Combinatorial Properties of the Hilbert Series of Macdonald Polynomials

Elizabeth M. Niese

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

Nicholas Loehr, Chair

Ezra Brown

Edward Green

Peter Haskell

March 30, 2010

Blacksburg, Virginia

Keywords: Macdonald polynomials, symmetric functions, tableaux, permutation statistics

Combinatorial Properties of the Hilbert Series of Macdonald Polynomials

Elizabeth M. Niese

(ABSTRACT)

The original Macdonald polynomials P_μ form a basis for the vector space of symmetric functions which specializes to several of the common bases such as the monomial, Schur, and elementary bases. There are a number of different types of Macdonald polynomials obtained from the original P_μ through a combination of algebraic and plethystic transformations one of which is the modified Macdonald polynomial $\widetilde{H}_\mu$. In this dissertation, we study a certain specialization $\widetilde{F}_\mu(q,t)$ which is the coefficient of $x_1 x_2 \cdots x_N$ in $\widetilde{H}_\mu$ and also the Hilbert series of the Garsia-Haiman module M_μ. Haglund found a combinatorial formula expressing $\widetilde{F}_\mu$ as a sum of $n!$ objects weighted by two statistics. Using this formula we prove a q,t-analogue of the hook-length formula for hook shapes. We establish several new combinatorial operations on the fillings which generate $\widetilde{F}_\mu$. These operations are used to prove a series of recursions and divisibility properties for $\widetilde{F}_\mu$.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

The Macdonald polynomials P_μ [17] occupy an important place in algebraic combinatorics and the theory of symmetric functions. The P_μ provide a basis for the vector space of symmetric functions depending on two parameters, q and t. By suitable choices of the parameter, this basis specializes to the elementary, Schur, and monomial symmetric functions, which are common bases for the symmetric functions. The modified Macdonald polynomials $\widetilde{H}_\mu$ and the associated Hilbert series $\widetilde{F}_\mu$ have connections to representation theory and combinatorics that have been studied by Garsia, Haglund, Haiman, Loehr and many others [5][11] [14]. The combinatorial descriptions of $\widetilde{H}_\mu$ and $\widetilde{F}_\mu$ are a recent development in the theory of Macdonald polynomials. In this dissertation we undertake a detailed study of the combinatorial properties of the polynomial $\widetilde{F}_\mu(q,t)$, which is a q,t-analogue of $n!$ parametrized by an integer partition μ. We prove a q,t-analogue of the hook-length formula for hook shapes, develop several combinatorial operations on the fillings which generate $\widetilde{F}_\mu$, and prove a number of new recursions for $\widetilde{F}_\mu$ based on removing the bottom row of the fillings.

1.1 Symmetric Polynomials

Let K be a field of characteristic zero. We define the ring of symmetric polynomials in N variables with coefficients in K to be

$$\Lambda_N = \{f \in K[x_1, x_2, \dots, x_N] : f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(N)}) = f(x_1, x_2, \dots, x_N) \text{ for all } \sigma \in S_N\}$$

where S_N is the group of permutations of $\{1, \dots, N\}$. If we restrict to only those $f \in \Lambda_N$ which are homogeneous of degree k, the result is a K-vector space

$$\Lambda_N^k = \{f \in \Lambda_N : f = 0 \text{ or } f \text{ is homogeneous of degree } k\}.$$

This gives a grading

$$\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k.$$

Since Λ_N^k is a vector space, there is interest in understanding what a basis might look like. In fact, there are many common bases of Λ_N^k that appear in many different areas of mathematics. For our purposes, the power-sum and Schur bases are most relevant and will be reviewed briefly.

1.1.1 Power-Sums

Definition 1.1.1. A *partition* of a positive integer k is a sequence of positive integers $\mu = (\mu_1, \mu_2, \cdots, \mu_j)$ with $\mu_1 + \mu_2 + \cdots + \mu_j = k$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_j > 0$. Let $\mathrm{Par}(k)$ be the set of all partitions of k. The *length* of μ is the number j of parts of μ and is denoted $l(\mu)$.

An alternate notation for $\mu \in \mathrm{Par}(k)$ is $\mu = (1^{m_1}2^{m_2}\cdots k^{m_k})$ where m_i is the number of parts of μ of size i.

Definition 1.1.2. For a positive integer m, the m-th *power-sum in N variables* is

$$p_m(x_1,\ldots,x_N) = x_1^m + x_2^m + \cdots + x_N^m.$$

Then for $\mu \in \mathrm{Par}(k)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_j)$, we define

$$p_\mu = p_{\mu_1}p_{\mu_2}\cdots p_{\mu_j}.$$

Example 1.1.3. For $\mu = (2,1)$ and $N = 3$,

$$p_{(2,1)}(x_1,x_2,x_3) = p_2(x_1,x_2,x_3)p_1(x_1,x_2,x_3) = (x_1^2+x_2^2+x_3^2)(x_1+x_2+x_3).$$

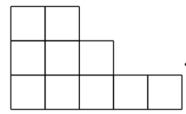
Theorem 1.1.4. *Let $N \geq k$. The set $\{p_\mu(x_1,\ldots,x_N) : \mu \in \mathrm{Par}(k)\}$ is a basis for Λ_N^k.*

Proof. See [20], p. 154. □

1.1.2 Schur Polynomials

Definition 1.1.5. For $\mu = (\mu_1,\ldots,\mu_l) \in \mathrm{Par}(n)$, the *Ferrers diagram* of μ is a collection of cells consisting of l left-justified rows with μ_i cells in row i. The *diagram* of μ is $\mathrm{dg}(\mu) = \{(i,j) : 1 \leq i \leq l(\mu), 1 \leq j \leq \mu_i\}$.

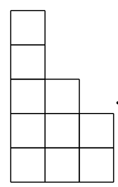
Example 1.1.6. The Ferrers diagram of $\mu = (5, 3, 2)$ is



.

Definition 1.1.7. The *conjugate* of $\mu \in \mathrm{Par}(n)$ is the partition $\mu' = (\mu'_1, \mu'_2, \ldots, \mu'_m)$ where μ'_i is the length of the i-th column in the Ferrers diagram of μ. Note that $\mu' \in \mathrm{Par}(n)$.

Example 1.1.8. The conjugate of $\mu = (5, 3, 2)$ is $\mu' = (3, 3, 2, 1, 1)$. The Ferrers diagram for μ' is



.

Definition 1.1.9. A *semistandard Young tableau* is a placement of integers drawn from the alphabet $\{1, 2, \ldots, N\}$ in the Ferrers diagram of a partition $\mu \in \mathrm{Par}(k)$ for some integer $k \leq N$ such that the entries weakly increase along rows, and strictly increase up columns. Let

$$\mathrm{SSYT}_N(\mu) = \{\text{semistandard Young tableaux of shape } \mu \text{ filled using letters in } 1, \ldots, N\}.$$

Example 1.1.10. For $\mu = (4, 2, 1)$ and $N = 8$,

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 6 \\ \cline{1-2} 5 & 8 \\ \hline 2 & 3 & 3 & 8 \\ \hline \end{array}$$

is a semistandard Young tableau.

Definition 1.1.11. A *standard Young tableau* of shape $\mu \in \mathrm{Par}(n)$ is a placement of the integers $1, 2, \ldots, n$ in the Ferrers diagram of μ (each used exactly once) such that the entries are strictly increasing along rows and up columns. Let

$$\mathrm{SYT}(\mu) = \{\text{standard Young tableaux of shape } \mu\}.$$

The number of standard Young tableaux of shape μ is denoted f^μ.

Example 1.1.12. For $\mu = (4, 2, 1)$,

$$S = \begin{array}{|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-2} 4 & 7 \\ \hline 1 & 2 & 3 & 6 \\ \hline \end{array}$$

is a standard Young tableau.

Definition 1.1.13. The *content monomial* of a semistandard Young tableau T of shape $\mu \in \mathrm{Par}(k)$ is

$$x^T = \prod_{i=1}^{N} x_i^{\text{number of } i\text{'s in } T}.$$

Example 1.1.14. The content monomial for the semistandard Young tableau in Example 1.1.10 is $x^T = x_2 x_3^2 x_5 x_6 x_8^2$.

Definition 1.1.15. For $\mu \in \mathrm{Par}(k)$, the *Schur polynomial* $s_\mu(x_1, \ldots, x_N) \in \Lambda_N^k$ is

$$s_\mu(x_1, \ldots, x_N) = \sum_{T \in \mathrm{SSYT}_N(\mu)} x^T.$$

Example 1.1.16. For $\mu = (2, 1)$ and $N = 3$, the elements of $\mathrm{SSYT}_3(2, 1)$ are listed along with their content monomials in Figure 1.1. Then

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2 x_3^2.$$

Theorem 1.1.17. *For integers N, k with $N \geq k$, $\{s_\mu : \mu \in \mathrm{Par}(k)\}$ is a basis for Λ_N^k.*

Proof. See [21], p. 315. □

$$\begin{array}{cccccccc}
\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline\end{array} &
\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline\end{array} &
\begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{|c|}{2} \\ \hline\end{array} &
\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array} &
\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array} &
\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline\end{array} &
\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline\end{array} &
\begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{|c|}{3} \\ \hline\end{array} \\
x_1^2x_2 & x_1^2x_3 & x_2^2x_3 & x_1x_2^2 & x_1x_2x_3 & x_1x_2x_3 & x_1x_3^2 & x_2x_3^2
\end{array}$$

Figure 1.1: The elements of $\mathrm{SSYT}_3(2,1)$ and their content monomials.

Several other symmetric polynomials can be defined as special cases of the Schur polynomials.

Definition 1.1.18. The *homogeneous symmetric polynomials* h_n can be defined as $h_n = s_{(n)}$ where $(n) \in \mathrm{Par}(n)$ is a single row of length n. Similarly, the *elementary symmetric polynomials* e_n can be defined by $e_n = s_{(1^n)}$ where $(1^n) \in \mathrm{Par}(n)$ is a single column with n boxes. As with the power sums, for $\mu = (\mu_1, \ldots, \mu_k) \in \mathrm{Par}(n)$ we write $h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_k}$ and $e_\mu = e_{\mu_1} e_{\mu_2} \cdots e_{\mu_k}$.

Theorem 1.1.19. *For integers N, k with $N \geq k$, $\{e_\mu : \mu \in \mathrm{Par}(k)\}$ and $\{h_\mu : \mu \in \mathrm{Par}(k)\}$ are both bases for Λ_N^k.*

Proof. See [20], p. 154. □

1.2 Plethysm

In order to understand much of the progress made in the theory of Macdonald polynomials, it is necessary to introduce plethystic notation. We start with a review of the universal mapping property for polynomial rings.

Theorem 1.2.1 (Universal Mapping Property, [16], p. 3)**.** *Let K be a field of characteristic zero. Let $K[z_1, \ldots, z_N]$ be a polynomial ring in N indeterminates. For every K-algebra S and* *every N-tuple $(a_1, \ldots, a_N)$ of elements of S, there exists a unique K-algebra homomorphism $\phi : K[z_1, \ldots, z_N] \to S$ such that $\phi(z_i) = a_i$ for $1 \leq i \leq N$.*

Definition 1.2.2. Let $K = \mathbb{Q}(q,t)$. The ring of *abstract symmetric functions* is $\Lambda = K[p_1, p_2, \ldots]$ where the p_i are indeterminates referred to as *abstract power sums.*

We now define the particular plethystic operations used in this thesis.

Definition 1.2.3. For any $f \in \Lambda$, $f[X(1-t)]$ is the image of f under the homomorphism $\phi_1 : \Lambda \to \Lambda$ defined on the power sums by $\phi_1(p_k) = p_k(1-t^k)$ for all $k \geq 1$. Similarly, $f[\frac{X}{1-t}]$ is the image of f under the homomorphism $\phi_2 : \Lambda \to \Lambda$ defined on the power sums by $\phi_2(p_k) = \frac{p_k}{1-t^k}$. We can also define $f[X(t-1)]$ as the image of f under the homomorphism $\phi_3 : \Lambda \to \Lambda$ defined on the power sums by $\phi_3(p_k) = p_k(t^k - 1)$. Similarly, $f[X(q-1)]$ is the image of f under the homomorphism $\phi_4 : \Lambda \to \Lambda$ defined on the power sums by $\phi_4(p_k) = p_k(q^k - 1)$.

Example 1.2.4. Let $f \in \Lambda$. To evaluate $f[X(1-t)]$, we first write f as a linear combination of the power-sums,

$$\begin{aligned} f &= \sum_{\mu} c_\mu p_\mu \qquad\qquad (c_\mu \in K) \\ &= \sum_{\mu} c_\mu \prod_i p_{\mu_i}. \end{aligned}$$

Then, by Definition 1.2.3,

$$\begin{aligned} f[X(1-t)] &= \phi_1(f) \\ &= \phi_1\left(\sum_\mu c_\mu \prod_i p_{\mu_i}\right) \\ &= \sum_\mu c_\mu \prod_i \phi_1(p_{\mu_i}) \\ &= \sum_\mu c_\mu \prod_i p_{\mu_i}(1-t^{\mu_i}) \\ &= \sum_\mu c_\mu p_\mu \prod_i (1-t^{\mu_i}). \end{aligned}$$

1.3 History of Macdonald Polynomials

Let K be the field $\mathbb{Q}(q,t)$. In [17], Macdonald introduced a basis $\{P_\mu(x_1,\ldots,x_N;q,t) : \mu \in \mathrm{Par}(k)\}$ for Λ_N^k. Before stating Macdonald's original definition, we discuss a partial ordering on partitions and some features of Ferrers diagrams.

Definition 1.3.1. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ are partitions of n. If $i > k$, set $\lambda_i = 0$. Similarly, if $i > m$, set $\mu_i = 0$. Then λ *dominates* μ, written $\lambda \trianglerighteq \mu$, if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$$

for all $i \geq 1$. It can be shown that $\trianglerighteq$ is a partial ordering on $\mathrm{Par}(n)$ for all $n \geq 0$.

Example 1.3.2. We have $(2,1) \trianglerighteq (1,1,1)$ since $2 \geq 1$, $2+1 \geq 1+1$, and $2+1+0 \geq 1+1+1$. The partitions $(3,3)$ and $(4,1,1)$ are not comparable since $4 \geq 3$ but $3+3 \geq 4+1$.

Definition 1.3.3. Let $\mu \in \mathrm{Par}(n)$ and $c = (i,j)$ be the cell in the i-th row and j-th column of the Ferrers diagram of μ.

(a) The *arm* of c is the set $ARM(c)$ of all cells in the same row as c and to the right of c. Let $a(c) = a_\mu(c) = |ARM(c)|$.

(b) The *coarm* of c is the set $COARM(c)$ of all cells in the same row as c and to the left of c. Let $a'(c) = |COARM(c)|$.

(c) The *leg* of c is the set $LEG(c)$ of all cells in the same column as c and above c. Let $l(c) = l_\mu(c) = |LEG(c)|$.

(d) The *coleg* of cell c is the set $COLEG(c)$ of all cells in the same column as c which are below c. Let $l'(c) = |COLEG(c)|$.

(e) The *hook* of c is the set $H(c) = ARM(c) \cup LEG(c) \cup \{c\}$. The *hook length* of cell c is $h_c = |H(c)|$.

Definition 1.3.4. For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathrm{Par}(n)$, define $n(\mu) = \sum_{i=1}^{k} (i-1)\mu_i$.

Example 1.3.5. For $\mu = (5, 5, 3, 2)$, and $c = (2, 3)$

$$\begin{array}{|c|c|c|c|c|} \hline & \\ \cline{1-3} & & \boldsymbol{L} \\ \hline A' & A' & \boldsymbol{c} & \boldsymbol{A} & \boldsymbol{A} \\ \hline & & L' & & \\ \hline \end{array}.$$

$ARM(c)$ is the set of cells marked by A, and $a(c) = 2$. $LEG(c)$ is marked by L and $l(c) = 1$. $COARM(c)$ is marked by A' and $a'(c) = 2$. $COLEG(c)$ is marked by L' and $l'(c) = 1$. The cells in $H(c)$ are in bold and $h_{(2,3)} = 4$. To determine $n(\mu)$, place $(i-1)$ in each of the cells of row μ_i and sum the entries in each cell as shown:

$$\begin{array}{|c|c|c|c|c|} \cline{1-2} 3 & 3 \\ \cline{1-3} 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}.$$

So $n(\mu) = 3 + 3 + 2 + 2 + 2 + 1 + 1 + 1 + 1 + 1 = 17$.

Definition 1.3.6. In order to define P_μ, we need the following:

(a) For $\mu \in \mathrm{Par}(n)$, define $z_\mu = 1^{m_1} 2^{m_2} \cdots n^{m_n} m_1! m_2! \cdots m_n!$ where m_i is the number of parts of μ of size i.

(b) For any statement A, let $\chi(A) = 1$ if A is true and 0 if A is false.

(c) We can define a scalar product $\langle \cdot, \cdot \rangle_{q,t}$ on the power-sum basis from Theorem 1.1.4 by setting

$$\langle p_\mu, p_\lambda \rangle_{q,t} = z_\mu \prod_i \frac{1 - q^{\mu_i}}{1 - t^{\mu_i}} \chi(\mu = \lambda).$$

Theorem 1.3.7 ([17], p. 140)**.** *For each $n \geq 0$, there are unique symmetric functions $\{P_\mu : \mu \in \mathrm{Par}(n)\}$ such that*

(a) $P_\mu = s_\mu + \sum_{\substack{\lambda \trianglelefteq \mu, \\ \lambda \neq \mu}} s_\lambda \xi_{\lambda\mu}$, *for some* $\xi_{\lambda\mu} \in \mathbb{Q}(q,t)$, *and*

(b) $\langle P_\mu, P_\lambda \rangle_{q,t} = 0$ *for all* $\mu \neq \lambda$.

One of the features of the original Macdonald polynomials is that they specialize to several different bases of Λ^k_N as seen in Table 1.1.

Macdonald also defined $\{Q_\mu(x_1, \ldots, x_N; q, t)\}$ which provides a dual basis relative to $\langle \cdot, \cdot \rangle_{q,t}$ to the $\{P_\mu\}$ by setting

$$Q_\mu(x_1, \ldots, x_N; q, t) = \frac{h_\mu(q,t)}{h'_\mu(q,t)} P_\mu(x_1, \ldots, x_N; q, t),$$

Table 1.1: Specializations of Macdonald Polynomials [17]

Specialization	Symmetric Function
$P_\mu\vert_{q=t} = s_\mu$	Schur functions
$P_\mu\vert_{q=0} = HL_\mu$	Hall-Littlewood Polynomials
$P_\mu\vert_{t=1} = m_\mu$	Monomial symmetric functions
$P_\mu\vert_{q=1} = e_{\mu'}$	Elementary symmetric functions

where

$$h_\mu(q,t) = \prod_{c\in \mathrm{dg}(\mu)} (1 - q^{a_\mu(c)} t^{l_\mu(c)+1}) \text{ and } h'_\mu(q,t) = \prod_{c\in \mathrm{dg}(\mu)} (1 - q^{a_\mu(c)+1} t^{l_\mu(c)}).$$

He goes on to define

$$J_\mu(x_1,\ldots,x_N;q,t) = h_\mu(q,t)P_\mu(x_1,\ldots,x_N;q,t) = h'_\mu(q,t)Q_\mu(x_1,\ldots,x_N;q,t).$$

The J_μ can also be written using plethystic notation [6]:

$$J_\mu(X;q,t) = \sum_\lambda s_\lambda[X(1-t)]K_{\lambda\mu},$$

where $K_{\lambda\mu} \in \mathbb{Q}(q,t)$ is a q,t-analogue of the Kostka coefficient. Yet another plethystic transformation gives us

$$H_\mu = J_\mu\left[\frac{X}{1-t}\right] = \sum_\lambda s_\lambda K_{\lambda\mu}.$$

The *modified Macdonald polynomial* is

$$\widetilde{H}_\mu(x_1,\ldots,x_N;q,t) = t^{n(\mu)}H(x_1,\ldots,x_N;q,1/t) = \sum_\lambda s_\lambda \widetilde{K}_{\lambda\mu}$$

where $\widetilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,1/t)$. The coefficient of $x_1x_2\ldots x_N$ in $\widetilde{H}_\mu(X;q,t)$ is given by

$$\widetilde{F}_\mu(q,t) = \sum_\lambda f^\lambda \widetilde{K}_{\lambda\mu}(q,t),$$

where f^λ is the number of standard tableaux of shape λ.

There is an axiomatic description of the modified Macdonald polynomials which was used in the proof of Haglund's combinatorial formula for Macdonald polynomials [14].

Theorem 1.3.8 ([14])**.** *Let $N, k \in \mathbb{N}$ with $N \geq k$. There exists a unique basis $\{\widetilde{H}_\mu : \mu \in \mathrm{Par}(k)\}$ for Λ_N^k satisfying the following axioms:*

(1) $\widetilde{H}_\mu[X(q-1);q,t] = \sum_{\lambda \trianglelefteq \mu'} c_{\mu,\lambda} m_\lambda$ *for some* $c_{\mu,\lambda} \in \mathbb{Q}(q,t)$,

(2) $\widetilde{H}_\mu[X(t-1);q,t] = \sum_{\lambda \trianglelefteq \mu} d_{\mu,\lambda} m_\lambda$ *for some* $d_{\mu,\lambda} \in \mathbb{Q}(q,t)$, *and*

(3) The coefficient of x_1^k in $\widetilde{H}_\mu$ is 1.

1.4 Garsia-Haiman Modules

$\widetilde{H}_\mu$ can also be defined as the Frobenius series of a particular S_n-module [20], denoted M_μ. These modules were introduced by Garsia and Haiman in [5]. In lieu of a general definition, we give an example of the construction.

Start with $\mu = (2,2,1)$. Consider the diagram of μ and label the cells with coordinates as shown here:

$$\begin{array}{|c|c|}\cline{1-1} (0,2) & \multicolumn{1}{c}{} \\ \hline (0,1) & (1,1) \\ \hline (0,0) & (1,0) \\ \hline \end{array}\ .$$

Create a determinant Δ_μ by using the coordinates of each cell as exponents of the variables $x_1, \ldots, x_5, y_1, \ldots, y_5$ in order from bottom to top, left to right as shown below:

$$\Delta_\mu = \det \begin{pmatrix} x_1^0 y_1^0 & x_2^0 y_2^0 & x_3^0 y_3^0 & x_4^0 y_4^0 & x_5^0 y_5^0 \\ x_1^1 y_1^0 & x_2^1 y_2^0 & x_3^1 y_3^0 & x_4^1 y_4^0 & x_5^1 y_5^0 \\ x_1^0 y_1^1 & x_2^0 y_2^1 & x_3^0 y_3^1 & x_4^0 y_4^1 & x_5^0 y_5^1 \\ x_1^1 y_1^1 & x_2^1 y_2^1 & x_3^1 y_3^1 & x_4^1 y_4^1 & x_5^1 y_5^1 \\ x_1^0 y_1^2 & x_2^0 y_2^2 & x_3^0 y_3^2 & x_4^0 y_4^2 & x_5^0 y_5^2 \end{pmatrix}.$$

To construct M_μ, now take all possible partial derivatives of $\Delta_\mu \in \mathbb{C}[x_1, \ldots, x_5, y_1, \ldots, y_5]$ and then take all $\mathbb{C}$-linear combinations of these partial derivatives. The result is M_μ. S_n acts on M_μ by the rule $\sigma \cdot x_i = x_{\sigma(i)}$ and $\sigma \cdot y_i = y_{\sigma(i)}$. The module can be written $M_\mu = \bigoplus_{a \geq 0} \bigoplus_{b \geq 0} M_\mu^{(a,b)}$ where $M_\mu^{(a,b)}$ is the submodule of homogeneous polynomials in M_μ of degree a in $\mathbf{x}$ and degree b in $\mathbf{y}$. It is known [11] that $\dim(M_\mu) = n!$. If the double grading of the module M_μ is ignored, the left regular representation of S_n is obtained [11].

We can write $M_\mu^{(a,b)} = \bigoplus_{\lambda \in \mathrm{Par}(n)} c_\lambda V^\lambda$ where the V^λ are the irreducible representations of S_n [20]. Then $\mathrm{Frob}(M_\mu^{(a,b)}) = \sum_\lambda c_\lambda s_\lambda$.

