

Chapter 2

A Comparison of Public and Private Education Systems in a General Equilibrium Model

2.1 Introduction

This paper addresses three basic questions concerning public education and the economy. First, how do democratic societies determine how much education to provide? Second, how does this level of education compare with the level provided by a private system? Third, how might public education affect the income distribution within a society? These questions have been asked by others, especially Loury (1981), Glomm & Ravikumar (1992), and Perotti (1993). I briefly review some of the theoretical work on this topic to establish motivation for this paper.

The role of education in an economy attracted increased interest with the introduction of endogenous growth models (Lucas, 1988). Barro (1991) showed empirically that higher education levels within a country were positively related to economic growth. In these models, education involves an externality, making the social returns higher than the private returns, so that encouraging human capital growth at a societal level is Pareto-improving. Stiglitz (1974) also argues that there may be social benefits to education, though he suggests that the marginal social benefit is zero

at the privately preferred level of education. This motivation for public education augments the traditional justification of redistributing income from wealthier individuals to poorer ones (Perotti, 1993; Saint-Paul & Verdier, 1993). Indeed, the importance of human capital in production implies that wages are directly related to a worker's human capital levels. Thus, the equality of the education system is related to equality of income.

In an overlapping-generations model, human capital can be augmented across generations, and thus provide an engine of growth, if the education abilities of members of each generation incorporate the human capital levels of their parents or the average of the human capital levels of their parents' generation. An important contribution of this literature is the development of human capital production functions. These functions relate certain inputs, such as time, education expenditures, innate ability, and, as just mentioned, human capital levels of parents and/or average human capital levels of society to the production of human capital. In turn, human capital is transformed directly into income or is combined with other inputs, such as physical capital, to produce a consumption/investment good(s).

These models also rely on parental decisions in funding education for their children. In Galor & Zeira (1993), imperfect credit markets make it prohibitive for the young to borrow for their education, while in other papers, borrowing is strictly prohibited. These papers rely on parental altruism towards their children, implying that parents choose to spend their resources on their children's education. Parents care only about the welfare of their own children. In Loury (1981), families need not even engage in any activity with any other family in the basic model. Public education is introduced in Saint-Paul & Verdier (1993) and Glomm & Ravikumar (1992) as a means to redistribute wealth from rich families to poorer families. In Owen (1994), the economy is interconnected by a production function that allows the wages of skilled and unskilled to be determined by the ratio of skilled to unskilled workers in the economy. In Perotti (1993), explicit externalities are used to generate redistribution through taxes to subsidize the education of poorer workers.

This paper uses an overlapping-generations model and traditional neoclassical production functions to show that members of a society may choose to educate their children, even without an explicit bequest motive. A higher average human capital level of workers increases the return to physical capital saved by the older generation.¹ Thus, the members of a generation care about the human

¹Since these are pecuniary effects, there is a parallel decrease in the wage rate. The effect of this on the decision of young agents is discussed below.

capital levels of the following generation. Unlike other models, where agents act to increase the wealth of their own children, in this model agents have no incentive to act alone. Assuming that labor and capital are perfectly mobile in the economy and that population is continuous, no single individual has any influence over the level of average human capital. Therefore, for an equilibrium to exist with positive education investment, all agents must be assured of receiving education. If localities decide on tax rates independently, all agents in the economy (except the young) will move to the localities with the lowest tax. There will be no locality with both a positive tax rate and a positive population. Thus, this model cannot incorporate the idea that neighborhood stratification is a result of education decisions as in Durlauf (1996) and Fernandez & Rogerson (1996).² This model makes sense in societies where centralized governments provide education funding, as in many countries in the world, and increasingly in the U.S. (Hanushek, 1988).

The model is similar to Boldrin (1992) and Soares(1993).³ In Boldrin (1992), there are 3 generations alive at a given time, preferences are Cobb-Douglas over consumption in the last two periods and there is a Cobb-Douglas production function with physical capital and human capital as inputs. Boldrin uses a different human capital production function, namely: $h_{t+1} = \frac{(\epsilon+g_t)^\gamma}{1+n} h_t$. This function is linear in parent's human capital and allows human capital to be transferred to the young with zero education spending (through the parameter ϵ).

Boldrin does not compare private and public education systems or the effect of the heterogeneity of the young agents on the education policy. But she discusses two issues which I find interesting. First, she considers the effect of allowing the young to trade off leisure for study time. Under these circumstances, higher education funding encourages students to spend more time studying. This leads to a higher level of education funding than the model where students provide study time inelastically.⁴ The second issue deals with heterogeneity of human and physical capital levels across families. Richer families may opt out of the public education system if the expenditure level is too low.⁵ This has two effects on the public school students. First, the available education funds are divided among a smaller number of students. Second, the tax rate to finance public education

²Likewise, if the production of human capital depended on independent parental action, such as time spent educating their children at home as in Tamura [1991], no such action would be provided.

³The similarities are coincidental since I had nearly completed this paper when these papers were brought to my attention.

⁴I discuss this issue more below.

⁵Parents have altruism for their children, which encourages them to provide funds for private education. Boldrin does not allow agents to borrow for education funds.

may fall, reducing the amount of public funds for each student. The reason that the tax rate may fall is because the median voter in the economy is likely to be a rich parent who sends his child to a private school.⁶ This parent still votes for some public education funds because of the general equilibrium effects on his return to capital.

Soares (1992) also uses a 3-generation model in which middle-aged agents take into account the effect of increasing the young's human capital on their returns to physical capital. By using a simulation to approximate the preferred tax rate, Soares is able to employ a more general model; for example, agents have CES preferences. Soares does not consider the case where young agents are heterogeneous. Also, Soares assumes that young agents cannot borrow for their educations. The result of this is that the private education system provides very low education spending levels (.5% of GDP compared with approximately 5% in the public system), and so any comparison between the two systems will obviously favor the public regime.⁷

Lin (1998) uses a general equilibrium argument to suggest that public education funds may decrease the amount that agents invest in human capital. He considers a two-period model where agents may invest in either human capital or physical capital. If there is an income tax to pay for public education, individuals save less because their first-period disposable income is less. This causes the return to physical capital to rise for the next period, reducing the incentive to invest in human capital.

In the next section, the basic model will be presented. A public education system will be described, and the preferred tax rate will be chosen as a function of the underlying parameters of the economy. I also show that income inequality will not decline over time since society, in order to maximize human capital, chooses to optimize education spending by providing more funds to young agents with greater learning ability. Section 2.3 derives a private education system equivalent to the model of Section 2.2. I assume that young agents can borrow from middle-aged agents at the equilibrium interest rate. In Section 2.4, the steady states are derived for each model assuming that there is a constant-returns human capital production function and there is no inequality in ability. I compare the two systems in Section 2.5. I find that the public education system is more likely to provide greater growth in the steady state if agents have a low rate of time preference for consumption and

⁶In this model, all old agents vote for a zero tax rate and the young agents cannot vote. If the population growth rate is small enough, it is highly likely that the median voter will be one of the lowest demanders of public education funds in the middle-aged group.

⁷Young agents are allowed to work, so they can earn funds to pay for education.

Table 2.1: Actions of Agents

Young	Middle-Aged	Old
<ul style="list-style-type: none"> • Go to school • (borrow money from the middle-aged)^a 	<ul style="list-style-type: none"> • Work • Consume/Save • (repay education loans) 	<ul style="list-style-type: none"> • Consume returns to capital

^aActions in parentheses denote possible actions in a non-publicly provided education system.

if the elasticity of human capital production with respect to education funds is not too low.

2.2 Human Capital Production with Public Education

The following model provides an economic justification for public education expenditures. Traditional neoclassical production functions are used to show that members of a society may choose to educate their children, even without an explicit bequest motive.

