectories must all still satisfy the same governing equations; however, such a decomposition allows for behaviors to be studied from a different point of view. In particular, they permit one to consider a behavior as an input/output system.

Note. From here on out, we use the above formalism implicitly; in particular, for a kernel representation R we write $R = \begin{bmatrix} -Q & P \end{bmatrix}$ and $w = (u, y)$ where it is understood that the decomposition is under a given i/o-structure. When it is necessary to discuss the i/o-structure, we use the notation $(\mathcal{I}, \mathcal{O})$ unless otherwise stated.

Consider the following lemma.

Lemma 2.1.50. *Let $\mathcal{B} \subset A^q$ be a behavior given as*

$$S = \{w \in A^q : Rw = 0\} \qquad\qquad R \in D^{p\times q}.$$

If we consider R as a matrix over the field K, then the rank $r = \operatorname{rank}(R)$ does not depend on any choice of R but only on $\mathcal{B}$.

Proof. Let $\mathcal{B} = \ker_A(R')$ where $R \neq R'$. By Corollary 2.1.49, there exists matrices X and X' of appropriate size with coefficients in $D \subset K$ satisfying $R' = XR$ and $R = X'R'$. As a consequence

$$\operatorname{rank}(R') = \operatorname{rank}(XR) \leq \operatorname{rank}(R) = \operatorname{rank}(X'R') \leq \operatorname{rank}(R').$$

We conclude $\operatorname{rank}(R) = \operatorname{rank}(R')$. $\square$

A consequence of the above lemma is that the following definition is well-defined.

Definition 2.1.51. For a behavior $\mathcal{B} = \ker_A(R) \subset A^q$, we define the **input dimension**, denoted by $n_\mathcal{I}$, of $\mathcal{B}$ as $n_\mathcal{I} = q - \operatorname{rank}(R)$. Similarly, we define the **output dimension**, denoted $n_\mathcal{O}$, as $n_\mathcal{O} = \operatorname{rank}(R)$.

Using the above definitions, we define a special type of i/o structure.

Definition 2.1.52. For a behavior $\mathcal{B}$ with input dimension $n_\mathcal{I}$ and output dimension $n_\mathcal{O}$ we say that an i/o-structure $(\mathcal{I}, \mathcal{O})$ is a **full i/o-structure** if $|\mathcal{I}| = n_\mathcal{I}$ and $|\mathcal{O}| = n_\mathcal{O}$.

It is important to note that the above *numbers* only reflect how large the input and output are – it does not claim that there is a unique choice of input or output. Consider the following example.

Example 2.1.53. Let the two-dimensional behavior $\mathcal{B} \subset \mathcal{A}^2$ be given by the kernel representation

$$R = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

This kernel representation states that the two components of $\mathcal{B}$ are *equal* to each other, i.e. $\mathcal{B} \cong \mathcal{A}$. As a result, we can choose either component to be input or output but continue to have a full i/o-structure in either case.

Theorem 2.1.54. *[39, pg. 38-40] Let $\mathcal{B} = \ker_A(R)$, $R \in D^{p\times q}$, be a behavior and $(\mathcal{I}, \mathcal{O})$ be an i/o-structure with associated decomposition $w = (y, u)$ and $R = [-Q \;\; P]$. The following are equivalent:*

1. $\operatorname{rank}(R) = \operatorname{rank}(P) = p$ *and the columns of P are K-linearly independent, i.e. $(\mathcal{I}, \mathcal{O})$ is a full i/o-structure.*
2. *There exists a* ***transfer matrix****, $H \in K^{|\mathcal{O}|\times|\mathcal{I}|}$ such that $PH = Q$.*
3. *For every $u \in A^\mathcal{I}$, there exist a $y \in A^\mathcal{O}$ such that $(u, y) \in \mathcal{B}$, i.e. projection onto the input components of $\mathcal{B}$ is a surjective map.*

The transfer matrix provided by the above theorem is of great importance to us. To study its role in behavioral systems, we now move our discussion to what Oberst refers to as "signal flow systems."

Let $\mathcal{B} = \ker_A(R) \subset A^q$ be a behavior with $R \in D^{p\times q}$. Since A is an injective cogenerator, one can study the dual module $\operatorname{coker}_D(R^T)$ rather than $\mathcal{B}$ to establish analogous associations. For the inclusion map $\iota : \mathcal{B} \hookrightarrow A^q$ define the exact sequence

$$0 \longrightarrow \mathcal{B} \xrightarrow{\iota} A^q \xrightarrow{R} A^p.$$

After applying $\operatorname{Hom}_D(-, A)$ we reach the exact sequence

$$D^p \xrightarrow{R^T} D^q \xrightarrow{\operatorname{Hom}_D(\iota,A)} \mathcal{M} := \operatorname{coker}_D(R^T) \longrightarrow 0.$$

Since fields are flat and $D \subset K$, $- \otimes_D K$ is an exact functor which we may apply to reach the exact sequence

$$K^p \xrightarrow{R^T\otimes 1_K} K^q \xrightarrow{\operatorname{Hom}_D(\iota,A)\otimes 1_K} \mathcal{M} \otimes_D K \longrightarrow 0.$$

Due to the fact that vector spaces are injective, we may apply the exact functor $\operatorname{Hom}_K(-, K)$ to reach the exact sequence of dual vector spaces

$$0 \longrightarrow (\mathcal{M} \otimes_D K)^* \longrightarrow K^q \xrightarrow{R\otimes 1_K} K^p.$$

Because $\mathcal{M}$ is a finitely generated D-module, $\mathcal{M} \otimes_D K$ is a finite dimensional K-vector space; as a consequence, we have the K-vector space isomorphism $(\mathcal{M} \otimes_D K)^* \cong \mathcal{M} \otimes_D K$. By following the above steps, we reach the exact functor

$$\widehat{(-)} : \mathbf{Syst}(A) \to \mathbf{Modf}(K) \qquad \mathcal{B} \mapsto \widehat{\mathcal{B}} = (\operatorname{coker}_D(R^T) \otimes_D K)^* = \ker_K(R).$$

This leads us to the following definition.

Definition 2.1.55. For a behavior $\mathcal{B} \subset A^q$ we define its associated **signal flow system** as $\widehat{\mathcal{B}}$ where

$$\widehat{\mathcal{B}} = \{\widehat{w} \in K^q : R\widehat{w} = 0\}.$$

Through our above discussion, we may make the following observations about the functor $\widehat{(-)}$.

Theorem 2.1.56. *The following are true.*

1. *The functor $\widehat{(-)}$ is exact.*
2. *For a behavior $\mathcal{B} \subset A^q$, $\widehat{\mathcal{B}}$ is well defined and only depends on $\mathcal{B}$ and not the choice of kernel representation R.*
3. *If $\mathcal{B} \subset A^q$ and $\mathcal{B}' \subset A^{q'}$ are two behaviors and $P : A^q \to A^{q'}$ is a system morphism so that $P\mathcal{B} \subset \mathcal{B}'$, then $P\widehat{\mathcal{B}} \subset \widehat{\mathcal{B}'}$*

4. *If the input dimension of $\mathcal{B}$ is $n_{\mathcal{I}}$, then* $\dim_K(\widehat{\mathcal{B}}) = n_{\mathcal{I}} = q - \mathrm{rank}(R)$ *and this does not depend on the choice of kernel representation R.*

Proof. (1). This follows from A being a large injective cogenerator and K being flat.

(2). This follows from $\mathrm{coker}_D(R^T)$ being invariant under choice of kernel representation R.

(3). This follows from $\widehat{(-)}$ being an exact covariant functor.

(4). This follows from Lemma 2.1.50 and the definition of rank. □

Consider a behavior $\mathcal{B} = \ker_A(R)$, $R \in D^{p\times q}$, with full i/o-structure $(\mathcal{I}, \mathcal{O})$ which induces the decomposition of the kernel representation $R = [P \quad -Q]$. We thus have

$$\mathcal{B} = \{(u, y) \in A^{\mathcal{I}} \times A^{\mathcal{O}} : Py = Qu\}$$

and the transfer matrix H such that $PH = Q$. After applying $\widehat{(-)}$ we reach

$$\mathcal{B} = \{(\widehat{u}, \widehat{y}) \in K^{\mathcal{I}} \times K^{\mathcal{O}} : P\widehat{y} = Q\widehat{u}\}.$$

Since the i/o-structure is full, we have $\mathrm{rank}(P) = \mathrm{rank}(R) = n_{\mathcal{O}}$ so that $P : K^{n_{\mathcal{O}}} \to K^p$ is injective. If we look at the transfer matrix we see

$$P\widehat{y} = Q\widehat{u} = PH\widehat{u}.$$

However, since P is injective, $\widehat{y} = H\widehat{u}$. From this observation we reach

$$\widehat{\mathcal{B}} = \{(\widehat{u}, \widehat{y}) \in K^{\mathcal{I}} \times K^{\mathcal{O}} : \widehat{y} = H\widehat{u}\} = \mathrm{Graph}(H : K^{\mathcal{I}} \to K^{\mathcal{O}}).$$

Let us instead not choose an i/o-structure but rather consider $\widehat{\mathcal{B}}$,

$$\widehat{\mathcal{B}} = \{\widehat{w} \in K^q : R\widehat{w} = 0\}.$$

Since R is a linear transformation of a vector space, we can always write

$$\widehat{\mathcal{B}} = \{\widehat{w} = (\widehat{u}, \widehat{y}) \in K^{\mathcal{I}} \times K^{\mathcal{O}} : P\widehat{y} = Q\widehat{u}\} = \mathrm{Graph}(\widehat{H})$$

for some choice of $(\mathcal{I}, \mathcal{O})$ and $\widehat{H} \in K^{|\mathcal{O}|\times|\mathcal{I}|}$ by using standard linear algebra. In particular, $P\widehat{H}\widehat{u} = Q\widehat{u}$ for all $\widehat{u} \in K^{\mathcal{I}}$. It is also easy to see that $\mathrm{rank}(R) = \dim(\widehat{\mathcal{B}})$. As a consequence, $(\mathcal{I}, \mathcal{O})$ is a full i/o-structure of $\mathcal{B}$ and $\widehat{H}$ is its transfer matrix. This leads us to the following observation.

Theorem 2.1.57. *For a behavior $\mathcal{B}$, a chosen i/o-structure is full if and only if the signal flow system $\widehat{\mathcal{B}}$ is given as the graph of a transfer function H with appropriate dimensions. In particular, the map*

$$\begin{bmatrix} I_{|\mathcal{I}|} \\ H \end{bmatrix} : K^{|\mathcal{I}|} \to \widehat{\mathcal{B}}$$

is a K-isomorphism.

Unfortunately, the signal flow system is useless in terms of a characterization of the trajectories. When we apply $-\otimes_D K$, it may be the case that $\ker(\mathcal{M} \to \mathcal{M} \otimes_D K)$ is non-trivial; as a result, the process is not "reversible." We now discuss situations when this not the case, i.e. when it is possible to directly associate the signal flow system to the behavior.

Definition 2.1.58. For two behaviors $\mathcal{B}$ and $\mathcal{B}'$, we define the equivalence relation $\mathcal{B} \sim \mathcal{B}'$ given by $\widehat{\mathcal{B}} = \widehat{\mathcal{B}}'$. In such a case we say the two behaviors are **transfer equivalent**. For a behavior $\mathcal{B}$ we denote by $[\mathcal{B}]$ the equivalence class of $\mathcal{B}$ under $\sim$ and call it the **transfer class** of $\mathcal{B}$.

Theorem 2.1.59. *[39, pg. 141-142] For behaviors $\mathcal{B} = \ker_A(R) \subset A^q$ and $\mathcal{B}' = \ker_A(R') \subset A^q$ the following are equivalent*

1. $\widehat{\mathcal{B}} = \widehat{\mathcal{B}}'$
2. $\mathrm{im}_{\mathcal{D}}(R^T) \otimes_{\mathcal{D}} K = \mathrm{im}_{\mathcal{D}}({R'}^T) \otimes_{\mathcal{D}} K$
3. $\mathrm{coker}_{\mathcal{D}}(R^T) \otimes_{\mathcal{D}} K = \mathrm{coker}_{\mathcal{D}}({R'}^T) \otimes_{\mathcal{D}} K$
4. *There are matrices X and X' of appropriate size with entries in K such that $R' = XR$ and $R = X'R'$.*

*For a subspace $V \subset K^q$, any behavior $\mathcal{B}$ for which $\widehat{\mathcal{B}} = V$ is called a **transfer realization** of V.*

Remark. It is instructive to compare (4) in Theorem 2.1.59 to Theorem 2.1.48.

We now turn our attention to recovering the behavior from the signal flow system. It turns out that there are a few particular properties which one can use to make this association. One of these properties requires the following definition.

Definition 2.1.60. We say that a matrix $R \in D^{p\times q}$ is **generalized factor left prime** (GFLP) if the following holds. If there are matrices $R' \in D^{p'\times q}$ and $X \in D^{p\times p'}$ such that $R = XR'$ where $\mathrm{rank}(R) = \mathrm{rank}(R')$, then there exists a polynomial matrix Y such that $R' = YR$.

For a subset $V \subset K^q$, we define the polar complement V^π as

$$V^\pi = \{x \in K^q : x \cdot y = 0 \text{ for all } y \in V\}.$$

Let $V \subset K^q$ be a K-subspace and let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a transfer realization of V. The submodule $\mathcal{D}^q \cap V^\pi$ is the unique largest submodule of $\mathcal{D}^q$ satisfying $V^\pi = (\mathcal{D}^q \cap V^\pi) \otimes K$. In particular, $\mathcal{B}_{min} = (\mathcal{D}^q \cap V^\pi)^\perp$ is the unique minimal transfer realization of V, i.e. $\mathcal{B}_{min}$ is the smallest, under set inclusion, element in its transfer class $[\mathcal{B}]$. The following theorem provides equivalent properties of $\mathcal{B}_{min}$.

Theorem 2.1.61. *[39, pg. 142-144] Let $\mathcal{B} = \ker_A(R) \subset A^q$ be a behavior with kernel representation $R \in D^{p\times q}$. Then the following are equivalent.*

1. *$\mathcal{B}$ is minimal, under set inclusion, in its transfer class.*

2. $\operatorname{coker}_D(R^T)$ *is torsion-free.*

3. R *is GFLP*

Remark. Signal flow systems and the properties listed in Theorem 2.1.61 are crucial in the discussion of controllable behaviors. We also emphasize that Theorem 2.1.61 requires only that D be an affine domain.

2.1.7 The Canonical Cauchy Problem

Notation. Define k be a field equipped with the discrete topology, $d \in \mathbb{N}$ to be a positive integer, $\mathcal{D} = k[z_1, \dots, z_d]$, and $\mathcal{A} = k^{\mathbb{N}^d}$. We also equip $\mathcal{A}$ with the weak-$*$ topology induced by $\mathcal{D}$.

This section discusses how one can directly compute the trajectories of a behavior from initial condition data. In particular, we show that there exists a unique solution to the canonical Cauchy problem. The abstract Cauchy problem is the starting point for this matter since it assumes that we already know the "initial conditions" of the problem. The canonical Cauchy problem involves the problem of identifying the initial conditions for a difference equation so as to prepare the abstract Cauchy problem.

For a behavior $\mathcal{B} = \ker_{\mathcal{A}}(R)$ where $R \in \mathcal{D}^{p \times q}$, we have that any $w \in \mathcal{B}$ must satisfy $Rw(t) = 0$ for all $t \in \mathbb{N}^d$. Since the signals in $\mathcal{A}$ form a $\mathcal{D}$-module under the backward shift, we may identify

$$w \in \mathcal{A} \mapsto \{(\mathbf{z}^i w)(0)\}_{i \in \mathbb{N}^d} = \{w(i)\}_{i \in \mathbb{N}^d} \in k^{\mathbb{N}^d} := \prod_{\mathbb{N}^d} k.$$

Then for any $p = a_1 \mathbf{z}^{i_1} + \cdots + a_r \mathbf{z}^{i_r}$ we have

$$pw(t) = a_1 w(t + i_1) + \cdots + a_r w(t + i_r) \qquad \text{for all } t \in \mathbb{N}^d.$$

If we were to ask for signals which lie in the kernel of p for every t, then alternatively we pose this problem via the following infinite dimensional Toeplitz matrix

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & a_1 & 0 & \cdots & 0 & a_2 & 0 & \cdots & 0 & \cdots & 0 & a_r & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & a_1 & 0 & \cdots & 0 & a_2 & 0 & \cdots & 0 & \cdots & 0 & a_r & 0 & \cdots \\ \vdots & & & & & \ddots & & & & \ddots & & & & & & \ddots & & \end{bmatrix} \begin{bmatrix} w(1) \\ w(2) \\ \vdots \end{bmatrix}$$

In this way, instead of finding the solutions of the system at each time $t \in \mathbb{N}^d$, we may consider the problem as an infinite dimensional linear system. For the case when we have a polynomial matrix R, the above reasoning can be extended to accommodate for multiple equations and vector-valued signals.

The abstract Cauchy problem, essentially, asks for when an infinite dimensional linear system of the form $Ax = b$, where x and b are infinite dimensional vectors, admits a unique solution. In the finite dimensional setting, if we have a non-zero matrix $A \in \mathbb{C}^{p \times q}$ and a vector $b \in \mathbb{C}^p$, then we have that there exists a solution if and only if $b \in \operatorname{im}_{\mathbb{C}}(A)$ and the solution is unique if and only if $\ker_{\mathbb{C}}(A) = \{0\}$. This is where we see the two places to look for existence and uniqueness of solutions: the kernel and the image.

If we allow arbitrary b values, then it must necessarily be true that $\mathrm{im}_{\mathbb{C}}(A) = \mathbb{C}^p$. A surjective linear map may have a kernel and, in this case, the solution is not unique. If, however, we assume $\mathbb{C}^q$ has a decomposition $\mathbb{C}^q = N \oplus \ker_{\mathbb{C}}(A)$ with associated projection operator $P_{\ker_{\mathbb{C}}(A)}$ onto $\ker_{\mathbb{C}}(A)$ along N, and we pose the problem as: find $x \in \mathbb{C}^q$ such that for a specified value x_0 in $\ker_{\mathbb{C}}(A)$ and $b \in \mathbb{C}^p$ we have $P_{\ker_{\mathbb{C}}(A)}x = x_0$ and $Ax = b$, then the solution is unique. In particular, we have that the problem induces the direct sum decomposition $\mathbb{C}^q = \mathbb{C}^q \oplus \ker_{\mathbb{C}}(A)$ where the initial condition lies in $\ker_{\mathbb{C}}(A)$ and the rest of the space comes from the image of A. If b is arbitrary and A is not surjective, then the problem is only defined on the image of A; hence, we replace $\mathbb{C}^p$ by $\mathrm{im}_{\mathbb{C}}(A)$.

In this way, we can take a problem which does not exhibit a unique solution and consider it as one which does by asking that the kernel component be specified. When one solves linear systems where initial conditions are specified, this is the same as specifying the value the solution takes on inside the kernel. The key observation here is that there is a direct sum decomposition $\mathbb{C}^q = \mathbb{C}^p \oplus \ker_{\mathbb{C}}(A)$. Since A is surjective, there exists a matrix $B \in \mathbb{C}^{q\times p}$ such that $AB = I_{p\times p}$. For any $y \in \mathbb{C}^q$ we have

$$y \mapsto (Ay, (I - BA)y) \in \mathbb{C}^p \oplus \ker_{\mathbb{C}}(A).$$

The part mapping into $\mathbb{C}^p$ corresponds to the image and the $(I - BA)y$ part corresponds to the kernel of A; this latter point follows since

$$A(I - BA)y = (A - ABA)y = (A - A)y = 0.$$

For any solution $A(y + y') = b$, where $y' \in \ker_{\mathbb{C}}(A)$ we have

$$\langle Ay, b\rangle = \langle A(y + y'), b\rangle = \langle y + y', A^T b\rangle = \langle Ay, b\rangle + \langle y', A^T b\rangle.$$

In particular,

$$\langle y', A^T b\rangle = 0.$$

So we must have that y' is orthogonal to the image of the adjoint problem $y = A^T b$. Similarly, the kernel of the adjoint problem is orthogonal to the image of A; take $b' \in \ker_{\mathbb{C}}(A^T)$ and $y = A^T b$ to see

$$\langle y, A^T(b + b')\rangle = \langle Ay, b + b'\rangle = \langle y, A^T b\rangle + \langle Ay, b'\rangle \quad \Rightarrow \quad \langle Ay, b'\rangle = 0.$$

The important observation is that, instead of specifying the solution on the kernel of A, we can ask that the solution be orthogonal to the image of the adjoint system. These observations continue to appear in the following infinite dimensional setting.

Problems of this sort (i.e., solving linear systems) have been extended to the case of infinite dimensions in several settings. A well known situation is when we have a vector space which is a Banach space and a linear operator which is compact. (Recall that a linear operator $T : X \to Y$, where X and Y are normed spaces, is compact if for any bounded subset of X the image is relatively compact in Y.) There are several ingredients that go into this problem. Let $T : X \to X$ be a compact operator on the Banach space X. The following holds.

1. The adjoint operator $T^* : \mathrm{Hom}_{\mathbb{C}}(Y, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C})$ is also compact.

2. For every $\lambda \in \mathbb{C} \setminus \{0\}$, $\ker_X(T - \lambda I)$ is finite dimensional.
3. For every $\lambda \in \mathbb{C} \setminus \{0\}$ the operator $T_\lambda := T - \lambda I$ has a closed image.
4. There exists a unique smallest integer n such that $\ker_X(T_\lambda^n) = \ker_X(T_\lambda^{n+m})$ for all $m \in \mathbb{N}$. We have a direct sum decomposition $X = \ker_X(T_\lambda^n) \oplus \mathrm{im}_X(T_\lambda^n)$. Note that, since T is bounded, $\ker_X(T_\lambda^n)$ is closed and by (4) we have $\mathrm{im}_X(T_\lambda^n)$ is closed.

Each piece is important and emulates properties demonstrated by the finite dimensional setting. (1) states (as we shall see) that the adjoint problem can be posed as well. (3) demonstrates that the image is once again a Banach space and by (2) that the kernel is a negligible (but useful!) subspace. (4) establishes a direct sum decomposition which is comparable to the one found in the finite dimensional setting.

We are concerned with the following problems.

1. (Non-homogeneous Problem) Given $y \in X$ and $\lambda \neq 0$ find a solution $x \in X$ which satisfies the equation
$$Tx - \lambda x = y.$$
2. (Homogeneous Problem) Given $\lambda \neq 0$ find a solution $x \in X$ which satisfies the equation
$$Tx - \lambda x = 0.$$

These two problems have analogues with the adjoint operator.

1. (Non-homogeneous Adjoint Problem) Given $g \in \mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C})$ and $\lambda \neq 0$ find a solution $f \in \mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C})$ which satisfies the equation
$$T^* f - \lambda f = g.$$
2. (Homogeneous Adjoint Problem) Given $\lambda \neq 0$ find a solution $f \in \mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C})$ which satisfies the equation
$$T^* f - \lambda f = 0.$$

The so-called Fredholm alternative is a sort of dichotomy of the possible outcomes.

Definition 2.1.62. A bounded linear operator $T : X \to X$ is said to satisfy the **Fredholm alternative** if T is such that either of the following holds.

1. The non-homogeneous equations
$$Tx = y \qquad\qquad T^* f = g$$
admit unique solutions $x \in X$ and $f \in \mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C})$ for any given $y \in X$ and $f \in \mathrm{Hom}_{\mathbb{C}}(X, \mathbb{C})$. The corresponding homogeneous equations
$$Tx = 0 \qquad\qquad T^* f = 0$$
have only the trivial solutions $x = 0$ and $f = 0$.

2. The homogeneous equations

$$Ax = 0 \qquad\qquad A^* f = 0$$

have the same number of linearly independent solutions and the non-homogeneous problem is not solvable for arbitrary values of y and g; however, they admit solutions y and g if and only if

$$f(y) = 0 \quad \text{and} \quad g(x) = 0 \qquad\qquad \text{for all } f \in \ker(T^*),\ x \in \ker(T).$$

If T is a compact operator, then for $\lambda \in \mathbb{C} \setminus \{0\}$ we have $T_\lambda := T - \lambda I$ satisfies the Fredholm alternative. In this case, we have that either λ is not an eigenvalue of T and thus T_λ is invertible or λ is an eigenvalue of T and thus T_λ has a kernel; in this latter case we observe that the solutions, when they exist, must be orthogonal (in terms of the dual space) to the solutions of the homogeneous adjoint problem. That is, for $X^* := \operatorname{Hom}_{\mathbb{C}}(X, \mathbb{C})$, we have the identities

$$\operatorname{im}_X(T_\lambda) = \ker_{X^*}(T_\lambda^*)^\perp \qquad\qquad \operatorname{im}_{X^*}(T_\lambda^*) = \ker_X(T_\lambda)^\perp.$$

These observations appear in different guises when we discuss the Cauchy problem. We now turn to the abstract Cauchy problem and its equivalent forms.

In the proof of the abstract Cauchy problem we employ a very standard result used in diagram chasing.

Lemma 2.1.63 (Splitting Lemma)**.** *Let $f : M \to N$ be a surjection between two modules and $g : N \to M$ a module homomorphism such that $f \circ g = 1_N$. Then M decomposes as $M = \ker(f) \oplus \operatorname{im}(g)$ via the map*

$$a \mapsto (a - (g \circ f)(a)) + (g \circ f)(a) = \pi(a) + (1_M - \pi)(a)$$

where $\pi = 1_M - (g \circ f)$ and $(1_M - \pi)$ are, respectively, the projections onto $\ker(f)$ and $\operatorname{im}(g)$. In particular, $\ker(f) = \operatorname{im}(\pi)$ and $\operatorname{im}(g) = \ker(\pi)$.

Theorem 2.1.64 (Abstract Cauchy Problem)**.** *Let $\mathcal{B} = \ker_{\mathcal{A}} \begin{bmatrix} -Q & P \end{bmatrix}$ be a behavior with a given* full *i/o-structure, $P \in \mathcal{D}^{p \times q}$, $Q \in \mathcal{D}^{p \times g}$ and $[q] \times \mathbb{N}^d = G \cup G^c$ be a disjoint decomposition. Then the following are equivalent:*

1. *The* ***inhomogeneous Cauchy problem***

$$Py = Qu \quad \textit{where} \quad y|_G = x \qquad\qquad u \in \mathcal{A}^g,\ x \in k^G$$

has a unique solution $y \in \mathcal{A}^q$ for any given input $u \in \mathcal{A}^g$ and initial condition $x \in k^G$.

2. *The* ***homogeneous Cauchy problem***

$$Py = 0 \qquad\qquad y|_G = x,\ x \in k^G$$

has a unique solution $y \in \mathcal{A}^q$ for any given initial condition $x \in k^G$.

3. *There exists a direct sum decomposition*

$$\mathcal{A}^q = \ker_{\mathcal{A}}(P) \oplus k^{G^c}.$$

Proof. (1) $\Rightarrow$ (2). It is clear that (2) is a special case of (1).

(2) $\Rightarrow$ (1). Let $x \in k^G$ and $u \in \mathcal{A}^g$ be given. Since the chosen i/o-structure is full, by Theorem 2.1.54 there exists $\widehat{y} \in \mathcal{A}^q$ such that $P\widehat{y} = Qu$ for the given u. By hypothesis, there exists a unique $\widetilde{y} \in \mathcal{A}^q$ satisfying $P\widetilde{y} = 0$ and $\widetilde{y}|_G = x - \widehat{y}|_G$. Define $y = \widetilde{y} + \widehat{y}$. We readily observe

$$Py = P\widetilde{y} + P\widehat{y} = 0 + Qu = Qu.$$

Furthermore,

$$y|_G = (x - \widehat{y}|_G) + \widehat{y}|_G = x,$$

so y has the desired initial conditions. We now argue that y is unique. For any other solution y' we have

$$P(y - y') = Qu - Qu = 0 \qquad\qquad (y - y')|_G = x - x = 0.$$

However, by uniqueness of the homogeneous problem this means $y = y'$ and hence y is a unique solution.

$(1), (2) \Rightarrow (3)$. First we show that $\ker_{\mathcal{A}}(P) \cap k^{G^c} = 0$. Choose $y \in \ker_{\mathcal{A}}(P) \cap k^{G^c}$. In this case we must have $Py = 0$. The map $\pi : \mathcal{A}^q \to k^G$ given by $y \mapsto y|_G$ has kernel k^{G^c}. As a result, $\pi(y) = y|_G = 0$. Since the only solution to the homogeneous problem with zero initial conditions is zero, (2) implies that $y = 0$.

Let $w \in \mathcal{A}^q$ be given. Define $x = w|_G$ and let y be a solution to the homogeneous problem so that $y|_G = x = w|_G$ and $Py = 0$. This provides the decomposition $w \mapsto y + (w|_{G^c} - y|_{G^c})$. In this case we have

$$(y + (w|_{G^c} - y|_{G^c}))|_G = y|_G + (w|_{G^c})|_G - (y|_{G^c})|_G = y|_G = x = w|_G$$

and

$$(y + (w|_{G^c} - y|_{G^c}))|_{G^c} = y|_{G^c} + w|_{G^c} - y|_{G^c} = w|_{G^c}.$$

We thus have that the map $\mathcal{A}^q \to \ker(P) \oplus k^{G^c}$ given by $w \mapsto y + (w|_{G^c} - y|_{G^c})$ is a bijection.

(3) $\Rightarrow$ (1). By the first isomorphism theorem, we have that P induces an isomorphism

$$P : \mathcal{A}^q / \ker_{\mathcal{A}}(P) \cong \operatorname{im}_{\mathcal{A}}(P).$$

The decomposition $\mathcal{A}^q \to \ker(P) \oplus k^{G^c}$, however, implies that we have the isomorphism $P|_{G^c} : k^{G^c} \cong \operatorname{im}_{\mathcal{A}}(P)$. Define $H : \operatorname{im}_{\mathcal{A}}(P) \to k^{G^c}$ as the inverse so that we have $P \circ H = 1_{\operatorname{im}_{\mathcal{A}}(P)}$. From the splitting lemma we reach

$$\mathcal{A}^q = \ker_{\mathcal{A}}(P) \oplus \operatorname{im}_{\mathcal{A}}(H) = \ker_{\mathcal{A}}(P) \oplus k^{G^c} \qquad\qquad y \mapsto \pi(y) + (1_{\mathcal{A}^q} - \pi)(y),$$

where $\pi = 1_{\mathcal{A}^q} - H \circ P$ and $1_{\mathcal{A}^q} - \pi = H \circ P$ are the projections onto, respectively, $\ker_{\mathcal{A}}(P)$ and $\operatorname{im}_{\mathcal{A}}(H) = k^{G^c}$.

Due to the assumption that the i/o-structure is full, we may appeal to Theorem 2.1.54 to see that for every $u \in \mathcal{A}^g$ there is a solution $(u, y) \in \mathcal{A}^{g+q}$ such that $Py = Qu$. In particular, $\operatorname{im}_{\mathcal{A}}(Q) \subset \operatorname{im}_{\mathcal{A}}(P)$. Since H is defined on $\operatorname{im}_{\mathcal{A}}(P)$, H is defined on $\operatorname{im}_{\mathcal{A}}(Q)$ and hence we have the well-defined map

$$\widetilde{H} : H \circ Q : \mathcal{A}^g \to \mathcal{A}^q \qquad\qquad u \mapsto \widetilde{H}(u) = (H \circ Q)(u).$$

Let $u \in \mathcal{A}^g$ and $x \in k^G$ be given. Define $y = \pi(x) + \widetilde{H}(u)$. Since π projects onto $\ker_{\mathcal{A}}(P)$, $P \circ H = 1_{\operatorname{im}_{\mathcal{A}}(P)}$, and $\operatorname{im}_{\mathcal{A}}(Q) \subset \operatorname{im}_{\mathcal{A}}(P)$,

$$Py = P\pi(x) + P\widetilde{H}u = 0 + P\widetilde{H}u = PHQu = Qu.$$

We also have

$$y|_G = \pi(x)|_G + \widetilde{H}(u)|_G \quad \Rightarrow \quad y|_G - \pi(x)|_G = \widetilde{H}(y)|_G$$

Because $1_{\mathcal{A}^q} - \pi = H \circ P$ projects onto k^{G^c} we have

$$y|_G - \pi(x)|_G = (H \circ P)(u)|_G \in k^{G^c}.$$

However, the restriction $(-)|_G$ has kernel equal to k^{G^c} we reach $y|_G - \pi(x)|_G = 0$ and hence $(y - \pi(x))|_G = 0$. Using that $\ker_{\mathcal{A}}(\pi) = k^{G^c}$, may reach $\pi(x|_G) = \pi(x)$

$$andthusy|_G = \pi(x)|_G = \pi(x|_G) = x|_G - (HPx)|_G = x|_G + 0 = x|_G = x.$$

This demonstrates that a solution exists. For uniqueness, let us assume that there exists another solution y'. Then $y|_G = y'|_G = x$ and

$$y' = \pi(y') + H \circ P(u) = \pi(y') + H \circ Q(u) = \pi(y') + \widetilde{H}(u) = \pi(x) + \widetilde{H}(u) = y.$$

We conclude that the solution is unique. □

At this point, we refer the reader to Section A.1 for a review of Gröbner bases. We diverge a bit from the original exposition of Oberst and provide a more "modern" treatment which leaves some of the machinery under the hood, so to speak. First, we pick an ordering.

Definition 2.1.65 (Lexicographic Order)**.** We define the **lexicographic ordering** on $\mathbb{N}^d$ as

$$m \leq_l n \iff \begin{cases} m(i) = n(i) & i < \ell < d \\ m(i) < n(i) & i = \ell \leq d \end{cases}$$

The lexicographic ordering of $\mathbb{N}^d$ is **strict** or **total** in the sense that for any two $m, n \in \mathbb{N}^d$, we have $m \leq_l n$ or $n \leq_l m$. It is also easy to see that $m \leq_l n$ implies that $m + t \leq_l n + t$ for all $t \in \mathbb{N}^d$ and that $0 \leq_l n$ for all $n \in \mathbb{N}^d$. Furthermore, for any set $G \subset \mathbb{N}^d$, it is clear that G admits a smallest element under $\leq_l$.

Recall that we may identify the multivariable polynomial ring

$$\mathcal{D} = k[z_1, \ldots, z_d] \leftrightarrow k^{(\mathbb{N}^d)} \qquad\qquad p = \sum \alpha_i \mathbf{z}^i \mapsto \{\alpha_i\}_{i\in\mathbb{N}^d} \text{ for } p \in k[\mathbf{z}].$$

Also recall that the notation $(\mathbb{N}^d)$ is used to imply that the components are zero for all but at most a finite number of elements. We now extend the above details to the case of polynomial vectors. Define $[q] = \{1, \ldots, q\}$ so that for a vector $p = (p_1, \ldots, p_q) \in k[\mathbf{z}]^q$ we may identify

$$k[\mathbf{z}]^q = \left(k^{(\mathbb{N}^d)}\right)^q = k^{[q]\times(\mathbb{N}^d)}$$

by

$$p = (p_1, \ldots, p_q) \mapsto \{(p_1(i), \ldots, p_q(i))\}_{i\in(\mathbb{N}^d)} = \{p_j(i)\}_{(j,i)\in[p]\times(\mathbb{N}^d)},$$

where we write $p_i = \sum_j p_i(j)\mathbf{z}^j$ so that $p_i(j)$ is a coefficient. Through the lexicographic ordering on $\mathbb{N}^d$, we may order $[p] \times \mathbb{N}^d$ by defining

$$(i, m) <_m (j, n) \iff \begin{cases} i < j \\ i = j \text{ and } m <_l n. \end{cases}$$

The above ordering is a total ordering on $[p] \times \mathbb{N}^d$ and thus its induced leading monomials form a k-vector space basis. Due to the definition of $<_m$, for any $p \in \mathcal{D}^q$ we have the leading exponents

$$\mathrm{LE}(p) = \max\{(i, m) \in [p] \times \mathbb{N}^d : p_i(m) \neq 0\} \in [p] \times \mathbb{N}^d.$$

The following lemma is a natural consequence of this observation.

Lemma 2.1.66. *A non-zero vector $p \in k[\mathbf{z}]^q$ has $\mathrm{LE}(p) = (i, \ell) \in [p] \times \mathbb{N}^d$ with respect to the ordering $<_m$ if and only if p has the form*

$$p = (p_1, \ldots, p_i, 0, \ldots, 0) \text{ where } p_i = p_i(\ell)\mathbf{z}^\ell + \sum_{m<\ell} p_i(m)\mathbf{z}^m \text{ and } p_i(\ell) \neq 0.$$

In particular, $\mathrm{LE}(p) \in \{i\} \times \mathbb{N}^d$ *if and only* $p_i \neq 0$ *but* $p_{i+1} = \cdots = p_q = 0$.

Under the ordering $<_m$ we have a monomial ordering (which is also a well-ordering) for free modules and thus are now able to discuss Gröbner bases for free modules. Recall that for a submodule $U \subset \mathcal{D}^q$ we define the **leading module** as

$$L(U) = \langle \mathrm{LM}(f) | f \in U \setminus \{0\} \rangle,$$

where $\mathrm{LM}(f)$ is the **leading monomial** of f (as defined in Section A.1.3.) For each index $i \in \{1, \ldots, q\}$ we may also consider the exponents of leading ideals for each basis element

$$E(i) = \{\alpha \in \mathbb{N}^d : \mathbf{z}^\alpha e_i \in L(U)\} \subset \mathbb{N}^d.$$

Following Hironaka in [27, pg. 244], we refer to each $E(i)$ as an E**-subset**.

For a behavior $\mathcal{B} = \ker_{\mathcal{A}}(R)$ where $R \in \mathcal{D}^{p\times q}$, the following lemma relates properties of R, $M := \mathrm{coker}_{\mathcal{D}}(R^T)$, $\mathcal{B}^\perp = \mathrm{im}_{\mathcal{D}}(R^T)$ and the sets $E(i)$ for $i = 1, \ldots, q$.

Lemma 2.1.67. *For $R \in \mathcal{D}^{g\times q}$ define the $\mathcal{D}$-submodule $U = \mathrm{im}_{\mathcal{D}}(R^T) \subset \mathcal{D}^q$. The following are equivalent*

1. $\mathrm{rank}(R) = q$
2. $M = \mathrm{coker}_{\mathcal{D}}(R^T)$ *is a torsion module.*
3. *Each* $E(i) = \{m \in \mathbb{N}^d : (i, m) \in \deg(U)\}$ *is non-empty for* $i = 1, \dots, q$.

Proof. (1) $\Rightarrow$ (2). If $\mathrm{rank}(R) = q$, then R has full column rank and so it must be the case that $g \geq q$. One may then observe that the 0^{th} Fitting ideal[4] of M is non-zero; as a consequence, we have $\mathrm{Ann}(M) \neq 0$ and hence M is a torsion module.

(2) $\Rightarrow$ (1). Similarly, if M is a torsion module, $\mathrm{Ann}(M)$ is non-zero. Since the 0^{th} Fitting ideal is a power of the annihilator, $\sqrt{\mathrm{Ann}(M)} = \sqrt{\mathfrak{F}_0(M)} \neq 0$. This, however, means that R has full column rank since the q^{th} determinantal ideal is non-zero.

(2) $\Rightarrow$ (3). Assume that M is a torsion module; as a consequence, there exists a non-constant $p \in \mathcal{D}$ such that $(p, \dots, p) \in \mathrm{im}_{\mathcal{D}}(R^T) = U$. As a result, each $E(i)$ is non-empty.

(3) $\Rightarrow$ (1). Assume that each $E(i)$ is non-empty. We may construct a $q \times q$ matrix Q such that the i^{th} column is comprised of an element of the form $(p_{i,1}, \dots, p_{i,i}, 0, \dots, 0)$ with $0 \neq \mathrm{LE}(Q_{i,i}) \in E(i)$. By construction, Q is upper triangular with non-trivial diagonal so $\mathrm{rank}(Q) = q$. Furthermore, $\mathrm{im}_{\mathcal{D}}(Q^T) \subset \mathrm{im}_{\mathcal{D}}(R^T)$ implies that there exists X such that $Q = PX$ so it must be the case that $\mathrm{rank}(R) \geq \mathrm{rank}(Q) = q$ and hence $\mathrm{rank}(R) = q$. □

Recall from Section 2.1.6 that a behavior $\mathcal{B} = \ker_{\mathcal{A}}(R)$ admits a decomposition $R = \begin{bmatrix} -Q & P \end{bmatrix}$ for a chosen i/o-structure. Furthermore, we may always choose a full i/o-structure so that $\mathrm{rank}(P) = \mathrm{rank}(R)$ and P has full column rank. Under the i/o-structure, we have that $\mathcal{B}$ is also given as

$$\mathcal{B} = \{(u, y) \in \mathcal{A}^{m+q} : Py = Qu\}.$$

For a full i/o-structure Theorem 2.1.54 asserts that P has full column rank; hence we may apply the results of Lemma 2.1.67 to conclude that, for the submatrix P, the associated E-subsets $E(i)$ are non-empty for all $i = 1, \dots, q$.

Lemma 2.1.68. *The column module $U = \mathrm{im}_{\mathcal{D}}(P^T)$ depends on $\mathcal{B}$ and its given i/o-structure, but not on the choice of the matrix P.*

Proof. Let $R' = \begin{bmatrix} -Q' & P' \end{bmatrix}$ be another kernel representation of $\mathcal{B}$. By Corollary 2.1.49, there exists polynomial matrices X_1 and X_2 such that

$$X_1 R = X_1 \begin{bmatrix} -Q & P \end{bmatrix} = \begin{bmatrix} -X_1 Q & X_1 P \end{bmatrix} = R' = \begin{bmatrix} -Q' & P' \end{bmatrix}.$$

The above implies $X_1 P = P'$; similarly, we also reach $X_2 P' = P$. This implies

$$U = \mathrm{im}_{\mathcal{D}}(P^T) = \mathrm{im}_{\mathcal{D}}(P'^T X_2^T) \subset \mathrm{im}_{\mathcal{D}}(P'^T) = \mathrm{im}_{\mathcal{D}}(P^T X_1^T) \subset \mathrm{im}_{\mathcal{D}}(P^T).$$

We conclude that $\mathrm{im}_{\mathcal{D}}(P'^T) = \mathrm{im}_{\mathcal{D}}(P)$. □

[4]See Appendix A.5 for more details on Fitting ideals

Definition 2.1.69. For a behavior $\mathcal{B} = \ker_{\mathcal{A}} \begin{bmatrix} -Q & P \end{bmatrix}$ with given i/o-structure, $P \in \mathcal{D}^{p\times q}$, $\mathcal{D}$-module $U = \operatorname{im}_{\mathcal{D}}(P^T)$ and orders $\leq_l$ on $\mathbb{N}^d$ and $\leq_m$ on $[q] \times \mathbb{N}^d$, we define its **leading subset** as

$$L^{\mathbb{N}}(U) = \{(i, \alpha) \in \{1, \ldots, q\} \times \mathbb{N}^d : \mathbf{z}^{\alpha} e_i \in L(U)\} \subset \{1, \ldots, q\} \times \mathbb{N}^d.$$

as defined in Section A.1.3. We also define the complementary set

$$G(U) = ([q] \times \mathbb{N}^d) \setminus L^{\mathbb{N}}(U)$$

as the set of **initial conditions**. When a choice of orderings and i/o-structure is understood, we write $\partial\mathcal{B}$ to denote the set G and $D(\mathcal{B})$ to denote the set $D(U)$.

With the above terminology, we reach the following corollary to Lemma 2.1.68.

Corollary 2.1.70. *For a behavior $\mathcal{B} = \ker_{\mathcal{A}} \begin{bmatrix} -Q & P \end{bmatrix}$ with given i/o-structure, $\mathcal{D}$-module $U = \operatorname{im}_{\mathcal{D}}(P^{\perp})$ and orders $\leq_l$ on $\mathbb{N}^d$ and $\leq_m$ on $[q] \times \mathbb{N}^d$, the sets $D(\mathcal{B})$ and $\partial\mathcal{B}$ depend on the i/o-structure used and the orderings, but not on the choice of P.*

Due to this corollary, the notation $\partial\mathcal{B}$ and $D(\mathcal{B})$ are well defined once an ordering and i/o-structure are chosen. We now move onto the Cauchy problem. Recall that we may identify the signal space $\mathcal{A}^q$ as

$$w = (w_1, \ldots, w_q) \in \mathcal{A}^q \mapsto \{((\mathbf{z}^i w_1)(0), \ldots, (\mathbf{z}^i w_q)(0))\}_{i\in\mathbb{N}^d} = \{(w_1(i), \ldots, w_q(i))\}_{i\in\mathbb{N}^d} \in k^{[q]\times\mathbb{N}^d}.$$

For any subset $G \subset [q] \times \mathbb{N}^d$ we may consider $k^G \subset \mathcal{A}^q$ as the k-subspace which has a zero extension of the trajectories. In particular, for $w = (w_1, \ldots, w_q) \in k^G$ we have $w_i(t) = 0$ for $(i, t) \notin G$. For any subset $G \subset [p] \times \mathbb{N}^d$ we may also discuss its complement

$$G^c = ([q] \times \mathbb{N}^d) \setminus G = \{(i, n) \in [q] \times \mathbb{N}^d : (i, n) \notin G\}.$$

By zero-extension, we have the direct sum decomposition

$$\mathcal{A}^q = k^G \oplus k^{G^c}$$

through the mapping

$$w \in \mathcal{A}^q \mapsto w|_G + w|_{G^c} \in k^G \oplus k^{G^c}.$$

Furthermore, we may also discuss the projection of $\mathcal{A}^q$ to k^G by restriction, i.e., for $w \in \mathcal{A}^q$ we have $w|_G \in k^G$.

The reasoning for the discussion of Gröbner bases and construction of the initial condition set becomes quite apparent once we relate Theorem 2.1.64 to the following theorem.

Theorem 2.1.71. *For a behavior $\mathcal{B} = \ker_{\mathcal{A}} \begin{bmatrix} -Q & P \end{bmatrix}$ with given i/o-structure, $P \in \mathcal{D}^{p\times q}$ and set $\partial\mathcal{B}$ we have the following decomposition*

$$\mathcal{D}^q = k^{(\partial\mathcal{B})} \oplus \operatorname{im}_{\mathcal{D}}(P^T).$$

Proof. This follows from the normal form. Let $G = \{g_1, \dots, g_r\}$ be a Gröbner basis for the submodule $U := \mathrm{im}_{\mathcal{D}}(P^T) \subset \mathcal{D}^q$. For any $f \in \mathcal{D}^q$ we have

$$f - \mathrm{NF}(f|G) = \sum_{i=1}^{r} a_i g_i \in U \qquad\qquad a_i \in \mathcal{D}.$$

We also have $\mathrm{NF}(f|G) = 0$ if and only if $f \in U$. If $\mathrm{NF}(f|G) \neq 0$, then $\mathrm{LM}(\mathrm{NF}(f|G)) \notin L(U)$ and thus $\mathrm{LE}(\mathrm{NF}(f|G)) \in \partial\mathcal{B}$ (note that $\mathrm{LE}(\mathrm{NF}(f|G))$ denotes the **leading exponent** of $\mathrm{NF}(f|G)$); by associating monomials to points in $\partial\mathcal{B}$ we have $\mathrm{NF}(f|G) \in k^{\partial\mathcal{B}}$. If $f \in U \cap k^{\partial\mathcal{B}}$ and $f \neq 0$, then $f \in U$ so $\mathrm{NF}(f|G) = 0$; however, since $f \in k^{\partial\mathcal{B}}$ we have $\mathrm{NF}(f|G) = f$ and $\mathrm{LM}(f) \notin L(U)$ which implies $f \notin U$. It follows that it must be the case that $f = 0$. We thus reach the direct sum decomposition

$$f \in \mathcal{D}^q \mapsto \mathrm{NF}(f|G) + \left(\sum_{i=1}^{r} a_i g_i\right) \in k^{(\partial\mathcal{B})} \oplus \mathrm{im}_{\mathcal{D}}(P^T).$$

□

Through the identifications $\mathcal{A}^q = k^{[q]\times\mathbb{N}^d}$ and $\mathcal{D}^q = k^{([q]\times\mathbb{N}^d)}$ we may consider the non-degenerate bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{D}^q \times \mathcal{A}^q \to k$$

given by

$$\langle \{f_i(m) : (i,m) \in [q]\times\mathbb{N}^d\}, \{g_j(n) : (j,n) \in [q]\times\mathbb{N}^d\} \rangle \mapsto \sum_{(i,m)\in[q]\times\mathbb{N}^d} f_i(m) g_i(m) \in k. \tag{2.16}$$

For the signal space $\mathcal{A}^q = k^{[q]\times\mathbb{N}^d}$ equipped with the weak-$*$ topology we have the identification

$$\mathrm{Hom}_k(\mathcal{D}^q, k) \leftrightarrow (\mathcal{D}^q)^* \leftrightarrow \left(k^{([q]\times\mathbb{N}^d)}\right)^* \leftrightarrow k^{[q]\times\mathbb{N}^d} \leftrightarrow \mathcal{A}^q.$$

Recall that, for any $\mathcal{D}$-submodule $U \subset \mathcal{D}^q$ and closed $\mathcal{D}$-submodule $V \subset \mathcal{A}^q$ there are the orthogonal complements

$$U^\perp = \{w \in \mathcal{A}^q : \langle f, w \rangle_q = 0 \text{ for all } f \in U\} \subset \mathcal{A}^q$$
$$V^\perp = \{f \in \mathcal{D}^q : \langle f, w \rangle_q = 0 \text{ for all } w \in V\} \subset \mathcal{D}^q.$$

As previously observed, the orthogonal complements induce an involution between their respective categories, i.e. $(U^\perp)^\perp = U$ and $(V^\perp)^\perp = V$.