Definition 1.4.1. Let $\mu \in \mathrm{Par}(n)$. The *Frobenius series* for M_μ is

$$\mathrm{Frob}(M_\mu) = \sum_{a \geq 0} \sum_{b \geq 0} q^a t^b \, \mathrm{Frob}(M_\mu^{(a,b)}).$$

Definition 1.4.2. Let $\mu \in \mathrm{Par}(n)$. The *Hilbert series* of M_μ is

$$\mathrm{Hilb}(M_\mu) = \sum_{a,b \geq 0} \dim(M_\mu^{(a,b)}) q^a t^b.$$

Theorem 1.4.3. *For $\mu \in \mathrm{Par}(n)$,*

(a) $\widetilde{H}_\mu(X; q, t) = \mathrm{Frob}(M_\mu)$ *and*

(b) $\widetilde{F}_\mu(q, t) = \mathrm{Hilb}(M_\mu)$.

Proof. See [11]. □

1.5 Combinatorial Formulas for Macdonald Polynomials

In [9] Haglund conjectured, and later in [14] Haglund, Haiman, and Loehr proved, a combinatorial formula for the $\widetilde{H}_\mu$. In this section, we review Haglund's formula for the $\widetilde{H}_\mu$ and $\widetilde{F}_\mu$.

Definition 1.5.1. A *filling* of shape $\mu \in \mathrm{Par}(n)$ is a placement of integers drawn from $\{1, 2, \cdots, N\}$ (integers can be repeated) in the Ferrers diagram of μ.

Example 1.5.2. For $\mu = (5, 3, 1)$ and $N = 8$, an example of a filling is:

$$T = \begin{array}{|c|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-3} 2 & 4 & 7 \\ \hline 4 & 7 & 8 & 2 & 5 \\ \hline \end{array} .$$

Definition 1.5.3. A *standard filling* T of shape $\mu \in \mathrm{Par}(n)$ is a placement of the integers $1, 2, \ldots, n$ in the Ferrers diagram of μ with each integer used exactly once. A filling can be identified with a permutation $w \in S_n$ by reading the rows of the diagram from left to right, beginning with the shortest row. The *column words* of a filling are obtained by reading the columns from top to bottom.

Example 1.5.4. For $\mu = (5, 3, 1)$, an example of a standard filling is:

$$T = \begin{array}{|c|c|c|c|c|}\cline{1-1} 4 \\ \cline{1-3} 3 & 1 & 6 \\ \hline 2 & 8 & 5 & 9 & 7 \\ \hline \end{array}\,.$$

This filling is identified with the permutation $w = 431628597 \in S_9$. The column words for this filling are 432, 18, 65, 9, and 7.

Definition 1.5.5. The *standardization* of a filling T, denoted $\mathrm{stdz}(T)$, is constructed by replacing the 1's in T with $1, 2, \cdots, k$ in reading order (left to right, top to bottom), then replacing the 2's in T with $k+1, \cdots, j$, etc. The result is a standard filling.

Example 1.5.6. The standardization of

$$T = \begin{array}{|c|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-3} 2 & 4 & 6 \\ \hline 4 & 7 & 8 & 2 & 5 \\ \hline \end{array}$$

is

$$\mathrm{stdz}(T) = \begin{array}{|c|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-3} 1 & 3 & 7 \\ \hline 4 & 8 & 9 & 2 & 6 \\ \hline \end{array}\,.$$

Haglund's formula involves statistics $\mathrm{inv}_\mu(T)$ and $\mathrm{maj}_\mu(T)$ on fillings. To define these, we first recall some classical permutation statistics defined for $w \in S_n$.

Definition 1.5.7. Fix $w = w_1 w_2 \cdots w_n \in S_n$. An *inversion* of w is a pair $i < j$ such that $w_i > w_j$. Note that i and j do not need to be consecutive. Let $\mathrm{inv}(w)$ be the number of inversions of w. A *descent* of w is an index $i < n$ with $w_i > w_{i+1}$. The *descent set* of w is $\mathrm{Des}(w) = \{i : w_i > w_{i+1}\}$. The *major index* of w is $\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i$. Similarly, define the *ascent set* $\mathrm{Asc}(w) = \{i : w_i < w_{i+1}\}$ and the *co-major index* $\mathrm{comaj}(w) = \sum_{i \in \mathrm{Asc}(w)} i$.

Example 1.5.8. For $w = 52341 \in S_5$, $\mathrm{inv}(w) = 7$, $\mathrm{Des}(w) = \{1, 4\}$, $\mathrm{maj}(w) = 5$, $\mathrm{Asc}(w) = \{2, 3\}$, and $\mathrm{comaj}(w) = 5$.

Definition 1.5.9. For a standard filling T of $\mu \in \mathrm{Par}(n)$, we define a *triple* to be three cells in T with entries a, b, c, arranged as shown here:

$$\begin{array}{|c|}\hline a \\ \hline c \\ \hline \end{array} \qquad \begin{array}{|c|}\hline b \\ \hline \end{array}$$

where a is directly above c and b is in the same row as a and to the right of a. If the cells containing a and b are in the lowest row, take $c = \infty$. This triple is an *inversion triple* of T iff $a < c < b$ or $c < b < a$ or $b < a < c$. We define

$$\mathrm{inv}_\mu(T) = \text{number of inversion triples of } T.$$

Definition 1.5.10. For a standard filling T of $\mu \in \mathrm{Par}(n)$, the *μ-major index* of T is

$$\mathrm{maj}_\mu(T) = \text{sum of the major indices of the column words of } T.$$

Example 1.5.11. For the standard filling T of $\mu = (3, 3, 2, 1)$ given by

$$T = \begin{array}{|c|c|c|}\cline{1-1} 6 \\ \cline{1-2} 2 & 5 \\ \hline 1 & 3 & 8 \\ \hline 7 & 4 & 9 \\ \hline \end{array} \, ,$$

$$\text{maj}_\mu(T) = \text{maj}(6217) + \text{maj}(534) + \text{maj}(89) = 3 + 1 + 0 = 4,$$

$n(\mu) = 10$, and $\text{inv}_\mu(T) = 3$ since the inversion triples are $4 < 7 < \infty$, $1 < 7 < 8$, and $3 < 4 < 8$.

Definition 1.5.12. For $\mu \in \text{Par}(n)$, let $\mathcal{F}_\mu^N$ be the set of all fillings U of shape μ which are filled using letters in $\{1, 2, \cdots, N\}$.

Definition 1.5.13. For all partitions μ, let $\mathcal{F}_\mu$ be the set of all standard fillings of the shape μ.

We can now state Haglund's combinatorial formula [9] for the modified Macdonald polynomial $\widetilde{H}_\mu(X; q, t)$:

Definition 1.5.14. For $X = (x_1, x_2, \cdots, x_N)$ and $\mu \in \text{Par}(n)$,

$$\widetilde{H}_\mu(X; q, t) = \sum_{U \in \mathcal{F}_\mu^N} q^{\text{inv}_\mu(\text{stdz}(U))} t^{\text{maj}_\mu(\text{stdz}(U))} X^U.$$

Example 1.5.15. Recall from Example 1.5.6 the filling

$$T = \begin{array}{|c|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-3} 2 & 4 & 6 \\ \hline 4 & 7 & 8 & 2 & 5 \\ \hline \end{array}$$

and

$$\text{stdz}(T) = \begin{array}{|c|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-3} 1 & 3 & 7 \\ \hline 4 & 8 & 9 & 2 & 6 \\ \hline \end{array}\ .$$

Then $\text{inv}_{(5,3,1)}(T) = 6$, $\text{maj}_{(5,3,1)} = 1$, and $X^T = x_2^2 x_4^2 x_5^2 x_6 x_7 x_8$. Thus, the term that T contributes to $\widetilde{H}_{(5,3,1)}$ is $q^6 t x_2^2 x_4^2 x_5^2 x_6 x_7 x_8$.

Haglund's combinatorial formula [9] for $\widetilde{F}_\mu(q,t)$ is

$$\widetilde{F}_\mu(q,t) = \sum_{T \in \mathcal{F}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{maj}_\mu(T)}.$$

$\widetilde{F}_\mu(q,t)$ is the coefficient of $x_1 x_2 \cdots x_N$ in the modified Macdonald polynomial $\widetilde{H}_\mu(X;q,t)$.

Definition 1.5.16. For $T \in \mathcal{F}_\mu$, define $\mathrm{comaj}_\mu(T) = n(\mu) - \mathrm{maj}_\mu(T)$. Note that $\mathrm{comaj}_\mu(T)$ can also be written as the sum of the co-major indices of the column words of T.

In the following, we will study the specialization

$$F_\mu(q,t) = t^{n(\mu)} \widetilde{F}_\mu(q, 1/t) = \sum_{T \in \mathcal{F}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{comaj}_\mu(T)}$$

which is the coefficient of $x_1 \cdots x_N$ in H_μ.

Chapter 2

Quantum analogues of the hook-length formula

2.1 Background

In [2], a formula to find the number of standard Young tableaux of shape $\mu \in \mathrm{Par}(n)$ was given. This formula, known as the hook-length formula, was an early result in what would eventually become the field of algebraic combinatorics. Recall from Definition 1.1.11 that f^{μ} is the number of standard Young tableaux of shape μ.

Theorem 2.1.1 (Hook-Length Formula)**.** *For $\mu \in \mathrm{Par}(n)$,*

$$f^{\mu} = \frac{n!}{\prod_{c \in \mathrm{dg}(\mu)} h_c}.$$

Theorem 2.1.1 was proved probabilistically by Greene, Nijenhuis, and Wilf in [8]. In [3],[15], and [19] there are several different bijective proofs.

Example 2.1.2. For $\mu = (2, 2, 1)$, fill each cell c with the hook-length, h_c to get

$$\begin{array}{|c|c}\hline 1 & \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline 4 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\,.$$

Then, by Theorem 2.1.1,

$$f^\mu = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5.$$

The five standard Young tableaux of shape μ are

$$\begin{array}{|c|c}\hline 5 & \\ \hline 3 & \multicolumn{1}{c|}{4} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 4 & \\ \hline 3 & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 4 & \\ \hline 2 & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 5 & \\ \hline 2 & \multicolumn{1}{c|}{4} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline\end{array}\,.$$

Definition 2.1.3. For an integer $n \geq 0$,

$$[n]_x = x^0 + x^1 + x^2 + \cdots + x^{n-1}$$

and

$$[n]!_x = [n]_x[n-1]_x \cdots [2]_x[1]_x.$$

In [18] the following q-analogue of the hook length formula using Macdonald polynomials was introduced.

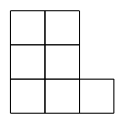
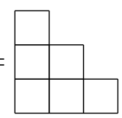
Theorem 2.1.4. *For* $\mu \in \mathrm{Par}(n)$,

$$F_\mu(q, q) = f^\mu \prod_{c \in \mathrm{dg}(\mu)} [h_c]_q.$$

Theorem 2.1.4 was established in [6] by a recursion, based on removing corners of the diagram of μ, which we now discuss.

Definition 2.1.5. For $\mu \in \mathrm{Par}(n)$, an *inner corner* of the Ferrers diagram of μ is a cell which is at the end of a row and a column. If the Ferrers diagram of $\nu \in \mathrm{Par}(n-1)$ is obtained from that of μ by removing one inner corner, we write $\nu \to \mu$.

Example 2.1.6. For $\mu = (3, 2, 2)$ and $\nu = (3, 2, 1)$, we have $\nu \to \mu$ as shown in the following diagrams.

$\mu =$  $\nu =$ 

Theorem 2.1.7 ([6],p.76-78)**.** *For $\mu \in \mathrm{Par}(n)$ and $\nu \in \mathrm{Par}(n-1)$ with $\nu \to \mu$, let $\mathcal{R}_{\mu/\nu}$ be the set of cells in ν that are in the same row as the square removed from μ to obtain ν. Similarly, let $\mathcal{C}_{\mu/\nu}$ be the set of cells in ν that are in the same column as the square removed from μ. Define*

$$c_{\mu\nu}(q,t) = \prod_{s \in \mathcal{R}_{\mu/\nu}} \frac{t^{l_\mu(s)} - q^{a_\mu(s)+1}}{t^{l_\nu(s)} - q^{a_\nu(s)+1}} \prod_{s \in \mathcal{C}_{\mu/\nu}} \frac{q^{a_\mu(s)} - t^{l_\mu(s)+1}}{q^{a_\nu(s)} - t^{l_\nu(s)+1}}.$$

Then

$$\widetilde{F}_\mu(q,t) = \sum_{\nu \to \mu} c_{\mu\nu}(q,t) \widetilde{F}_\nu(q,t)$$

with initial condition $\widetilde{F}_{(1)}(q,t) = 1$.

The proof of Theorem 2.1.4 presented in [6] is based on the algebraic definition of $\widetilde{F}_\mu$ and does not give insight into what is happening with the fillings. A brief overview of this proof is given here.

Recalling that $F_\mu(q,t) = t^{n(\mu)}\widetilde{F}_\mu(q,1/t)$, the recursion in Theorem 2.1.7 can be rewritten as

$$F_\mu(q,q) = \sum_{\nu\to\mu} \prod_{s\in\mathcal{R}_{\mu/\nu}} \frac{1-q^{h_\mu(s)}}{1-q^{h_\nu(s)}} \prod_{s\in\mathcal{C}_{\mu/\nu}} \frac{1-q^{h_\mu(s)}}{1-q^{h_\nu(s)}} F_\nu(q,q). \tag{2.1}$$

If s is not in $\mathcal{R}_{\mu/\nu}$ or $\mathcal{C}_{\mu/\nu}$, $h_\mu(s) = h_\nu(s)$. This allows simplification of (2.1), which can then be rewritten as

$$\frac{F_\mu(q,q)}{\prod_{s\in\mu}[h_\mu(s)]_q} = \sum_{\nu\to\mu} \frac{F_\nu(q,q)}{\prod_{s\in\nu}[h_\nu(s)]_q}.$$

Since $f^\mu = \sum_{\nu\to\mu} f^\nu$ with $f^{(1)} = 1$,

$$F_\mu(q,q) = f^\mu \prod_{c\in\mathrm{dg}(\mu)} [h_c]_q \ .$$

2.2 Permutation Statistics

We will give a bijective proof of Theorem 2.1.4 in the special case where μ itself is a hook shape, i.e., $\mu = (k, 1^{n-k})$ for some k. First, we define some bijections on permutations and review some well-known facts about the statistics inv, maj, and comaj.

Definition 2.2.1. Define a *cyclic shift down* for a permutation $w = w_1w_2\dots w_n \in S_n$ as $\mathrm{cyc}{\downarrow}(w) = (w_1-1)(w_2-1)\cdots(w_n-1) \mod n$ and $(\mathrm{cyc}{\downarrow})^k(w) = (w_1-k)(w_2-k)\cdots(w_n-k) \mod n$. Likewise, a *cyclic shift up* for a permutation $w = w_1w_2\dots w_n \in S_n$ is $\mathrm{cyc}{\uparrow}(w) = (w_1+1)(w_2+1)\cdots(w_n+1) \mod n$ and $(\mathrm{cyc}{\uparrow})^k(w) = (w_1+k)(w_2+k)\cdots(w_n+k) \mod n$.

Theorem 2.2.2. *Let $w \in S_n$. Then, for $k \le n$,*

(a) $\mathrm{maj}((\mathrm{cyc}{\uparrow})^k(w)) = \mathrm{maj}(w) - k + n\chi(w_n > n-k)$,

(b) $\mathrm{maj}((\mathrm{cyc}{\downarrow})^k(w)) = \mathrm{maj}(w) + k - n\chi(w_n \leq k)$,

(c) $\mathrm{comaj}((\mathrm{cyc}{\uparrow})^k(w)) = \mathrm{comaj}(w) + k - n\chi(w_n > n - k)$, *and*

(d) $\mathrm{comaj}((\mathrm{cyc}{\downarrow})^k(w)) = \mathrm{comaj}(w) - k + n\chi(w_n \leq k)$.

Proof. Let $w \in S_n$. Then $\mathrm{cyc}{\uparrow}(w) = (w_1 + 1)(w_2 + 1) \cdots (w_n + 1) \mod n$. For some i, $w_i = n$, so $(w_i + 1) \mod n = 1$. Thus, if $i < n$, then $i \in \mathrm{Des}(w)$, $i - 1 \in \mathrm{Asc}(w)$ or $i - 1 = 0$, but $i - 1 \in \mathrm{Des}(\mathrm{cyc}{\uparrow}(w))$ and $i \in \mathrm{Asc}(\mathrm{cyc}{\uparrow}(w))$. If $i = n$, then $n - 1 \in \mathrm{Asc}(w)$ and $n - 1 \in \mathrm{Des}(\mathrm{cyc}{\uparrow}(w))$. These are the only changes in the ascent and descent sets. Therefore,

$$\mathrm{maj}(\mathrm{cyc}{\uparrow}(w)) = \mathrm{maj}(w) + \begin{cases} -1 & \text{if } w_n \neq n \\ n - 1 & \text{if } w_n = n \end{cases}$$

and

$$\mathrm{comaj}(\mathrm{cyc}{\uparrow}(w)) = \mathrm{comaj}(w) + \begin{cases} 1 & \text{if } w_n \neq n \\ -(n - 1) & \text{if } w_n = n \end{cases} .$$

Iteration of this result gives

$$\mathrm{maj}((\mathrm{cyc}{\uparrow})^k(w)) = \mathrm{maj}(w) - k + n\chi(w_n > n - k)$$

and

$$\mathrm{comaj}((\mathrm{cyc}{\uparrow})^k(w)) = \mathrm{comaj}(w) + k - n\chi(w_n > n - k) .$$

Cyclic shifting down is handled similarly. □

Example 2.2.3. Consider $w = 624513 \in S_6$. Then $(\mathrm{cyc}{\uparrow})^3(w) = 351246$ and $(\mathrm{cyc}{\uparrow})^4(w) = 462351$. Note that $\mathrm{maj}(w) = 5$, $\mathrm{maj}((\mathrm{cyc}{\uparrow})^3(w)) = 2$, and $\mathrm{maj}((\mathrm{cyc}{\uparrow})^4(w)) = 7$.

We now define a bijection which will be used in the proof of Theorem 2.1.4 when $\mu = (k, 1^{n-k})$.

Definition 2.2.4. Define $f : S_n \to (S_{n-1} \times \{1, 2, \ldots, n\})$ in the following way:

1. Let $w \in S_n$ with $w = w_1 w_2 \ldots w_{n-1} c$ for some $c \in \{1, 2, \ldots, n\}$.

2. Rewrite $w_1 w_2 \ldots w_{n-1}$ as $v = v_1 v_2 \ldots v_{n-1} \in S_{n-1}$ by setting $v_i = w_i$ for all $w_i < c$ and $v_i = w_i - 1$ for all $w_i > c$.

3. Set $f(w) = ((\mathrm{cyc}{\downarrow})^{c-1}(v), c)$.

Example 2.2.5. For $w = 235614 \in S_6$, $c = 4$ and $v = 23451$,

$$f(w) = ((\mathrm{cyc}{\downarrow})^{3}(23451), 4) = (45123, 4).$$

Definition 2.2.6. Define a function $g : (S_{n-1} \times \{1, 2, \ldots, n\}) \to S_n$ as follows:

1. Let $v = v_1 v_2 \ldots v_{n-1} \in S_{n-1}$ and $c \in \{1, 2, \ldots, n\}$.

2. Compute $(\mathrm{cyc}{\uparrow})^{c-1}(v)$ to obtain a permutation $v' = v'_1 v'_2 \ldots v'_{n-1} \in S_{n-1}$.

3. Construct a permutation $w \in S_n$ by setting $w_i = v_i$ for $v_i < c$ and $w_i = v_i + 1$ for $v_i \geq c$, and set $w_n = c$.

Then $g(v, c) = w$.

Example 2.2.7. Let $v = 231564 \in S_6$ and $c = 3$. Then $v' = (\mathrm{cyc}{\uparrow})^2(231564) = 453126$. Then $w = 5641273$ and thus $g(231564, 3) = 5641273$.

Theorem 2.2.8. *f and g are inverse functions.*

Proof. Let $w = w_1w_2\ldots w_{n-1}c \in S_n$. Let $v = v_1v_2\ldots v_{n-1}$ where $v_i = w_i$ for $w_i < c$ and $v_i = w_i - 1$ for $w_i > c$. Then $f(w) = ((\text{cyc}\!\downarrow)^{c-1}(v), c)$. To find $g(f(w))$, notice that $(\text{cyc}\!\uparrow)^{c-1}((\text{cyc}\!\downarrow)^{c-1}(v)) = v$. Now, construct a permutation $u \in S_n$ by setting $u_i = v_i$ for $v_i < c$ and $u_i = v_i + 1$ for $v_i \geq c$ and $u_n = c$. Then $w = u$. Thus $g(f(w)) = w$. A similar argument is used to show $f(g(v, c)) = (v, c)$. □

Theorem 2.2.9. *Let $n \in \mathbb{N}$ and $c \in \{1, 2, \ldots, n\}$. Then*

(a) $\displaystyle\sum_{\substack{w\in S_n\\ w_n=c}} t^{\text{comaj}(w)} = t^{c-1}[n-1]!_t.$

(b) $\displaystyle\sum_{w\in S_n} t^{\text{comaj}(w)} = [n]!_t.$

Proof. We use induction on n to prove (a) and (b) simultaneously. These formulas hold when $n = 1$. Let $w = w_1w_2\ldots w_{n-1}c \in S_n$. Then

$$\text{comaj}(w) = \text{comaj}(w_1w_2\ldots w_{n-1}) + (n-1)\chi(w_{n-1} < c). \tag{2.2}$$

Let $f(w) = (v, c)$. Then

$$\text{comaj}(v) = \text{comaj}(w_1w_2\ldots w_{n-1}) - (c-1) + (n-1)\chi(w_{n-1} < c) \tag{2.3}$$

by Theorem 2.2.2. Combining (2.2) and (2.3) yields

$$\begin{aligned} \text{comaj}(w) &= \text{comaj}(v) + (c-1) - (n-1)\chi(w_{n-1} < c) + (n-1)\chi(w_{n-1} < c) \\ &= \text{comaj}(v) + (c-1). \end{aligned} \tag{2.4}$$

Then by induction,

$$\begin{aligned}
\sum_{\substack{w\in S_n\\ w_n=c}} t^{\mathrm{comaj}(w)} &= \sum_{v\in S_{n-1}} t^{\mathrm{comaj}(v)+(c-1)} && \text{by (2.4)}\\
&= t^{c-1}\sum_{v\in S_{n-1}} t^{\mathrm{comaj}(v)}\\
&= t^{c-1}[n-1]!_t && \text{by induction.}
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{w\in S_n} t^{\mathrm{comaj}(w)} &= \sum_{c=1}^{n}\left(\sum_{\substack{w\in S_n\\ w_n=c}} t^{\mathrm{comaj}(w)}\right)\\
&= \sum_{c=1}^{n}\left(t^{c-1}[n-1]!_t\right)\\
&= (t^0+t^1+\cdots+t^{n-1})[n-1]!_t\\
&= [n]!_t.
\end{aligned}$$

□

Theorem 2.2.10. *For $n \in \mathbb{N}$,*

(a) $\sum_{w\in S_n} t^{\mathrm{maj}(w)} = [n]!_t$ *and*

(b) $\sum_{w\in S_n} t^{\mathrm{inv}(w)} = [n]!_t$.

Proof. The proof of part (a) is similar to the proof of Theorem 2.2.9.

To prove part (b), first define a bijection $h : S_n \to S_{n-1} \times \{1, 2, \ldots, n\}$ as follows: for $w = w_1w_2\ldots w_n \in S_n$, let $c = w_n$ and $v = v_1v_2\ldots v_{n-1}$ where v_i is w_i when $w_i < w_n$ and

$w_i - 1$ when $w_i > w_n$. Then $h(w) = (v, c)$. Note that $\mathrm{inv}(w) = \mathrm{inv}(v) + n - c$. Then

$$\begin{aligned}
\sum_{\substack{w \in S_n \\ w_n = c}} t^{\mathrm{inv}(w)} &= \sum_{v \in S_{n-1}} t^{\mathrm{inv}(v)+n-c} \\
&= t^{n-c} \sum_{v \in S_{n-1}} t^{\mathrm{inv}(v)} \\
&= t^{n-c}[n-1]!_t \qquad \text{by induction.}
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{w \in S_n} t^{\mathrm{inv}(w)} &= \sum_{c=1}^{n} \left(\sum_{\substack{w \in S_n \\ w_n = c}} t^{\mathrm{inv}(w)} \right) \\
&= \sum_{c=1}^{n} \left(t^{n-c}[n-1]!_t \right) \\
&= (t^{n-1} + t^{n-2} + \cdots + t^0)[n-1]!_t \\
&= [n]!_t.
\end{aligned}$$

$\square$

2.3 Bijective Proof of q, t-Hook-length Formula for Hook Shapes

The main result of this section is a bijective proof of a q, t-analogue of the hook length formula for $\mu = (k, 1^{n-k})$, which specializes to 2.1.4. A similar result was obtained independently by Meesue Yoo [22].