2.2.1 The Economic Environment

Consider an overlapping-generations model where individuals live for three periods. In any period, there are three generations alive. Population growth is assumed to be zero, and the population of each generation is normalized to one. Table 2.1 summarizes the actions of each individual in each period. When young, agents build up their levels of human capital. In middle age, agents supply their human capital inelastically to the production sector. Agents use their income in this period to either consume or invest in physical capital. When old, agents sell their physical capital (if any) to the production sector. They consume all their third-period income. Members of a particular generation are indexed by the time period in which they are middle-aged.

The utility of each individual in generation t is given by a logarithmic utility function:

$$U_{it} = \ln c_{it,t} + b \ln c_{it,t+1} \quad (2.1)$$

The first subscript on consumption is the individual's generation and the second is the time period in which the consumption takes place. Agents care about consumption in the second and third periods of their lives, and they weight the logarithm of consumption in the third period by the nonnegative fraction b .

There is one aggregate good produced in the economy, using a standard Cobb-Douglas aggregate production function:

$$Y_t = AH_t^\alpha K_t^{1-\alpha} \quad (2.2)$$

where H_t is the aggregate amount of human capital of generation t , and K_t is the amount of physical capital available in period t . The amount of K_t is equal to the aggregate savings of generation $t-1$. A is a parameter reflecting total factor productivity, perhaps incorporating measures of natural resources.

The production function has constant returns to scale, thus supporting a competitive economy. The factors of production are paid their marginal product, and total profits of the firms in the economy are zero. The good produced in this economy can be used for consumption or investment, where the coefficient of transformation into each is normalized to unity.

The return to physical capital r_t , which the old agents in period t receive on their investments, is positively affected by the level of human capital of the middle-aged generation in period t , as shown here:

$$r_t = (1 - \alpha)AH_t^\alpha K_t^{-\alpha} \quad (2.3)$$

$$\frac{\partial r_t}{\partial H_t} = \alpha(1 - \alpha)AH_t^{\alpha-1} K_t^{-\alpha} \quad (2.4)$$

Since the value of (2.4) is positive, each generation has an interest in the level of human capital of the following generation, due to purely selfish reasons.

Human capital in this economy is accumulated using a generalized Cobb-Douglas production function:

$$h_{it} = \theta_i H_{t-1}^\gamma \delta_{it}^\mu \quad (2.5)$$

The level of human capital of an individual i of generation t is a function of the average human capital of the previous generation⁸ and an indicator of the quality of education, δ_i . This variable

⁸Since the population of each generation is normalized to 1, average human capital is equal to aggregate human capital.

will be represented by expenditures on education. The variable θ_i represents individual innate ability. It affects the marginal productivity of additional expenditures (see Loury, 1981). It can also be thought of as a measure of the cost of transformation of the consumption good into school quality.⁹ Note that δ is subscripted by the period in which the individual receives the education (youth), although his generation is subscripted by the following period, so, for instance, δ_t is the quality of education received by generation- $(t + 1)$ individuals in period t .

I assume that θ_i is a random variable, with bounded support. The distribution is represented by a cumulative distribution function (c.d.f.) $G(\theta)$. Average ability of a generation is given by

$$\theta = \int_{\theta_l}^{\theta_u} \theta_i dG(\theta) \quad (2.6)$$

where $\theta_l > 0$.

2.2.2 Public Education With Majority Voting

The political environment

A public education system exists in this economy with the following structure. Education quality δ_t is funded by a proportional tax τ on all income in period t , where $\tau \in [0, 1]$. As shown below, δ^i is a function of individual i 's ability θ_i . I assume that each student's ability is readily observable. The government's budget constraint is thus:

$$\int_{\theta_l}^{\theta_u} \delta_i(\theta_i) dG(\theta) = \tau_t Y_t \quad (2.7)$$

This equation is equivalent to assuming that the government has a balanced budget in each period. The specific tax rate is chosen by majority vote¹⁰. Every member of each generation is allowed one vote. All possible tax rates are considered, so that a stable tax rate chosen by this system will be the first choice of the median voter.

Due to the framework of the model, the old face a trivial decision when choosing their preferred tax rates. The old prefer $\tau = 0$ since they derive no benefit but pay a proportionate cost.

⁹In this case, the price of δ relative to consumption is $\frac{1}{\theta^\mu}$. Note this price decreases when μ increases.

¹⁰Or a variation of this where elected legislators represent the interests of their constituencies.

The preferred tax rate of the middle-aged

For the middle generation, equation (2.4) gives the key incentive for public provision of education for the young. Members of the middle generation face a two-part problem. First, they must decide on the level of savings, given the tax rate and returns to capital. Then they must choose a tax rate to optimize utility. Each agent faces the problem: maximize (2.1) subject to

$$\begin{aligned} c_{it,t} &= (1 - \tau_t)w_t H_{it} - K_{it+1} \\ c_{it,t+1} &= (1 - \tau_{t+1})r_{t+1}K_{it+1} \end{aligned}$$

given $\tau_t, \tau_{t+1}, r_{t+1}, w_t$. This problem gives the preferred levels of consumption in both periods and the capital accumulation:

$$\begin{aligned} c_{t,t}^* &= \frac{1}{1+b}(1 - \tau_t)w_t H_{it} \\ c_{t,t+1}^* &= \frac{b}{1+b}(1 - \tau_t)(1 - \tau_{t+1})r_{t+1}w_t H_{it} \\ K_{t+1}^* &= (1 - \tau_t)\frac{b}{1+b}w_t H_{it} \end{aligned} \quad (2.8)$$

Considering equations (2.3), (2.4), and (2.7), we see that r_{t+1} depends on τ_t through the level of H_{t+1} :

$$r_{t+1} = (1 - \alpha)AK_{t+1}^{-\alpha} \left[\int_{\theta^l}^{\theta^u} \theta_i (\delta_{it}(\tau_t))^\mu dG(\theta) \right]^\alpha H_t^{\alpha\gamma} \quad (2.9)$$

An increase in the tax rate increases the average level of human capital in period $t + 1$ through the quality of education in the previous period, thus increasing the returns to K_{t+1} . Note that τ_t will also affect r_{t+1} through the level of K_{t+1} , as seen from (2.8). This effect is positive also, since K_{t+1} declines as the tax rate increases.¹¹

Before the tax rate is chosen, agents must decide how to distribute education quality across students in order to maximize average human capital levels, given any level of taxes. This can be set up as:

$$\max_{\delta_1, \delta_2, \dots} H_{t+1} = \int_{\theta^l}^{\theta^u} \theta_i \delta_{it}^\mu H_t^\gamma dG(\theta) \quad (2.10)$$

subject to:

$$\int_{\theta^l}^{\theta^u} \delta_{it} dG(\theta) = T \quad (2.11)$$

¹¹The absence of a direct effect of r_{t+1} on K_{t+1} in (2.8) is due to the logarithmic preferences.

where T is a given level of government revenue. The first-order conditions for this problem can be written as:

$$\theta_i \delta_{it}^{\mu-1} = \theta_j \delta_{jt}^{\mu-1} = \frac{\lambda}{\mu H_t^\gamma} \quad (2.12)$$

for all individuals i, j . The term λ is the shadow price of tax revenues. This condition implies that the marginal return of δ_i is equalized for all agents. This condition is identical to the distribution of education spending in a private education system discussed below. As long as there are decreasing returns to education quality (i.e., $\mu < 1$), equation (2.12) implies that $\delta_i < \delta_j$ if $\theta_i < \theta_j$; less able students receive less education quality.