Now consider any k-linear map $\psi : \mathcal{D}^g \to \mathcal{D}^q$; through the pairing we may induce the adjoint map $\psi^* : \mathcal{A}^q \to \mathcal{A}^g$ given weakly as

$$\langle \psi(f), w \rangle_q = \langle f, \psi^*(w) \rangle_g \qquad\qquad f \in \mathcal{D}^g,\ w \in \mathcal{A}^q.$$

For such a map, we have

$$\begin{aligned}\ker_{\mathcal{A}}(\psi^*) &= \{w \in \mathcal{A}^q : \langle f, \psi^*(w)\rangle = 0 \text{ for all } f \in \mathcal{D}^g\} = \{w \in \mathcal{A}^q : \langle \psi(f), w\rangle = 0 \text{ for all } f \in \mathcal{D}^g\} \\ &= \{w \in \mathcal{A}^q : \langle f', w\rangle = 0 \text{ for all } f' \in \operatorname{im}_{\mathcal{D}}(\psi)\} = \operatorname{im}_{\mathcal{D}}(\psi)^{\perp}.\end{aligned}$$

Similarly, for $w \in \operatorname{im}_{\mathcal{A}}(\psi^*)$ there exists $w' \in \mathcal{A}^q$ such that $w = \psi^*(w')$ and hence for any $f \in \ker_{\mathcal{D}}(\psi)$,

$$\langle f, w\rangle_g = \langle f, \psi^*(w')\rangle_g = \langle \psi(f), w'\rangle_p = \langle 0, w'\rangle_p = 0.$$

Since $f \in \ker_{\mathcal{D}}(\psi)$ was arbitrary, we have the inclusion $\operatorname{im}_{\mathcal{A}}(\psi^*) \subset \ker_{\mathcal{D}}(\psi)^{\perp}$. By Lemma 2.1.41, $\operatorname{im}_{\mathcal{A}}(\psi^*)$ is a closed linear subspace of $\mathcal{A}^g$. Say that the two sets are not equal. By the Hahn-Banach theorem for this setting (see Theorem 2.1.43), for $v \in \ker_{\mathcal{D}}(\psi)^{\perp}$ where $v \notin \operatorname{im}_{\mathcal{A}}(\psi^*)$ there exists (since $\mathcal{A}$ is given the weak-$*$ topology, we have the continuous dual space $\widehat{\operatorname{Hom}_k}(\mathcal{A}, k) = \mathcal{D}$) $f \in \mathcal{D}$ such that $\langle f, w\rangle = 0$ for all $w \in \operatorname{im}_{\mathcal{A}}(\psi^*)$ and $\langle f, v\rangle = 1$. We thus have

$$0 = \langle f, \psi^*(w')\rangle_g = \langle \psi(f), w'\rangle_p$$

for all $w' \in \mathcal{A}^q$ and, hence, $\psi(f) = 0$. However, since $f \in \ker_{\mathcal{D}}(\psi)$ we have $\langle f, v\rangle = 0$ since $v \in \ker_{\mathcal{D}}(\psi)^{\perp}$; because this contradicts the equality $\langle f, v\rangle = 1$, we reach that such a v cannot exist and thus we have the equality $\operatorname{im}_{\mathcal{A}}(\psi^*) = \ker_{\mathcal{D}}(\psi)^{\perp}$.

We have just demonstrated that any direct sum decomposition $\mathcal{D}^q = U \oplus V$ with projection map $\phi : \mathcal{D}^q \to V$ such that $\ker(\phi) = U$ and $\operatorname{im}(\phi) = U$ induces the decomposition $\mathcal{A}^q = U^{\perp} \oplus V^{\perp}$ via the adjoint map ϕ^* where $\phi^* : \mathcal{A}^q \to U^{\perp}$ is also a projection map. We now proceed to the main theorem of this section.

Theorem 2.1.72 (Canonical Cauchy Problem)**.** *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$, $R \in \mathcal{D}^{p\times q}$ be a behavior with a given i/o-structure through the decomposition $R = [-Q \quad P]$ and initial condition set $\partial\mathcal{B}$ provided by the above orderings. Then the* ***canonical Cauchy problem***

$$Py = Qu, \quad y|_G = x \qquad\qquad \textit{where } u \in A^{n_{\mathcal{I}}},\ x \in k^{\partial\mathcal{B}} \textit{ given},\ y \in \mathcal{A}^{n_{\mathcal{O}}} \textit{ unknown}$$

is uniquely solvable.

Proof. By Theorem 2.1.71 we have the direct sum decomposition

$$\mathcal{D}^q = k^{(\partial\mathcal{B})} \oplus \operatorname{im}_{\mathcal{D}}(P^T).$$

Let $\phi : \mathcal{D}^{1\times q} \to k^{(\partial\mathcal{B})}$ be the projection map onto $k^{(\partial\mathcal{B})}$. Through the bilinear form (2.16) we reach

$$(k^{(\partial\mathcal{B})})^{\perp} = \operatorname{im}_{\mathcal{D}}(\phi)^{\perp} = \ker(\phi^*) = (\operatorname{im}_{\mathcal{D}}(P^T))^{\perp} = k^{\partial\mathcal{B}^c}.$$

We also have the map $P^T : \mathcal{D}^p \to \mathcal{D}^q$ provides the equality $\operatorname{im}_{\mathcal{D}}(P^T)^{\perp} = \ker_{\mathcal{A}}(P)$. This leads us to the direct sum decomposition

$$\mathcal{A}^q = \ker_{\mathcal{A}}(P) \oplus k^{\partial\mathcal{B}^c}$$

with the projection map ϕ^* onto $\ker_{\mathcal{A}}(P)$. By Theorem 2.1.64 we conclude with the desired result. $\square$

Corollary 2.1.73. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a non-zero behavior with dual module $\mathcal{M} = \operatorname{coker}_{\mathcal{D}}(R^T)$. If $\mathcal{M}$ is a torsion module, then there exists a non-empty proper subset $J \subset \mathbb{N}^d$ such that J is the smallest subset of $\mathbb{N}^d$ for which any two trajectories $w_1, w_2 \in \mathcal{B}$ where $w_1|_J = w_2|_J$ necessarily are equal on $\mathbb{N}^d$.*

Proof. By Lemma 2.1.67, each $E(i)$ is a non-empty cw-ideal (see Definition A.1.6.) Define $J' = \cap_{i=1}^{1} E(i)$ and note that $J' \neq \emptyset$ due to the fact that each $E(i)$ is a cw-ideal. Also define the set $J = \mathbb{N}^d \setminus J'$. Since

$$(\mathbb{N}^d \setminus E(i)) \subset \mathbb{N}^d \setminus \left(\bigcap_{i=1}^{q} E(i) \right) = J \qquad \text{for all } i = 1, \dots, q.$$

We have $\partial\mathcal{B} \subset [q] \times J$ and thus, by uniqueness of the canonical Cauchy problem, equality on J implies equality on $\mathbb{N}^d$. $\square$

Corollary 2.1.74. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a behavior with an initial condition set $\partial\mathcal{B}$. It follows that the projection onto $\partial\mathcal{B}$ is surjective, i.e. $k^{\partial\mathcal{B}} = \mathcal{B}|_{\partial\mathcal{B}}$.*

Corollary 2.1.75. *Let $\mathcal{B} \subset \mathcal{B}'$ be two behaviors. Then $\mathcal{B}|_{\partial\mathcal{B}} = \mathcal{B}'|_{\partial\mathcal{B}}$ and $\mathcal{B}|_{\partial\mathcal{B}'} \subset \mathcal{B}'|_{\partial\mathcal{B}'}$.*

It is important to emphasize that the initial condition set is *not* unique. Nevertheless, once an ordering and i/o-structure are chosen, then the solution is the same under any other choice. Before concluding, we consider the following example.

Example 2.1.76. A simple application of Singular or Macaulay2 brings to light how the Cauchy problem can be solved via Gröbner bases. We consider the scalar problem where we have the kernel representation consisting of generators of the ideal $I = \langle -xy + y^3 - 2z, x^5 - y^2 \rangle$. The following is an example of a Singular session.

```
// First we define the ring. Note that 'lp' denotes lexicographic ordering.
> ring R = 0, (x, y, z), lp;
// Then we define the ideal we are considering.
> ideal I = -x*y + y^3 - 2z, x^5 - y^2;
// We then proceed to compute its standard basis.
> ideal S = groebner(I);
// We can also identify the initial condition set by
// computing the generators of the leading monomial ideal.
> ideal G = lead(S);
> G;
G[1]=y15
G[2]=16xz4
G[3]=xy
G[4]=8x2z3
G[5]=4x3z2
G[6]=2x4z
G[7]=x5
```

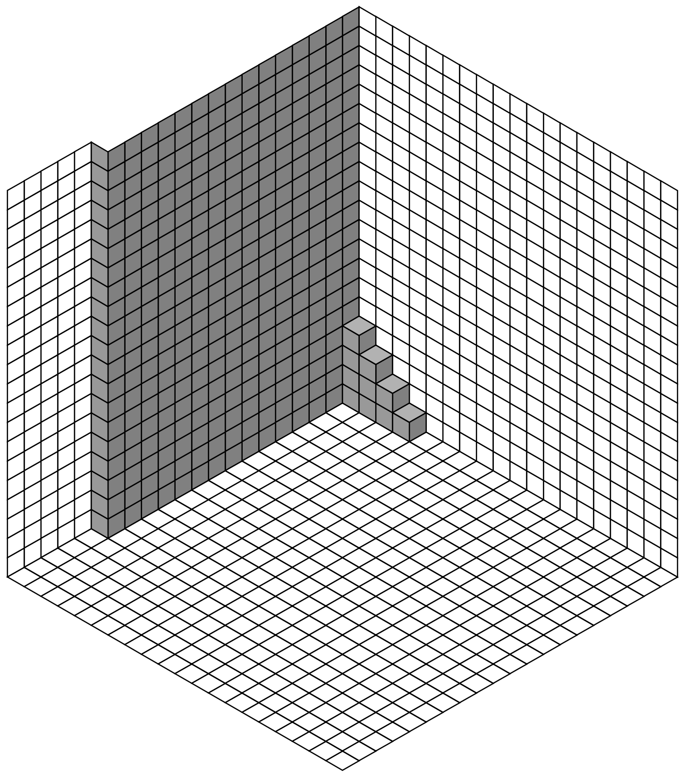

Figure 2.1: Example of ∂I for $I = \langle -xy + y^3 - 2z, x^5 - y^2 \rangle$.

```
// For a point outside of the initial condition set
// the normal form reduces the point to a polynomial
// which can be evaluated to determine the signal value
// at the specified point.
> reduce(y^16, S);
10y13z-40y10z2+y8+80y7z3-80y4z4+32yz5
> reduce(x*z^4, S);
-1/16y14+5/8y11z-5/2y8z2+1/16y6+5y5z3-4y2z4
// For points in the initial condition set the normal form
// just returns the provided monomial
> reduce(x, S);
x
> reduce(z^6, S);
z6
```

A graphical representation of the initial condition set for this example can be seen in Figure 2.1. See [18, 20] for more details on computer algebra systems.

Remark. We note that the presented approach for solving the canonical Cauchy problem continues to work under any global monomial ordering.

2.2 Structural Properties of Behaviors

Notation. Let k be a field and define the operator ring $\mathcal{D} = k[z_1, \ldots, z_d]$ and signal space $\mathcal{A} = k^{\mathbb{N}^d}$.

The previous section focused on a rather abstract treatment of behaviors which was mainly algebraic. Although most of the results on behavioral systems are algebraic in nature, there are usually "trajectory based" analogues which tie the theory to practical matters in systems theory. In this section we connect the results of the previous section to results on properties of trajectories. In most cases, the following definitions and conclusions are restatements of the contents of the previous section.

2.2.1 Free Variables

Free variables are the components of the behavior which can be arbitrarily specified. In the input/state/output setting, these correspond to inputs since they have no restrictions on their values. When we discussed the i/o-structure of a behavior, the dimension of the input space for the behavior $\mathcal{B} \subset \mathcal{A}^q$ was defined as $q - \text{rank}(R)$. In a tautological way, we define free variables.

Definition 2.2.1. Let $\mathcal{B} \subset \mathcal{A}^q$ be a behavior with kernel representation $R = \mathcal{D}^{p \times q}$. We define the number of **free variables** of $\mathcal{B}$ to be $q - \text{rank}(R)$.

Even though the definition of inputs and free variables are the same, free variables are *not* discussed in terms of an i/o-structure. They are instead a property of the behavior that is independent of i/o-structure; while this distinction is minute, it should be emphasized since behaviors are usually not discussed in terms of an i/o-structure.

To understand what a free variable is, we must determine what the trajectories look like. For a subset of indices $\mathcal{I} = \{j_1, \ldots, j_\ell\} \subset \{1, \ldots, q\}$ we define the projection map associated to $\mathcal{I}$ as

$$\pi_{\mathcal{I}} : \mathcal{B} \to \mathcal{A}^\ell \qquad\qquad \pi_{\mathcal{I}} : (w_1, \ldots, w_q) \mapsto (w_{j_1}, \ldots, w_{j_\ell}).$$

With this notation we have the following.

Theorem 2.2.2. *Let $\mathcal{B} \subset \mathcal{A}^q$ be a behavior with kernel representation $R = \mathcal{D}^{p \times q}$. The following are equivalent:*

1. *The largest integer ℓ such that there exists $\mathcal{I} \subset \{1, \ldots, q\}$, $|\mathcal{I}| = \ell$, and $\pi_{\mathcal{I}}$ is a surjective map.*
2. *The number of free variables of $\mathcal{B}$.*

Proof. This is merely a restatement of Theorem 2.1.54 with slightly different terminology. □

Remark. We note that free variables do not always exist. Indeed, when free variables do not exist we reach a special class of behaviors referred to as **autonomous behaviors**. In Section 2.2.3 we discuss these behaviors in more detail.

2.2.2 Controllability

Understanding the controllability of behaviors is an ongoing area of interest in behavioral systems theory. In [51] Willems introduced controllability of one-dimensional behaviors as the ability to patch or concatenate trajectories. Ever since this characterization, it has been a central theme in behavioral systems theory. In [43], Rocha extended this idea to two-dimensional behaviors over $\mathbb{Z}^2$. Wood and Zerz in [55] provided the most general form of this trajectory patching property, and thus obtained a unifying definition of controllability through trajectory patching for discrete-time behaviors. In this section we discuss controllability and its equivalent forms.

Consider the one-dimensional input/state/ouput system of the form

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \qquad\qquad t \in \mathbb{N} \end{aligned} \tag{2.17}$$

where x is the state, u is the input, y is the output, and each takes values in a finite dimensional vector space. Controllability in this context involves point-to-point control of the state as follows.

Definition 2.2.3 (First Order Linear i/s/o Controllability)**.** A system given by (2.17) is controllable if for the initial state $x(0) = 0$ and any desired state value v, there exists $s \in \mathbb{N}$ and a series of input values $u(0), \ldots, u(s)$ so that $x(s) = v$.

When Willems presented one-dimensional behaviors, he developed axioms which allowed behaviors to have a "state." The state variables were then used to relate behaviors with their input/state/output analogues. The point Willems made was that behaviors could handle all of the classical questions posed by first-order linear systems. However, in the two-dimensional setting, there are several models one can consider, each having its own causality. For the behavioral community, this has been resolved by picking one model (output-nulling) which is commonly used to relate behaviors to the input/state/output systems.

Before stating the definition of controllability for behaviors, we must define distance on $\mathbb{N}^d$.

Definition 2.2.4. For any two points $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in \mathbb{N}^d$ we define their **distance**

$$d(a, b) = \sum_{i=1}^{d} |a_i - b_i|.$$

For any two subsets $A, B \subset \mathbb{N}^d$ we define their **distance** as

$$d(A, B) = \min_{a \in A, b \in B} d(a, b).$$

The accepted definition of controllability is as follows.

Definition 2.2.5. We say that the behavior $\mathcal{B}$ is controllable if the following holds. There exists $\rho \geq 0$ such that for any $T_1, T_2 \subset \mathbb{N}^d$ with $d(T_1, T_2) > \rho$, $b_1, b_2 \in T$, and $w_1, w_2 \in \mathcal{B}$, there exists $w \in \mathcal{B}$ where

$$w(t) = \begin{cases} w_1(t - b_1) & t - b_1 \in \mathbb{N}^d, \quad t \in T_1 \\ w_2(t - b_2) & t - b_2 \in \mathbb{N}^d, \quad t \in T_2 \end{cases}$$

The significant piece of the above conditions, which took 10 years to discover, was that controllable behaviors must be permanent.

Definition 2.2.6. A d-dimensional behavior $\mathcal{B}$ is said to be **permanent** if $z_i\mathcal{B} = \mathcal{B}$ for all $i = 1, \ldots, d$.

In the definition of controllability, this property is observed through the variables b_1 and b_2 which emulate forward shifting. In particular, for any trajectory $w \in \mathcal{B}$ (where $\mathcal{B}$ is some permanent behavior), there exists $w' \in \mathcal{B}$ such that $z_i w' = w$. As seen in [43] for the two-dimensional setting, the other pieces of the condition were well known from the study of behaviors over $\mathbb{Z}^d$. An interesting observation is that this has a very strong connection to the operator ring used. When we discuss the controllability of reduced behaviors this observation becomes essential.

In [51] and [43], the concept of an **image representation** was shown to be equivalent to controllability in various settings. This is often used to define controllability in general settings before an ample theory can be developed.

Definition 2.2.7. Let $\mathcal{B}$ be a behavior. We say that $M \in \mathcal{D}^{q\times g}$ is an **image representation** if $\mathcal{B} = \mathrm{im}_{\mathcal{A}}(M)$. In this case,

$$\mathcal{B} = \{w \in \mathcal{A}^q : \text{ there exists } v \in \mathcal{A}^g \text{ such that } w = Mv\}.$$

Note that an image representation, in general, may have a kernel.

It is easy to translate image representations and kernel representations to a matter of exactness of chain complexes. Consider a behavior $\mathcal{B}$ with kernel representation $R \in \mathcal{D}^{p\times q}$. There may be a polynomial matrix $M \in \mathcal{D}^{q\times g}$ such that we have a chain complex.

$$\mathcal{A}^g \xrightarrow{M} \mathcal{A}^q \xrightarrow{R} \mathcal{A}^p.$$

For instance, we can always take $M = 0$. However, an image representation is equivalent to having exactness; in particular, if $\mathcal{B}$ has an image representation M, then the above sequence is exact. Conversely, if the above sequence is exact, then $\mathcal{B}$ has an image representation given by M. This point is repeated when minimal left annihilators are discussed below.

Example 2.2.8. Consider the kernel representations

$$R_{grad} = \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix}$$
$$R_{curl} = \begin{bmatrix} 0 & -(z_3-1) & (z_2-1) \\ (z_3-1) & 0 & -(z_1-1) \\ -(z_2-1) & (z_1-1) & 0 \end{bmatrix}$$
$$R_{div} = \begin{bmatrix} x-1 & y-1 & z-1 \end{bmatrix}$$

and the chain complex

$$\mathcal{A}^1 \xrightarrow{R_{grad}} \mathcal{A}^3 \xrightarrow{R_{curl}} \mathcal{A}^3 \xrightarrow{R_{div}} \mathcal{A}^1.$$

One can observe that the above complex is exact. It follows that the image representation of R_{curl} is R_{grad} and the image representation of R_{div} is R_{curl}; however, R_{grad} does not have an image representation.

Both Willems and Rocha established connections between controllability and the existence of an image representation to factor left primeness of the kernel representation.

Definition 2.2.9. We say that $R \in \mathcal{D}^{p\times q}$ is **factor left prime** (FLP) if any left factor is unimodular, i.e., if there exists a square matrix D and matrix R' so that the equality $R = DR'$ holds, then $\det(D)$ is a non-zero constant.

It turns out that for 1D and 2D systems the connection between FLP and controllability was just happenstance, i.e., there is an implicit hypothesis which is naturally satisfied by the operator ring that is used in the 1D and 2D setting. In particular, in the 1D setting, every kernel representation can be taken to have full row rank. In the 2D setting, every kernel representation can be factored so that it is a product of a full row rank matrix and a full column rank matrix. In more than two variables, we have no way of reaching full row rank matrices. The underlying reason deals with a combination of the global dimension of the ring $\mathcal{D}$ and the minimal number of generators needed to make a prime ideal; in this latter case, one can make prime ideals which have many generators and the syzygies are non-trivial. The Hilbert Syzygy Theorem states that the global dimension of $\mathcal{D} = k[z_1, \dots, z_d]$ is d; in particular, the relations on the generators of an ideal can give rise to a series of higher order syzygies. The connection between FLP and image representations is observable only if the global dimension of $\mathcal{D}$ is less than three. A more precise account of the above statements can be found in [39, pg. 147-152].

Due to the insufficiency of factor left primeness in the multidimensional setting, in [39, pg. 142-144] Oberst defines **generalized factor left primeness** (GFLP). A polynomial matrix which is GFLP was then shown, almost by definition, to have an image representation. The definition puts no size conditions on the left factor, as it only demands that it have a right inverse. We now restate the definition.

Definition 2.2.10. We say that a polynomial matrix $R \in \mathcal{D}^{p\times q}$ is **generalized factor left prime** (GFLP) if the following holds. If there are polynomial matrices such that $R = ER'$ where $\operatorname{rank}(R) = \operatorname{rank}(R')$, then there exists a polynomial matrix Y such that $R' = YR$.

At this point, it is appropriate to discuss minimal left annihilators (MLAs).

Definition 2.2.11. For a polynomial matrix $M \in \mathcal{D}^{q\times g}$ we say that a matrix $R \in \mathcal{D}^{p\times q}$ is a **minimal left annihilator** (MLA) of M if

1. $RM = 0$, i.e. $\ker_{\mathcal{D}}(R) = \operatorname{im}_{\mathcal{D}}(M)$.
2. For any matrix $R' \in \mathcal{D}^{p'\times q}$ for which $R'M = 0$ there exists $X \in \mathcal{D}^{p\times p'}$ such that $R' = XR$.

We say that R is a **minimal right annihilator** (MRA) if R^T is a minimal left annihilator.

Lemma 2.2.12. *Let $R \in \mathcal{D}^{p\times q}$ and $M \in \mathcal{D}^{q\times g}$ be two matrices. Then R is a minimal left annihilator of M if and only if* $\ker_{\mathcal{D}}(M^T) = \operatorname{im}_{\mathcal{D}}(R^T)$.

Proof. ($\Rightarrow$). Let R be an MLA of M. Because $RM = 0$, we have $M^T R^T = 0$ and so $\mathrm{im}_{\mathcal{D}}(R^T) \subset \ker_{\mathcal{D}}(M^T)$. Choose $y^T \in \ker_{\mathcal{D}}(M^T)$ and notice $0 = M^T y^T$ or $0 = yM$; since R is an MLA of M there exists x such that $y = xR$ or $y^T = R^T x^T$. In particular, $y^T \in \mathrm{im}_{\mathcal{D}}(R^T)$.

($\Leftarrow$). Note that, since $\ker_{\mathcal{D}}(M^T) = \mathrm{im}_{\mathcal{D}}(R^T)$ we have $(RM)^T = M^T R^T = 0$ and thus $RM = 0$. Let $\widetilde{R} \in \mathcal{D}^{p' \times q}$ be any other such matrix so that $\widetilde{R}M = 0$. Denote the rows of $\widetilde{R}$ by $\{r_1, \ldots, r_{p'}\}$. By exactness, we have $0 = r_i M$ (or, equivalently, $M^T r_i^T = 0$) implies that there exists $y_1, \ldots, y_{p'}$ such that $y_i R = r_i$ (or, equivalently, $R^T y_i^T = r_i^T$). Define the matrix X so that the rows are comprised of $y_1, \ldots, y_{p'}$. We then have $XR = \widetilde{R}$. Since $\widetilde{R}$ was arbitrary, we conclude that R is an MLA of M. □

Proposition 2.2.13. *Let $M \in \mathcal{D}^{q \times g}$, then there exist a matrix $R \in \mathcal{D}^{p \times q}$ such that R is an MLA of M.*

Proof. Since $\mathcal{D}$ is Noetherian and $\ker_{\mathcal{D}}(M^T) \subset \mathcal{D}^q$ is a submodule, $\ker_{\mathcal{D}}(M^T)$ is finitely generated. Let $\{g_1, \ldots, g_n\} \subset \mathcal{D}^q$ be the generators of $\ker_{\mathcal{D}}(M^T)$ and define the matrix R so that R^T has columns consisting of the generators $\{g_1, \ldots, g_n\}$; we then have

$$(RM)^T = M^T R^T = M^T g_1 + \cdots + M^T g_n = 0$$

and hence $RM = 0$. By construction, for any $y^T \in \ker_{\mathcal{D}}(M^T)$ there exists $x^T \in \mathcal{D}^n$ such that $R^T x^T = y^T$ and so $\mathrm{im}_{\mathcal{D}}(R^T) = \ker_{\mathcal{D}}(M^T)$. By Lemma 2.2.12 we may conclude that R is an MLA of M. □

Corollary 2.2.14. *Let $R \in \mathcal{D}^{p \times q}$, then there exist a matrix $M \in \mathcal{D}^{q \times g}$ such that M is an MRA of R.*

Proof. Apply Proposition 2.2.13 to R^T. Then there exists a matrix M^T which is an MLA of R^T and, by definition, we have M is an MRA of R. □

One important property about matrices which are GFLP is that they are MLAs of some matrix.

Lemma 2.2.15. *Let $R \in \mathcal{D}^{p \times q}$ be a matrix. There exists a matrix $M \in \mathcal{D}^{q \times g}$ such that R is the MLA of M if and only if R is GFLP.*

Proof. ($\Rightarrow$) Let R be an MLA of M and $X, \widetilde{R}$ be matrices such that $R = X\widetilde{R}$ and $\mathrm{rank}(R) = \mathrm{rank}(\widetilde{R})$. This leads to the following chain of inclusions

$$\ker_{\mathcal{D}}(R) = \ker_{\mathcal{D}}(X\widetilde{R}) \supset \ker_{\mathcal{D}}(\widetilde{R})$$

Define the inclusion map $\iota : \ker_{\mathcal{D}}(\widetilde{R}) \hookrightarrow \ker_{\mathcal{D}}(R)$ and canonical quotient map $\phi : \ker_{\mathcal{D}}(R) \to \mathrm{coker}_{\mathcal{D}}(\iota)$ to reach the exact sequence

$$0 \longrightarrow \ker_{\mathcal{D}}(\widetilde{R}) \xrightarrow{\iota} \ker_{\mathcal{D}}(R) \xrightarrow{\phi} \mathrm{coker}_{\mathcal{D}}(\iota) \longrightarrow 0.$$

Since $Q(\mathcal{D})$ is flat we reach the exact sequence

$$0 \longrightarrow \ker_{\mathcal{D}}(\widetilde{R}) \otimes Q(\mathcal{D}) \xrightarrow{\iota \otimes 1_{Q(\mathcal{D})}} \ker_{\mathcal{D}}(R) \otimes Q(\mathcal{D}) \xrightarrow{\phi \otimes 1_{Q(\mathcal{D})}} \operatorname{coker}_{\mathcal{D}}(\iota) \otimes Q(\mathcal{D}) \longrightarrow 0.$$

Since each of the above $Q(\mathcal{D})$-vector spaces is finite dimensional and $\operatorname{rank}(R) = \operatorname{rank}(\widetilde{R})$ we have

$$q - \dim(\ker_{\mathcal{D}}(\widetilde{R}) \otimes Q(\mathcal{D})) = \operatorname{rank}(\widetilde{R}) = \operatorname{rank}(R) = q - \dim(\ker_{\mathcal{D}}(R) \otimes Q(\mathcal{D}))$$

Since dimension is additive, this implies $\dim(\operatorname{coker}_{\mathcal{D}}(\iota) \otimes Q(\mathcal{D})) = 0$ and by exactness,

$$\ker_{\mathcal{D}}(R) \otimes Q(\mathcal{D}) = \ker_{\mathcal{D}}(\widetilde{R}) \otimes Q(\mathcal{D}).$$

From the inclusion $\mathcal{D} \subset Q(\mathcal{D})$ we have that, for any $w \in \mathcal{D}^q$, $Rw = 0$ if and only if $\widetilde{R}w = 0$; as a consequence, we reach $\widetilde{R}M = 0$. We may now use that R is an MLA of M to construct a matrix Y such that $\widetilde{R} = YR$ and conclude that R is GFLP.

($\Leftarrow$) Assume that R is GFLP. Let M be an MRA of R (which exists by Proposition 2.2.14) and $\widetilde{R}$ be an MLA of M (which exists by Corollary 2.2.13.) Since $\widetilde{R}$ is an MLA of M we may use ($\Rightarrow$) to see that $\widetilde{R}$ is GFLP. Also note that, since M is an MRA of R, we have $\ker_{\mathcal{D}}(R) = \operatorname{im}_{\mathcal{D}}(M)$ since the columns of M are comprised of the generators of $\ker_{\mathcal{D}}(R)$. (It is important to emphasize that, even though $\ker_{\mathcal{D}}(R) = \operatorname{im}_{\mathcal{D}}(M)$, it is *not necessarily the case* that $\ker_{\mathcal{A}}(R) = \operatorname{im}_{\mathcal{A}}(M)$.)

Since M is an MRA of R we have $(M^T R^T) = 0$ and thus $RM = 0$. Due to the fact that $\widetilde{R}$ is an MLA of M, we have that there exists a matrix X with entries in $\mathcal{D}$ such that $R = X\widetilde{R}$. This leads to the inclusion $\ker_{\mathcal{D}}(\widetilde{R}) \subset \ker_{\mathcal{D}}(R)$ and the inequality

$$\dim(\ker_{\mathcal{D}}(\widetilde{R}) \otimes Q(\mathcal{D})) \leq \dim(\ker_{\mathcal{D}}(R) \otimes Q(\mathcal{D})).$$

From the property $\widetilde{R}M = 0$ it is also true that $\operatorname{im}_{\mathcal{D}}(M) \subset \ker_{\mathcal{D}}(\widetilde{R})$ and hence

$$\dim(\operatorname{im}_{\mathcal{D}}(M) \otimes Q(\mathcal{D})) \leq \dim(\ker_{\mathcal{D}}(\widetilde{R}) \otimes Q(\mathcal{D})).$$

We may also observe that, since $\ker_{\mathcal{D}}(R) = \operatorname{im}_{\mathcal{D}}(M)$, we have the equality

$$\dim(\operatorname{im}_{\mathcal{D}}(M) \otimes Q(\mathcal{D})) = \dim(\ker_{\mathcal{D}}(R) \otimes Q(\mathcal{D})).$$

Combining these observations yields

$$\dim(\operatorname{im}_{\mathcal{D}}(M) \otimes Q(\mathcal{D})) \leq \dim(\ker_{\mathcal{D}}(\widetilde{R}) \otimes Q(\mathcal{D})) \leq \dim(\ker_{\mathcal{D}}(R) \otimes Q(\mathcal{D})) = \dim(\operatorname{im}_{\mathcal{D}}(M) \otimes Q(\mathcal{D}))$$

and thus

$$\dim(\ker_{\mathcal{D}}(\widetilde{R}) \otimes Q(\mathcal{D})) = \dim(\ker_{\mathcal{D}}(R) \otimes Q(\mathcal{D})).$$

It follows that $\operatorname{rank}(\widetilde{R}) = \operatorname{rank}(R)$.

From the equality $R = X\widetilde{R}$, the assumption that R is GFLP and the equality $\operatorname{rank}(\widetilde{R}) = \operatorname{rank}(R)$, there exists a matrix Y with entries in $\mathcal{D}$ such that $\widetilde{R} = YR$. In particular, we have

$$\ker_{\mathcal{D}}(\widetilde{R}) \subset \ker_{\mathcal{D}}(R) \subset \ker_{\mathcal{D}}(\widetilde{R})$$

and thus the equality $\ker_{\mathcal{D}}(\widetilde{R}) = \ker_{\mathcal{D}}(R)$. This equality combined with the fact $\widetilde{R}$ is an MLA of M proves that R is also an MLA of M. $\square$

A simple application of Oberst duality yields the following result.

Lemma 2.2.16. *For a given matrix $R \in \mathcal{D}^{p\times q}$, $\mathcal{B} = \ker_{\mathcal{D}}(R)$ has an image representation $M \in \mathcal{D}^{q\times g}$ if and only if R is an MLA of M.*

Proof. ($\Rightarrow$) Let $\mathcal{B}$ have the image representation M. By Corollay 2.1.9, the exact sequence

$$\mathcal{A}^g \xrightarrow{M} \mathcal{A}^q \xrightarrow{R} \mathcal{A}^p$$

implies that the following sequence is also exact

$$\mathcal{D}^p \xrightarrow{R^T} \mathcal{D}^q \xrightarrow{M^T} \mathcal{D}^g \ .$$

By Lemma 2.2.12 we conclude that R is an MLA of M.

($\Leftarrow$) Let $M \in \mathcal{D}^{q\times g}$ be such that R is an MLA of M. By Lemma 2.2.12 we have the exact sequence

$$\mathcal{D}^p \xrightarrow{R^T} \mathcal{D}^q \xrightarrow{M^T} \mathcal{D}^g \ .$$

By Corollary 2.1.9 this implies that we also have the exact sequence

$$\mathcal{A}^g \xrightarrow{M} \mathcal{A}^q \xrightarrow{R} \mathcal{A}^p.$$

In particular, $\ker_{\mathcal{A}}(R) = \mathrm{im}_{\mathcal{A}}(M)$; we conclude that R has M as an image representation. □

Corollary 2.2.17. *For a given matrix $R \in \mathcal{D}^{p\times q}$, $\mathcal{B} = \ker_{\mathcal{D}}(R)$ has an image representation $M \in \mathcal{D}^{q\times g}$ if and only if R is GFLP.*

Proof. This follows directly from Lemma 2.2.16 and Lemma 2.2.15. □

Remark 2.2.1. Lemma 2.2.15, Lemma 2.2.12, Lemma 2.2.16, Proposition 2.2.13, Corollary 2.2.14 and Corollary 2.2.17 continue to hold in the setting where $\mathcal{D}$ is replaced by a Noetherian domain. Excluding Lemma 2.2.15 and Corollary 2.2.17, the results continue to hold even in the setting where $\mathcal{D}$ is a commutative Noetherian ring. Note that, in the case that $\mathcal{D}$ has zero-divisors, the definition of rank used is not suitable. (See [35, Chapter 1.G] for discussion on the McCoy rank as an alternative.)

Our final characterization of controllability, which first appeared in [54, page 48], is the notion of divisibility.

Definition 2.2.18. We say that the behavior $\mathcal{B}$ is **divisible** if for any $v \in \mathcal{B}$ and non-zero $p \in \mathcal{D}$, there exists $w \in \mathcal{B}$ such that $v = pw$. In other words, for any non-zero $p \in \mathcal{D}$, $p\mathcal{B} = \mathcal{B}$.

This leads us to the numerous equivalent definitions of controllability.

Theorem 2.2.19. *Let $\mathcal{B}$ be a behavior with kernel representation R. Then the following are equivalent:*

1. *$\mathcal{B}$ is controllable.*

2. *$\mathcal{B}$ has an image representation.*

3. *$\mathcal{M} = \operatorname{coker}_{\mathcal{D}}(R^T)$ is torsion-free.*

4. *$\mathcal{B}$ is minimal in its transfer class.*

5. *R is GFLP.*

6. *$\mathcal{B}$ is divisible.*

Proof. The equivalence of (3), (4), and (5) is established in Theorem 2.1.61. The remaining equivalences are established for a more general setting in Theorems 5.3.1, 5.3.7, and 5.3.8. Note that the equivalence (2) $\iff$ (5) can be directly observed from Corollary 2.2.17. $\square$

2.2.3 Autonomy

For behaviors over $\mathbb{N}^d$ it is common to discuss subsets of $\mathbb{N}^d$ which are cw-ideals (component-wise ideals). A **cw-ideal** is a subset $G \subset \mathbb{N}^d$ such that for any $p \in \mathbb{N}^d$, $p + G \subset G$. Such sets are crucial when discussing autonomous behaviors and the Canonical Cauchy Problem as seen in Section 2.1.7.

Definition 2.2.20. We say that a non-zero behavior $\mathcal{B}$ is **autonomous** if there exists a non-empty region $J \subset \mathbb{N}^d$ such that $\mathbb{N}^d \setminus J$ is a cw-ideal and for any two $w_1, w_2 \in \mathcal{B}$, $w_1|_J = w_2|_J$ implies $w_1 = w_2$ on $\mathbb{N}^d$.

The above is a trajectory-based definition; as such, it is difficult to demonstrate that it is an E-isomorphism invariant. To reach an invariant definition of autonomous behaviors we must introduce some language from commutative algebra.

Definition 2.2.21. We define the **annihilator** of $\mathcal{B}$ to be the following

$$\operatorname{Ann}(\mathcal{B}) = \{p \in \mathcal{D} : pw = 0 \text{ for all } w \in \mathcal{B}\} \subset \mathcal{D}.$$

Note that a condition of the form $pw = 0$ for $p \in \mathcal{D}$ involves no coupling between the components of $w \in \mathcal{B} \subset \mathcal{A}^q$ (in the case $q > 1$). We therefore say that $\mathcal{B}$ satisfies a **non-coupling condition** if $\operatorname{Ann}(\mathcal{B}) \neq 0$.

The annihilator of a behavior is a very interesting object of study. The following example explains this through a well known kernel representation.

Example 2.2.22. Consider the three-dimensional behavior $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^3$ where

$$R_{grad} = \begin{bmatrix} z_1 - 1 \\ z_2 - 1 \\ z_3 - 1 \end{bmatrix}.$$

Then $\operatorname{Ann}(\mathcal{B}) = \langle z_1 - 1, z_2 - 1, z_3 - 1 \rangle$. In this case $\mathcal{B} \cong k$ since the only vector fields with trivial gradient are constants; this is also seen by $\operatorname{Ann}(\mathcal{B})$ being a maximal ideal in $\mathcal{D}$.

There has been extensive work on characterizing autonomous behaviors through properties exhibited by both the behavior $\mathcal{B}$ and the dual module $\mathcal{M}$. The following theorem summarizes these equivalences.

Theorem 2.2.23. *Let $\mathcal{B}$ be a behavior with kernel representation $R \in \mathcal{D}^{p\times q}$. The following are equivalent:*

1. *$\mathcal{B}$ is autonomous*
2. *$\mathcal{B}$ has no free variables.*
3. *For $\mathcal{B} = \ker_{\mathcal{A}}(R)$, R has full column rank.*
4. *$\mathrm{Ann}(\mathcal{B}) \neq 0$, i.e. $\mathcal{B}$ is a torsion module.*

Proof. This is a restatement of Lemma 2.1.67 combined with Corollary 2.1.73. □

Conditions (2) and (4) can be used to demonstrate that autonomy is an E-isomorphism invariant. The following well-known lemma is used to demonstrate that (4) is an E-module invariant.

Lemma 2.2.24. *Let $\mathcal{B}$ be a behavior and let $\mathcal{M} = D(\mathcal{B})$ denote its dual module. Then $\mathrm{Ann}(\mathcal{B}) = \mathrm{Ann}(\mathcal{M})$.*

Proof. Let $p \in \mathrm{Ann}(\mathcal{B})$ be given; by definition, every trajectory in $\mathcal{B}$ is annihilated by p, i.e. $p(\mathcal{B}) = 0$. As a consequence, $pe_i \in \mathrm{im}_{\mathcal{D}}(R^T)$ for each standard basis vector $e_1, \ldots, e_q$. Since $\mathcal{M} = \mathrm{coker}_{\mathcal{D}}(R^T)$ we see $(pI_q)\mathcal{M} = 0$ and hence $\mathrm{Ann}(\mathcal{B}) \subset \mathrm{Ann}(\mathcal{M})$.

Similarly, if $p\mathcal{M} = 0$, then $pe_i \in \mathrm{im}_{\mathcal{D}}(R^T)$ and thus $pe_i \in \mathcal{B}^{\perp}$ for each standard basis vector $e_1, \ldots, e_q$; it follows that $\mathrm{Ann}(\mathcal{M}) \subset \mathrm{Ann}(\mathcal{B})$. □

Corollary 2.2.25. *If $\mathcal{B} = \ker_{\mathcal{A}}(R)$ and $\mathcal{B}' = \ker_{\mathcal{A}}(R')$ are E-isomorphic behaviors, then $\mathrm{Ann}(\mathcal{B}) = \mathrm{Ann}(\mathcal{B}')$.*

Since the annihilator of a module is preserved under $\mathcal{D}$-isomorphism, both autonomy and the annihilator are E-isomorphism invariants.

2.2.4 Controllable-Autonomous Decomposition

Notation. Let D be a commutative Noetherian domain.

The concept of a controllable-autonomous decomposition started in the one-dimensional setting in [51]; it then became clear that in the multidimensional setting (i.e. when $\mathcal{D}$ is not a principal ideal domain) that the result is much more difficult to achieve.

Definition 2.2.26. Let $\mathcal{B}$ be a behavior; we say that $\mathcal{B}$ has a **controllable-autonomous decomposition** if there are sub-behaviors $\mathcal{B}_c$ and $\mathcal{B}_a$ such that

$$\mathcal{B} = \mathcal{B}_c \oplus \mathcal{B}_a,$$

where $\mathcal{B}_c$ is a controllable behavior and $\mathcal{B}_a$ is an autonomous behavior.

We first consider the case when $d = 1$ and then move onto the more general multidimensional setting.

Case $d = 1$**.**

When $d = 1$, we have that $D = k[z]$ is a principle ideal domain (PID). There is a classic result that is useful in these situations.

Theorem 2.2.27 (Structure Theorem for Finitely Generated Modules over a PID)**.** *Let D be a PID and let M be finitely generated D-module. Then there exists a finite set of elements $q_1, \ldots, q_\ell$ and an integer $n \geq 0$ such that*

$$M \cong D^n \oplus D/(q_1) \oplus \cdots \oplus D/(q_\ell).$$

In particular, M can be decomposed into a direct sum with free part and torsion part.

Proof. (Sketch) Let M be presented by the matrix R with full row rank. By the Hilbert syzygy theorem, such a matrix always exists. It follows that R has a Smith normal form (see [17]). Let $q_1, \ldots, q_\ell$ be the non-zero diagonal elements. One can easily check that

$$M \cong D^n \oplus D/(q_1) \oplus \cdots \oplus D/(q_\ell).$$

□

As previously discussed, controllable behaviors correspond to torsion-free D-modules and autonomous behaviors correspond to torsion D-modules. We thus have the following corollary.

Corollary 2.2.28. *Every one-dimensional behavior exhibits a controllable-autonomous decomposition.*

Case $d > 1$**.**

For a D-module M we denote by tM the **torsion submodule** of M given by

$$tM = \{p \in M : \text{ there exists non-zero } \alpha \in D \text{ such that } \alpha p = 0\}.$$

The search for a controllable-autonomous decomposition can also take place with the dual module of a behavior. The following results from [54] provides us with another view at the decomposition.

Lemma 2.2.29. *Let D be a Noetherian domain and M be a finitely generated D-module. Then there exists a torsion-free submodule $Y \subset M$ such that Y has non-trivial intersection with every torsion-free submodule of M.*

Proof. Let Y be a maximal torsion-free submodule of M. Due to the fact that D is Noetherian, such a module necessarily exists. Such a Y must have a non-trivial intersection with any torsion-free submodule of M. Say that this is not the case. Then there exist a torsion-free submodule $Z \subset M$ such that $Z \cap Y = \{0\}$. We then have $Y \subset Z + Y$ and $Z + Y$ is torsion-free – a contradiction to the maximality of Y. □

Theorem 2.2.30 (Torsion/Torsion-Free Decomposition)**.** *Let D be a Noetherian domain and M be a finitely generated D-module. Then the following holds:*

1. *There exists submodules X and Y of M such that $X \cap Y = 0$, M/X is torsion-free and M/Y is a torsion module.*
2. *A module Y satisfies the conditions of (1) for some X precisely when Y is torsion-free and has the property:*

$$Y \text{ has a non-trivial intersection with every torsion-free submodule of } M. \tag{2.18}$$

   In particular, Y is uniquely determined if and only if M is a torsion module or D is a field.

3. *The unique X satisfying (1) for some Y is the torsion submodule tM of M.*

Proof. **(1)**. Let Y be a submodule of M defined by Lemma 2.2.29. Choose $X = tM$. We now show that $X \cap Y = \{0\}$. Say that $p \in X \cap Y$ and $p \neq 0$. Then there exists $\alpha \in D$ such that $\alpha p = 0$. But then Y contains a torsion element – a contradiction. This demonstrates $X \cap Y = \{0\}$. It is also clear that M/X is torsion free by the definition of X.

We now show that for any $m \in M$ there exists non-zero $r \in D$ such that $rm \in Y$. There are two possibilities: (a) either $m \in tM$ in which case there exists non-zero $r \in D$ such that $rm = 0$, or (b) m is a torsion-free element and thus by Lemma 2.2.29 generates a torsion-free submodule which has a non-zero intersection with Y. We conclude with M/Y as a torsion module.

(2). The argument in **(1)** shows that such a Y works. We now argue the other direction.

Suppose that we have submodules X and Y such that $X \cap Y = 0$, M/X is torsion-free and M/Y is a torsion module. Let $N \subset M$ be an arbitrary torsion-free submodule and select $n \in$N and observe $Dn \subset N$. Since M/Y is a torsion module there exists non-zero $r \in D$ such that $rn \in Y$. On the other hand, any such r and n has $rn \neq 0$; if this were not the case then n would be a torsion element which would contradict N being torsion-free. This implies that $rn \in Y \cap N$ and thus Y has a non-trival intersection with N. Because N was arbitrary, Y satisfies (2.18).

If $p \in tM \cap Y$ and $p \neq 0$, then, due to the assumption $X \cap Y = \{0\}$, p is a non-zero element of M/X which is a torsion element. However, this contradicts that M/X is torsion-free. We conclude that $tM \cap Y = \{0\}$ and thus Y is torsion-free.

When M is a torsion module, then it is clear that $Y = 0$ is the only choice possible. If D is a field, then, because a field has no torsion, M/Y is not a torsion module unless $Y = M$. In both of these cases we have Y is uniquely defined. We now wish to show that these are the only two cases for which Y can be uniquely determined.

Take $X = tM$ and suppose that $tM \neq 0$. Let Y be any maximal torsion-free submodule of M. If Y is uniquely determined, then Y consists of all torsion-free elements of M; otherwise, we could choose a different Y by arranging for the new Y to contain this additional element. For $x \in X$ and $y \in Y$ both non-zero there exists non-zero $r \in D$ such that $rx = 0$. Since either $x + y$ has torsion or does not have torsion, either $x + y \in X$ or $x + y \in Y$. In the first case, this implies that $y \in X$;

in the latter case, since $rx = 0$ we may observe

$$r(x+y) = rx + ry = ry \qquad \Rightarrow \qquad x + y \in Y,$$

since Y is torsion-free. Since $y \in Y$ it follows that $x+y-y = x \in Y$. This implies that $X \cap Y \neq \{0\}$ – a contradiction. We thus reach that Y is not uniquely determined.

We now consider the other situation when $tM = 0$. Choose a maximal linearly independent subset $\{m_1, \ldots, m_\ell\}$ of M and let $r \in D$ be a non-zero non-unit (because D is not a field, this must exist). Let N be the module spanned by $\{rm_1, m_2, \ldots, m_\ell\}$. For any αm_1 we have that $r\alpha m_1 \in N$; as a result, M/N is a torsion module and $m_1 \notin N$. Then both pairs $(X = 0, Y = M)$ and $(X = 0, Y = N)$ satisfy condition **(1)**.

(3). Let $X \subset M$ be a module satisfying the conditions in **(1)** for some Y. For any non-zero $m \in M/X$, $r(m + X) = X$ implies that $r = 0$ since M/X is torsion-free. As a result, X must contain all torsion elements of M. Now suppose that X contains a non-torsion element m. It must be the case that X contains the torsion-free submodule Dm. By condition (2.18) it must be the case that $X \cap Y \neq \{0\}$ – thus contradicting the assumption that **(1)** is satisfied. We conclude that it must be the case that $X = tM$. $\square$

Chapter 3

Autonomy Degree

Notation. Let k be an algebraically closed field, $\mathcal{D} = k[z_1, \ldots, z_d]$ for some $d \in \mathbb{N}$ and $\mathcal{A} = k^{\mathbb{N}^d}$.

Autonomy degree is quite possibly the most important tool used in the following arguments to connect discrete-time systems to algebraic geometry. It builds on the fact that the dimension of an affine ring is given by the transcendence degree of the field extension $k \subset Q(\mathcal{D}/I)$. Recall the contents of Section 2.1.7 where we discussed the role of the initial condition set in the solution of canonical Cauchy problem. In particular, the initial condition set provides a convenient parametrization of system trajectories even in the absence of free variables. It becomes crucial to understand "how big" the initial condition set is and what it means in terms of the system's memory. Autonomy degree provides characterization of the size of this set.

In this section, we present a way of relating the size of a behavior's initial condition set to the height of its annihilator, i.e. the size of a *discrete* object is related to the codimension of a possibly *infinite* algebraic set. While Wood, Rogers and Owens [53] introduced and studied to some extent the notion of autonomy degree, as far as the author is aware, the notion has not reappeared in the literature with the exception of an occasional recall of the definition or a remark. One of the contributions of this dissertation is to further develop the behavioral structure associated with autonomy degree. Once we finish reviewing the existing theory, we indicate the inadequacies of the original definitions due to the lack of a connection to the geometry of behaviors; to alleviate these shortcomings, we provide a notion of **pure autonomy degree** which is more suitable for the geometric approach toward behaviors which we take here. (This is analogous to a pure dimensional ideal in the context of primary decomposition.) We then provide a decomposition of behaviors according to pure autonomy degree and conclude with some conditions for the decomposition to be a direct sum decomposition.

3.1 Autonomy Degree

In [53] **primeness degree** is defined and studied in two ways. Left primeness degree is used to relate primeness aspects of kernel representations to the height of their determinantal ideals. Right primeness degree is used to relate the size of the initial condition set of a behavior to the height of its annihilator. Since this section is focused on autonomous behaviors, we only discuss right primeness degree; however, we choose a different manner of presentation than [53]. Instead of defining dimension via the size of a sublattice on which a component is free, we relate the discussion to cw-ideals and their associated monomial ideals. This approach provides us with a way of defining dimension for arbitrary sets in a consistent way – even when they don't properly contain a lattice. We begin with the following definition (which the author hopes does not cause confusion in the context of a "d-dimensional system").

We refer the reader to Appendix A.3 for details on the dimension of a ring and height of an ideal.

Definition 3.1.1. For a behavior $\mathcal{B}$, we define the **dimension** of $\mathcal{B}$, denoted as $\dim(\mathcal{B}) = \dim(\mathcal{D}/\mathrm{Ann}(\mathcal{B}))$.

The next goal is to relate a behavior's dimension (an algebro-geometric concept) with the size of an initial condition set for the behavior. Recall from Section 2.2.3 that a subset $G \subset \mathbb{N}^d$ is a cw-ideal if for any $p \in G$ we have $p + \mathbb{N}^d \subset G$. We also need a special name for the complement of a cw-ideal.

Definition 3.1.2. We call a subset $G \subset \mathbb{N}^d$ a **staircase** if $\mathbb{N}^d \setminus G$ is a cw-ideal.

In the literature, cw-ideals are normally referred to as staircases; however, staircase seems appropriate for this setting and so we refer to them as such. To handle arbitrary subsets, we have the following.

Definition 3.1.3. For a subset $G \subset \mathbb{N}^d$ we define the **staircase closure** of G, denoted $\overline{G}$, as

$$\overline{G} = \mathbb{N}^d \setminus \left(\bigcup_{g \in \{p \in \mathbb{N}^d : (p+\mathbb{N}^d) \cap G = \emptyset\}} g + \mathbb{N}^d \right).$$

The following result demonstrates that the staircase closure is consistent with properties one expects with a closure operation.

Lemma 3.1.4. *For $G \subset \mathbb{N}^d$ we have that $\overline{G}$ is a staircase, $G \subset \overline{G}$ and if G is staircase then $G = \overline{G}$.*

Proof. By definition

$$\overline{G}^c := \mathbb{N}^d \setminus \overline{G} = \bigcup_{g \in \{p \in \mathbb{N}^d : (p+\mathbb{N}^d) \cap G = \emptyset\}} g + \mathbb{N}^d.$$

For any $p \in \overline{G}^c$ there exists $g \in \mathbb{N}^d$ such that $(g + \mathbb{N}^d) \cap G = \emptyset$ and $p \in g + \mathbb{N}^d$; as a consequence, $p + \mathbb{N}^d \subset g + \mathbb{N}^d \subset \overline{G}^c$, thus demonstrating that $\overline{G}^c$ is a cw-ideal.