Theorem 2.3.1. *For* $\mu = (k, 1^{n-k})$,

$$F_\mu(q,t) = [k-1]!_q[n-k]!_t \sum_{S \in SYT(\mu)} \sum_{c=1}^{n} q^{\mathrm{qpow}(S,c)} t^{\mathrm{tpow}(S,c)},$$

where

$$\mathrm{tpow}(S,c) = \begin{cases} (\#\text{ of entries in column 1 of } S) < c & \text{if } c \text{ is in column 1;} \\ (\#\text{ of entries in col. 1 of } S \text{ except corner cell}) < c & \text{if } c \text{ is in row 1;} \end{cases} \tag{2.5}$$

$$\mathrm{qpow}(S,c) = \begin{cases} (\#\text{ of entries in row 1 of } S \text{ except corner cell}) < c & \text{if } c \text{ is in column 1;} \\ (\#\text{ of entries in row 1 of } S) < c & \text{if } c \text{ is in row 1.} \end{cases} \tag{2.6}$$

Proof. To prove Theorem 2.3.1, we construct a bijection from $\mathcal{F}_\mu$ to $\{(S,\sigma,\tau,c) : S \in \mathrm{SYT}(\mu), \sigma \in S_{n-k}, \tau \in S_{k-1}, \text{ and } c \in \{1,2,\ldots,n\}\}$ such that for $T \in \mathcal{F}_\mu$ corresponding to (S,σ,τ,c), $\mathrm{inv}_\mu(T) = \mathrm{inv}(\tau) + \mathrm{qpow}(S,c)$ and $\mathrm{comaj}_\mu(T) = \mathrm{comaj}(\sigma) + \mathrm{tpow}(S,c)$.

- For a filling $T \in \mathcal{F}_\mu$, let c denote the entry in cell $(1,1)$.

- Denote the entries in the first row by the word $cu_1u_2\ldots u_{k-1}$. We can rewrite $u = u_1u_2\ldots u_{k-1}$ as an element $\tau = \tau_1 \ldots \tau_{k-1} \in S_{k-1}$ by replacing the smallest u_i with $\tau_i = 1$, the next smallest with $\tau_j = 2$, and so on.

- Denote the entries in the first column by the word $w_1w_2\ldots w_{n-k}c = w$. We can rewrite w as an element of S_{n-k+1} by replacing the smallest w_i with 1, the next smallest with

2, and so on. Denote this element of S_{n-k+1} by $w' = w'_1 w'_2 \dots w'_{n-k-1} c'$. Notice that $c' = \{$ number of $w_i < c\} + 1$.

- Apply the function f from Definition 2.2.4 to w' to obtain $f(w') = (\sigma, c')$ where $\sigma \in S_{n-k}$.

- Construct a filling T_c as follows:

  – Place c in cell $(1, 1)$.

  – Place the remaining entries of the first row of T into the first row of T_c in increasing order from left to right.

  – Place the remaining entries of the first column of T into the first column of T_c in increasing order from bottom to top.

- To obtain $S \in \mathrm{SYT}(\mu)$,

  – If $c = 1$, then $S = T_c$.

  – If 1 is in some cell in $LEG(1, 1)$ of T_c, place 1 into cell $(1, 1)$ of S, and place c in the unique cell in the first column of S to make the column increasing (from bottom to top), shifting the entries in other cells down if needed.

  – If 1 is in some cell in $ARM(1, 1)$ of T_c, place 1 into cell $(1, 1)$ of S, and place c in the unique cell in the first row of S to make the row increasing (from left to right), shifting the entries in other cells to the left if needed.

Recall that $c' - 1$ is the number of $w_i < c$. If c is in the first column of S, then 1 is in the first column of T, so the first column of S is a reordering of the first column of T, so $c' - 1$ is also the number of entries in column 1 of S which are less than c. If c is in the first row of S, the entries in cells of $LEG(1,1)$ of S and c were the entries in the first column of T, so $c' - 1$ is the number of entries in cells of $LEG(1,1)$ which are less than c. Thus, $\text{tpow}(S,c) = c' - 1$. Let $d =$ number of entries of row 1 of T which are less than c. By a similar argument, $\text{qpow}(S,c) = d$. Then

$$\begin{aligned}
\text{comaj}_\mu(T) &= \text{comaj}(w) \\
&= \text{comaj}(\sigma) + c' - 1 && \text{by Theorem 2.2.9} \\
&= \text{comaj}(\sigma) + \text{tpow}(S,c)
\end{aligned}$$

and

$$\text{inv}_\mu(T) = \text{inv}(\tau) + \text{qpow}(S,c).$$

The inverse map is computed as follows:

Given (S, σ, τ, c) where $S \in \text{SYT}(\mu)$, $\sigma \in S_{n-k}$, $\tau \in S_{k-1}$ and $c \in \{1, 2, \ldots, n\}$, perform the following steps to obtain a filling $T \in \mathcal{F}_\mu$.

- Starting with S, build a new filling S_c as follows. Move c to the cell $(1,1)$. If c was in the first column of S, move entries in the first column which are less than c up one position in the column. If c was in the first row of S, move entries in the first row which are less than c to the right one box in the row.

- Set $c' = $ (number of entries in $LEG(1,1)$ of S_c which are less than c) $+ 1$.

- Find $g(\sigma, c') = w \in S_{n-k+1}$.

- Let $w'_i = w_i$th smallest entry in $\{\{1,\ldots,n\} - \{\text{entries of row 1 in } S_c\}\}$. Let the entries in the first column of T from top to bottom be $w'_1, w'_2, \ldots, w'_{n-k+1}c'$. Let the entries in $ARM(1,1)$ in T be $u_{\tau(1)}, \ldots, u_{\tau(k-1)}$ where $u_1, \ldots, u_{k-1}$ are the entries in $ARM(1,1)$ in S_c.

- The result is a filling T of shape $(k, 1^{n-k})$.

This establishes the bijection, so

$$\begin{aligned}
F_{(k,1^{n-k})}(q,t) &= \sum_{T \in \mathcal{F}_{(k,1^{n-k})}} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{comaj}_\mu(T)} \\
&= \sum_{\sigma \in S_{n-k}} t^{\mathrm{comaj}(\sigma)} \sum_{\tau \in S_{k-1}} q^{\mathrm{inv}(\tau)} \sum_{S \in \mathrm{SYT}((k,1^{n-k}))} \sum_{c=1}^{n} q^{\mathrm{qpow}(S,c)} t^{\mathrm{tpow}(S,c)} \\
&= [n-k]!_t [k-1]!_q \sum_{S \in \mathrm{SYT}((k,1^{n-k}))} \sum_{c=1}^{n} q^{\mathrm{qpow}(S,c)} t^{\mathrm{tpow}(S,c)}
\end{aligned}$$

□

We prove Theorem 2.1.4 by setting $t = q$. First note that for $S \in \mathrm{SYT}((k,1^{n-k}))$ and $c \in \{1,\ldots,n\}$, $\mathrm{qpow}(S,c) + \mathrm{tpow}(S,c) = c - 1$, the total number of entries in S which are less than c. Then

$$\begin{aligned}
F_{(k,1^{n-k})}(q,q) &= [n-k]!_q[k-1]!_q \sum_{S\in \mathrm{SYT}((k,1^{n-k}))} \sum_{c=1}^{n} q^{\mathrm{qpow}(S,c)+\mathrm{tpow}(S,c)} \\
&= [n-k]!_q[k-1]!_q \sum_{S\in \mathrm{SYT}((k,1^{n-k}))} [n]_q \\
&= f^{(k,1^{n-k})}[n-k]!_q[k-1]!_q[n]_q \\
&= f^{(k,1^{n-k})} \prod_{c\in \mathrm{dg}((k,1^{n-k}))} [h(c)]_q.
\end{aligned}$$

Example 2.3.2. Given the following filling T of $\mu = (5,1^4)$

$$\begin{array}{|c|c|c|c|c|}
\cline{1-1} 3 \\
\cline{1-1} 2 \\
\cline{1-1} 6 \\
\cline{1-1} 8 \\
\hline 5 & 4 & 7 & 1 & 9 \\
\hline
\end{array}$$

with $\mathrm{comaj}_\mu(T) = 5$ and $\mathrm{inv}_\mu(T) = 4$, apply the above map to obtain (S,σ,τ,c).

We compute $\tau = 2314$ and $w = 32685$, $w' = 21453$ and $f(w') = ((\mathrm{cyc}\!\downarrow)^2(2134),3) = (4312,3)$, thus $\sigma = 4312$, and

$$T_5 = \begin{array}{|c|c|c|c|c|}
\cline{1-1} 8 \\
\cline{1-1} 6 \\
\cline{1-1} 3 \\
\cline{1-1} 2 \\
\hline 5 & 1 & 4 & 7 & 9 \\
\hline
\end{array}.$$

Then

$$S = \begin{array}{|c|c|c|c|c|}
\cline{1-1} 8 \\
\cline{1-1} 6 \\
\cline{1-1} 3 \\
\cline{1-1} 2 \\
\hline 1 & 4 & 5 & 7 & 9 \\
\hline
\end{array}.$$

Notice that $\mathrm{comaj}_\mu(T) = \mathrm{comaj}(\sigma) + \mathrm{tpow}(S,5)$ and $\mathrm{inv}_\mu(T) = \mathrm{inv}(\tau) + \mathrm{qpow}(S,5)$.

Example 2.3.3. To illustrate the inverse map, start with $(S, 213, 2134, 4)$ where

$$S = \begin{array}{|c|c|c|c|c|}
\cline{1-1} 6 \\
\cline{1-1} 4 \\
\cline{1-1} 3 \\
\hline 1 & 2 & 5 & 7 & 8 \\
\hline
\end{array}.$$

Then $\text{tpow}(S,4) = 2$, $\text{qpow}(S,4) = 1$, and

$$S_4 = \begin{array}{|c|cccc}\hline 6 & & & & \\ \hline 3 & & & & \\ \hline 1 & & & & \\ \hline 4 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \hline \end{array}\ .$$

Note that $c' = 3$, and $(\text{cyc}\uparrow)^2(213) = 132$, so $g(v, c') = w = 1423$. Thus $w' = 1634$ and

$$T = \begin{array}{|c|cccc}\hline 1 & & & & \\ \hline 6 & & & & \\ \hline 3 & & & & \\ \hline 4 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

with $\text{comaj}_\mu(T) = 4 = \text{comaj}(\sigma) + \text{tpow}(S,4)$ and $\text{inv}_\mu(T) = 2 = \text{inv}(\tau) + \text{qpow}(S,4)$.

Theorem 2.3.4. *Let $n, k \in \mathbb{N}$ with $k \leq n$. Then $F_{(k,1^{n-k})}(q,t) = F_{(n-k+1,1^{k-1})}(t,q)$.*

Proof. For $S \in \text{SYT}((k,1^{n-k}))$, let S' denote the standard Young tableau of shape $(n-k+1, 1^{k-1})$ obtained by conjugation. Notice that

$$\text{qpow}(S,c) = \text{tpow}(S',c) \quad \text{and}$$
$$\text{tpow}(S,c) = \text{qpow}(S',c).$$

Then

$$\begin{aligned} F_{(k,1^{n-k})}(q,t) &= [n-k]!_t[k-1]!_q \sum_{S\in\text{SYT}_{(k,1^{n-k})}} \sum_{c=1}^{n} q^{\text{qpow}(S,c)} t^{\text{tpow}(S,c)} \\ &= [n-k]!_t[k-1]!_q \sum_{S\in\text{SYT}((k,1^{n-k}))} \sum_{c=1}^{n} q^{\text{tpow}(S',c)} t^{\text{qpow}(S',c)} \\ &= [n-k]!_t[k-1]!_q \sum_{T\in\text{SYT}((n-k+1,1^{k-1}))} \sum_{c=1}^{n} q^{\text{tpow}(T,c)} t^{\text{qpow}(T,c)} \\ &= F_{(n-k+1,1^{k-1})}(t,q). \end{aligned}$$

□

This calculation along with the bijection in the proof of Theorem 2.3.1 gives a bijection on fillings as seen in the following example.

Example 2.3.5. Let $\mu = (3, 1^4)$, and let $T \in \mathcal{F}_\mu$ be the filling

$$T = \begin{array}{|c|cc}\cline{1-1} 7 \\ \cline{1-1} 1 \\ \cline{1-1} 6 \\ \cline{1-1} 2 \\ \hline 4 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{3} \\ \hline \end{array},$$

with $\text{comaj}_\mu(T) = 6$ and $\text{inv}_\mu(T) = 2$. Under the bijection from the proof of Theorem 2.3.1 T maps to $(S, 2314, 21, 4)$ where

$$S = \begin{array}{|c|cc}\cline{1-1} 7 \\ \cline{1-1} 6 \\ \cline{1-1} 4 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array}.$$

From the above calculation, take the conjugate of S to obtain

$$S' = \begin{array}{|c|cccc}\cline{1-1} 5 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{7} \\ \hline \end{array}.$$

For every $n \in \mathbb{N}$, there is a bijection $h : S_n \to S_n$ such that $\text{inv}(h(\sigma)) = \text{comaj}(\sigma)$. Using this bijection we get $h(2314) = 4132$ and $h^{-1}(21) = 12$. Note that $\text{comaj}(2314) = \text{inv}(4132)$ and $\text{inv}(21) = \text{comaj}(12)$. By the map in Theorem 2.3.1, $(S', 12, 4132, 4)$ maps to

$$T' = \begin{array}{|c|cccc}\cline{1-1} 5 \\ \cline{1-1} 3 \\ \hline 4 & \multicolumn{1}{|c|}{7} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{2} \\ \hline \end{array}.$$

Notice that $\text{comaj}_{\mu'}(T') = 2 = \text{inv}_\mu(T)$ and $\text{inv}_{\mu'}(T') = 6 = \text{comaj}_\mu(T)$.

Chapter 3

Combinatorial Operations on Standard Fillings

In this chapter we define and analyze a number of combinatorial operations on standard fillings which will be used to establish the results in Chapter 4.

3.1 The Inversion Flip Operation

This section describes a combinatorial operation on fillings called the *inversion flip* that will allow us to partially sort the bottom row of a filling $T \in \mathcal{F}_\mu$. This operation preserves the maj_μ and comaj_μ statistics while resulting in a change of ± 1 in the inv_μ statistic.

Definition 3.1.1. Suppose $\mu \in \mathrm{Par}(n)$ and $i \in \mathbb{N}^+$ satisfy $\mu'_i = \mu'_{i+1}$. Define the *inversion* *flip* move $s_i : \mathcal{F}_\mu \to \mathcal{F}_\mu$ as follows:

- Given $T \in \mathcal{F}_\mu$, let a (resp. b) be the entry of T at the bottom of column i (resp. $i+1$).

- Switch entries a and b in the bottom row as shown here:

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline c & d \\ \hline a & b \\ \hline\end{array} \;\to\; \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline c & d \\ \hline b & a \\ \hline\end{array}\,.$$

- If a, c, d and b, c, d are either both inversion triples or both not inversion triples, the move is complete. Otherwise, apply s_i recursively to the filling T' of $(\mu_2, \mu_3, \ldots)$ obtained by ignoring the bottom row of T.

Theorem 3.1.2. *Given $\mu \in \mathrm{Par}(n)$ and $i, j \in \mathbb{N}^+$ with $\mu'_i = \mu'_{i+1}$ and $\mu'_j = \mu'_{j+1}$,*

(a) $s_i^2 = s_i \circ s_i = \mathrm{id}_{\mathcal{F}_\mu}$;

(b) $s_i \circ s_j = s_j \circ s_i$ *when* $|i - j| \geq 2$.

Proof. Both properties follow directly from the definition of s_i. □

Remark 3.1.3. The s_i's do not satisfy the braid relations in general; i.e., when $\mu'_i = \mu'_{i+1} = \mu'_{i+2}$, we may have $s_i \circ s_{i+1} \circ s_i \neq s_{i+1} \circ s_i \circ s_{i+1}$. This can be seen in Figure 3.1.

Theorem 3.1.4. *Given $\mu \in \mathrm{Par}(n)$ and $i \in \mathbb{N}^+$ with $\mu'_i = \mu'_{i+1}$, let $T \in \mathcal{F}_\mu$ have entries a and b in the bottom row of columns i and $i+1$, respectively. Then:*

(a) $\mathrm{comaj}_\mu(s_i(T)) = \mathrm{comaj}_\mu(T)$;

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6&4\\\hline\end{array} \xrightarrow{s_2} \begin{array}{|c|c|c|}\hline 1&5&3\\\hline 2&4&6\\\hline\end{array} \xrightarrow{s_1} \begin{array}{|c|c|c|}\hline 1&5&3\\\hline 4&2&6\\\hline\end{array} \xrightarrow{s_2} \begin{array}{|c|c|c|}\hline 1&5&3\\\hline 4&6&2\\\hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6&4\\\hline\end{array} \xrightarrow{s_1} \begin{array}{|c|c|c|}\hline 3&1&5\\\hline 6&2&4\\\hline\end{array} \xrightarrow{s_2} \begin{array}{|c|c|c|}\hline 3&1&5\\\hline 6&4&2\\\hline\end{array} \xrightarrow{s_1} \begin{array}{|c|c|c|}\hline 3&1&5\\\hline 4&6&2\\\hline\end{array}$$

Figure 3.1: $s_i \circ s_{i+1} \circ s_i \neq s_{i+1} \circ s_i \circ s_{i+1}$ in general.

(b) $\mathrm{maj}_\mu(s_i(T)) = \mathrm{maj}_\mu(T)$*;*

(c) $\mathrm{inv}_\mu(s_i(T)) = \mathrm{inv}_\mu(T) + \begin{cases} 1 & \text{if } a < b; \\ -1 & \text{if } b < a. \end{cases}$

Proof. Since every column contributes independently to comaj_μ and maj_μ, to prove (a) and (b) it is sufficient to consider a filling T of shape $\mu = (2^n)$ and $i = 1$. By Theorem 3.1.2(a) we may also assume that the bottom row of T is increasing. We must show that $\mathrm{comaj}_\mu(s_1(T)) = \mathrm{comaj}_\mu(T)$, which automatically yields $\mathrm{maj}_\mu(s_1(T)) = \mathrm{maj}_\mu(T)$. This result is true when $n = 1$.

When $n = 2$, we can write

$$T = \begin{array}{|c|c|}\hline c&d\\\hline a&b\\\hline\end{array}\,.$$

If

$$s_1(T) = \begin{array}{|c|c|}\hline d&c\\\hline b&a\\\hline\end{array}\,,$$

column words are preserved, so $\mathrm{comaj}_\mu(s_1(T)) = \mathrm{comaj}_\mu(T)$ and $\mathrm{maj}_\mu(s_1(T)) = \mathrm{maj}_\mu(T)$.

It is also possible that

$$s_1(T) = \begin{array}{|c|c|}\hline c&d\\\hline b&a\\\hline\end{array}.$$

In this case, if a, c, d formed an inversion triple in T, then b, c, d forms an inversion triple in $s_1(T)$. If $a < b < d < c$, $d < c < a < b$, or $c < a < b < d$, then the location and number of the ascents are preserved, so $\text{comaj}_\mu(s_1(T)) = \text{comaj}_\mu(T)$. If $a < d < c < b$, the number of ascents is preserved, but the column in which the ascent is located changes. However, since the columns are of equal height, $\text{comaj}_\mu(s_1(T)) = \text{comaj}_\mu(T)$. Similarly, if a, c, d is not an inversion triple in T, then b, c, d is not an inversion triple in $s_1(T)$. If $a < b < c < d$, $c < d < a < b$, or $d < a < b < c$, the location and number of ascents are preserved. If $a < c < d < b$, the number of ascents are preserved, but the column in which the ascent is located changes. As before, since the columns are of equal height, $\text{comaj}_\mu(s_1(T)) = \text{comaj}_\mu(T)$. By induction, this result holds for any number of rows.

To show that $\text{inv}_\mu(s_i(T)) = \text{inv}_\mu(T) + 1$, we must show that the inversion flip does not affect the total number of inversion triples, excluding the triple a, b, ∞. By the definition of the inversion flip, it is sufficient to consider triples positioned as shown:

$$\begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline\end{array} \cdots \begin{array}{|c|}\hline z \\ \hline \\ \hline\end{array}$$

since all other triples in T will be preserved. Once again, there are two possibilities. First, if

$$s_i(T) = \begin{array}{|c|c|}\hline d & c \\ \hline b & a \\ \hline\end{array} \cdots \begin{array}{|c|}\hline z \\ \hline \\ \hline\end{array}$$

the inversion triples themselves are preserved. On the other hand, if

$$s_i(T) = \begin{array}{|c|c|}\hline c & d \\ \hline b & a \\ \hline\end{array} \cdots \begin{array}{|c|}\hline z \\ \hline \\ \hline\end{array},$$

to show that the total number of inversion triples is preserved requires a tedious case analysis. We present several cases here and leave the remainder to the reader. First, suppose $z < a <$

$b < c < d$. Then none of $z < a < c$, $z < b < c$, $z < a < d$, and $z < b < d$ are inversion triples. Next, if $a < z < b < c < d$, then $a < z < c$ is an inversion triple in T, and $z < b < c$ is not an inversion triple in $s_i(T)$. On the other hand, $z < b < d$ is not an inversion triple in T, but $a < z < d$ is an inversion triple in $s_i(T)$. Thus, the total number of inversion triples is preserved. The remaining cases are similar. $\square$

3.2 The Cyclic Shift Operation

The cyclic shift on fillings is a bijection on standard fillings which preserves inv_μ while increasing comaj_μ by 1 in most cases.

Definition 3.2.1. Given $\mu \in \mathrm{Par}(n)$ and $T \in \mathcal{F}_\mu$, define the *cyclic shift* of T, denoted $\mathrm{cyc}(T)$, by replacing each entry c in T by $(c+1) \bmod n$. Here we use the convention that $a \bmod n \in \{1, 2, \ldots, n\}$.

Theorem 3.2.2. *For all* $\mu \in \mathrm{Par}(n)$, $\mathrm{cyc}^n = \mathrm{cyc} \circ \mathrm{cyc} \circ \cdots \circ \mathrm{cyc} = \mathrm{id}_{\mathcal{F}_\mu}$.

Proof. Let $T \in \mathcal{F}_\mu$. Each entry $c \in T$ will be replaced by the entry $(c+n) \bmod n = c$ in $\mathrm{cyc}^n(T)$. Thus $\mathrm{cyc}^n(T) = T$. $\square$

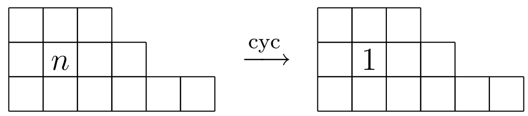
Theorem 3.2.3. *Let* $\mu \in \mathrm{Par}(n)$*. For* $T \in \mathcal{F}_\mu$*, where* n *is not in the bottom row,*

(a) $\mathrm{comaj}_\mu(\mathrm{cyc}(T)) = \mathrm{comaj}_\mu(T) + 1$*;*

(b) $\mathrm{maj}_\mu(\mathrm{cyc}(T)) = \mathrm{maj}_\mu(T) - 1$*;*

(c) $\mathrm{inv}_\mu(\mathrm{cyc}(T)) = \mathrm{inv}_\mu(T)$.

Proof. This can be seen by considering what happens to the cell c containing n in a filling.



The ascent between n and the cell above (if any) is shifted downward by a unit, while the descent between n and the cell below are shifted up one unit. All other ascents and descents are unaffected. All triples in T which do not involve n are preserved. Triples in T which include n have the form $a < b < n$. The corresponding triple in T' will be $1 < a+1 < b+1$. Thus, the status of the triple will be preserved. The assumption that n is not in the bottom row is needed to ensure that no triple involving n also involves an ∞ below the bottom row. $\square$

Example 3.2.4. For $\mu = (3,3,2)$ and $n = 8$,

$$T = \begin{array}{|c|c|c|}\hline 3 & 8 & \\ \hline 5 & 2 & 1 \\ \hline 4 & 7 & 6 \\ \hline \end{array} \qquad \text{and} \qquad \mathrm{cyc}(T) = \begin{array}{|c|c|c|}\hline 4 & 1 & \\ \hline 6 & 3 & 2 \\ \hline 5 & 8 & 7 \\ \hline \end{array},$$

the statistics are:

$\mathrm{inv}_\mu(T) = 3$; $\qquad$ $\mathrm{inv}_\mu(\mathrm{cyc}(T)) = 3$;

$\mathrm{comaj}_\mu(T) = 4$; $\qquad$ $\mathrm{comaj}_\mu(\mathrm{cyc}(T)) = 5$;

$\mathrm{maj}_\mu(T) = 3$; $\qquad$ $\mathrm{maj}_\mu(\mathrm{cyc}(T)) = 2$.