Now, the middle-aged agent chooses τ_t to maximize indirect utility:

$$\max_{\tau_t} V_{it} = \ln \left[\frac{1}{1+b} (1-\tau_t) w_t H_{it} \right] \quad (2.13)$$

$$+ b \ln \left[\frac{b}{1+b} (1-\tau_t) (1-\tau_{t+1}) r_{t+1} w_t H_{it} \right] \quad (2.14)$$

To simplify this equation, all variables that are not affected by τ_t are subsumed into constant terms:¹²

$$V_{it} = \ln(1-\tau_t) + b \ln(1-\tau_t) - \alpha b \ln(1-\tau_t) + \mu \alpha b \ln(\tau_t) + \text{constants} \quad (2.15)$$

Each of the above terms in (2.15) involving τ has an economic interpretation. The first term represents the loss in first period consumption due to the income tax; the second term represents the loss of K_{t+1} due to the tax; the third term represents the increase in the interest rate due to the lower aggregate level of K_{t+1} ; and the fourth term reflects the gain from the increased level of skill among the second period's young. Note that the fourth term is increasing in μ , the parameter which measures the elasticity of education quality in the human capital production function. Also, the presence of the discount factor b reflects the fact that as the future is discounted more, the more the first term of (2.15) dominates the individual's decision.

Maximizing the above equation with respect to τ gives the preferred tax rate:¹³

$$\tau_t^* = \frac{\alpha \mu}{1 + \frac{1}{b} - \alpha(1-\mu)} \quad (2.16)$$

¹²Using (2.12), it can be shown that average human capital in the economy is proportional to the tax rate, τ

¹³Since $V'' = - \left[\frac{1+(1-\alpha)b}{(1-\tau)^2} + \frac{\mu \alpha b}{\tau^2} \right] < 0$ for all possible parameter values, this is a global maximum.

which satisfies $\tau \in [0, 1]$ for all parameter values. The tax rate is increasing in the level of α and μ . Intuitively, the higher the value of α , the more the level of children's human capital affects the return to K_{t+1} ; and the higher the value of μ , the more school quality affects human capital accumulation. The preferred tax rate is also increasing in b . Figure 1 graphs the relationship between τ^* and b for given values of the other parameters. Note that $\lim_{b \rightarrow 0} \tau^* = 0$, since middle-aged agents would have no incentive to increase the human capital levels of the young. Due to the technology and the form of preferences, the exponent on the level of human capital in the previous period does not affect the preferred tax rate. Also, the tax rate does not depend on any level variables directly, so it is time-invariant.

An increase in the tax rate in period t will reduce capital holdings in period $t + 1$ and will increase the aggregate level of human capital. Each of these effects reduces the wage rate in period $t + 1$. The young in period t must therefore weigh off the increase in H_{t+1} with the decrease in w_{t+1} . This may induce them to prefer a lower tax rate than τ^* . The elderly will also prefer the lower tax rate, and τ^* will be defeated.

The preferred tax rate of the young

Young agents will look at the total effect of the tax rate on their earnings, $w_{t+1}H_{t+1}$. If all agents are homogeneous (they possess the same ability), then

$$\text{sign} \left[\frac{\partial w_{t+1}H_{t+1}}{\partial \tau_t} \right] = \text{sign} \left[\alpha \mu \frac{(1 - \tau)}{\tau} - (1 - \alpha) \right]. \quad (2.17)$$

Since $\frac{\partial(w_{t+1}H_{t+1})^2}{\partial \tau_t} < 0$, $w_{t+1}H_{t+1}$ is a concave function of τ_t , and there exists a maximum of the function. It is sufficient to show here that at the tax rate chosen by the middle-aged τ^* , $\frac{\partial w_{t+1}H_{t+1}}{\partial \tau_t}$ is positive, which is indeed the case. This result suggests that the young will vote with the middle-aged against any alternative preferred by the elderly, and τ^* will be chosen.

If agents differ by ability, the result above may not hold. The effect of an increase in the tax rate on the young's earnings is given by the following relation:

$$\begin{aligned} \text{sign} \left[\frac{\partial w_{t+1}H_{it+1}}{\partial \tau_t} \right] = \\ \text{sign} \left[(\alpha - 1) \frac{\partial H_{t+1}}{\partial \tau_t} \frac{1}{H_{t+1}} + \frac{\partial H_{it+1}}{\partial \tau_t} \frac{1}{H_{it+1}} + (1 - \alpha) H_{t+1}^{\alpha-1} K_{t+1}^{-\alpha} \frac{\partial K_{t+1}}{\partial \tau_t} \right] \end{aligned} \quad (2.18)$$

It can be shown from (2.18) that if the elasticity of human capital with respect to τ for the individual is equal to the elasticity for the "average" agent, then (2.18) reduces to (2.17). This will be true for an average agent, or for all young agents if education quality is equally distributed to every student. However, if education funding is distributed according to (2.12), then the preferred tax rate for individuals will be an increasing function of ability. Less able students are less likely to prefer a tax rate greater than τ^* . What is important now is that the median young person prefers a higher tax rate. I assume that this holds for the rest of the discussion, so that τ^* is in fact the tax rate chosen for the economy.¹⁴

The distribution of ability across the economy ensures an unequal distribution of income. One may wonder how changing the distribution of ability will affect the aggregate accumulation of education in this model. From (2.16), the distribution of income among middle-aged agents does not affect the level of the preferred tax. However, aggregate human capital levels will depend on the distribution of ability:

$$H_{t+1} = \lambda^{\frac{\mu}{\mu-1}} \int_{\theta_l}^{\theta_u} \theta_i \frac{1}{1-\mu} H_t^\gamma dG(\theta) \quad (2.19)$$

The aggregate amount of human capital is sensitive to the c.d.f. $G(\theta)$. Consider the case of a mean-preserving spread. If

$$\int_{\theta_l}^x [G(\theta) - F(\theta)] dG(\theta) \leq 0 \quad (2.20)$$

for all $x \in [\theta_l, \theta_u]$, then G is said to be larger than F according to the definition of second-degree stochastic dominance. Since the means of the two distributions are assumed to be equal, we have:

$$\int_{\theta_l}^{\theta_u} [1 - F(t)] dt = \int_{\theta_l}^{\theta_u} [1 - G(t)] dt \quad (2.21)$$

$$\int_{\theta_l}^{\theta_u} [G(t) - F(t)] dt = 0 \quad (2.22)$$

If we also assume that θ_u is finite (and that the c.d.f.'s are continuous), then we get the following result:

$$\int_{\theta_l}^{\theta_u} \theta^{\frac{1}{1-\mu}} dG(\theta) \leq \int_{\theta_l}^{\theta_u} \theta^{\frac{1}{1-\mu}} dF(\theta) \quad (2.23)$$

¹⁴In the above discussion, I ignore the additional benefit of a higher tax rate in period t on the return to physical capital in period $t + 1$ which is also of interest to the young in period t .

since $\theta^{\frac{1}{1-\mu}}$ is convex. The proof of this (see the appendix) follows that of Theorem 2 in Hadar & Russell (1971). The implications of this is that an economy with a more unequal distribution of ability, holding the mean constant, will grow faster. The additional human capital generated by the additional agents with higher ability outweighs that lost by the increased proportion of low-ability agents.

2.2.3 Public Education and Inequality Over Time

In most of the research on public education (Stiglitz, 1974; Glomm & Ravikumar, 1992), income inequality across agents in an economy falls over time. However, due to the form of the human capital production function and the unequal distribution of education quality in this model, one does not get this result. If one measures inequality by the relative earnings of any two middle-aged workers, i, j , one gets:

$$\frac{w_t H_i}{w_t H_j} = \left(\frac{\theta_i}{\theta_j} \right)^{\frac{1}{1-\mu}} \quad (2.24)$$

Income inequality depends on the relative innate learning abilities of workers. The parameter μ also affects income inequality. A larger μ increases the difference in wage income, due to the fact that individuals with higher ability will be given higher quality education. Also, income inequality in this model is time-invariant. However, aggregate measures of inequality, such as the Gini coefficient, would change over time if distribution of ability in the economy changed across generations.

