Essays in Revenue Management and Dynamic Pricing

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Abstract

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In this dissertation, I study two topics in the context of revenue management. The first topic involves building a mathematical model to analyze the competition between many retailers who can change the price of their respective products in real time. I develop a game-theoretic model for the dynamic price competition where each retailer's objective is to maximize its own expected total revenue. I use the Nash equilibrium to predict market equilibrium and provide managerial insights into how each retailer should take into account its competitors' behavior when setting the price.

The second topic involves working with Amtrak, the national railroad passenger corporation, to develop a revenue management model. The revenue management department of Amtrak provides the sales data of Auto Train, a service of Amtrak that allows passengers to bring their vehicles on the train. I analyze the demand structure from sales data and build a mathematical model to describe the sales process for Auto Train. I further develop an algorithm to calculate the optimal pricing strategy that yields the maximum revenue. Because of the distinctive service provided by Auto Train, my findings make important contribution to the revenue management literature.

To

Mansoureh

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Chapter 1

Introduction and Literature Review

Revenue management is a scientific method that helps firms to improve profitability of their business. For many years, firms use revenue management to predict demand, to replenish inventory, and to set the product price. The benefit of revenue management can been found in a variety of industries, including airlines, hotels, and electric utilities.

Dynamic pricing is a popular method of revenue management, especially when a firm needs to sell a given stock by a deadline. The goal of dynamic pricing is to increase the revenue by discriminating customers who arrive at different times. For instance, if a firm faces a high level of demand, it has an incentive to increase the price to reserve some products for later customers who may be willing to pay more. On the other hand, if the demand is low the firm wishes to lower the price in order to induce the demand, because the product has no value (or very low value) after the deadline. Therefore, with dynamic pricing, a firm sets different prices depending on time-dependent variables such as demand, inventory, and customer type.

Dynamic pricing strategy often discounts excessive items, and charges a higher price for scarce items. A firm can often increase its revenue by carefully adjusting its product price over time.

The rest of this chapter is organized as follows. In Section 1.1, we discuss the applications of dynamic pricing. In Section 1.2, we review relevant literature. In Section 1.3, we present the outline of the rest of this dissertation.

1.1 Applications of Dynamic Pricing

Dynamic pricing practices appear in industries where it is impossible (or very costly) to increase the inventory level and there exists a deadline to sell the products. These industries include fashion goods, airlines, hotel rooms, rental cars, and highway congestion control. The vendor of a fashion apparel, which perishes when the season ends, has to set its product price during the season. The consumers of the fashion goods enjoy more if they have the item in the earlier points of the season. For example, an air conditioner is more valuable at the beginning of the summer, so the consumers are willing to pay more in price earlier in summer. Using this fact, a retailer can potentially increase the revenue by charging a higher price at the beginning of the season and a lower price at the end of the season.

Airline industries often sell the airline tickets at a lower price to those consumers who buy their tickets early in advance, in order to reduce the risk of the airplane taking off with many vacant seats. If too many seats are sold at a low price, the airlines often would raise its ticket price, not only because the risk of having vacant seats is low but also because the airline wants to reserve seats for potential last-minute travellers.

Recently, dynamic pricing has been used in transportation planning in order to control the congestion level of a highway. This type of pricing deals with the time value of passengers who are willing to pay a fee to reduce the delay on a congested highway. This technique is called value pricing, congestion pricing, or peak-period pricing. Value pricing is a multi objective dynamic pricing problem where both the social welfare and the expected revenue are two potentially conflicting objectives. It charges different fees, or tolls, for the use of a highway based on the congestion level. Fees are typically assessed electronically (such as EZ-pass used in LBJ highway, Texas) to reduce delays associated with mechanical toll collections.

1.2 Literature Review

There is a substantial literature on dynamic pricing under different assumptions. Research in this area has focused mostly on airline and hotel industries. Elmaghraby and Keskinkocak [11] define three categories for dynamic pricing models including replenishment policy, consumer behavior, and time dependency. We add the number of retailers in the market as another characteristic and next explain these four categories in more detail.

- *Replenishment or no replenishment of inventory.*

  Inventory policy plays an important role in revenue management models. If inventory replenishment is allowed during the time horizon, the retailer should make a joint inventory and pricing decision during the time horizon; if the replenishment is not allowed, the retailer should make the pricing decision based on the given inventory. Examples of products with no replenishment policies include fashion goods, seasonal products, airline seats, and hotel rooms. We will explore the literature of pricing policies with no replenishment in more detail in Chapter 2. In this study, we do not review the dynamic pricing policies with replenishment. However, interested readers can review dynamic pricing models with inventory replenishment in Federguen and Heching [12] and Berstein and Federguen [2].