It is possible to cyclic shift when n is in the first row for any $\mu \in \mathrm{Par}(n)$, but inv_μ will not be preserved. We will see in certain cases that we can use the inversion flip operation with the cyclic shift to maintain inv_μ.

Theorem 3.2.5. *Let $\mu \in \mathrm{Par}(n)$. Suppose $T \in \mathcal{F}_\mu$ with first row $\sigma(\mathbf{a}) = a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(\mu_1)}$ for some $\sigma \in S_{\mu_1}$ and $\mathbf{a} = \langle a_1, a_2, \ldots, a_{\mu_1}\rangle$ with $a_1 < a_2 < \cdots < a_{\mu_1} = n$. Then n is in the cell $c = (1, \sigma^{-1}(\mu_1))$ and*

$$\begin{aligned}
\mathrm{comaj}_\mu(\mathrm{cyc}(T)) &= \mathrm{comaj}_\mu(T) - l(c); \\
\mathrm{inv}_\mu(\mathrm{cyc}(T)) &= \mathrm{inv}_\mu(T) + a'(c) - a(c).
\end{aligned}$$

Proof. Let $T \in \mathcal{F}_\mu$ with first row $\sigma(\mathbf{a}) = \langle a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(\mu_1)}\rangle$ for some $\sigma \in S_{\mu_1}$ and $\mathbf{a} = \langle a_1, \ldots, a_{\mu_1}\rangle$ where $a_1 < a_2 < \cdots < a_{\mu_1} = n$. Let $c = (1, \sigma^{-1}(\mu_1))$ be the cell in μ containing n. The entry in cell c in $\mathrm{cyc}(T)$ is 1. Thus the ascent in T between the first two rows in column $\sigma^{-1}(\mu_1)$ becomes a descent in $\mathrm{cyc}(T)$, whereas all other ascents are unchanged, so $\mathrm{comaj}_\mu(\mathrm{cyc}(T)) = \mathrm{comaj}_\mu(T) - l(c)$. All inversion triples, except those in just the first row, are preserved by the cyclic shift. In the first row, T has $a(c)$ inversions between n and each entry to the right of n. In $\mathrm{cyc}(T)$, these inversions are lost, but each entry to the left of cell c creates a new inversion with the 1 in cell c. Thus, $\mathrm{inv}_\mu(\mathrm{cyc}(T)) = \mathrm{inv}_\mu(T) + a'(c) - a(c)$. $\square$

Example 3.2.6. Let $\mu = (5, 4, 2)$ and let $T \in \mathcal{F}_\mu$ be given by

$$\begin{array}{|c|c|c|c|c|}
\cline{1-2} 3 & 2 \\
\cline{1-4} 1 & 7 & 8 & 10 \\
\hline 6 & 11 & 4 & 5 & 9 \\
\hline
\end{array},$$

where $\mathrm{comaj}_\mu(T) = 5$ and $\mathrm{inv}_\mu(T) = 9$. Then

$$\mathrm{cyc}(T) = \begin{array}{|c|c|c|c|c|}
\cline{1-2} 4 & 3 \\
\cline{1-4} 2 & 8 & 9 & 11 \\
\hline 7 & 1 & 5 & 6 & 10 \\
\hline
\end{array}$$

with $\mathrm{comaj}_\mu(\mathrm{cyc}(T)) = 3$ and $\mathrm{inv}_\mu(\mathrm{cyc}(T)) = 7$.

3.3 Going Over the Cliff

We now work toward a series of operations which, in certain cases, allow cyclic shifting when n is in the first row and preserve inv_μ. The first step is to show that s_i and cyc commute.

Theorem 3.3.1. *Given $\mu \in \mathrm{Par}(n)$ and $i \in \mathbb{N}^+$ with $\mu'_i = \mu'_{i+1}$, $s_i \circ \mathrm{cyc} = \mathrm{cyc} \circ s_i$.*

Proof. Since s_i only modifies two adjacent columns in the partition, we are reduced to considering $T \in \mathcal{F}_{(2^n)}$. We argue by induction on n. If $2n$ is not in the bottom row of T, then, since cyc preserves both the number and location of inversion triples, $(s_1 \circ \mathrm{cyc})(T) = (\mathrm{cyc} \circ s_1)(T)$ follows by the recursive definition of s_1 and induction.

If $2n$ is in the bottom row of T, we can assume the bottom row is increasing by Theorem 3.1.2 (a). Then

$$T = \begin{array}{|c|c|} \hline \vdots & \vdots \\ \hline c & d \\ \hline a & 2n \\ \hline \end{array} \quad \text{and} \quad \mathrm{cyc}(T) = \begin{array}{|c|c|} \hline \vdots & \vdots \\ \hline c' & d' \\ \hline a' & 1 \\ \hline \end{array}$$

where $a' = a + 1$, $c' = c + 1$, and $d' = d + 1$. If $c < a < d$ or $d < a < c$, then

$$s_1(T) = \begin{array}{|c|c|} \hline \vdots & \vdots \\ \hline d & c \\ \hline 2n & a \\ \hline \end{array}, \quad s_1(\mathrm{cyc}(T)) = \begin{array}{|c|c|} \hline \vdots & \vdots \\ \hline d' & c' \\ \hline 1 & a' \\ \hline \end{array}, \quad \text{and} \quad \mathrm{cyc}(s_1(T)) = \begin{array}{|c|c|} \hline \vdots & \vdots \\ \hline d' & c' \\ \hline 1 & a' \\ \hline \end{array},$$

where the portions of the fillings above the bottom 2 rows agree by induction. Otherwise,

$$s_1(T) = \begin{array}{|c|c|} \hline \vdots & \vdots \\ \hline c & d \\ \hline 2n & a \\ \hline \end{array}, \quad s_1(\mathrm{cyc}(T)) = \begin{array}{|c|c|} \hline \vdots & \vdots \\ \hline c' & d' \\ \hline 1 & a' \\ \hline \end{array}, \quad \text{and} \quad \mathrm{cyc}(s_1(T)) = \begin{array}{|c|c|} \hline \vdots & \vdots \\ \hline c' & d' \\ \hline 1 & a' \\ \hline \end{array}.$$

Therefore, $s_1 \circ \mathrm{cyc} = \mathrm{cyc} \circ s_1$. □

Theorem 3.3.2. *Let $T \in \mathcal{F}_{(2^n)}$ for some $n \geq 1$ where $2n$ is in the bottom row of T. Let $S = s_1(\mathrm{cyc}(T))$. Then $\mathrm{comaj}_{(2^n)}(S) = \mathrm{comaj}_{(2^n)}(T) - (n-1)$ and $\mathrm{inv}_{(2^n)}(S) = \mathrm{inv}_{(2^n)}(T)$.*

Proof. First notice that when $2n$ is in the bottom row of T,

$$\mathrm{inv}_{(2^n)}(\mathrm{cyc}(T)) = \mathrm{inv}_{(2^n)}(T) + \begin{cases} 1 & \text{if } 2n \text{ is in column } 2; \\ -1 & \text{if } 2n \text{ is in column } 1. \end{cases}$$

Since the entry $2n$ in T becomes 1 in $\mathrm{cyc}(T)$,

$$\mathrm{comaj}_{(2^n)}(\mathrm{cyc}(T)) = \mathrm{comaj}_{(2^n)}(T) - (n-1)$$

because the ascent which occurs in T between $2n$ and the entry above it is lost. It follows from Theorem 3.1.4 that

$$\mathrm{inv}_{(2^n)}(S) = \mathrm{inv}_{(2^n)}(s_1(\mathrm{cyc}(T))) = \mathrm{inv}_{(2^n)}(T)$$

and

$$\mathrm{comaj}_{(2^n)}(S) = \mathrm{comaj}_{(2^n)}(s_1(\mathrm{cyc}(T))) = \mathrm{comaj}_{(2^n)}(T) - (n-1)\ . \qquad \square$$

Example 3.3.3. For $n = 4$ and

$$T = \begin{array}{|c|c|} \hline 6 & 1 \\ \hline 7 & 5 \\ \hline 4 & 2 \\ \hline 3 & 8 \\ \hline \end{array}\ ,$$

we have

$$\mathrm{cyc}(T) = \begin{array}{|c|c|} \hline 7 & 2 \\ \hline 8 & 6 \\ \hline 5 & 3 \\ \hline 4 & 1 \\ \hline \end{array}\ ,$$

and

$$S = s_1(\mathrm{cyc}(T)) = \begin{array}{|c|c|} \hline 7 & 2 \\ \hline 8 & 6 \\ \hline 3 & 5 \\ \hline 1 & 4 \\ \hline \end{array}\ .$$

Then $\mathrm{comaj}_{(2^4)}(T) = 5$, $\mathrm{comaj}_{(2^4)}(S) = 2$, and $\mathrm{inv}_{(2^4)}(T) = \mathrm{inv}_{(2^4)}(S) = 2$.

Similar moves are available for fillings with three or more columns of equal height. We call these cliff moves and state the three column version first.

Definition 3.3.4. Let $T \in \mathcal{F}_{(3^n)}$. The first row of T can be written as $\sigma(\mathbf{a}) = a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}$ where $\sigma \in S_3$ and $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ with $a_1 < a_2 < a_3$. Define $\text{cliff} : \mathcal{F}_{(3^n)} \to \mathcal{F}_{(3^n)}$ by

$$\text{cliff}(T) = \begin{cases} \text{cyc}(T) & \text{if } 3n \text{ is not in row 1 of } T; \\ s_1 \circ s_2 \circ \text{cyc}(T) & \text{if } 3n \text{ is in row 1 and } \sigma = 123, 231, \text{ or } 312; \\ s_2 \circ s_1 \circ \text{cyc}(T) & \text{if } 3n \text{ is in row 1 and } \sigma = 132, 213, \text{ or } 321. \end{cases}$$

Theorem 3.3.5. *Let $T \in \mathcal{F}_{(3^n)}$. Let the first row of T be $\sigma(\mathbf{a})$ where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\sigma \in S_3$. Then for $0 \le i < 3n$,*

(a) $\text{comaj}_{(3^n)}(\text{cliff}^i(T)) = \text{comaj}_{(3^n)}(T) + i - n(\chi(i > 3n - a_1) + \chi(i > 3n - a_2)$
$+ \chi(i > 3n - a_3))$, *and*

(b) $\text{inv}_{(3^n)}(\text{cliff}^i(T)) = \text{inv}_{(3^n)}(T)$.

Prior to the proof of Theorem 3.3.5 we offer a clarifying example.

Example 3.3.6. Let $T = \begin{array}{|c|c|c|}\hline 3 & 8 & 9 \\ \hline 7 & 2 & 4 \\ \hline 6 & 1 & 5 \\ \hline \end{array}$. Then $\text{cliff}^5(T) = \begin{array}{|c|c|c|}\hline 4 & 8 & 5 \\ \hline 7 & 3 & 9 \\ \hline 6 & 1 & 2 \\ \hline \end{array}$. Each step is shown in Figure 3.2.

$$\begin{array}{ccccccc}
\begin{array}{|c|c|c|}\hline 3 & 8 & 9 \\ \hline 7 & 2 & 4 \\ \hline 6 & 1 & 5 \\ \hline \end{array} & \xrightarrow{\text{cyc}} & \begin{array}{|c|c|c|}\hline 4 & 9 & 1 \\ \hline 8 & 3 & 5 \\ \hline 7 & 2 & 6 \\ \hline \end{array} & \xrightarrow{\text{cyc}} & \begin{array}{|c|c|c|}\hline 5 & 1 & 2 \\ \hline 9 & 4 & 6 \\ \hline 8 & 3 & 7 \\ \hline \end{array} & \xrightarrow{\text{cyc}} & \\
q^4t^3 & & q^4t^4 & & q^4t^5 & & \\
\begin{array}{|c|c|c|}\hline 6 & 2 & 3 \\ \hline 1 & 5 & 7 \\ \hline 9 & 4 & 8 \\ \hline \end{array} & \xrightarrow{s_1 \circ s_2 \circ \text{cyc}} & \begin{array}{|c|c|c|}\hline 7 & 3 & 4 \\ \hline 2 & 6 & 8 \\ \hline 9 & 1 & 5 \\ \hline \end{array} & \xrightarrow{s_1 \circ s_2 \circ \text{cyc}} & \begin{array}{|c|c|c|}\hline 4 & 8 & 5 \\ \hline 7 & 3 & 9 \\ \hline 6 & 1 & 2 \\ \hline \end{array} & & \\
q^4t^6 & & q^4t^4 & & q^4t^2 & &
\end{array}$$

Figure 3.2: Example of cliff^i.

Proof. Let $T \in \mathcal{F}_{(3^n)}$ with first row $\sigma(\mathbf{a})$ as above. By Theorem 3.3.1, $\text{cliff}^i(T) = s \circ \text{cyc}^i(T)$ where s is some sequence of s_1's and s_2's. For $0 < j \leq i$, when $3n$ is not in the first row of $\text{cyc}^{j-1}(T)$, $\text{comaj}_{(3^n)}(\text{cyc}^j(T)) = \text{comaj}_{(3^n)}(\text{cyc}^{j-1}(T)) + 1$. If $3n$ is in the first row of $\text{cyc}^{j-1}(T)$ for some j, then $a_1 + (j-1) = 3n, a_2 + (j-1) = 3n$, or $a_3 + (j-1) = 3n$ and $\text{comaj}_{(3^n)}(\text{cyc}^j(T)) = \text{comaj}_{(3^n)}(\text{cyc}^{j-1}(T)) - (n-1)$. The sequence s of inversion flips does not affect the $\text{comaj}_{(3^n)}$ statistic, so

$$\text{comaj}_{(3^n)}(\text{cliff}^i)(T) = \text{comaj}_{(3^n)}(T) + i - n(\chi(i > 3n - a_1) + \chi(i > 3n - a_2) + \chi(i > 3n - a_3)).$$

To show that $\text{inv}_{(3^n)}(\text{cliff}^i(T)) = \text{inv}_{(3^n)}(T)$ by Theorem 3.2.3(c) it is sufficient to show

(i) $\text{inv}_{(3^n)}(s_1 \circ s_2 \circ \text{cyc})(T) = \text{inv}_{(3^n)}(T)$ when $\sigma = 123, 231$, or 312 and $a_3 = 3n$, and

(ii) $\text{inv}_{(3^n)}(s_2 \circ s_1 \circ \text{cyc})(T) = \text{inv}_{(3^n)}(T)$ when $\sigma = 132, 213$, or 321 and $a_3 = 3n$.

Since $a_3 = 3n$, the first row of $\text{cyc}(T)$ is $\text{cyc}{\uparrow}(\sigma)(\widetilde{a})$ where $\widetilde{a} = \langle 1, a_1 + 1, a_2 + 1 \rangle$. All inversion triples above the first row remain unchanged after cyc, so it is sufficient to show that the inversions in the first rows of the fillings T and $(s \circ \text{cyc})(T)$ are the same, where s is either $s_1 \circ s_2$ or $s_2 \circ s_1$. First consider $s_1 \circ s_2 \circ \text{cyc}(T))$. The first row of $\text{cyc}(T)$ is $\boxed{a'_1}\boxed{a'_2}\boxed{a'_3}$ where $a'_j = (a_{\sigma(j)} + 1) \mod 3n$. Then the first row of $s_1 \circ s_2 \circ \text{cyc}(T)$ is $\boxed{a'_3}\boxed{a'_1}\boxed{a'_2}$. When $\sigma = 123$, the first row of T is $\boxed{a_1}\boxed{a_2}\boxed{a_3}$, which has no inversions, and $\boxed{a'_3}\boxed{a'_1}\boxed{a'_2} = \boxed{1}\boxed{\widetilde{a}_1}\boxed{\widetilde{a}_2}$ where $\widetilde{a}_j = a_j + 1$, so the first row of $s_1 \circ s_2 \circ \text{cyc}(T)$ also has no inversions. When $\sigma = 231$, the first row of T is $\boxed{a_2}\boxed{a_3}\boxed{a_1}$, which has two inversions, and $\boxed{a'_3}\boxed{a'_1}\boxed{a'_2} = \boxed{\widetilde{a}_1}\boxed{\widetilde{a}_2}\boxed{1}$, which also has two inversions. Finally, when $\sigma = 312$, the first row of T is $\boxed{a_3}\boxed{a_1}\boxed{a_2}$, which has two inversions, and

$\boxed{a_3'}\boxed{a_1'}\boxed{a_2'} = \boxed{\widetilde{a}_2}\boxed{1}\boxed{\widetilde{a}_1}$, which also has two inversions. Thus $\mathrm{inv}_{(3^n)}(s_1 \circ s_2 \circ \mathrm{cyc}(T)) = \mathrm{inv}_{(3^n)}(T)$. A similar case analysis of $s_2 \circ s_1 \circ \mathrm{cyc}(T)$ leads to $\mathrm{inv}_{(3^n)}(s_2 \circ s_1 \circ \mathrm{cyc}(T)) = \mathrm{inv}_{(3^n)}(T)$. $\square$

We now look at cliff for fillings of shape (m^n) which have increasing first rows.

Definition 3.3.7. For $\mu \in \mathrm{Par}(n)$ and $\mathbf{a} = \langle a_1, \ldots, a_{\mu_1} \rangle$, let

$$\mathcal{F}_{\mu,\mathbf{a}} = \{T \in \mathcal{F}_\mu : \text{ first row of } T \text{ is } \mathbf{a}\}.$$

When μ has only two columns, let

$$\mathcal{F}_{\mu,a} = \{T \in \mathcal{F}_\mu : \text{ first row of } T \text{ is } \boxed{1}\boxed{a}\}.$$

Definition 3.3.8. Define $\mathrm{cliff} : \mathcal{F}_{(m^n),\mathbf{a}} \to \mathcal{F}_{(m^n)}$ when $\mathbf{a} = \langle a_1, \ldots, a_m \rangle$ with $1 \leq a_1 < a_2 < \cdots < a_m \leq mn$ by

$$\mathrm{cliff}(T) = \begin{cases} \mathrm{cyc}(T) & \text{if } a_m \neq mn \\ s_1 \circ s_2 \circ \ldots \circ s_{m-1} \circ \mathrm{cyc}(T) & \text{if } a_m = mn \end{cases}.$$

Theorem 3.3.9. *For all $m, n, \mathbf{a}, T, i$ where $m, n \in \mathbb{N}, \mathbf{a} = \langle a_1, \ldots, a_m \rangle$ with $1 \leq a_1 < \cdots < a_m \leq mn$, $T \in \mathcal{F}_{(m^n),\mathbf{a}}$, and $0 \leq i < mn$,*

$$\begin{aligned} \mathrm{inv}_{(m^n)}(\mathrm{cliff}^i(T)) &= \mathrm{inv}_{(m^n)}(T) \textit{ and} \\ \mathrm{comaj}_{(m^n)}(\mathrm{cliff}^i(T)) &= \mathrm{comaj}_{(m^n)}(T) + i - n\left(\sum_{j=1}^{m} \chi(i > mn - a_j)\right). \end{aligned}$$

Proof. Let $T \in \mathcal{F}_{(m^n),\mathbf{a}}$. By Theorem 3.3.1, $\mathrm{cliff}^i(T) = (s \circ \mathrm{cyc}^i)(T)$ where s is some composition of s_i's. For $0 < j \leq i$, when mn is not in the first row of $\mathrm{cyc}^{j-1}(T)$,

$$\mathrm{comaj}_{(m^n)}(\mathrm{cyc}^j(T)) = \mathrm{comaj}_{(m^n)}(\mathrm{cyc}^{j-1}(T)) + 1.$$

If mn is in the first row of $\text{cyc}^{j-1}(T)$ for some j, then $a_k+(j-1)=mn$ for some k, $1 \leq k \leq m$ and

$$\text{comaj}_{(m^n)}(\text{cyc}^j(T)) = \text{comaj}_{(m^n)}(\text{cyc}^{j-1}(T)) - (n-1).$$

Since the sequence s of inversion flips does not affect the $\text{comaj}_{(m^n)}$ statistic,

$$\text{comaj}_{(m^n)}(\text{cliff}^i(T)) = \text{comaj}_{(m^n)}(T) + i - n\left(\sum_{j=1}^{m} \chi(i > mn - a_j)\right).$$

To see that $\text{inv}_{(m^n)}(\text{cliff}^i(T)) = \text{inv}_{(m^n)}(T)$, it is sufficient, by Theorem 3.2.3 (c), to show that if $a_m = mn$,

$$\text{inv}_{(m^n)}(s_1 \circ s_2 \circ \cdots \circ s_{m-1} \circ \text{cyc})(T) = \text{inv}_{(m^n)}(T).$$

Since $a_m = mn$, the first row of $\text{cyc}(T)$ is $\boxed{a'_1}\boxed{a'_2}\cdots\boxed{1}$ where $a'_i = a_i + 1$. Notice that $\text{inv}_{(m^n)}(\text{cyc}(T)) = \text{inv}_{(m^n)}(T) + m - 1$. The s_i's have the effect of moving the 1 back to the first box in the row, thus eliminating all the extra inversions. Therefore, by Theorems 3.1.4 and 3.2.5

$$\text{inv}_{(m^n)}(s_1 \circ s_2 \circ \cdots \circ s_{m-1} \circ \text{cyc})(T) = \text{inv}_{(m^n)}(T).$$

□

Example 3.3.10. Let $T \in \mathcal{F}_{(5^3),\langle 2,3,6,9,12\rangle}$ as shown below:

4	1	8	15	14
13	7	11	5	10
2	3	6	9	12

.

Figure 3.3 shows each step required to obtain $\text{cliff}^4(T)$.

$$
\underset{q^{16}t^{7}}{\begin{array}{|c|c|c|c|c|}\hline 4&1&8&15&14\\\hline 13&7&11&5&10\\\hline 2&3&6&9&12\\\hline\end{array}}
\xrightarrow{\text{cyc}^4}
\underset{q^{20}t^{8}}{\begin{array}{|c|c|c|c|c|}\hline 8&5&12&4&3\\\hline 2&11&15&9&14\\\hline 6&7&10&13&1\\\hline\end{array}}
\xrightarrow{s_4}
\underset{q^{19}t^{8}}{\begin{array}{|c|c|c|c|c|}\hline 8&5&12&4&3\\\hline 2&11&15&14&9\\\hline 6&7&10&1&13\\\hline\end{array}}
\xrightarrow{s_3}
$$

$$
\underset{q^{18}t^{8}}{\begin{array}{|c|c|c|c|c|}\hline 8&5&12&4&3\\\hline 2&11&15&14&9\\\hline 6&7&1&10&13\\\hline\end{array}}
\xrightarrow{s_2}
\underset{q^{17}t^{8}}{\begin{array}{|c|c|c|c|c|}\hline 8&5&12&4&3\\\hline 2&11&15&14&9\\\hline 6&1&7&10&13\\\hline\end{array}}
\xrightarrow{s_1}
\underset{q^{16}t^{8}}{\begin{array}{|c|c|c|c|c|}\hline 8&5&12&4&3\\\hline 11&2&15&14&9\\\hline 1&6&7&10&13\\\hline\end{array}}
$$

Figure 3.3: Example of cliff for (5^3).

3.4 Augmentation for (2^n)

In this section we explore what happens to a filling when we fix the first row and perform inversion flips and cyclic shifts on the remaining rows.

Definition 3.4.1. For $n \geq 1$, $T \in \mathcal{F}_{(2^n)}$ and $2 \leq a \leq 2n+2$, we define the *a-augmentation* of T to be the filling $A_a(T) \in \mathcal{F}_{(2^{n+1}),a}$ which is obtained by first relabeling the entries of T as follows: $c \in T$ is replaced by $c+1$ if $c < a-1$ and by $c+2$ if $c \geq a-1$; and then placing the relabeled filling over the new bottom row $\begin{array}{|c|c|}\hline 1 & a\\\hline\end{array}$.

Example 3.4.2. For $a = 3$ and $T \in \mathcal{F}_{(2^3)}$ given by

$$T = \begin{array}{|c|c|}\hline 3&5\\\hline 6&4\\\hline 1&2\\\hline\end{array}\,,$$

we have

$$A_3(T) = \begin{array}{|c|c|}\hline 5&7\\\hline 8&6\\\hline 2&4\\\hline 1&3\\\hline\end{array}\,.$$

We consider next what A_a does to inv_μ and comaj_μ and how it interacts with cyc and s_1.