We now show $\overline{G}^c \subset G^c$. For any $p \in \overline{G}^c$ there exists $g \in \mathbb{N}^d$ such that $(g + \mathbb{N}^d) \cap G = \emptyset$ and $p \in g + \mathbb{N}^d$. Since $\{p\} \subset g + \mathbb{N}^d$ we have $\{p\} \cap G \subset (g + \mathbb{N}^d) \cap G = \emptyset$, thus demonstrating $p \notin G$ and, hence, $p \in G^c$. By demonstrating $\overline{G}^c \subset G^c$ by complementing we reach $G \subset \overline{G}$.

If G is already a staircase, then for every $p \in G^c$ we have $p + \mathbb{N}^d \subset G^c$ and thus $(p + \mathbb{N}^d) \cap G \subset G^c \cap G = \emptyset$. This demonstrates $p \in \overline{G}^c$, $G^c \subset \overline{G}^c$ and hence $\overline{G} \subset G$. From the previously observed inclusion $G \subset \overline{G}$ we conclude that $G = \overline{G}$. $\square$

Definition 3.1.5. Let $u \subset \{z_1, \dots, z_d\}$ be a non-empty subset of the indeterminates of $\mathcal{D}$. For $u = \{z_{i_1}, \dots, z_{i_{|u|}}\}$ we define the **sublattice**

$$\mathcal{L}[u] = \mathrm{span}_{\mathbb{N}}\{e_{i_1}, \dots, e_{i_{|u|}}\} \subset \mathbb{N}^d,$$

were $e_1, \dots, e_d$ are the standard basis vectors of $\mathbb{N}^d$. In this case we say that $\mathcal{L}[u]$ is an **$|u|$-dimensional sublattice** of $\mathbb{N}^d$. For $u = \emptyset$ we define the 0-dimensional lattice as being the point $0 \in \mathbb{N}^d$.

Recall from Section 2.1.7 that an initial condition set $\partial\mathcal{B}$ for a behavior $\mathcal{B}$, by construction, has a complement in $\mathbb{N}^d$ which is a cw-ideal and hence is *always* a staircase.

Definition 3.1.6. For a staircase $G \subset \mathbb{N}^d$ we define the **dimension** of G to be the largest value ℓ for which there is an ℓ-dimensional sublattice contained in G. We denote the dimension of G by $\dim(G)$. For an arbitrary set $G' \subset \mathbb{N}^d$, we define $\dim(G') = \dim(\overline{G'})$ where $\overline{G'}$ is the staircase closure as in Definition 3.1.3. For $G'' = \cup_{i=1}^q \{i\} \times G''_i \subset (\mathbb{N}^d)^q$, we define $\dim(G'')$

$$\dim(G'') = \max_{1 \le i \le q} \dim(G''_i).$$

The reasoning for this definition is the following connection to monomial ideals. We refer the reader to Appendix A.1 for a review of some terminology on Dickson bases and associated terminology. We also refer the reader to Appendix A.3 for background information on the dimension of an ideal; in particular, Lemma A.3.4.

Lemma 3.1.7. *Let $G \subset \mathbb{N}^d$ be a staircase and $B = \{b_1, \dots, b_r\}$ be the Dickson basis of $\mathbb{N}^d \setminus G$. For the monomial ideal $I = \langle \mathbf{z}^{b_1}, \dots, \mathbf{z}^{b_r} \rangle$ we have $\dim(I) = \dim(G)$.*

Proof. Let $u \subset \{z_1, \dots, z_d\}$ be a set of indeterminates for which $\mathcal{L}[u] \subset G$ is a maximal sublattice contained in G so that $\dim(G) = |u|$. We now argue that $k[u] \cap I = 0$. Say that this is not the case. Then, there exists a polynomial $f \in I$ such that $\mathrm{LM}(f) \in L(I)$ and $\mathrm{LM}(f) \in k[u]$. By the construction of I, we thus have $\mathrm{LE}(f) \notin G$ and hence $\mathcal{L}[u] \not\subset G$. This demonstrates that u is also an independent set of I and hence $\dim(I) \ge \dim(G)$.

Choose a maximal independent set of indeterminates $v \subset \{z_1, \dots, z_d\}$ so that $k[v] \cap I = 0$ and $\dim(I) = |v|$. Then for every $f \in I$ we have $\mathrm{LM}(f) \notin k[v]$. However, this then means $\mathrm{LE}(f) \notin \mathcal{L}[v]$ for all $f \in I$. By construction of I, this means $\mathcal{L}[v] \subset G$. We conclude that $\dim(I) \le \dim(G)$. From the previous inequality this leads to $\dim(I) = \dim(G)$. $\square$

Definition 3.1.8 (Autonomy Degree)**.** For a non-trivial d-dimensional behavior $\mathcal{B} \subset \mathcal{A}^q$ we define its **autonomy degree** as $d - \dim(\partial\mathcal{B})$. We denote the autonomy degree of $\mathcal{B}$ as $\mathrm{adeg}(\mathcal{B})$. Since $\partial\mathcal{B}$ is the disjoint union of cw-ideals, we have that $\mathrm{adeg}(\mathcal{B})$ is equal to d minus the dimension of the largest sublattice onto which some component of $\mathcal{B}$ projects freely.

The following theorem of Wood, Rogers and Owens establishes the equality of $\mathrm{adeg}(\mathcal{B})$ (the size of a discrete object—a subset of $\mathbb{N}^d$) and $\dim(\mathcal{B})$ (the size of a continuous object—the zero set of a collection of polynomials). In light of the canonical Cauchy problem, one can regard this as an extension of Lemma 3.1.7 to the module setting.

Lemma 3.1.9. *[53, pg. 67-69] Let $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^q$ be a non-trivial d-dimensional behavior over $\mathcal{D} = k[z_1, \ldots, z_d]$ with $R \in \mathcal{D}^{p\times q}$. It follows that $\mathcal{B}$ has autonomy degree greater than j, $0 \leq j \leq d$, if and only if for all $u \subset \{z_1, \ldots, z_d\}$ with $|u| = d - j$, $\mathrm{Ann}(\mathcal{B}) \cap k[u] \neq 0$.*

Proof. ($\Leftarrow$). For this argument we choose an arbitrary global ordering of $\mathcal{D}$. Define $\ell = d - j$ and let L be an ℓ-dimensional sub-lattice of $\{1, \ldots, q\} \times \mathbb{N}^d$. Also let $u \subset \{z_1, \ldots, z_d\}$ be the set of indeterminates so that $L = \mathcal{L}[u]$.

Since $\mathrm{Ann}(\mathcal{B}) \cap k[u] \neq 0$, there exists a non-zero element p contained in both sets. By definition, $p(\mathcal{B}) = 0$, so p defines a dependence relation on all components of $\mathcal{B}$, regardless of their particular index. In particular, for $w = (w_1, \ldots, w_q) \in \mathcal{B}$ we have $pw_i = 0$ and thus $\mathrm{LM}(p)w_i = \mathrm{tail}(p)w_i$ for all $i = 1, \ldots, q$. Since p only consists of elements from $k[u]$, $\mathcal{B}$ cannot be free on L due to the fact that any trajectory which does not satisfy $\mathrm{LM}(p)w = \mathrm{tail}(p)w$ cannot be contained in $\mathcal{B}$. In particular, $L \not\subset \partial\mathcal{B}$. Since L was arbitrary, we may conclude $\dim(\partial\mathcal{B}) < \ell$. Since $\ell = d-j$ we reach

$$j < d - \dim(\partial\mathcal{B}) = \mathrm{adeg}(\mathcal{B}).$$

($\Rightarrow$). Suppose $\mathrm{adeg}(\mathcal{B}) > s$ and let $u \subset \{z_1, \ldots, z_d\}$ be such that $|u| = \ell = d-s$. We now define the monomial ordering $\geq_0$ on $(k[u])^q$. Let $e_1, \ldots, e_q$ be the standard basis vectors for the free module $(k[u])^q$. For any $x_1, x_2 \in \mathcal{D}$ and $1 \leq i, j \leq q$, define

$$x_1 e_i \geq_0 x_2 e_j \qquad \Longleftrightarrow \qquad \begin{cases} \deg(x_1) > \deg(x_2) \\ \deg(x_1) = \deg(x_2) \text{ and } i > j \\ \deg(x_1) = \deg(x_2) \text{ and } i = j \text{ and } x_1 \geq_l x_2 \end{cases}$$

where $\geq_l$ is the lexicographic ordering on $k[u]$.

We will now construct the monomial ordering $\geq_e$ on $\mathcal{D}^q$ as follows. For any two monomials y_1x_1, $y_2x_2 \in \mathcal{D}^q$ where $y_1, y_2 \in k[\{z_1, \ldots, z_d\} \setminus u]$ and $x_1, x_2 \in (k[u])^q$, we define

$$y_1x_1 \geq_e y_2x_2 \qquad \Longleftrightarrow \qquad \begin{cases} \deg(y_1) > \deg(y_2) \\ \deg(y_1) = \deg(y_2) \text{ and } x_1 \geq_0 x_2. \end{cases}$$

Let G be a Gröbner basis of $\mathrm{im}_{\mathcal{D}}(R^T)$ under the ordering $\geq_e$. Recall from Section 2.1.7 that the monomials of $\mathcal{D}^q$ which are *not* multiples of the leading terms of the elements of G correspond to points in $\partial\mathcal{B}$.

Let L be the sub-lattice of $\mathbb{N}^d$ corresponding to the chosen indeterminates u. Define for each $1 \leq i \leq q$ the sub-lattice $L_i = \{i\} \times L$. Since $\mathrm{adeg}(\mathcal{B}) > s$, by definition $\dim(\partial\mathcal{B}) < d-s$. However, since $\dim(L) = d - s$ we have $L \not\subseteq \partial\mathcal{B}$. In particular, for each i there exists a point in L_i for which there exists an element $x_i \in G$ which has a leading term corresponding to this point. It follows that the initial term of each x_i is contained in $(k[u])^q$. But by the used monomial ordering, all of the terms of x_i must then be in $(k[u]^q)$.

For the x_i as discussed, let Q be a $q \times q$ matrix with the i^{th} column consisting of $x_i \in (k[u]^q)$. Since each x_i is an element of G, we have $\mathrm{im}_{\mathcal{D}}(Q) \subset \mathrm{im}_{\mathcal{D}}(R^T)$ and, as a consequence, $\mathrm{Ann}_{\mathcal{D}}(Q) \subset \mathrm{Ann}_{\mathcal{D}}(R^T)$. By Lemma A.5.3, we have $\det(Q) \in \mathrm{Ann}_{\mathcal{D}}(Q)$ (note that, in this case, the 0^{th} Fitting ideal is the determinant) and thus $\det(Q) \in \mathrm{Ann}_{\mathcal{D}}(R^T)$. By construction, every entry of Q is in $k[u]$, so we have $\det(Q) \in k[u]$. Provided $\det(Q) \neq 0$, we are finished.

Since all of the entries in Q are contained in $k[u]$, we have that the initial terms of the column vectors under the ordering $\geq_0$ are equal the initial terms under the ordering $\geq_e$. By construction the diagonal entries of Q are non-zero; as result, if we consider the Laplace expansion of Q, one term is equal to the produce of the diagonal entries. Furthermore, due to the ordering, the terms corresponding to the remaining cofactors have strictly smaller degree. It follows that the product of the diagonal term cannot be canceled in the Laplace expansion. As a result, $\det(Q) \neq 0$. We thus conclude that $k[u] \cap \mathrm{Ann}_{\mathcal{D}}(R^T) \neq 0$. By Lemma 2.2.24, this means $k[u] \cap \mathrm{Ann}_{\mathcal{D}}(\mathcal{B}) \neq 0$. $\square$

This leads us to the following.

Theorem 3.1.10. *[53, page 67-69] Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a d-dimensional behavior over $k[z_1, \ldots, z_d]$. It follows that $\mathrm{adeg}(\mathcal{B}) = d - \dim(\mathcal{B})$.*

Proof. Let $\mathrm{adeg}(\mathcal{B}) = n$. Then there exists a maximal independent set $u \subset \{z_1, \ldots, z_d\}$ for which $|u| = d - n$ and $k[u] \cap \mathrm{Ann}(\mathcal{B}) = 0$. (Otherwise, we would have $\mathrm{adeg}(\mathcal{B}) > n$.) By Lemma A.3.4 it follows that $\mathrm{adeg}(\mathcal{B}) = d - \dim(\mathcal{B})$. $\square$

As an alternative to the proof of Theorem 3.1.9, one could prove the result by applying Corollary 5.2.4, Lemma 5.2.5, and (since this reduced the problem to the scalar case) the identity $\dim(\mathcal{D}/I) = \dim(\mathcal{D}/L(I))$.

A natural corollary is that autonomy degree is a numerical invariant associated with a behavior. Note, however, that we demonstrate this through its dual module by using the height of its annihilator. It is only in this way that the connection is easily established.

Corollary 3.1.11. *If $\mathcal{B}$ and $\mathcal{B}'$ are two E-isomorphic behaviors then $\mathrm{adeg}(\mathcal{B}) = \mathrm{adeg}(\mathcal{B}')$.*

Proof. This follows from Corollary 2.2.25 and Theorem 3.1.10. $\square$

Before concluding, we have the following examples.

Example 3.1.12. Consider the two-dimensional behavior $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^2$ with kernel representation

$$R = \begin{bmatrix} x-1 & 0 \\ 0 & y-1 \end{bmatrix}.$$

It is clear that the trajectories are

$$\begin{aligned} \mathcal{B} = \{w = (w_1, w_2) \in \mathcal{A}^q : {} & w_1(t_1, t_2) = w_1(t_1+1, t_2) \\ & \text{and } w_2(t_1, t_2) = w_2(t_1, t_2+1) \text{ for all } (t_1, t_2) \in \mathbb{N}^2\}. \end{aligned}$$

The first component is free on the y-axis and the second component is free on the x-axis. In this way, we see that the autonomy degree of $\mathcal{B}$ is one.

Since the annihilator of $\mathcal{B}$ is $\langle -xy+x+y-1 \rangle = \langle x-1 \rangle \cap \langle y-1 \rangle$, it follows that $\mathrm{height}(\mathrm{Ann}(\mathcal{B})) = 1$.

Example 3.1.13. Consider the two-dimensional behavior $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^2$ with kernel representation

$$R = \begin{bmatrix} 5x^2+4xy+y^2 & 9x^2+3xy+4y^2 \\ 7x & 7x+2y \end{bmatrix}.$$

The annihilator of $\mathcal{B}$ is $\langle 28x^3 - 17x^2y^2 + 13xy^2 - 2y^3 \rangle$, which is a prime ideal given by an irreducible polynomial. As a result, $\mathcal{B}$ has autonomy-degree-one. The initial condition set is

$$\begin{aligned} 1^{st} \text{ Component} &: \{(a,b) \in \mathbb{N}^2 : b = 0\} \\ 2^{nd} \text{ Component} &: \{(a,b) \in \mathbb{N}^2 : b \leq 1\}. \end{aligned}$$

Remark. The notion of dimension for subsets of $\mathbb{N}^d$ is motivated by the dimension of monomial ideals. When we take the staircase closure, the Dickson basis for the complement provides a monomial ideal. The dimension of this ideal is the dimension which we assign to the subset. This way, we have a notion of definition which is motivated by the dimension of monomial ideals. In Section 5.3 we use this general form of dimension so that we can relate initial condition sets between the Cauchy problem for two different scalar behaviors.

3.2 Pure Autonomy Degree

Recall that every behavior exhibits what is known as a **controllable-autonomous** decomposition, i.e., for a behavior $\mathcal{B}$ there exists a unique controllable behavior $\mathcal{B}_c$ and an autonomous behavior $\mathcal{B}_a$ such that $\mathcal{B} = \mathcal{B}_c + \mathcal{B}_a$. The autonomous behavior provided by the decomposition may exhibit various degrees of torsion, i.e., it may be composed of ideals of various dimensions. From the previous section, we saw that the autonomy degree of the behavior characterizes how much torsion is present; for instance, an autonomy degree one behavior is associated with an algebraic set of codimension one. In this section we provide a method for decomposing a behavior into pieces corresponding to these various degrees of autonomy. In this way, we can study behaviors over irreducible hypersurfaces or at least hypersurfaces of the same dimension.

Definition 3.2.1. Let $\mathcal{B}$ be a given behavior which is also a torsion module. We say that $\mathcal{B}$ has pure autonomy degree k if in a minimal primary decomposition of $\mathrm{Ann}(\mathcal{B})$, every ideal has codimension k. If $\mathcal{B}$ is a torsion-free module, then $\mathcal{B}$ is defined as having pure autonomy degree 0. If $\mathcal{B} = 0$, then we say that $\mathcal{B}$ has pure autonomy degree ∞.

Thus $\mathcal{B}$ has pure autonomy degree k if and only if $\mathcal{V}(\mathrm{Ann}(\mathcal{B}))$ is comprised of irreducible varieties of codimension k. We now break the behavior up into pieces corresponding to each irreducible component of the characteristic variety.

For $\mathrm{Ann}(\mathcal{B}) = \mathrm{Ann}(\mathcal{M})$ we may use the Lasker-Noether Theorem to produce a finite number of primary ideals $U_1, \ldots, U_k \subset \mathcal{D}$ such that

$$\mathrm{Ann}(\mathcal{B}) = U_1 \cap \cdots \cap U_k$$

The primary ideals in the above decomposition are not necessarily unique. As pointed out in Section A.4, there may be "embedded primes." Since each U_i has an associated dimension we may define

$$J_i = \bigcap_{\text{height}(U_j)=i} U_j. \tag{3.1}$$

In this way we may write

$$\text{Ann}(\mathcal{B}) = J_0 \cap J_1 \cap \cdots \cap J_d$$

where $\text{height}(J_i) = i$. This provides a decomposition of the annihilator into pieces of distinct dimension. We then define

$$\mathcal{B}_i = \{w \in \mathcal{B} : pw = 0 \text{ for all } p \in J_i\}. \tag{3.2}$$

By construction, we have that each $\mathcal{B}_i$ has pure autonomy degree i. The next result demonstrates that $\mathcal{B}_i$ is a behavior.

Lemma 3.2.2. *The set $\mathcal{B}_i$ defined as in* (3.2) *is a behavior with a kernel representation given by*

$$R_i = \begin{bmatrix} R \\ p_1^{(i)} I_{q\times q} \\ \vdots \\ p_s^{(i)} I_{q\times q} \end{bmatrix},$$

where the set of polynomials $\{p_1^{(i)}, \ldots, p_{s_i}^{(i)}\}$ is a generating set for the ideal J_i. As a direct consequence of this construction, $\text{Ann}(\mathcal{B}_i) = J_i$.

Proof. Notice that $\ker(R_i) \subset \ker(R)$ because the row module of R_i contains the row module of R. Clearly $\widetilde{\mathcal{B}} := \ker(R_i)$ is a behavior. We now argue that $\widetilde{\mathcal{B}} = \mathcal{B}_i$.

First we prove that $\widetilde{\mathcal{B}} \subset \mathcal{B}_i$. By construction, for $w \in \widetilde{\mathcal{B}}$ we have $Rw = 0$ since R_i appears as the first row of R, so $w \in \mathcal{B}$. Furthermore, $p_j^{(i)} I_{q\times q} w = p_j^{(i)} w = 0$ for all $j = 1, \ldots, s_i$. Let $p \in J_i$ be given. Since $J_i = (p_1^{(i)}, \ldots, p_{s_i}^{(i)})$, there exists $\alpha_1, \ldots, \alpha_s$ such that

$$p = \alpha_1 p_1^{(i)} + \cdots \alpha_s p_{s_i}^{(i)}.$$

Since $p_j^{(i)} w_j = 0$ for all $j = 1, \ldots, s_i$, it follows that $pw = 0$ for every $p \in J_i$. This shows that $\widetilde{\mathcal{B}} \subset \mathcal{B}_i$.

Conversely, for $w \in \mathcal{B}_i$, $p_j^{(i)} w = 0$ for all $j = 1, \ldots, s_i$ since each $p_j^{(i)} \in J_i$ and $w \in \mathcal{B}_i$; this demonstrates $R_i w = 0$. We conclude that $\mathcal{B}_i = \widetilde{\mathcal{B}}$. Since $\widetilde{\mathcal{B}}$ is a behavior, we see that $\mathcal{B}_i$ is a behavior with a kernel representation as stated. By construction $\text{Ann}(\mathcal{B}_i) = J_i$.

∎

We now show show that every behavior exhibits a decomposition into behaviors of pure autonomy degree. For an ideal $I = \langle p_1, \ldots, p_k \rangle \subset \mathcal{D}$ and signal space $\mathcal{A}^q$ define the matrix

$$\Lambda_q(I) = \begin{bmatrix} p_1 I_{q\times q} \\ \vdots \\ p_k I_{q\times q} \end{bmatrix}. \tag{3.3}$$

Theorem 3.2.3. *Let $\mathcal{B} \subset \mathcal{A}^q$ be a non-trivial n-dimensional behavior. There exists a set of sub-behaviors $\mathcal{B}_0, \ldots, \mathcal{B}_d$ such that*

$$\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 + \cdots + \mathcal{B}_d,$$

where each $\mathcal{B}_i$ has pure autonomy degree i. If $\mathcal{B}_0 = 0$ and $\mathrm{Ann}(\mathcal{B})$ *has no embedded primes, then this decomposition is unique.*

Proof. **Case 1 ($\mathcal{B}$ is a torsion module):** Because $\mathcal{B}$ is a torsion module, $\mathrm{Ann}(\mathcal{B}) \neq 0$. Let $U_1, \ldots, U_k$ be the primary decomposition and $J_1, \ldots, J_d$ be defined as in (3.1) so that we have the decomposition

$$\mathrm{Ann}(\mathcal{B}) = J_1 \cap \cdots \cap J_d,$$

where each J_i has dimension i. With this primary decomposition in place, define $\mathcal{B}_1, \ldots, \mathcal{B}_d$ as in the construction of Lemma 3.2.2 so that each $\mathcal{B}_i$ has pure autonomy degree i. It remains to show that $\mathcal{B} = \mathcal{B}_1 + \cdots + \mathcal{B}_d$.

By linearity, $\mathcal{B}_1 + \cdots + \mathcal{B}_d \subset \mathcal{B}$ since each $\mathcal{B}_j$ is a sub-behavior of $\mathcal{B}$ for $j = 1, \ldots, n$. We now show that $\mathcal{B}_1 + \cdots + \mathcal{B}_d = \mathcal{B}$ by using the interconnection of systems[1]. Recall that for two behaviors $\mathcal{B}'$ and $\mathcal{B}''$ the following relationships hold:

$$\begin{aligned}(\mathcal{B}' + \mathcal{B}'')^\perp &= (\mathcal{B}')^\perp \cap (\mathcal{B}'')^\perp, \\ (\mathcal{B}' \cap \mathcal{B}'')^\perp &= (\mathcal{B}')^\perp + (\mathcal{B}'')^\perp.\end{aligned} \tag{3.4}$$

An application of (3.4) implies that

$$\mathcal{B}_i^\perp = (\mathcal{B} \cap \ker_{\mathcal{A}}(\Lambda_q(J_i)))^\perp = \mathcal{B}^\perp + \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_i)) \qquad i = 1, \ldots, d.$$

As a result,

$$\begin{aligned}(\mathcal{B}_1 + \cdots + \mathcal{B}_d)^\perp = \mathcal{B}_1^\perp \cap \cdots \cap \mathcal{B}_d^\perp &= (\mathcal{B}^\perp + \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_1))) \cap \cdots \cap (\mathcal{B}^\perp + \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_d))) \\ &= \mathcal{B}^\perp + \left(\bigcap_{i=1}^n \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_i))\right) = \mathcal{B}^\perp + \mathrm{im}_{\mathcal{D}}(\Lambda_q(\mathrm{Ann}(\mathcal{B}))).\end{aligned}$$

Since $\mathcal{B}$ is a torsion module, $\mathcal{B} \subset \ker_{\mathcal{A}}(\Lambda_q(\mathrm{Ann}(\mathcal{B})))$ and hence

$$(\mathcal{B}^\perp + \mathrm{im}_{\mathcal{D}}(\Lambda_q(\mathrm{Ann}(\mathcal{B}))))^\perp = \mathcal{B} \cap \ker_{\mathcal{A}}(\Lambda_q(\mathrm{Ann}(\mathcal{B}))) = \mathcal{B}.$$

[1] See [57] for a thorough treatment of this topic.

We conclude that

$$(\mathcal{B}_1 + \cdots + \mathcal{B}_d)^\perp = \mathcal{B}^\perp \qquad \Rightarrow \qquad \mathcal{B} = \mathcal{B}_1 + \cdots + \mathcal{B}_d.$$

Case 2 ($\mathcal{B}$ is a torsion-free): If $\mathcal{B}$ is torsion-free, then $\mathrm{Ann}(\mathcal{B}) = 0$. Take $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{B}_1, \ldots, \mathcal{B}_d = 0$ to arrive at the desired decomposition.

Case 3 ($\mathcal{B}$ is a general behavior): First we handle the $\mathcal{B}_0$ piece. Every behavior exhibits a controllable-autonomous decomposition $\mathcal{B} = \mathcal{B}_c + \mathcal{B}_a$ where $\mathcal{B}_c$ is controllable and $\mathcal{B}_a$ is autonomous. Let $\mathcal{B}_0 = \mathcal{B}_c$. Then since $\mathcal{B}_0$ is torsion-free, $\mathrm{Ann}(\mathcal{B}_0) = 0$. We may apply **Case 1** to $\mathcal{B}_a$ to find the sub-behaviors $\mathcal{B}_1, \ldots, \mathcal{B}_d$ to reach the desired decomposition. □

The pure autonomy degree of a behavior is easily computed through the primary decomposition of the annihilator. Consider the following example.

Example 3.2.4. Consider the three-dimensional behavior $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^3$ where

$$R = \begin{bmatrix} xyz - x & yz & -1 \\ y^2 + x & y - 2 & 0 \\ x + y & 0 & z^3 \\ 0 & xy & 1 \end{bmatrix}.$$

The annihilator is given by the intersection of the prime ideals

$$\begin{aligned} I_1 = \big\langle & xy^3 - xy^2z + y^3z + x^2y + 3xyz + xy - 2x,\ x^2yz^4 - x^2z^3 + x^2 + xy + xz + yz, \\ & xy^2z^3 + x^2z^3 - xy^2 + xyz - y^2z - x^2 - 3xz + y - 2\big\rangle \\ I_2 = \langle & x, y\rangle\,. \end{aligned}$$

We can verify that $\dim(I_1) = 1$ and $\dim(I_2) = 1$; it follows that $\mathcal{B}$ has pure autonomy degree 1 and in fact, as an ideal, $\mathrm{Ann}(\mathcal{B})$ is pure dimensional.

Remark. Note that the pure-autonomy-degree decomposition of a behavior is a generalization of the controllable-autonomous decomposition. A natural consequence of the primary decomposition is that the absence of embedded primes is a sufficient condition for the decomposition to be unique.

3.3 Direct Sum Decomposition

We wish to derive sufficient conditions for the pure autonomy degree decomposition in Theorem 3.2.3 to be a direct sum. Let us begin with the following consequence of (3.4) (here we use the notation (3.3)):

$$\begin{aligned} (\mathcal{B}_i \cap \mathcal{B}_j)^\perp &= (\mathcal{B}^\perp + \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_i))) + (\mathcal{B}^\perp + \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_j))) \\ &= \mathcal{B}^\perp + (\mathrm{im}_{\mathcal{D}}(\Lambda_q(J_i)) + \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_j))). \end{aligned}$$

By general duality principles, $(\mathcal{B}_i \cap \mathcal{B}_j) = 0$ if and only if $(\mathcal{B}_i \cap \mathcal{B}_j)^\perp = \mathcal{D}^q$. If $\mathcal{B}$ is a non-zero behavior, then $\mathcal{B}^\perp$ is a proper subset of $\mathcal{D}^q$. It follows that $(\mathcal{B}_i \cap \mathcal{B}_j) = 0$ if and only if

$$\mathcal{D}^q = \mathcal{B}^\perp + (\mathrm{im}_{\mathcal{D}}(\Lambda_q(J_i)) + \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_j))) \tag{3.5}$$

This leads us to a necessary and sufficient condition for a direct sum decomposition.

Lemma 3.3.1. *For a behavior $\mathcal{B}$, two non-trivial sub-behaviors $\mathcal{B}_i$ and $\mathcal{B}_j$, $i \neq j$, in the decomposition*

$$\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 + \cdots + \mathcal{B}_d,$$

have trivial intersection if and only if

$$\mathcal{D}^q = \mathcal{B}^\perp + (\mathrm{im}_{\mathcal{D}}(\Lambda_q(J_i)) + \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_j))).$$

Computationally, the above condition is easy to verify; however, it is very difficult to characterize. In particular, we wish to find a geometric condition for a direct sum decomposition. Let us begin with the following result.

Lemma 3.3.2. *Let $J_i, J_j \subset \mathcal{D}$ be two coprime ideals respectively associated with the behaviors $\mathcal{B}_i$ and $\mathcal{B}_j$. Then $\mathcal{B}_i \cap \mathcal{B}_j = 0$.*

Proof. If J_i and J_j are coprime, then $J_i + J_j = \mathcal{D}$. We have that the images of $\Lambda_q(J_i)$ and $\Lambda_q(J_j)$ are

$$\mathrm{im}_{\mathcal{D}}(\Lambda_q(J_i)) = \bigoplus_{k=1}^{q} J_i \qquad \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_j)) = \bigoplus_{k=1}^{q} J_j.$$

It easily follows that

$$\mathrm{im}_{\mathcal{D}}(\Lambda_q(J_i)) + \mathrm{im}_{\mathcal{D}}(\Lambda_q(J_j)) = \bigoplus_{k=1}^{q} (J_i + J_j) = \mathcal{D}^q.$$

Using this in (3.5) we reach the desired result. □

We now show that the coprime condition has an interesting geometric interpretation. Recall that $\mathcal{V}(J_i) \cap \mathcal{V}(J_j) = \emptyset$ if and only if $\mathcal{V}(J_i + J_j) = \emptyset$. By the Weak Nullstellensatz, this means that $J_i + J_j = \mathcal{D}$, i.e. J_i and J_j are coprime. This observation leads to the following corollary of Lemma 3.3.2.

Corollary 3.3.3. *If $\mathcal{V}(J_i) \cap \mathcal{V}(J_j) = \emptyset$ for $i \neq j$, then $\mathcal{B}_i \cap \mathcal{B}_j = 0$.*

We also have the converse direction.

Lemma 3.3.4. *If $\mathcal{B}_i \cap \mathcal{B}_j = 0$ for $i \neq j$, then $\mathcal{V}(J_i) \cap \mathcal{V}(J_j) = \emptyset$.*

Proof. From (3.5) we have the explicit form of kernel representation $\mathcal{B}_i \cap \mathcal{B}_j = \ker_{\mathcal{A}} R_{i,j}$, where

$$R_{i,j} := \begin{bmatrix} R \\ \Lambda_q(J_i) \\ \Lambda_q(J_j) \end{bmatrix}.$$

Say that there exists $\boldsymbol{\lambda} \in \mathcal{V}(J_i) \cap \mathcal{V}(J_j)$; since $\mathcal{V}(\mathrm{Ann}(\mathcal{B})) = \mathcal{V}(J_1) \cup \cdots \cup \mathcal{V}(J_d)$, we have $\boldsymbol{\lambda} \in \mathcal{V}(\mathrm{Ann}(\mathcal{B}))$. It follows that there exists an exponential trajectory (see Theorem 4.1.4) $\alpha\boldsymbol{\lambda}^t \in \mathcal{B}$ where $\alpha \in k^q$ is non-zero (hence, even if $\boldsymbol{\lambda} = 0$, $\alpha\boldsymbol{\lambda}^t$ is still a non-trivial trajectory.) But then we have $R_{i,j}\alpha\boldsymbol{\lambda}^t = 0$ and thus $\alpha\boldsymbol{\lambda}^t \in \mathcal{B}_i \cap \mathcal{B}_j$, which contradicts the assumption $\mathcal{B}_i \cap \mathcal{B}_j = 0$. We conclude that it must be the case $\mathcal{V}(J_i) \cap \mathcal{V}(J_j) = \emptyset$ $\square$

We now summarize our results concerning the existence of a direct-sum pure autonomy degree decomposition of a general behavior.

Theorem 3.3.5. *Let $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 + \cdots + \mathcal{B}_d$ be a pure autonomy degree decomposition of an arbitrary behavior. If $\mathrm{Ann}(\mathcal{B}) \neq \langle 0 \rangle$, then the decomposition is a direct sum decomposition if and only if $\mathcal{V}(\mathrm{Ann}(\mathcal{B}_i)) \cap \mathcal{V}(\mathrm{Ann}(\mathcal{B}_j)) = \emptyset$ for non-zero $i \neq j$.*

Remark. Wood, Rogers, Owens, and Oberst in [52, page 657] provide a primary decomposition for behaviors which deals with associated primes rather than the annihilator. Because autonomy degree is defined by the annihilator, our pure-autonomy degree decomposition does not produce coprimary ideals. However, the motivation for our decomposition becomes apparent in the next section, where we consider exponential trajectories over the characteristic variety. In this setting, we wish to consider an autonomous behavior as exponential trajectories which reside over an algebraic set.

Chapter 4

Exponential Trajectories

In this section we demonstrate that certain geometric properties of behaviors are behavioral invariants; the main tool in this section is exponential trajectories. The exponential trajectories associated with behaviors have been extensively studied in [52, 58, 40]; outside of [51], exponential trajectories have not been attached to the characteristic variety in a systematic way. However, we do note that the approach we present here is well known in topics related to Livšic systems and determinantal representations as seen in [7, 47, 8, 48, 49, 34, 1].

4.1 Exponential Trajectories

Notation. Let k be an algebraically closed field, $\mathcal{D} = k[z_1, \dots, z_d]$, $\mathcal{A} = k^{\mathbb{N}^d}$, and $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^q$ be a behavior.

Note. We refer the reader to (the rather meager) Section A.2 for references to terminology on vector bundles and related matters.

We now discuss a special class of trajectories parametrized by an amplitude vector $\alpha \in \mathbb{C}^q$ and a multivariate frequency $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d$.

Definition 4.1.1. We define an **exponential trajectory** with **frequency** $\boldsymbol{\lambda} \in k^d$ and **amplitude** $\alpha \in k^q$ to be a trajectory of the form

$$w(t_1, \dots, t_d) = \alpha \lambda_1^{t_1} \cdots \lambda_d^{t_d}.$$

A useful characterization of an exponential trajectory is given by the following lemma.

Lemma 4.1.2. *A trajectory w is an exponential trajectory with frequency $\boldsymbol{\lambda}$ if and only if for every $p \in \mathcal{D}$, $p(\boldsymbol{\lambda})w - p(\mathbf{z})w = 0$.*

Proof. ($\Rightarrow$). It is clear that every exponential trajectory satisfies this condition.

($\Leftarrow$). Let $\boldsymbol{\lambda}$ be such that for every $p \in \mathcal{D}$, $p(\boldsymbol{\lambda})w - p(\mathbf{z})w = 0$. Then for every $t \in \mathbb{N}^d$,

$$w(t) = (\mathbf{z}^t w)(0) = w(0)\boldsymbol{\lambda}^t.$$

We conclude that w is an exponential trajectory. $\square$

Definition 4.1.3. Let $\mathcal{B}$ be a behavior. We call $\boldsymbol{\lambda} \in k^d$ a **characteristic point** if $\mathcal{B}$ contains an exponential trajectory with frequency $\boldsymbol{\lambda}$. We denote by $\mathcal{V}(\mathcal{B})$ the set of all characteristic points associated with $\mathcal{B}$ and refer to it as the **characteristic variety** of $\mathcal{B}$.

The following theorem provides several interpretations for the characteristic points of a behavior.

Theorem 4.1.4. *[52, page 639-640] Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$, $\mathcal{M} = \mathrm{Hom}(\mathcal{B}, \mathcal{A})$, $\mathcal{D} = k[\mathbf{z}]$, and $\boldsymbol{\lambda} \in k^d$ be a given point. The following are equivalent:*

- $\boldsymbol{\lambda} \in \mathcal{V}(\mathrm{Ann}(\mathcal{B})) = \mathcal{V}(\mathrm{Ann}(\mathcal{M}))$.
- $R(\boldsymbol{\lambda})$ *has less than full column rank.*
- $\boldsymbol{\lambda}$ *is a characteristic point of* $\mathcal{B}$.

Example 4.1.5. Consider the two-dimensional behavior $\mathcal{B} = \ker_{\mathcal{A}}(R)$ where $R = [z_1]$. It is easy to see that $\mathcal{B} \cong k^{\mathbb{N}}$ since every trajectory must vanish off of the y-axis. The characteristic variety of $\mathcal{B}$ is $\mathcal{V}(z_1) = \{(\lambda_1, \lambda_2) \in k^2 : \lambda_1 = 0\}$.

4.2 Family of Vector Spaces over the Characteristic Variety

Notation. In this section we specialize to the case when $k := \mathbb{C}$, the complex numbers, and continue to use $\mathcal{D} = k[z_1, \ldots, z_d]$ and $\mathcal{A} = k^{\mathbb{N}^d}$.

Note. Parts of this section involve the use of **Fitting invariants**. We refer the reader to Appendix A.5 for definitions and rudimentary background material.

When we discussed the characteristic variety, there were several equivalent conditions which defined characteristic points; in particular, if $\boldsymbol{\lambda} \in \mathcal{V}(\mathcal{B})$ then $\ker(R(\boldsymbol{\lambda})) \neq \{0\}$. Note that the characteristic variety only records the existence of exponential trajectories. However, the actual wave trajectories, i.e., frequency as well as amplitude-vector information, provide important information concerning a behavior. In this section we demonstrate that the amplitudes of the exponential trajectories naturally form a family of vector spaces over the characteristic variety. We then provide conditions for the behavior to exhibit the structure of a holomorphic vector bundle with sections consisting of the amplitude vectors. The results in this section lead to a frequency domain theory of behaviors which is characterized by families of vector spaces over an algebraic set.

Let $\boldsymbol{\lambda} \in \mathcal{V}(\mathcal{B})$ be a characteristic point of the behavior $\mathcal{B}$ with kernel representation $R \in \mathcal{D}^{p \times q}$. By Theorem 4.1.4, there exists linearly independent $w_1, \ldots, w_r \in k^q$ such that

$$\ker(R(\boldsymbol{\lambda})) = \mathrm{span}\,\{w_1, \ldots, w_r\} \subset k^q.$$

Each element of $\ker(R(\boldsymbol{\lambda}))$ is an amplitude vector associated with the frequency $\boldsymbol{\lambda}$. In other words, we may choose any $\alpha \in \ker(R(\boldsymbol{\lambda}))$ and define $w(\mathbf{t}) = \alpha\boldsymbol{\lambda}^{\mathbf{t}}$ to arrive at

$$(R(\boldsymbol{\sigma})w)(\mathbf{t}) = R(\mathbf{t})\alpha\boldsymbol{\lambda}^{\mathbf{t}} = 0 \text{ for all } \mathbf{t}.$$

We now introduce additional structure above $\mathcal{V}(\mathcal{B})$ so that we may study amplitudes and frequencies in tandem.

Definition 4.2.1. For a behavior $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^q$ we define the **family of amplitude vector spaces** as

$$\mathfrak{G}(\mathcal{B}) = \{(\boldsymbol{\lambda}, \alpha) : \boldsymbol{\lambda} \in \mathcal{V}(\mathcal{B}) \text{ and } \alpha \in \ker(R(\boldsymbol{\lambda}))\}$$

with projection map $\pi : \mathfrak{G}(\mathcal{B}) \to \mathcal{V}(\mathcal{B})$ defined as $\pi(\boldsymbol{\lambda}, \alpha) = \boldsymbol{\lambda}$ with topology on $\mathfrak{G}(\mathcal{B})$, respectively $\mathcal{V}(\mathcal{B})$, equal to the subspace topology induced by $k^d \times k^q$, respectively k^d.

Definition 4.2.2. For a family of amplitude vector spaces $\mathfrak{G}(\mathcal{B})$ we define the **minimal rank** as the smallest dimension among the fibers of $\mathfrak{G}(\mathcal{B})$, i.e., $\min_{\boldsymbol{\lambda}\in\mathcal{V}(\mathcal{B})} \dim(\pi^{-1}(\boldsymbol{\lambda}))$.

Note that the minimal rank is computed over $\mathcal{V}(\mathcal{B})$ and, in general, not over k^d.

Locally there is no guarantee that the dimension is constant; however, for future use, it is important to develop a hypothesis for behaviors to exhibit constant rank fibers. Fitting ideals are the tool to determine the rank of the kernel representation at every point in $\mathcal{V}(\mathcal{B})$.

Let $\mathcal{M}$ be the dual module of the behavior $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^q$ where $R \in \mathcal{D}^{p\times q}$. The Fitting filtration provides us with information about the rank of the fibers of $\mathfrak{G}(\mathcal{B})$. Recall that $\mathfrak{F}_0(\mathcal{M}) = \mathfrak{I}_q(R)$ so that we have the filtration

$$\mathfrak{I}_q(R) \subset \cdots \subset \mathfrak{I}_{q-(r-1)}(R) \subset \mathfrak{I}_{q-r}(R) \subset \mathfrak{I}_r(R) \subset \cdots \subset \mathfrak{I}_0(R) = \mathcal{D}.$$

For each point $\boldsymbol{\lambda} \in \mathcal{V}(\mathcal{B})$, we can determine the rank of $R(\boldsymbol{\lambda})$ by finding the location $0 \leq \ell \leq q$ where $\boldsymbol{\lambda} \notin \mathcal{V}(\mathfrak{I}_\ell(R))$. At this particular point, at least one of the $\ell \times \ell$ minors of R has full rank when considered as a matrix with entries in k. As a result, $\dim(\ker(R(\boldsymbol{\lambda}))) = q - \ell$.

Recall that the Fitting ideals of $\mathcal{M}$ do not depend on the choice of presentation matrix. Furthermore, isomorphic finitely generated modules have *equal* Fitting ideals; this observation leads us to the following result.

Lemma 4.2.3. *Let $\mathcal{B} \subset \mathcal{A}^q$ and $\mathcal{B}' \subset \mathcal{A}^{q'}$ be two E-isomorphic behaviors. Then the ranks of the fibers of $\mathfrak{G}(\mathcal{B})$ and $\mathfrak{G}(\mathcal{B}')$ are equal for all points of $\mathcal{V}(\mathcal{B})$. In particular, the common ranks of the fibers is a behavioral invariant. Furthermore, the E-isomorphism $A \in \mathcal{D}^{q'\times q} : \mathcal{B} \to \mathcal{B}'$ induces a k-vector space isomorphism on the fibers of $\mathfrak{G}(\mathcal{B})$.*

Since the rank of the fibers is a behavioral invariant, it is natural to ask if any additional structure remains invariant. For instance, what conditions guarantee that $\mathfrak{G}(\mathcal{B})$ exhibits the structure of a holomorphic vector bundle and is this additional structure a behavioral invariant? Clearly, a necessary condition is that $\mathfrak{G}(\mathcal{B})$ have constant rank. The following lemma presents conditions for this to be true.

Lemma 4.2.4. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a behavior with kernel representation $R \in \mathcal{D}^{p\times q}$ and dual module $\mathcal{M}$ with q generators. If $\sqrt{\mathfrak{F}(M)}$ is type III with integer r such that*

$$\sqrt{\mathfrak{F}_0(\mathcal{M})} = \cdots = \sqrt{\mathfrak{F}_{r-1}(\mathcal{M})} \neq \sqrt{\mathfrak{F}_r(\mathcal{M})} \qquad \text{and} \qquad \mathfrak{F}_r(\mathcal{M}) = \cdots \mathfrak{F}_\ell(\mathcal{M}) = \mathcal{D}$$

and $\operatorname{Ann}(\mathcal{B})$ *is a prime ideal, then* $\dim(\ker(R(\boldsymbol{\lambda}))) = r$ *for all* $\boldsymbol{\lambda} \in \mathcal{V}(\mathcal{B})$.

Proof. Recall that the exponential trajectories are precisely the points $\boldsymbol{\lambda} \in \mathcal{V}(\operatorname{Ann}(\mathcal{B}))$, or equivalently, the points where $\operatorname{rank}(R(\boldsymbol{\lambda})) < \operatorname{rank}(R)$. For the integral domain $\mathcal{D}' = \mathcal{D}/\operatorname{Ann}(\mathcal{B})$, the Fitting filtration associated with $\mathfrak{F}(M \otimes \mathcal{D}')$ is

$$\mathfrak{F}(M \otimes \mathcal{D}') : \mathfrak{F}_0(M \otimes \mathcal{D}') = 0 \subset \cdots \subset \mathfrak{F}_{r-1}(M \otimes \mathcal{D}') = 0 \subset \mathfrak{F}_r(M \otimes \mathcal{D}') = \mathcal{D}' \subset \cdots .$$

Because the transition takes place at r, we see that $\ker(R(\boldsymbol{\lambda})) = r$ on $\mathcal{V}(\mathcal{B})$. □

Recall that any variety may be considered as a topological space with topology inherited from k^d; provided that the variety has no singularities, then it also has the structure of a complex manifold (see [22] for more details on this connection.) If $\mathcal{V}(\mathcal{B})$ is a nonsingular variety, then the additional complex manifold structure combined with the above lemma provides the necessary conditions for $\mathfrak{G}(\mathcal{B})$ to be a holomorphic vector bundle.

Theorem 4.2.5. *Assume the conditions of Lemma 4.2.4 are satisfied and that $\mathcal{V}(\mathcal{B})$ is a connected smooth complex manifold. Then $\mathfrak{G}(\mathcal{B})$ is a holomorphic vector bundle of rank r over $V(\mathcal{B})$. Furthermore, $\mathfrak{G}(\mathcal{B})$ can be viewed as a subbundle of the trivial bundle $\mathcal{V}(\mathcal{B}) \times k^q$.*

Proof. Define $X = \mathcal{V}(\mathcal{B})$ and let $\{\phi_i : U_i \to V_i\}$ be a parametrizing cover for the complex manifold X. Let $\boldsymbol{\lambda} \in U_i$ be a point on X and define $\boldsymbol{\gamma} = \phi_i(\boldsymbol{\lambda})$. Also define $R'(z) = R(\phi_i^{-1}(z))$ to be the local parametrization of the kernel representation. We find a submatrix of $R'(z)$ which provides us with the desired map to k^r.

Because the Fitting filtration is type III, by Lemma 4.2.4, the fibers of $\mathfrak{G}(\mathcal{B})$ have constant rank r over the curve $\mathcal{V}(\mathcal{B})$. We may switch the Fitting filtration to a filtration of determinantal ideals to see that

$$\mathfrak{I}_q(R) = \cdots = \mathfrak{I}_{q-(r-1)}(R) = \operatorname{Ann}(\mathcal{B}) \qquad\qquad \mathfrak{I}_{q-r}(R) = \cdots = \mathfrak{I}_1(R) = \mathcal{D}.$$

The above filtration states that for every point in $\mathcal{V}(\mathcal{B})$, there exists an $(q-r)$-order minor of R which does not vanish at $\boldsymbol{\lambda}$; it follows that there is a $(q-r)$-order minor $h(z)$ of R' which does not vanish at $\boldsymbol{\gamma}$. Choose a $(q-r)\times(q-r)$ submatrix $H(z)$ of $R'(z)$ such that $h(z) = \det(H(z))$. Because $h(z)$ is a holomorphic function, there is an open neighborhood of $\boldsymbol{\gamma}$, $N(\boldsymbol{\gamma})$, for which $h(z)$ does not vanish.

There exists row permutations Λ and column permutations Ω, so that

$$\Lambda R'(z)\Omega = \begin{bmatrix} H(z) & V_2(z) \\ V_3(z) & V_4(z). \end{bmatrix}.$$

Without loss of generality, assume that such permutations are not necessary. Denote the submatrix $V(z) = [H(z) \;\; V_2(z)]$. Because $H(z)$ has full rank on $N(\boldsymbol{\gamma})$, there exists a holomorphic left inverse $H^{-1}(z)$ so that $H^{-1}(z)H(z) = I$. Furthermore, $\det H^{-1}(z) = (h(z))^{-1}$ is a holomorphic function on $N(\boldsymbol{\gamma})$; it follows that $H^{-1}(z)$ is a holomorphic matrix on $N(\boldsymbol{\gamma})$.

On $N(\boldsymbol{\gamma})$, consider the decomposition $w = (y, u) \in \mathfrak{G}(\mathcal{B})$ such that the following is true.

$$H(z)y = V_2(z)u.$$

In particular, $y = H^{-1}(z)V_2(z)u$. We thus have the map

$$\left(p, \begin{bmatrix} I \\ H^{-1}(\phi_i(p))V_2(\phi_i(p)) \end{bmatrix}\right) : \phi_i^{-1}(B(\gamma)) \times k^r \to \phi_i^{-1}(B(\gamma)) \times k^q.$$

Because the fibers of $\mathfrak{G}(\mathcal{B})$ have constant rank r, the above map injects into the fibers of $\mathfrak{G}(\mathcal{B})$ for all points in $N(\boldsymbol{\gamma})$. There is some concern about whether the above mapping maps onto the fibers of $\mathfrak{G}(\mathcal{B})$; however, if there is a zero-input, non-zero output fiber of $\mathfrak{G}(\mathcal{B})$, then the rank of the fiber at this point would be greater than r. It follows that the above map is a bijective map between $\phi^{-1}(B(\gamma)) \times k^r$ and the fibers of $\mathfrak{G}(\mathcal{B})$. We now construct the inverse map.

Consider the map

$$\left(p, \begin{bmatrix} I & 0 \end{bmatrix}\right) : \Gamma(\mathfrak{G}(\mathcal{B}), \phi_i^{-1}(N(\boldsymbol{\gamma}))) \to \phi_i^{-1}(N(\boldsymbol{\gamma})) \times k^r.$$

It follows that

$$\begin{bmatrix} I \\ V_1^{-1}(z)V_2(z) \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ V_1^{-1}(z)V_2(z) & 0 \end{bmatrix}$$

with $z = \phi_i(p)$. We thus have a k-linear biholomorphic map which trivializes the fibers of $\mathfrak{G}(\mathcal{B})$ on $\phi_i^{-1}(N(\boldsymbol{\gamma}))$. We may continue in this fashion to provide $\mathfrak{G}(\mathcal{B})$ with a trivializing cover. □

It turns out that the Fitting invariant condition in the above theorem is a necessary requirement for reaching a vector bundle structure.