For a closer comparison to other models of education, assume that education expenditures are divided up equally among young agents, regardless of ability. Then it can be shown that wage inequality is given by the ratio $\frac{\theta_i}{\theta_j}$. Again, inequality persists. This result is similar to the private education model of Glomm & Ravikumar (1992).

2.3 Human Capital Production with Private Education

The above regime will now be compared to an education system where agents finance their own education through borrowing. The basic setup of this model will be the same as above. That is, agents will have the same type of utility function and agents will live for three periods in an

overlapping-generations framework with zero population growth (see Table 2.1). In addition, the production functions for output and education will remain unchanged.

2.3.1 Determining Levels of Education Quality

There is no public education system. Individuals face no taxes, but must finance their own educations. Individuals must borrow when young the funds needed to cover their educational expenses. These loans are borrowed from the generation ahead, and must be repaid in the next period. The individual maximizes (2.1) subject to:

$$\begin{aligned} K_{it+1} + \Delta_{it} &= w_t H_{it} - c_{it,t} - \rho_t \delta_{it-1} \\ c_{it,t+1} &= r_{t+1} K_{it+1} + \rho_{t+1} \Delta_{it} \\ H_{it} &= \theta_i \delta_{t-1}^\mu H_{t-1}^\gamma \end{aligned}$$

where ρ_t is the interest rate on education loans and Δ_t is the amount of educational loans that the individual, when middle-aged, lends to the next generation.

From the maximization problem, one can derive the levels of consumption, education quality, and saving chosen by the agents:

$$c_{it,t} = \frac{(1-\mu)}{1+b} w_t H_{it} \quad (2.25)$$

$$\delta_{it-1} = \mu \frac{w_t}{\rho_t} H_{it} \quad (2.26)$$

$$K_{it+1} + \Delta_{it} = w_t H_{it} - \frac{1+b\mu}{1+b} w_t H_{it} = \frac{b(1-\mu)}{1+b} w_t H_{it} \quad (2.27)$$

An arbitrage argument can be employed to show that in any equilibrium, $\rho_t = r_t$ (see the appendix for a proof). Henceforth, r_t will be used in place of ρ_t . The quantities are a function of each individual's expected level of human capital. Since individuals are fully informed about the form of the human capital production function and their own abilities, expected human capital levels will equal actual levels:

$$H_{it} = [\delta_{it}]^{\frac{\mu}{1-\mu}} H_{t-1}^{\frac{\gamma}{1-\mu}} \theta_i^{\frac{1}{1-\mu}} \quad (2.28)$$

Substituting (2.26) into (2.28) and solving for r_t gives the following:

$$r_t = \theta_i \mu w_t H_{t-1}^\gamma \delta_{t-1}^{\mu-1} \quad (2.29)$$

Note that this equation implies a common optimization criterion: individuals equate the marginal cost of education borrowing and the marginal return to education¹⁵. Both cost and benefit accrue in the second period, so that the discount rate, b , cancels. Education expenditures increase with μ for two reasons. First, a higher μ increases expenditures for a given level of expected income. Second, the level of expected income increases, all else equal, when μ increases. An income effect increases expenditures. Also, the income effect implies that individuals with higher ability will have higher human capital levels than less able individuals. In fact, a comparison of the wage income of any two individuals in the same generation gives a ratio exactly the same as (2.24). Inequality is the same with either public or private education.

It is also obvious how agents respond to an expected increase in the parameter α . An individual's desired level of education quality is a function of the ratio $\frac{w}{r}$. This ratio is an increasing function of α . Individuals always get more education when α rises since the wage they will receive always rises relative to the interest rate they have to pay for loans.

2.3.2 Aggregation of Education Spending and Inequality

Assuming a distribution of ability, $G(\theta)$, over the population as in the previous section, aggregate levels of consumption, education expenditures, and saving can be derived as follows. Summing equation (2.28) over all individuals, one obtains:

$$\int_{\theta_l}^{\theta_u} H_{it} dG(\theta) = \left[\mu \frac{w_t}{r_t} \right]^{\frac{\mu}{1-\mu}} H_{t-1}^{\frac{\gamma}{1-\mu}} \int_{\theta_l}^{\theta_u} \theta_i^{\frac{1}{1-\mu}} dG(\theta) \quad (2.30)$$

Solving for the aggregate levels of consumption, education expenditures, and saving now gives us:

$$\begin{aligned} \int_{\theta_l}^{\theta_u} c_{1it} dG(\theta) &= \frac{(1-\mu)}{1+b} w_t \int_{\theta_l}^{\theta_u} H_{it} dG(\theta) \\ &= \frac{(1-\mu)}{1+b} w_t \left[\mu \frac{w_t}{r_t} \right]^{\frac{\mu}{1-\mu}} H_{t-1}^{\frac{\gamma}{1-\mu}} \int_{\theta_l}^{\theta_u} \theta_i^{\frac{1}{1-\mu}} dG(\theta) \end{aligned} \quad (2.31)$$

¹⁵Individuals may not be able to borrow at the market interest rate due to market imperfections in education borrowing. Lenders might have to monitor borrowers; this increases the cost of borrowing. For instance, banks may choose a cost of defaulting per unit of monitoring, $\pi > 1$, where π is chosen by the banks to satisfy an incentive compatibility constraint. Then the equilibrium repayment per unit of the loan will be $\tilde{r} = \frac{1+\pi r}{\pi-1} > r$. See Galor & Zeira (1993). If this holds, then the preferred δ will be smaller since the right hand side of (2.26) is decreasing in δ^i .

$$\int_{\theta_l}^{\theta_u} \delta_{it-1} dG(\theta) = \left[\mu \frac{w_t}{r_t} \right]^{\frac{1}{1-\mu}} H_{t-1}^{\frac{\gamma}{1-\mu}} \int_{\theta_l}^{\theta_u} \theta_i^{\frac{1}{1-\mu}} dG(\theta) \quad (2.32)$$

$$\int_{\theta_l}^{\theta_u} K_{it+1} + \Delta_{it} dG(\theta) = \frac{b(1-\mu)}{1+b} w_t \left[\mu \frac{w_t}{r_t} \right]^{\frac{\mu}{1-\mu}} H_{t-1}^{\frac{\gamma}{1-\mu}} \int_{\theta_l}^{\theta_u} \theta_i^{\frac{1}{1-\mu}} dG(\theta) \quad (2.33)$$

As in the public education model, equation (2.23) suggests that human capital levels and education expenditures are greater under c.d.f. $F(\theta)$. Increasing the dispersion of θ , holding the mean constant, increases the human capital levels of the agents in the upper-tail of the distribution more than the decrease in the human capital levels of the lower-tail agents.

2.4 Comparing Steady States

In this section I derive steady state levels of the two regimes, where the steady state implies a constant physical capital-human capital ratio. I make two assumptions in order to solve for the steady state. First, all agents are assumed to be identical. Second, the human capital production function will now be constant returns to scale:

$$H_t = \theta \delta_{t-1}^{1-\gamma} H_{t-1}^\gamma \quad (2.34)$$

2.4.1 Steady State with Public Education

Using the above model, I solve for difference equations showing the path of accumulation of the two level variables, H and K :

$$K_{t+1} = \frac{b}{1+b} (1-\tau) \alpha A H_t^\alpha K_t^{1-\alpha} \quad (2.35)$$

$$H_{t+1} = \theta \tau^{1-\gamma} A^{1-\gamma} H_t^{\alpha-\alpha\gamma+\gamma} K_t^{1-\gamma-\alpha+\alpha\gamma} \quad (2.36)$$

It is necessary to define a new variable in order to solve for a steady state. In the long run, accumulation of both forms of capital will continue. The new steady state variable allows us to solve for the ratio of physical capital to human capital that is constant in each period in the long run.