- *Dependent or independent demand over time.*

  If a retailer has a durable product to sell, the demand for the product might be a dependent function across multiple periods of time. For this type of product, the benefit duration of the product is longer than the time horizon of the sale. On the other hand, if costumers' knowledge about the product plays an important role in their decision to buy the product, the demand would also be dependent over time. For such products, the current sale has a negative impact on the sale in future. In contrast, for some non-durable products the demand is independent over time and the current sale has no impact on the future sale. We consider time-independent demand models in this study and refer readers to Bitran and Mondschein [4] and Zhao and Zheng [38] for models with dependent demand.

- *Myopic or strategic customers.*

  The retailers should take into account the purchasing behavior of the customers in order to have an efficient pricing policy. If a customer makes his decision based only on the price he sees when he arrives, we call this customer a *myopic* customer. On the contrary, we call it a *strategic* customer if the customer takes into account the price history and the possible price movement in the future when deciding whether to buy the product. In this study, we focus on models with myopic customers.

- *Single-retailer or multiple-retailer models.*

  The majority of research in the literature assumes that a retailer enjoys a monopoly. An important assumption of such a monopolistic model is that the distribution of the random demand depends only on the price of the monopolist. Although this assumption is quite realistic in the past, when it was difficult for a customer to compare prices, it can be problematic nowadays in the presence of the Internet. With a few minutes of surfing on the Internet, a customer can easily extract the real-time price of different airlines. This information has a huge impact on the customers' behavior and on the entire demand. Therefore, in such cases, the demand for a product depends not only on its own price, but also on the price of similar products.

In this study, we focus on dynamic pricing models with no inventory replenishment, independent demand over time, and myopic customers. In the first part of our study, we consider a model with multiple retailers. In the second part, we consider a monopolistic retailer—Amtrak. We next review related dynamic-pricing literature for both monopolistic and oligopolistic markets.

1.2.1 Monopolistic Dynamic Pricing Literature

The literature of monopolistic dynamic pricing is substantial. Research papers on this topic use different methods to model the demand, pricing policies, and the customer response to price. In this section, we introduce these characteristics and discuss different solution methods studied in the literature. Demand modeling—specifically, the customer response with respect to price—is one of the most important issues in dynamic pricing. In other words, the impact of price on the arrival process and on the customer willingness to pay needs to be considered in dynamic pricing models. Some papers including Lazear [18] and Elmaghraby et al. [10] assume that the potential number of customers is deterministic and known to the retailers. These papers assume that the customers have a reservation price for the and this reservation price is a random variable with a known distribution function. Other researches such as Gallego and van Ryzin [14] and Feng and Gallego [13] model the demand as a Poisson process whose rate is non-increasing in price. Some other papers, such as Zhao and Zheng [38] and Bitran et al. [3] consider non-homogenous Poisson processes. Lin [20] assumes that the total number of potential customers is a random variable and customers arrive sequentially.

Another important characteristic of dynamic pricing models is the number of allowable price changes during the sales horizon. Gallego and van Ryzin [14] and Bitran and Mondschein [4] develop continuous-time dynamic pricing models. Feng and Gallego [13] address models with limited number of price changes during the sales period. Lin [20] studies a model with sequential arrivals when price changes are allowed for each new customer. In Feng and Gallego [13], the price can only be changed once; the new price is predetermined and the objective is to find the optimal time of the price change. They consider both the markdown and markup pricing policy and determine a time threshold to change the price. Bitran and Mondschein [4] study a model where the price can be changed many times. Papers that study pricing policies of a monopolist with no inventory replenishment and independent demand over time include Bitran et al. [3], Bitran and Monschein [4], Feng and Gallego [13], Gallego and van Ryzin [14], Zhao and Zheng [38], and Smith and Achabal [28]. A complete review of existing literature in dynamic pricing can be found in McGill and van Ryzin [23] and Elmaghraby and Keskinocak [11]. We review a selective number of papers in more detail in what follows.

Gallego and van Ryzin [14] consider a monopolistic retailer who can influence the demand by changing its product price. The demand arrives according to a Poisson process with a known rate that depends only on the current product price. At the beginning of the sales horizon, the retailer starts to sell a given number of items in a finite time horizon. They change the price continuously to control the intensity of the Poisson demand process. For some special cases, they show that the price increases after each sale, and decreases until the next sale. They introduce several structural properties based on the optimality conditions: (1) at any time, the optimal price is non-increasing in inventory level; (2) for a given inventory level, the optimal price is non-decreasing in remaining time; (3) the expected revenue is non-decreasing in remaining time; (4) the expected revenue is non-decreasing in inventory level. The optimal pricing policy in their model is continuous in time, which is costly and impractical in the real world. To resolve this problem, they introduce a heuristic which is asymptotically optimal as the initial inventory level increases to infinity.