Lemma 4.2.6. *If $\mathfrak{G}(\mathcal{B})$ is a holomorphic vector bundle and $\mathcal{V}(\mathcal{B})$ is connected, then it follows that $\sqrt{\mathfrak{F}(\mathcal{M})}$ is type III.*

Proof. If $\mathfrak{G}(\mathcal{B})$ is a vector bundle of rank r, then $\dim(\ker(R(\boldsymbol{\lambda})) = r$ for every $\boldsymbol{\lambda} \in V(\mathcal{B})$. Clearly, this requires the condition $\sqrt{\mathfrak{F}_0(\mathcal{M})} = \cdots = \sqrt{\mathfrak{F}_{r-1}(\mathcal{M})}$ and $\mathfrak{F}_r(\mathcal{M}) = \mathcal{D}$. □

The above demonstrates that the conditions in Theorem 4.2.5 are necessary and sufficient for $\mathfrak{G}(\mathcal{B})$ to be a vector bundle for the case where $\mathcal{V}(\mathcal{B})$ is a complex manifold. We have, in essence, proved the Serre-Swan Theorem for behaviors (see [45, Proposition 4, page 242]). However, the construction of the above vector bundle is done with exponential trajectories rather than sheaves over the affine scheme $\mathrm{Spec}(\mathcal{D}/\mathrm{Ann}(\mathcal{B}))$ as in the Serre-Swan Thoerem. Nevertheless, the following corollary states that the hypotheses and results are the same since in [loc. cit.] it is stated that the sheafification of a module by the structure sheaf is locally free if and only if it the module is projective. In [loc. cit.] it is required that the ring have no nilpotent elements, hence our requirement that the annihilator be a prime ideal; however, if one is willing to take advantage of the full power of schemes, this condition can be relaxed.

Corollary 4.2.7. *Let $\mathcal{B}$ be a behavior with dual module $\mathcal{M}$. Assume that $\mathcal{V}(\mathcal{B})$ is a complex manifold and* $\mathrm{Ann}(\mathcal{B})$ *is a prime ideal. Then $\mathfrak{G}(\mathcal{B})$ is a holomorphic vector bundle of rank r over $V(\mathcal{B})$ if and only if $\mathcal{M} \otimes \mathcal{D}/\mathrm{Ann}(\mathcal{B})$ is a projective $\mathcal{D}/\mathrm{Ann}(\mathcal{B})$-module.*

Proof. This result directly follows from Corollary A.5.10 and Theorem 4.2.5. □

Note that it is necessary (and well-defined) to change the ring for the above condition to be satisfied. In the following section we elaborate more on behaviors over varieties and, equivalently, the dual module viewed as a module over the coordinate ring of the variety.

It is interesting to ask for conditions for the vector bundle to be trivial.

Lemma 4.2.8. *Let $\mathcal{B} \subset \mathcal{A}^q$ be a d-dimensional behavior with dual module $\mathcal{M}$. Then $\mathcal{M}$ is a projective $\mathcal{D}$-module if and only if $\mathfrak{G}(\mathcal{B})$ is a trivial vector bundle over k^d.*

Proof. By the Quillen-Suslin Theorem, $\mathcal{M}$ is projective if and only if $\mathcal{M}$ is free. One can now verify that $\mathcal{M}$ is free if and only if $\mathfrak{G}(\mathcal{B})$ is a trivial vector bundle over k^d. (See [15, page 622-623].) □

Because $\mathcal{D}/\mathrm{Ann}(\mathcal{B})$ is not a projective-free ring, projective does not provide us with a trivial vector bundle as in the the above situation. However, there are some simple situations when this does occur.

Lemma 4.2.9. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a behavior and let $\mathcal{M}$ be its dual module, $\phi : \mathcal{D} \to \mathcal{D}/\mathrm{Ann}(\mathcal{B})$ denote the canonical quotient map, and $\mathrm{Ann}(\mathcal{B})$ be a prime ideal. The following are equivalent.*

1. *$\mathfrak{F}(\mathcal{M} \otimes \mathcal{D}/\mathrm{Ann}(\mathcal{B}))$ is type I.*
2. *$\mathcal{M} \otimes \mathcal{D}/\mathrm{Ann}(\mathcal{B})$ is a free $\mathcal{D}/\mathrm{Ann}(\mathcal{B})$-module.*
3. *Provided $\mathcal{V}(\mathcal{B})$ is a complex manifold, then $\mathfrak{G}(\mathcal{B})$ is a trivial vector bundle.*

Proof. The equivalence of (1) and (2) is is immediate from Proposition A.5.5 and Lemma A.5.9. The relation between (1) and (3) follows as in the proof of Lemma 4.2.8. □

Remark. One shortcoming of the above results is that they all require $\mathcal{V}(\mathcal{B})$ to be a complex manifold. It is well known that any projective algebraic variety can be repeatedly blown up to produce a smooth biholomorphically equivalent non-singular algebraic variety $\widehat{\mathcal{V}(\mathcal{B})}$. In this situation it is more reasonable to ask for $\mathfrak{G}(\mathcal{B})$ to *lift* to a holomorphic vector bundle over $\widehat{\mathcal{V}(\mathcal{B})}$. In the case $d = 2$, the construction of $\widehat{\mathcal{V}(\mathcal{B})}$ (called the normalizing Riemann surface) is particularly explicit via utilization of the local Weierstrass polynomial for the defining polynomial of the curve at the various singular points of $\mathcal{V}(\mathcal{B})$ (see [23, Chapter 2]). The construction should be computationally practical in view of the fact that SINGULAR has a library specifically for the Weierstrass preparation theorem [21, page 361] or [20]. To perform desingularization algebraically (i.e. computationally), one can instead (only in the codimension one case) compute the integral closure of the coordinate ring inside its field of fractions; this is also known as the **normalization** of the coordinate ring.

For the case when $d = 2$, this equates to the affine part of the normalizing Riemann surface since the non-normal locus consists of the $(d - \dim(\mathrm{Ann}(\mathcal{B})))$-minors of the Jacobian ideal. We leave details on these matters to another occasion. Recommended sources for a more detailed exposition include [23, 26, 28, 29, 19].

4.3 Vector Bundle Isomorphism

The main point of this section is to show that *isomorphic behaviors have isomorphic vector bundles.* We first resolve a couple of preliminary questions.

Lemma 4.3.1. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two isomorphic behaviors with a connected characteristic variety. If $\mathfrak{G}(\mathcal{B})$ is a vector bundle then so is $\mathfrak{G}(\mathcal{B}')$. The vector bundles also have the same rank and base space.*

Proof. This follows from Lemma 4.2.6, Lemma 2.2.24, Lemma 4.2.3, and the fact that the Fitting ideals of both behaviors are equal. $\square$

Lemma 4.3.2. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two isomorphic behaviors with E-isomorphism $\mathbf{A} : \mathcal{B} \to \mathcal{B}'$ implemented via multiplication by the matrix $A \in \mathcal{D}^{p\times q}$. Then for every $\boldsymbol{\lambda} \in \mathcal{V}(\mathcal{B})$, $A(\boldsymbol{\lambda})\mathfrak{G}(\mathcal{B}, \boldsymbol{\lambda}) = \mathfrak{G}(\mathcal{B}', \boldsymbol{\lambda})$.*

Proof. Let $w(t) = \alpha\boldsymbol{\lambda}^t \in \mathcal{B}$ be an exponential trajectory with frequency $\boldsymbol{\lambda}$. We easily see that

$$(\mathbf{A}w)(t) = (A(\boldsymbol{\lambda})\alpha)\boldsymbol{\lambda}^t.$$

This demonstrates that $A(\boldsymbol{\lambda})\mathfrak{G}(\mathcal{B}, \boldsymbol{\lambda}) \subset \mathfrak{G}(\mathcal{B}', \boldsymbol{\lambda})$.

Let $w'(t) = \alpha\boldsymbol{\lambda}^t \in \mathcal{B}'$ be a non-zero exponential trajectory with frequency $\boldsymbol{\lambda}$. There exists a unique $w \in \mathcal{B}$ such that $\mathbf{A}w = w'$. However, for every $p \in \mathcal{D}$,

$$\mathbf{A}(pw - p(\boldsymbol{\lambda})w) = pw' - p(\boldsymbol{\lambda})w' = 0.$$

Since $\ker(\mathbf{A}) = 0$, by Lemma 4.1.2 it follows that w is an exponential trajectory of $\mathcal{B}$ with frequency $\boldsymbol{\lambda}$. We conclude that $A(\boldsymbol{\lambda})\mathfrak{G}(\mathcal{B}, \boldsymbol{\lambda}) = \mathfrak{G}(\mathcal{B}', \boldsymbol{\lambda})$ $\square$

The preceding result tells us that the vector bundle structure is retained through isomorphism. The next result shows that associated vector bundles are actually isomorphic as vector bundles from which we conclude that frequency-amplitude vector bundle associated with a behavior is an E-module invariant.

Theorem 4.3.3. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ and $\mathcal{B}' = \ker_{\mathcal{A}}(R')$ be isomorphic behaviors with kernel representations $R \in \mathcal{D}^{p\times q}$ and $R' \in \mathcal{D}^{p'\times q'}$. If $\mathfrak{G}(\mathcal{B})$ is a holomorphic vector bundle over $V(\mathcal{B})$, then $\mathfrak{G}(\mathcal{B}')$ is a holomorphic vector bundle and $\mathfrak{G}(\mathcal{B})$ is vector-bundle isomorphic to $\mathfrak{G}(\mathcal{B}')$.*

Proof. By Lemma 4.3.1 we have that $\mathfrak{G}(\mathcal{B}')$ is also a vector bundle of rank m. Let $\{(V_i, U_i, \phi_i : V_i \to U_i)\}$ be a trivializing cover subordinate to both $\mathfrak{G}(\mathcal{B})$ and $\mathfrak{G}(\mathcal{B}')$. Let $\mathbf{A} : \mathcal{B} \to \mathcal{B}'$ be the given behavioral isomorphism with inverse $\mathbf{A}' \colon \mathcal{B}'' \to \mathcal{B}$ implemented by multiplication on the left by the respective matrices $A \in \mathcal{D}^{p' \times p}$ and $A' \in \mathcal{D}^{p' \times p}$. By Lemma 4.3.2, both A and A' induce linear isomorphisms on the sections of $\mathfrak{G}(\mathcal{B})$ and $\mathfrak{G}(\mathcal{B}')$. Since both A and A' are holomorphic, the desired result follows. $\square$

Chapter 5

Overdetermined Systems

As we saw in Theorem 2.2.23, autonomous behaviors admit a non-trivial annihilator. In view of the results in the previous section, it seems natural to consider autonomous behaviors as ones which are associated with an algebraic set. The rationale behind this is that the rank of the kernel representation drops only on an algebraic set rather than on the whole affine space. This means that a frequency domain theory is only viable on an algebraic set, where the set of admissible frequencies is a *proper* subset of k^d. However, since this subset of frequencies is an algebraic set, it admits a rich structure that, as demonstrated below, is naturally tied to the time domain setting by working over a different ring and signal space.

In this section, we discuss a framework which allows us to relate certain properties of autonomous behaviors, once we switch to a new ring and signal space, with their controllable analogues. We shall say that the behavior is *reduced* by its annihilator. In essence, we are restricting the signal space to only the necessary signals and also "removing" operators from the ring which are already satisfied by this new signal space. If a behavior is a torsion module, then necessarily every trajectory is annihilated by each scalar polynomial in the annihilator of the behavior. As a consequence, one can think of the signal space $\mathcal{A}$ is "too big" since many of its trajectories are excluded from the behavior; equivalently, the operator ring is not suitable for performing linear algebra[1] since the functor $- \otimes Q(\mathcal{D})$ annihilates completely the dual module of the behavior. The idea is to work over an ambient signal space with trajectories already satisfying all non-coupled kernel conditions which are satisfied by the behavior. By working with a smaller signal space which is "minimal" in this sense, we are still able to employ linear algebra but over a different field and thereby recover results analogous to those in the classical setting. Furthermore, when one looks at the frequency domain theory, it becomes apparent that this new setting is not only compatible, but also quite natural.

We note that most of the results in this section are restatements of those obtained by Oberst in [39]. While we provide the formalism for treating autonomous behaviors with the same tools as behaviors with free variables, it is only because of Oberst's foresight and non-restrictive hypotheses when constructing the theory of multidimensional behaviors in [loc. cit.] that we are able to present the material below. The new contributions of this section include the introduction of a trajectory-

[1] Recall the construction of the signal flow space in Section 2.1.6.

based definition and characterization of controllability for autonomous behaviors, the machinery introduced for working with autonomous behaviors, and the connection between frequency-domain results and controllable behaviors.

5.1 Reduced Behaviors

The key process introduced here is that of **reduction** of a behavior by an ideal. Although the process is quite simple, under certain assumptions, it allows autonomous behaviors to be studied by the same tools which are commonly employed in the analysis of behaviors with free variables.

Definition 5.1.1. Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be an autonomous behavior and $I \subset \mathcal{D}$ be an ideal such that $I \subset \mathrm{Ann}(\mathcal{B})$. Let $\mathcal{D}_r$ denote the **reduced** ring $\mathcal{D}_r = \mathcal{D}/I$ and let $\phi : \mathcal{D} \to \mathcal{D}_r$ be the canonical quotient map. The signal space associated with $\mathcal{D}_r$ is

$$\mathcal{A}_r = \{w \in \mathcal{A} : pr = 0 \text{ for all } p \in I\}.$$

We define the behavior **reduced** by I, denoted by $\mathrm{Red}_I(\mathcal{B})$, as

$$\mathrm{Red}_I(R) = \mathcal{B}_r = \{w \in \mathcal{A}_r : \phi(R)w = 0\} = \ker_{\mathcal{A}_r}(\widetilde{R}), \qquad \widetilde{R} = \phi(R).$$

If $\mathcal{B}$ is reduced by its annihilator, then we simply write $\mathrm{Red}(\mathcal{B})$ or $\mathcal{B} \otimes \mathcal{D}_r$. We reserve the symbols $\mathcal{B}_r$, $\mathcal{A}_r$, and $\mathcal{D}_r$ to denote, respectively, the reduced behavior, signal space, and ring of a behavior.

Since $I \subset \mathrm{Ann}(\mathcal{B})$, we have that the reduced behavior consists of the same trajectories as the original behavior. The only difference is that we have changed the ring of operators so that it is consistent with the new signal space. Because of the nature of reduction, we usually only reduce a behavior by its annihilator. It is easy to observe that $\mathcal{B}$ is equal to $\mathrm{Red}(\mathcal{B})$ as $\mathcal{D}_r$-modules.

In [39, page 46-47], Oberst shows that $\mathcal{A}_r$, as stated above, is a large injective cogenerator over finitely generated $\mathcal{D}_r$-modules. For a reduced behavior $\mathcal{B}_r = \ker_{\mathcal{A}_r}(R)$ where $R \in \mathcal{D}_r^{p\times q}$, through the same pairing (2.6), we have that its dual module is given by $\mathrm{coker}_{\mathcal{D}_r}(R^T)$. Furthermore, Oberst duality continues to hold in this new setting since $\mathcal{A}_r$ is a large injective cogenerator.

5.2 Free Variables

Notation. For this section let $\mathcal{D}_r$ denote an affine domain, i.e. $\mathcal{D}_r = \mathcal{D}/I$ where $I \subset \mathcal{D}$ is a prime ideal. Let $\mathcal{A}_r$ be the signal space associated with $\mathcal{D}_r$. Note that, by Theorem 2.1.32 $\mathcal{A}_r$ is a large injective cogenerator (and hence, also an injective cogenerator).

Recall that an equivalent definition for a behavior to be autonomous is that it has no free variables. However, over an algebraic set a behavior's kernel representation may lose some rank; this property is indicative of free variables of a new sort. We show that autonomous behaviors do exhibit free variables in this new sense. Once we define free variables for autonomous behaviors we prove that the definition is analogous to the classical setting.

Definition 5.2.1. For $R \in \mathcal{D}_r^{p \times q}$, we define the rank of the matrix R as the largest nonnegative integer ℓ with $1 \leq \ell \leq q$ such that $\mathfrak{I}_\ell(R) \neq 0$ (recall that $\mathfrak{I}_\ell(R)$ denotes the ideal generated by the determinant of the set of all order-ℓ minors of R).

It is well known (see [35]) that the above definition, provided that the ring is an integral domain, is equivalent to the usual definition of rank by working over the quotient field.

The following lemma shows that, once a behavior is reduced by its annihilator, the new kernel representation no longer has full column rank. This sets the stage for defining free variables.

Lemma 5.2.2. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a nonzero behavior with $R \in \mathcal{D}^{p \times q}$. Then $\operatorname{rank}_{\mathcal{D}_r}(R) < q$, i.e., R necessarily has less than full column rank when considered as a matrix over $\mathcal{D}_r = \mathcal{D}/\operatorname{Ann}(\mathcal{B})$.*

Proof. Since $\mathfrak{F}_0(\mathcal{M}) \subset \operatorname{Ann}(\mathcal{B})$ and $\mathfrak{F}_0(\mathcal{M}) \neq 0$, we have

$$\mathfrak{F}_0(\mathcal{M} \otimes_{\mathcal{D}} (\mathcal{D}/\operatorname{Ann}(\mathcal{B}))) = \mathfrak{F}_0(\mathcal{M})\mathcal{D}/\operatorname{Ann}(\mathcal{B}) = 0.$$

In particular, $\operatorname{rank}(R) < q$. Since the Fitting ideals are invariant under any choice of presentation matrix, this holds for any kernel representation of the reduced behavior. □

The above states that, once we switch to the reduced behavior, the rank of the kernel representation must drop. As in the classical case, we use the rank of the kernel representation to define the number of free variables in the behavior.

Definition 5.2.3. For a reduced behavior $\mathcal{B}_r = \ker_{\mathcal{A}_r}(R) \subset \mathcal{A}_r^q$, we define the number of **free variables** as $q - \operatorname{rank}(R)$.

A direct consequence of Lemma 5.2.2 is the following.

Corollary 5.2.4. *If a nontrivial behavior is reduced by its annihilator and the annihilator is a prime ideal, then free variables always exist.*

Remark. In Corollary 5.2.4 we assume that the annihilator is prime in order to have a meaningful notion of rank. In the case where this is not true, then a more suitable version of rank is the **McCoy rank** mentioned earlier in a different context (see Remark 2.2.1).

We now show a connection between our definition of free variables and the trajectory-based definition of free variables. Consider the map

$$\pi_i : \mathcal{B} \longrightarrow \mathcal{A}_r \qquad\qquad 1 \leq i \leq 1.$$

For an index set $\mathcal{I} = \{j_1, \dots, j_n\} \subset \{1, 2, \dots, q\}$ we similarly define $\pi_{\mathcal{I}}$ as

$$\pi_{\mathcal{I}} : \mathcal{B} \longrightarrow \mathcal{A}^{|\mathcal{I}|} \qquad\qquad (w_1, \dots, w_q) \mapsto (w_{j_1}, \dots, w_{j_n}) \in \mathcal{A}_r^{|\mathcal{I}|}.$$

The following lemma is essentially a restatement of Theorem 2.1.54. We refer the reader to [39, page 38–40] for complete details.

Lemma 5.2.5. *Assume that $\mathcal{D}_r$ is a commutative Noetherian domain. Let $\mathcal{B}_r = \mathrm{Red}(\mathcal{B})$ be a reduced behavior and $\mathcal{I} \subset \{1, 2, \ldots, q\}$ be the maximal subset such that $\pi_{\mathcal{I}} : \mathcal{B}_r \to \mathcal{A}_r^{|\mathcal{I}|}$ is surjective. Then $|\mathcal{I}|$ is equal to the number of free variables of $\mathcal{B}_r$.*

The reduced signal space $\mathcal{A}_r$ has "freedom" on a subset of $\mathbb{N}^d$ given by its initial condition set. In particular, for an ideal $I = \langle p_1, \ldots, p_k \rangle \subset \mathcal{D}_r$ we can interpret $\mathcal{A}_r$ as the behavior $\mathcal{A}_r = \ker_{\mathcal{A}}(I)$. From the canonical Cauchy problem (i.e., Theorem 2.1.72) we have that the only free part of $\mathcal{A}_r$ is ∂I; that is, on ∂I we have that any trajectory $w \in \mathcal{A}_r$ can be prescribed with arbitrary values from which the values of w on $\mathbb{N}^d$ are derived.

The reduced signal space has properties similar to those of $\mathcal{A}$. For instance, once the ring has been changed, the new signal space is divisible.

Lemma 5.2.6. *If $\mathcal{D}_r$ is an affine domain, then $\mathcal{A}_r$ is divisible over $\mathcal{D}_r$.*

Proof. Let non-trivial $p \in \mathcal{D}_r$ be given. Since $\mathcal{D}_r$ is a domain, $p : \mathcal{D}_r \to \mathcal{D}_r$ is injective, i.e.

$$0 \longrightarrow \mathcal{D}_r \xrightarrow{p} \mathcal{D}_r$$

is exact. By Proposition 2.1.8,

$$\mathcal{A}_r \xrightarrow{p} \mathcal{A}_r \longrightarrow 0$$

is exact. In particular, p is surjective. We conclude that $\mathcal{A}_r$ is divisible. □

There is a very useful result which can be found in [21, page 281].

Lemma 5.2.7. *Let $k[z_1, \ldots, z_s] \subset \mathcal{D}_r$ be a Noether normalization. If $\mathcal{D}_r$ is equidimensional (i.e. the associated primes all have the same dimension) then $k[z_1, \ldots, z_s]$ has no zero-divisors in $\mathcal{D}_r$.*

This leads us to the sharpened version of Lemma 5.2.6.

Corollary 5.2.8. *Let $\mathcal{D}_r$ satisfy the hypothesis of Lemma 5.2.7. Then for any non-zero $p \in k[z_1, \ldots, z_s]$ we have $p\mathcal{A}_r = \mathcal{A}_r$.*

Proof. This is identical to the proof of Lemma 5.2.6. By Lemma 5.2.7 we have $fp \neq 0$ for all non-zero $f \in \mathcal{D}_r$ and hence $\ker(p : \mathcal{D}_r \to \mathcal{D}_r) = 0$, i.e., p is injective. By Proposition 2.1.8 we conclude that $p : \mathcal{A}_r \to \mathcal{A}_r$ is surjective. □

Note that, in this case, we do not necessarily have an integral domain. We may also see the following.

Lemma 5.2.9. *Let $\mathcal{A}_r^q$ be the signal space associated with a Noetherian domain $\mathcal{D}_r = \mathcal{D}/I$. Provided that $\mathcal{V}(I)$ is a complex manifold, then $\mathfrak{G}(\mathcal{A}_r^q)$ is a trivial rank q holomorphic vector bundle over $\mathcal{V}(I)$.*

Proof. By definition,

$$\mathfrak{G}(\mathcal{A}_r^q) = \{(\boldsymbol{\lambda}, w_0) : \boldsymbol{\lambda} \in \mathcal{V}(I) \text{ and } w_0 \in k^q\}.$$

In particular, $\mathfrak{G}(\mathcal{A}_r^q) = \mathcal{V}(I) \times k^q$. Because $\mathcal{V}(I)$ is a complex manifold, this is clearly a trivial holomorphic vector bundle. □

We may also relate the number of free variables of a reduced behavior $\mathcal{B}_r$ to the minimal rank of $\mathfrak{G}(\mathcal{B}_r)$.

Lemma 5.2.10. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a behavior, $I = \operatorname{Ann}(\mathcal{B})$ be a prime ideal, and $\mathcal{B}_r = \operatorname{Red}(\mathcal{B})$ be its reduced behavior. Then the minimal rank of $\mathfrak{G}(\mathcal{B})$ is equal to the number of free variables.*

Proof. By Lemma 4.2.4, the minimal rank of $\mathfrak{G}(\mathcal{B})$ is precisely $q - \operatorname{rank}(R \otimes 1_{\mathcal{D}_r})$. □

Remarks. One can remove the condition that $\mathcal{D}_r$ be an integral domain by putting conditions on the Fitting ideals associated with $\mathcal{M}$. This, however, requires using linear algebra over arbitrary commutative rings. See [35] for more exposition on this topic.

5.3 Controllability

Let $\mathcal{B}_r = \operatorname{Red}(\mathcal{B}) \subset \mathcal{A}_r^q$ be a reduced behavior and assume that the associated ring $\mathcal{D}_r$ is an affine domain. Under this assumption, we now describe how autonomous behaviors may be considered as controllable along a sublattice.

Since we have changed the operator ring, it is now possible that $\mathcal{B}_r$ has an **image representation** over $\mathcal{D}_r$. The heavy constraints make this impossible over $\mathcal{D}$ since $\mathcal{B}$ has a trivial controllable part. However, since every trajectory in the signal space already satisfies the non-coupled annihilator conditions, a behavior may have an image representation over this smaller signal space. As in the classical case, we define an image representation of a reduced behavior $\mathcal{B}_r$ as a polynomial matrix $M \in \mathcal{D}_r^{q \times g}$ such that

$$\mathcal{B}_r = \operatorname{im}_{\mathcal{A}_r}(M) = \{w \in \mathcal{A}_r^q : \text{ there exists } v \in \mathcal{A}_r^g \text{ such that } Mv = w\}.$$

Note that the change of both the ring and signal space enables this definition to be well-defined.

As pointed out earlier, the idea of working over a different ring and signal space has been developed by Oberst in [39]; as a consequence we have the following result.

Theorem 5.3.1. *Let $\mathcal{B}_r = \ker_{\mathcal{A}_r}(R)$ be a reduced behavior, where the associated ring $\mathcal{D}_r$ is an affine domain, and $\mathcal{M}_r = \operatorname{coker}_{\mathcal{D}_r}(R^T)$ be its dual module. The following are equivalent.*

1. *$\mathcal{M}_r$ is torsion-free*
2. *$\mathcal{B}_r$ has an image representation.*
3. *R is GFLP over $\mathcal{D}_r$*

4. $\mathcal{B}_r$ is minimal in its transfer class.

5. $\mathcal{B}_r$ is divisible.

Proof. (1) $\iff$ (3) $\iff$ (4). This has been demonstrated in [39, page 142-144] and by Theorem 2.1.61.

(2) $\iff$ (3). This follows from Corollary 2.2.17. Note that the results leading to the proof of Corollary 2.2.17 continue to hold in the setting where $\mathcal{D}_r$ is an integral domain.

(2) $\Rightarrow$ (5): Let $\mathcal{B}_r$ have image representation $M \in \mathcal{D}_r^{q \times g}$. Then for any $w \in \mathcal{B}_r$, there exists $v \in \mathcal{A}_r^g$ such that $w = Mv$. Let a non-zero $p \in \mathcal{D}_r$ be given. By Lemma 5.2.6, $\mathcal{A}_r$ is divisible. It follows that there exists $v' \in \mathcal{A}_r$ such that $pv' = v$. For $w' = Mv'$ we have

$$pw' = p(Mv') = M(pv') = Mv = w.$$

we conclude that $\mathcal{B}_r$ is divisible.

(5) $\Rightarrow$ (1). If $\mathcal{B}_r$ is divisible, then for any non-zero $p \in \mathcal{D}_r$, $p : \mathcal{B}_r \to \mathcal{B}_r$ is surjective or, equivalently,

$$\mathcal{B}_r \xrightarrow{p} \mathcal{B}_r \longrightarrow 0$$

is exact. By Proposition 2.1.8,

$$0 \longrightarrow \mathcal{M}_r \xrightarrow{p} \mathcal{M}_r$$

is exact. Say that $\mathcal{M}_r$ has a torsion element $v \in \mathcal{M}_r$ such that for non-zero $p' \in \mathcal{D}_r$, $p'v = 0$; existence of such a p' or v would contradict the above exactness since $p' : \mathcal{M}_r \to \mathcal{M}_r$ would no longer be injective. We conclude that $\mathcal{M}_r$ is torsion-free. $\square$

Definition 5.3.2. Let $\mathcal{B}_r = \ker_{\mathcal{A}_r}(R)$ be a reduced behavior, where the associated ring $\mathcal{D}_r$ is an affine domain. We say that $\mathcal{B}_r$ is **$\mathcal{D}_r$-controllable** if any of the equivalent properties in Theorem 5.3.1 is satisfied.

The next goal is to introduce a trajectory-based definition of controllability. Before doing so, we must bring up a few technical points. We refer the reader to Appendix A.1 for preparation towards the following material.

When we work over the reduced ring, there are issues with associating points in $\mathbb{N}^d$ with monomials. Under the conventional association, a monomial corresponds to many points when considered in its residue class; to rectify this, we use a normal form. For any total monomial ordering $\leq$ we say that the ordering is **global** if $1 < z_i$ for $i = 1, \ldots, d$. For the problems addressed in behavioral systems theory, it is most common to work with a global ordering (such as lexicographic ordering or degree ordering). For any polynomial $f \in \mathcal{D}$, under the given global ordering, we define $\mathrm{LM}(f)$ as the leading monomial of f. For an ideal $I \subset \mathcal{D}$ we define its associated monomial ideal as

$$L(I) = \langle \mathrm{LM}(f) : f \in I \rangle.$$

For a given global ordering, we may also compute a unique reduced (or minimal) Gröbner basis $G = \{g_1, \ldots, g_r\}$ for the ideal I. For any $f \in \mathcal{D}$, we define the **reduced normal form**

$$\mathrm{NF}(f|G) : \mathcal{D} \to \mathcal{D}$$

where $\mathrm{NF}(f|G) = 0$ if and only if $f \in I$ (i.e., no remainder), $f - \mathrm{NF}(f|G) \in I$, and $\mathrm{NF}(f|G) \neq 0$ implies that $\mathrm{LM}(\mathrm{NF}(f|G)) \notin L(I)$. In particular, $\mathrm{NF}(f|G)$ is a unique representative of its residue class in $\mathcal{D}/I$ given under the monomial ordering and Gröbner basis.

Example 5.3.3. Consider the affine domain $A = k[x, y]/\langle x^2 - y + 1\rangle$ with lexicographic ordering. The monomial x^3 is equivalent to $xy - x$ in its residue class and thus is not minimal. However, the monomial xy is minimal in its residue class and $\mathrm{NF}(xy|G) = xy$.

By using Buchberger's algorithm we have a way of specifying a unique representative from any residue class in a quotient ring. Naturally, we may extend this notion to the case of matrices as seen in the following definition.

Definition 5.3.4. Let $\mathcal{D}_r = \mathcal{D}/I$ be an affine domain where $R \in \mathcal{D}_r^{p\times q}$ and G is a reduced Gröbner basis of I under a provided global ordering. We define the **support** of R, denoted by $\mathrm{supp}(R)$ as follows. Let $\mathrm{NF}(R|G)$ be the matrix consisting of the normal form of the entries of R, i.e., for $R = \{r_{i,\ell}\}$ we have $\mathrm{NF}(R|G)_{i,\ell} = \mathrm{NF}(r_{i,\ell}|G)$. We may write

$$\mathrm{NF}(R|G) = \sum R_i \mathbf{z}^i$$

where each R_i is a constant matrix. From this we define

$$\mathrm{supp}(R) = \{i \in \mathbb{N}^d : R_i \neq 0\}.$$

We define the **closed support**, $\overline{\mathrm{supp}}(R)$, to be the smallest ($d$-dimensional) interval containing $\mathrm{supp}(R)$. We also define the **diameter** of $\mathrm{supp}(R)$ as

$$\rho(\mathrm{supp}(R)) = \max_{a,b\in \mathrm{supp}(R)} d(a, b),$$

where $d(\cdot, \cdot)$ is given by Definition 2.2.2

Under the prescribed ordering, we have that elements $f \in \mathcal{D}$ such that $\mathrm{NF}(f|G) = f$ have support in the initial condition set for the canonical Cauchy problem. Hence, we always have $\mathrm{supp}(R) \subset \partial I$.

We present a stripped down version of controllability which is based on the Cauchy problem instead of its most general form; in particular, the definition requires the choice of a global monomial ordering. See the remarks at the end of the section for discussion on the choice of definition.

Definition 5.3.5. For $j \in \{1, \ldots, d\}$ and $\mathcal{B}$ a d-dimensional behavior we say that $\mathcal{B}$ is **j-controllable** if there exists a j-dimensional lattice $\mathcal{L}[u]$, $u \subset \{z_1, \ldots, z_d\}$ and $|u| = j$, (the **motion sublattice**) such that $\mathcal{L}[u] \subset \partial\mathrm{Ann}(\mathcal{B})$ and, for any $T_1 \subset \partial\mathrm{Ann}(\mathcal{B})$ such that $\dim(T_1) < j$ and *finite* subset $J \subset \mathbb{N}^d$, there exists an integer (the **separation distance**) $\rho(T_1 + J) \geq 0$ such that, for any $w_1, w_2 \in \mathcal{B}$ and $b \in \mathcal{L}[u]$ with $d(T_1 + J, b + \mathbb{N}^d) > \rho(T_1 + J)$, there exists $w \in \mathcal{B}$ such that

$$w(t) = \begin{cases} w_1(t) & t \in T_1 + J \\ w_2(t - b) & t \in b + \mathbb{N}^d. \end{cases}$$

By the definition of j-controllability, we have the following chain of relations

$$d\text{-controllable} \quad \Rightarrow \quad (d-1)\text{-controllable} \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad 0\text{-controllable.}$$

As a consequence, for any d-dimensional system, there are d different forms of controllability available and each determines the extent of the system's memory. Natural candidates for j-controllability include behaviors with pure autonomy degree $d-j$.

The following example indicates why motion is allowed only along a given lattice in the Definition 5.3.5.

Example 5.3.6. Let $R = [z_1]$ be a kernel representation for the 2-dimensional behavior $\mathcal{B} \subset \mathcal{A}$ with ring $\mathcal{D} = k[z_1, z_2]$. The annihilator of $\mathcal{B}$ is the prime ideal $\langle z_1 \rangle \subset \mathcal{D}$. The trajectories of $\mathcal{B}$ are arbitrary values on the z_2-axis and zero off of the z_2-axis. However, the reduced behavior is completely free; in particular, it has an image representation given by the identity. Note that $\mathcal{B}$ cannot take non-zero values off of the z_1-axis. The reasoning is that the new ring is $\mathcal{D}_r \cong k[z_2]$ and thus any shift in the z_1-axis is equivalent to multiplying by zero. Asking that $\mathcal{B}_r$ allow shifts in this direction is far too demanding since it has an image representation over its reduced signal space.

Theorem 5.3.7. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a given behavior. Provided $\dim(\mathcal{B}) = j$, $\mathcal{B}$ is j-controllable, and $\mathrm{Ann}(\mathcal{B})$ is a prime ideal, then $\mathcal{B}_r = \mathrm{Red}(\mathcal{B})$ has an image representation over its reduced ring.*

Proof. We note that this proof is similar to that of Zerz and Wood in [55]; we only adjust a few points to make it hold in this new setting.

Let $\mathcal{B}_r = \mathrm{Red}(\mathcal{B})$ be the behavior reduced by its annihilator $I = \mathrm{Ann}(\mathcal{B})$ with reduced signal space $\mathcal{A}_r$ and with reduced ring $\mathcal{D}_r = \mathcal{D}/I$, and $\mathcal{B}_r^c = \ker_{\mathcal{A}_r}(R^c)$ be the behavior minimal in $\mathcal{B}_r$'s transfer class (see Theorem 2.1.61 with D replaced by $\mathcal{D}_r$). By Theorem 2.1.59, there exists a rational matrix D of appropriate size with entries in $Q(\mathcal{D}_r)$ so that $R^c = DR$. By taking the common denominators, we may write $[h]R^c = XR$ where $[h]$ a nonzero element of $\mathcal{D}_r$ and where X has entries in $\mathcal{D}_r$.

If $[h]$ is a unit in $\mathcal{D}_r$, then dividing through we have $R^c = X'R$ where X' has entries in $\mathcal{D}_r$. This implies that $\mathcal{B}_r \subset \mathcal{B}_r^c$; however, since $\mathcal{B}_r^c \subset \mathcal{B}_r$, this implies $\mathcal{B}_r = \mathcal{B}_r^c$. Since $\mathcal{B}_r^c$ has an image representation over $\mathcal{D}_r$, we can conclude that $\mathcal{B}_r$ has an image representation over $\mathcal{D}_r$. We now assume that $[h] \neq 0$ is not a unit in $\mathcal{D}_r$.

By Lemma A.3.7, since I is a prime ideal, $\mathrm{height}(h+I) > \mathrm{height}(I)$ and by Lemma 3.1.7, $\dim(\partial(h+I)) < \dim(\partial I)$. Define $G = \partial(h+I)$,

$$\begin{aligned} T_1 &= \{a + s : a \in \mathrm{supp}(R^c) \text{ and } s \in G\} \\ T_2 &= b + \mathbb{N}^d, \end{aligned}$$

where $b \in \mathcal{L}[u]$ ($\mathcal{L}[u]$ as in Definition 5.3.5 for j-controllability of $\mathcal{B}$) is chosen such that the distance between T_1 and T_2 is greater than $\rho(T_1)$. By Corollary A.3.8, such a choice of b is always possible. By hypothesis, there exists a trajectory $v \in \mathcal{B}$ such that

$$w(t) = \begin{cases} 0 & t \in T_1 \\ v(t-b) & t \in T_2 \end{cases}$$

where $v \in \mathcal{B}$ is an arbitrary trajectory. Since $w \in \mathcal{B}$ and $[h]R^c = XR$, we have $[h]R^c w(t) = XRw(t) = 0$ for all $t \in \mathbb{N}^d$. We now prove that $R^c w = 0$.

Clearly $R^c w = 0$ on G since w is the zero trajectory on G augmented by the support of R^c; however, because G contains the initial condition set of $[h]$ and $[h](R^c w) = 0$ on all of $\mathbb{N}^d$ (i.e. $R^c w$ is a solution to $[h]$ and by Theorem 2.1.72 this means $R^c w = 0$ on $\mathbb{N}^d$ since $R^c w = 0$ on G), it must be the case that $R^c w = 0$ on $\mathbb{N}^d$ since the only trajectory in $\ker_{\mathcal{A}_r}([h])$ with a zero initial condition is the zero trajectory. This demonstrate that $w \in \mathcal{B}_r^c$. However, since $\mathcal{B}_r^c$ is shift-invariant, $\mathbf{z}^b w = v \in \mathcal{B}_r^c$ (Note that, because $b \in \mathcal{L}[u] \subset \partial I$, we have $k[u] \cap I = 0$ and hence $\mathbf{z}^b \neq 0$ in $\mathcal{D}_r$.) Since v was arbitrary, we conclude that $\mathcal{B}_r = \mathcal{B}_r^c$. Using that $\mathcal{B}_r^c$ is minimal in its transfer class, by Theorem 2.1.61 and Theorem 5.3.1 we conclude that $\mathcal{B}_r$ has an image representation over $\mathcal{D}_r$. □

When working over a reduced signal space, the "free" signals are strangely behaved. Certain properties that one takes for granted over $\mathcal{A}$ may be no longer true in $\mathcal{A}_r$. For showing the reverse direction of the above result, we must assume[2] that free variables are j-controllable. If this is satisfied, then $\mathcal{A}_r$ behaves enough like $\mathcal{A}$ to lend itself to solving problems in a fashion similar to the classical setting.

Theorem 5.3.8. *Let $\mathcal{B} \subset \mathcal{A}^q$ be a behavior. If $\mathrm{Ann}(\mathcal{B})$ is a prime ideal, the reduced signal space $\mathcal{A}_r$ is j-controllable with motion sublattice $\mathcal{L}[u]$, $u \subset \{z_1, \ldots, z_d\}$, $|u| = j$, $\mathcal{L}[u] \subset \partial\mathrm{Ann}(\mathcal{B})$, and $\mathcal{B}_r = \mathrm{Red}(\mathcal{B})$ has image representation M over its reduced ring, then $\mathcal{B}$ is j-controllable with motion sublattice $\mathcal{L}[u]$.*

Proof. We note that this proof is practically identical to the proof of Rocha in [43, page 178-179].

Let $T_1 \subset \partial I$ be given so that $\dim(T_1) < j$, $J \subset \mathbb{N}^d$ be a finite set and let $\rho'(T_1 + J)$ be the separation distance given by $\mathcal{A}_r$. Define $\rho(T_1 + J) = \rho'(T_1 + J) + \rho(\mathrm{supp}(M))$ where $\rho(\mathrm{supp}(M))$ is given by Definition 5.3.4. Let $w_1, w_2 \in \mathcal{B}_r$ and $b \in L$ such that $d(T_1 + J, b + \mathbb{N}^d) > \rho$. For ease of notation, define the set $T_2 = b + \mathbb{N}^d$. For any $s_1, s_2 \in \mathrm{supp}(M)$, $t_1 \in T_1$, and $t_2 \in T_2$ we have

$$|t_1 - t_2| > \rho = \rho(T_1 + J) = \rho'(T_1 + J) + \rho(\mathrm{supp}(M)) \geq \rho(\mathrm{supp}(M)) \geq |s_1 - s_2|.$$

It follows that $(T_1 + J + \mathrm{supp}(M)) \cap (T_2 + \mathrm{supp}(M)) = \emptyset$. Since M is an image representation of $\mathcal{B}_r$, there exits $v_1, v_2 \in \mathcal{A}_r^q$ such that $w_1 = Mv_1$ and $w_2 = Mv_2$. Because $\mathcal{A}_r$ is j-controllable and $d(T_1 + J, T_2) > \rho > \rho'$, there exists $w \in \mathcal{A}_r^q$ such that

$$w = \begin{cases} v_1(t) & t \in T_1 + J + \mathrm{supp}(M) \\ v_2(t - b) & t \in T_2 + \mathrm{supp}(M) \end{cases}$$

It follows that

$$Mw = \begin{cases} w_1(t) & t \in T_1 + J \\ w_2(t - b) & t \in T_2 \end{cases}$$

We conclude that $\mathcal{B}$ is j-controllable. □

[2]This assumption is not entirely outlandish. In the following section we construct a class of signal spaces which are 1-controllable. See the remarks at the end of this section for more discussion on the topic.

Remarks. (1) The definition of j-controllability differs from conventional behavioral controllability in two ways. One way is that the separation distance is not constant, but depends on the set $T_1 + J$. To understand why this is necessary, one can look at Lemma 6.5.4. It turns out that the separation distance should not really depend on $T_1 + J$, but the size of J. One can also see that the definition of d-controllability is not as accommodating as the classical definition of controllability as found in Definition 2.2.5. The reason is that the definition provided is the bare minimum required for proving the relation to having an image representation.

(2) Theorem 5.3.8 leads to the question: *when are signal spaces j-controllable?*. Currently, this seems to require that $\partial\mathrm{Ann}(\mathcal{B})$ be closed under addition on the motion sublattice. A sketch is as follows: use Noether normalization to construct a k-algebra automorphism so that, for a maximal independent set $u = \{z_{s+1}, \ldots, z_d\}$, we have $k[u] \subset \mathcal{D}/\mathrm{Ann}(\mathcal{B})$ is a finite extension (i.e. $\mathcal{D}/\mathrm{Ann}(\mathcal{B})$ is a finitely generated $k[u]$-module); once this step is performed, if the ring is equidimensional (i.e. the behavior has pure autonomy degree) then $k[u]$ has no zero-divisors in $\mathcal{D}/\mathrm{Ann}(\mathcal{B})$ – even if $\mathrm{Ann}(\mathcal{B})$ is the intersection of primary ideals. There is then a somewhat explicit form of a subset of the Gröbner basis which allows us to provide a construction via lexicographic ordering without knowing the Gröbner basis. This last point follows because the Noether normalization states that each $z_i \notin u$ is integral over $\mathcal{D}/\mathrm{Ann}(\mathcal{B})$ and thus we have that one element of the Gröbner basis divides z_i^ℓ for some ℓ. The implication is that the initial condition set does not break into multiple pieces of the same dimension. Provided that $\partial\mathrm{Ann}(\mathcal{B})$ is closed under addition by $\mathcal{L}[u]$ (what provides this is currently unknown), one can use the normal form to identify where initial conditions should be specified for the two trajectories and then conclude by using the existence of a solution to the canonical Cauchy problem. As far as $\partial\mathrm{Ann}(\mathcal{B})$ being closed under addition by $\mathcal{L}[u]$, we at least know that for principal ideals this condition is always satisfied by Noether normalization.

(3) Removing the requirement of a monomial ordering for Definition 5.3.5 (as required to define $\partial\mathrm{Ann}(\mathcal{B})$) requires a significantly different presentation than the one given. One must associate staircases to monomial ideals and work at the level of independent sets rather than initial conditions. This approach gives results consistent with material presented here, however, requires much more space to discuss.

5.4 Frequency Domain Analysis

Notation. Let k be a field, $\mathcal{D} = k[z_1, \ldots, z_d]$, and $\mathcal{A} = k^{\mathbb{N}^d}$; we also equip $\mathcal{A}$ with the weak-$*$ topology as discussed in Section 2.1.4. Define $\widehat{\mathbf{Mod}(\mathcal{D})}$ as the category of $\mathcal{D}$-submodules of $\mathcal{A}^q$ for some q which are closed in the linearly compact topology inherited from $\mathcal{A}^q$ with morphisms consisting of continuous $\mathcal{D}$-homomorphims.

As pointed out by Oberst in [40], in general, exponential trajectories are not dense in a behavior. However, under certain conditions the amplitude FVS uniquely determines the behavior; in such a situation, we have a connection between the frequency-domain theory and time-domain theory. The ideas are motivated by the one-dimensional development obtained by Willems in [51]; however, we extend the concept to d-dimensional systems which are reduced by a prime ideal.

Definition 5.4.1. For a behavior $\mathcal{B} \subset \mathcal{A}^q$, we define its exponential closure as the E-$\mathcal{D}$-module

$$\mathfrak{E}(\mathcal{B}) = \overline{\operatorname{span}_E}\{w_0\boldsymbol{\lambda}^t \in \mathcal{A}^q : w_0\boldsymbol{\lambda}^t \in \mathcal{B}\},$$

where the closure is taken over the weak-$*$ topology on $\mathcal{A}^q$. In particular, $\mathfrak{E}(\mathcal{B})$ is the smallest (under set inclusion) closed element of $\widehat{\mathbf{Mod}(\mathcal{D})}$ which has the same exponential trajectories as $\mathcal{B}$. If $\mathcal{B} = \mathfrak{E}(\mathcal{B})$ we say that $\mathcal{B}$ is **exponentially closed**. (Note that, by Theorem 2.1.45, $\mathfrak{E}(\mathcal{B})$ is a behavior.)

First we emphasize that the exponential closure is unique and always a subbehavior.

Proposition 5.4.2. *Let $\mathcal{B}$ be a behavior and $\mathcal{B}^e := \mathfrak{E}(\mathcal{B})$ be its exponential closure. Then $\mathcal{B}^e$ is the smallest behavior for which $\mathfrak{G}(\mathcal{B}) = \mathfrak{G}(\mathcal{B}^e)$. Furthermore, $\mathfrak{E}(\mathcal{B}) \subset \mathcal{B}$.*

Proof. We demonstrate that $\mathfrak{E}(\mathcal{B}') = \mathfrak{E}(\mathcal{B}^e)$ implies $\mathcal{B}^e \subset \mathcal{B}'$. By definition, if $\mathfrak{E}(\mathcal{B}') = \mathfrak{E}(\mathcal{B}^e)$ then a dense subset of $\mathcal{B}^e$ is contained in $\mathcal{B}'$; since $\mathcal{B}'$ is closed, it follows that $\mathcal{B}^e \subset \mathcal{B}'$. In the case where $\mathcal{B}' = \mathcal{B}$ we reach $\mathcal{B}^e \subset \mathcal{B}'$. □

Another interesting observation is that the exponential closure removes all multiplicity from the behavior.

Proposition 5.4.3. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^q$ be a behavior and $\mathcal{B}^e = \mathfrak{E}(\mathcal{B})$. Then* $\operatorname{Ann}(\mathcal{B}^e) = \sqrt{\operatorname{Ann}(\mathcal{B}^e)}$.

Proof. Let $R^e \in \mathcal{D}^{p\times q}$ be a kernel representation for $\mathcal{B}^e$ and $J = \langle f_1, \ldots, f_r\rangle = \sqrt{\operatorname{Ann}(\mathcal{B})}$ where $f_1, \ldots, f_r \in \mathcal{D}$ are generators of J. By the Hilbert Nullstellensatz we have $\mathcal{V}(J) = \mathcal{V}(\operatorname{Ann}(\mathcal{B}))$. Define the matrix

$$\widehat{R} = \begin{bmatrix} R^e \\ f_1 I_{q\times q} \\ \vdots \\ f_r I_{q\times q} \end{bmatrix}.$$

For any point $\boldsymbol{\lambda} \in \mathcal{V}(\mathcal{B})$ we may observe $\widehat{R}(\boldsymbol{\lambda}) = [R^e(\boldsymbol{\lambda}) \ \ 0]^T$ and hence every exponential trajectory of $\mathcal{B}^e$ is also an exponential trajectory of $\ker_{\mathcal{A}}(\widehat{R})$; by noting that $\mathcal{B}^e$ is the closure of the span of exponential trajectories and taking limits, we get $\mathcal{B}^e \subset \ker_{\mathcal{A}}(\widehat{R})$. However, we also have

$$R^e = \begin{bmatrix} I_{p\times p} & 0_{p\times(rq)} \end{bmatrix} \begin{bmatrix} R^e \\ f_1 I_{q\times q} \\ \vdots \\ f_r I_{q\times q} \end{bmatrix}$$

and so by Theorem 2.1.48 we have the reverse containment $\ker_{\mathcal{A}}(\widehat{R}) \subset \mathcal{B}^e$. In this way, we arrive at $\mathcal{B}^e = \ker_{\mathcal{A}}(\widehat{R})$. The equality $\mathcal{V}(\mathcal{B}^e) = \mathcal{V}(\mathcal{B})$ yields the inclusion $\operatorname{Ann}(\mathcal{B}) \subset \sqrt{\operatorname{Ann}(\mathcal{B})}$. Conversely, for any $f \in \sqrt{\operatorname{Ann}(\mathcal{B})}$ we may write $f = a_1 f_1 + \cdots + a_r f_r$ to see

$$\begin{bmatrix} 0_{q\times p} & a_1 I_{q\times q} & \cdots & a_r I_{q\times q} \end{bmatrix} \begin{bmatrix} R' \\ f_1 I_{q\times q} \\ \vdots \\ f_r I_{q\times q} \end{bmatrix} = f I_{q\times q},$$

and thus $f \in \operatorname{Ann}(\mathcal{B}^e)$. Since $f \in \sqrt{\operatorname{Ann}(\mathcal{B})}$ was arbitrary, we conclude that $\operatorname{Ann}(\mathcal{B}^e) = \sqrt{\operatorname{Ann}(\mathcal{B})}$. □

The main goal of this section is address exponential closure and conditions for a behavior to be exponentially closed. In the end, we reach that $\mathcal{D}_r$-controllability is sufficient for reaching this equality.

Lemma 5.4.4. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^q$, $R \in \mathcal{D}^{p\times q}$, be a behavior with $\operatorname{Ann}(\mathcal{B}) = 0$. Then the exponentially closed behavior $\widetilde{\mathcal{B}} := \mathfrak{E}(\mathcal{B})$ has the same number of free variables as $\mathcal{B}$; in particular, the kernel representations of each behavior have the same rank.*

Proof. First note that since $\operatorname{Ann}(\mathcal{B}) = 0$, $\mathcal{V}(\operatorname{Ann}(\mathcal{B})) = k^d$; in particular, R loses rank over all affine space. Since $\mathfrak{G}(\mathcal{B}) = \mathfrak{G}(\widetilde{\mathcal{B}})$, the rank of the fibers are equal over k^d. As a result, for any kernel representation $\widetilde{\mathcal{B}} = \ker_{\mathcal{A}}(R')$, $R' \in \mathcal{D}^{p'\times q}$, we have the equality

$$\sqrt{\mathfrak{F}_i(R)} = \sqrt{\mathfrak{F}_i(R')} \qquad 1 \le i \le q.$$

As a result, $\operatorname{rank}(R) = \operatorname{rank}(R')$ and both $\widetilde{\mathcal{B}}$ and $\mathcal{B}$ have the same number of free variables. □

When a behavior is all torsion and its annihilator is prime, then the annihilator of the exponential closure is *equal* to the annihilator of the original behavior and the exponential closure maintains the same number of free variables as the original behavior. If the first condition were not the case, then the reduced behaviors would not be comparable since then they would be over different reduced signal spaces and reduced rings; however, the following lemma establishes that under certain conditions the exponential closure maintains the same annihilator.