Theorem 3.4.3. *For all $n \geq 2$, all a with $2 \leq a \leq 2n$, and all $T \in \mathcal{F}_{(2^{n-1})}$ with bottom row* $\boxed{x\,|\,z}$,

(a) $\operatorname{comaj}_{(2^n)}(A_a(T)) = \operatorname{comaj}_{(2^{n-1})}(T) + (n-1)\chi(z < a-1)$;

(b) $\operatorname{inv}_{(2^n)}(A_a(T)) = \operatorname{inv}_{(2^{n-1})}(T)$.

Proof. Let $T \in \mathcal{F}_{(2^{n-1})}$ have bottom row $\boxed{x\,|\,z}$, and let $2 \leq a \leq 2n$. The bottom two rows of $A_a(T)$ look like:

$$\begin{array}{|c|c|} \hline x' & z' \\ \hline 1 & a \\ \hline \end{array}$$

where x' is $x+1$ if $x < a-1$ or $x+2$ if $x \geq a-1$, and z' is $z+1$ if $z < a-1$ or $z+2$ if $z \geq a-1$. Now $\operatorname{inv}_{(2^n)}(A_a(T)) = \operatorname{inv}_{(2^{n-1})}(T)$ since the two entries in the bottom row of $A_a(T)$ do not form an inversion triple, and $1, x', z'$ form an inversion triple in $A_a(T)$ iff ∞, x, z form an inversion triple in T. The ascents in the top $n-1$ rows of $A_a(T)$ are the same as the ascents in T. There cannot be an ascent between the two lowest cells in column 1 of $A_a(T)$ since the lowest cell in this column contains 1. If a is greater than the entry directly above it in $A_a(T)$, that entry is $z+1$, giving a new ascent, so $\operatorname{comaj}_{(2^n)}(A_a(T)) = \operatorname{comaj}_{(2^{n-1})}(T)+n-1$. If a is less than the entry directly above it in $A_a(T)$, that entry must be $z+2$, and thus $\operatorname{comaj}_{(2^n)}(A_a(T)) = \operatorname{comaj}_{(2^{n-1})}(T)$. □

Definition 3.4.4. For all $n \geq 2$ and all a with $2 \leq a \leq 2n$, define $\overline{\operatorname{cyc}} : \mathcal{F}_{(2^n),a} \to \mathcal{F}_{(2^n),a}$ by $\overline{\operatorname{cyc}} = A_a \circ \operatorname{cyc} \circ A_a^{-1}$. Similarly, define $\overline{s}_1 : \mathcal{F}_{(2^n),a} \to \mathcal{F}_{(2^n),a}$ by $\overline{s}_1 = A_a \circ s_1 \circ A_a^{-1}$.

Theorem 3.4.5. *For all n, a, b, j, T with $n \geq 2$, $2 \leq a \leq 2n$, $2 \leq b \leq 2n-2$, $0 \leq j \leq 2n-2-b$, and $T \in \mathcal{F}_{(2^{n-1}),b}$:*

(a) $\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^j(A_a(T))) = \mathrm{comaj}_{(2^{n-1})}(T) + j + (n-1)\chi(j < a-b-1)$*;*

(b) $\mathrm{inv}_{(2^n)}(\overline{\mathrm{cyc}}^j(A_a(T))) = \mathrm{inv}_{(2^{n-1})}(T)$*;*

(c) $\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^j(\overline{s}_1(A_a(T)))) = \mathrm{comaj}_{(2^{n-1})}(T) + j + (n-1)\chi(j < a-2)$*;*

(d) $\mathrm{inv}_{(2^n)}(\overline{\mathrm{cyc}}^j(\overline{s}_1(A_a(T)))) = \mathrm{inv}_{(2^{n-1})}(T) + 1$*.*

Proof. Let $T \in \mathcal{F}_{(2^{n-1}),b}$. Then, by Theorems 3.2.3 and 3.4.3,

$$\begin{aligned}\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^j(A_a(T))) &= \mathrm{comaj}_{(2^n)}(A_a(\mathrm{cyc}^j(T))\\ &= \mathrm{comaj}_{(2^{n-1})}(\mathrm{cyc}^j(T)) + (n-1)\chi(b+j < a-1)\\ &= \mathrm{comaj}_{(2^{n-1})}(T) + j + (n-1)\chi(j < a-b-1).\end{aligned}$$

By Theorem 3.4.3, $\mathrm{inv}_{(2^n)}(\overline{\mathrm{cyc}}^j(A_a(T))) = \mathrm{inv}_{(2^{n-1})}(T)$.

Similarly, using Theorems 3.3.1, 3.4.3, and 3.2.3, we find that

$$\begin{aligned}\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^j(\overline{s}_1(A_a(T)))) &= \mathrm{comaj}_{(2^n)}(A_a(\mathrm{cyc}^j(s_1(T))))\\ &= \mathrm{comaj}_{(2^n)}(A_a(s_1(\mathrm{cyc}^j(T))))\\ &= \mathrm{comaj}_{(2^{n-1})}(\mathrm{cyc}^j(T)) + (n-1)\chi(1+j < a-1)\\ &= \mathrm{comaj}_{(2^{n-1})}(T) + j + (n-1)\chi(j < a-2).\end{aligned}$$

By Theorems 3.4.3, 3.1.4, and 3.2.3, $\mathrm{inv}_{(2^n)}(\overline{\mathrm{cyc}}^j(\overline{s}_1(A_a(T)))) = \mathrm{inv}_{(2^{n-1})}(T) + 1$. $\square$

Example 3.4.6. Let $\mu = (2^4)$ and $T \in \mathcal{F}_{(2^4),3}$

$$T = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 8 & 7 \\ \hline 5 & 6 \\ \hline 1 & 3 \\ \hline\end{array}$$

with $\mathrm{comaj}_{(2^4)}(T) = 2$ and $\mathrm{inv}_{(2^4)}(T) = 1$. Then

$$\overline{\mathrm{cyc}}(T) = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 2 & 8 \\ \hline 6 & 7 \\ \hline 1 & 3 \\ \hline \end{array}$$

where $\mathrm{comaj}_{(2^4)}(\overline{\mathrm{cyc}}(T)) = 3$ and $\mathrm{inv}_{(2^4)}(\overline{\mathrm{cyc}}(T)) = 1$.

3.5 Augmentation for (m^n)

Definition 3.5.1. Let $n \geq 1$, $m \geq 2$, and $T \in \mathcal{F}_{(m^n)}$. Given $\mathbf{a} = 1 < a_2 < \cdots < a_m$, with $a_m \leq m(n+1)$, the $\mathbf{a}$-*augmentation* of T is the filling $A_{\mathbf{a}}(T) \in \mathcal{F}_{(m^{n+1})}$ obtained as follows: for each entry c in T, replace c by $c + j$ where j is the smallest index such that $a_j \leq c + j < a_{j+1}$; then place this filling over a new first row $\begin{array}{|c|c|}\hline a_1 & a_2 \\ \hline\end{array} \cdots \begin{array}{|c|}\hline a_m \\ \hline\end{array}$.

Before examining what $A_{\mathbf{a}}$ does to inv_μ and comaj_μ, we first look at a basic fact about the relabeling of the entries of T. This fact holds for partitions of any shape.

Theorem 3.5.2. *Let $n \geq 1$, $m \geq 2$, $T \in \mathcal{F}_{(m^n)}$, and $\mathbf{a} = 1 < a_2 < \cdots < a_m$, with $a_m \leq m(n+1)$. Suppose the first row of T is $\begin{array}{|c|c|}\hline b_1 & b_2 \\ \hline\end{array} \cdots \begin{array}{|c|}\hline b_m \\ \hline\end{array}$. Let the second row of $A_{\mathbf{a}}(T)$ be denoted $\begin{array}{|c|c|}\hline b'_1 & b'_2 \\ \hline\end{array} \cdots \begin{array}{|c|}\hline b'_m \\ \hline\end{array}$. Then for all i, j with $1 \leq i, j \leq m$, $b'_j < a_i$ if and only if $b_j + (i-1) < a_i$.*

Proof. Suppose $b'_j > a_i$. Then, there exists some $k \geq i$ such that $a_k < b'_j < a_{k+1}$. So $b'_j = b_j + k$ by Definition 3.5.1. Thus, $a_k < b_j + k < a_{k+1}$. Note that $a_k \geq a_i + (k - i)$. Now we have $a_i + k - i < b_j + k$ and hence, $a_i < b_j + i$.

Next, suppose $b_j + i - 1 \geq a_i$. Then, $b_j + i > a_i$ and there exists some $k \geq i$ such that $a_i \leq a_k \leq b_j + i < a_{k+1}$. By Definition 3.5.1 $b'_j = b_j + k$, so $b'_j \geq b_j + i \geq a_k \geq a_i$. $\square$

Theorem 3.5.3. *Given $n \geq 2$, $m \geq 2$, $T \in \mathcal{F}_{(m^{n-1})}$, and $\mathbf{a} = 1 < a_2 < \cdots < a_m$ where $a_m \leq mn$, suppose the first row of T is* $\boxed{b_1}\boxed{b_2}\cdots\boxed{b_m}$.

$$\text{(a) } \operatorname{inv}_{(m^n)}(A_{\mathbf{a}}(T)) = \operatorname{inv}_{(m^{n-1})}(T) + \sum_{i=2}^{m-1}\sum_{j=i+1}^{m}(\chi(b_i < a_i - (i-1) \leq b_j) - \chi(b_j < a_i - (i-1) \leq b_i)).$$

$$\text{(b) } \operatorname{comaj}_{(m^n)}(A_{\mathbf{a}}(T)) = \operatorname{comaj}_{(m^{n-1})}(T) + (n-1)\sum_{i=2}^{m}\chi(b_i < a_i - (i-1)).$$

Proof. We start by proving part (b). Let $m \geq 2, n \geq 2, T \in \mathcal{F}_{(m^{n-1})}$, and $\mathbf{a} = 1 < a_2 < \cdots < a_m$ where $a_m \leq mn$. Denote the first row of T by $\boxed{b_1}\boxed{b_2}\cdots\boxed{b_m}$ and the second row of $A_{\mathbf{a}}(T)$ by $\boxed{b'_1}\boxed{b'_2}\cdots\boxed{b'_m}$. Notice that all ascents above the first two rows of $A_{\mathbf{a}}(T)$ are the same as the ascents in T. There is an ascent in $A_{\mathbf{a}}(T)$ between row one and two in column i, iff $b'_i < a_i$. By Theorem 3.5.2, this holds iff $b_i + i - 1 < a_i$.

For part (a), consider a triple

$$\begin{array}{|c|}\hline b'_i \\ \hline a_i \\ \hline \end{array} \qquad \begin{array}{|c|}\hline b'_j \\ \hline \end{array}.$$

Note that if $i = 1$, $b_i > b_j$ iff $b'_i > b'_j$, so we need only consider $2 \leq i \leq m-1$. If $b_i > b_j$ in T, then $b'_i > b'_j$ in $A_{\mathbf{a}}(T)$. If $a_i < b'_j < b'_i$ or $b'_j < b'_i < a_i$, then the triple is an inversion triple in $A_{\mathbf{a}}(T)$. However, $b'_j < a_i < b'_i$ iff $b_j < a_i - (i-1) \leq b_i$ and the triple is not an inversion triple, so there is one less inversion triple in $A_{\mathbf{a}}(T)$ than in T. Similarly, if $b_i < b_j$ in T, $b'_i < b'_j$ in $A_{\mathbf{a}}(T)$. If $a_i < b'_i < b'_j$ or $b'_i < b'_j < a_i$, the triple is not an inversion triple.

If $b'_i < a_i < b'_j$, equivalently $b_i < a_i - (i-1) \leq b_j$ and the triple is an inversion triple, and there is one more inversion triple in $A_{\mathbf{a}}(T)$ than in T. $\square$

Definition 3.5.4. For all $n \geq 2, m \geq 3$, and all $\mathbf{a} = \langle a_1, \ldots, a_m \rangle$ with $1 \leq a_1 < a_2 < \cdots < a_m \leq mn$, define $\overline{\text{cyc}} : \mathcal{F}_{(m^n),\mathbf{a}} \to \mathcal{F}_{(m^n),\mathbf{a}}$ by $\overline{\text{cyc}} = A_{\mathbf{a}} \circ \text{cyc} \circ A_{\mathbf{a}}^{-1}$.

Theorem 3.5.5. *For all $m, n, \mathbf{a}, \mathbf{b}, j, T$ with $n \geq 2, m \geq 3, \mathbf{a} = \langle a_1, \ldots, a_m \rangle$ with $1 = a_1 < a_2 < \cdots < a_m \leq mn, \mathbf{b} = \langle b_1, \ldots, b_m \rangle$ with $b_k \leq m(n-1)$ for all $1 \leq k \leq m, 0 \leq j \leq m(n-1) - \max(b_k)$, and $T \in \mathcal{F}_{(m^{n-1}),\mathbf{b}}$:*

(a) $\text{comaj}_{(m^n)}(\overline{\text{cyc}}^j(A_{\mathbf{a}}(T))) = \text{comaj}_{(m^{n-1})}(T) + j + (n-1)\sum_{i=2}^{m} \chi(j < a_i - b_i - (i-1))$*;*

(b) $$\text{inv}_{(m^n)}(\overline{\text{cyc}}^j(A_{\mathbf{a}}(T))) = \text{inv}_{(m^{n-1})}(T) + \sum_{i=2}^{m-1}\sum_{k=i+1}^{m} (\chi(b_i + j < a_i - (i-1) \leq b_k + j) - \chi(b_k + j < a_i - (i-1) \leq b_i + j)).$$

Proof. Let $m, n, \mathbf{a}, \mathbf{b}, j$, and T be as above. Then

$$\begin{aligned}
\text{comaj}_{(m^n)}(\overline{\text{cyc}}^j(A_{\mathbf{a}}(T))) &= \text{comaj}_{(m^n)}(A_{\mathbf{a}}(\text{cyc}^j(T))) \\
&= \text{comaj}_{(m^{n-1})}(\text{cyc}^j(T)) + (n-1)\sum_{i=2}^{m} \chi(b_i + j < a_i - (i-1)) \\
&\qquad \text{by Theorem 3.5.3} \\
&= \text{comaj}_{(m^{n-1})}(T) + j + (n-1)\sum_{i=2}^{m} \chi(j < a_i - b_i - (i-1)) \\
&\qquad \text{by Theorem 3.2.3.}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{inv}_{(m^n)}(\overline{\mathrm{cyc}}^j(A_{\mathbf{a}}(T))) &= \mathrm{inv}_{(m^n)}(A_{\mathbf{a}}(\mathrm{cyc}^j(T))) \\
&= \mathrm{inv}_{(m^{n-1})}(\mathrm{cyc}^j(T)) + \sum_{i=2}^{m-1}\sum_{k=i+1}^{m}(\chi(b_i+j<a_i-(i-1)\le b_k+j) \\
&\qquad -\chi(b_k+j<a_i-(i-1)\le b_i+j)) \\
&\qquad\qquad \text{by Theorem 3.5.3} \\
&= \mathrm{inv}_{(m^{n-1})}(T) + \sum_{i=2}^{m-1}\sum_{k=i+1}^{m}(\chi(b_i+j<a_i-(i-1)\le b_k+j) \\
&\qquad -\chi(b_k+j<a_i-(i-1)\le b_i+j)) \\
&\qquad\qquad \text{by Theorem 3.1.4.} \quad \square
\end{aligned}$$

Example 3.5.6. Let $\mathbf{a} = \langle 1,2,4,7\rangle$ and $U \in \mathcal{F}_{(4^3),\mathbf{a}}$, where

$$U = \begin{array}{|c|c|c|c|}\hline 6 & 3 & 11 & 12 \\ \hline 10 & 5 & 8 & 9 \\ \hline 1 & 2 & 4 & 7 \\ \hline\end{array} \quad \text{and} \quad A_{\mathbf{a}}^{-1}(U) = T = \begin{array}{|c|c|c|c|}\hline 3 & 1 & 7 & 8 \\ \hline 6 & 2 & 4 & 5 \\ \hline\end{array}$$

with $\mathrm{comaj}_{(4^2)}(T) = 2$ and $\mathrm{inv}_{(4^2)}(T) = 8$. Then

$$\begin{aligned}
\overline{\mathrm{cyc}}^2(U) &= A_{\mathbf{a}}(\mathrm{cyc}^2(T)) \\
&= A_{\mathbf{a}}\left(\begin{array}{|c|c|c|c|}\hline 5 & 3 & 1 & 2 \\ \hline 8 & 4 & 6 & 7 \\ \hline\end{array}\right) \\
&= \begin{array}{|c|c|c|c|}\hline 9 & 6 & 3 & 5 \\ \hline 12 & 8 & 10 & 11 \\ \hline 1 & 2 & 4 & 7 \\ \hline\end{array}.
\end{aligned}$$

Notice that $\mathrm{comaj}_{(4^3)}(\overline{\mathrm{cyc}}^2(U)) = 4 = \mathrm{comaj}_{(4^2)}(T)+2$ and $\mathrm{inv}_{(4^3)}(\overline{\mathrm{cyc}}^2(U)) = 8 = \mathrm{inv}_{(4^2)}(T)$.

3.6 Augmentation for $\mu \in \mathrm{Par}(n)$

Definition 3.6.1. Let $\mu = (\mu_1, \ldots, \mu_k) \in \mathrm{Par}(n)$, $T \in \mathcal{F}_\mu$, $\mathbf{a} = \langle a_1, \ldots, a_m \rangle$ for some $m \geq \mu_1$ where $a_1 < a_2 < \cdots < a_m$, and let $\sigma \in S_m$. Let $\nu = (m, \mu_1, \mu_2, \ldots, \mu_k) \in \mathrm{Par}(m+n)$. Define the $\sigma(\mathbf{a})$-augmentation of T to be the filling $A_{\sigma(\mathbf{a})}(T) \in \mathcal{F}_\nu$ obtained as follows: for each entry $c \in T$, replace c by $c + i$ where i is the smallest index such that $a_i \leq c + i < a_i + 1$; place the result over a new first row with entries from left to right $a_{\sigma(1)}, \ldots, a_{\sigma(m)}$.

Theorem 3.6.2. *Let $\mu = (\mu_1, \ldots, \mu_k) \in \mathrm{Par}(n)$, $T \in \mathcal{F}_\mu$, $\mathbf{a} = \langle a_1, \ldots, a_m \rangle$ for some $m \geq \mu_1$ where $a_1 < a_2 < \cdots < a_m$, and let $\sigma \in S_m$. Let $c_1, \ldots c_m$ be the cells in the first row of $A_{\sigma(\mathbf{a})}(T)$. Suppose $\boxed{b_1}\boxed{b_2}\cdots\boxed{b_{\mu_1}}$ is the first row of T. Let $\nu = (m, \mu_1, \mu_2, \ldots, \mu_k) \in$ $\mathrm{Par}(n+m)$. Then*

(a) $\mathrm{comaj}_\nu(A_{\sigma(\mathbf{a})}(T)) = \mathrm{comaj}_\mu(T) + \sum_{i=1}^{\mu_i} l(c_i)\chi(b_i < a_{\sigma(i)} - (\sigma(i) - 1))$ *and*

(b) $\mathrm{inv}_\nu(A_{\sigma(\mathbf{a})}(T)) = \mathrm{inv}_\mu(T) + \mathrm{inv}(\sigma) + \sum_{i=1}^{\mu_1 - 1} \sum_{j=i+1}^{\mu_1} (\chi(b_i < a_{\sigma(i)} - (\sigma(i) - 1) \leq b_j)$
$-\chi(b_j < a_{\sigma(i)} - (\sigma(i) - 1) \leq b_i))$.

Proof. Let $c_1, \ldots c_m$ be the cells in the first row of $A_{\sigma(\mathbf{a})}(T)$. Suppose $\boxed{b_1}\boxed{b_2}\cdots\boxed{b_{\mu_1}}$ is the first row of T. Let b'_i be the entry in the second row of $A_{\sigma(\mathbf{a})}(T)$ corresponding to b_i in T.

Each ascent in T is present in $A_{\sigma(\mathbf{a})}(T)$. Additional ascents may be present between the first two rows of $A_{\sigma(\mathbf{a})}(T)$ when $b'_i < a_{\sigma(i)}$ for $1 \leq i \leq \mu_1$. By extending Theorem 3.5.2, $b'_i < a_{\sigma(i)}$ if and only if $b_i < a_{\sigma(i)} - (\sigma(i) - 1)$. When this occurs, the ascent contributes $l(c_i)$, the length of the leg of the cell in which $a_{\sigma(i)}$ is located, to $\mathrm{comaj}_\nu(A_{\sigma(\mathbf{a})}(T))$. Thus

$$\mathrm{comaj}_\nu(A_{\sigma(\mathbf{a})}(T)) = \mathrm{comaj}_\mu(T) + \sum_{i=1}^{\mu_1} l(c_i)\chi(b_i < a_{\sigma(i)} - (\sigma(i)-1)).$$

The location and number of inversions in the first row of T are the same as the locations and number of the inversions in σ. The inversions in T which do not involve the first row are also preserved in $A_{\sigma(\mathbf{a})}(T)$. Looking at a triple in the first two rows of $A_{\sigma(\mathbf{a})}(T)$

$$\begin{array}{|c|}\hline b'_i \\ \hline a'_i \\ \hline \end{array} \qquad\qquad \begin{array}{|c|}\hline b'_j \\ \hline \end{array}$$

where $a'_i = a_{\sigma(i)}$, if $b_i > b_j$ in T and $b'_i > b'_j > a_{\sigma(i)}$ or $a_{\sigma(i)} > b'_i > b'_j$ in $A_{\sigma(\mathbf{a})}(T)$, the inversion is present in both T and $A_{\sigma(\mathbf{a})}(T)$, but if $b'_i > a_{\sigma(i)} > b'_j$ an inversion is lost when passing from T to $A_{\sigma(\mathbf{a})}(T)$. Similarly, if $b_i < b_j$ in T and $b'_i < b'_j < a_{\sigma(i)}$ or $a_{\sigma(i)} < b'_i < b'_j$ in $A_{\sigma(\mathbf{a})}(T)$, the inversion status is preserved when passing from T to $A_{\sigma(\mathbf{a})}(T)$, but if $b'_i < a_{\sigma(i)} < b'_j$ a new inversion is formed in $A_{\sigma(\mathbf{a})}(T)$. Using Theorem 3.5.2, we get

$$\begin{aligned}\mathrm{inv}_\nu(A_{\sigma(\mathbf{a})}(T)) &= \mathrm{inv}_\mu(T) + \mathrm{inv}(\sigma) + \sum_{i=1}^{\mu_1-1}\sum_{j=i+1}^{\mu_1} (\chi(b_i < a_{\sigma(i)} - (\sigma(i)-1) \leq b_j) \\ &\quad - \chi(b_j < a_{\sigma(i)} - (\sigma(i)-1) \leq b_i)). \quad \square\end{aligned}$$

Example 3.6.3. Let $\mu = (4,3,1)$, $\mathbf{a} = \langle 1,3,4,5,8 \rangle$, $\sigma = 31254 \in S_5$ and T as shown below

$$\begin{array}{|c|c|c|c|}\cline{1-1} 1 \\ \cline{1-3} 3 & 7 & 6 \\ \hline 8 & 5 & 2 & 4 \\ \hline \end{array}\ .$$

Then

$$A_{\sigma(\mathbf{a})}(T) = \begin{array}{|c|c|c|c|c|}\cline{1-1} 2 \\ \cline{1-3} 7 & 12 & 11 \\ \cline{1-4} 13 & 10 & 6 & 9 \\ \hline 4 & 1 & 3 & 8 & 5 \\ \hline \end{array}$$

with $\mathrm{inv}_{(5,4,3,1)}(A_{\sigma(\mathbf{a})}(T)) = 9 = \mathrm{inv}_{(4,3,1)}(T) + 3$ and

$\mathrm{comaj}_{(5,4,3,1)}(A_{\sigma(\mathbf{a})}(T)) = 3 = \mathrm{comaj}_{(4,3,1)}(T)$.

Chapter 4

Recursions and Divisibility Properties

In this chapter, we examine a divisibility property of F_μ as a first step toward a bijective proof of Theorem 2.1.4. In order to prove the divisibility property combinatorially, we define a number of recursions which are based on removing the first row of a filling.