$$Z_{t+1} = \frac{K_{t+1}}{H_{t+1}} = \frac{\alpha A^\gamma}{\theta} \frac{b}{1+b} \frac{1-\tau}{\tau^{1-\gamma}} Z_t^{\gamma-\alpha\gamma} \quad (2.37)$$

In the steady state, Z will be constant. This constant is given by:

$$\bar{Z} = \left[\frac{\alpha A^\gamma}{\theta} \frac{b}{1+b} \frac{1-\tau}{\tau^{1-\gamma}} \right]^{\frac{1}{1-\gamma+\alpha\gamma}} \quad (2.38)$$

A necessary condition for a stable non-cyclical steady state is that the slope of the phase diagram at the steady state level is positive and less than one. This condition is satisfied for any allowable values of γ and α :

$$\left. \frac{\partial Z_{t+1}}{\partial Z_t} \right|_{Z=\bar{Z}} = \gamma(1-\alpha) \quad (2.39)$$

A more visual way of expressing the steady state is to derive the growth rates (factors, actually) of human and physical capital as functions of the ratio Z :

$$g_H^p = \frac{H_{t+1}}{H_t} = \theta \tau^{1-\gamma} A^{1-\gamma} Z^{(1-\alpha)(1-\gamma)} \quad (2.40)$$

$$g_K^p = \frac{K_{t+1}}{K_t} = \frac{b}{1+b} (1-\tau) \alpha A Z^{-\alpha} \quad (2.41)$$

These equations can be graphed against the level of Z . The graph of g_H^p is upward-sloping and the graph of g_K^p is downward-sloping. The values of the growth rates when $Z = 0$ and $Z \rightarrow \infty$ ensures that the two graphs intersect. This intersection is the steady state value of Z . Starting from any initial Z , the economy always converges to the steady state. The out-of-steady state implications are evident. Economies with low physical-to-human capital, such as western Europe after World War II, should have higher growth rates of physical capital and lower growth rates of human capital than economies with a relatively high physical-to-human capital ratio.

2.4.2 Steady State with Private Education

As in the economy with public education, there exists a steady state with a constant capital-labor ratio, Z . This is found as follows:

$$\frac{K_t}{H_t} = \frac{\frac{b\gamma}{1+b} w_{t-1} H_{t-1} - \delta_{t-1}}{\theta \delta_{t-1}^{1-\gamma} H_{t-1}^\gamma} \quad (2.42)$$

From the first-order condition, δ can be rewritten:

$$\delta_{t-1} = \left[(1 - \gamma)\theta \frac{\alpha}{1 - \alpha} \right]^{\frac{1}{\gamma}} Z_t^{\frac{1}{\gamma}} H_{t-1} \quad (2.43)$$

From here, one can solve for Z_t in terms of Z_{t-1} :

$$Z_t = \frac{\left[\frac{b\gamma}{1+b} \alpha A \right]^{\gamma} \left[\frac{\Omega}{1+\Omega} \right]^{\gamma} Z_{t-1}^{\gamma(1-\alpha)}}{[\theta\Omega]} \quad (2.44)$$

where $\Omega = (1 - \gamma) \frac{\alpha}{1 - \alpha}$. One can solve for the steady state level \bar{Z} to get:

$$\bar{Z} = \frac{\left[\frac{b\gamma}{1+b} \alpha A \frac{\Omega}{1+\Omega} \right]^{\frac{\gamma}{1-\gamma+\alpha\gamma}}}{[\theta\Omega]^{\frac{1}{1-\gamma+\alpha\gamma}}} \quad (2.45)$$

A necessary condition for a stable non-cyclical steady state is that the slope of the phase diagram at the steady state level is positive and less than one.

$$\left. \frac{\partial Z_{t+1}}{\partial Z_t} \right|_{Z=\bar{Z}} = \gamma(1 - \alpha) \quad (2.46)$$

This is the same as in section 2.4.1.

As in the previous section, one can graph the growth rates of human and physical capital against levels of Z .

$$g_H^I = \frac{H_{t+1}}{H_t} = \theta \left[\frac{\alpha b \gamma}{1+b} A \frac{\Omega}{1+\Omega} \right]^{1-\gamma} Z_t^{(1-\gamma)(1-\alpha)} \quad (2.47)$$

$$g_K^I = \frac{K_{t+1}}{K_t} = \frac{\alpha b \gamma}{1+b} A \left[\frac{1}{1+\Omega} \right] Z_t^{-\alpha} \quad (2.48)$$

This analysis is similar to the previous section, showing that the economy converges to \bar{Z} beginning from any level of Z .

2.4.3 Growth Or Decline in the Steady State?

Although the preceding analysis suggests that steady states will exist in each regime, it is not assured that the levels of human and physical capital will be growing over time. It is possible that

the values of g^P and g^I are less than one, so that the economy is converging to nonexistence over time.

I take special note of the productivity parameters θ and A , since they have been somewhat ignored to this point. If these parameters are too small, the economy will be imploding. A large A signifies a greater productivity of the economy's resources. This allows greater investment in both forms of capital, and thus greater growth.

The parameter θ measures the innate abilities of students. This parameter has two countervailing forces on the growth rates of capital. A larger value for θ obviously increases the amount of human capital available in the economy *ceteris paribus*; this increases the growth of human capital. However, the steady state physical capital-human capital ratio is decreasing in θ in both regimes. Since the growth rate of human capital is *increasing* in this ratio, the indirect effect of a larger θ is to decrease the growth rate of human capital. In the private regime, a lower Z increases the relative cost of education, and in the public regime, tax revenues collected from physical capital help to propel human capital in the next period beyond the depressed current level of human capital.

Despite the countervailing implications of a higher θ , the positive benefit outweighs the negative, so g^P and g^I are increasing in θ . In fact, the marginal effect of an increase in θ on the growth rate in either regime is

$$\alpha\theta^{\alpha-1} \tag{2.49}$$

The more important human capital is in production, the greater the effect on output θ will have. Figure 2.2 shows the growth rates for different levels of θ . These curves are monotonically increasing and convex.

2.5 Comparing Public and Private Regimes

In this section, I will compare the two regimes from several perspectives. First, I compare the quality of education in a period in which incomes are equal. Second, I compare the steady state growth rates of physical and human capital. And third, I make a welfare comparison by determining if a movement from one system to the other is Pareto-improving.

2.5.1 Comparisons of Physical Capital Accumulation

In this section, I compare the effect of increasing δ by one unit on the amount of physical capital that will be accumulated. I compare the public and private regimes to a Social Planner's problem to investigate the efficiency of incentives to save in each regime.

In the public regime, a one-unit increase in δ can be achieved by an increase in τ such that disposable income is decreased by 1 unit. The disposable income of the middle-aged will decrease by α units since this is the share of national income that goes to workers. The marginal propensity of the middle-aged to save is equal to $\frac{b}{1+b}$ (see equation (2.8)). Therefore, physical capital will decline by $\alpha \frac{b}{1+b}$ for an additional unit of education funding.

In the private regime, an extra unit of education spending will decrease physical capital in the next period by one unit. This is so because middle-aged agents continue to save the same amount, but they must increase their holdings of education loans by one unit. Thus, physical capital purchases must decrease by one unit.

This leads to the *static* optimal equilibrium condition in the private education model that the value of an additional unit of education spending, $w_{t+1}\mu\theta H_t^\gamma \delta_t^{\mu-1}$, must be equal to the cost, r_{t+1} .

The private model is only statically efficient because it does not take into account the gains to future generations from an increase in H_{t+1} . Consider the effect of increasing δ_0 (the education spending for generation 1) on generation 2. There are two benefits:

- The higher human capital levels of generation 1 will increase their savings, which will increase the physical capital available in period 2.
- Members of generation 2 will be able to accumulate more human capital for a given level of δ_1 .

If the gains to generation 2 and beyond were taken into account by generation 1, then generation 1 would invest more in education than they do in the private regime above.