Lemma 5.4.5. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R) \subset \mathcal{A}^q$, $R \in \mathcal{D}^{p\times q}$, be a behavior with $\operatorname{Ann}(\mathcal{B})$ a nonzero prime ideal and $\mathcal{B}_r$ be the behavior reduced by its annihilator. Let $\widetilde{\mathcal{B}}$ denote the behavior $\widetilde{\mathcal{B}} := \mathfrak{E}(\mathcal{B})$. Then $\operatorname{Ann}(\widetilde{\mathcal{B}}) = \operatorname{Ann}(\mathcal{B})$ and $\operatorname{Red}(\widetilde{\mathcal{B}})$ and $\mathcal{B}_r$ have the same number of free variables over the reduced ring $\mathcal{D}_r := \mathcal{D}/\operatorname{Ann}(\mathcal{B})$; in particular, their kernel representations have the same rank when considered over the reduced ring.*

Proof. Let $R' \in \mathcal{D}^{p'\times q}$ be a kernel representation of $\widetilde{\mathcal{B}}$ over the ring $\mathcal{D}$. Since $\mathfrak{G}(\widetilde{\mathcal{B}}) = \mathfrak{G}(\mathcal{B})$, by Theorem 4.1.4 and the Hilbert Nullstellensatz, $\sqrt{\operatorname{Ann}(\mathcal{B})} = \sqrt{\operatorname{Ann}(\widetilde{\mathcal{B}})} \neq 0$; as a consequence, Theorem 2.2.23 states R' has full column rank and thus $q \le p'$. Furthermore, equality of the fibers provides us with the equalities $\sqrt{\mathfrak{I}_i(R')} = \sqrt{\mathfrak{I}_i(R)}$ for $1 \le i \le q$.

Since $\widetilde{\mathcal{B}} \subset \mathcal{B}$, $\operatorname{Ann}(\mathcal{B}) \subset \operatorname{Ann}(\widetilde{\mathcal{B}})$; however, the equality $\sqrt{\operatorname{Ann}(\mathcal{B})} = \sqrt{\operatorname{Ann}(\widetilde{\mathcal{B}})}$ and assumption $\operatorname{Ann}(\mathcal{B})$ is prime implies that $\operatorname{Ann}(\mathcal{B}) = \operatorname{Ann}(\widetilde{\mathcal{B}})$. As a result, the reduced behaviors of $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are over the same ring and signal space. We may pass to the ring $\mathcal{D}_r$ and use that the determinantal ideals are, up to a power, equal to conclude that $\operatorname{rank}(R' \otimes 1_{\mathcal{D}_r}) = \operatorname{rank}(R \otimes 1_{\mathcal{D}_r})$. It follows that $\operatorname{Red}(\widetilde{\mathcal{B}})$ and $\mathcal{B}_r$ have the same number of free variables. $\square$

The above lemmas show that in both the classical and reduced settings, the exponentially closed behavior is a sub-behavior with the same number of free variables as the original behavior. We now establish conditions for determining equality of a behavior to its exponential closure.

Lemma 5.4.6. *Let $\mathcal{B} \subset \mathcal{A}^q$ be controllable and $\widetilde{\mathcal{B}} := \mathfrak{E}(\mathcal{B})$ be its exponential closure. Then $\mathcal{B} = \widetilde{\mathcal{B}}$.*

Proof. By Lemma 5.4.4, for kernel representations $R \in \mathcal{D}^{p\times q}$ and $\widetilde{R} \in \mathcal{D}^{p'\times q}$ of, respectively, $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ it follows that $\operatorname{rank}(R) = \operatorname{rank}(\widetilde{R})$. Since $\widetilde{\mathcal{B}} \subset \mathcal{B}$, there exists $X \in \mathcal{D}^{p\times p'}$ such that $R = X\widetilde{R}$. However, because $\mathcal{B}$ is controllable, by Theorem 2.2.19 R is GFLP; as a consequence, there exists $Y \in \mathcal{D}^{p'\times p}$ such that $\widetilde{R} = YR$. This implies that $\mathcal{B} \subset \widetilde{\mathcal{B}}$ and thus the equality $\mathcal{B} = \widetilde{\mathcal{B}}$. $\square$

We arrive at a similar property for the reduced setting.

Lemma 5.4.7. *Let $\mathcal{B} \subset \mathcal{A}^q$ is a behavior such that* $\operatorname{Ann}(\mathcal{B})$ *is a prime ideal and the reduced behavior $\mathcal{B}_r = \operatorname{Red}(\mathcal{B}) \subset \mathcal{A}_r^q$ is $\mathcal{D}_r$-controllable, then $\mathcal{B}_r = (\mathfrak{E}(\mathcal{B}))_r$.*

Proof. Set $\widetilde{\mathcal{B}} = \mathfrak{E}(\mathcal{B})$. By Lemma 5.4.5, $\operatorname{Ann}(\widetilde{\mathcal{B}}) = \operatorname{Ann}(\mathcal{B})$ and so we may reduce both $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ by $\operatorname{Ann}(\mathcal{B})$ to reach, respectively, $\mathcal{B}_r$ and $\widetilde{\mathcal{B}_r}$. By the same lemma, after passing to the reduced ring the kernel representations $R \in \mathcal{D}_r^{p\times q}$ and $R' \in \mathcal{D}_r^{p'\times q}$ of, respectively, $\mathcal{B}_r$ and $\widetilde{\mathcal{B}_r}$ have the same rank. Since $\widetilde{\mathcal{B}_r} \subset \mathcal{B}_r$, there exists $X \in \mathcal{D}_r^{p\times p'}$ such that $R = XR'$. However, because $\mathcal{B}_r$ is $\mathcal{D}_r$-controllable, by Theorem 5.3.1 R is GFLP over $\mathcal{D}_r$; as a consequence, there exists $Y \in \mathcal{D}_r^{p'\times p}$ such that $R' = YR$. This implies that $\mathcal{B}_r \subset \widetilde{\mathcal{B}_r}$ and hence $\mathcal{B}_r = \widetilde{\mathcal{B}_r}$. $\square$

Lemma 5.4.7 is quite useful when the only access we have to the behaviors is through their exponential trajectories. In the classical setting, we get a containment between the behaviors provided that at least one of them is controllable.

Lemma 5.4.8. *Let $\mathcal{B}, \mathcal{B}' \subset \mathcal{A}^q$ be two behaviors for which $\mathfrak{E}(\mathcal{B}) = \mathfrak{E}(\mathcal{B}')$. If $\mathcal{B}$ is controllable, then $\mathcal{B} \subset \mathcal{B}'$. If both $\mathcal{B}$ and $\mathcal{B}'$ are controllable, then $\mathcal{B} = \mathcal{B}'$.*

Proof. Let $R \in \mathcal{D}^{p\times q}$, $R' \in \mathcal{D}^{p'\times q}$, and $\widetilde{R} \in \mathcal{D}^{g\times q}$ be kernel representation of, respectively, $\mathcal{B}$, $\mathcal{B}'$, and $\mathfrak{E}(\mathcal{B})$. Since $\mathfrak{E}(\mathcal{B}) \subset \mathcal{B}$ and $\mathfrak{E}(\mathcal{B}') \subset \mathcal{B}'$, there exists matrices A and B with entries in $\mathcal{D}$ such that

$$R = A\widetilde{R} \qquad\qquad R' = B\widetilde{R}$$

with R, R' and $\widetilde{R}$ all of the same rank. However, since $\mathcal{B}$ is controllable, by Theorem 2.2.19 R is GFLP; as consequence, there exists a matrix X such that $\widetilde{R} = XR$. This implies

$$R' = BXR \quad \Rightarrow \quad \mathcal{B} \subset \mathcal{B}'.$$

If $\mathcal{B}'$ is also controllable, the reverse inclusion similarly follows. $\square$

We may extend the above argument to the reduced setting for a similar result. First we have the following.

Lemma 5.4.9. *Let $\mathcal{B}, \mathcal{B}' \subset \mathcal{A}^q$ be two behaviors for which $\mathfrak{C}(\mathcal{B}) = \mathfrak{C}(\mathcal{B}')$. If both* $\mathrm{Ann}(\mathcal{B})$ *and* $\mathrm{Ann}(\mathcal{B}')$ *are prime ideals, then* $\mathrm{Ann}(\mathcal{B}) = \mathrm{Ann}(\mathcal{B}')$.

Proof. By Lemma 5.4.5, $\mathrm{Ann}(\mathcal{B}) = \mathrm{Ann}(\mathfrak{C}(\mathcal{B}))$ and $\mathrm{Ann}(\mathcal{B}') = \mathrm{Ann}(\mathfrak{C}(\mathcal{B}'))$. Since $\mathfrak{C}(\mathcal{B}) = \mathfrak{C}(\mathcal{B}')$, we conclude that $\mathrm{Ann}(\mathcal{B}) = \mathrm{Ann}(\mathcal{B}')$. □

Lemma 5.4.10. *Let $\mathcal{B}, \mathcal{B}' \subset \mathcal{A}^q$ be two behaviors for which $\mathfrak{C}(\mathcal{B}) = \mathfrak{C}(\mathcal{B}')$ and both* $\mathrm{Ann}(\mathcal{B})$ *and* $\mathrm{Ann}(\mathcal{B}')$ *are prime ideals. If $\mathcal{B}_r := \mathrm{Red}(\mathcal{B})$ is $\mathcal{D}_r$-controllable, then $\mathcal{B}_r \subset \mathcal{B}'_r = \mathrm{Red}(\mathcal{B}')$. If both $\mathcal{B}$ and $\mathcal{B}'$ are $\mathcal{D}_r$-controllable, then $\mathcal{B}_r = \mathcal{B}'_r$.*

Proof. By Lemma 5.4.9, $\mathrm{Ann}(\mathcal{B}) = \mathrm{Ann}(\mathcal{B}')$. As a consequence, we may reduce both $\mathcal{B}$ and $\mathcal{B}'$ to the ring $\mathcal{D}_r = \mathcal{D}/\mathrm{Ann}(\mathcal{B})$ and reduced signal space $\mathcal{A}_r = \{w \in \mathcal{A}^q : hw = 0 \text{ for all } h \in \mathrm{Ann}(\mathcal{B})\}$.

Let $R \in \mathcal{D}_r^{p\times q}$, $R' \in \mathcal{D}_r^{p'\times q}$, and $\widetilde{R} \in \mathcal{D}_r^{g\times q}$ be kernel representation of, respectively, $\mathcal{B}_r$, $\mathcal{B}'_r$, and $\mathrm{Red}(\mathfrak{C}(\mathcal{B}))$. Since $\mathfrak{C}(\mathcal{B}) \subset \mathcal{B}$ and $\mathfrak{C}(\mathcal{B}') \subset \mathcal{B}'$, there exists matrices A and B with entries in $\mathcal{D}_r$ such that

$$R = A\widetilde{R} \qquad\qquad R' = B\widetilde{R}.$$

However, since $\mathcal{B}_r$ is $\mathcal{D}_r$-controllable, there exists a matrix X with entries in $\mathcal{D}_r$ such that $\widetilde{R} = XR$. This implies

$$R' = BXR \qquad \Rightarrow \qquad \mathcal{B}_r \subset \mathcal{B}'_r.$$

If $\mathcal{B}'_r$ is also $\mathcal{D}_r$-controllable, the reverse inclusion similarly follows. □

The results in this section demonstrate that controllability in both the classical and reduced setting allows one to connect frequency-domain analysis directly to time-domain analysis. However, absence of controllability allows one to only infer properties of a subbehavior from the frequency-domain.

5.5 Oberst-Zerz Lifting Theorem

In [59], Oberst and Zerz demonstrate that the canonical Cauchy Problem admits a solution over the integer lattice $\mathbb{Z}^d$. In part of their construction, they develop some machinery for changing the ring/signal space of a behavior to produce an isomorphic copy. While the ultimate goal of [loc. cit.] was to relate trajectories over $\mathbb{Z}^d$ to those over $\mathbb{N}^{2d}$, we may reapply their results to a different class of problems.

Notation. k is a field, D and D' are affine k-algebras, and A and A' are k-vector spaces. Assume also that we are given non-degenerate[3] k-bilinear forms

$$\begin{aligned} \langle -,-\rangle &: D \times A \to k & (d,a) &\mapsto \langle d,a\rangle \\ \langle -,-\rangle &: D' \times A' \to k & (d',a') &\mapsto \langle d',a'\rangle. \end{aligned} \tag{5.1}$$

Let $\phi : D' \to D$ be an k-algebra epimorphism. Under the pairings (5.1), we have that ϕ induces the adjoint injection

$$\phi^* : A \to A' \qquad \langle d', \phi^*(a)\rangle = \langle \phi(d'), a\rangle \qquad d' \in D', a \in A \tag{5.2}$$

Due to the non-degeneracy of the pairings, we have the isomorphisms

$$\begin{aligned} A &\cong \mathrm{Hom}_k(D,k) & a &\mapsto \langle -,a\rangle \\ A' &\cong \mathrm{Hom}_k(D',k) & a' &\mapsto \langle -,a'\rangle. \end{aligned}$$

We may impose a D-module structure on both A and A' by using the given parings as is normally done. However, with ϕ we may also consider A as a D'-module through the adjoint map ϕ^*. Define

$$d' \cdot a = \phi^*(d')a \qquad d' \in D', a \in A. \tag{5.3}$$

Because ϕ is only an epimorphism, it may have a kernel. Define the D'-submodule of A'

$$(0 : \ker(\phi)) = \langle a' \in A' : da' = 0 \text{ for all } d \in \ker(\phi)\}.$$

In particular, $\mathrm{Ann}_{D'}((0 : \ker(\phi))) = \ker(\phi)$.

The goal is to start with a behavior $\mathcal{B} \subset A^q$ and construct an isomorphic $\mathcal{B}' \subset A'^q$-system, where each behavior is defined over its associated ring. In this way, we are able to change a signal space or ring and reach an isomorphic system. The necessary ingredients are as follows.

Lemma 5.5.1. *[59, page 253] The map ϕ^* induces a D'-isomorphism $A \cong (0 : \ker(\phi))$.*

Proof. Because ϕ is surjective and the pairings are non-degenerate, ϕ^* is injective. Let $\iota : \ker(\phi) \hookrightarrow D'$ be the canonical injection map and consider the exact sequence

$$0 \longrightarrow \ker(\phi) \longrightarrow D' \xrightarrow{\phi} 0.$$

We may apply $\mathrm{Hom}_k(-,k)$ to reach the following commutative diagram with exact first row

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_k(D,k) \arrow[r, "\mathrm{Hom}(\phi,k)"] \arrow[d, "\cong"] & \mathrm{Hom}_k(D',k) \arrow[r, "\mathrm{Hom}(\iota,k)"] \arrow[d, "\cong"] & \mathrm{Hom}_k(\ker(\phi),k) \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r, "\phi^*"] & A' & & \end{tikzcd}$$

[3] Recall that we defined non-degenerate pairings in Section 2.1.3. This differs from [31, page 85] where Köthe uses a separation property instead. In our setting non-degenerate means $\{\langle -,a\rangle : a \in A\} \cong \mathrm{Hom}_k(D,k)$.

Exactness in the first row implies

$$\operatorname{im}(\operatorname{Hom}(\phi, k)) = \ker(\operatorname{Hom}(\iota, k)) = \{a' \in \operatorname{Hom}_k(D', k) : a' \circ \iota = 0\}.$$

By nondegeneracy and $a' \circ \iota = a'|_{\ker(\phi)}$, we reach

$$\operatorname{im}(\phi^*) = \{a \in A' : \langle -, a\rangle|_{\ker(\phi)} = 0\} = \{a' \in A' : \langle \ker(\phi), a'\rangle = 0\}.$$

The above demonstrates that $\operatorname{im}(\phi^*) = (0 : \ker(\phi))$, i.e. ϕ^* is surjective. Because ϕ^* is defined through the pairings, a routine calculation using (5.2) demonstrates that ϕ^* is D'-linear, i.e. for $d' \in \mathcal{D}'$, $\phi^*(d'a) = d'\phi^*(a)$. □

Theorem 5.5.2. *[59, page 253-254] Let $R \in D^{p\times q}$ be an arbitrary matrix and $R' \in D'^{p\times q}$ be a matrix such that $\phi(R') = R$. Let $\ker(\phi) = \langle p_1, \ldots, p_n\rangle$ where the p_* are generators of $\ker(\phi)$. Define*

$$\widehat{R} = \begin{bmatrix} R' \\ \Lambda_q(p_1) \\ \vdots \\ \Lambda_q(p_n) \end{bmatrix} \in D'^{(k+nq)\times q}.$$

Then the behaviors $\mathcal{B} = \ker_A(R)$ and $\mathcal{B}' = \ker_{A'}(\widehat{R})$ are D'-isomorphic via

$$w = (w_1, \ldots, w_q) \mapsto \phi^*(w) = (\phi^*(w_1), \ldots, \phi^*(w_q)).$$

Proof. By Lemma 5.5.1, we have the isomorphism $A^q \cong (0 : \ker(\phi))$ given by

$$w \mapsto \phi^*(w) \qquad\qquad w \in A^q.$$

Since $\ker(\phi) = \langle p_1, \ldots, p_n\rangle$ we have

$$(0 : \ker(\phi)) = \{w' \in A' : p_i' w' = 0 \text{ for all } i = 1, \ldots, n\}$$

and thus

$$(0 : \ker(\phi))^q = \ker_{A'} \begin{bmatrix} \Lambda_q(p_1) \\ \vdots \\ \Lambda_q(p_n) \end{bmatrix}.$$

As a result, we have

$$\mathcal{B}' = \{w' \in A'^q : \widehat{R}w' = 0\} = \{w' \in (0 : \ker(\phi))^q : R'w' = 0\}.$$

Since $\operatorname{im}_A(\phi^*)^q = (0 : \ker(\phi))^q$,

$$\mathcal{B}' = \{\phi^*(w) : \text{ there exist } w \in A^q \text{ where } R'\phi^*(w) = 0\}.$$

Noting that A is a D'-module, by (5.3) we have

$$Rw = \phi(R')w = R'w.$$

If $Rw = 0$, then

$$0 = \phi^*(Rw) = \phi^*(R'w) = R'\phi^*(w).$$

Similarly, since ϕ^* is injective,

$$0 = R'\phi^*(w) = \phi^*(R'w) = \phi^*(Rw)$$

this implying $Rw = 0$. We conclude that $\mathcal{B}' = \phi^*(\mathcal{B})$. $\square$

In the following section we use Theorem 5.5.2 when ϕ is a k-algebra *isomorphism* between polynomial rings. In this context, we may significantly simplify the preceding arguments; instead of considering two distinct signal spaces and rings, we may instead consider ϕ as a k-algebra automorphism.

If ϕ is a k-algebra isomorphism $\phi : D' \to D$ then for any scalar polynomial $h \in D$ and signal $w \in A$ we readily observe

$$\langle d, \phi^*(hw)\rangle = \langle \phi(d), hw\rangle = \langle h\phi(d), w\rangle = \langle \phi(\phi^{-1}(h)d), w\rangle = \langle \phi^{-1}(h)d, \phi^*(w)\rangle = \langle d, \phi^{-1}(h)\phi^*(w)\rangle$$

from which we conclude that

$$\phi^*(hw) = \phi^{-1}(h)\phi^*(w). \tag{5.4}$$

Let us now consider the case when $D = \mathcal{D}$ and $A = \mathcal{A}$. For a matrix R over $\mathcal{D}$, we let $\phi(R)$ be the matrix over D defined entry-wise.

Lemma 5.5.3. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ and $\mathcal{B}' = \ker_{\mathcal{A}}(R')$ be two given behaviors, $\phi : \mathcal{D} \to \mathcal{D}$ a k-algebra automorphism, and ϕ^* its induced adjoint isomorphism. If $\phi(R') = R$, then $\mathrm{Ann}(\mathcal{B}') = \phi^{-1}(\mathrm{Ann}(\mathcal{B}))$.*

Proof. Since $\phi(R') = R$, we may apply Theorem 5.5.2 to reach $\phi^*(\mathcal{B}) = \mathcal{B}'$. For $h \in \mathrm{Ann}(\mathcal{B})$ and non-trivial $w \in \mathcal{B}'$ we may apply (5.4) to observe

$$\phi^{-1}(h)w = \phi^{-1}(h)\phi^*(\phi^{*-1}(w)) = \phi^*(h\phi^{*-1}(w)) = 0$$

since $\phi^{*-1}(\mathcal{B}') = \mathcal{B}$. This demonstrates that $\phi^{-1}(\mathrm{Ann}(\mathcal{B})) \subset \mathrm{Ann}(\mathcal{B}')$. For the other inclusion, let $h' \in \mathrm{Ann}(\mathcal{B}')$ be given and let $w \in \mathcal{B}$ be given. By (5.4) we have

$$\phi^*(\phi(h)w) = \phi^{-1}(\phi(h))\phi^*(w) = h\phi^*(w) = 0.$$

Because ϕ^* is an isomorphism, we have $\phi(h)w = 0$, thus implying $\mathrm{Ann}(\mathcal{B}') \subset \phi^{-1}(\mathrm{Ann}(\mathcal{B}))$. We conclude that $\mathrm{Ann}(\mathcal{B}') = \phi^{-1}(\mathrm{Ann}(\mathcal{B}))$. $\square$

Corollary 5.5.4. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ and $\mathcal{B}' = \ker_{\mathcal{A}}(R')$ be two given behaviors, $\phi : \mathcal{D} \to \mathcal{D}$ a k-algebra automorphism, and ϕ^* its induced adjoint isomorphism. If $\phi(R') = R$ and $\mathrm{Ann}(\mathcal{B})$ is a prime ideal, then $\mathrm{Ann}(\mathcal{B}') = \phi^{-1}(\mathrm{Ann}(\mathcal{B}))$ and $\mathrm{Ann}(\mathcal{B}')$ is also a prime ideal.*

We are now concerned with $\mathcal{D}_r$-controllability being preserved in this context. We start with a simple preparatory proposition.

Proposition 5.5.5. *Let $\phi : \mathcal{D} \to \mathcal{D}$ be a k-algebra automorphism, $I \subset \mathcal{D}$ be an ideal and $I' := \phi(I) \subset \mathcal{D}$ be its inverse. The induced map $\phi \otimes 1_{\mathcal{D}/I'} : \mathcal{D}/I \to \mathcal{D}/I'$ is a well-defined k-algebra isomorphism.*

Proof. For any $[h] \in \mathcal{D}/I$ with representatives $h_1, h_2 \in \mathcal{D}$ such that $h_1 - h_2 \in I$, since $I' = \phi(I)$. we have

$$\phi(h_1 - h_2) \otimes 1_{\mathcal{D}/I'} = 0.$$

As a consequence $\phi \otimes 1_{\mathcal{D}/I'}$ is well defined. Let $h \in \mathcal{D}/I$ be such that $\phi(h) \otimes 1_{\mathcal{D}/I'} = 0$. It follows that $\phi(h) \in I'$, thus implying $h \in I$ and making $\phi \otimes 1_{\mathcal{D}/I'}$ injective. For any $h' \in \mathcal{D}/I'$, let $\ell \in \mathcal{D}$ be a representative. It follows that

$$\phi(\phi^{-1}(\ell)) \otimes 1_{\mathcal{D}/I'} = \ell \otimes 1_{\mathcal{D}/I'} = h'.$$

For any other representative ℓ' such that $\ell - \ell' \in I'$ we have

$$\phi(\phi^{-1}(\ell - \ell')) \otimes 1_{\mathcal{D}/I'} = (\ell - \ell') \otimes 1_{\mathcal{D}/I'} = 0.$$

Since $\phi \otimes 1_{\mathcal{D}/I'}$ is injective, $\phi^{-1}(\ell - \ell') \in I$ thus demonstrating $\phi^{-1}(\ell) - \phi^{-1}(\ell') \in I$. We conclude that $\phi \otimes 1_{\mathcal{D}/I'}$ is a well-defined k-algebra isomorphism. □

Lemma 5.5.6. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ and $\mathcal{B}' = \ker_{\mathcal{A}}(R')$ be two given behaviors, $\phi : \mathcal{D} \to \mathcal{D}$ a k-algebra automorphism and ϕ^* its induced adjoint isomorphism. If $\phi(R') = R$, $\mathrm{Ann}(\mathcal{B})$ is a prime ideal, and $\mathrm{Red}(\mathcal{B})$ is $\mathcal{D}/\mathrm{Ann}(\mathcal{B})$-controllable then $\mathrm{Red}(\mathcal{B}')$ is $\mathcal{D}/\mathrm{Ann}(\mathcal{B}')$-controllable.*

Proof. Define $\mathcal{B}_r = \mathrm{Red}(\mathcal{B})$, $\mathcal{B}'_r = \mathrm{Red}(\mathcal{B}')$, $\mathcal{D}_r = \mathcal{D}/\mathrm{Ann}(\mathcal{B})$, $\mathcal{D}'_r = \mathcal{D}/\mathrm{Ann}(\mathcal{B}')$, and $\mathcal{A}_r$ and $\mathcal{D}'_r$ as the respective signal spaces of $\mathcal{D}_r$ and $\mathcal{D}'_r$. From Corollary 5.5.4 we see that $\mathcal{D}'_r$ is an integral domain. By Proposition 5.5.5, $\widehat{\phi} := \phi \otimes 1_{\mathcal{D}_r}$ is a well-defined k-algebra isomorphism. Let $\widehat{R} = R \otimes 1_{\mathcal{D}_r}$ and $\widehat{R'} = R' \otimes 1_{\mathcal{D}'_r}$. Since $\phi(R') = R$, we have

$$\widehat{\phi}(\widehat{R'}) = \phi(R') \otimes 1_{\mathcal{D}_r} = R \otimes 1_{\mathcal{D}_r} = \widehat{R}.$$

By Theorem 5.5.2 we have $\mathcal{B}'_r = \widehat{\phi}^*(\mathcal{B}_r)$. Since $\mathcal{B}_r$ is $\mathcal{D}_r$-controllable, there exists a matrix $M \in \mathcal{D}_r^{q \times g}$ such that $\ker_{\mathcal{A}_r}(\widehat{R}) = \mathrm{im}_{\mathcal{A}_r}(M)$. We now argue that $M' := \widehat{\phi}^{-1}(M)$ is an image representation for $\mathcal{B}'_r$. Since $\widehat{\phi}$ is an isomorphism,

$$\widehat{\phi}(\widehat{R'}M') = \widehat{R}M = 0$$

implies that $\widehat{R'}M' = 0$, thus leading to the inclusion $\mathrm{im}_{\mathcal{A}'_r}(M) \subset \ker_{\mathcal{A}'_r}(R)$. For $w' \in \mathcal{B}'_r$, there exists $v \in \mathcal{A}_r{}^g$ such that

$$Mv = \widehat{\phi^*}^{-1}(w').$$

By (5.4) we reach

$$w' = \widehat{\phi}^*(Mv) = \widehat{\phi}^{-1}(M)\widehat{\phi}^*(v) = M'\widehat{\phi}^*(v),$$

thus demonstrating the equality $\ker_{\mathcal{A}'_r}(\widehat{R'}) = \mathrm{im}_{\mathcal{D}'_r}(M')$. We conclude that $\mathcal{B}'_r$ is $\mathcal{D}'_r$-controllable. □

The consequences of Lemma 5.5.6 are rather interesting since it demonstrates that under any automorphism of the ring, a behavior will retain $\mathcal{D}_r$-controllability. In particular, one can perform a "change of coordinates" of the behavior to reduce the kernel representation to a more favorable form; once this is done, then certain structural properties are preserved.

Chapter 6

Livšic Systems and Vessels

The most common types of input/state/output linear systems considered in the multidimensional system theory literature are those of Fornasini-Marchesini or Givone-Roesser types where there is a well-defined state and output trajectory for any choice of input signal and appropriate state-vector initial condition. Another type of input/state/output system originating in the work of Livšic has appeared in the mathematical literature, where the state-space equations are overdetermined; the constraint that the determination of the state vector be well defined leads to compatibility conditions on the state operators and the input signal; such compatibility conditions are exactly what one encounters when looking at autonomous behaviors.

In this section, we provide a brisk overview of Livšic systems and their associated properties. Once a solid footing has been established, we proceed to discuss how Livšic systems can be considered as two-dimensional behaviors with autonomy degree one. We then proceed to the discussion of determinantal representations and conditions for a behavior to have a determinantal representation as a kernel representation. Once the main ingredients are provided, we reach the ultimate goal of this section: to relate 2D behaviors with degree one autonomy to Livšic systems. To provide a mechanism for this connection, we must discuss controllability of both types: behavioral $\mathcal{D}_r/j$-controllability and Livšic controllability. Our previous discussion of j-controllability proves to be invaluable in showing the equivalence of the two. Once this has been established, we may relate two behaviors by their reduced signal flow systems and transfer functions.

The main tool we use to construct the state space representation of a degree one autonomy behavior is an analogue of the Hankel realization for the classical case. This operates on the level of bundles; as such, we use the frequency domain to link behaviors and Livšic systems and use the equivalence of controllability to ensure equality. When this is combined with the pure autonomy degree decomposition, we reach a complete state-space representation theory for two-dimensional behaviors.

6.1 Overview of 2D Discrete Livšic Systems and Vessels

Throughout this section we shall be concerned with discrete-time 2D input/state/output colligations of the form

$$\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{E}_* \end{bmatrix}.$$

which are given by

$$\begin{aligned} x(t+e_1) &= A_1 x(t) + B_1 u(t) \\ x(t+e_2) &= A_2 x(t) + B_2 u(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{6.1}$$

In the above, $t = (t_1, t_2) \in \mathbb{N}^2$ and the coordinates e_1 and e_2 denote the standard basis vectors for the free $\mathbb{N}$-module $\mathbb{N}^2$. In this way, $t + e_1$ represents a backward shift in the horizontal direction and $t + e_2$ represents a backward shift in the vertical direction. The vector $x(t)$ takes values in the state space $\mathcal{X}$, $u(t)$ takes values in the input space $\mathcal{U}$, and $y(t)$ takes values in the output space $\mathcal{Y}$. The state, input, and output spaces are all finite dimensional complex vector spaces. References for Livšic systems include [7, 47, 8, 48, 49, 34, 1, 9, 33].

Compatibility Conditions

The system given by (6.1) is **overdetermined** in the sense that there are two ways to compute $x(t + e_1 + e_2)$ from $x(t)$:

$$\begin{aligned} x(t) &\mapsto x(t+e_1) \mapsto x(t+e_1+e_2) \\ x(t) &\mapsto x(t+e_2) \mapsto x(t+e_2+e_1). \end{aligned}$$

This leads us to the discussion of **compatibility conditions** for the system to exhibit solutions.

(Zero Input Evolution). Let $u(t) = 0$ for all $t \in \mathbb{N}^2$ with an arbitrary initial state $x(0)$. By the state-evolution equations in (6.1), we reach the following.

$$x(e_1 + e_2) = A_2 x(e_1) = A_2 A_1 x(0) \qquad x(e_1 + e_2) = A_1 x(e_1) = A_1 A_2 x(0).$$

Because the above relation must hold for every choice of $x(0) \in \mathcal{X}$, it must be the case that A_1 and A_2 commute. This leads us to the first requirement.

$$A_1 A_2 = A_2 A_1 \tag{6.2}$$

(Input Signal Compatibility). If a nonzero input signal is present the two paths for state evolution lead us to the following equations.

$$\begin{aligned} x(t+e_1+e_2) &= A_1 x(t+e_2) + B_1 u(t+e_2) = A_1(A_2 x(t) + B_2 u(t)) + B_1 u(t+e_2) \\ x(t+e_1+e_2) &= A_2 x(t+e_1) + B_2 u(t+e_1) = A_2(A_1 x(t) + B_1 u(t)) + B_2 u(t+e_1). \end{aligned}$$

After subtracting the the top equation from the bottom equation and using the commutativity condition (6.2), we reach

$$B_2u(t+e_1) - B_1u(t+e_2) + (A_2B_1 - A_1B_2)u(t) = 0. \tag{6.3}$$

We now assume that we have factorizations of B_1, B_2, and $A_2B_1 - A_1B_2$ of the form

$$\begin{cases} B_1 = \widetilde{B}\sigma_1 \\ B_2 = \widetilde{B}\sigma_2 \\ A_2B_1 - A_1B_2 = A_2\widetilde{B}\sigma_1 - A_1\widetilde{B}\sigma_2 = \widetilde{B}\gamma \end{cases} \tag{6.4}$$

In the above, σ_1, σ_2, and γ are continuous linear mappings from the input space $\mathcal{U}$ into some auxiliary input space $\widetilde{\mathcal{U}}$; the continuous linear map $\widetilde{B} : \widetilde{\mathcal{U}} \to \mathcal{X}$ maps to the state space from the auxiliary input space. By using (6.4) in (6.3) we see that

$$\widetilde{B}\left(\sigma_2u(t+e_1) - \sigma_1u(t+e_2) + \gamma u(t)\right) = 0.$$

One way for the above to hold is to assume that the input signal satisfies the following **input compatibility condition**:

$$\sigma_2u(t+e_1) - \sigma_1u(t+e_2) + \gamma u(t) = 0. \tag{6.5}$$

(Output Signal Compatibility). We now seek conditions similar to (6.5) for the output signal. Let σ_{1*}, σ_{2*}, and γ_* be continuous linear maps from the output space $\mathcal{Y}$ to an auxiliary output space $\widetilde{\mathcal{Y}}$. Define the **output compatibility condition**

$$\sigma_{2*}y(t+e_1) - \sigma_{1*}y(t+e_2) + \gamma_*y(t) = 0. \tag{6.6}$$

After substituting $y(t) = Cx(t) + Du(t)$ into (6.6) we see that

$$\begin{aligned} &(\sigma_{2*}CA_1 - \sigma_{1*}CA_2 + \gamma_*C)\,x(t) + \\ &\qquad \sigma_{2*}Du(t+e_1) + \sigma_{1*}Du(t+e_2) + \left(\sigma_2C\widetilde{B}\sigma_1 - \sigma_{1*}C\widetilde{B}\sigma_2 + \gamma_*D\right)u(t) = 0. \end{aligned}$$

For the above identity to hold (under the assumption that the input compatibility condition is satisfied) we require that there exist a continuous linear map $\widetilde{D} : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}_*$ so that we have the following factorizations:

$$\begin{cases} \sigma_{1*}CA_2 - \sigma_{2*}CA_1 = \gamma_*C \\ \sigma_{2*}D = \widetilde{D}\sigma_2 \\ \sigma_{1*}D = \widetilde{D}\sigma_1 \\ \sigma_{2*}C\widetilde{B}\sigma_1 - \sigma_{1*}C\widetilde{B}\sigma_2 + \gamma_*D = \widetilde{D}\gamma. \end{cases} \tag{6.7}$$

The stated factorizations and compatibility conditions lead us to the definition of a vessel.

Definition 6.1.1. We define a **vessel** to be a collection of operators

$$\mathfrak{V} = (A_1, A_2, \widetilde{B}, C, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*)$$

where

$$A_1, A_2 \in \mathcal{L}(\mathcal{X}), \qquad \widetilde{B} \in \mathcal{L}(\widetilde{\mathcal{U}}, \mathcal{X}), \qquad C \in \mathcal{L}(\mathcal{X}, \mathcal{Y}), \qquad D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$$
$$\widetilde{D} \in \mathcal{L}(\widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}}), \qquad \sigma_1, \sigma_2, \gamma \in \mathcal{L}(\mathcal{U}, \widetilde{\mathcal{U}}), \qquad \sigma_{1*}, \sigma_{2*}, \gamma_* \in \mathcal{L}(\mathcal{Y}, \widetilde{\mathcal{Y}})$$

are subject to conditions

$$\begin{cases} A_1 A_2 = A_2 A_1 & \text{(commutativity)} \\ A_2 \widetilde{B} \sigma_1 - A_1 \widetilde{B} \sigma_2 = \widetilde{B} \gamma & \text{(input vessel condition)} \\ \sigma_{1*} C A_2 - \sigma_{2*} C A_1 = \gamma_* C & \text{(output vessel condition)} \\ \sigma_{2*} C \widetilde{B} \sigma_1 - \sigma_{1*} C \widetilde{B} \sigma_2 + \gamma_* D = \widetilde{D} \gamma & \text{(linkage conditions)} \\ \sigma_{1*} D = \widetilde{D} \sigma_1 & \\ \sigma_{2*} D = \widetilde{D} \sigma_2 & \end{cases}$$

The derivation of the various compatibility conditions and factorizations leading up to the definition of a vessel naturally suggest the following properties of a vessel.

Proposition 6.1.2. *Let $\mathfrak{V}$ be a vessel associated with the input/state/output system*

$$\begin{aligned} x(t + e_1) &= A_1 x(t) + B_1 u(t) \\ x(t + e_2) &= A_2 x(t) + B_2 u(t) \\ y(t) &= C x(t) + D u(t). \end{aligned}$$

*The above system is consistent over $\mathbb{N}^2$ for any choice of initial state $x(0)$ as long as the input signal satisfies the **input compatibility condition***

$$\sigma_2 u(t + e_1) - \sigma_1 u(t + e_2) + \gamma u(t) = 0.$$

*In this case, the output signal satisfies the **output compatibility condition***

$$\sigma_{2*} y(t + e_1) - \sigma_{1*} y(t + e_2) + \gamma_* y(t) = 0.$$

For the remainder of this paper we make the following two additional assumptions about vessels.

Nondegeneracy Assumption (ND) *There are complex scalars ξ_1 and ξ_2 such that*

$$\text{im } \xi_1 \sigma_1 + \xi_2 \sigma_2 = \widetilde{\mathcal{U}} \qquad\qquad \text{im } \xi_1 \sigma_{1*} + \xi_2 \sigma_{2*} = \widetilde{\mathcal{Y}}.$$

Finite-Dimensionality (FD) *The input spaces $\mathcal{U}$ and $\widetilde{\mathcal{U}}$, the state space $\mathcal{X}$, and the output spaces $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are finite dimensional with $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ of the same finite dimension and $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ of the same finite dimension.* We define

$$\mathcal{U} = \widetilde{\mathcal{U}} = \mathbb{C}^m \qquad\qquad \mathcal{X} = \mathbb{C}^n \qquad\qquad \mathcal{Y} = \widetilde{\mathcal{Y}} = \mathbb{C}^{m_*}.$$

When the two assumptions are combined, we have the following.

ND-FD. The pencils $\xi_i\sigma_1 + \xi_2\sigma_2$ and $\xi_1\sigma_{1*} + \xi_2\sigma_{2*}$ are nonsingular.

To simplify matters, we also have the following assumption. Note that we use the notation $\mathcal{V}(\cdot)$ to denote the **algebraic set** or **zero locus** of an ideal in $\mathcal{D}$.

Irreducibility/Maximality Assumption (IMA) *For the matrix pencils $U(z_1, z_2)$ and $U_*(z_1, z_2)$ we have $\det U(z_1, z_2) = f^r$ and $\det U_*(z_1, z_2) = f^{r_*}$ where f is an irreducible polynomial over $\mathbb{C}[z_1, z_2]$ and*

$$\dim \ker_{\mathbb{C}}(U(\boldsymbol{\lambda})) = r \text{ and } \dim \ker_{\mathbb{C}}(U_*(\boldsymbol{\lambda})) = r_* \qquad \text{for all } \boldsymbol{\lambda} \in \mathcal{V}(f) \setminus \mathbf{S}$$

where $\mathbf{S}$ consists of the finite set of singularities of the curve $\mathcal{V}(f)$.

Assumption 6.1.1. For the vessel $\mathfrak{V}$ we assume that it satisfies the nondegeneracy assumption, the finite-dimensionality assumption, and the maximality assumption.

6.2 Frequency Domain Analysis

Consider a **discrete wave trajectory**

$$u(t) = u_0\lambda_1^{t_1}\lambda_2^{t_2} \qquad x(t) = x_0\lambda_1^{t_1}\lambda_2^{t_2} \qquad y(t) = y_0\lambda_1^{t_1}\lambda_2^{t_2}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, $u_0 \in \mathcal{E}$, $x_0 \in \mathcal{X}$, and $y_0 \in \mathcal{E}_*$. When $u(t)$ has the form as stated, then the input compatibility condition becomes

$$(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)\, u_0 = 0.$$

For $u_0 \neq 0$, the above matrix must exhibit a non-trivial kernel. As a result, for

$$p(\lambda_1, \lambda_2) = \det\left(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma\right),$$

it must be the case that $p(\lambda_1, \lambda_2) = 0$. Define the family of vector spaces

$$\boldsymbol{\mathcal{E}} = \{(\boldsymbol{\lambda}, u_0) : p(\boldsymbol{\lambda}) = 0 \text{ and } u_0 \in \boldsymbol{\mathcal{E}}(\boldsymbol{\lambda})\}$$

where the fibers are defined as

$$\boldsymbol{\mathcal{E}}(\boldsymbol{\lambda}) = \ker\left(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma\right).$$

We call $\boldsymbol{\mathcal{E}}$ the **input kernel bundle** and p the **input discriminant polynomial**. Similarly, we define $\boldsymbol{\mathcal{E}}_*$ as the **output kernel bundle** and p_* as the **output discriminant polynomial**:

$$\begin{aligned} p_*(\boldsymbol{\lambda}) &= \det\left(\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*\right) \\ \boldsymbol{\mathcal{E}}_* &= \{(\boldsymbol{\lambda}, y_0) : p_*(\boldsymbol{\lambda}) = 0 \text{ and } u_0 \in \boldsymbol{\mathcal{E}}(\boldsymbol{\lambda})\} \\ \boldsymbol{\mathcal{E}}_*(\boldsymbol{\lambda}) &= \ker\left(\lambda_1\sigma_{2*} - \lambda_2\sigma_{1*} + \gamma_*\right). \end{aligned}$$

By considering signals which are discrete wave trajectories, we may rewrite the system equations (6.1) to reach

$$\begin{aligned} \lambda_1 x_0 &= A_1 x_0 + \widetilde{B}\sigma_1 u_0 \\ \lambda_2 x_0 &= A_2 x_0 + \widetilde{B}\sigma_2 u_0 \\ y_0 &= C x_0 + D u_0. \end{aligned}$$

Let $(\xi_1, \xi_2) \in (\mathbb{C} \setminus \{0\})^2$ be any given point. We may combine the two state equations to arrive at

$$(\xi_1\lambda_1 + \xi_2\lambda_2)x_0 = (\xi_1 A_1 + \xi_2 A_2)x_0 + \widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)u_0.$$

If the frequency (λ_1, λ_2) is not contained in the joint spectrum of (A_1, A_2), then we can solve for x_0 and y_0 to arrive at (recall that the **FD-ND** assumption is being made)

$$\begin{aligned} x_0 &= ((\xi_1\lambda_1 + \xi_2\lambda_2)I - (\xi_1 A_1 + \xi_2 A_2))^{-1}\widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)u_0 \\ y_0 &= S_{\mathfrak{V}}(\boldsymbol{\lambda})u_0 \end{aligned}$$

where $S_{\mathfrak{V}}(\boldsymbol{\lambda}) : \boldsymbol{\mathcal{E}}(\boldsymbol{\lambda}) \to \mathcal{E}_*(\boldsymbol{\lambda})$ is given by

$$S_{\mathfrak{V}}(\boldsymbol{\lambda}) = D + C((\xi_1\lambda_1 + \xi_2\lambda_2)I - (\xi_1 A_1 + \xi_2 A_2))^{-1}\widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2)\Big|_{\boldsymbol{\mathcal{E}}(\boldsymbol{\lambda})}. \tag{6.8}$$

We call $S(\boldsymbol{\lambda})$ the **joint transfer function** of the system at the frequency (λ_1, λ_2). If we introduce the **complete transfer function** $W_{\mathfrak{V}}(\xi_1, \xi_2, z)$ according to

$$W_{\mathfrak{V}}(\xi_1, \xi_2, z) = D + C(zI - (\xi_1 A_1 + \xi_2 A_2))^{-1}\widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2), \tag{6.9}$$

then we may rewrite (6.8) as

$$S_{\mathfrak{V}}(\boldsymbol{\lambda}) = W(\xi_1, \xi_2, \xi_1\lambda_1 + \xi_2\lambda_2)|_{\boldsymbol{\mathcal{E}}(\boldsymbol{\lambda})}.$$

Note that $W(c\xi_1, c\xi_2, z) = cW(\xi_1, \xi_2, z)$ for any scalar $c \in \mathbb{C}$, so W can be viewed as a function of two complex variables (one projective and one affine). Given a vessel as in Definition 6.1.1, the **dual vessel** $\mathfrak{V}^{\sim}$ is given by

$$\mathfrak{V}^{\sim} = (A_1^T, A_2^T, C^T, \widetilde{B}^T, \widetilde{D}^T, D^T, \sigma_{1*}^T, \sigma_{2*}^T, \gamma_*^T, \sigma_1^T, \sigma_2^T, \gamma^T).$$

It is routine to check that the collection $\mathfrak{V}^{\sim}$ satisfies all the vessel axioms exactly when $\mathfrak{V}$ does. If we introduce the **dual complete transfer function** $W_{\mathfrak{V}^{\sim}}$ according to

$$W_{\mathfrak{V}^{\sim}}(\xi_1, \xi_2, z) = \widetilde{D}^T + \widetilde{B}^T(zI - (\xi_1 A_1^T + \xi_2 A_2^T))^{-1}C^T(\xi_1\sigma_{1*}^T + \xi_2\sigma_{2*}^T), \tag{6.10}$$

then the **dual joint transfer function** $S_{\mathfrak{V}^{\sim}}(\boldsymbol{\lambda})$ is given by

$$S_{\mathfrak{V}^{\sim}}(\boldsymbol{\lambda}) = W_{\mathfrak{V}^{\sim}}(\xi_1, \xi_2, \xi_1\lambda_1 + \xi_2\lambda_2)|_{\boldsymbol{\mathcal{E}}^{\sim}(\boldsymbol{\lambda})} \tag{6.11}$$

where $\boldsymbol{\mathcal{E}}^{\sim}(\boldsymbol{\lambda}) = \ker U(\boldsymbol{\lambda})^T$. Note the form for $W_{\mathfrak{V}^{\sim}}^T$:

$$W_{\mathfrak{V}^{\sim}}(\xi_1, \xi_2, z)^T = \widetilde{D} + (\xi_1\sigma_{1*} + \xi_2\sigma_{2*})C(zI - (\xi_1 A_1 + \xi_2 A_2))^{-1}\widetilde{B}.$$

Given any vessel $\mathfrak{V}$ as in Definition 6.1.1, we let U and U_* denote the matrix pencils associated with the input/output compatibility conditions:

$$U(z_1, z_2) = \sigma_2 z_1 - \sigma_1 z_2 + \gamma \tag{6.12}$$

$$U_*(z_1, z_2) = z_1\sigma_{2*} - z_2\sigma_{1*} + \gamma_*. \tag{6.13}$$

The following useful intertwining property between complete transfer functions and determinantal representations will prove useful in Section 6.6 below; we refer to [6] for the proof and a discussion of the history.

Proposition 6.2.1. *For a vessel* $\mathfrak{V}$ *and* $(\xi_1, \xi_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ *we have the identity*

$$U_*(z_1, z_2)W_{\mathfrak{V}}(\xi_1, \xi_2, \xi_1 z_1 + \xi_2 z_2) = W_{\mathfrak{V}^\sim}(\xi_1, \xi_2, \xi_1 z_1 + \xi_2 z_2)^T U(z_1, z_2).$$

6.3 Vessels as 2D Behaviors with Degree One Autonomy

We now demonstrate how vessels can be interpreted as 2D behaviors with degree one autonomy. Define the (tall) kernel representation

$$R^{i/s/o} = \begin{bmatrix} A_1 - z_1 I & \widetilde{B}\sigma_1 & 0 \\ A_2 - z_2 I & \widetilde{B}\sigma_2 & 0 \\ C & D & -I \\ 0 & U(z_1, z_2) & 0 \\ 0 & 0 & U_*(z_1, z_2) \end{bmatrix} \tag{6.14}$$

where U and U_* are as in (6.12) and (6.13). We then define the **augmented behavior associated with the vessel** as

$$\mathcal{B}^{i/s/o} = \left\{ w = \mathrm{col}(x, u, y) \in (\mathcal{X} \oplus \mathcal{U} \oplus \mathcal{Y})^{\mathbb{N}^2} : R^{i/s/o} w = 0 \right\}.$$

It is a consequence of Proposition 6.1.2 that one can drop the last row of $R^{i/s/o}$ without changing the behavior but it is more convenient for our purposes to work with $R^{i/s/o}$ as it is in (6.14). The time domain is $\mathbb{N}^2$, the associated ring of operators is $\mathcal{D} = \mathbb{C}[z_1, z_2]$, the signal space is $\mathcal{A} = \mathbb{C}^{\mathbb{N}^2}$, $d = \dim(\mathcal{X}) + \dim(\mathcal{U}) + \dim(\mathcal{Y})$, and $\mathcal{B}^{i/s/o} \subset \mathcal{A}^d$.

Before continuing, we introduce the role of minimal left annihilators (MLAs) as a mechanism for projecting a behavior onto a subset of its components. Recall from Definition 2.2.11 that a matrix $R \in \mathcal{D}^{p \times q}$ is an MLA of a matrix $M \in \mathcal{D}^{q \times g}$ if $RM = 0$ and for any $R' \in \mathcal{D}^{p' \times q}$ such that $R'M = 0$ there exists a matrix $X^{p' \times p}$ such that $R' = XR$.

Let $R \in \mathcal{D}^{p \times q}$ be an arbitrary matrix and let us write $R = [-Q \quad P]$. Note that any solution to the problem $Rw = 0$ must then satisfy $Qu = Py$ for $w = (u, y)$ and vice versa. We are interested in projecting the behavior $\mathcal{B} = \ker_{\mathcal{A}}(R)$ on the the components corresponding to u; in other words there is a subset $\mathcal{I} \subset \{1, \ldots, q\}$ for which we wish to project $\pi_{\mathcal{I}} : \mathcal{B} \to \mathcal{A}^{|\mathcal{I}|}$ so that for any $u \in \pi_{\mathcal{I}}(\mathcal{B})$ there exists y such that $Qu = Py$.

By Proposition 2.2.13, P has a minimal left annihilator which we denote by $X \in \mathcal{D}^{s \times p}$. Furthermore, by Corollary 2.2.17 we have that, since X is a minimal left annihilator of P, it is a controllable behavior and P is an image representation for X over the signal space $\mathcal{A}$. Thus for any signal $v \in \mathcal{A}^p$ where $Xv = 0$, there exists $v' \in \mathcal{A}^{|\mathcal{I}|}$ such that $v = Pv'$. In particular, for any $u \in \mathcal{A}^{q-|\mathcal{I}|}$ such that $XQu = 0$ we have $X(Qu) = 0$ and, since X is an MLA of P, there exist $y \in \mathcal{A}^{|\mathcal{I}|}$ such that $Qu = Py$. We conclude that $\ker_{\mathcal{A}}(XQ)$ is precisely the behavior we desire. We summarize this discussion in the following proposition.