4.1 Divisibility Properties

Theorem 4.1.1. *Let $\mu = (1^{n_1} 2^{n_2} \cdots k^{n_k})$ be an integer partition with $\mu' = (1^{m_1} 2^{m_2} \cdots p^{m_p})$.*

(a) $\widetilde{F}_\mu(q,t)$ and $F_\mu(q,t)$ are divisible by $[m_1]!_q [m_2]!_q \cdots [m_p]!_q$.

(b) $\widetilde{F}_\mu(q,t)$ and $F_\mu(q,t)$ are divisible by $[n_1]!_t [n_2]!_t \cdots [n_k]!_t$.

Proof of (a). Begin by considering a filling T of shape $\mu = (m^n)$. For $1 \leq i \leq m$, let k_i be the ith smallest entry in the bottom row of T, and let $c_1, c_2, \ldots, c_m$ be the cells in the bottom row of μ. If k_m is in cell c_i, applying $s_{m-1} \circ s_{m-2} \circ \cdots \circ s_i$ to T will move k_m to cell c_m. Continue likewise to move each k_i into cell c_i in decreasing order. The sequence of inversion flips used to sort the k_i's gives rise to a permutation $w \in S_m$, by replacing each s_j in the sequence by the basic transposition $(j, j+1)$. Let T' be the filling that results from sorting T in this way. By Theorem 3.1.4(c), $\mathrm{inv}_\mu(T) = \mathrm{inv}_\mu(T') + \mathrm{inv}(w)$ and $\mathrm{comaj}_\mu(T) = \mathrm{comaj}_\mu(T')$. Hence,

$$\begin{aligned}
F_\mu(q,t) &= \sum_{T \in \mathcal{F}_\mu} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{comaj}_\mu(T)} \\
&= \sum_{\substack{T' \in \mathcal{F}_\mu \text{ with} \\ \text{increasing row 1}}} \sum_{w \in S_m} q^{\mathrm{inv}_\mu(T') + \mathrm{inv}(w)} t^{\mathrm{comaj}_\mu(T')} \\
&= \left(\sum_{w \in S_m} q^{\mathrm{inv}(w)} \right) \left(\sum_{\substack{T' \in \mathcal{F}_\mu \text{ with} \\ \text{increasing row 1}}} q^{\mathrm{inv}_\mu(T')} t^{\mathrm{comaj}_\mu(T')} \right) \\
&= [m]!_q \left(\sum_{\substack{T' \in \mathcal{F}_\mu \text{ with} \\ \text{increasing row 1}}} q^{\mathrm{inv}_\mu(T')} t^{\mathrm{comaj}_\mu(T')} \right).
\end{aligned}$$

This result can be extended to $\mu \in \mathrm{Par}(n)$ with $\mu' = (1^{m_1} 2^{m_2} \cdots p^{m_p})$ by applying the above argument to each set of m_i columns of equal height. Thus $F_\mu(q,t)$ is divisible by $[m_1]!_q \cdots [m_p]!_q$. It follows that $\widetilde{F}_\mu(q,t) = t^{n(\mu)} F_\mu(q, 1/t)$ is divisible by the same factors. $\square$

Part (b) of Theorem 4.1.1 follows from (a) and the fact that $\widetilde{H}_\mu(X;q,t) = \widetilde{H}_{\mu'}(X;t,q)$ by [14] and hence $\widetilde{F}_\mu(q,t) = \widetilde{F}_{\mu'}(t,q)$. This is not a bijective proof of part (b) and offers no insight into what is happening with the fillings. The following recursions have each been developed in an effort to prove part (b) of Theorem 4.1.1 bijectively.

4.2 Recursions for Two-Column Rectangles

In this section, we consider what happens when we fix the first row of the fillings of (2^n) to be $\boxed{1}\boxed{a}$. This leads to a recursion characterizing the polynomials $F_{(2^n)}(q,t)$.

Definition 4.2.1. For all $\mu, \mathbf{a}$, and σ with $\mu \in \mathrm{Par}(n)$, $\mathbf{a} = \langle a_1, \ldots, a_{\mu_1}\rangle$ where $1 = a_1 < a_2 < \cdots < a_{\mu_1} \leq n$ and $\sigma \in S_{\mu_1}$ define

$$R_{\mu,\sigma(\mathbf{a})}(q,t) = \sum_{T \in \mathcal{F}_{\mu,\sigma(\mathbf{a})}} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{comaj}_\mu(T)}.$$

When μ has only two columns we write

$$R_{\mu,a} = \sum_{T \in \mathcal{F}_{\mu,a}} q^{\mathrm{inv}_\mu(T)} t^{\mathrm{comaj}_\mu(T)}.$$

Before we show how to calculate $R_{(2^n),a}$, we first look at an important property of the recursion we call *folding*.

Theorem 4.2.2. *For all $n \geq 1$ and $2 \leq a \leq 2n$, $R_{(2^n),2n-a+2} = t^{n-a+1} R_{(2^n),a}$.*

Proof. It suffices to prove the result for $2 \leq a \leq n$. Note that $s_1 \circ \mathrm{cyc}^{2n-a+1}$ is a bijection from $\mathcal{F}_{(2^n),a}$ onto $\mathcal{F}_{(2^n),2n-a+2}$. For $T \in \mathcal{F}_{(2^n),a}$, Theorems 3.2.3 and 3.3.2 give

$$\mathrm{comaj}_{(2^n)}(s_1(\mathrm{cyc}^{2n-a+1}(T))) = \mathrm{comaj}_{(2^n)}(T) + 2n - a - (n-1) = \mathrm{comaj}_{(2^n)}(T) + n - a + 1;$$

$$\mathrm{inv}_{(2^n)}(s_1(\mathrm{cyc}^{2n-a+1}(T))) = \mathrm{inv}_{(2^n)}(T). \qquad \square$$

We can compute $R_{(2^n),a}$ for $2 \leq a \leq n+1$ by the recursion below.

Theorem 4.2.3. *For all $n \geq 2$ and $2 \leq a \leq n+1$,*

$$R_{(2^n),a} = [n-1]_t \Bigg(\sum_{b=2}^{n-1} (t^{(a-b-1)^+} + t^{n-b} + qt^{a-2} + qt^{n-1-(b+1-a)^+}) R_{(2^{n-1}),b} + (1+qt^{a-2}) R_{(2^{n-1}),n} \Bigg)$$

where $R_{(2),2} = 1$ and $x^+ = \max\{x, 0\}$.

Proof. To prove the theorem bijectively, we will decompose $\mathcal{F}_{(2^n),a}$ into a collection of disjoint sets $X_2, \ldots, X_n, Y_2, \ldots, Y_n$, such that the generating function for X_b is $[n-1]_t(t^{(a-b-1)^+} + t^{n-b})R_{(2^{n-1}),b}$ when $2 \leq b < n$, and $[n-1]_t R_{(2^{n-1}),n}$ when $b = n$. Similarly, the generating function for Y_b is $[n-1]_t(qt^{a-2} + qt^{n-1-(b+1-a)^+})R_{(2^{n-1}),b}$ when $2 \leq b < n$, and $[n-1]_t qt^{a-2} R_{(2^{n-1}),n}$ when $b = n$.

For $2 \leq b \leq n$, define

$$X_b = \{T \in \mathcal{F}_{(2^n),a} : \text{ for some } U \in \mathcal{F}_{(2^{n-1}),b}, T = \overline{\mathrm{cyc}}^i(A_a(U)) \text{ where } 0 \leq i \leq 2n-2-b \text{ or } T = \overline{\mathrm{cyc}}^j(\overline{s}_1(A_a(U))) \text{ for } 2n-1-b \leq j \leq 2n-3\}$$

and

$$Y_b = \{T \in \mathcal{F}_{(2^n),a} : \text{ for some } U \in \mathcal{F}_{(2^{n-1}),b}, T = \overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U))) \text{ where } 0 \leq i \leq 2n-2-b \text{ or } T = \overline{\mathrm{cyc}}^j(A_a(U)) \text{ for } 2n-1-b \leq j \leq 2n-3\}.$$

One sees, using Theorems 3.1.2, 3.2.2, and 3.3.1, that each $T \in \mathcal{F}_{(2^n),a}$ has the form $\overline{\mathrm{cyc}}^j(\overline{s}_1^{\epsilon}(A_a(U)))$ for exactly one choice of $j \in \{0, 1, \ldots, 2n-3\}$, $\epsilon \in \{0, 1\}$, $b \in \{2, \ldots, n\}$, and $U \in \mathcal{F}_{(2^{n-1}),b}$. It follows that $\mathcal{F}_{(2^n),a}$ is the disjoint union of the sets X_b and Y_b just defined.

By Theorems 3.3.2 and 3.4.5, $\mathrm{inv}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U))) = \mathrm{inv}_{(2^n)}(\overline{\mathrm{cyc}}^j(\overline{s}_1(A_a(U)))) = \mathrm{inv}_{(2^{n-1})}(U)$ for all $0 \leq i \leq 2n-2-b$ and all $2n-1-b \leq j \leq 2n-3$. Similarly, $\mathrm{inv}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U)))) = \mathrm{inv}_{(2^n)}(\overline{\mathrm{cyc}}^j(A_a(U))) = \mathrm{inv}_{(2^{n-1})}(U)+1$ for all i, j in the indicated ranges. Before continuing the proof, we give an example that illustrates the main ideas in the calculations that follow.

Example 4.2.4. For $n = 4$, $a = 4$, and $b = 3$, consider

$$U = \begin{array}{|c|c|}\hline 2 & 6 \\ \hline 5 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \quad \text{and } A_4(U) = \begin{array}{|c|c|}\hline 3 & 8 \\ \hline 7 & 6 \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline \end{array}.$$

Notice that $q^{\mathrm{inv}_{(2^3)}(U)} t^{\mathrm{comaj}_{(2^3)}(U)} = q^2t^1$. In Figure 4.1, the fillings in bold are elements of X_3 derived from U, while the rest are elements of Y_3 derived from U. The generating functions for these elements in X_3 and Y_3 are $[3]_t(1+t)q^2t^1$ and $[3]_t(qt^2+qt^3)q^2t^1$, respectively.

$$\begin{array}{ccccccccccc}
\begin{array}{|c|c|}\hline \mathbf{3} & \mathbf{8} \\ \hline \mathbf{7} & \mathbf{6} \\ \hline \mathbf{2} & \mathbf{5} \\ \hline \mathbf{1} & \mathbf{4} \\ \hline \end{array} & \xrightarrow{\overline{\mathrm{cyc}}} &
\begin{array}{|c|c|}\hline \mathbf{5} & \mathbf{2} \\ \hline \mathbf{8} & \mathbf{7} \\ \hline \mathbf{3} & \mathbf{6} \\ \hline \mathbf{1} & \mathbf{4} \\ \hline \end{array} & \xrightarrow{\overline{\mathrm{cyc}}} &
\begin{array}{|c|c|}\hline \mathbf{6} & \mathbf{3} \\ \hline \mathbf{2} & \mathbf{8} \\ \hline \mathbf{5} & \mathbf{7} \\ \hline \mathbf{1} & \mathbf{4} \\ \hline \end{array} & \xrightarrow{\overline{\mathrm{cyc}}} &
\begin{array}{|c|c|}\hline \mathbf{7} & \mathbf{5} \\ \hline \mathbf{3} & \mathbf{2} \\ \hline \mathbf{6} & \mathbf{8} \\ \hline \mathbf{1} & \mathbf{4} \\ \hline \end{array} & \xrightarrow{\overline{\mathrm{cyc}}} &
\begin{array}{|c|c|}\hline 8 & 6 \\ \hline 5 & 3 \\ \hline 7 & 2 \\ \hline 1 & 4 \\ \hline \end{array} & \xrightarrow{\overline{\mathrm{cyc}}} &
\begin{array}{|c|c|}\hline 2 & 7 \\ \hline 6 & 5 \\ \hline 8 & 3 \\ \hline 1 & 4 \\ \hline \end{array} \\
q^2t & & q^2t^2 & & q^2t^3 & & q^2t^4 & & q^3t^5 & & q^3t^6 \\
\updownarrow \overline{s}_1 & & \updownarrow \overline{s}_1 & & \updownarrow \overline{s}_1 & & \updownarrow \overline{s}_1 & & \updownarrow \overline{s}_1 & & \updownarrow \overline{s}_1 \\
\begin{array}{|c|c|}\hline 3 & 8 \\ \hline 7 & 6 \\ \hline 5 & 2 \\ \hline 1 & 4 \\ \hline \end{array} & \xrightarrow{\overline{\mathrm{cyc}}} &
\begin{array}{|c|c|}\hline 5 & 2 \\ \hline 8 & 7 \\ \hline 6 & 3 \\ \hline 1 & 4 \\ \hline \end{array} & \xrightarrow{\overline{\mathrm{cyc}}} &
\begin{array}{|c|c|}\hline 6 & 3 \\ \hline 2 & 8 \\ \hline 7 & 5 \\ \hline 1 & 4 \\ \hline \end{array} & \xrightarrow{\overline{\mathrm{cyc}}} &
\begin{array}{|c|c|}\hline 7 & 5 \\ \hline 3 & 2 \\ \hline 8 & 6 \\ \hline 1 & 4 \\ \hline \end{array} & \xrightarrow{\overline{\mathrm{cyc}}} &
\begin{array}{|c|c|}\hline \mathbf{8} & \mathbf{6} \\ \hline \mathbf{5} & \mathbf{3} \\ \hline \mathbf{2} & \mathbf{7} \\ \hline \mathbf{1} & \mathbf{4} \\ \hline \end{array} & \xrightarrow{\overline{\mathrm{cyc}}} &
\begin{array}{|c|c|}\hline \mathbf{2} & \mathbf{7} \\ \hline \mathbf{6} & \mathbf{5} \\ \hline \mathbf{3} & \mathbf{8} \\ \hline \mathbf{1} & \mathbf{4} \\ \hline \end{array} \\
q^3t^4 & & q^3t^5 & & q^3t^3 & & q^3t^4 & & q^2t^2 & & q^2t^3
\end{array}$$

Figure 4.1: The elements of X_3 and Y_3 generated by U.

Returning to the proof, we first consider the case where $2 \leq b < n$ and $a \leq b+1$. Let $U \in \mathcal{F}_{(2^{n-1}),b}$. By Theorem 3.4.5, $\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U))) = \mathrm{comaj}_{(2^{n-1})}(U) + i$ for $0 \leq i \leq$

$2n-2-b$, so

$$\sum_{i=0}^{n-2} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U)))} = [n-1]_t t^{\mathrm{comaj}_{(2^{n-1})}(U)} = [n-1]_t t^{(a-b-1)^+} t^{\mathrm{comaj}_{(2^{n-1})}(U)} .$$

For $2n-1-b \le i \le 2n-3$, $\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U)))) = \mathrm{comaj}_{(2^{n-1})}(U) + i - (n-1)$ by Theorems 3.4.5 and 3.3.2, since $2n-b > a$. Then

$$\begin{aligned}
&\sum_{i=n-1}^{2n-2-b} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U)))} + \sum_{i=2n-1-b}^{2n-3} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U))))} \\
&\quad = t^{\mathrm{comaj}_{(2^{n-1})}(U)} \left(\sum_{i=n-1}^{2n-2-b} t^i + \sum_{i=2n-1-b}^{2n-3} t^{i-(n-1)} \right) = t^{\mathrm{comaj}_{(2^{n-1})}(U)} \left(\sum_{i=n-1}^{2n-2-b} t^i + \sum_{i=n-b}^{n-2} t^i \right) \\
&\quad = [n-1]_t t^{n-b+\mathrm{comaj}_{(2^{n-1})}(U)} .
\end{aligned}$$

Thus, the generating function for X_b is $[n-1]_t(t^{(a-b-1)^+} + t^{n-b})R_{(2^{n-1}),b}$. Similarly, we can use Theorems 3.4.5 and 3.3.2 to obtain:

$$\begin{aligned}
&\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U)))) = \mathrm{comaj}_{(2^{n-1})}(U) + i + n - 1 && \text{if } 0 \le i < a-2; \\
&\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U)))) = \mathrm{comaj}_{(2^{n-1})}(U) + i && \text{if } a-2 \le i \le 2n-2-b; \\
&\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U))) = \mathrm{comaj}_{(2^{n-1})}(U) + i && \text{if } 2n-1-b \le i < 2n+a-b-3; \\
&\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U))) = \mathrm{comaj}_{(2^{n-1})}(U) + i - (n-1) && \text{if } a-2+(2n-1-b) \le i \le 2n-3.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{i=0}^{n-2} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U))))} &= \sum_{i=0}^{a-3} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i+n-1} + \sum_{i=a-2}^{n-2} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} \\
&= \sum_{i=a-2}^{a+n-4} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} = [n-1]_t t^{(a-2)+\mathrm{comaj}_{(2^{n-1})}(U)}
\end{aligned}$$

and

$$\sum_{i=n-1}^{2n-2-b} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U))))} + \sum_{i=2n-1-b}^{a+2n-b-4} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U)))} + \sum_{i=a+2n-b-3}^{2n-3} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U)))}$$

$$= \sum_{i=n-1}^{2n-2-b} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} + \sum_{i=2n-b-1}^{a+2n-b-4} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} + \sum_{i=a+2n-b-3}^{2n-3} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i-(n-1)}$$

$$= \sum_{i=n-1}^{a+2n-b-4} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} + \sum_{i=a+n-b-2}^{n-2} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i}$$

$$= [n-1]_t t^{(n-1)-(b+1-a)+\mathrm{comaj}_{(2^{n-1})}(U)}.$$

So the generating function for Y_b in this case is $[n-1]_t q(t^{a-2} + t^{n-1-(b+1-a)^+})R_{(2^{n-1}),b}$.

Next we look at what happens when $b+1 < a$, $2 \leq b < n$, and $2 \leq a \leq n+1$. Let $U \in \mathcal{F}_{(2^{n-1}),b}$. Then, by Theorems 3.4.5 and 3.3.2,

$$\begin{aligned}
&\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U))) = \mathrm{comaj}_{(2^{n-1})}(U) + i + n - 1 && \text{if } 0 \leq i < a-b-1;\\
&\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U))) = \mathrm{comaj}_{(2^{n-1})}(U) + i && \text{if } a-b-1 \leq i \leq 2n-b-2;\\
&\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U)))) = \mathrm{comaj}_{(2^{n-1})}(U) + i - (n-1) && \text{if } 2n-b-1 \leq i \leq 2n-3.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{i=0}^{n-2} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U)))} &= \sum_{i=0}^{a-b-2} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i+n-1} + \sum_{i=a-b-1}^{n-2} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i}\\
&= \sum_{i=n-1}^{n+a-b-3} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} + \sum_{i=a-b-1}^{n-2} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i}\\
&= [n-1]_t t^{(a-b-1)+\mathrm{comaj}_{(2^{n-1})}(U)}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=n-1}^{2n-b-2} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U)))} + \sum_{i=2n-b-1}^{2n-3} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U))))} \\
&= \sum_{i=n-1}^{2n-b-2} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} + \sum_{i=2n-b-1}^{2n-3} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i-(n-1)} \\
&= \sum_{i=n-1}^{2n-b-2} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} + \sum_{i=n-b}^{n-2} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} \\
&= [n-1]_t t^{n-b+\mathrm{comaj}_{(2^{n-1})}(U)} \ .
\end{aligned}$$

Thus, the generating function for X_b in this case is $[n-1]_t(t^{(a-b-1)^+} + t^{n-b})R_{(2^{n-1}),b}$.

Similarly, for Y_b, we have

$$\begin{aligned}
&\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U)))) = \mathrm{comaj}_{(2^{n-1})}(U) + i + n - 1 && \text{if } 0 \le i < a-2; \\
&\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U)))) = \mathrm{comaj}_{(2^{n-1})}(U) + i && \text{if } a-2 \le i \le 2n-2-b; \\
&\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U))) = \mathrm{comaj}_{(2^{n-1})}(U) + i && \text{if } 2n-1-b \le i \le 2n-3.
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{i=0}^{n-2} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U))))} &= \sum_{i=0}^{a-3} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i+n-1} + \sum_{i=a-2}^{n-2} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} \\
&= \sum_{i=n-1}^{a+n-4} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} + \sum_{i=a-2}^{n-2} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} \\
&= [n-1]_t t^{a-2+\mathrm{comaj}_{(2^{n-1})}(U)}
\end{aligned}$$

and

$$\begin{aligned}\sum_{i=n-1}^{2n-2-b} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U))))} + \sum_{i=2n-1-b}^{2n-3} t^{\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U)))}&\\ = \sum_{i=n-1}^{2n-2-b} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i} + \sum_{i=2n-1-b}^{2n-3} t^{\mathrm{comaj}_{(2^{n-1})}(U)+i}&\\ = [n-1]_t t^{n-1+\mathrm{comaj}_{(2^{n-1})}(U)}.&\end{aligned}$$

So the generating function for Y_b in this case is $[n-1]_t q(t^{a-2}+t^{n-1})R_{(2^{n-1}),b}$. Combining all of the above gives us that, for $2 \leq b < n$, the generating function for X_b is $[n-1]_t(t^{(a-b-1)^+} + t^{n-b})R_{(2^{n-1}),b}$ and the generating function for Y_b is $[n-1]_t q(t^{a-2}+t^{n-1-(b+1-a)^+})R_{(2^{n-1}),b}$.

When $b=n$ and $U \in \mathcal{F}_{(2^{n-1}),n}$,

$$A_a(U), \overline{\mathrm{cyc}}(A_a(U)), \ldots, \overline{\mathrm{cyc}}^{n-2}(A_a(U)) \in X_n$$

and

$$\overline{s}_1(A_a(U)), \overline{\mathrm{cyc}}(\overline{s}_1(A_a(U))), \ldots, \overline{\mathrm{cyc}}^{n-2}(\overline{s}_1(A_a(U))) \in Y_n.$$

Since $a-n-1 \leq 0$,

$$\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(A_a(U))) = \mathrm{comaj}_{(2^{n-1})}(U)+i,$$

so the generating function for X_n is $[n-1]_t R_{(2^{n-1}),n}$. To obtain the generating function for Y_n, we note that

$$\begin{aligned}\mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U)))) &= \mathrm{comaj}_{(2^{n-1})}(U)+i+n-1 && \text{if } 0 \leq i < a-2\\ \mathrm{comaj}_{(2^n)}(\overline{\mathrm{cyc}}^i(\overline{s}_1(A_a(U)))) &= \mathrm{comaj}_{(2^{n-1})}(U)+i && \text{if } a-2 \leq i \leq n-2.\end{aligned}$$

Then, as above, the generating function for Y_n is $[n-1]_t(qt^{a-2})R_{(2^{n-1}),n}$. □

We can use Theorem 4.2.2 to obtain the following more streamlined version of the recursion for two-column rectangles.

Theorem 4.2.5. *For all $n \geq 2$ and $2 \leq a \leq 2n$,*

$$R_{(2^n),a} = [n-1]_t \sum_{b=2}^{2n-2} \left(t^{(a-b-1)^+} + qt^{n-1-(b+1-a)^+} \right) R_{(2^{n-1}),b},$$

where $R_{(2),2} = 1$ and $x^+ = \max(x, 0)$.