The public education system sets a lower cost on education funding than the private system. However, the public regime undervalues the benefits of additional education funding because it only takes into account the benefits to the middle-aged of educating young people in each period.

The difference between costs and benefits depends on the parameters of the production functions and the discount factor. In the next section, I derive conditions under which the public system provides more education funding than the private system.

2.5.2 Comparisons of Education Quality

Consider two economies at some time period t with identical endowments of physical and human capital and identical technologies. If one of these economies has a public education system and the other has a private education system as described above, then which of these economies will provide the next generation with more education? Education quality in the public-education economy is given by the following equation:

$$\delta_t^P = \tau^* Y_t = \frac{\alpha(1-\gamma)}{1 + \frac{1}{b} - \alpha\gamma} Y_t \quad (2.50)$$

Education quality in the private-education economy is given by:

$$\delta_t^I = \frac{1-\gamma}{r_{t+1}} w_{t+1} H_{t+1} = \frac{1-\gamma}{r_{t+1}} \alpha Y_{t+1} \quad (2.51)$$

A key difference in these equations is the fact that public education expenditures are determined by the income of the economy one period ahead of the private expenditures. This is because in the public education regime, each generation is deciding education expenditures for their children, whereas in the private system, individuals are determining their own education expenditures. When I compare these two regimes at any time, I have to project the private-education economy one period ahead. This is simple, provided the parameters do not change over the period. Starting with (2.51), one can derive δ_t^I in terms of period t income:

$$\begin{aligned} \delta_t^I &= \frac{1-\gamma}{r_{t+1}} \alpha Y_{t+1} = (1-\gamma) \frac{\alpha}{1-\alpha} K_{t+1} \\ \delta_t^I &= (1-\gamma) \frac{\alpha}{1-\alpha} \frac{b\gamma}{1+b} [w_t H_t - \delta_t^i] \\ \delta_t^I &= \frac{\Omega}{1+\Omega} \frac{\alpha b \gamma}{1+b} \alpha Y_t \end{aligned}$$

using $\Omega = (1-\gamma) \frac{\alpha}{1-\alpha}$ as in the previous section. Note that Ω is a measure of the wage/return-to-capital ratio times the elasticity of education quality on human capital accumulation. Since

$\frac{\partial \frac{\Omega}{1+\Omega}}{\partial \Omega} = \frac{1}{(1+\Omega)^2} > 0$, education quality increases in either $\frac{w}{r}$ or $(1 - \gamma)$. In effect, individuals invest more when the returns to human capital rise relative to the cost of borrowing funds to finance that education or when the elasticity of spending on human capital increases.

The value of K_{t+1} is the amount that generation t had saved. This depends on the discount factor and the amount of education loans taken out by generation t .

Given the same Y_t in the two regimes, $\delta_t^P > \delta_t^I$ iff $\frac{1-\gamma}{1+\frac{1}{b}-\alpha\gamma} > \frac{b\gamma}{1+b} \frac{\Omega}{1+\Omega}$ ¹⁶. First, I consider two extreme cases: Case 1 assumes that the term b approaches ∞ and Case 2 assumes that $b = 0$. In Case 1, one can show that the public education regime will give a higher level of education financing than the private system. In this case, $\delta^I = \alpha\gamma\delta^P < \delta^P$ since α and γ are both less than 1. In case 1, agents only care about third period consumption. It is important to note here why the private system does poorly as α and γ fall. For low values of α , agents in the private system receive lower wages but have to pay higher interest payments on education loans. In the public education system, middle-aged agents are not so concerned about their lower incomes from working (the elderly pay a higher share of the taxes), but the increased return to physical capital will induce them to vote for higher tax rates. In the private regime, a lower γ (higher μ) implies that the previous generation had saved less, so that w_{t+1} is lower and r_{t+1} is higher.

In Case 2, agents only care about middle-aged consumption. The amount of education funding in both regimes goes to zero. The middle-aged in the public education system will obviously vote for zero taxes. In the private system, the young will be unable to borrow funds from the middle-aged and therefore they will have no education funding.

Aside from these special cases, one can compare the two regimes by looking at the combination (α, γ) where the two regimes give equal amounts of education funding, given a value for b . The result is given by the following equation:

$$\gamma = \frac{B}{\alpha} \tag{2.52}$$

where $B = \frac{1+b-(1+b)^{\frac{1}{2}}}{b}$. $\delta^I > \delta^P$ if $\gamma > \frac{B}{\alpha}$. For a given b , this equation defines a rectangular hyperbola in (γ, α) -space. Given the restrictions on the values of the parameters, when $\alpha \in [B, 1]$, it is possible to have a γ such that $\delta^I \geq \delta^P$. Otherwise, $\delta^P > \delta^I$. Note that the function B is monotonically increasing with $\lim_{b \rightarrow 0} = .5$ and $\lim_{b \rightarrow \infty} = 1$. Therefore, it is possible for $\delta^I > \delta^P$

¹⁶A reduced form of this condition is $(1 - \alpha\gamma)(1 + b) > \alpha b\gamma(1 + \frac{1}{b} - \alpha\gamma)$.

for any value of b , but only if α and γ are sufficiently large. For example, if $b = 1$ and $\alpha = .7$, then $\delta^I > \delta^P$ only if γ is greater than approximately .84.

The importance of a large value of γ for the superiority of the private regime can be ascertained from the above discussion. When γ is large, investments in education are relatively unimportant. People borrow less for education, therefore more physical capital is produced. It is somewhat ironic that the private regime is less likely to be dynamically efficient as compared to a regime with a Social Planner when γ is larger.¹⁷

2.5.3 A Comparison of Steady State Growth Rates

In this section, I compare the growth rates of human capital and physical capital in the steady state. The steady state in both of these models is characterized by a constant physical capital per effective worker, so that physical capital and human capital should grow at the same rate. The growth factors for the two regimes are expressed in the following form:

$$\frac{H_t}{H_{t-1}} = G^q(\bar{Z}^q; \theta, \alpha, \gamma, b)$$

for $q = P, I$. This specification shows that the growth factors depend on the steady state levels of the ratio of physical capital to human capital.

For the public education regime, we have:

$$G^P = \theta A^{1-\gamma} \tau^{1-\gamma} \bar{Z}_P^{(1-\alpha)(1-\gamma)} \quad (2.53)$$

And for the private education regime:

$$G^I = \theta A^{1-\gamma} \left[\frac{\alpha b \gamma}{(1+b)} \frac{\Omega}{(1+\Omega)} \right]^{1-\gamma} \bar{Z}_I^{(1-\alpha)(1-\gamma)} \quad (2.54)$$

Which regime gives higher steady-state growth rates?

First, consider the special case where $b \rightarrow \infty$. This greatly simplifies the comparison of the two growth rates due to the fact that as b approaches infinity, τ converges to $\frac{\Omega}{1+\Omega}$ and $\frac{1-\tau}{\tau}$ converges to $\frac{1}{\Omega}$. In this case, G^P is greater than G^I if and only if:

$$\alpha^{-\alpha} > \gamma \quad (2.55)$$

¹⁷See the discussion in the previous section.

There are no permissible values of α and γ which do not satisfy this inequality (see the appendix for proofs). Thus, the public regime always gives higher growth rates when individuals only care about consumption in the last period of life.

For smaller values of b , one can show that $G^P > G^I$ if the following inequality holds:

$$\alpha^{-\alpha} > \gamma \frac{\Upsilon}{\Lambda} \quad (2.56)$$

where Λ and Υ are functions of the parameters.¹⁸ In the above case where $b \rightarrow \infty$, Λ and Υ are equal. It can be shown that Υ is independent of b and that Λ is an increasing function of b .