Proposition 6.3.1. *Let P and Q be polynomial matrices of size $p \times |\mathcal{O}|$ and $p \times |\mathcal{I}|$, $R = \begin{bmatrix} -Q & P \end{bmatrix}$ and $\mathcal{B} = \ker_{\mathcal{A}}(R)$. Then the projected behavior*

$$\pi_{\mathcal{I}}(\mathcal{B}) = \{u \in \mathcal{A}^{|\mathcal{I}|} : \text{ there exists } y \in \mathcal{A}^{|\mathcal{O}|} \text{ so that } (u, y) \in \mathcal{B}\}$$

has a kernel representation XQ:

$$\pi_{\mathcal{I}}(\mathcal{B}) = \ker_{\mathcal{A}}(XQ)$$

where X is an MLA of P.

This approach of using MLAs for portions of the kernel representation is a standard approach in behavioral systems theory for removing latent variables such as state variables. For some more examples one may consult [51, 43]. We now return to our discussion of vessels and their associated kernel representations.

We may rewrite (6.14) in ARMA form to reach the following equations

$$\underbrace{\begin{bmatrix} \widetilde{B}\sigma_1 & 0 \\ \widetilde{B}\sigma_2 & 0 \\ D & -I \\ U(z_1, z_2) & 0 \\ 0 & U_*(z_1, z_2) \end{bmatrix}}_{P(z_1,z_2)} \begin{bmatrix} u \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -(A_1 - z_1 I) \\ -(A_2 - z_2 I) \\ -C \\ 0 \\ 0 \end{bmatrix}}_{Q(z_1,z_2)} x. \tag{6.15}$$

Let $N(z_1, z_2)$ be the minimal left annihilator (MLA) of the Hautus pencil $Q(z_1, z_2)$. By Proposition 6.3.1, given (u, y) there exists x so that (6.15) holds if and only if

$$N(z_1, z_2) P(z_1, z_2) \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] = 0.$$

We define the **external behavior associated with the vessel** as

$$\mathcal{B}^{i/o} = \{\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in (\mathcal{U} \oplus \mathcal{Y})^{\mathbb{N}^2} : N(z_1, z_2) P(z_1, z_2) \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] = 0\}.$$

It is of interest to identify $N(z_1, z_2)$ more explicitly. Since we know that the MLA of the zero matrix is the identity matrix, we know that a minimal left annihilator of $N(z_1, z_2)$ is of the form

$$N(z_1, z_2) = \begin{bmatrix} 0 & 0 & 0 & I_{\mathcal{U}} & 0 \\ 0 & 0 & 0 & 0 & I_{\mathcal{Y}} \\ V_1(z_1, z_2) & V_2(z_1, z_2) & V_3(z_1, z_2) & 0 & 0 \end{bmatrix}. \tag{6.16}$$

In the above $I_{\mathcal{U}}$ and $I_{\mathcal{Y}}$ are identity operators on the input space $\mathcal{U}$ and output space $\mathcal{Y}$ respectively. The kernel representation for the external behavior becomes

$$R^{i/o} = \begin{bmatrix} U(z_1,z_2) & 0 \\ 0 & U_*(z_1,z_2) \\ V_1(z_1,z_2)\widetilde{B}\sigma_1 + V_2(z_1,z_2)\widetilde{B}\sigma_2 + V_3(z_1,z_2)D & -V_3(z_1,z_2) \end{bmatrix}. \tag{6.17}$$

Note. From here on out we suppress z_1 and z_2 and simply write V_1, V_2 and V_3 while keeping in mind these are in fact polynomial matrices. We also reserve V_1, V_2 and V_3 so that they always refer back to the matrix (6.16).

Our next goal is to obtain the analogue of Corollary 6.4.5 (to come) for the external behavior associated to a vessel, namely, the fact that the annihilator of the external behavior of a vessel satisfying Assumption 6.1.1 is also the prime ideal $\langle f\rangle$ (see Theorem 6.3.5 below). Toward this goal we develop a few identities in the following sequence of propositions and lemmas.

Proposition 6.3.2. *For any vessel $\mathfrak{V}$ the identity*

$$(z_2 I_{\mathcal{X}} - A_2)\widetilde{B}\sigma_1 = (z_1 I_{\mathcal{X}} - A_1)\widetilde{B}\sigma_2 - \widetilde{B}U(z_1,z_2)$$

holds

Proof. From the input vessel condition (i.e., $A_2\widetilde{B}\sigma_1 = A_1\widetilde{B}\sigma_2 + \widetilde{B}\gamma$) and the definition of $U(z_1,z_2)$ as in (6.12) it follows

$$\begin{aligned}(z_2 I_{\mathcal{X}} - A_2)\widetilde{B}\sigma_1 &= \widetilde{B}\sigma_1 z_2 - A_1\widetilde{B}\sigma_2 - \widetilde{B}\gamma \\ &= \widetilde{B}\sigma_1 z_2 - A_1\widetilde{B}\sigma_2 - \widetilde{B}U(z_1,z_2) + \widetilde{B}\sigma_2 z_1 - \widetilde{B}\sigma_1 z_2 \\ &= \widetilde{B}\sigma_2 z_1 - A_1\widetilde{B}\sigma_2 - \widetilde{B}U(z_1,z_2) \\ &= (z_1 I_{\mathcal{X}} - A_1)\widetilde{B}\sigma_2 - \widetilde{B}U(z_1,z_2).\end{aligned}$$

□

In light of the fact we do not know the entries of (6.16), the following proposition allows us to still use the matrices therein in calculations.

Proposition 6.3.3. *For any vessel $\mathfrak{V}$ the identity*

$$V_3 W = (V_3 D + V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2) - V_2(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U(z_1,z_2)$$

holds, where we let

$$W = W(z_1) = D + C(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}\sigma_1 \tag{6.18}$$

be the $(\xi_1,\xi_2) = (1,0)$ section of the Complete Characteristic Function for the vessel $\mathfrak{V}$ defined as in (6.9)*:*

$$W(z_1) = W_{\mathfrak{V}}(1,0,z_1).$$

Proof. Note that, by the construction of (6.16), we have the equality

$$\begin{aligned}&-V_1(A_1 - z_1 I_{\mathcal{X}}) - V_2(A_1 - z_2 I_{\mathcal{X}}) - C \\ &\qquad = V_1(z_1 I - A_1) + V_2(z_2 I - A_2) - V_3 C = 0.\end{aligned} \tag{6.19}$$

After applying the identity (6.19) we may observe the following

$$\begin{aligned}V_3 W &= V_3 D + V_3 C(z_1 I - A_1)^{-1}\widetilde{B}\sigma_1 \\ &= V_3 D + (V_1(z_1 I_{\mathcal{X}} - A_1) + V_2(z_2 I - A_2))(z_1 I - A_1)^{-1}\widetilde{B}\sigma_1) \\ &= V_3 D + V_1\widetilde{B}\sigma_1 + V_2(z_2 I - A_2)(z_1 I - A_1)^{-1}\widetilde{B}\sigma_1.\end{aligned} \tag{6.20}$$

From the vessel commutativity condition $A_1 A_2 = A_2 A_1$ one may observe

$$(z_2 I_{\mathcal{X}} - A_2)(z_1 I_{\mathcal{X}} - A_1) = (z_1 I - A_1)(z_2 I_{\mathcal{X}} - A_2).$$

and consequently also

$$(z_2 I_{\mathcal{X}} - A_2)(z_1 I_{\mathcal{X}} - A_1)^{-1} = (z_1 I - A_1)^{-1}(z_2 I - A_2). \tag{6.21}$$

After applying (6.21) and Proposition 6.3.2 to (6.20) it follows

$$\begin{aligned}V_3 W &= V_3 D + V_1\widetilde{B}\sigma_1 + V_2(z_2 I_{\mathcal{X}} - A_2)(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}\sigma_1 \\ &= V_3 D + V_1\widetilde{B}\sigma_1 + V_2(z_1 I_{\mathcal{X}} - A_1)^{-1}(z_2 I_{\mathcal{X}} - A_2)\widetilde{B}\sigma_1 \\ &= V_3 D + V_1\widetilde{B}\sigma_1 + V_2(z_1 I_{\mathcal{X}} - A_1)^{-1}((z_1 I_{\mathcal{X}} - A_1)\widetilde{B}\sigma_2 - \widetilde{B}U(z_1, z_2)) \\ &= V_3 D + V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 - V_2(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U(z_1, z_2).\end{aligned}$$

□

From the original Livšic state-space definition of $\mathcal{B}^{i/o}_{\mathfrak{V}}$, we know that $\pi_{\mathcal{I}}(\mathcal{B}^{i/o}_{\mathfrak{V}}) = \ker(U)$. We now derive this as a consequence of the kernel representation.

Lemma 6.3.4. *For a given vessel* $\mathfrak{V}$*, the external behavior always satisfies the identity* $\pi_{\mathcal{I}}(\mathcal{B}^{i/o}_{\mathfrak{V}}) = \ker_{\mathcal{A}}(U)$ *where* $\pi_{\mathcal{I}} : \mathcal{B}^{i/o}_{\mathfrak{V}} \to \mathcal{A}^{|\mathcal{I}|}$ *is the projection map to the components which correspond to the input of the system.*

Proof. Recall from (6.17) that a kernel representation of $\mathcal{B}^{i/o}_{\mathfrak{V}}$ is

$$R^{i/o} = \begin{bmatrix} U(z_1, z_2) & 0 \\ 0 & U_*(z_1, z_2) \\ V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3 D & -V_3 \end{bmatrix}.$$

Writing the above in ARMA form we reach

$$\underbrace{\begin{bmatrix} U(z_1, z_2) \\ 0 \\ V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3 D \end{bmatrix}}_{Q(z_1,z_2)} u = \underbrace{\begin{bmatrix} 0 \\ -U_*(z_1, z_2) \\ V_3 \end{bmatrix}}_{P(z_1,z_2)} y.$$

Let

$$N(z_1,z_2) = \begin{bmatrix} I_{\mathcal{U}} & 0 & 0 \\ 0 & L_1(z_1,z_2) & L_2(z_1,z_2) \end{bmatrix}$$

be an MLA of $P(z_1,z_2)$. We then have the identity

$$L_1(z_1,z_2)U_*(z_1,z_2) = L_2(z_1,z_2)V_3(z_1,z_2). \tag{6.22}$$

By Proposition 6.3.1, a kernel representation for the input behavior is then given by

$$R^i(z_1,z_2) = N(z_1,z_2)Q(z_1,z_2) = \begin{bmatrix} U(z_1,z_2) \\ L_2(z_1,z_2)(V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3D) \end{bmatrix}.$$

To conserve space, we write L_1 and L_2 while keeping in mind they depend on z_1 and z_2. By Proposition 6.3.3 we have

$$R^i(z_1,z_2) = \begin{bmatrix} U(z_1,z_2) \\ L_2(V_3W + V_2(z_1I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U(z_1,z_2)) \end{bmatrix}$$

From the identity (6.22) we reach

$$R^i(z_1,z_2) = \begin{bmatrix} U(z_1,z_2) \\ L_1U_*(z_1,z_2)W(z_1) + L_2V_2(z_1I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U(z_1,z_2) \end{bmatrix}.$$

Proposition 6.2.1 then yields

$$\begin{aligned} R^i(z_1,z_2) &= \begin{bmatrix} U(z_1,z_2) \\ L_1W_{\mathfrak{V}^\sim}(1,0,z_1)^TU(z_1,z_2) + L_2V_2(z_1I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U(z_1,z_2) \end{bmatrix} \\ &= \begin{bmatrix} I_{\mathcal{U}} \\ L_1W_{\mathfrak{V}^\sim}(1,0,z_1)^T + L_2V_2(z_1I_{\mathcal{X}} - A_1)^{-1}\widetilde{B} \end{bmatrix} U(z_1,z_2). \end{aligned}$$

By noting

$$\ker_{\mathcal{A}} \begin{bmatrix} I_{\mathcal{U}} \\ L_1W_{\mathfrak{V}^\sim}(1,0,z_1)^T + L_2V_2(z_1I_{\mathcal{X}} - A_1)^{-1}\widetilde{B} \end{bmatrix} = \{0\}$$

it follows that $\pi_{\mathcal{I}}(\mathcal{B}^{i/o}_{\mathfrak{V}}) = \ker_{\mathcal{A}}(U(z_1,z_2))$. □

We now arrive at one of the main result of this section: the annihilator of the external behavior of a vessel satisfying Assumption 6.1.1 is a prime ideal. This point is crucial for the remainder of this dissertation since it states that we may reduce the behavior and work over a ring that is an integral domain.

Theorem 6.3.5. *For any vessel $\mathfrak{V}$ which satisfies Assumption 6.1.1 we have* $\mathrm{Ann}(\mathcal{B}^{i/o}_{\mathfrak{V}}) = \langle f \rangle$ *and thus the annihilator is a prime ideal.*

Proof. By Corollary 6.4.5 (to come) it follows

$$\operatorname{Ann}(U(z_1,z_2)) = \operatorname{Ann}(U_*(z_1,z_2)) = \langle f\rangle, \tag{6.23}$$

where f is the irreducible polynomial in Assumption IMA.

Recall from (6.17) we have that a kernel representation associated to $\mathcal{B}^{i/o}_{\mathfrak{V}}$ is given by

$$R^{i/o} = \begin{bmatrix} U(z_1,z_2) & 0 \\ 0 & U_*(z_1,z_2) \\ V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3 D & -V_3 \end{bmatrix}.$$

From (6.23), there exists matrices X and X_* such that

$$XU = fI_{\mathcal{U}} \qquad\qquad X_*U_* = fI_{\mathcal{Y}}.$$

As a consequence, we have

$$\begin{bmatrix} X & 0 & 0 \\ 0 & X_* & 0 \end{bmatrix} \begin{bmatrix} U(z_1,z_2) & 0 \\ 0 & U_*(z_1,z_2) \\ V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3 D & -V_3 \end{bmatrix} = f\begin{bmatrix} I_{\mathcal{U}} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix}.$$

This demonstrates that $\langle f\rangle \subset \operatorname{Ann}(\mathcal{B}^{i/o}_{\mathfrak{V}})$.

We now show equality. Choose a non-zero $g \in \operatorname{Ann}(\mathcal{B}^{i/o}_{\mathfrak{V}})$. Then for every $w = (u,y) \in \mathcal{B}^{i/o}_{\mathfrak{V}}$ we have $gw = (gu, gy) = 0$. However, by Lemma 6.3.4 for all $u' \in \ker_{\mathcal{A}}(U)$ there exists y' such that $(u',y') \in \mathcal{B}^{i/o}_{\mathfrak{V}}$; hence, $gu' = 0$ for all $u' \in \ker_{\mathcal{A}}(U)$. By definition, then this means $g \in \operatorname{Ann}(\operatorname{coker}_{\mathcal{D}}(U)) = \langle f\rangle$. We thus conclude with the desired equality $\operatorname{Ann}(\mathcal{B}^{i/o}_{\mathfrak{V}}) = \langle f\rangle$. $\square$

A direct consequence of Theorem 6.3.5 is the following.

Corollary 6.3.6. *For any vessel $\mathfrak{V}$ which satisfies Assumption 6.1.1 the associated external behavior $\mathcal{B}^{i/o}_{\mathfrak{V}}$ has pure autonomy degree one.*

We now introduce an analogue, the **reduced Livšic transfer matrix** $S_{\mathfrak{V},r}$, of the Joint Characteristic Function $S_{\mathfrak{V}}$ and establish its role in determining the signal flow system of the external behavior $\mathcal{B}^{i/o}_{\mathfrak{V}}$ of a vessel $\mathfrak{V}$. The following first preparatory result allows us to reduce the proof in the main theorem to a special case so as to ease the complexity of the calculations.

Lemma 6.3.7. *For any $(\xi_1,\xi_2) \in \mathbb{C}^2 \setminus \{(0,0\}$ we have the identity*

$$W(z_1)|_{\ker_{K_r}(U)} = W_{\mathfrak{V}}(\xi_1,\xi_2,\xi_1 z_1 + \xi_2 z_2)|_{\ker_{K_r}(U)}\,.$$

where $W_{\mathfrak{V}}$ is the Livšic Complete Characteristic Function (6.9) *and where $W(z_1) := W_{\mathfrak{V}}(1,0,z_1)$ is as stated in (6.18) of Proposition 6.3.3.*

Proof. We first collect some needed identities. Recall from Proposition 6.3.2 that we have the identity

$$\widetilde{B}\sigma_2 = (z_1 I_{\mathcal{X}} - A_1)^{-1}(z_2 I_{\mathcal{X}} - A_2)\widetilde{B}\sigma_1 + (z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U(z_1, z_2) \tag{6.24}$$

An application of (6.24) provides the following

$$\begin{aligned}
&\widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2) \\
&\quad = \xi_1\widetilde{B}\sigma_1 + \xi_2(z_1 I_{\mathcal{X}} - A_1)^{-1}(z_2 I_{\mathcal{X}} - A_2)\widetilde{B}\sigma_1 + \xi_2(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U \\
&\quad = \left(\xi_1 I_{\mathcal{X}} + \xi_2(z_1 I_{\mathcal{X}} - A_1)^{-1}(z_2 I_{\mathcal{X}} - A_2)\right)\widetilde{B}\sigma_1 + \xi_2(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U
\end{aligned} \tag{6.25}$$

We also employ the following

$$\begin{aligned}
&C(\xi_1(z_1 I_{\mathcal{X}} - A_1) + \xi_2(z_2 I_{\mathcal{X}} - A_2))^{-1} \\
&\qquad\qquad = C(z_1 I_{\mathcal{X}} - A_1)^{-1}(\xi_1 I_{\mathcal{X}} + \xi_2(z_1 I_{\mathcal{X}} - A_1)^{-1}(z_2 I_{\mathcal{X}} - A_2))^{-1}
\end{aligned} \tag{6.26}$$

Through the combination of (6.25) and (6.26) we then observe

$$\begin{aligned}
&W_{\mathfrak{V}}(\xi_1, \xi_2, \xi_1 z_1 + \xi_2 z_2) - D \\
&\quad = C\left(\xi_1(z_1 I_{\mathcal{X}} - A_1) + \xi_2(z_2 I_{\mathcal{X}} - A_2)\right)^{-1}\widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2) \\
&\quad = C(z_1 I_{\mathcal{X}} - A_1)^{-1}\left(\xi_1 I_{\mathcal{X}} + \xi_2(z_1 I_{\mathcal{X}} - A_1)^{-1}(z_2 I_{\mathcal{X}} - A_2)\right)^{-1} \\
&\qquad \times\left[\left(\xi_1 I_{\mathcal{X}} + \xi_2(z_1 I_{\mathcal{X}} - A_1)^{-1}(z_2 I_{\mathcal{X}} - A_2)\right)\widetilde{B}\sigma_1 + \xi_2(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U\right] \\
&\quad = \left[C(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}\sigma_1\right] \\
&\qquad + \xi_2 C(\xi_1(z_1 I_{\mathcal{X}} - A_1) + \xi_2(z_2 I_{\mathcal{X}} - A_2))^{-1}(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U
\end{aligned} \tag{6.27}$$

For any $u \in \ker_{K_r}(U)$ (note that the following calculations are now performed over the reduced quotient field $Q(\mathcal{D}/\mathrm{Ann}(\mathfrak{V}^{i/o}_{\mathfrak{V}}))$ we may apply (6.27) as follows

$$\begin{aligned}
&W_{\mathfrak{V}}(\xi_1, \xi_2, \xi_1 z_1 + \xi_2 z_2)u \\
&\quad = \left[D + C(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}\sigma_1\right]u \\
&\qquad + \left[\xi_2 C(\xi_1(z_1 I_{\mathcal{X}} - A_1) + \xi_2(z_2 I_{\mathcal{X}} - A_2))^{-1}(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U\right]u \\
&\quad = \left[D + C(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}\sigma_1\right]u \\
&\quad = W(z_1)u
\end{aligned}$$

to conclude with the desired equality. □

In view of Lemma 6.3.7 it makes sense to define the **reduced Livšic transfer matrix** $S_{\mathfrak{V},r}$ of the vessel $\mathfrak{V}$ to be the restriction of the Complete Transfer Function to the kernel of U over the quotient field K_r of the reduced ring $\mathcal{D}_r$:

$$S_{\mathfrak{V},r}(z_1, z_2) = W_{\mathfrak{V}}(\xi_1, \xi_2, \xi_1 z_1 + \xi_2 z_2)|_{\mathrm{Ker}_{K_r}(U)}\,. \tag{6.28}$$

The next result also plays a key role in the proof of Theorem 6.3.9.

Lemma 6.3.8. *Let $\mathfrak{V}$ be a vessel satisfying Assumption 6.1.1 and $\mathcal{B}_{\mathfrak{V}}^{i/o}$ be its external behavior, and set*

$$R^s := \begin{bmatrix} A_1 - z_1 I_{\mathcal{X}} \\ A_2 - z_2 I_{\mathcal{X}} \end{bmatrix}. \tag{6.29}$$

Then R^s has trivial kernel when viewed over the the reduced quotient field $K_r = Q(\mathcal{D}/\mathrm{Ann}(\mathcal{B}_{\mathfrak{V}}^{i/o}))$:

$$\ker_{K_r}(R^s \otimes 1_{\mathcal{D}_r}) = 0.$$

Proof. Let $I \subset \mathcal{D}$ be the ideal generated by the $(\dim(\mathcal{X}) \times \dim(\mathcal{X}))$-order minors of R^s. Notice that $f_1(z_1) := \det(A_1 - z_1 I_{\mathcal{X}}) \in I$ and $f_2(z_2) := \det(A_2 - z_2 I_{\mathcal{X}}) \in I$. For the quotient ring $\mathcal{D}_I = \mathcal{D}/I$ we have (after dividing through by the leading coefficient if necessary) the monic relations $f_1(z_1) \equiv 0$ and $f_2(z_2) \equiv 0$. It follows that $\mathcal{D}_I$ has zero dimension and I has height equal to two. (We refer the reader to Appendix A.3 for more details on dimension of rings and quotient rings.)

Define the ideal $J = \mathrm{Ann}(\mathcal{B}_{\mathfrak{V}}^{i/o})$ and note by Theorem 6.3.5 that J is a prime ideal. We have $I \subset I + J$ and thus $\mathrm{height}(I) = \mathrm{height}(I + J) = 2$ or $I + J = \mathcal{D}$; hence, $J \subset I + J$ and so $\mathrm{height}(I + J) = 1$ in $\mathcal{D}/J$ or $I + J = \mathcal{D}/J$. This demonstrates that the ideal generated by $(\dim(\mathcal{X}) \times \dim(\mathcal{X}))$-order minors of $R^s \otimes 1_{\mathcal{D}_r}$ is non-zero in the integral domain $\mathcal{D}/J$ and hence $\ker_{K_r}(R^s \otimes 1_{\mathcal{D}_r}) = 0$. □

We now demonstrate the role of the Livšic transfer function in determining the signal flow system of the reduced behavior $\mathcal{B}_{\mathfrak{V}}^{i/o} \otimes \mathcal{D}_r$.

Theorem 6.3.9. *Let $\mathfrak{V}$ be a vessel satisfying Assumption 6.1.1 and $\mathcal{B}_{\mathfrak{V}}^{i/o}$ be its external behavior. Then the following equality holds*

$$\widehat{\mathcal{B}_{\mathfrak{V}}^{i/o} \otimes \mathcal{D}_r} := \ker_{K_r}(R_{\mathfrak{V}}^{i/o} \otimes 1_{\mathcal{D}_r}) = \mathrm{im}_{K_r} \left. \begin{bmatrix} I_{\mathcal{U}} \\ S_{\mathfrak{V},r}(z_1, z_2) \end{bmatrix} \right|_{\ker_{K_r}(U(z_1,z_2))}, \tag{6.30}$$

where $K_r = Q(\mathcal{D}_r)$ is the quotient field of $\mathcal{D}_r = \mathcal{D}/\mathrm{Ann}(\mathcal{B}_{\mathfrak{V}}^{i/o})$, $S_{\mathfrak{V},r}$ is the reduced Livšic transfer matrix of the vessel $\mathfrak{V}$ as defined in (6.28) and $K_r = Q(\mathcal{D}_r)$ is the quotient field of the reduced ring $\mathcal{D}_r$.

Proof. We first show the result for a special case and then apply Lemma 6.3.7 to conclude that the result holds in general. Let $(u, y) \in K_r^{\dim(\mathcal{U})+\dim(\mathcal{Y})}$ be such that $R^{i/o}(u, y)(u, y)^T = 0$. Then

$$\begin{bmatrix} U & 0 \\ 0 & U_* \\ V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3 D & -V_3 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} Uu \\ U_* y \\ (V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3 D)u - V_3 y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

from which we see that $u \in \ker_{K_r}(U(z_1, z_2))$. Proposition 6.3.3 (and W as defined by (6.18) therein) furnishes us with the identity

$$\begin{aligned} V_3 W u &= \left((V_3 D + V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2) - V_2(z_1 I_{\mathcal{X}} - A_1)^{-1}\widetilde{B}U \right) u \\ &= (V_3 D + V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2)u, \end{aligned}$$

and hence

$$(V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3D)u - V_3y = V_3(Wu - y) = 0.$$

Noting that

$$\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ Wu-y \end{bmatrix} = 0$$

and $\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$ is an MLA of the Hautus pencil given in (6.16), there exists $x \in K_r^{\dim(\mathcal{X})}$ such that

$$\begin{bmatrix} A_1 - z_1 I_{\mathcal{X}} \\ A_2 - z_2 I_{\mathcal{X}} \\ C \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ Wu - y \end{bmatrix}.$$

However, by Lemma 6.3.8, we have $\begin{bmatrix} A_1 - z_1 I_{\mathcal{X}} \\ A_2 - z_2 I_{\mathcal{X}} \end{bmatrix}$ has full column rank over K_r; it follows that it must be the case that $x = 0$ and thus $Wu = y$. In this way, we arrive at the desired equality

$$\widehat{\mathcal{B}_{\mathfrak{V}}^{i/o} \otimes \mathcal{D}_r} = \ker_{K_r}(R_{\mathfrak{V}}^{i/o} \otimes 1_{\mathcal{D}_r}) = \operatorname{im}_{K_r} \left.\begin{bmatrix} I_{\mathcal{U}} \\ W(z_1) \end{bmatrix}\right|_{\ker_{K_r}(U(z_1,z_2))}.$$

By Lemma 6.3.7 we may sharpen the result to

$$\widehat{\mathcal{B}_{\mathfrak{V}}^{i/o} \otimes \mathcal{D}_r} = \operatorname{im}_{K_r} \left.\begin{bmatrix} I_{\mathcal{U}} \\ W_{\mathfrak{V}}(\xi_1, \xi_2, \xi_1 z_1 + \xi_2 z_2) \end{bmatrix}\right|_{\ker_{K_r}(U(z_1,z_2))}$$

and conclude that the equality holds for any ξ_1, ξ_2. By the definition of the reduced Livšic transfer matrix (6.28), we conclude with the desired equality (6.30). □

We now consider the initial condition set associated to the external behavior of a vessel with respect to the canonical Cauchy problem.

Lemma 6.3.10. *Let $U(z_1, z_2)$ be a $q \times q$ non-degenerate matrix pencil of the form $U(z_1, z_2) = \sigma_2 z_1 - \sigma_1 z_2 + \gamma$. Under the lexicographic ordering $z_1 > z_2 > 1$, we have for the behavior $\mathcal{B} = \ker_{\mathcal{A}}(U(z_1, z_2))$ that*

$$\partial\mathcal{B} = \left(\bigcup_{i \in I_1} \{i\} \times (\mathbb{N} \times \{0\})\right) \cup \left(\bigcup_{i \in I_2} \{i\} \times (\{0\} \times \mathbb{N})\right)$$

where I_1 consists of the indices of the maximal set of linearly independent columns of σ_1, using the assumed ordering on $\{1, \ldots, q\}$, and $I_2 = \{1, \ldots, q\} \setminus I_1$.

Proof. This follows because the standard basis algorithm is Gaussian elimination when we have only linear polynomials. □

As shown by Fornasini, Rocha, and Zampieri in [16], a zero-input Livšic system corresponds to the state space representation of a two-dimensional $\mathbb{C}$-finite dimensional behavior. This leads to the following corollary.

Corollary 6.3.11. *For the external behavior $\mathcal{B}_{\mathfrak{V}}^{i/o}$ we have that in the initial condition set $\mathcal{B}_{\mathfrak{V}}^{i/o}$, the input components are specifiable on a set containing a one-dimensional lattice and the output components are specifiable on a set consisting of at most a finite number of points.*

6.4 Determinantal Representations

In this section we discuss conditions for a behavior to have a kernel representation which is a determinantal representation. We extend the results from this section in the following section where we discuss the relationship between Livšic controllability and j-controllability on the one hand and Hankel realizations on the other.

In our previous discussion, we encountered the two determinantal representations given by the input and output compatibility pencils

$$U(z_1, z_2) = \sigma_2 z_1 - \sigma_1 z_2 + \gamma \qquad\qquad U_*(z_1, z_2) = \sigma_{2*} z_1 - \sigma_{1*} z_2 + \gamma_*.$$

In general, a two-dimensional **(affine) determinantal representation** is given by a matrix pencil of the form

$$V(z_1, z_2) = Az_1 + Bz_2 + C$$

where, $A, B, C \in \mathbb{C}^{q\times q}$. Note that every determinantal representation yields a family of vector spaces

$$\mathfrak{G}(V) = \{(\boldsymbol{\lambda}, \alpha) \in \mathbb{C}^2 \times \mathbb{C}^q : \boldsymbol{\lambda} \in \mathbf{C} \text{ and } \alpha \in \ker_{\mathbb{C}} V(\boldsymbol{\lambda})\},$$

where $\mathbf{C} = \mathcal{V}(\det(V))$. One can get a compact version of these notions by homogenizing the matrix pencil V to obtain a **(projective) determinantal representation**

$$\mathbf{V}(z_0, z_1, z_2) = Az_1 + Bz_2 + Cz_0, \tag{6.31}$$

by noting that $\mathfrak{G}(V(z_0, z_1, z_2))$ is well-defined over $\mathbb{P}^2$. Then the kernel FVS associated with $\mathbf{V}$ (6.31) is over the compact topological space in $\mathbb{P}^2$.

In [47], Vinnikov provides a thorough treatment of the case where $\det(V)$ is an irreducible, non-singular polynomial. In this case it turns out that necessarily $\mathfrak{G}(V)$ is a holomorphic line bundle. Further results include necessary and sufficient conditions for a given line bundle to be isomorphic to the kernel bundle of a determinantal representation of the curve. This is appealing since it demonstrates a connection between exponential trajectories and kernel representations of behaviors. However, since the restriction to having a line bundle is rather confining, we instead focus on the vector bundle setting.

In [7], Ball and Vinnikov discuss determinantal representations and extend the above setting to the case where $\det(V)$ is a power of an irreducible and (possibly) singular polynomial. For such situations, $\mathfrak{G}(V)$ may lift[1] to a rank r holomorphic vector bundle over the normalizing Riemann surface of the curve $\mathcal{V}(\det(V))$. Naturally, if $\mathcal{V}(\det(V))$ is already a smooth and irreducible curve, then by Corollary 4.2.7, we have conditions for $\mathfrak{G}(V)$ to be a holomorphic vector bundle. The question addressed in [loc. cit.] was: when is a vector bundle *isomorphic* to a vector bundle arising from a determinantal representation. For our purposes here, however, the relevant question is when is a given vector subbundle of a trivial bundle *equal* to a vector bundle arising from a determinantal representation. By using the results in Section 5.4, we now provide conditions for the amplitude

[1] Recall that we discussed methods at the end of Section 4.2 for determining when this occurs.

FVS $\mathfrak{G}(\mathcal{B})$ associated with a behavior $\mathcal{B}$ not only to be a vector bundle but also to be equal to a bundle defined as the kernel bundle of a determinantal representation of the curve $\mathcal{V}(\mathcal{B})$.

We assume that the reader is familiar with divisors on complex manifolds, the Riemann-Roch theorem and Serre Duality. In Section A.2 we present some rudimentary definitions; however, more detailed results are not stated due to the extensive background required. More complete details can be found in [28, 14, 25, 24].

The following lemma can essentially be found in [47, pages 114–115]; we include a proof for completeness.

Lemma 6.4.1. *Let $U(z_1, z_2)$ be a $q \times q$ matrix polynomial such that for some $\ell < q$, every minor of order $\ell \times \ell$ is divisible by f^s. Then every minor of order $(\ell+1) \times (\ell+1)$ is divisible by f^{s+1}.*

Proof. Let $V(z_1, z_2)$ be an $(\ell+1) \times (\ell+1)$ minor of $U(z_1, z_2)$. Let p be the greatest power of f which divides $\det(V(z_1, z_2))$. Since every minor of $V(z_1, z_2)$ of order $\ell \times \ell$ is divisible by f^s, every entry of $\operatorname{Adj}(V)$ is divisible by f^s. Furthermore, $\det \operatorname{Adj}(V) = (\det V)^\ell$ is divisible by $f^{s(\ell+1)}$. As a result, the greatest power of f that divides $(\det V)^\ell$, namely ps, must be at least $s(\ell+1)$, i.e., $p\ell \geq s(\ell+1)$, from which we get

$$p \geq \frac{\ell+1}{\ell} s.$$

It follows that $p \geq s+1$ and thus $\det(V(z_1, z_2))$ is divisible by f^{s+1}. Since V was an arbitrary minor, the desired result follows. □

By induction, we reach the following corollary.

Corollary 6.4.2. *Let U be a $q \times q$ matrix such that for some $\ell < q$, the determinant of every $\ell \times \ell$ minor is divisible by f^s. Then for $j \leq q - \ell$ the determinant of every $(\ell+j) \times (\ell+j)$ minor is divisible by f^{s+j}.*

Theorem 6.4.3. *Let $\mathbf{C}$ be an irreducible projective plane curve defined as the zero-locus of an irreducible homogeneous polynomial F of degree m, X be the normalizing Riemann surface of $\mathbf{C}$, g be the genus of X, E be a family of vector spaces which lifts to a rank r holomorphic vector bundle over X. Then E is the kernel bundle of a determinantal representation of F^r if and only if*

1. *$c(\det((E \otimes \mathcal{O}_X(m-2)(-D))) = r(g-1)$,*
2. *$h^0(E \otimes \mathcal{O}_X(m-2)(-D)) = 0$.*

In the above, D is the divisor of singularities for $\mathbf{C}$, $\mathcal{O}_X(m-2)$ is the line bundle with sections consisting of homogenous polynomials of degree $m-2$ which vanish on D, and $h^0(E \otimes \mathcal{O}_X(m-2)(-D))$ is the dimension of the vector space of global holomorphic sections.

Proof. $(\Rightarrow)$. See Theorem 3.2 in [7, page 273-275] for a proof of this direction.

($\Leftarrow$). By the Riemann-Roch theorem,

$$h^0(E\otimes\mathcal{O}_X(m-2)(-D)) - h^1(E\otimes\mathcal{O}_X(m-2)(-D)) = c(\det(E\otimes\mathcal{O}_X(m-2)(-D))) + r(1-g).$$

From the hypotheses $h^0(E\otimes\mathcal{O}_X(m-2)(-D)) = 0$ and $c(\det(E\otimes\mathcal{O}_X(m-2)(-D))) = r(g-1)$, we have $h^1(E\otimes\mathcal{O}_X(m-2)(-D)) = 0$.

Define the bundle E_ℓ so that

$$E_\ell\otimes\mathcal{O}_X(m-2) \cong (E\otimes\mathcal{O}_X(m-2))^*\otimes K_X$$

By Serre duality, the assumption $h^0(E\otimes\mathcal{O}_X(m-2)(-D)) = 0$ and the above observation that $h^1(E\otimes\mathcal{O}_X(m-2)(-D)) = 0$,

$$\begin{aligned} h^0(E_\ell\otimes\mathcal{O}_X(m-2)) &= h^0((E\otimes\mathcal{O}_X(m-2))^*\otimes K_X) = h^1(E\otimes\mathcal{O}_X(m-2)(-D)) = 0\\ h^1(E_\ell\otimes\mathcal{O}_X(m-2)) &= h^1((E\otimes\mathcal{O}_X(m-2))^*\otimes K_X) = h^0(E\otimes\mathcal{O}_X(m-2)(-D)) = 0. \end{aligned}$$

By Bézout's theorem and the assumption $c(\det(E\otimes\mathcal{O}_X(m-2)(-D))) = r(g-1)$,

$$\begin{aligned} c(\det(E\otimes\mathcal{O}_X(m-1)(-D))) - r(g-1) &= c(\det(E\otimes\mathcal{O}_X(m-2)(-D))\otimes\mathcal{O}_X(r)) - r(g-1)\\ &= r(g-1) + mr - r(g-1) = mr. \end{aligned}$$

On the other hand as a consequence of the Riemann-Roch theorem we have

$$c(\det(E\otimes\mathcal{O}_X(m-1)(-D))) - r(g-1) \le h^0(E\otimes\mathcal{O}_X(m-1)(-D)).$$

Combining these gives

$$mr \le h^0(E\otimes\mathcal{O}_X(m-1)(-D)).$$

A similar application of the Riemann-Roch theorem and Bézout's theorem gives

$$\begin{aligned} h^0(E\otimes\mathcal{O}_X(m-1)(-D)) &\le c(\det(E\otimes\mathcal{O}_X(m-1)(-D))\otimes\mathcal{O}_X(r)) + r(1-g)\\ &= c(\det(E\otimes\mathcal{O}_X(m-2)(-D))\otimes\mathcal{O}_X(r)) + r(1-g)\\ &= c(\det(E\otimes\mathcal{O}_X(m-2)(-D))) + rm + r(1-g) = rm \end{aligned}$$

and we conclude that $h^0(E_\ell\otimes\mathcal{O}_X(m-1)(-D)) = rm$.

By the construction of E_ℓ, we have the pairing induced by the dual bundle of E,

$$\langle -,-\rangle : E_\ell\otimes\mathcal{O}_X(m-1)(-D)\times E\otimes\mathcal{O}_X(m-1)(-D) \to \mathcal{O}_X(m-1)(-D) \tag{6.32}$$

Define $M = rm$ and choose a basis $\{F_1,\dots,F_M\}$ for $H^0(X, E\otimes\mathcal{O}_X(m-1)(-D))$ and a basis $\{G_1,\dots,G_M\}$ for $H^0(X, E_\ell\otimes\mathcal{O}_X(m-1)(-D))$. By the pairing (6.32), define the $M\times M$ matrix V with $V_{i,j} = \langle G_i, F_j\rangle$ for $0\le i,j\le M$. Also define the holomorphic matrices

$$S = \begin{bmatrix} F_1 & \cdots & F_M \end{bmatrix} \qquad\qquad S_\ell = \begin{bmatrix} G_1 & \cdots & G_M \end{bmatrix}.$$

By the construction of E_ℓ, we may take $S_\ell = S^T$, so $V = SS_\ell = SS^T$. By the completeness of adjoint plane curves[2], each entry $V_{i,j}$ is an $(m-1)$-degree homogeneous polynomial that vanishes on D or is

[2]See [2, Appendix A].

the zero section. As stated in [7, page 275], due to the assumption $h^0(E \otimes \mathcal{O}_X(m-2)(-D)) = 0$ it follows that the pairing (6.32) is nondegenerate and hence $\det(V) \not\equiv 0$ (despite vanishing identically on $\mathbf{C}$).

Along $\mathbf{C}$, V has rank r, so the determinant of every $(r+1) \times (r+1)$ minor of V vanishes along $\mathbf{C}$ and thus is divisible by F. By Corollary 6.4.2, $\det(V)$ is divisible by F^{M-r}. However, since each entry $V_{i,j}$ has degree $m-1$, is homogeneous and $V \not\equiv 0$,

$$\deg \det(V) = rm(m-1) = m(rm-r) = \deg F^{M-r}.$$

It follows that $\det(V) = cF^{M-r}$ for some $c \in \mathbb{C}$. Because $\det(V) \not\equiv 0$, we have $c \neq 0$.

Once again, by Corollary 6.4.2 the every minor of V of order$(M-1) \times (M-1)$ is divisible by F^{M-r-1}. We now demonstrate that

$$U = \frac{1}{F^{M-r-1}} \mathrm{Adj}(V)$$

is the determinantal representation we seek. Since F is a degree m polynomial, $\deg(F^{r-1}) = m(r-1)$. However, since every entry of $\mathrm{Adj}(U)$ has degree $(m-1)(rm-1)$, it follows that every entry of U has degree $(m-1)(rm-1)-m(rm-r-1) = 1$; this demonstrates that U is homogeneous and linear. By construction, we also have

$$\det(U) = \frac{1}{F^{M(M-r-1)}} \det(\mathrm{Adj}(V)) = \frac{c^{M-1}F^{(M-1)(M-r)}}{F^{M(M-r-1)}} = c^{M-1}F^r,$$

which demonstrates that U is a determinantal representation of F^r.

Since

$$UV = \frac{1}{F^{M-r-1}} \mathrm{Adj}(V)V = \frac{cF^{M-r}}{F^{M-r-1}} I_{q\times q} = cFI_{q\times q},$$

it follows that $UV = 0$ along $\mathbf{C}$. Since $V = SS^T$, we have

$$US = cF(S^T)^{-1}$$

which is zero along $\mathbf{C}$. As a consequence, the kernel bundle of U is equal to E. We conclude that U is the desired determinantal representation. □

Consider the two-dimensional behavior $\mathcal{B}$ and its amplitude FVS $\mathfrak{G}(\mathcal{B})$. As mentioned earlier, we may consider $\mathfrak{G}(\mathcal{B})$ as an FVS over a compact complex manifold if we homogenize the kernel representation by its rows as in (6.31) and lift to the normalizing Riemann surface. We now discuss applying the above theorem to demonstrate that behaviors with pure autonomy degree one can have kernel representations which are determinantal representations. Before stating this result, we have some preparatory lemmas. Recall that we say a $q \times q$ matrix pencil $Az_1 + Bz_2 + C$ is **nondegenerate** if there exists $(\xi_1, \xi_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ such that $\mathrm{im}_{\mathbb{C}}(\xi_1 A + \xi_2 B) = \mathbb{C}^q$.

Our first concern is that the reduction of the behavior is over a reduced ring which is an integral domain. The following lemma establishes conditions for this to occur.

Lemma 6.4.4. *Let U be a $q \times q$ matrix (not necessarily a determinantal representation) such that $\det(U) = f^r$, where f is an irreducible polynomial, and the minimal rank of $\mathfrak{G}(U)$ is r over the curve $\mathcal{V}(f)$. Then* $\mathrm{Ann}(\ker_{\mathcal{A}}(U)) = \langle f \rangle$*, i.e., the annihilator is a prime ideal.*

Proof. Define $\mathcal{B} = \ker_{\mathcal{A}}(U) \subset \mathcal{A}^q$. Recall that $h \in \mathrm{Ann}(\mathcal{B})$ if and only if there exists a matrix X with entries in $\mathcal{D}$ such that

$$XU = hI_{q\times q}.$$

If $\mathfrak{G}(U)$ has minimal rank r over the curve $\mathcal{V}(f)$, $\mathcal{V}(\mathfrak{I}_i(U)) = \mathcal{V}(f)$ for $q - r < i \leq q$. In particular, f divides the determinant of every $(q - r + 1) \times (q - r + 1)$ minor of U. By Corollary 6.4.2, every entry of $\mathrm{Adj}(U)$ is divisible by f^{r-1}. As a result,

$$\frac{1}{f^{r-1}}\mathrm{Adj}(U)U = \frac{\det(U)}{f^{r-1}}I_{q\times q} = \frac{f^r}{f^{r-1}}I_{q\times q} = fI_{q\times q},$$

thus implying $f \in \mathrm{Ann}(\mathcal{B})$. Since f is an irreducible polynomial, and $\sqrt{\mathrm{Ann}(\mathcal{B})} = \langle f \rangle$, we conclude that $\mathrm{Ann}(\mathcal{B}) = \langle f \rangle$. □

Corollary 6.4.5. *Let $U(z_1, z_2)$ be a determinantal representation which satisfies the conditions of Theorem 6.4.3. Then* $\mathrm{Ann}(\ker_{\mathcal{A}}(U)) = \langle f \rangle$ *is a prime ideal.*

Theorem 6.4.6. *Let $\mathcal{B} \subset \mathcal{A}^q$ be a two-dimensional behavior whose amplitude FVS $\mathfrak{G}(\mathcal{B})$ satisfies the conditions of Theorem 6.4.3. Then there exists a determinantal representation U such that* $\ker_{\mathcal{A}}(U) \subset \mathcal{B}$*. Provided that* $\mathrm{Ann}(\mathcal{B})$ *is a prime ideal and the reduced behavior is $\mathcal{D}_r$-controllable, then* $\ker_{\mathcal{A}}(U) = \mathcal{B}$*.*

Proof. The existence of the determinantal representation $U(z_1, z_2)$ is given by Theorem 6.4.3. The construction provides that $U(z_1, z_2)$ is nondegenerate and by Corollary 6.4.5 $\mathrm{Ann}(\ker_{\mathcal{A}}(U))$ is a prime ideal, so by Lemma 6.5.9 (to come), $\mathcal{B}' = \ker_{\mathcal{A}}(U)$ has an image representation over its reduced ring. Because the $\mathcal{B}$ and $\mathcal{B}'$ have the same amplitude fibers and $\mathcal{B}'$ has an image representation over its reduced ring (which is an affine domain), by Lemma 5.4.7, $\mathcal{B}' \subset \mathcal{B}$. If $\mathrm{Ann}(\mathcal{B})$ is a prime ideal and $\mathcal{B}$ when reduced has an image representation, we may apply Lemma 5.4.10 to conclude that $\mathcal{B} = \mathcal{B}'$. □

6.5 Controllability

In this section we show that a vessel $\mathfrak{V}$ is Livšic controllable (see Definition 6.5.2 below) if and only if its associated augmented and external behaviors have image representations over the reduced ring. A first step is to establish this correspondence for a special class of vessels; we then use a change of variables to show that any vessel satisfying the nondegeneracy condition (ND) is isomorphic to a vessel in this special class. We use the connection between $\mathcal{D}_r$-controllability and Livšic controllability in the next section to show that an abstract behavior satisfying certain additional properties is equal to the external behavior associated with a Livšic system.

Before moving to the main results of this section, we require more preliminary results on vessels. See [6, 9] for a more detailed treatment. For $A = (A_1, A_2)$ a pair of commuting operators on a finite-dimensional linear space and $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$, we use the standard multivariable notation $A^\gamma = A_1^{\gamma_1} A_2^{\gamma_2}$.

Definition 6.5.1. Given a vessel $\mathfrak{V}$ as in Definition 6.1.1, we define the **controllability operator** $\mathcal{C}$ by

$$\mathcal{C} = \mathrm{row}_{\gamma \in \mathbb{N}^2}[A^\gamma \widetilde{B}]. \tag{6.33}$$

Definition 6.5.2. A vessel $\mathfrak{V}$ satisfying the nondegeneracy condition (ND) is said to be **Livšic controllable** if the controllability operator $\mathcal{C}$ (6.33) has image equal to the entire state space $\mathcal{X}$, i.e., for any point $T \in \mathbb{N}^2$ suitably far into the future and state $x_0 \in \mathcal{X}$, there exists an input trajectory $u(t)$ such that u drives the zero initial state to x_0 at time T.

There is also an analogue of the Hautus test for vessels.

Theorem 6.5.3 (Hautus Test for Vessels)**.** *[6] For the vessel $\mathfrak{V}$ the following are equivalent.*

1. *$\mathfrak{V}$ is Livšic controllable,*
2. $\mathrm{im}_{\mathbb{C}} \begin{bmatrix} A_1 - \lambda_1 & A_2 - \lambda_2 & \widetilde{B} \end{bmatrix} = \mathcal{X}$ *for all* $(\lambda_1, \lambda_2) \in \mathbb{C}^2$,
3. $\mathrm{im}_{\mathbb{C}} \begin{bmatrix} A_1 - \lambda_1 & A_2 - \lambda_2 & \widetilde{B} \end{bmatrix} = \mathcal{X}$ *for all* (λ_1, λ_2) *in the joint spectrum of the commuting pair* (A_1, A_2).

We next analyze controllability properties for the behavior defined by a matrix pencil (6.12) with one of σ_1 and σ_2 invertible.

Lemma 6.5.4. *Assume that, in the input compatibility pencil, σ_1 is invertible. Then for any $a \in \mathbb{N}$ and any two trajectories $u_1, u_2 \in \ker_{\mathcal{A}}(U) \subset \mathcal{A}^q$, there exists a unique trajectory $w \in \ker_{\mathcal{A}}(U)$ such that*

$$w(t) = \begin{cases} u_1(t_1, t_2) & t_1 + t_2 < a \\ u_2(t_1 - a, t_2) & t_1 \geq a \end{cases}$$

Proof. Since σ_1 is invertible, we may rewrite the input compatibility condition to be

$$z_2 = \sigma_1^{-1} \sigma_2 z_1 + \sigma_1^{-1} \gamma.$$

As a result, the initial condition set for $U(z_1, z_2)$ can be taken as the x-axis, i.e., we may freely assign values on the x-axis and there exists a unique extension off to $\mathbb{N}^2$. Define the values of w on the x-axis as

$$w(t, 0) = \begin{cases} u_1(t, 0) & t < a \\ u_2(t - a, 0) & t \geq a \end{cases}$$

By construction, w has the same initial conditions as u_1 on $[0, a)$ and the same initial conditions as u_2 on $[a, \infty)$. Let w be the unique extension of w in $\ker_{\mathcal{A}}(U(z_1, z_2))$. We now argue that w provides the desired trajectory.

Clearly $w(0,0) = u_1(0,0)$. We induct on the diagonals in $\mathbb{N}^2$. Let $n \in \mathbb{N}$ be such that $n < a$ and assume that $w(t_1, t_2) = u_1(t_1, t_2)$ for $t_1 + t_2 < n$. Define the matrices

$$A = \begin{bmatrix} -\sigma_1 & \sigma_2 & 0 & 0 & 0 \\ 0 & -\sigma_1 & \sigma_2 & \ddots & 0 \\ 0 & 0 & \ddots & \sigma_2 & 0 \\ 0 & \ddots & 0 & -\sigma_1 & \sigma_2 \\ & 0 & 0 & 0 & I \end{bmatrix} \qquad B = \begin{bmatrix} -\gamma & 0 & 0 & 0 & 0 \\ 0 & -\gamma & 0 & \ddots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & \ddots & 0 & 0 & -\gamma \\ 0 & \ddots & 0 & 0 & 0 \end{bmatrix}$$

In the above, A is an $(n+1) \times (n+1)$ matrix and B is an $(n+1) \times n$ matrix. Since σ_1 is invertible, A is invertible. Define the block column vectors

$$v_{n-1} = \begin{bmatrix} w(0, n-1) \\ w(1, n-2) \\ \vdots \\ w(n-2, 1) \\ w(n-1, 0) \end{bmatrix} \quad v_n = \begin{bmatrix} w(0, n) \\ w(1, n-1) \\ \vdots \\ w(n-1, 1) \\ w(n, 0) \end{bmatrix} \quad v'_{n-1} = \begin{bmatrix} u_1(0, n-1) \\ u_1(1, n-2) \\ \vdots \\ u_1(n-2, 1) \\ u_1(n-1, 0) \end{bmatrix} \quad v'_n = \begin{bmatrix} u_1(0, n) \\ u_1(1, n-1) \\ \vdots \\ u_1(n-1, 1) \\ u_1(n, 0) \end{bmatrix}$$

Since u_1 and w satisfy the input compatibility condition, we have

$$v_n = A^{-1}(Bv_{n-1} + e_{n+1}w(n, 0)) \qquad v'_n = A^{-1}(Bv'_{n-1} + e_{n+1}u_1(n, 0))$$

By the inductive hypothesis, $v_{n-1} = v'_{n-1}$ and, by construction, $w(n, 0) = u_1(n, 0)$. It follows that, $v_n = v'_n$. We thus have $w(t_1, t_2) = u_1(t_1, t_2)$ for $t_1 + t_n \leq n$. We may continue in this way for $n < a$.