Proof. First, we use Theorem 4.2.2 to rewrite the recursion from Theorem 4.2.3 for $2 \leq a \leq n+1$. We see that $R_{(2^n),a}$ equals

$$\begin{aligned}
& [n-1]_t \left(\sum_{b=2}^{n-1} (t^{(a-b-1)^+} + t^{n-b} + qt^{a-2} + qt^{n-1-(b+1-a)^+}) R_{(2^{n-1}),b} + (1+qt^{a-2}) R_{(2^{n-1}),n} \right) \\
= \; & [n-1]_t \left(\sum_{b=2}^{n-1} (t^{(a-b-1)^+} + qt^{n-1-(b+1-a)^+}) R_{(2^{n-1}),b} + \sum_{b=2}^{n-1} t^{b-n} (t^{n-b} + qt^{a-2}) R_{(2^{n-1}),2n-b} \right. \\
& \left. \quad + (1+qt^{a-2}) R_{(2^{n-1}),n} \right) \\
= \; & [n-1]_t \left(\sum_{b=2}^{n} (t^{(a-b-1)^+} + qt^{n-1-(b+1-a)^+}) R_{(2^{n-1}),b} + \sum_{b=2}^{n-1} (1+qt^{a+b-n-2}) R_{(2^{n-1}),2n-b} \right) \\
= \; & [n-1]_t \left(\sum_{b=2}^{n} (t^{(a-b-1)^+} + qt^{n-1-(b+1-a)^+}) R_{(2^{n-1}),b} + \sum_{b=n+1}^{2n-2} (1+qt^{n-1-(b+1-a)}) R_{(2^{n-1}),b} \right) \\
= \; & [n-1]_t \sum_{b=2}^{2n-2} (t^{(a-b-1)^+} + qt^{n-1-(b+1-a)^+}) R_{(2^{n-1}),b}.
\end{aligned}$$

To see that this same recursion works for $n+2 \leq a \leq 2n$, apply Theorem 4.2.2 again. For

a with $n+2 \leq a \leq 2n$, note that $2 \leq 2n-a+2 \leq n+1$. Then

$$\begin{aligned} R_{(2^n),a} &= t^{a-n-1} R_{(2^n),2n-a+2} \\ &= t^{a-n-1}[n-1]_t \sum_{b=2}^{2n-2} (t^{(2n-a+2-b-1)^+} + qt^{n-1-(b+1-(2n-a+2))^+}) R_{(2^{n-1}),b} \\ &= [n-1]_t \sum_{b=2}^{2n-2} (t^{a-n-1+(2n-a+2-b-1)^+} + qt^{a-2-(b+a-2n-1)^+}) R_{(2^{n-1}),b}. \end{aligned}$$

One final application of Theorem 4.2.2, this time to $R_{(2^{n-1}),b}$ for $2 \leq b \leq 2n-2$, leads to

$$\begin{aligned} R_{(2^n),a} &= [n-1]_t t^{n-b} \sum_{b=2}^{2n-2} (t^{a-n-1+(2n-a+1-(2n-b))^+} + qt^{a-2-(2n-b+a-2n-1)^+}) R_{(2^{n-1}),b} \\ &= [n-1]_t \sum_{b=2}^{2n-2} (t^{a-b-1+(b+1-a)^+} + qt^{n-b+a-2-(a-b-1)^+}) R_{(2^{n-1}),b} \\ &= [n-1]_t \sum_{b=2}^{2n-2} (t^{(a-b-1)^+} + qt^{n-1-(b+1-a)^+}) R_{(2^{n-1}),b}. \qquad \square \end{aligned}$$

Theorem 4.2.6. *For* $n \geq 1$, $F_{(2^n)}(q,t) = R_{(2^{n+1}),2}(q,t)$.

Proof. Note A_2 is a bijection from $\mathcal{F}_{(2^n)}$ onto $\mathcal{F}_{(2^{n+1}),2}$. By Theorem 3.4.3, for $U \in \mathcal{F}_{(2^n)}$,

$$\mathrm{comaj}_{(2^{n+1})}(A_2(U)) = \mathrm{comaj}_{(2^n)}(U) \text{ and } \mathrm{inv}_{(2^{n+1})}(A_2(U)) = \mathrm{inv}_{(2^n)}(U). \qquad \square$$

Remark 4.2.7. A different recursion for $\widetilde{F}_\mu$ when μ has two columns was found recently by Garsia and Haglund [4]. Their recursion was based on removing inner corners of μ to obtain a smaller shape ν. They proved the recursion using representation-theoretical techniques rather than combinatorially.

4.3 Recursion for Rectangles

Theorem 4.3.1. *For all* $m \geq 1, n \geq 1, \mathbf{a}$ *where* $\mathbf{a} = \langle a_1, a_2, \ldots, a_m\rangle$ *with* $1 = a_1 < a_2 < \cdots < a_m \leq mn$,

$$R_{(m^n),\mathbf{a}} = \sum_{\substack{\mathbf{b}=\langle b_1,\ldots,b_m\rangle \text{ with} \\ 1=b_1<\cdots<b_m\leq m(n-1)}} R_{(m^{n-1}),\mathbf{b}} \sum_{\sigma\in S_m} \sum_{i=0}^{m(n-1)-b_m} q^{\mathrm{inv}(\sigma)+A} t^{i+B}$$

where

$$\begin{aligned} A &= \sum_{j=2}^{m-1}\sum_{k=j+1}^{m} (\chi(i < a_j - b_{\sigma(j)} - (j-1))\chi(i > a_j - b_{\sigma(k)} - j) \\ &\qquad -\chi(i < a_j - b_{\sigma(k)} - (j-1))\chi(i > a_j - b_{\sigma(j)} - j)) \text{ and} \\ B &= (n-1)\sum_{j=2}^{m} \chi(i < a_j - b_{\sigma(j)} - (j-1)) \end{aligned}$$

with initial condition $R_{(m),\langle 1,2,\ldots,m\rangle} = 1$.

Proof. Let $T \in \mathcal{F}_{(m^n),\mathbf{a}}$ where $\mathbf{a} = \langle a_1, a_2, \ldots, a_m\rangle$ with $1 = a_1 < a_2 < \cdots < a_m \leq mn$. There exist unique $U, i, \sigma, \mathbf{b}$ where $U \in \mathcal{F}_{(m^{n-1})}$, $\mathbf{b} = \langle b_1, \ldots, b_m\rangle$ with $1 = b_1 < b_2 < \cdots < b_m \leq m(n-1)$, $\sigma \in S_m$, and $0 \leq i \leq m(n-1) - b_m$ such that $T = A_{\mathbf{a}}(\mathrm{cyc}^i(U))$ and the first row of U is $\boxed{b_{\sigma(1)}}\boxed{b_{\sigma(2)}}\boxed{\ldots}\boxed{b_{\sigma(m)}}$. Note that $U = (\mathrm{cyc}^{-1})^i(A_{\mathbf{a}}^{-1}(T))$ where i is the number of times $A_{\mathbf{a}}^{-1}(T)$ must be shifted down to get 1 in the first row. By Theorem 3.5.5,

$$\begin{aligned}
\mathrm{inv}_{(m^n)}(T) &= \mathrm{inv}_{(m^{n-1})}(U) + \sum_{j=2}^{m-1} \sum_{k=j+1}^{m} (\chi(b_{\sigma(j)} + i < a_j - (j-1) \leq b_{\sigma(k)} + i) \\
&\qquad - \chi(b_{\sigma(k)} + i < a_j - (j-1) \leq b_{\sigma(j)} + i) \\
&= \mathrm{inv}_{(m^{n-1})}(U) + \sum_{j=2}^{m-1} \sum_{k=j+1}^{m} (\chi(i < a_j - b_{\sigma(j)} - (j-1))\chi(i > a_j - b_{\sigma(k)} - j)) \\
&\qquad - \chi(i < a_j - b_{\sigma(k)} - (j-1))\chi(i > a_j - b_{\sigma(j)} - j))
\end{aligned}$$

and

$$\mathrm{comaj}_{(m^n)}(T) = \mathrm{comaj}_{(m^{n-1})}(U) + i + (n-1) \sum_{j=2}^{m} \chi(i < a_j - b_{\sigma(j)} - (j-1)).$$

Using the inversion flip, the first row of U can be sorted to obtain a filling $U' \in \mathcal{F}_{(m^{n-1}),\mathbf{b}}$. Note that $\mathrm{inv}_{(m^{n-1})}(U) = \mathrm{inv}_{(m^{n-1})}(U') + \mathrm{inv}(\sigma)$ and $\mathrm{comaj}_{(m^{n-1})}(U) = \mathrm{comaj}_{(m^{n-1})}(U')$. $\square$

Example 4.3.2. Let $T \in \mathcal{F}_{(4^4),\langle 1,3,5,10\rangle}$ as shown:

$$\begin{array}{|c|c|c|c|}\hline 8 & 2 & 6 & 13 \\ \hline 12 & 15 & 4 & 7 \\ \hline 16 & 9 & 11 & 14 \\ \hline 1 & 3 & 5 & 10 \\ \hline \end{array}.$$

To obtain U, i, σ, and $\mathbf{b}$, first find $A_{\mathbf{a}}^{-1}(T)$ and then cyclic shift down i times until 1 is in the first row:

$$A^{-1}_{\langle 1,3,5,10\rangle}(T) = \begin{array}{|c|c|c|c|}\hline 5 & 1 & 3 & 9 \\ \hline 8 & 11 & 2 & 4 \\ \hline 12 & 6 & 7 & 10 \\ \hline \end{array}.$$

Here $i = 5$ and

$$U = (\mathrm{cyc}^{-1})^5(A^{-1}_{\langle 1,3,5,10\rangle}(T)) = \begin{array}{|c|c|c|c|}\hline 12 & 8 & 10 & 4 \\ \hline 3 & 6 & 9 & 11 \\ \hline 7 & 1 & 2 & 5 \\ \hline \end{array}.$$

The first row of this filling is $\mathbf{b} = \langle 1,2,5,7\rangle$ and $\sigma = 4123$.

As with the two-column case, there is a folding theorem for $R_{(m^n),\mathbf{a}}$.

Theorem 4.3.3. *For all $m, n, k, \mathbf{a}$, where $m, n, k \in \mathbb{N}$, $1 \leq k \leq m-1$ and $\mathbf{a} = \langle a_1, \ldots, a_m \rangle$ with $1 = a_1 < a_2 < \cdots < a_m \leq mn$, let $\mathbf{a}^{(k)} = \mathrm{cyc}^{mn - a_{m-(k-1)}+1}(\mathbf{a})$ with the entries sorted in increasing order. Then*

$$R_{(m^n),\mathbf{a}^{(k)}} = t^{(m-k)n - a_{m-(k-1)}+1} R_{(m^n),\mathbf{a}}.$$

Proof. Let $T \in \mathcal{F}_{(m^n),\mathbf{a}}$. Note that $\mathrm{cliff}^{mn-a_{m-(k-1)}+1}$ is a bijection from $\mathcal{F}_{(m^n),\mathbf{a}}$ to $\mathcal{F}_{(m^n),\mathbf{a}^{(k)}}$. By Theorem 3.3.9,

$$\begin{aligned}
\mathrm{inv}_{(m^n)}(\mathrm{cliff}^{mn-a_{m-(k-1)}+1}(T)) &= \mathrm{inv}_{(m^n)}(T) \text{ and} \\
\mathrm{comaj}_{(m^n)}(\mathrm{cliff}^{mn-a_{m-(k-1)}+1}(T)) &= \mathrm{comaj}_{(m^n)}(T) + mn - a_{m-(k-1)} + 1 \\
&\quad -n \sum_{j=1}^{m} \chi(a_j + mn - a_{m-(k-1)} + 1 > mn) \\
&= \mathrm{comaj}_{(m^n)}(T) + mn - a_{m-(k-1)} + 1 - n(k) \\
&= \mathrm{comaj}_{(m^n)}(T) + (m-k)n - a_{m-(k-1)} + 1. \quad \square
\end{aligned}$$

Theorem 4.3.3 allows us to rewrite Theorem 4.3.1 as

$$\begin{aligned}
R_{(m^n),\mathbf{a}} &= \sum_{\substack{\mathbf{b}=\langle b_1,\ldots,b_m\rangle \text{ with} \\ 1=b_1<\cdots<b_m \text{ and} \\ b_i \leq (i-1)(n-1)+1 \text{ when } i<m \\ \text{and } b_m \leq (m-1)(n-1)}} \sum_{i=0}^{m(n-1)-b_m} \sum_{k=1}^{m} \sum_{\sigma \in S_m} R_{(m^{n-1}),\mathbf{b}} q^{\mathrm{inv}(\sigma)+A} t^{i+B} \\
&\quad + R_{(m^{n-1}),\mathbf{c}} \sum_{i=0}^{m(n-1)-b_m} \sum_{\sigma \in S_m} q^{\mathrm{inv}(\sigma)+A'} t^{i+B'} \qquad (4.1)
\end{aligned}$$

where

$$
\begin{aligned}
\mathbf{c} &= \langle c_1, \ldots, c_m\rangle \text{ where } c_i = (i-1)(n-1)+1 \text{ for } i < m \text{ and } c_m = (m-1)(n-1), \\
A &= \sum_{j=2}^{m-1} \sum_{p=j+1}^{m} (\chi(i < a_j - b^{(k)}_{\sigma(j)} - (j-1))\chi(i > a_j - b^{(k)}_{\sigma(p)} - j) \\
&\qquad\qquad -\chi(i < a_j - b^{(k)}_{\sigma(p)} - (j-1))\chi(i > a_j - b^{(k)}_{\sigma(j)} - j)), \\
B &= (m-k)(n-1) - b_{m-(k-1)} + 1 + (n-1)\sum_{j=2}^{m} \chi(i < a_j - b^{(k)}_{\sigma(j)} - (j-1)), \\
A' &= \sum_{j=2}^{m-1} \sum_{p=j+1}^{m} (\chi(i < a_j - c_{\sigma(j)} - (j-1))\chi(i > a_j - c_{\sigma(p)} - j) \\
&\qquad\qquad -\chi(i < a_j - c_{\sigma(p)} - (j-1))\chi(i > a_j - c_{\sigma(j)} - j)), \text{ and} \\
B' &= (n-1)\sum_{j=2}^{m} \chi(i < a_j - c_{\sigma(j)} - (j-1))
\end{aligned}
$$

with the j^{th} entry of $\mathbf{b}^{(k)}$ denoted $b^{(k)}_j$.

4.3.1 Three Column Recursion

When $m = 3$ the preceding result can be algebraically simplified to show the factor of $[n-1]_t$.

Theorem 4.3.4. *For $\mu = (3^n)$ and $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ with $1 = a_1 < a_2 < a_3 \leq 3n$,*

$$R_{(3^n),\mathbf{a}} = [n-1]_t \sum_{\substack{\mathbf{b}=\langle b_1,b_2,b_3\rangle \text{ with} \\ 1=b_1<b_2<b_3\leq 3(n-1)}} R_{(3^{n-1}),\mathbf{b}}(t^{A_1} + qt^{A_2} + qt^{A_3} + q^2t^{A_4} + q^2t^{A_5} + q^3t^{A_6})$$

where

$$\begin{aligned}
A_1 &= (a_2-b_2-1)^+ + (a_3-b_3-2-(a_2-b_2-1)^+)^+,\\
A_2 &= (a_3-b_2-2)^+ - (a_3-b_2-2-(3n-2-b_3)-(a_2-2-(3n-2-b_3))^+)^+,\\
A_3 &= a_2-2+(a_3-b_3-2-(a_2-2))^+ - (a_2-2-(3n-2-b_3))^+\\
&\quad +(a_2-b_2-1-(3n-2-b_3))^+,\\
A_4 &= a_3-3+(a_3-b_2-2-(3n-2-b_3))^+ - (a_3-3-(3n-2-b_3))^+\\
&\quad -(a_3-b_2-2-(3n-2-b_3)-(a_2-2))^+,\\
A_5 &= a_2-2+(a_3-b_2-2-(a_2-2))^+ - (a_3-b_2-2-(3n-2-b_3))^+\\
&\quad -(a_2-2-(3n-2-b_2))^+ + (a_2-2-(3n-2-b_3))^+ - (a_2-2-(3n-2-b_3)\\
&\quad -(a_3-b_2-2-(3n-2-b_3))^+)^+, \text{ and}\\
A_6 &= a_3-3-(a_3-3-(3n-2-b_3)-(a_2-2))^+ - (a_2-(b_3-b_2+1)-1)^+\\
&\quad +(a_2-(b_3-b_2+1)-1-(a_3-3-(3n-2-b_3))^+)^+,
\end{aligned}$$

with initial condition $R_{(3),\langle 1,2,3\rangle} = 1$.

Proof. We begin by stating (4.1) for $m=3$.

$$\begin{aligned}
R_{(3^n),\mathbf{a}} &= \sum_{\substack{\mathbf{b}=\langle b_1,b_2,b_3\rangle \text{ where}\\ b_1=1,b_2\le n,b_3\le 2(n-1)}} \sum_{i=0}^{3(n-1)-b_3} \sum_{k=1}^{3} \sum_{\sigma\in S_3} R_{(3^{n-1}),\mathbf{b}} q^{\mathrm{inv}(\sigma)+A} t^{i+B}\\
&\quad + R_{(3^{n-1}),\mathbf{c}} \sum_{i=0}^{n-2} \sum_{\sigma\in S_3} q^{\mathrm{inv}(\sigma)+A'} t^{i+B'} \qquad (4.2)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{c} &= \langle 1, n, 2n-1 \rangle, \\
A &= \chi(i < a_2 - b^{(k)}_{\sigma(2)} - 1)\chi(i > a_2 - b^{(k)}_{\sigma(3)} - 2) \\
&\quad -\chi(i < a_2 - b^{(k)}_{\sigma(3)} - 1)\chi(i > a_2 - b^{(k)}_{\sigma(2)} - 2), \\
B &= (3-k)(n-1) - b_{3-(k-1)} + 1 \\
&\quad +(n-1)(\chi(i < a_2 - b^{(k)}_{\sigma(2)} - 1) + \chi(i < a_3 - b^{(k)}_{\sigma(3)} - 2)), \\
A' &= \chi(i < a_2 - c_{\sigma(2)} - 1)\chi(i > a_2 - c_{\sigma(3)} - 2) \\
&\quad -\chi(i < a_2 - c_{\sigma(3)} - 1)\chi(i > a_2 - c_{\sigma(2)} - 2), \text{ and} \\
B' &= (n-1)(\chi(i < a_2 - c_{\sigma(2)} - 1) + \chi(i < a_3 - c_{\sigma(3)} - 2).
\end{aligned}$$

By Theorem 4.3.3 we need only consider $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ with $a_1 = 1, a_2 \leq n+1$, and $a_3 \leq 2n+1$. The full algebraic proof requires a large number of cases, followed by using Theorem 4.3.3 to sum over the full range of $\mathbf{b}$. Here we show only the factor of $[n-1]_t$ in $\sum_{i=0}^{n-2} \sum_{\sigma \in S_3} q^{\mathrm{inv}(\sigma)+A'} t^{i+B'}$. *Case 1:* $\sigma = 123$. In this case $A' = 0$ so

$$\begin{aligned}
\sum_{i=0}^{n-2} q^{\mathrm{inv}(\sigma)+A'} t^{i+B'} &= \sum_{i=0}^{n-2} t^{i+(n-1)(\chi(i<a_2-n-1)+\chi(i<a_3-(2n-1)-2))} \\
&= \sum_{i=0}^{n-2} t^i \\
&= [n-1]_t.
\end{aligned}$$

Case 2: $\sigma = 132$. In this case $A' = 0$ so

$$\begin{aligned}\sum_{i=0}^{n-2} q^{\mathrm{inv}(\sigma)+A'} t^{i+B'} &= \sum_{i=0}^{n-2} q t^{i+(n-1)(\chi(i<a_2-(2n-1)-1)+\chi(i<a_3-n-2))} \\ &= \sum_{i=0}^{n-2} q t^{i+(n-1)\chi(i<a_3-n-2)} \\ &= q t^{(a_3-n-2)^+}[n-1]_t.\end{aligned}$$

When $A' \neq 0$, we pair certain permutations to find the factor of $[n-1]_t$. *Case 3:* $\sigma = 213$ and $\sigma = 231$. Notice that $\chi(i < a_2 - (2n-1) - 1) = 0, \chi(i > a_2 - (2n-1) - 2) = 1$, and $\chi(i < a_3 - (2n-1) - 2) = 0$. Then

$$\begin{aligned}\sum_{i=0}^{n-2}(q^{\mathrm{inv}(213)+A'}t^{i+B'} + q^{\mathrm{inv}(231)+A'}t^{i+B'}) &= \sum_{i=0}^{n-2}(q^{1+\chi(i<a_2-2)}t^{i+(n-1)\chi(i<a_2-2)} \\ &\qquad + q^{2-\chi(i<a_2-2)}t^{i+(n-1)\chi(i<a_3-3)}) \\ &= \begin{cases} (qt^{a_2-2} + q^2 t^{a_3-3})[n-1]_t & \text{if } a_3 \leq n+2 \\ (qt^{a_2-2} + q^2 t^{n-1})[n-1]_t & \text{if } a_3 > n+2 \end{cases} \\ &= (qt^{a_2-2} + q^2 t^{a_3-3-(a_3-n-2)^+})[n-1]_t.\end{aligned}$$

Case 4: $\sigma = 312$ and $\sigma = 321$. Notice that $\chi(i > a_2 - n - 2) = 1$ and $\chi(i < a_2 - n - 1) = 0$.

Then

$$\begin{aligned}
\sum_{i=0}^{n-2}(q^{\mathrm{inv}(312)+A'}t^{i+B'} &+ q^{\mathrm{inv}(321)+A'}t^{i+B'}) \\
&= \sum_{i=0}^{n-2}(q^{2+\chi(i<a_2-2)}t^{i+(n-1)(\chi(i<a_2-2)+\chi(i<a_3-n-2))} \\
&\qquad +q^{3-\chi(i<a_2-2)}t^{i+(n-1)\chi(i<a_3-3)}) \\
&= \begin{cases} (q^2t^{a_2-2}+q^3t^{a_3-3})[n-1]_t & \text{if } a_3 \leq n+2 \\ (q^2t^{a_3-n-2}+q^3t^{n-1+a_2-2})[n-1]_t & \text{if } a_3 > n+2 \\ & \text{and } a_2 \leq a_3-n \\ (q^2t^{a_2-2}+q^3t^{a_3-3})[n-1]_t & \text{if } a_3 > n+2 \\ & \text{and } a_2 > a_3-n \end{cases} \\
&= (q^2t^{a_2-2+(a_3-n-a_2)^+}+q^3t^{a_3-3-(a_3-n-a_2)^+})[n-1]_t.
\end{aligned}$$

□

Remark 4.3.5. There is likely a combinatorial proof of Theorem 4.3.4 which is analogous to the proof for (2^n). This will be a topic of future study.

4.4 Recursion for $\lambda \in \mathrm{Par}(n)$

We can define a recursion for fillings of partitions of any shape.

Definition 4.4.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathrm{Par}(n)$. Define the partition $\lambda\setminus\lambda_1 \in \mathrm{Par}(n-\lambda_1)$ by $\lambda \setminus \lambda_1 = (\lambda_2, \ldots, \lambda_k)$.

Theorem 4.4.2. *For all λ, σ, and $\mathbf{a}$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathrm{Par}(n)$, $\sigma \in S_{\lambda_1}$, and $\mathbf{a} = \langle a_1, a_2, \ldots, a_{\lambda_1}\rangle$ with $1 = a_1 < a_2 < \cdots < a_{\lambda_1} \leq n$,*

$$R_{\lambda,\sigma(\mathbf{a})} = \sum_{\substack{\mathbf{b}=\langle b_1,\cdots,b_{\lambda_2}\rangle \\ \text{with } 1=b_1<\cdots<b_{\lambda_2}\leq n-\lambda_1}} \sum_{\tau\in S_{\lambda_2}} \sum_{i=0}^{n-\lambda_1-b_{\lambda_2}} R_{\lambda\setminus\lambda_1,\tau(\mathbf{b})} q^{\mathrm{inv}(\sigma)+A} t^{i+B}$$

where

$$\begin{aligned} A &= \sum_{j=1}^{\lambda_2-1}\sum_{k=j+1}^{\lambda_2} (\chi(i < a_{\sigma(j)} - b_{\tau(j)} - (\sigma(j)-1))\chi(i > a_{\sigma(j)} - b_{\tau(k)} - \sigma(j)) \\ &\qquad -\chi(i < a_{\sigma(j)} - b_{\tau(k)} - (\sigma(j)-1))\chi(i > a_{\sigma(j)} - b_{\tau(j)} - \sigma(j))) \text{ and} \\ B &= \sum_{j=1}^{\lambda_2} l(a_{\sigma(j)})\chi(i < a_{\sigma(j)} - b_{\tau(j)} - (\sigma(j)-1)). \end{aligned}$$

with initial condition $R_{(k),\sigma(\langle 1,2,\ldots,k\rangle)} = q^{\mathrm{inv}(\sigma)}$.