As b gets smaller, the right hand side of equation (2.56) gets larger. So the smaller is b , the more likely it is that the steady state growth rate in the private system will be larger than the growth rate in the public system steady state. Figure 3 shows possible values of b for which the private system gives higher growth. The parameter values are highly restrictive and not likely to represent the values found in an actual economy.

2.5.4 A Comparison of Welfare

In an overlapping-generations model, the concept of comparing welfare in a Pareto sense is rather restrictive. It requires that every generation must be made better off, since there is no way to trade off welfare gains and losses across generations. Suppose that at some point in time the economy moves from a public education system to a private education system, or vice versa. Can we say that the move will be welfare-improving?

First, we can dismiss a move from a private education system to a public education system as Pareto-improving by looking just at the old generation's welfare at the switch. In a private system, the old consume all their rental income. But in a public system, this income would be subject to taxation. Therefore, the initial old generation would be made worse off.

I analyze welfare differences by comparing the utilities of comparable generations beginning with the initial middle-aged generation. Once a steady state has been reached, the change in the utility of subsequent generations is constant. In particular,

$$\Delta u_t^j = (1 + b) \ln(G^j), \quad j = \{P, I\} \quad (2.57)$$

¹⁸ $\Lambda = \tau^\alpha \left[\frac{b}{1+b} \right]^{-\alpha} (1 - \tau)^{1-\alpha}$ and $\Upsilon = \frac{\Omega^\alpha}{1+\Omega}$.

Table 2.2: Summary of Welfare Gains from Movement from a Public Education System to a Private System

α	.7	.7	.7	.7
b	1	.5	10	1
γ	.8	.8	.8	.5
A	10	10	10	10
θ	1	1	1	1
$I > P$	1st Period	Never	1st-4th Periods	Never

where Δu_t is the change in the utilities of subsequent generations. This equation reiterates the importance of the analysis of the previous section. Since the public regime will usually have a higher growth rate, individuals will eventually attain higher utility than in the private regime.

In order to determine the effect of a move from a public system to a private system, I ran simulations of an economy from an initial state to the steady state. In most of these simulations, the economy converged to the steady state within a very few iterations. To see the effect of a move from a public system to a private system, I compared utility levels in each of the two regimes once a steady state was attained in the public system. For most of the periods, the public system gave higher utilities. However, the private system had higher utility in at least one generation for many parameter values. If I assume α equals .7, then I am able to concentrate my attention on changing the values of b and γ . In general, the private system is more likely to give higher utility levels for at least the first generation if b and γ are large. Table 2.2 summarizes some of the results.

2.6 Concluding Remarks

This paper has studied a public education system in which the motive of agents is to maximize their consumption. Unlike previous models of public education, there are no bequest motives or externalities to encourage investment in education. Here, educating children increases the future returns to physical capital for retired agents. Under this system, there is no incentive for neighborhood stratification, but society does prefer an education policy which provides higher quality education to students with higher ability. Thus, public education still does not lead to convergence

of earnings over time.

A comparison of the public education model with a private education model suggests that there are similarities between the two systems. However, I have derived the conditions on the parameters of the model that are necessary for one system to (a) give higher education quality, (b) give higher steady state growth rates for physical and human capital, and (c) provide higher utility levels. The public education system provides higher growth rates and higher education quality under most (including the most realistic) parameter specifications.

Appendix A

A.1 Proof that the level of aggregate human capital depends on the distribution of ability.

$$\begin{aligned} \int_{\theta^l}^{\theta^u} \theta^{\frac{1}{1-\mu}} dG(\theta) - \int_{\theta^l}^{\theta^u} \theta^{\frac{1}{1-\mu}} dF(\theta) = \\ \int_{\theta^l}^{\theta^u} \theta^{\frac{1}{1-\mu}} g(\theta) dG(\theta) - \int_{\theta^l}^{\theta^u} \theta^{\frac{1}{1-\mu}} f(\theta) dG(\theta) \end{aligned} \quad (\text{A.1})$$

Using integration by parts:

$$\begin{aligned} \int_{\theta^l}^{\theta^u} \theta^{\frac{1}{1-\mu}} g(\theta) dG(\theta) - \int_{\theta^l}^{\theta^u} \theta^{\frac{1}{1-\mu}} f(\theta) dG(\theta) = \\ \theta^{\frac{1}{1-\mu}} [G(\theta) - F(\theta)] \Big|_{\theta^l}^{\theta^u} - \int_{\theta^l}^{\theta^u} \frac{1}{1-\mu} \theta^{\frac{\mu}{1-\mu}} [G(\theta) - F(\theta)] dG(\theta) \end{aligned}$$

The first term on the R.H.S. vanishes since $(\theta^u)^{\frac{1}{1-\mu}}$ is bounded and $[G - F]_{\theta^l}^{\theta^u} = 0$. Integrating by parts the second term gives:

$$\begin{aligned} \int_{\theta^l}^{\theta^u} \frac{1}{1-\mu} \theta^{\frac{\mu}{1-\mu}} [G(\theta) - F(\theta)] dG(\theta) = \\ \left[\frac{1}{1-\mu} \theta^{\frac{\mu}{1-\mu}} \int_{\theta^l}^x [G(x) - F(x)] dx \right] \Big|_{\theta^l}^{\theta^u} - \int_{\theta^l}^{\theta^u} \frac{\mu}{(1-\mu)^2} \theta^{\frac{2\mu-1}{1-\mu}} \int_{\theta^l}^x [G(x) - F(x)] dx dG(\theta) \end{aligned}$$

The first term on the R.H.S. vanishes given equation (2.22) and that $\frac{1}{1-\mu}(\theta^u)^{\frac{\mu}{1-\mu}}$ is finite. One can sign the remaining term, considering equation (2.20), so that the inequality in equation (2.23) is proved. Q.E.D.

A.2 Proof that r_t must equal ρ_t .

Note from the budget constraints that the costs of transforming savings into either capital or education quality are identical. Therefore, individuals will buy no Δ_t if $r_{t+1} > \rho_{t+1}$ and no K_{t+1} if $r_{t+1} < \rho_{t+1}$. If $r_{t+1} = \rho_{t+1}$, then individuals are indifferent between how much of each they will buy, and the quantity of education provided will be determined by demand.

In equilibrium, the aggregate amount of education quality demand by generation t must be equal to the amount supplied by generation $t + 1$ for all t . If $r_{t+1} > \rho_{t+1}$, the supply of Δ_t will be zero. This can only be an equilibrium if demand equals zero also. This requires from the first order conditions that $\mu \frac{w_{t+1}}{r_{t+1}} H_{t+1}^i = 0$ for all i . But this never occurs unless production in the economy is zero.

If $r_{t+1} < \rho_{t+1}$, then $\Delta_t = \frac{b(1-\mu)}{1+b} w_t \int_{\theta^l}^{\theta^u} H_t^i dG(\theta)$ and $K_{t+1} = 0$. For this to be an equilibrium, the quantity of loans supplied must be equal to the quantity demanded, that is ρ_{t+1} solves for $\Delta_t = \delta_t$:

$$\rho_{t+1} = \frac{\mu w_{t+1} \int_{\theta^l}^{\theta^u} H_{t+1}^i dG(\theta)}{\frac{b(1-\mu)}{1+b} w_t \int_{\theta^l}^{\theta^u} H_t^i dG(\theta)} \quad (\text{A.2})$$

But if $K_{t+1} = 0$, then $w_{t+1} = 0$. This would imply that in equilibrium, $\rho_{t+1} = 0$. However, $r_{t+1} \rightarrow \infty$, so that the initial assumption that $\rho_{t+1} > r_{t+1}$ would be violated. Thus, only $r_t = \rho_t$ for all t can hold in equilibrium. Q.E.D.

A.3 Proof of equation (2.55) and the subsequent statement.