To reach that $w(t_1, t_2) = u_2(t_1, t_2)$ for $t_1 \geq a$, all we do is consider $z_1^a w$ and notice that, by construction, the shifted trajectory has the same initial values as u_2; as a consequence $z^a w = u_2$. It follows that $w(t_1, t_2) = u_2(t_1 - a, t_2)$ for all $t_1 \geq a$. We conclude with the desired result. □

Corollary 6.5.5. *Assume that, in the input compatibility pencil, σ_2 is invertible. Then for any $a \in \mathbb{N}$ and any two trajectories $u_1, u_2 \in \ker_{\mathcal{A}}(U) \subset \mathcal{A}^q$, there exists a unique trajectory $w \in \ker_{\mathcal{A}}(U)$ such that*

$$w(t) = \begin{cases} u_1(t_1, t_2) & t_1 + t_2 < a \\ u_2(t_1, t_2 - a) & t_2 \geq a \end{cases}$$

Proof. Change the A matrix in the proof of the above Lemma to

$$A = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ -\sigma_1 & \sigma_2 & 0 & 0 & \ddots \\ 0 & -\sigma_1 & \sigma_2 & \ddots & 0 \\ 0 & 0 & \ddots & \sigma_2 & 0 \\ 0 & \ddots & 0 & -\sigma_1 & \sigma_2 \end{bmatrix}.$$

One can then verify that the approach used in Lemma 6.5.4 naturally extends to this setting. □

The concatenation of input trajectories can be used to relate Livšic controllability to 1-controllability.

Theorem 6.5.6. *If $\mathfrak{V}$ is Livšic controllable vessel and either σ_1 or σ_2 is invertible, then $\mathcal{B}_{\mathfrak{V}}^{i/s/o}$ is 1-controllable.*

Proof. Without loss of generality, we assume that σ_1 is invertible. Since σ_1 is invertible, by Lemma 6.5.4 we may patch input trajectories on the one-dimensional sublattice $L = \{(a, b) \in \mathbb{N}^2 : b = 0\}$.

By the assumption that $\mathfrak{V}$ is Livšic controllable, we know that

$$\mathrm{im}_{\mathbb{C}}\ \mathrm{row}_{(n_1,n_2)\in\mathbb{N}^2}[A_1^{n_1}A_2^{n_2}\widetilde{B}] = \mathcal{X}.$$

Since σ_1 is invertible, we see that

$$\mathrm{im}_{\mathbb{C}}\widetilde{B} = \mathrm{im}_{\mathbb{C}}\widetilde{B}\sigma_1 = \mathrm{im}_{\mathbb{C}}B_1$$

and, from the input vessel condition in Definition 6.1.1

$$\mathrm{im}_{\mathbb{C}}A_2\widetilde{B} = \mathrm{im}_{\mathbb{C}}(A_1\widetilde{B}\sigma_2\sigma_1^{-1} + \widetilde{B}\gamma\sigma_1^{-1})) \subset \mathrm{im}_{\mathbb{C}}\begin{bmatrix}\widetilde{B} & A_1\widetilde{B}\end{bmatrix}.$$

and we conclude that

$$\mathcal{X} = \mathrm{im}_{\mathbb{C}}\ \mathrm{row}_{(n_1,n_2)\in\mathbb{N}^2}[A_1^{n_1}A_2^{n_2}\widetilde{B}] \subset \mathrm{im}_{\mathbb{C}}\ \mathrm{row}_{n\in\mathbb{N}}[A_1^nB_1].$$

From the classical Cayley-Hamilton Theorem, we conclude that there is a finite $N \in \mathbb{N}$ so that

$$\mathcal{X} = \mathrm{im}_{\mathbb{C}}\begin{bmatrix}B_1 & A_1B_1 & \cdots & A_1^NB_1\end{bmatrix},$$

(where $N + 1 \leq \dim\mathcal{X}$), i.e., (A_1, B_1) is controllable in the classical single-variable Kalman sense.

Let $(x_1, u_1, y_1), (x_2, u_2, y_2) \in \mathcal{B}_{\mathfrak{V}}^{i/s/o}$ be two given trajectories and $T_1 \subset \mathbb{N}^d$ be a given finite set. Let $[0, a_1] \times [0, a_2]$ be the smallest interval containing T_1 and define $\rho(T_1) = N + a$ where $a = a_1 + a_2$. Since (A_1, B_1) is controllable, for any $b \in \mathbb{N}$ where $b > N+a$, there exists $u'(0, 0), \ldots, u'(b-a-2, 0) \in \mathcal{U}$ such that

$$\sum_{i=0}^{b-a-2} A_1^{b-a-2-i}B_1u'(i, 0) = x_2(0, 0) - A_1^bx_1(0) - \left(\sum_{i=0}^{a} A_1^{b-1-i}B_1u_1(i, 0)\right). \tag{6.34}$$

Since u' is defined on the initial condition set of U, there exists some non-unique extension to $\mathbb{N}^2$. Although, it is only important that u' has values as stated.

By Lemma 6.5.4, we may patch u_1 and u' at $(a, 0)$ to form u'' and we may then patch u'' with u_2 at $(b, 0)$ to form the trajectory u having values

$$u(t_1, t_2) = \begin{cases} u_1(t_1, t_2) & t_1 + t_2 \leq a \\ u'(t_1 - a_1, t_2) & t_1 > a,\ t_1 + t_2 < b \\ u_2(t_1 - b, t_2) & t_1 \geq b, t_2 > 0 \end{cases} \tag{6.35}$$

and satisfying the input compatibility condition. Define $x(0,0) = x_1(0,0)$ and let x and y be, respectively, the state and output trajectories for the input trajectory u and initial state $x(0,0)$.

Under the convention $A_1^0 = I_{\mathcal{X}}$ we may apply the state evolution equations and (6.34) to get

$$\begin{aligned}
x(b_2,0) &= A_1^b x(0,0) + \left(\sum_{i=0}^{b-1} A_1^{b-1-i} B_1 u(i,0)\right) \\
&= A_1^b x_1(0,0) + \left(\sum_{i=0}^{a} A_1^{b-1-i} B_1 u_1(i,0)\right) + \left(\sum_{i=a+1}^{b-1} A_1^{b-1-i} B_1 u'(i-a-1,0)\right) \\
&= A_1^b x_1(0,0) + \left(\sum_{i=0}^{a} A_1^{b-1-i} B_1 u_1(i,0)\right) + \left(\sum_{i=0}^{b-a-2} A_1^{b-a-2-i} B_1 u'(i,0)\right) \\
&= A_1^b x_1(0,0) + \left(\sum_{i=0}^{a} A_1^{b-1-i} B_1 u_1(i,0)\right) + x_2(0,0) - A_1^b x_1(0,0) - \left(\sum_{i=0}^{a} A_1^{b-1-i} B_1 u_1(i,0)\right) \\
&= x_2(0,0).
\end{aligned}$$

By the output equation and our construction,

$$y(b,0) = Cx(b,0) + Du(b,0) = Cx_2(0,0) + Du_2(0,0) = y_2(0,0).$$

By (6.35), the equalities

$$x(0,0) = x_1(0,0), \quad y(0,0) = y_1(0,0), \quad x(b,0) = x_2(0,0), \quad y(b,0) = y_2(0,0),$$

and shift-invariance demonstrates

$$x(t) = \begin{cases} x_1(t) & t \in T_1 \\ x_2(t-(b,0)) & t \in (b,0) + \mathbb{N}^2 \end{cases} \qquad y(t) = \begin{cases} y_1(t) & t \in T_1 \\ y_2(t-(b,0)) & t \in (b,0) + \mathbb{N}^2 \end{cases}$$

We conclude that $\mathcal{B}_{\mathfrak{V}}^{i/s/o}$ 1-controllable. □

Corollary 6.5.7. *If $\mathfrak{V}$ is Livšic controllable vessel and either σ_1 or σ_2 is invertible, then $\mathcal{B}_{\mathfrak{V}}^{i/o}$ is 1-controllable.*

Proof. For any two trajectories $(u_1, y_1), (u_2, y_2) \in \mathcal{B}_{\mathfrak{V}}^{i/o}$ there exist state trajectories x_1 and x_2 such that $(x_1, u_1, y_1), (x_2, u_2, y_2) \in \mathcal{B}_{\mathfrak{V}}^{i/s/o}$. We may then apply same construction as in Theorem 6.5.6 to show that $\mathcal{B}_{\mathfrak{V}}^{i/o}$ is 1-controllable. □

Corollary 6.5.8. *If in the matrix pencil $U(z_1, z_2) = \sigma_2 z_1 - \sigma_1 z_2 + \gamma$, either σ_1 or σ_2 is invertible, then $\ker_{\mathcal{A}}(U)$ is 1-controllable. If σ_1 is invertible, then the motion sublattice can be taken to be the x-axis, and if σ_2 is invertible, then the motion sublattice can be taken to be the y-axis.*

If the input compatibility pencil has neither σ_1 or σ_2 invertible but does satisfy the nondegeneracy condition (ND), then we may perform a "change of coordinates" to bring it into a form where one of the matrices is invertible. We now go through how such a change of coordinates can be implemented.

Assume that $U(z_1, z_2) = \sigma_2 z_1 - \sigma_1 z_2 + \gamma$ is nondegenerate and hence there exists $\xi_1, \xi_2 \in \mathbb{C}$ such that $\xi_1\sigma_1 + \xi_2\sigma_2$ is invertible. Define $\mathcal{B} = \ker_{\mathcal{A}}(U)$. Without loss of generality, assume that $\xi_1^2 + \xi_2^2 = 1$. Define the ring $\mathcal{D}' = \mathbb{C}[w_1, w_2]$, associated signal space $\mathcal{A}' = (\mathbb{C}^{\mathbb{N}^2})^m$, the matrices

$$\sigma_1' = \xi_1\sigma_1 + \xi_2\sigma_2 \qquad\qquad \sigma_2' = \xi_1\sigma_2 - \xi_2\sigma_1,$$

the matrix pencil

$$U'(w_1, w_2) = \sigma_2' w_1 - \sigma_1' w_2 + \gamma,$$

and its associated behavior $\mathcal{B}' = \ker_{\mathcal{A}'}(U')$. We would like to solve this new problem and have a way to relate it back to the original problem. To make this connection we change the ring.

Let ψ be the $\mathbb{C}$-algebra isomorphism of $\mathcal{D}$ defined on the generators z_1 and z_2 by

$$\psi(z_1) = \xi_1 z_1 - \xi_2 z_2 \qquad\qquad \psi(z_2) = \xi_2 z_1 + \xi_1 z_2. \tag{6.36}$$

or, via a matrix presentation, by

$$\begin{bmatrix} \psi(z_1) \\ \psi(z_2) \end{bmatrix} = \begin{bmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Then it is easily verified that ψ is invertible with inverse ϕ on generators given by

$$\begin{bmatrix} \phi(z_1) \\ \phi(z_2) \end{bmatrix} = \begin{bmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

or equivalently

$$\phi(z_1) = \xi_1 z_1 + \xi_2 z_2 \qquad\qquad \phi(z_2) = \xi_2 z_1 + \xi_1 z_2.$$

Thus we have

$$\psi \circ \phi = \mathrm{id}_{\mathcal{D}'} : \mathcal{D}' \to \mathcal{D}' \qquad\qquad \phi \circ \psi = \mathrm{id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}.$$

Then ψ extends to polynomials in z_1 and z_2 (with scalar, vector, or matrix coefficients) via

$$\psi(U)(z_1, z_2) = U(\psi(z_1), \psi(z_2)) =: U'(z_1, z_2).$$

Then we have the relations

$$\begin{aligned} \phi(U'(z_1, z_2)) &:= U'(\xi_1 z_1 + \xi_2 z_2, \xi_1 z_2 - \xi_2 z_1) = U(z_1, z_2) \\ \psi(U(z_1, z_2)) &:= U(\xi_1 z_1 - \xi_2 z_2, \xi_2 z_1 + \xi_1 z_2) = U'(z_1, z_2). \end{aligned} \tag{6.37}$$

Recall from the discussion in Section 5.5 that, through the canonical pairing, ϕ and ψ induce adjoint maps $\phi^* : \mathcal{A} \to \mathcal{A}$ and $\psi^* : \mathcal{A} \to \mathcal{A}$ given by

$$\begin{aligned} \langle d', \phi^*(w) \rangle &= \langle \phi(p'), w \rangle \qquad && w \in \mathcal{A},\ d' \in \mathcal{D}'^m \\ \langle d, \psi^*(w') \rangle &= \langle \psi(d), w' \rangle && w' \in \mathcal{A}',\ d \in \mathcal{D}^m. \end{aligned}$$

For p a scalar polynomial in $\mathcal{D}$, we compute

$$\langle d, \phi^*(pw')\rangle = \langle \phi(d), pw'\rangle = \langle p\phi(d), w'\rangle = \langle \phi(\phi^{-1}(p)d), w'\rangle = \langle \phi^{-1}(p)d, \phi^*(w')\rangle = \langle d, \phi^{-1}(p)\phi^*(w')\rangle$$

from which we conclude (as stated in (5.4)) that

$$\phi^*(pw') = \phi^{-1}(p)\phi^*(w').$$

These observations are key for the next lemma.

Lemma 6.5.9. *Assume that, the matrix pencil $U(z_1, z_2) = \sigma_2 z_1 - \sigma_1 z_2 + \gamma$ is nondegenerate and $\mathrm{Ann}(\ker_{\mathcal{A}}(U))$ is a prime ideal. Then $\ker_{\mathcal{A}}(U)$ has an image representation over its reduced ring.*

Proof. Define $\mathcal{B} = \ker_{\mathcal{A}}(U)$. By the above procedure, we may construct an isomorphic matrix pencil $U'(w_1, w_2) = \sigma'_2 z_1 - \sigma'_1 z_2 + \gamma$ which has σ'_1 invertible. By Corollary 6.5.8, $\mathcal{B}' := \ker_{\mathcal{A}}(U')$ is 1-controllable. Define the affine domain $\mathcal{D}'_r = \mathcal{D}/\mathrm{Ann}(\mathcal{B}')$. By Theorem 5.3.7 and Theorem 5.3.1 we have that $\mathrm{Red}(\mathcal{B}')$ is $\mathcal{D}'_r$-controllable. Since ϕ is a $\mathbb{C}$-algebra automorphism and $\phi(U') = U$, Lemma 5.5.6 provides that $\mathrm{Red}(\mathcal{B})$ is $\mathcal{D}_r$-controllable. □

We now show that the the change-of-variable procedure provides an isomorphism between vessels.

Theorem 6.5.10. *Let*

$$\mathfrak{V} = (A_1, A_2, \widetilde{B}, C, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*)$$

be a vessel with $\xi_1, \xi_2 \in \mathbb{C}$ such that $\xi_1^2 + \xi_2^2 = 1$ and $\xi_1\sigma_1 + \xi_2\sigma_2$ an invertible matrix. Then the set of operators

$$\begin{aligned}\mathfrak{V}' = (&A'_1 = \xi_1 A_1 + \xi_2 A_2, A'_2 = \xi_1 A_2 - \xi_2 A_1, \widetilde{B}, C, D, \widetilde{D},\\ &\sigma'_1 = \xi_1\sigma_1 + \xi_2\sigma_2, \sigma'_2 = \xi_1\sigma_2 - \xi_2\sigma_1, \gamma, \sigma'_{1*} = \xi_1\sigma_{1*} + \xi_2\sigma_{2*}, \sigma'_{2*} = \xi_1\sigma_{2*} - \xi_2\sigma_{1*}, \gamma_*)\end{aligned}$$

forms a vessel with σ'_1 invertible. Furthermore, $\mathcal{B}^{i/s/o}_{\mathfrak{V}}$ is $\mathcal{D}$-isomorphic to $\mathcal{B}^{i/s/o}_{\mathfrak{V}'}$ and $\mathcal{B}^{i/o}_{\mathfrak{V}}$ is $\mathcal{D}$-isomorphic to $\mathcal{B}^{i/o}_{\mathfrak{V}'}$. If $\mathfrak{V}$ is Livšic controllable, then $\mathfrak{V}'$ is also Livšic controllable.

Proof. Without loss of generality, assume that $\xi_1 \neq 0$. By using that $\mathfrak{V}$ is a vessel, we first verify that

$$\mathfrak{V}' = (A'_1, A'_2, \widetilde{B}, C, D, \widetilde{D}, \sigma'_1, \sigma'_2, \gamma, \sigma'_{1*}, \sigma'_{2*}, \gamma_*)$$

satisfies the necessary conditions so that $\mathfrak{V}'$ is a vessel. Note that since $A_1A_2 = A_2A_1$, it follows that $A'_1A'_2 = A'_2A'_1$. We now verify that $A'_2\widetilde{B}\sigma'_1 - A'_1\widetilde{B}\sigma'_2 = \widetilde{B}\gamma$:

$$\begin{aligned}&(\xi_1A_2 - \xi_2A_1)\widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2) - (\xi_1A_1 + \xi_2A_2)\widetilde{B}(\xi_1\sigma_2 - \xi_2\sigma_1)\\ &= (\xi_1^2 + \xi_2^2)A_2\widetilde{B}\sigma_1 - (\xi_1^2 + \xi_2^2)A_1\widetilde{B}\sigma_2 = \widetilde{B}\gamma.\end{aligned}$$

Similarly, we verify $\sigma'_{1*}CA'_2 - \sigma_{2*}CA'_1 = \gamma_*C$:

$$\begin{aligned}&(\xi_1\sigma_{1*} + \xi_2\sigma_{2*})C(\xi_1A_2 - \xi_2A_1) - (\xi_1\sigma_{2*} - \xi_2\sigma_{2*})C(\xi_1A_1 + \xi_2A_2)\\ &= (\xi_1^2 + \xi_2^2)\sigma_{1*}CA_2 - (\xi_1^2 + \xi_2^2)\sigma_{2*}CA_1 = \gamma_*C.\end{aligned}$$

By the linkage conditions on $\mathfrak{V}$, we get the last two linkage conditions for $\mathfrak{V}'$:

$$\begin{aligned}\sigma'_{2*}D - \widetilde{D}\sigma'_2 &= \xi_1(\sigma_{2*}D - \widetilde{D}\sigma_2) - \xi_2(\sigma_{1*}D - \widetilde{D}\sigma_1) = 0\\ \sigma'_{1*}D - \widetilde{D}\sigma'_1 &= \xi_1(\sigma_{1*}D - \widetilde{D}\sigma_1) + \xi_2(\sigma_{2*}D - \widetilde{D}\sigma_2) = 0.\end{aligned}$$

For the remaining $\mathfrak{V}'$ linkage condition, note that

$$\begin{aligned}&(\xi_1\sigma_{2*} - \xi_2\sigma_{1*})C\widetilde{B}(\xi_1\sigma_1 + \xi_2\sigma_2) - (\xi_1\sigma_{1*} + \xi_2\sigma_{2*})C\widetilde{B}(\xi_1\sigma_2 - \xi_2\sigma_1) + \gamma_* D\\ &\quad = (\xi_1^2 + \xi_2^2)\sigma_{2*}C\widetilde{B}\sigma_1 - (\xi_1^2 + \xi_2^2)(\sigma_{1*}C\widetilde{B}\sigma_2) + \gamma_* D = \widetilde{D}\gamma.\end{aligned}$$

We have thus verified that $\mathfrak{V}'$ is a vessel.

If $\mathfrak{V}$ is a Livšic controllable vessel, we have that it satisfies the Hautus test for vessels

$$\mathrm{im}_{\mathbb{C}} \begin{bmatrix} A_1 - \lambda_1 & A_2 - \lambda_2 & \widetilde{B} \end{bmatrix} = \mathcal{X} \qquad \text{for all } (\lambda_1, \lambda_2) \in \mathbb{C}^2.$$

Define the invertible matrix

$$X' = \begin{bmatrix} \xi_1 I & -\xi_2 I & 0 \\ \xi_2 I & \xi_1 I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Then

$$\begin{aligned}\begin{bmatrix} A_1 - \lambda_1 & A_2 - \lambda_2 & \widetilde{B} \end{bmatrix} X' &= \begin{bmatrix} (\xi_1 A_1 + \xi_2 A_2) - (\xi_1\lambda_1 + \xi_2\lambda_2) & (\xi_1 A_2 - \xi_2 A_1) - (\xi_1\lambda_2 - \xi_2\lambda_1) & \widetilde{B} \end{bmatrix}\\ &= \begin{bmatrix} A'_1 - (\xi_1\lambda_1 + \xi_2\lambda_2) & A'_2 - (\xi_1\lambda_2 - \xi_2\lambda_1) & \widetilde{B} \end{bmatrix}\end{aligned}$$

Since X' is invertible and $\mathfrak{V}$ is Livšic controllable,

$$\mathrm{im}_{\mathbb{C}} \begin{bmatrix} A'_1 - (\xi_1\lambda_1 + \xi_2\lambda_2) & A'_2 - (\xi_1\lambda_2 - \xi_2\lambda_1) & \widetilde{B} \end{bmatrix} = \mathcal{X} \qquad \text{for all } (\lambda_1, \lambda_2) \in \mathbb{C}^2.$$

We conclude that, $\mathfrak{V}'$ is a Livšic controllable vessel.

Define the kernel representations

$$R = \begin{bmatrix} A_1 - z_1 & \widetilde{B}\sigma_1 & 0 \\ A_2 - z_2 & \widetilde{B}\sigma_2 & 0 \\ D & C & -I \\ 0 & U(z_1, z_2) & 0 \\ 0 & 0 & U_*(z_1, z_2) \end{bmatrix} \quad R' = \begin{bmatrix} A_1 - (\xi_1 z_1 - \xi_2 z_2) & \widetilde{B}\sigma_1 & 0 \\ A_2 - (\xi_2 z_1 + \xi_1 z_2) & \widetilde{B}\sigma_2 & 0 \\ D & C & -I \\ 0 & U'(z_1, z_2) & 0 \\ 0 & 0 & U'_*(z_1, z_2) \end{bmatrix}$$

where $U(z_1, z_2)$ and $U_*(z_1, z_2)$ are as in (6.12), (6.13) and

$$\begin{aligned}U'(z_1, z_2) &= (\xi_1\sigma_2 - \xi_2\sigma_1)z_1 - (\xi_1\sigma_1 + \xi_2\sigma_2)z_2 + \gamma,\\ U'_*(z_1, z_2) &= (\xi_1\sigma_{2*} - \xi_2\sigma_{1*})z_1 - (\xi_1\sigma_{1*} + \xi_2\sigma_{2*})z_2 + \gamma_*.\end{aligned}$$

and the polynomial matrices

$$R'' = \begin{bmatrix} A_1' - z_1 & \widetilde{B}\sigma_1' & 0 \\ A_2' - z_2 & \widetilde{B}\sigma_2' & 0 \\ D & C & -I \\ 0 & U'(z_1, z_2) & 0 \\ 0 & 0 & U_*'(z_1, z_2) \end{bmatrix} \qquad X = \begin{bmatrix} \xi_1 I_{\mathcal{X}} & \xi_2 I_{\mathcal{X}} & 0 & 0 & 0 \\ -\xi_2 I_{\mathcal{X}} & \xi_1 I_{\mathcal{X}} & 0 & 0 & 0 \\ 0 & 0 & I_{\mathcal{Y}} & 0 & 0 \\ 0 & 0 & 0 & I_{\mathcal{U}} & 0 \\ 0 & 0 & 0 & 0 & I_{\mathcal{Y}} \end{bmatrix}.$$

A straightforward computation demonstrates that $XR' = R''$. Since X is invertible, we have that R' and R'' are kernel representations of the same behavior. One can verify that R'' corresponds to the vessel $\mathfrak{V}'$.

We now show that $\mathcal{B}_{\mathfrak{V}}^{i/s/o}$ is $\mathcal{D}$-isomorphic to $\mathcal{B}_{\mathfrak{V}'}^{i/s/o}$. Let $\phi : \mathbb{C}[z_1, z_2] \to \mathbb{C}[z_1, z_2]$ be the $\mathbb{C}$-algebra isomorphism defined on generators by

$$\phi(z_1) = \xi_1 z_1 - \xi_2 z_2 \qquad \phi(z_2) = \xi_2 z_1 + \xi_1 z_2.$$

By our previous discussion, $\phi(U) = U'$; similarly, one can verify $\phi(R) = R'$. Appealing to Theorem 5.5.2, we thus have that $\mathcal{B}_{\mathfrak{V}}^{i/s/o}$ is $\mathcal{D}$-isomorphic to $\mathcal{B}_{\mathfrak{V}'}^{i/s/o}$ via a ring automorphism.

Consider the following ARMA form of R, R' and R''.

$$\underbrace{\begin{bmatrix} \widetilde{B}\sigma_1 & 0 \\ \widetilde{B}\sigma_2 & 0 \\ C & -I \\ U(z_1, z_2) & 0 \\ 0 & U_*(z_1, z_2) \end{bmatrix}}_{P(z_1,z_2)} \begin{bmatrix} u \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -(A_1 - z_1) \\ -(A_2 - z_2) \\ D \\ 0 \\ 0 \end{bmatrix}}_{Q(z_1,z_2)} x$$

$$\underbrace{\begin{bmatrix} \widetilde{B}\sigma_1 & 0 \\ \widetilde{B}\sigma_2 & 0 \\ C & -I \\ U'(z_1, z_2) & 0 \\ 0 & U_*'(z_1, z_2) \end{bmatrix}}_{P'(z_1,z_2)} \begin{bmatrix} u \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -(A_1 - (\xi_1 z_1 - \xi_2 z_2)) \\ -(A_2 - (\xi_2 z_1 + \xi_1 z_2)) \\ D \\ 0 \\ 0 \end{bmatrix}}_{Q'(z_1,z_2)} x$$

$$\underbrace{\begin{bmatrix} \widetilde{B}\sigma_1' & 0 \\ \widetilde{B}\sigma_2' & 0 \\ C & -I \\ U'(z_1, z_2) & 0 \\ 0 & U_*'(z_1, z_2) \end{bmatrix}}_{P''(z_1,z_2)} \begin{bmatrix} u \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -(A_1' - z_1) \\ -(A_2' - z_2) \\ D \\ 0 \\ 0 \end{bmatrix}}_{Q''(z_1,z_2)} x$$

Since $XR' = R''$ we have $XP'(z_1, z_2) = P''(z_1, z_2)$ and $XQ'(z_1, z_2) = Q''(z_1, z_2)$. Furthermore, the $\mathbb{C}$-algebra isomorphism ϕ provides the equalities $\phi(P(z_1, z_2)) = P'(z_1, z_2)$ and $\phi(Q(z_1, z_2)) = Q'(z_1, z_2)$. Let $N''(z_1, z_2)$ be a minimal left annihilator (MLA) of $Q''(z_1, z_2)$. Then

$$N''(z_1, z_2)Q''(z_1, z_2) = N''(z_1, z_2)XQ'(z_1, z_2) = 0.$$

Define $N'(z_1, z_2) = N''(z_1, z_2)X$. We now verify that $N'(z_1, z_2)$ is a MLA of $Q'(z_1, z_2)$. It is clear that $N'(z_1, z_2)$ is a left annihilator of $Q'(z_1, z_2)$. Let $M(z_1, z_2)$ be another left annihilator of $Q'(z_1, z_2)$. We then have

$$M(z_1, z_2)Q'(z_1, z_2) = M(z_1, z_2)X^{-1}Q''(z_1, z_2) = 0.$$

Since $N''(z_1, z_2)$ is a minimal left annihilator of $Q''(z_1, z_2)$, we have that there exists $L(z_1, z_2)$ such that

$$L(z_1, z_2)N''(z_1, z_2) = M(z_1, z_2)X^{-1} \qquad \Rightarrow \qquad L(z_1, z_2)N'(z_1, z_2) = M(z_1, z_2).$$

It follows that $N'(z_1, z_2)$ is an MLA of $Q'(z_1, z_2)$. Define $N(z_1, z_2) = \phi^{-1}(N'(z_1, z_2))$. We now argue that $N(z_1, z_2)$ is an MLA of $Q(z_1, z_2)$. Observe that, since ϕ is a $\mathbb{C}$-algebra isomorphism,

$$\phi(N(z_1, z_2)Q(z_1, z_2)) = N'(z_1, z_2)Q'(z_1, z_2) = 0,$$

implies that $N(z_1, z_2)Q(z_1, z_2) = 0$. Let $M(z_1, z_2)$ be another left annihilator of $Q(z_1, z_2)$. We then have

$$\phi(M(z_1, z_2)Q(z_1, z_2)) = \phi(M(z_1, z_2))Q'(z_1, z_2) = 0.$$

Since $N'(z_1, z_2)$ is an MLA of $Q'(z_1, z_2)$, it follows that there exists $L(z_1, z_2)$ such that

$$\phi(M(z_1, z_2)) = L(z_1, z_2)N'(z_1, z_2).$$

In particular,

$$\phi(M(z_1, z_2) - \phi^{-1}(L(z_1, z_2))N(z_1, z_2)) = 0.$$

Since ϕ is a $\mathbb{C}$-algebra isomorphism, we conclude that $M(z_1, z_2) = \phi^{-1}(L(z_1, z_2))N(z_1, z_2))$ and so $N(z_1, z_2)$ is an MLA of $Q(z_1, z_2)$. As a consequence of the above arguments, we have

$$\phi(N(z_1, z_2)P(z_1, z_2)) = N'(z_1, z_2)P'(z_1, z_2) = N''(z_1, z_2)XP'(z_1, z_2) = N''(z_1, z_2)P''(z_1, z_2).$$

By Theorem 5.5.2, we conclude that $\mathcal{B}^{i/o}_{\mathfrak{V}}$ is $\mathcal{D}$-isomorphic to $\mathcal{B}^{i/o}_{\mathfrak{V}'}$ via a ring automorphism. □

Theorem 6.5.11. *Let $\mathfrak{V}$ and $\mathfrak{V}'$ be as stated in Theorem 6.5.10. If $\mathfrak{V}$ is Livšic controllable and* $\mathrm{Ann}(\mathcal{B}^{i/o}_{\mathfrak{V}})$ *is a prime ideal, then $\mathcal{B}^{i/o}_{\mathfrak{V}}$ has an image representation over its reduced ring.*

Proof. First note that, by Theorem 6.5.10, $\mathcal{B}^{i/o}_{\mathfrak{V}}$ is $\mathcal{D}$-isomorphic via ϕ to $\mathcal{B}^{i/o}_{\mathfrak{V}'}$. As a consequence, $\mathrm{Ann}(\mathcal{B}^{i/o}_{\mathfrak{V}}) = \phi^{-1}(\mathrm{Ann}(\mathcal{B}^{i/o}_{\mathfrak{V}'}))$. It follows that we may reduce the behaviors by isomorphic annihilators and, since each annihilator is a prime ideal, the reduced rings $\mathcal{D}_r$ $\mathcal{D}'_r$ are affine domains.

By Theorem 6.5.10, $\mathfrak{V}'$ is a Livšic controllable vessel with σ'_1 invertible in its input compatibility pencil. By Corollary 6.5.7 it follows that $\mathcal{B}^{i/o}_{\mathfrak{V}'}$ is 1-controllable and by Theorem 5.3.7, that $\mathcal{B}^{i/o}_{\mathfrak{V}'}$ is $\mathcal{D}_r$-controllable. Since $\mathcal{B}^{i/o}_{\mathfrak{V}}$ is $\mathcal{D}$-isomorphic to $\mathcal{B}^{i/o}_{\mathfrak{V}'}$ via the $\mathbb{C}$-algebra automorphism, we may appeal to Lemma 5.5.6 to conclude that $\mathcal{B}^{i/o}_{\mathfrak{V}}$ is also $\mathcal{D}_r$-controllable. □

The above process shows how one may work with the trajectory form of j-controllability. Under a change of coordinates, there may be an isomorphic behavior whose reduced signal space is j-controllable. This process can be employed in other situations to demonstrate that an image representation exists for the original behavior.

For completeness, we show the other direction.

Lemma 6.5.12. *Let $\mathfrak{V}$ be a vessel and $\mathcal{B}_{\mathfrak{V}}^{i/s/o}$ be its associated behavior. If $\operatorname{Ann}(\mathcal{B}_{\mathfrak{V}}^{i/s/o})$ is a prime ideal of height one and $\mathcal{B}_{\mathfrak{V}}^{i/s/o}$, when reduced, has an image representation, then $\mathfrak{V}$ is Livšic controllable.*

Proof. Define the reduced ring $\mathcal{D}_r = \mathcal{D}/\operatorname{Ann}(\mathcal{B}_{\mathfrak{V}}^{i/s/o})$. By Theorem 5.3.1, the $\operatorname{Red}(\mathcal{B}_{\mathfrak{V}}^{i/s/o})$ is divisible over its reduced ring $\mathcal{D}_r$. Since $\dim(\operatorname{Ann}(\mathcal{B}_{\mathfrak{V}}^{i/s/o})) = 1$, we have either $\mathbb{C}[z_1] \subset \mathcal{D}_r$ or $\mathbb{C}[z_2] \subset \mathcal{D}_r$. Without loss of generality, assume $\mathbb{C}[z_1] \subset \mathcal{D}_r$ is non-trivial, then for any $(t_1, 0) \in \mathbb{N}^2$ we have $z^{t_1}\mathcal{B}_{\mathfrak{V}}^{i/s/o} = \mathcal{B}_{\mathfrak{V}}^{i/s/o}$, i.e. all states are obtainable on $\mathbb{N} \times \{0\}$. This demonstrates

$$\operatorname{im}_{\mathbb{C}} \operatorname{row}_{n\in\mathbb{N}} \left[A_1^n \widetilde{B}\right] = \mathcal{X}$$

and hence

$$\operatorname{im}_{\mathbb{C}} \operatorname{row}_{(n_1,n_2)\in\mathbb{N}^2} \left[A_1^{n_1} A_2^{n_2} \widetilde{B}\right] = \mathcal{X}.$$

We conclude that, $\mathfrak{V}$ is Livšic controllable. □

6.6 Vessel I/O-Structure and Induced Bundles

Assumption 6.6.1. Let $\mathcal{B} \subset \mathcal{A}^q$ be a two-dimensional behavior with pure autonomy degree one and $\operatorname{Ann}(\mathcal{B})$ a prime ideal. We define $\mathcal{B}_r = \operatorname{Red}(\mathcal{B})$, $\mathcal{D}_r = \mathcal{D}/\operatorname{Ann}(\mathcal{B})$, and $\mathcal{A}_r$ is the associated reduced signal space. We also define the quotient fields $K = Q(\mathcal{D})$ and $K_r = Q(\mathcal{D}_r)$.

In [39], Oberst discusses signal flow systems and transfer functions for behaviors with a full i/o-structure. Clearly vessels do not use a full i/o-structure. In this section we present some modifications to signal flow systems and provide a procedure for identifying an i/o-structure which is more like that of a vessel.

Let $\mathcal{B}$ be a behavior satisfying Assumption 6.6.1. By Corollary 5.2.4, $\mathcal{B}_r$ has at least one free variable. However, it may be the case that multiple components (more than the number of free variables) have freedom on a sublattice of dimension one. Consider the following example.

Example 6.6.1. Define the matrix

$$R = \begin{bmatrix} z_1 - 1 & z_2 & 0 \\ z_1 & z_2 - 1 & z_1 \\ z_2 & 0 & z_2 - 1 \end{bmatrix},$$

and consider the behavior $\mathcal{B} = \ker_{\mathcal{A}}(R)$. We may compute $\partial\mathcal{B}$ to see

$$(\partial\mathcal{B})^c = \left(\{1\} \times (1,0) + \mathbb{N}^2\right) \cup \left(\{2\} \times (0,1) + \mathbb{N}^2\right) \cup \left(\{3\} \times (1,0) + \mathbb{N}^2\right).$$

That is, each component of $\mathcal{B}$ is free on a sublattice of dimension one.

One can verify that

$$\operatorname{Ann}(\mathcal{B}) = \langle xy^2 - xy - y^2 + x + 2y - 1\rangle$$

is a prime ideal. As a result, the reduced ring is an affine domain so we may compute rank as defined; however, over $\mathcal{D}_r$ we have that $\mathcal{B}_r$ has one free variable since $R \otimes 1_{\mathcal{D}_r}$ has rank equal to two.

Let $\mathfrak{V}$ be a vessel satisfying Assumption 6.1.1 and $\mathcal{B}_{\mathfrak{V}}^{i/o}$ be the external behavior associated with $\mathfrak{V}$ with its induced i/o-structure. A consequence of Theorem 6.3.5 is that $\mathcal{B}_{\mathfrak{V}}^{i/o}$ satisfies Assumption 6.6.1. By Corollary 6.3.11 we have that the input components are free on a sublattice of dimension one. This leads us to the following definition for a general 2-D behavior with autonomy degree one.

Definition 6.6.2. Let $\mathcal{B} \subset \mathcal{A}^q$ be an autonomous two-dimensional behavior with autonomy degree one. We define the **vessel i/o-structure** to be the i/o-structure where

$$\begin{aligned} \mathcal{I} =& \{i \in \{1,\dots,q\} : \text{the } i^{th} \text{ component of } \mathcal{B} \text{ has initial condition set} \\ & \quad \text{containing a one-dimensional lattice}\}, \\ \mathcal{O} =& \{1,\dots,q\} \setminus \mathcal{I}. \end{aligned}$$

Note that a vessel i/o-structure does not depend on the choice of kernel representation but does depend on the monomial ordering used to derive the initial condition set $\partial\mathcal{B}$.

Let $\mathcal{B}$ be a behavior satisfying Assumption 6.6.1 which has a kernel representation of the form

$$R_1 = \begin{bmatrix} -Q & P \end{bmatrix}$$

where the decomposition is subordinate to a given vessel i/o-structure. Let $\widehat{P}$ be an MLA of P so that, by Proposition 6.3.1, the following is also a kernel representation

$$R_2 = \begin{bmatrix} \widehat{P}Q & 0 \\ -Q & P \end{bmatrix}$$

and $\widehat{P}Q$ is a kernel representation for the associated input behavior. Assume that there exists a matrix M so that $\ker_{\mathcal{A}_r} \widehat{P}Q = \operatorname{im}_{\mathcal{A}_r} M$, i.e., the input behavior has an image representation over the reduced ring.

The reduced signal flow system of $\mathcal{B}$ is given by

$$\widehat{\mathcal{B} \otimes \mathcal{D}_r} := \ker_{K_r} \begin{bmatrix} -Q & P \end{bmatrix} = \ker_{K_r} \begin{bmatrix} \widehat{P}Q & 0 \\ -Q & P \end{bmatrix}$$

where $K_r = Q(\mathcal{D}_r)$ is the quotient field of the reduced ring. Assume P has full column rank over K_r and there exists a matrix H with entries in K_r such that

$$\widehat{\mathcal{B} \otimes \mathcal{D}}_r = \mathrm{im}_{K_r} \begin{bmatrix} I_{|\mathcal{I}|} \\ H \end{bmatrix}\Bigg|_{\ker_{K_r} \widehat{P}Q} = \mathrm{im}_{K_r} \begin{bmatrix} M \\ HM \end{bmatrix}$$

As a consequence we have

$$\begin{bmatrix} -Q & P \end{bmatrix} \begin{bmatrix} M \\ HM \end{bmatrix} = 0 \qquad \Rightarrow \qquad PHM = QM$$

If H' is another matrix satisfying this relation then

$$0 = PHM - PH'M = P(HM - H'M).$$

Since P has full column rank, we have $HM = H'M$ and hence $(H - H')M = 0$; in particular, we cannot distinguish the matrices H and H' by the input.

For the output, assume that there exists a non-degenerate determinantal representation X of appropriate size such that

$$\pi_{\mathcal{O}}(\mathcal{B}) \subset \ker_{\mathcal{A}}(X).$$

As a consequence, we have that the matrix

$$R_3 = \begin{bmatrix} 0 & X \\ \widehat{P}Q & 0 \\ -Q & P \end{bmatrix}$$

is also a kernel representation of $\mathcal{B}$. Let $\widehat{Q}$ be an MLA of Q so that, by Proposition 6.3.1, we have $\pi_{\mathcal{O}}(\mathcal{B}) = \ker_{\mathcal{A}}(\widehat{Q}P)$. Since $\ker_{\mathcal{A}}(\widehat{Q}P) \subset \ker_{\mathcal{A}}(X)$, there exists a polynomial matrix Y of appropriate size such that

$$X = Y\widehat{Q}P.$$

By the construction of H

$$\begin{bmatrix} X \\ 0 \\ P \end{bmatrix} HM = \begin{bmatrix} Y\widehat{Q}PHM \\ 0 \\ PHM \end{bmatrix} = \begin{bmatrix} Y\widehat{Q}QM \\ 0 \\ QM \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ QM \end{bmatrix}.$$

In particular, H remains a valid transfer function for the kernel representation R_3. In this process, we may observe that the identification of $\widehat{P}Q$ is unique up to a representative, and that H is unique up to its values on inputs; however, in general X is not uniquely determined. Nevertheless, the resulting behavior is the same and the choice of X does not change $\widehat{P}Q$ or H. This discussion sufficiently motives the following definition.

Definition 6.6.3. For a behavior $\mathcal{B} = \ker_{\mathcal{A}}(R)$ satisfying Assumption 6.6.1 and with vessel i/o-structure $(\mathcal{I}, \mathcal{O})$ inducing the decomposition $R = \begin{bmatrix} -Q & P \end{bmatrix}$, we say that $\mathcal{B}$ **agrees** with is vessel i/o-structure if the following are satisfied:

1. There exists a non-degenerate determinantal representation Y such that $\ker_{\mathcal{A}}(Y) = \ker_{\mathcal{A}}(\widehat{P}Q)$, where $\widehat{P}$ is an MLA of P.

2. There exists a non-degenerate determinantal representation X such that $\ker_{\mathcal{A}}(\widehat{Q}P) \subset \ker_{\mathcal{A}}(X)$, where $\widehat{Q}$ is an MLA of Q.

3. P has full column rank over K_r and there exists a matrix H with entries in K_r such that

$$\widehat{\mathcal{B} \otimes \mathcal{D}_r} = \operatorname{im}_{K_r} \begin{bmatrix} I_{|\mathcal{I}|} \\ H \end{bmatrix} \Bigg|_{\ker_{K_r}(\widehat{P}Q)}$$

In this context, we call H the **vessel transfer matrix**.

This leads to the following lemma.

Lemma 6.6.4. *Let $\mathcal{B}_r$ and $\mathcal{B}'_r$ be two $\mathcal{D}_r$-controllable reduced behaviors satisfying Assumption 6.6.1. Assume that $\mathcal{B}_r$ and $\mathcal{B}'_r$ are defined over the same ring $\mathcal{D}_r$, agree with the same vessel i/o-structure $(\mathcal{I}, \mathcal{O})$, have the same input, and have the same vessel transfer matrix. Then $\mathcal{B}_r = \mathcal{B}'_r$.*

Proof. By Lemma 6.5.9, there exists an image representation M for the input behavior $\mathcal{B}_r^{\mathcal{I}}$. As a consequence

$$\widehat{\mathcal{B}_r} = \operatorname{im}_{K_r} \begin{bmatrix} M \\ HM \end{bmatrix}.$$

Since $\mathcal{B}'_r$ has the same input trajectories and vessel transfer function, we also have

$$\widehat{\mathcal{B}'_r} = \operatorname{im}_{K_r} \begin{bmatrix} M \\ HM \end{bmatrix}.$$

Because $\mathcal{B}_r$ and $\mathcal{B}'_r$ are both $\mathcal{D}_r$-controllable, by Theorem 2.1.61 they are both minimal in their (same) transfer class; we conclude with the equality $\mathcal{B}_r = \mathcal{B}'_r$. □

We note that, in light of Lemma 6.6.4, for $\mathcal{D}_r$-controllable behaviors it is clear that the selection of X in Definition 6.6.3 has no bearing on the result.

To justify Definition 6.6.3, we have the following observation.

Lemma 6.6.5. *Let $\mathcal{V}$ be a vessel which has external behavior $\mathcal{B}_{\mathfrak{V}}^{i/o}$ satisfying Assumption 6.6.1. Then $\mathcal{B}_{\mathfrak{V}}^{i/o}$ agrees with its inherent vessel i/o-structure. In particular, there exists a vessel i/o-structure on which $\mathcal{B}_{\mathfrak{V}}^{i/o}$ agrees.*

Proof. By Corollary 6.3.11, we have that the choice of inputs and outputs provided by the vessel form a vessel i/o-structure. One may verify that $Y = U(z_1, z_2)$, $X = U_*(z_1, z_2)$ in the definition work. By Lemma 6.3.9, the reduced joint transfer matrix $S_{\mathfrak{V},r}(z_1, z_2)$ satisfies the remaining conditions. □

Finally we note that the vessel transfer matrix H from Definition 6.6.3 induces a well-defined meromorphic bundle map

$$\widehat{H} : \mathfrak{G}(\ker_{\mathcal{A}}(\widehat{P}Q)) \to \mathfrak{G}(\ker_{\mathcal{A}}(X)).$$

That H is well defined follows from the fact that it was constructed over the quotient field K_r; since the base space of both vector bundles in question is the variety associated with the coordinate ring $\mathcal{D}_r$, we have that H is well-defined on $\mathcal{V}(\mathrm{Ann}(\mathcal{B}))$.

Remark. Note that, in general, one cannot implicitly determine the output compatibility conditions from the external behavior of a vessel. For the case of a vessel $\mathfrak{V}$ and its external behavior $\mathcal{B}^{i/o}_{\mathfrak{V}}$ we have the ARMA system

$$\begin{bmatrix} 0 \\ U_* \\ -V_3 \end{bmatrix} y = \begin{bmatrix} U \\ 0 \\ V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3 D \end{bmatrix} u.$$

If we take an MLA of the right matrix so that

$$\begin{bmatrix} 0 & I_{\mathcal{Y}} & 0 \\ L_1 & 0 & L_2 \end{bmatrix} \begin{bmatrix} U \\ 0 \\ V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3 D \end{bmatrix} = 0$$

we get the kernel representation for the output behavior

$$\mathcal{B}^{o}_{\mathfrak{V}} = \ker_{\mathcal{A}} \begin{bmatrix} U_* \\ -L_2 V_3 \end{bmatrix}.$$

Unlike the case with the input, the inclusion

$$\pi_{\mathcal{O}}(\mathcal{B}^{i/o}_{\mathfrak{V}}) \subset \ker_{\mathcal{A}}(U_*),$$

is, in general, strict. However, if the vessel satisfies additional conditions (e.g., D invertible) then it is possible to achieve equality.

6.7 Hankel Realization for Meromorphic Bundle Maps

We are now ready to provide a missing piece in the state space realization of behaviors. Let $\mathcal{B}$ be a two-dimensional behavior. Recall that by Theorem 3.2.3, we are able to decompose $\mathcal{B}$ as

$$\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{B}_2.$$

In [16], Fornasini, Rocha, and Zampieri derived results for constructing a state space representation of finite dimensional behaviors. In particular, they use a zero-input Livšic system as the state space representation of $\mathcal{B}_2$. Since $\mathcal{B}_0$ has pure autonomy degree zero, it is controllable. In [43, Chapter 3], Rocha provides results for realizing $\mathcal{B}_0$ by an output-nulling type model. In this section, we discuss the final piece: showing that $\mathcal{B}_1$ can be represented by a Livšic system.

The development in the previous sections focused on affine bundles associated with behaviors. It was later mentioned that this can be extended to a compact complex manifold by homogenizing the kernel representation. In both cases, we demanded that $\mathcal{V}(\mathrm{Ann}(\mathcal{B}))$ be smooth; however, it was also pointed out that it is reasonable to validate when the family of amplitude vectors associated with a behavior lifts to a vector bundle over its normalizing Riemann surface. In the following results, we only ask that the amplitude FVS associated with an i/o-structure lift to a vector bundle over its normalizing Riemann surface. The following theorem provides conditions for obtaining a vessel consistent with a given input/output bundle pair and a bundle map between the two.

Theorem 6.7.1 (Hankel Realization)**.** *[6] Suppose that we are given two vector bundles H and H_* on the Riemann surface X which satisfy the conditions of Theorem 6.4.3. Further suppose that we are also given a meromorphic bundle map $T : H \to H_*$ with no poles on the line at infinity. Then there exists a controllable and observable vessel $\mathfrak{V}$ that satisfies Assumption 6.1.1 and has T as its joint transfer function.*

As an application of this meromorphic bundle-map realization theorem, we construct a state-space representation for two-dimensional behaviors with pure autonomy degree one.

Theorem 6.7.2. *Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a two-dimensional behavior where $\mathrm{Ann}(\mathcal{B})$ is a prime ideal, $\mathcal{B}$ is $\mathcal{D}_r$-controllable behavior, $\mathcal{B}$ has pure autonomy degree one, and $\mathcal{B}$ has a vessel i/o-structure $(\mathcal{I}, \mathcal{O})$ on which it agrees, i.e., there exists determinantal representations U and U_* such that*

$$\mathcal{B} = \ker_{\mathcal{A}} \begin{bmatrix} U_* & 0 \\ 0 & U \\ P & -Q \end{bmatrix},$$

where $R = \begin{bmatrix} P & -Q \end{bmatrix}$ is the decomposition subordinate to the i/o-structure $(\mathcal{I}, \mathcal{O})$, and a vessel transfer function H which provides the signal flow system of $\mathcal{B}$. Provided that the induced bundle map $\widehat{H}$ has no poles on the line at infinity, then it follows that there exists a controllable and observable vessel $\mathfrak{V}$, with associated input/output behavior $\mathcal{B}^{i/o}_{\mathfrak{V}}$, such that $\mathcal{B}^{i/o}_{\mathfrak{V}} = \mathcal{B}$.