Proof. Let $T \in \mathcal{F}_{\lambda,\sigma(\mathbf{a})}$. For each $1 \leq i \leq \lambda_1$ let c_i be the cell in the Ferrers diagram of λ with entry $a_{\sigma(i)}$ in T. Then there exist unique $U, \mathbf{b}, \tau, i$ such that $U \in \mathcal{F}_{\lambda\setminus\lambda_1}$, $T = A_{\sigma(\mathbf{a})}(\mathrm{cyc}^i(U))$ and the first row of U is $\boxed{b_{\tau(1)}}\boxed{b_{\tau(2)}}\boxed{\ldots}\boxed{b_{\tau(\lambda_2)}}$ for some $\mathbf{b} = \langle b_1, b_2, \ldots, b_{\lambda_2}\rangle$ where $1 = b_1 < b_2 < \cdots < b_{\lambda_2} \leq n - \lambda_1$, $\tau \in S_{\lambda_2}$, and $0 \leq i \leq n - \lambda_1 - b_{\lambda_2}$. Note that $U = (\mathrm{cyc}^{-1})^i(A_{\sigma(\mathbf{a})}(T))$ where i is the number of cyc^{-1} required to obtain 1 in the first row of $A_{\sigma(\mathbf{a})}(T)$. The first row of U has entries $\tau(\mathbf{b}) = \langle b_{\tau(1)}, b_{\tau(2)}, \ldots, b_{\tau(\lambda_2)}\rangle$. Then by Theorem 3.6.2

$$\begin{aligned} \mathrm{inv}_\lambda(T) &= \mathrm{inv}_{\lambda\setminus\lambda_1}(U) + \mathrm{inv}(\sigma) \\ &\quad + \sum_{j=1}^{\lambda_2-1}\sum_{k=j+1}^{\lambda_2} (\chi(i < a_{\sigma(j)} - b_{\tau(j)} - (\sigma(j)-1))\chi(i > a_{\sigma(j)} - b_{\tau(k)} - \sigma(j)) \\ &\quad -\chi(i < a_{\sigma(j)} - b_{\tau(k)} - (\sigma(j)-1))\chi(i > a_{\sigma(j)} - b_{\tau(j)} - \sigma(j))) \text{ and} \\ \mathrm{comaj}_\lambda(T) &= \mathrm{comaj}_{\lambda\setminus\lambda_1}(U) + i + \sum_{j=1}^{\lambda_2} l(c_j)\chi(i < a_{\sigma(j)} - b_{\tau(j)} - (\sigma(j)-1)). \end{aligned}$$

□

Example 4.4.3. Let $\lambda = (3,2,1)$, $\mathbf{a} = \langle 1,3,5\rangle$, and $\sigma = 213$. Then

$$\begin{aligned} A &= \chi(i < a_{\sigma(1)} - b_{\tau(1)} - (\sigma(1)-1))\chi(i > a_{\sigma(1)} - b_{\tau(2)} - \sigma(1)) \\ &\quad -\chi(i < a_{\sigma(1)} - b_{\tau(2)} - (\sigma(1)-1))\chi(i > a_{\sigma(1)} - b_{\tau(1)} - \sigma(1)) \\ &= \chi(i < 2 - b_{\tau(1)})\chi(i > 1 - b_{\tau(2)}) - \chi(i < 2 - b_{\tau(2)})\chi(i > 1 - b_{\tau(1)}) \end{aligned}$$

and

$$\begin{aligned} B &= l(c_1)\chi(i < a_{\sigma(1)} - b_{\tau(1)} - (\sigma(1)-1)) + l(c_2)\chi(i < a_{\sigma(2)} - b_{\tau(2)} - (\sigma(2)-1)) \\ &= 2\chi(i < 2 - b_{\tau(1)}). \end{aligned}$$

Thus

$$\begin{aligned} R_{\lambda,\sigma(a)} = &\sum_{\substack{\mathbf{b}=\langle b_1,b_2\rangle \\ \text{with} \\ 1=b_1<b_2\le 3}} \sum_{\tau\in S_2} \sum_{i=0}^{3-b_2} R_{(2,1),\tau(\mathbf{b})} q^{\mathrm{inv}(213)+A} t^{i+2\chi(i<2-b_{\tau(1)})} \\ &= R_{(2,1),\langle 1,2\rangle}(q^2t^2+qt) + R_{(2,1),\langle 2,1\rangle}(1+qt) + R_{(2,1),\langle 1,3\rangle}(q^2t^2) + R_{(2,1),\langle 3,1\rangle}(1) \\ &= (q^2t^2+qt) + q(1+qt) + (q^2t^2) + qt(1) \\ &= q + q^2t + 2qt + 2q^2t^2. \end{aligned}$$

The fillings of $(3,2,1)$ with first row $\boxed{3\,|\,1\,|\,5}$ are:

$$\begin{array}{ccc} \begin{array}{|c|c|c|} \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & 6 & \multicolumn{1}{c}{} \\ \hline 3 & 1 & 5 \\ \hline \end{array} & \begin{array}{|c|c|c|} \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-2} 6 & 4 & \multicolumn{1}{c}{} \\ \hline 3 & 1 & 5 \\ \hline \end{array} & \begin{array}{|c|c|c|} \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 6 & \multicolumn{1}{c}{} \\ \hline 3 & 1 & 5 \\ \hline \end{array} \\ qt & q^2t & q^2t^2 \\ \begin{array}{|c|c|c|} \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 6 & 2 & \multicolumn{1}{c}{} \\ \hline 3 & 1 & 5 \\ \hline \end{array} & \begin{array}{|c|c|c|} \cline{1-1} 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 4 & \multicolumn{1}{c}{} \\ \hline 3 & 1 & 5 \\ \hline \end{array} & \begin{array}{|c|c|c|} \cline{1-1} 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & 2 & \multicolumn{1}{c}{} \\ \hline 3 & 1 & 5 \\ \hline \end{array} \\ qt & q^2t^2 & q \end{array}$$

4.5 Recursion for Two Unequal Columns

When we restrict to two unequal columns, the recursion can be rewritten using only increasing first rows. Since the columns are not of equal height, the inversion flip is not available to sort. Instead, we use the cyclic shift and careful accounting to obtain an increasing bottom row. This is really a special case of Theorem 3.2.5.

Theorem 4.5.1. *Let $m \geq 1$, $n \geq 0$ and $T \in \mathcal{F}_{(1^m 2^n)}$, where T has bottom row $\boxed{x \,|\, z}$ for some $x > z$. Then $\mathrm{cyc}^{m+2n-x+1}(T)$ will have bottom row $\boxed{1 \,|\, b}$ where $b = m+2n-x+1+z$,*

$$\mathrm{comaj}_{(1^m2^n)}(\mathrm{cyc}^{m+2n-x+1}(T)) = \mathrm{comaj}_{(1^m2^n)}(T) + n - x + 1$$

and

$$\mathrm{inv}_{(1^m2^n)}(\mathrm{cyc}^{m+2n-x+1}(T)) = \mathrm{inv}_{(1^m2^n)}(T) - 1.$$

Proof. Let $T \in \mathcal{F}_{(1^m2^n)}$ have bottom row $\boxed{x \,|\, z}$ with $x > z$. By Theorem 3.2.3,

$$\mathrm{comaj}_{(1^m2^n)}(\mathrm{cyc}^{m+2n-x}(T)) = \mathrm{comaj}_{(1^m2^n)}(T) + m + 2n - x.$$

The bottom row of $\mathrm{cyc}^{m+2n-x}(T)$ is $\boxed{x' \,|\, z'}$ where $x' = m + 2n$ and $z' = m + 2n - x + z$. Then $\mathrm{cyc}^{m+2n-x+1}(T)$ has bottom row $\boxed{1 \,|\, b}$ with $b = m + 2n - x + z + 1$. The ascent involving x' and the entry directly above it in $\mathrm{cyc}^{m+2n-x}(T)$ is lost when cyc is applied, so

$$\mathrm{comaj}_{(1^m2^n)}(\mathrm{cyc}^{m+2n-x+1}(T)) = \mathrm{comaj}_{(1^m2^n)}(\mathrm{cyc}^{m+2n-x}(T)) - (m + n - 1)$$

and thus

$$\mathrm{comaj}_{(1^m2^n)}(\mathrm{cyc}^{m+2n-x+1}(T)) = \mathrm{comaj}_{(1^m2^n)}(T) + n + 1 - x.$$

By Theorem 3.2.3,

$$\mathrm{inv}_{(1^m2^n)}(\mathrm{cyc}^{m+2n-x}(T)) = \mathrm{inv}_{(1^m2^n)}(T).$$

Since $m+2n$ is in the bottom row of $\mathrm{cyc}^{m+2n-x}(T)$,

$$\mathrm{inv}_{(1^m2^n)}(\mathrm{cyc}^{m+2n-x+1}(T)) = \mathrm{inv}_{(1^m2^n)}(\mathrm{cyc}^{m+2n-x}(T)) - 1,$$

so

$$\mathrm{inv}_{(1^m2^n)}(\mathrm{cyc}^{m+2n-x+1}(T)) = \mathrm{inv}_{(1^m2^n)}(T) - 1. \qquad \square$$

We can calculate $R_{(1^m2^n),a}$ for $2 \leq a \leq m+2n$ by the recursion below.

Theorem 4.5.2. *For all $m \geq 1$, $n \geq 2$, and $2 \leq a \leq m+2n$,*

$$\begin{aligned} R_{(1^m2^n),a} &= \sum_{b=2}^{m+2n-2} \sum_{i=0}^{m+2n-2-b} \Big(t^{i+(n-1)\chi(i<a-b-1)} R_{(1^m2^{n-1}),b} \\ &\qquad + q t^{b-n+i+(n-1)\chi(i<a-2)} R_{(1^m2^{n-1}),m+2n-b} \Big). \end{aligned}$$

The initial conditions are $R_{(1^m2),a} = F_{(1^m)} = [m]!_t$ for $m \geq 1$ and $2 \leq a \leq m+1$.

Proof. Let $T \in \mathcal{F}_{(1^m2^n),a}$. Then there exists a unique $U \in \mathcal{F}_{(1^m2^{n-1})}$ such that $T = A_a(\mathrm{cyc}^i(U))$ and the bottom row of U is either $\boxed{1\,|\,b}$ or $\boxed{b\,|\,1}$ for some b with $2 \leq b \leq m+2n-2$ and some i with $0 \leq i \leq m+2n-2-b$. If the bottom row of U is $\boxed{1\,|\,b}$, then by Theorems 3.6.2 and 3.2.3

$$\begin{aligned} \mathrm{comaj}_{(1^m2^n)}(T) &= \mathrm{comaj}_{(1^m2^n)}(A_a(\mathrm{cyc}^i(U))) \\ &= \mathrm{comaj}_{(1^m2^{n-1})}(U) + i + (n-1)\chi(i < a-b-1) \end{aligned}$$

and

$$\mathrm{inv}_{(1^m2^n)}(T) = \mathrm{inv}_{(1^m2^{n-1})}(U).$$

If the bottom row of U is $\boxed{b}\boxed{1}$, then by Theorems 3.6.2 and 3.2.3,

$$\begin{aligned}\mathrm{comaj}_{(1^m2^n)}(T) &= \mathrm{comaj}_{(1^m2^n)}(A_a(\mathrm{cyc}^i(U))) \\ &= \mathrm{comaj}_{(1^m2^{n-1})}(U) + i + (n-1)\chi(i < a-2).\end{aligned}$$

Let $U' = \mathrm{cyc}^{m+2n-1-b}(U)$, then, by Theorem 4.5.1,

$$\begin{aligned}\mathrm{comaj}_{(1^m2^n)}(T) &= \mathrm{comaj}_{(1^m2^{n-1})}(U) + i + (n-1)\chi(i < a-2) \\ &= \mathrm{comaj}_{(1^m2^{n-1})}(U') + i + b - n + (n-1)\chi(i < a-2)\end{aligned}$$

and

$$\mathrm{inv}_{(1^m2^n)}(T) = \mathrm{inv}_{(1^m2^{n-1})}(U') + 1.$$

Thus the second term in the recursion accounts for fillings T arising from those U having a decreasing bottom row. □

4.6 Factor of $[n]_t$ when $\mu = (m^n)$

While we do not yet have a bijective proof of part (b) of Theorem 4.1.1 for arbitrary shapes, we do have a bijective way to prove that $F_{(m^n)}(q,t)$ is divisible by $[n]_t$.

Definition 4.6.1. A word consisting of left and right parentheses is a *Dyck word* if the parentheses are balanced, i.e., every prefix has at least as many ('s as)'s.

Example 4.6.2. The word (()(())) is a Dyck word, but ())(((is not.

Theorem 4.6.3. *Any word consisting of an equal number of left and right parentheses can be shifted to obtain a Dyck word.*

Proof. Let w be a word with k left parentheses and k right parentheses. If the parentheses are balanced, w is a Dyck word. Assume the parentheses are unbalanced. Let w' be the subword of w consisting of only the unbalanced parentheses in w. Then w' consists of j right parentheses followed by j left parentheses. Shift w so that the first left parenthesis in w' is the first symbol in the shifted word. The result has balanced parentheses and is thus a Dyck word. □

Example 4.6.4. The word (())()))**(((** is unbalanced. The left parentheses in bold can be moved to the beginning of the word to obtain **(((**(())())) which is a Dyck word.

Theorem 4.6.5. *For all $m, n \in \mathbb{N}, n \geq 1$,*

$$
\begin{aligned}
F_{(m^n)}(q,t) &= \sum_{T \in \mathcal{F}_{(m^n)}} q^{\mathrm{inv}_{(m^n)}(T)} t^{\mathrm{comaj}_{(m^n)}(T)} \\
&= [m]!_q [n]_t \sum_{\substack{\mathbf{a}=\langle a_1,\ldots,a_m\rangle \\ \textit{with} \\ 1=a_1<\cdots<a_m\leq mn}} \sum_{T \in \mathcal{F}_{(m^n),\mathbf{a}}} q^{\mathrm{inv}_{(m^n)}(T)} t^{\mathrm{comaj}_{(m^n)}(T)}.
\end{aligned}
$$

Proof. Start with a filling $T \in \mathcal{F}_{(m^n),\mathbf{a}}$ with $\mathbf{a} = \langle a_1, a_2, \ldots, a_m\rangle$ where $1 = a_1 < a_2 < \cdots < a_m \leq mn$.

Generate a sequence of fillings $A_0, A_1, \ldots, A_{mn-1}$, where $A_i = \mathrm{cliff}^i(T)$. Generate a word by associating to each A_i either a left parenthesis or right parenthesis using the following rules:

- for $1 \leq i \leq mn-1$ use a left parenthesis if $\mathrm{comaj}_\mu(A_i) = \mathrm{comaj}_\mu(A_{i-1}) + 1$.
- for $1 \leq i \leq mn-1$ use $n-1$ right parentheses if $\mathrm{comaj}_\mu(A_i) = \mathrm{comaj}_\mu(A_{i-1}) - (n-1)$.
- for $i = 0$, associate A_0 with a left parenthesis if $\mathrm{comaj}_\mu(A_0) = \mathrm{comaj}_\mu(A_{mn-1}) + 1$ and with $n-1$ right parentheses if $\mathrm{comaj}_\mu(A_0) = \mathrm{comaj}_\mu(A_{mn-1}) - (n-1)$.

Since there will be an equal number of left and right parentheses, by Theorem 4.6.3, the word that results can be shifted to obtain a Dyck word.

Claim 4.6.6. *Parenthesis-matching decomposes* $\{A_0, A_1, \ldots, A_{mn-1}\}$ *into* m *disjoint sets* $\{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}$, $1 \leq i \leq m$, *of fillings with* $\mathrm{comaj}_\mu(S_{i_{j+1}}) = \mathrm{comaj}_\mu(S_{i_j}) + 1$.

Proof. Due to Theorem 4.6.3, we can assume the word starts with A_0. Note that any A_i associated with $n-1$ right parentheses must have 1 as its first entry. In $\{A_0, A_1, \ldots, A_{mn-1}\}$ there will be m distinct A_j with 1 as the first entry. Given a pair of parentheses the difference in comaj between the two fillings represented by them is given by

$$\Delta\,\mathrm{comaj} = -[(n-1-\#\text{ right parentheses between}) \mod (n-1)].$$

Let S_{i_1} be a filling with a 1 in the first row. Then in the word, S_1 is represented by $n-1$ right parentheses, so there are $n-1$ fillings represented by left parentheses paired with S_{i_1}. Of these fillings, label them from left to right: $S_{i_2}, S_{i_3}, \ldots, S_{i_n}$. □

□

Example 4.6.7. Using the fillings in Figure 4.2 we obtain the word:

$$\begin{array}{cccccccccccc})) & (& (& (& (& (& (&)) & (&)) &)) & (\\ A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_{11} \end{array}.$$

This word can be shifted to obtain the Dyck word

$$\begin{array}{cccccccccccc} (& (& (& (& (& (&)) & (&)) &)) & (&)) \\ A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_{11} & A_0 \end{array}.$$

Balancing parentheses gives the following sets of fillings:

- A_0, A_1, A_{11};
- A_{10}, A_2, A_3;
- A_9, A_4, A_8; and
- A_7, A_5, A_6.

$$\underset{A_0(q^8t^3)}{\begin{array}{|c|c|c|c|}\hline 9&7&11&12\\\hline 10&2&5&8\\\hline 1&3&4&6\\\hline\end{array}} \xrightarrow{\text{cliff}} \underset{A_1(q^8t^4)}{\begin{array}{|c|c|c|c|}\hline 10&8&12&1\\\hline 11&3&6&9\\\hline 2&4&5&7\\\hline\end{array}} \xrightarrow{\text{cliff}} \underset{A_2(q^8t^5)}{\begin{array}{|c|c|c|c|}\hline 11&9&1&2\\\hline 12&4&7&10\\\hline 3&5&6&8\\\hline\end{array}} \xrightarrow{\text{cliff}} \underset{A_3(q^8t^6)}{\begin{array}{|c|c|c|c|}\hline 12&10&2&3\\\hline 1&5&8&11\\\hline 4&6&7&9\\\hline\end{array}} \xrightarrow{\text{cliff}}$$

$$\underset{A_4(q^8t^7)}{\begin{array}{|c|c|c|c|}\hline 1&11&3&4\\\hline 2&6&9&12\\\hline 5&7&8&10\\\hline\end{array}} \xrightarrow{\text{cliff}} \underset{A_5(q^8t^8)}{\begin{array}{|c|c|c|c|}\hline 2&12&4&5\\\hline 3&7&10&1\\\hline 6&8&9&11\\\hline\end{array}} \xrightarrow{\text{cliff}} \underset{A_6(q^8t^9)}{\begin{array}{|c|c|c|c|}\hline 3&1&5&6\\\hline 4&8&11&2\\\hline 7&9&10&12\\\hline\end{array}} \xrightarrow{\text{cliff}} \underset{A_7(q^8t^7)}{\begin{array}{|c|c|c|c|}\hline 4&2&6&7\\\hline 9&5&3&12\\\hline 1&8&10&11\\\hline\end{array}} \xrightarrow{\text{cliff}}$$

$$\underset{A_8(q^8t^8)}{\begin{array}{|c|c|c|c|}\hline 5&3&7&8\\\hline 10&6&4&1\\\hline 2&9&11&12\\\hline\end{array}} \xrightarrow{\text{cliff}} \underset{A_9(q^8t^6)}{\begin{array}{|c|c|c|c|}\hline 6&4&8&9\\\hline 11&7&5&2\\\hline 1&3&10&12\\\hline\end{array}} \xrightarrow{\text{cliff}} \underset{A_{10}(q^8t^4)}{\begin{array}{|c|c|c|c|}\hline 7&5&9&10\\\hline 12&8&6&3\\\hline 1&2&4&11\\\hline\end{array}} \xrightarrow{\text{cliff}} \underset{A_{11}(q^8t^5)}{\begin{array}{|c|c|c|c|}\hline 8&6&10&11\\\hline 1&9&7&4\\\hline 2&3&5&12\\\hline\end{array}}$$

Figure 4.2: Fillings to illustrate parenthesis-matching.

Chapter 5

Future Work

Some problems suggested by the present work appear below.

- **Bijective proof of the recursion for three equal columns.**

  In the future, we would like to prove Theorem 4.3.4 bijectively. This will require a new combinatorial move $\overline{\text{cliff}} : \mathcal{F}_{(3^n),\mathbf{a}} \to \mathcal{F}_{(3^n),\mathbf{a}}$ which preserves $\text{inv}_{(3^n)}$ while increasing $\text{comaj}_{(3^n)}$ by $1 \mod (n-1)$.

- **Bijective proof of the recursion for m equal columns.**

  It is expected that a bijective proof of Theorem 4.3.4 will give us the tools necessary to find a version of Theorem 4.3.1 which exhibits the factor $[n-1]_t$ without resorting to algebraic manipulations. This will also require a move $\overline{\text{cliff}} : \mathcal{F}_{(m^n),\mathbf{a}} \to \mathcal{F}_{(m^n),\mathbf{a}}$ which preserves $\text{inv}_{(m^n)}$ and increases $\text{comaj}_{(m^n)}$ by $1 \mod (n-1)$.

- **Factor of $[n-1]_t$ in the recursion for two unequal columns.**

The recursion for $(1^m 2^n)$ in Theorem 4.5.2 does not show the factor of $[n-1]_t$. The data suggest that there may be other combinatorial operations needed to rewrite the recursion in a form that does exhibit this factor.

- **More q-analogues and q,t-analogues of the hook-length formula.**

  We would like to find bijective proofs of more cases of the q-analogue of the hook length formula in Theorem 2.1.4. The recursions may assist with this, since the factors they exhibit are present in the hook-length formula, but it would be necessary to guess a formula for $R_{\mu,\mathbf{a}}(q,q)$. It is possible that, as with the hook case, we may find more q,t-analogues of the hook-length formula as well.

Bibliography

[1] J. Bandlow, *Combinatorics of Macdonald polynomials and extensions,* UC San Diego doctoral dissertation (2007).

[2] Frame, J. S., G. de B. Robinson, and R. M. Thrall, "The hook graphs of the symmetric group", *Canadian Journal of Mathematics* **6** (1954), 316—324.

[3] Franzblau, D. S. and D. Zeilberger, "A bijective proof of the hook-length formula", *Journal of Algorithms* **3** (1982), 317—343.

[4] A. Garsia and J. Haglund, "A new recursion in the theory of Macdonald polynomials," preprint (2009).

[5] A. Garsia and M. Haiman, "A graded representation model for Macdonald's polynomials," *Proc. Natl. Acad. Sci. USA* **90** (1993), 3607–3610.

[6] A. Garsia and M. Haiman, "A random q, t-hook walk and a sum of Pieri coefficients," *J. Combin. Theory Ser. A* **82** (1998), 74–111.

[7] A. Garsia and M. Haiman, "A remarkable q,t-Catalan sequence and q-Lagrange inversion," *J. Algebraic Combinatorics* **5** (1996), 191–244.

[8] Greene, C., A. Nijenhuis, and H. S. Wilf,"A probabilistic proof of a formula for the number of Young tableaux of a given shape", *Advances in Mathematics* **31** (1979), 104—109.

[9] J. Haglund, "A combinatorial model for the Macdonald polynomials," *Proc. Natl. Acad. Sci. USA* **101** (2004), 16127–16131.

[10] J. Haglund, *The q,t-Catalan Numbers and the Space of Diagonal Harmonics, with an Appendix on the Combinatorics of Macdonald Polynomials,* AMS University Lecture Series (2008).

[11] M. Haiman, "Hilbert schemes, polygraphs, and the Macdonald positivity conjecture." *J. Amer. Math. Soc.* **14** (2001), 941–1006.

[12] M. Haiman, "Notes on Macdonald polynomials and the geometry of Hilbert schemes," *Symmetric Functions 2001: Surveys of Developments and Perspectives, NATO Sci. Ser. II Math. Phys. Chem.* **74** (2002), 1–64.

[13] J. Haglund, M. Haiman, and N. Loehr, "A combinatorial formula for Macdonald polynomials," *J. Amer. Math. Soc.* **18** (2005), 735–761.

[14] J. Haglund, M. Haiman, and N. Loehr, "Combinatorial theory of Macdonald polynomials I: proof of Haglund's formula," *Proc. Natl. Acad. Sci. USA* **102** #8 (2005), 2690–2696.

[15] Krattenthaler, C., "Bijective proofs of the hook formulas for the number of the standard Young tableaux, ordinary and shifted", *Electronic Journal of Combinatorics* **2** (1995), #R13.

[16] N. Loehr and J. Remmel, "A computational and combinatorial expose of plethystic calculus," preprint.

[17] I. Macdonald, "A new class of symmetric functions," *Actes du 20e Séminaire Lotharingien*, vol. 372/S-20, Publications I.R.M.A., Strasbourg (1988), 131–171.

[18] I. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed. Oxford University Press (1995).

[19] Novelli, Jean-Christophe, Igor Pak, and Alexander V. Stoyanovskii, "A direct bijective proof of the hook-length formula", *Discrete Mathematics and Theoretical Computer Science* **1** (1997), 53—67.

[20] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd ed. Springer, 2000.

[21] R. Stanley, *Enumerative Combinatorics* (2 volumes). Cambridge University Press, Cambridge, United Kingdom (1999).

[22] M. Yoo, "Combinatorial formula for the Hilbert series of bigraded S_n-modules," Proceedings of FPSAC 2009, *Discrete Math. Theor. Comput. Sci.* proc. **AK** (2009), 903–914.