The sign of the difference between G^P and G^I is the same as the difference between the two terms once all common terms are factored out. This difference is:

$$\alpha \frac{\alpha(\gamma-1)}{1-\gamma+\alpha\gamma} - \gamma \frac{1-\gamma}{1-\gamma+\alpha\gamma} \quad (\text{A.3})$$

Likewise, the sign of this is equivalent to the sign of any increasing monotonic transformation of the two terms. We can therefore cancel the denominators of the exponents of the two terms and simplify to get (2.55).

For the proof that inequality (2.55) always holds for permissible values of γ and α , we show that $\alpha^{-\alpha}$ is always greater than 1. Consider the contrary: $\alpha^{-\alpha} < 1$. Thus,

$$-\alpha \ln(\alpha) < \ln 1 = 0 \tag{A.4}$$

But for $\alpha < 1$, $\ln(\alpha) < 0$, and the above inequality can never hold. Q.E.D.

A.4 Figures

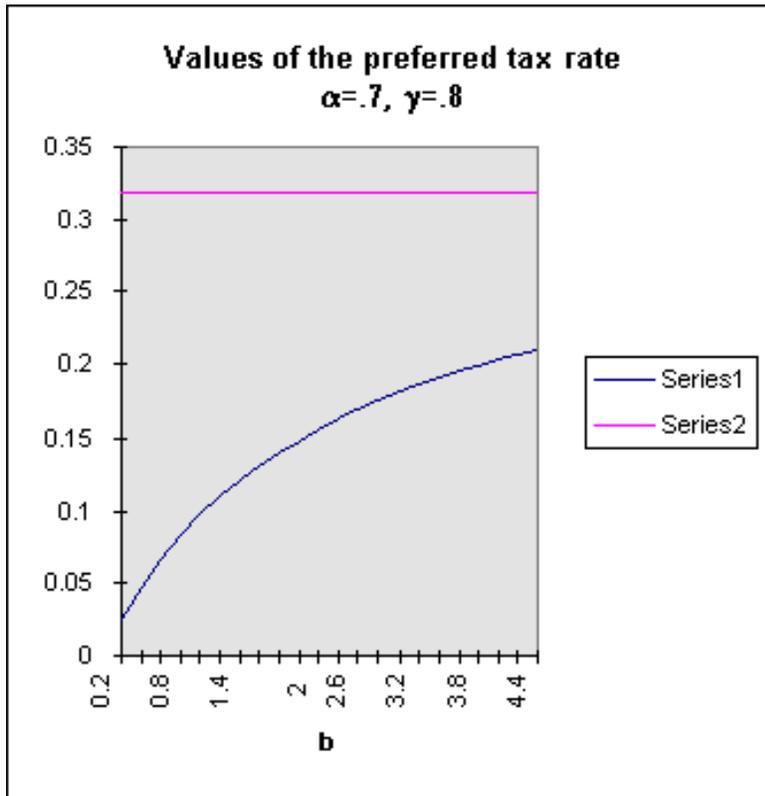


Figure 2.1

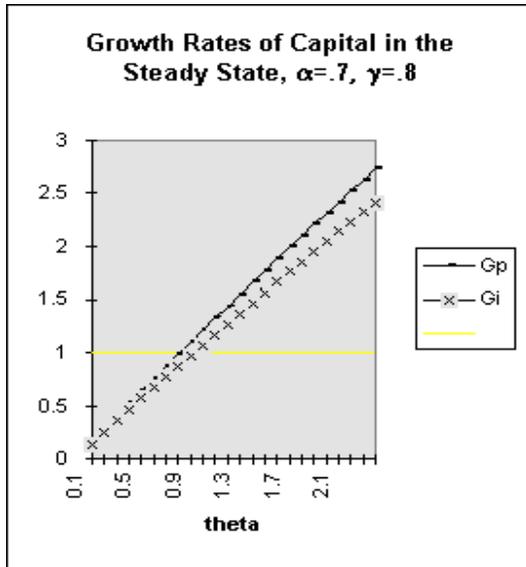


Figure 2.2a

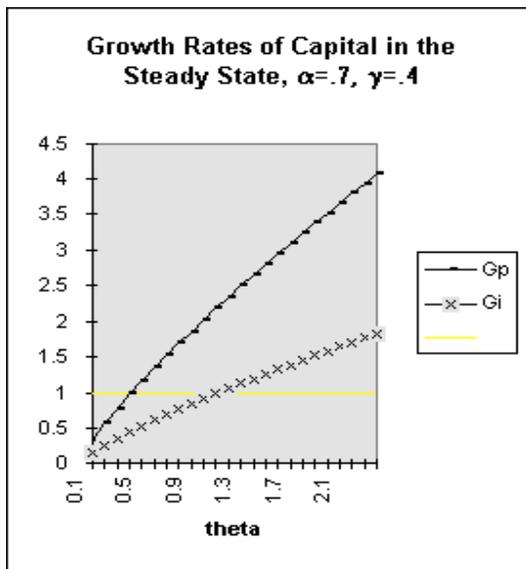


Figure 2.2b

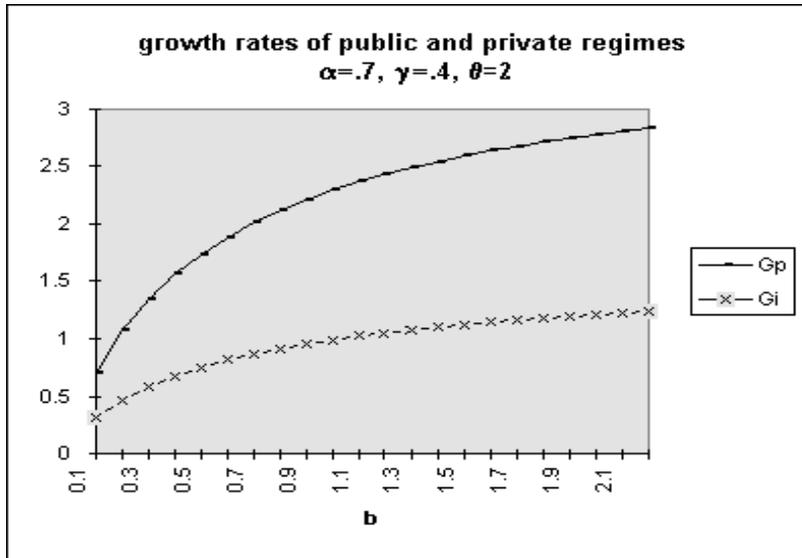


Figure 2.3a

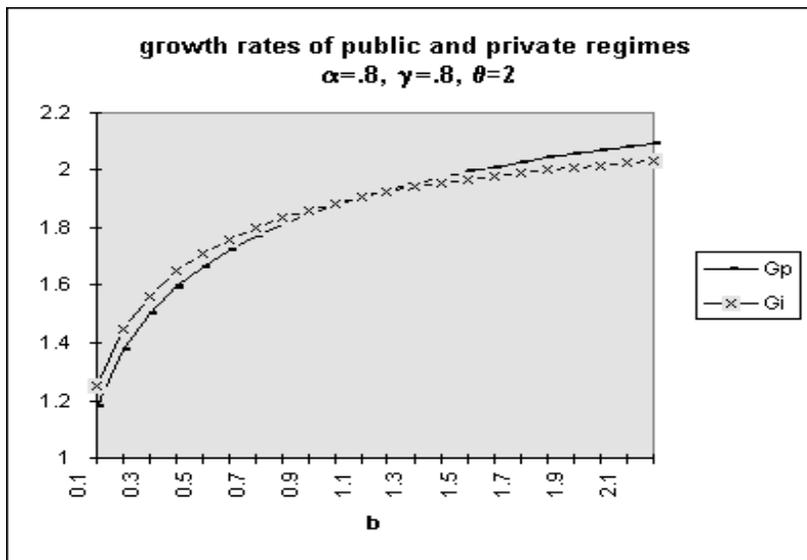


Figure 2.3b