Proof. By Theorem 6.4.3, we have that U and U_* provide holomorphic vector bundles satisfying the necessary conditions. We may apply Theorem 6.7.1 to construct the controllable and observable vessel $\mathfrak{V}$ that has $\widehat{H}$ as the joint transfer function and also satisfies Assumption 6.1.1. By Theorem 6.3.5, $\mathrm{Ann}(\mathcal{B}^{i/o}_{\mathfrak{V}})$ is a prime ideal. Since $\mathrm{Ann}(\mathcal{B})$ is a prime ideal and $\sqrt{\mathrm{Ann}(\mathcal{B})} = \sqrt{\mathrm{Ann}(\mathcal{B}^{i/o}_{\mathfrak{V}})}$, we have the equality $\mathrm{Ann}(\mathcal{B}) = \mathrm{Ann}(\mathcal{B}^{i/o}_{\mathfrak{V}})$; it follows that we may reduce both behaviors to the same reduced ring. Since $\mathfrak{V}$ is Livšic controllable, by Theorem 6.5.11, $\mathcal{B}^{i/o}_{\mathfrak{V}}$ is $\mathcal{D}_r$-controllable. Because the reduced Livšic transfer matrix of the vessel $\mathfrak{V}$ is the same as the vessel transfer matrix for the behavior $\mathcal{B}$, both $\mathcal{B}^{i/o}_{\mathfrak{V}}$ and $\mathcal{B}$ have the same vessel i/o-structure, both $\mathcal{B}^{i/o}_{\mathfrak{V}}$ and $\mathcal{B}$ are $\mathcal{D}_r$-controllable and both $\mathcal{B}^{i/o}_{\mathfrak{V}}$ and $\mathcal{B}$ have the same input trajectories, we may appeal Lemma 6.6.4 to conclude $\mathcal{B} = \mathcal{B}^{i/o}_{\mathfrak{V}}$. □

Remarks. (1) If $\mathfrak{G}(\mathcal{B})$ is a holomorphic vector bundle and $\overline{\mathcal{V}(\mathrm{Ann}(\mathcal{B}))}$ is smooth, then the reduced behavior has a projective dual module over its reduced ring. Provided we are willing to allow a projective change of coordinates, $\widehat{H}$ can be taken to not have any poles at infinity. The remaining parts of the hypothesis of Theorem 6.7.2 deal with the base space of the bundle and the structure of the input-output bundle pair.

(2) If we apply Theorem 6.7.2 to the behavior associated with a (possibly not Livšic controllable) vessel $\mathfrak{V}'$, we only reach the inclusion $\mathcal{B}^{i/o}_{\mathfrak{V}} \subset \mathcal{B}^{i/o}_{\mathfrak{V}'}$. The interesting observation is that if $\mathfrak{V}'$ is Livšic controllable, then Lemma 5.4.10 provides equality. However, if $\mathfrak{V}'$ is not Livšic controllable, then $\mathfrak{V}$ corresponds to Livšic controllable subsystem of $\mathfrak{V}$. In particular, Theorem 6.7.2 performs a crude Kalman decomposition of the vessel (see [6]).

Appendix A

Background Material

A.1 Gröbner Bases

In this section we discuss **standard bases** and some of their properties.

A.1.1 Monomial Orderings

Notation. Let k be a field and D be a polynomial ring of the form $D = k[z_1, \ldots, z_d]$.

In order to discuss monomial orderings it is necessary to define monomials.

Definition A.1.1. For a polynomial ring $D = k[z_1, \ldots, z_d]$ we define its set of **monomials** as

$$\mathrm{Mon}(D) = \{\mathbf{z}^\alpha : \alpha \in \mathbb{N}^d\}.$$

Under the induced action from D, we have that $\mathrm{Mon}(D)$ forms a semi-group under the multiplication operation. That is, for any two monomials $\mathbf{z}^\alpha, \mathbf{z}^\beta \in \mathrm{Mon}(D)$ we have $\mathbf{z}^\alpha \cdot \mathbf{z}^\beta = \mathbf{z}^{\alpha+\beta} \in \mathrm{Mon}(D)$. Via this operation, we have that there is a semi-group isomorphism $\mathrm{Mon}(D) \cong \mathbb{N}^d$ given by the exponent

$$\mathbf{z}^\alpha \in \mathrm{Mon}(D) \mapsto \alpha \in \mathbb{N}^d.$$

One can easily see that such a map is a bijection and preserve the semi-group operation on $\mathrm{Mon}(D)$ and $\mathbb{N}^d$ (where in the latter case, the semi-group operation is vector addition.) By building on this ordering, we have the following.

Definition A.1.2. We define a **monomial ordering** $\leq$ on $\mathrm{Mon}(D)$ as a total ordering which also satisfies

$$\mathbf{z}^\alpha \leq \mathbf{z}^\beta \quad \Rightarrow \quad \mathbf{z}^\alpha \mathbf{z}^\gamma \leq \mathbf{z}^\beta \mathbf{z}^\gamma \qquad \text{for all } \alpha, \beta, \gamma \in \mathbb{N}^d.$$

We write $<$ instead of $\leq$ when the order is strict, i.e. $\mathbf{z}^\alpha < \mathbf{z}^\beta$ if $\mathbf{z}^\alpha \leq \mathbf{z}^\beta$ and $\mathbf{z}^\alpha \neq \mathbf{z}^\beta$.

For a monomial ordering, we are concerned with specific properties of the ordering.

Definition A.1.3. Let $\leq$ be a monomial ordering on $\mathrm{Mon}(D)$. We say that $\leq$ is

1. a **global ordering** if $\mathbf{z}^\alpha > 1$ for all $\alpha \neq 0$.
2. a **local ordering** if $\mathbf{z}^\alpha < 1$ for all $\alpha \neq 0$.
3. a **mixed ordering** if it is neither local nor global.

For most simple applications of monomial orderings, use of global orderings is most natural. The use of local orderings tend to be most useful when discussing the localization of rings and related matters.

There is a "natural" *partial* ordering of monomials $\leq_{cw}$ we define as the **component-wise ordering** or **cw-ordering** given by

$$\mathbf{z}^\alpha \leq_{cw} \mathbf{z}^\beta \quad \iff \quad \beta - \alpha \in \mathbb{N}^d.$$

The cw-ordering is based on monomial division. In particular, if $\mathbf{z}^\alpha \leq_{cw} \mathbf{z}^\beta$ then there exists $\gamma \in \mathbb{N}^d$ such that $\beta - \alpha = \gamma$, $\beta = \alpha + \gamma$ and hence

$$\mathbf{z}^\alpha \mathbf{z}^\gamma = \mathbf{z}^{\alpha+\gamma} = \mathbf{z}^\beta.$$

We thus have the notion of divisibility given by $\mathbf{z}^\alpha | \mathbf{z}^\beta$. One interesting property of the cw-ordering is that *every* global ordering is a refinement of the cw-ordering.

Lemma A.1.4. *[21, page 11] For a global ordering $\leq$ we have $\mathbf{z}^\alpha \leq_{cw} \mathbf{z}^\beta$ implies $\mathbf{z}^\alpha \leq \mathbf{z}^\beta$. If $\alpha \neq \beta$, then $\mathbf{z}^\alpha < \mathbf{z}^\beta$.*

One further proper of the cw-ordering is the following.

Lemma A.1.5. *[21, page 12] Let $M \subset \mathbb{N}^d$ be any subset. Then there exist a* finite *subset $B \subset M$ such that for all $\alpha \in M$ there exists $\beta \in B$ such that $\beta \leq_{cw} \alpha$. By choosing B to be minimal under set inclusion, it is unique. The set B is known as the **Dickson basis** of G. For a subset $G \subset \mathbb{N}^d$ we define the set valued map* $\mathrm{Dickson}(G) = B$.

This leads us to the following definition.

Definition A.1.6. We say a subset $G \subset \mathbb{N}^d$ is a **cw-ideal** if $G + \mathbb{N}^d = G$ where

$$G + \mathbb{N}^d := \{p + q : p \in G,\ q \in \mathbb{N}^d\}.$$

In terms of the Dickson basis, we have the following observation.

Lemma A.1.7. *The subset $G \subset \mathbb{N}^d$ is a cw-ideal if and only if $G = \mathrm{Dickson}(G) + \mathbb{N}^d$.*

Proof. ($\Rightarrow$) Define $B = \text{Dickson}(G)$. For any $\alpha \in G$ there exists $\beta \in B \subset G$ and $\gamma \in \mathbb{N}^d$ such that $\beta + \gamma = \alpha$. This demonstrates $G \subset B + \mathbb{N}^d$. Since G is a cw-ideal, we have $B \subset G$ implies $B + \mathbb{N}^d \subset G + \mathbb{N}^d = G$. We conclude $G = B + \mathbb{N}^d$.

($\Leftarrow$) Define $B = \text{Dickson}(G)$. If $G = \text{Dickson}(G) + \mathbb{N}^d$ then

$$G + \mathbb{N}^d = \text{Dickson}(G) + \mathbb{N}^d + \mathbb{N}^d = \text{Dickson}(G) + \mathbb{N}^d = G.$$

□

Let us now return to monomial orderings and properties of monomial orderings. We may also use the monomial ordering to discuss certain pieces of polynomials in D.

Definition A.1.8. For $f \in D$ and a global monomial ordering $\leq$ we may write f as

$$f(\mathbf{z}) = a_{\alpha_1}\mathbf{z}^{\alpha_1} + a_{\alpha_2}\mathbf{z}^{\alpha_2} + \cdots + a_{\alpha_r}\mathbf{z}^{\alpha_r} \qquad\qquad \mathbf{z}^{\alpha_1} > \cdots > \mathbf{z}^{\alpha_r},$$

where $a_{\alpha_1}, \ldots, a_{\alpha_r} \in k$. In this situation, we define the following.

1. The **leading monomial** of f is defined as $\text{LM}(f) = \mathbf{z}^{\alpha_1}$.
2. The **leading term** of f is defined as $\text{LT}(f) = a_{\alpha_1}\mathbf{z}^{\alpha_1}$.
3. The **leading coefficient** of f is defined as $\text{LC}(f) = a_{\alpha_1}$.
4. The **leading exponent** of f is defined as $\text{LE}(f) = \alpha_1$.
5. The **tail** of f is defined as $\text{tail}(f) = f - \text{LT}(f)$.

For any monomial ordering, we are concerned with a particular set which is used to discuss Gröbner bases.

Definition A.1.9. For any monomial ordering $\leq$ on $\text{Mon}(D)$ we define the localization of D with respect to $\leq$ as

$$D_{\leq} = k[z_1, \ldots, z_d]_{\leq} = \{u \in D \setminus \{0\} : \text{LM}(u) = 1\}^{-1}D = \left\{ \frac{f}{g} : f, g \in D,\ \text{LM}(g) = 1 \right\}.$$

This ring is also known as the **localization of** D **with respect to** $\leq$.

One can see that, if $\leq$ is a global ordering, then $D_{\leq} = D$ since $\mathbf{z}^{\alpha} > 1$ for all non-zero $\alpha \in \mathbb{N}^d$; in particular, when $\leq$ is a global ordering

$$k \setminus \{0\} = \{u \in D \setminus \{0\} : \text{LM}(u) = 1\},$$

and thus localization by $k \setminus \{0\}$ does not change D since k is the scalar field. However, if $\leq$ is a local ordering, one can see that $D_{\leq}$ corresponds to localizing by the maximal ideal $\langle z_1, \ldots, z_d \rangle$.

From here, we reach an ideal which is crucial when discussing Gröbner bases.

Definition A.1.10. For a monomial ordering $\leq$ and any subset $G \subset D_{\leq}$ we define the **leading ideal** of G as

$$L_{\leq}(G) := L(G) := \langle \mathrm{LM}(f) : f \in G \setminus \{0\} \rangle_D \subset D.$$

We also define

$$L^{\mathbb{N}}(G) := \{\mathrm{LE}(f) : f \in G \setminus \{0\}\} \subset \mathbb{N}^d.$$

Important Note. *The leading ideal is, in general, not generated by the leading monomials of the generators of an ideal but is generated by the leading monomials of all elements of the ideal.*

One property of the leading ideal is the following.

Lemma A.1.11. *For a global ordering $\leq$ and any subset $G \subset D$ the set*

$$S := \{\mathrm{LE}(f) : f \in L(G)\} \subset \mathbb{N}^d$$

is a cw-ideal.

Proof. For any $\alpha \in S$ there exists $f \in L(G)$ such that $\mathrm{LE}(f) = \alpha$. Let $\beta \in \mathbb{N}^d$ be any given point. Since $L(G)$ is an ideal, we have $\mathbf{z}^\beta f \in L(G)$ and thus $\mathrm{LE}(\mathbf{z}^\beta f) = \beta + \mathrm{LE}(f) = \beta + \alpha \in S$. Since β was arbitrary, we have S is a cw-ideal. □

Corollary A.1.12. *Under the notation of Lemma A.1.11, the Dickson basis $B := \{b_1, \ldots, b_r\} =$ Dickson(S) provides the generators of the ideal $L(G)$. In particular,*

$$L(G) = \langle \mathbf{z}^{b_1}, \ldots, \mathbf{z}^{b_r} \rangle.$$

Proof. Since $B \subset S$, we have $\{\mathbf{z}^{b_1}, \ldots, \mathbf{z}^{b_r}\} \subset L(G)$ and thus $\langle \mathbf{z}^{b_1}, \ldots, \mathbf{z}^{b_r} \rangle \subset L(G)$. For any $f \in L(G)$ we have, by definition, that there exists $f_1, \ldots, f_n \in D$, $\alpha_1, \ldots, \alpha_n \in S$ and $g_1, \ldots, g_n \in G$ such that

$$f = f_1 \mathrm{LM}(g_1) + \cdots + f_n \mathrm{LM}(g_n) = f_1 \mathbf{z}^{\alpha_1} + \cdots + f_n \mathbf{z}^{\alpha_n}.$$

Because B is a Dickson basis of S, we have that there exists $\beta_1, \ldots, \beta_n \in B$ (with allowed repetition) and $\gamma_1, \ldots, \gamma_n \in \mathbb{N}^d$ such that

$$\alpha_1 = \beta_1 + \gamma_1, \alpha_2 = \beta_2 + \gamma_2, \ldots, \alpha_n = \beta_n + \gamma_n.$$

In particular, we have

$$f = (f_1 \mathbf{z}^{\gamma_1}) \mathbf{z}^{\beta_1} + \cdots + (f_n \mathbf{z}^{\gamma_n}) \mathbf{z}^{\beta_n} \in \langle \mathbf{z}^{b_1}, \ldots, \mathbf{z}^{b_r} \rangle.$$

We conclude that the Dickson basis induces a set of generators for the leading ideal $L(G)$. □

To demonstrate the importance of this corollary, we now turn to normal forms and Gröbner bases.

A.1.2 Gröbner Bases and Normal Forms for Ideals

Let us begin with the following definition.

Definition A.1.13. Let $\leq$ be a monomial ordering and $I \subset D_{\leq}$ be an ideal. We define

1. A *finite* set $G \subset D_{\leq}$ is a **standard basis** of I if $G \subset I$ and $L(I) = L(G)$. In other words, G is comprised of elements from I and for any $f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $\mathrm{LM}(g)|\mathrm{LM}(f)$.
2. If $\leq$ is a global ordering then we call a standard basis a **Gröbner basis**.

In the case that $\leq$ is a global ordering, then the existence of a Gröbner basis is quite clear; in any case, the existence follows from the fact D is a Noetherian ring.

Lemma A.1.14. *For every non-zero ideal $I \subset D_{\leq}$ there exists a standard basis G.*

Proof. If $\leq$ is a global ordering, then by Corollary A.1.12, there exists a finite set of monomials $\mathbf{z}^{b_1}, \ldots, \mathbf{z}^{b_r}$ such that

$$\langle \mathbf{z}^{b_1}, \ldots, \mathbf{z}^{b_r} \rangle = L(I).$$

By construction, for each $\mathbf{z}^{b_i}$ there exists $g_i \in I$ such that $\mathrm{LM}(g_i) = \mathbf{z}^{b_i}$. (Refer to Corollary A.1.12 and its proof to see why this is true.) We have that the set $G := \{g_1, \ldots, g_r\}$ define a Gröbner basis of I.

If $\leq$ is not a global ordering, then, since D is a Noetherian ring, there exists elements $m_1, \ldots, m_r \in D$ such that $L(I) = \langle m_1, \ldots, m_r \rangle$. Since $L(I)$ is generated by monomials, we have that $m_1, \ldots, m_r$ are monomials. By definition of $L(I)$, we have there exists $g_i \in I$ such that $\mathrm{LM}(g_i) = m_i$. One may then verify that $G := \{g_1, \ldots, g_r\}$ define a standard basis of I. $\square$

Standard bases are almost always never unique. As a result, it is necessary to further characterize them.

Definition A.1.15. Let $G \subset D_{\leq}$ be any subset.

1. G is said to be **interreduced** or **minimal** if $0 \notin G$ and $\mathrm{LM}(f) \nmid \mathrm{LM}(g)$ for distinct elements $f, g \in G$.
2. An element $f \in D_{\leq}$ is said to be **completely reduced with respect to** G if no monomial term of the power series expansion of f is contained in $L(G)$.
3. G is said to be **reduced** if G is interreduced and if for any $g \in G$ we have $\mathrm{LC}(g) = 1$ and $\mathrm{tail}(g)$ is completely reduced with respect to G.

When one usually refers to a Gröbner basis, it is under the assumption that it is reduced.

Definition A.1.16. Let $\mathcal{G}$ be the set of all finite lists (i.e., ordered sets) of $D_{\leq}$. We define a **normal form** as a map

$$\mathrm{NF} : D_{\leq} \times \mathcal{G} \to D_{\leq} \qquad\qquad (f, G) \mapsto \mathrm{NF}(f|G)$$

which satisfies the following properties for all $G \in \mathcal{G}$ and $f \in D_{\leq}$.

1. $\mathrm{NF}(0|G) = 0$
2. $\mathrm{NF}(f|G) \neq 0$ implies that $\mathrm{LM}(\mathrm{NF}(f|G)) \notin L(G)$.
3. For $G = \{g_1, \ldots, g_r\}$ as a **standard representation**

$$f - \mathrm{NF}(f|G) = \sum_{i=1}^{r} a_i g_i \qquad\qquad a_i \in D_{\leq}$$

   where $\mathrm{LM}(\sum_{i=1}^{r} a_i g_i) \geq \mathrm{LM}(a_i g_i)$ for all i such that $a_i \neq 0$.

In the case that $\mathrm{NF}(f|G)$ is reduced with respect to G we say that the normal form is a **reduced normal form**.

When one is working with an ordering which is not global, then condition (3) is usually relaxed to allow a **weak normal form**. The reason being that, since one is working with fractions rather than polynomials, condition (3) usually can never be satisfied. For global orderings (which is of main concern to us) Definition A.1.16 is what we seek.

The best way to think of a normal form is that it is just a division algorithm. For any element $f \in D_{\leq}$ there exists a representation

$$f = \mathrm{NF}(f|G) + \sum_{i=1}^{r} a_i g_i.$$

If we consider the ideal $G' = \langle g_1, \ldots, g_r \rangle$ then we have a direct sum decomposition

$$(\cdot - \mathrm{NF}(\cdot|G), \mathrm{NF}(\cdot|G)) : D_{\leq} \to L(G')^c \oplus G'$$

where $L(G')^c = \mathrm{span}_k\{\mathbf{z}^\alpha \in \mathrm{Mon}(D_{\leq}) : \mathbf{z}^\alpha \notin L(G')\}$ is the set complement of $L(G')$ given the structure of a k-vector space. Because both $D_{\leq}$ and G' are k-vector spaces, we have that the normal form produces an algebraic complementary basis for G' such that $D_{\leq} = L(G')^c \oplus G'$ is a direct sum of vector spaces.

Some useful properties of normal forms and Gröbner bases are summed in up the following lemma. We leave the notation $D_{\leq}$ rather than D (even though they are the same due to the assumption that the ordering is global) since the result also holds in the more general context of weak normal forms and standard bases.

Lemma A.1.17. *[21, page 48] Let $I \subset D_{\leq}$ be an ideal, $G = \{g_1, \ldots, g_r\} \subset I$ a Gröbner basis of I, and $\mathrm{NF}(-|G)$ a normal form on $D_{\leq}$. Then the following hold.*

1. *For any* $f \in D_{\leq}$ *we have* $f \in D_{\leq}$ *if and only if* $\mathrm{NF}(f|G) = 0$.

2. *If* $J \subset D_{\leq}$ *is an ideal such that* $I \subset J$, *then* $L(I) = L(J)$ *implies* $I = J$.

3. $I = \langle g_1, \ldots, g_r \rangle$, *i.e. the Gröbner basis serves as a set of generators for* I.

4. *If* $\mathrm{NF}(-|G)$ *is a reduced normal form, then it is unique.*

Before concluding the section, we present Buchberger's algorithm for computing the normal form. A necessary ingredient is the following.

Definition A.1.18. For $f, g \in D_{\leq} \setminus \{0\}$ with $\mathrm{LM}(f) = \mathbf{z}^{\alpha}$ and $\mathrm{LM}(g) = \mathbf{z}^{\beta}$ where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\beta = (\beta_1, \ldots, \beta_d)$ define

$$\gamma := (\max(\alpha_1, \beta_1), \ldots, \max(\alpha_d, \beta_d)).$$

We define the **s-polynomial** of f and g as

$$\mathrm{spoly}(f, g) = \mathbf{z}^{\gamma-\alpha} f - \frac{\mathrm{LC}(f)}{\mathrm{LC}(g)} \mathbf{z}^{\gamma-\beta} g.$$

We may now present Buchberger's algorithm for *global* monomial orderings.

Algorithm A.1.19 (Buchberger's Algorithm)**.**

Input: $f \in D$, $G \in \mathcal{G}$.

Output: $h \in D$, *a normal form of* f *with respect to* G.

1. *Set* $h = f$.
2. *While* $h \neq 0$ *and* $G_h := \{g \in G : \mathrm{LM}(g) | \mathrm{LM}(h)\} \neq \emptyset$,
3.   *Choose any* $g \in G_h$.
4.   *Set* $h = \mathrm{spoly}(h, g)$.
5. *Return* h.

A.1.3 Gröbner Bases and Normal Forms for Modules

The concept of a standard bases and normal forms can easily be extended to work with finitely generated D-modules. In this setting, the ring is replaced with a free module and the ideal is replaced by a submodule. Since every ideal can be regarded as a finitely generated D-module (recall that D is Noetherian), we have that the results from the previous section are a special case of the development in this section.

Definition A.1.20. Let $\leq$ be a monomial ordering on D. We define a **module ordering** on the free module D^q with basis $e_1, \ldots, e_q$ as a total ordering $\leq_m$ on the set of monomials

$$\mathrm{Mon}(D^q) := \{\mathbf{z}^\alpha e_i : \alpha \in \mathbb{N}^d, \quad i \in \{1, \ldots, q\}\}$$

which satisfies the following conditions.

1. If $\mathbf{z}^\alpha e_i \leq_m \mathbf{z}^\beta e_j$ then $\mathbf{z}^\gamma \mathbf{z}^\alpha e_i \leq_m \mathbf{z}^\gamma \mathbf{z}^\beta e_j$ for all $\gamma \in \mathbb{N}^d$.
2. If $\mathbf{z}^\alpha \leq \mathbf{z}^\beta$ then $\mathbf{z}^\alpha e_i \leq_m \mathbf{z}^\beta e_i$ for $i = 1, \ldots, q$.

Let $\leq_m$ be a module ordering with monomial ordering $\leq$. We have that any vector $f \in D^q \setminus \{0\}$ can be written uniquely as

$$f = c\mathbf{z}^\alpha e_i + \widehat{f}$$

where, as in the scalar setting, we define

1. The **leading monomial** as $\mathrm{LM}(f) = \mathbf{z}^\alpha e_i$.
2. The **leading coefficient** as $\mathrm{LC}(f) = c$.
3. The **leading term** as $\mathrm{LT}(f) = c\mathbf{z}^\alpha e_i$.
4. The **tail** as $\mathrm{tail}(f) = f - \mathrm{LT}(f)$.

For a subset $G \subset D^q$ we define the **leading submodule** as

$$L_{\leq_m} := L(G) := \langle \mathrm{LM}(g) | g \in G \setminus \{0\} \rangle \subset D^q.$$

When $G \subset D^q$ is a submodule, then $L(G)$ is the **leading module** of G.

For $f \in D^q = k[z_1, \ldots, z_d]^q$ we may write

$$f = a_{\alpha_1} \mathbf{z}^{\alpha_1} e_{n_1} + \cdots + a_{\alpha_r} \mathbf{z}^{\alpha_r} e_{n_r} \qquad \alpha_1, \ldots, \alpha_r \in \mathbb{N}^d, \; n_1, \ldots, n_r \in \{1, \ldots, q\}$$

For each "monomial" we have the identification

$$\mathbf{z}^{\alpha_i} e_{n_i} \mapsto (\alpha_i, n_i) \in \mathbb{N}^d \times \{1, \ldots, q\}.$$

For each $i \in \{1, \ldots, q\}$ we may identify $i \mapsto e_i \in \mathbb{N}^q$, where $e_1, \ldots, e_q$ are the standard basis of $\mathbb{N}^d$. In this way, we may identify, $\{1, \ldots, q\} \subset \mathbb{N}^q$ and thus

$$\mathbf{z}^{\alpha_i} e_{n_i} \mapsto (\alpha_i, e_{n_i}) \in \mathbb{N}^d \times \mathbb{N}^q = \mathbb{N}^{d+q}.$$

The cw-ordering on $\mathbb{N}^{d+q}$ provides us with the partial ordering

$$\mathbf{z}^\alpha e_i \leq_{cw} \mathbf{z}^\beta e_j \quad \Longleftrightarrow \quad i = j \text{ and } \mathbf{z}^\alpha \leq_{cw} \mathbf{z}^\beta.$$

Hence, when we speak of "divisibility" it is only under the condition that the basis elements are the same for the monomial pairs. Under this identification, for a D-module I we define the **E-subsets** as (leaving out I when it is understood)

$$E_I(i) := E(i) := \{\alpha \in \mathbb{N}^d : \mathbf{z}^\alpha e_i \in L(I)\} \subset \mathbb{N}^d.$$

and the **leading subset**

$$L^{\mathbb{N}}(I) = \{(i, \alpha) \in \{1, \dots, q\} \times \mathbb{N}^d : \mathbf{z}^\alpha e_i \in L(I)\} \subset \{1, \dots, q\} \times \mathbb{N}^d.$$

Definition A.1.21. Let $\leq_m$ be a module ordering with monomial ordering $\leq$ and $I \subset (D_{\leq})^q$ be a submodule.

1. We call a finite set $G \subset I$ a **standard basis** of I if $L(G) = L(I)$.
2. If the ordering $\leq_m$ is a well-ordering, then we call a standard basis a **Gröbner basis** of I. Note that, in this case, we have $D_{\leq} = D$ and thus $G \subset I \subset D^q$.
3. A set $G \subset (D_{\leq})^q$ is said to be **interreduced** or **minimal** if $0 \notin G$ and if $\mathrm{LM}(f) \notin L(G \setminus \{g\})$ for each $g \in G$.
4. An element $f \in (D_{\leq})^q$ is said to be **reduced with respect to** G if not monomial in the (vector valued) power series expansion of f is contained in $L(G)$.
5. A set $G \subset (D_{\leq})^q$ is said to be **reduced** if $0 \notin G$ and if each $g \in G$ is reduced with respect to $G \setminus \{g\}$, $\mathrm{LC}(g) = 1$, and $\mathrm{tail}(g)$ is reduced with respect to G.

Note that when $\leq_m$ is a well-ordering (the analogue of global in this setting) then a Gröbner basis is reduced if $\mathrm{LM}(g)$ is not divisible by any other element of G and has $\mathrm{LC}(g) = 1$.

This is also an analogue of normal forms for free modules.

Definition A.1.22. Let $\mathcal{G}$ be the set of all finite lists $G \subset (D_{\leq})^q$. We define a **normal form** as a map

$$\mathrm{NF} : (D_{\leq})^q \times \mathcal{G} \to (D_{\leq})^q \qquad (f, G) \mapsto \mathrm{NF}(f|G)$$

if the following hold for all $G \in \mathcal{G}$ and $f \in (D_{\leq})^q$.

1. $\mathrm{NF}(0|G) = 0$.
2. If $\mathrm{NF}(f|G) \neq 0$ then $\mathrm{LM}(\mathrm{NF}(f|G)) \notin L(G)$.
3. If $G = \{g_1, \dots, g_r\}$ then $f \in (D_{\leq})^q$ has a **standard representation**

   $$f - \mathrm{NF}(f|G) = \sum_{i=1}^{r} a_i g_i \qquad a_i \in D_{\leq}.$$

   where $\mathrm{LM}(\sum_{i=1}^r a_i g_i) \geq \mathrm{LM}(a_i g_i)$ for all i such that $a_i g_i \neq 0$.

If $\mathrm{NF}(f|G)$ is reduced with respect to G for all $G \in \mathcal{G}$, then we say that the normal form is a **reduced normal form**.

We also have the following properties of normal forms and Gröbner bases of modules.

Lemma A.1.23. *[21, pages 139-140] Let $I \subset (D_{\leq})^q$ be a submodule, $G = \{g_1, \dots, g_r\} \subset I$ be a Gröbner basis of I and $\mathrm{NF}(-|G)$ be a normal form on $(D_{\leq})^q$ with respect to G.*

1. *For any $f \in (D_{\leq})^q$ we have $f \in I$ if and only if $\mathrm{NF}(f|G) = 0$.*
2. *If $J \subset (D_{\leq})^q$ is a submodule with $I \subset J$, then $L(I) = L(J)$ implies $I = J$.*
3. *$I = D_{\leq}g_1 + \cdots + D_{\leq}g_r$, i.e., G generates the $D_{\leq}$-module I.*
4. *If $\mathrm{NF}(-|G)$ is a reduced normal form then it is unique.*

A.2 Holomorphic Vector Bundles

The following sections sketch what we need concerning holomorphic vector bundles and families of vector spaces; however, beyond the simple definitions presented here is a significant theory that we use but do not cover; more detailed references for this material include [3, 37, 24, 25].

Definition A.2.1. [3] We define a **family of (complex) vector spaces** (FVS) as the triple (E, X, π) where E and X are topological spaces and the continuous map $\pi : E \to X$ is the **projection map** onto the **base space** X. For a point $x \in X$ we call $\pi^{-1}(x) \in E$ the **fiber** of E over the point X. We require that $\pi^{-1}(x) \cong \mathbb{C}^q$ for some q; however the dimension of the fibers is not necessarily constant. When the X and π are understood, we write E instead of (E, X, π).

A particularly nice kind of FVS where the fiber dimension dim $\pi^{-1}(x)$ is independent of x is the particular case of a vector bundle as defined next.

Definition A.2.2. We call a **family of vector spaces** (E, X, π) a **vector bundle** if there exists an open cover $\mathcal{U} = \{U_i\}$ of E such that

$$\pi^{-1}(U_i) \cong U_i \times \mathbb{C}^q \qquad\qquad U_i \in \mathcal{U},$$

where q is fixed on the connected components of X. (Note that $\cong$ refers to a homeomorphism.) If X is connected, then we say that E is a vector bundle of **rank** q. If $q = 1$ then E is often called a **line bundle**. The cover $\mathcal{U}$, as stated above, is called a **trivializing cover** of E and $\pi^{-1}(U_i)$ is called a **trivial restriction** of E.

If the base space of the vector bundle has the structure of a smooth/complex manifold, then we may discuss whether or not functions on the manifold are differentiable/analytic.

Definition A.2.3. Let X be a smooth ℓ-dimensional complex manifold and E be a vector bundle over X with fibers in $\mathbb{C}^q$ and trivializing cover $\mathcal{U} = \{\psi_i : V_i \subset \mathbb{C}^\ell \to U_i \subset X\}$. We say that X is a **holomorphic vector bundle** if, for each i, there exists a biholomorphic map $\phi_i^{-1}(U_i) \cong U_i \times \mathbb{C}^q$ which preserves fibers and is linear on fibers.

If E is a rank q holomorphic vector bundle then, given two trivializing sets U_α and U_β such that $U_\alpha \cap U_\beta \neq \emptyset$, we may consider the **transition map** between the fibers as an element of $GL(q, \mathcal{O}) = \{D \in \mathcal{O}^{q\times q} : \det(D) \neq 0\}$ where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on X. This construction is useful for deriving new bundles from old ones. When we discuss determinantal representations in Section 6.4, we define some new bundles by adjusting the transition maps.

It turns out that one can define a holomorphic vector bundle by defining just the transition maps. To make the construction independent of choice of trivializing open cover for X, one usually works with the sheaf of germs of holomorphic functions and the transition maps are tied directly to Čech cohomology as in [25].

Let E be a rank r holomorphic vector bundle over the base space X and $X \times \mathbb{C}$ be the trivial line bundle. We define the **dual bundle** of E as the bundle $E^* := \mathrm{Hom}(E, X \times \mathbb{C})$ which is linear on the fibers. For each point $\boldsymbol{\lambda} \in X$, the fiber of E^* at $\boldsymbol{\lambda}$ is the dual vector space of the fiber of E at $\boldsymbol{\lambda}$. Similarly, let E be a holomorphic vector bundle over the Riemann surface X with transition matrices $\{\phi_{i,j} : U_i \cap U_j \to \mathrm{GL}(q, \mathcal{O}_X)\}$ where $\{U_i\}$ are a parametrizing cover of X and $\mathcal{O}_X$ is the sheaf of germs of holomorphic functions on X. We may equivalently define its dual bundle E^* as the holomorphic vector bundle given by the transition matrices $\{(\phi_{i,j}^T)^{-1} : U_i \cap U_j \to \mathrm{GL}(q, \mathcal{O}_X)\}$, where $\mathcal{O}_X$ is the sheaf of germs of holomorphic functions on X.

Definition A.2.4. We define the **tautological line bundle** $\mathcal{O}(1)$ over $\mathbb{P}^d$ as follows. Let $\mathbb{P}^d = \cup_{i=0}^n U_i$ be the standard coordinate open cover. We define $\mathcal{O}(1)$ by the transition maps $\{\phi_{i,j} : w \in \mathbb{C} \mapsto \frac{z_j}{z_i} w\}$. Similarly, we may define its dual $\mathcal{O}(-1)$ as the line bundle with transition maps $\{\phi_{i,j} : w \in \mathbb{C} \mapsto \frac{z_i}{z_j} w\}$. We denote the **trivial line bundle** as $\mathcal{O}(0)$. We let $\mathcal{O}(k)$ denote the k-fold tensor product $\otimes_k \mathcal{O}(1)$ and similarly $\mathcal{O}(-k) = \otimes_k \mathcal{O}(-1)$.

The line bundles on a complex manifold X form a group under the tensor product operation with inverses equal to the dual bundles. This group is known as the **Picard group** of X and is denoted by $\mathrm{Pic}(X)$. If we twist the holomorphic vector bundles E, then its dual becomes

$$(E \otimes \mathcal{O}(k))^* = E^* \otimes \mathcal{O}(-k).$$

Definition A.2.5. For a rank q holomorphic vector bundle E over the complex manifold X, the **determinantal line bundle** $\det(E)$ is defined by the transition maps $\{\det(\phi_{i,j}) : U_i \cap U_j \to \mathbb{C}\}$, where $\{\phi_{i,j} : U_i \cap U_j \to \mathbb{C}^q\}$ are the transition maps of E. Furthermore, we define the **Chern class** of E, denoted by $c(\det(E))$, is the Chern class of its determinantal line bundle, i.e., the degree of its associated divisor.

Definition A.2.6. Let X be a d-dimensional complex manifold. The **holomorphic tangent bundle**, T_X, is the bundle with transition maps given by $\{J(\phi_{i,j})(\phi_j(\mathbf{z})) : U_i \cap U_j \to \mathbb{C}^d\}$, where $\{\phi_{i,j} : U_i \cap U_j \to U_j\}$ are the transition maps of the complex manifold X and $J(\cdot)$ is the Jacobian matrix. The **holomorphic cotangent bundle**, Ω_X, is defined as the dual bundle of T_X. The holomorphic p-forms are defined as $\Omega_X^p = \bigwedge^p \Omega_X$ for $0 \leq p \leq d$. We define K_X to be the **canonical line bundle** of X to be $\det(\Omega_X) = \Omega_X^d$. If X has degree m and $X \subset \mathbb{P}^{d+1}$, then $K_X \cong \mathcal{O}_X(m-d-1)$.

A.3 Height and Codimension

Notation. Let k be an algebraically closed field and A be an affine ring.

Definition A.3.1 (Height)**.** [15, page 227] Let $I \subsetneq A$ be a prime ideal. A **descending chain of prime ideals** from I is a set of ideals $\{I_0, I_1, \ldots, I_{n-1}\}$ such that

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{n-1} \subsetneq I.$$

In this situation, we say that the chain has **length** equal to n. We define the **height** of I, denoted by $\text{height}(I)$, to be the supremum of lengths over all descending chains of prime ideals from I. If A is a domain, then we may take $I_0 = \langle 0 \rangle$ since this is the smallest prime ideal.

It is also common to refer to the height of I as the **codimension** of I. Due to the Hilbert Nullstellensatz, there is a one-to-one correspondence between prime ideals and algebraic varieties. One defines the codimension of a variety to be d (the dimension of the affine space) minus the dimension of the variety. In the special case when $A = k[z_1, \ldots, z_d]$ we also have that the dimension of the coordinate ring is given as

$$\dim(A/I) = d - \text{height}(I).$$

Unfortunately, since not all ideals are prime, we must also define the height of an arbitrary ideal of A.

Definition A.3.2 (Height, Dimension)**.** [15, page 227] Let $I \subsetneq A$ be an ideal. We define the **height** of I, denoted by $\text{height}(I)$, to be the infimum of the height of prime ideals containing I. We also define the dimension of I as $\dim(I) = \dim(A/I)$ as the Krull dimension of the ring A/I.

Geometrically, the above definition says that, even though an algebraic set may consist of pieces of variable dimension, we choose the component which has the largest dimension (hence smallest codimension). When we discuss the primary decomposition of ideals, the rationale for the above definition becomes more apparent. Note that in the case when A is not a domain, then this correspondence between height and dimension is, in general, false; one must consider dimension in a local sense in this setting.

Under our assumptions on A, there is another way of defining dimension. Since this different perspective is quite advantageous in this dissertation, it merits further discussion.

Definition A.3.3 (Independent Sets)**.** Let $I \subset A$ be an ideal. A subset $u = \{z_{i_1}, \ldots, z_{i_s}\} \subset \{z_1, \ldots, z_d\}$ is called an **independent set** if $I \cap k[z_{i_1}, \ldots, z_{i_r}] = 0$. For an independent set u we say that u is **maximal** if $\dim(A/I) = |u|$.

A simple application of Noether normalization (see [21, page 231]) provides the following result.

Lemma A.3.4. *[21, page 235] For an ideal $I \subset A$ we have $\dim(A/I) \geq |u|$ for any independent set $u \subset \{z_1, \ldots, z_d\}$; furthermore, there always exists a maximal independent set. In the case that u is empty (which is admissible) we have $|u| = 0$.*

Definition A.3.5. We define $\text{IndSets}(I) \subset 2^{\{z_1,\ldots,z_d\}}$ as the set of all independent sets of an ideal I.

We also have the following both simple and useful observations.

Lemma A.3.6. *For ideals $I \subset J \subset A$ we have $\text{IndSets}(J) \subset \text{IndSets}(I)$.*

Proof. For any $u \in \text{IndSets}(J)$ the inclusion $I \subset J$ implies

$$(I \cap k[u]) \subset (J \cap k[u]) = 0.$$

As a consequence, we have $u \in \text{IndSets}(I)$. □

Lemma A.3.7. *If $I \subset A$ is a prime ideal and $h \in \mathcal{D}/I$ is a nonzero element which is not a unit, then $\text{height}(h + I) > \text{height}(I)$ in A and hence $\dim(A/(h + I)) < \dim(A/I)$.*

Proof. By Krull's principle ideal theorem (See Corollary 11.17 in [4]), every minimal prime containing $h + I$ has height one in A/I; as a consequence, $\text{height}(h + I) = 1$ in A/I. Define $\pi : A \to A/I$ as the canonical quotient map. Let $P \supset h + I$ be any minimal prime containing $h + I$; then there exists a unique prime ideal $P' \in A$ such that $\pi(P') = P$ and $I \subset P'$.

Let

$$0 = I_0 \subset I_1 \subset \cdots \subset I_n \subset I$$

be a maximal chain of prime ideals containing I. Since $I \subset P'$ and I is a prime ideal,

$$0 = I_0 \subset I_1 \subset \cdots \subset I_n \subset I \subset P'$$

is a chain of prime ideals contained in P'. Since $\text{height}(P')$ is the length of a maximal chain of prime ideals, $\text{height}(P') > \text{height}(I)$. Because P was an arbitrary minimal prime containing $h + I$, we may conclude that $\text{height}(h + I) > \text{height}(I)$ in A. □

By combining Lemmas A.3.4, A.3.6, and A.3.7 we have the following.

Corollary A.3.8. *If $I \subset J \subset A$ are ideals and $\dim(A/J) < \dim(A/I)$ then for any maximal $u_J \in \text{IndSets}(J)$ and $u_I \in \text{IndSets}(I)$ we have that there exists some $z_i \in \{z_1, \ldots, z_d\}$ such that $z_i \in u_I$ and $z_i \notin u_J$.*

Recall that the **radical** of an ideal $I \subset A$ is defined as

$$\sqrt{I} = \{a \in A : a^n \in I \text{ for some } n \in \mathbb{N}_+\}.$$

The Hilbert Nullstellensatz states that there is a one-to-one correspondence between algebraic sets and radical ideals.

A.4 Primary Decomposition

The following terms and results are commonplace in commutative algebra. An excellent source for the material in this section is Atiyah and Macdonald [4, Chapter 4]. For understanding the computational aspects, Greuel and Pfister provide a beautiful expository treatment in [21].

Definition A.4.1. Let $N \subset M$ be a finitely generated A-module.

- M is **irreducible** if it cannot be written as the intersection of two strictly larger submodules.
- An **associated prime** of M is a prime ideal that is the annihilator of some element of M. The set of all associated primes of M is denoted by $\mathrm{Ass}(M)$.
- M is called **coprimary** if every zero divisor of M is nilpotent.
- A submodule N of a module M is called a **primary submodule** if M/N is coprimary.
- An ideal $I \subset A$ is **primary** if every zero-divisor of the ring A/I is nilpotent, or equivalently, if $ab \in I$ then either $a \in I$ or $b^n \in I$ for some n. Note that an equivalent more symmetric definition is : I is primary if whenever a and b are two elements of $A \setminus I$ such that $ab \in I$, then there is an $n \in \mathbb{N}$ so that $a^n \in I$ and $b^n \in I$.

Lemma A.4.2. *If $I \subset A$ is a primary ideal, then $\sqrt{I}$ is the smallest prime ideal containing I.*

Lemma A.4.3. *Any submodule of a finitely generated module over a Noetherian ring is the intersection of a finite number of irreducible submodules.*

Theorem A.4.4 (Lasker-Noether)**.** *Every submodule of a finitely generated module over a Noetherian ring is a finite intersection of irreducible primary submodules.*

When we consider the ring A as a module over itself, Theorem A.4.4 assumes the following form.

Corollary A.4.5. *Every ideal I of a Noetherian ring A is a finite intersection of primary ideals. We call this a **primary decomposition** of the ideal I.*

In the case of algebraic sets, the corollary is a generalization of: *every algebraic set can be written as the union of a finite number of varieties.* Since varieties correspond to prime ideals, we have that every radical ideal can be written as the finite intersection of prime ideals. The Lasker-Noether Theorem allows us to work with an ideal that is not radical since it performs the decomposition algebraically rather than geometrically.

The following terminology is quite useful when referring to a primary decomposition and its components.

Definition A.4.6. [4, page 52][21, page 259] Let $I \subset A$ be an ideal and $I = Q_1 \cap \cdots Q_r$ be a primary decomposition of I.

1. We say that the decomposition is **irredundant** if no Q_i can be omitted and $\sqrt{Q_i} \neq \sqrt{Q_j}$ for all $i \neq j$.

2. We call I **pure dimensional** if $\dim(A/Q_i) = \dim(A/Q_j)$ for all $i \neq j$.

3. We call a prime ideal P with $I \subset P$ a **minimal associated prime ideal** if I if for any prime ideal $Q \subset P$ with $I \subset Q \subset P$ we have the equality $Q = P$. We denote by $\mathrm{minAss}(I)$ the set of all minimal associated prime ideals associated to I.

4. If $\sqrt{Q_i} \subsetneq \sqrt{Q_j}$ then Q_j is called an **embedded prime ideal**.

5. If Q_i is not an embedded prime ideal, then it is called a **minimal prime ideal of the primary decomposition** of I.

Lemma A.4.7. *[4, page 54] The set of minimal prime ideals of the primary decomposition of an ideal is unique; however, in general the set of embedded primes in a minimal primary decomposition is not unique.*

A.5 Fitting Ideals

Fitting ideals, also known as Fitting's Invariants, provide a useful tool for studying finitely presented A-modules. We present an overview of the results on Fitting ideals needed in the sequel. A more thorough treatment of the material in this section can be found in [38, 13].

Definition A.5.1. Let $R \in A^{p\times q}$ be a matrix. For $1 \leq k \leq \min(p, q)$, we define the k^{th} **determinantal ideal** as the ideal generated by the $k \times k$ minors of R; we denote this ideal by $\mathfrak{I}_k(R)$. For $k > \min(p, q)$ we define $\mathfrak{I}_k(R) = 0$ and for $k \leq 0$ we define $\mathfrak{I}_k(R) = A$.

Let $R \in A^{p\times q}$ be a matrix and define $\ell = \min(p, q)$. It is clear that the determinantal ideals associated with a matrix form the following filtration.

$$\mathfrak{I}(R) : \cdots \mathfrak{I}_{\ell+1}(R) = 0 \subset \mathfrak{I}_\ell(R) \subset \mathfrak{I}_{\ell-1}(R) \subset \cdots \subset \mathfrak{I}_1(R) \subset \mathfrak{I}_0(R) = A \subset \cdots . \tag{A.1}$$

The above filtration, the **filtration of determinantal ideals**, appears several times in the following material. It is also useful to consider the **radical of the Fitting filtration**

$$\sqrt{\mathfrak{I}(R)} : \cdots \sqrt{\mathfrak{I}_{\ell+1}(R)} = 0 \subset \sqrt{\mathfrak{I}_\ell(R)} \subset \sqrt{\mathfrak{I}_{\ell-1}(R)} \subset \cdots \subset \sqrt{\mathfrak{I}_1(R)} \subset \sqrt{\mathfrak{I}_0(R)} = A \subset \cdots . \tag{A.2}$$

The following lemma assures us that determinantal ideals are invariant under unimodular matrix equivalence.

Lemma A.5.2. *[38, page 7] Let $R, R' \in A^{p\times q}$ be two equivalent matrices, i.e. there exists unimodular matrices U and V of suitable sizes so that $R = UR'V$. Then the determinantal ideals for R and R' are equal.*

Analogous ideas hold in the context of modules. . Let M be a finitely generated module which is generated by p elements. We say that M is **presented** by the matrix $R \in A^{p\times q}$ if M is a submodule of the free module A^p and $M \cong A^p/(RA^q)$. We define the **Fitting ideals** of M as

$$\mathfrak{F}_k(M) = \begin{cases} \mathfrak{I}_{p-k}(R) & 0 \leq k \leq p \\ A & k > \ell \end{cases}$$

In [38, Section 3.1], it is proved that the Fitting ideals are invariant under any choice of presentation matrix. Furthermore, they are *equal* under isomorphism.

From the filtrations (A.1) and (A.2) one easily arrives at the **filtration of Fitting ideals**

$$\mathfrak{F}(M)\colon \mathfrak{F}_0(M) \subset \mathfrak{F}_1(M) \subset \cdots \subset \mathfrak{F}_{\ell-1}(M) \subset \mathfrak{F}_\ell(M) \subset \cdots . \tag{A.3}$$

and the **radical of the filtration of Fitting ideals**

$$\sqrt{\mathfrak{F}(M)} : \sqrt{\mathfrak{F}_0(M)} \subset \sqrt{\mathfrak{F}_1(M)} \subset \cdots \subset \sqrt{\mathfrak{F}_{\ell-1}(M)} \subset \sqrt{\mathfrak{F}_\ell(M)}. \tag{A.4}$$

In fact, if M is generated by ℓ elements, then $\mathfrak{F}_\ell(M) = A$.

There are a couple of useful properties of modules characterized by the Fitting ideals. The first of these relates the 0-th Fitting ideal with the annihilator of the module.

Lemma A.5.3. *If M is a finitely generated A-module which is generated by ℓ elements, then*

$$(\mathrm{Ann}(M))^\ell \subset \mathfrak{F}_0(M) \subset \mathrm{Ann}(M).$$

In particular, $\sqrt{\mathrm{Ann}(M)} = \sqrt{\mathfrak{F}_0(M)}$.

The following three types of Fitting filtrations will be key for some of the discussion to come.

Definition A.5.4. Let M be a finitely generated A-module which is generated by ℓ elements. We say that $\mathfrak{F}(M)$ is **type I** if

$$\mathfrak{F}(M) : 0 = \mathfrak{F}_0(M) = \mathfrak{F}_1(M) = \cdots = \mathfrak{F}_{\ell-1}(M) \subset \mathfrak{F}_\ell(M) = A.$$

We say that $\mathfrak{F}(M)$ is **type II** if for some $k \leq \ell$,

$$\mathfrak{F}(M) : 0 = \mathfrak{F}_0(M) = \mathfrak{F}_1(M) = \cdots = \mathfrak{F}_{k-1}(M) \subset A = \mathfrak{F}_k(M) = \cdots = \mathfrak{F}_\ell(M).$$

We say that $\mathfrak{F}(M)$ is **type III** if for some $k \leq \ell$ and ideal $I \subset A$,

$$\mathfrak{F}(M) : I = \mathfrak{F}_0(M) = \mathfrak{F}_1(M) = \cdots = \mathfrak{F}_{k-1}(M) \subset A = \mathfrak{F}_k(M) = \cdots = \mathfrak{F}_\ell(M).$$

It is easy to see that the following relationships hold:

$$\text{type I} \quad \Rightarrow \quad \text{type II} \quad \Rightarrow \quad \text{type III}.$$

The following results provide very nice characterizations of modules through their Fitting ideals.

Proposition A.5.5. *[13, page 98] Let M be a finitely generated A-module and $\ell \geq 0$ be an integer. Then M is a free A-module of rank ℓ if and only if $\mathfrak{F}(M)$ is type I.*

Proposition A.5.6. *[13, page 101] Let M be a finitely generated A-module which is generated by ℓ elements. If $\mathfrak{F}(M)$ is type II, then M is a projective A-module.*

Recall that $\mathrm{Spec}(A)$ is the set of prime ideals of A with the Zariski topology (see [26, page 70]).

Lemma A.5.7. *[38, page 122]* Spec(A) *is connected if and only if A has no non-trivial idempotent elements.*

We thus reach the following useful relation between M and its Fitting ideals.

Proposition A.5.8. *[13, page 101] Let M be a finitely generated and* Spec(A) *be connected. Then $\mathfrak{F}(M)$ is of type II if and only if M is a projective A-module.*

The above result has a geometric consequence that we employ later on; however, we want to use it for a type III filtration.

Lemma A.5.9. *Let M be a finitely generated A-module and $I \subset A$ be an ideal. Then*

$$\mathfrak{F}_i(M \otimes_A (A/I)) = \mathfrak{F}_i(M) \otimes A/I.$$

Proof. This follows from the definitions. □

When we switch to working over a coordinate ring for a variety the following result becomes incredibly useful.

Corollary A.5.10. *Let M be a finitely generated A-module. If $\mathfrak{F}(M)$ is type III, $I = \mathfrak{F}_0(M)$, and I is a prime ideal, then $\mathfrak{F}(M \otimes A/I)$ is type II; in particular, $M \otimes (A/I)$ is a projective A/I-module.*

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