Bitran and Mondeschein [4] model a periodic review pricing policy, where only a fixed number of price changes are allowed during the sales horizon and the time length between these price changes are fixed. By assuming the reservation price distribution does not change over time, they show that posting a fixed price is asymptotically optimal when the inventory level goes to infinity . Using a numerical example, they compare periodic pricing and continuous pricing and show that the performance of these two policies are very close to each other.

Lin [20] considers a sequential dynamic pricing model where a retailer sells a given number of items to a random number of customers who show up one at a time. The retailer's objective is to post a potentially different price to each arriving customer in order to maximize the expected total revenue. He uses a stochastic dynamic programming model to formulate the optimal policy. He also applies the results from this sequential-arrival model to develop a heuristic policy for a continuous-time model where customers arrive according to a general point process.

Other models of monopolistic dynamic pricing can be found in Lee and Hersh [19], Weatherford et al. [35], Feng and Gallego [13], Petruzzi and Dada [26], Robinson [27], Chatwin [7], and Subramanian et. al [29].

1.2.2 Competitive Dynamic Pricing Literature

To the best of our knowledge, there are few game-theoretic models in operations management literature that address the real-time competitive dynamic pricing of perishable products. However, a considerable literature exists on inventory management under completion. Lippman and McCardle [21], Mahajan and van Ryzin [33], Bernstein and Federgruen [2], Besanko et al. [8] consider competition when inventory replenishment is allowed, while Netessine and Shumsky [25] consider competition between two airlines where they compete by controlling seat inventory levels. We next review these papers in detail.

Lippman and Cardle [21] model a dynamic inventory competition where each retailer chooses its own inventory level. They use the Kakutani fixed point theorem to show the existence of a pure strategy Nash equilibrium. They consider both the duopoly model and the oligopoly model. They assume that the initial demand of the market is allocated among the retailers according to a splitting rule that is known by all retailers. They show the existence of a unique Nash equilibrium in pure strategies when the initial splitting rule is deterministic and strictly increasing in the total demand. They compare the total inventory levels under different market settings. They show that the competition leads each firm to choose a higher inventory level.

Mahajan and van Ryzin [33] establish an inventory competition model among many firms who produce substitutable goods. They show that the inventory level of each retailer affects the demand of other retailers. The demand is a sequence of choices made by a stochastic number of utility-maximizer customers. They show the existence of a unique Nash equilibrium in the case of a symmetric game.

Besanko et al. [8] use an empirical study to model a brand choice between substitutable products when the prices are endogenous. They develop a game theoretic model between many retailers and a single manufacturer. In this model the price levels are the outcome of a Nash equilibrium among the players. Besides price levels, they argue that some unobservable parameters, such as coupon availability, also affect the consumers' willingness to pay, and that these unobservable parameters cause bias in the estimation of the price. They use a scanner data of a two-product category to support their argument.

Netessine and Shumsky [25] study seat inventory control of the airline industry under competition. They examine a seat inventory control problem under different competition settings such as horizontal competition, where two airlines compete on the same flight leg, and vertical competition, where two airlines fly different legs in a multiple-leg itinerary. Specifically, they consider two airlines with a fixed capacity with two fare classes available for passengers. Each passenger has an initial preference and if a passenger is denied a ticket at her preferable airline, she attempts to buy the ticket from another airline. They find Nash equilibrium solutions for different settings of their problem when the firms maximize expected profit by setting booking limits for different booking classes.

Bernstein and Federgruen [2] address an infinite horizon model for firms competing in an oligopolistic market. In each period, the firms have decisions to make about their production levels and prices. In each period, the firms face a random demand, whose distribution depends on the price of all retailers. They characterize the equilibrium behavior under a wholesale pricing scheme in a periodic-review and infinite-horizon model for a two-echelon system, where a single manufacturer serves a network of retailers. The retailers carry the items across the periods with a holding cost. They derive a vector of infinite horizon stationary strategy under which each retailer sets a stationary price and acquires a base stock in each period. They show that this type of polices forms a Nash equilibrium between the retailers.

Bernstein and Federgruen [2] use different demand models such as multiplicative and non-multiplicative demand structure. In the multiplicative structure, the retailer's demand in each period is the product of the expected demand, given by a general demand function of the retailer's price, and a random component whose distribution is independent of the price. In this case, the coefficient of variations of the demands are constant. In the non-multiplicative demand structure, the random demand is the output of a finite number of choice processes made by the customers and that the coefficient of variations are endogenous. In both cases, they show that the underlying game between the retailers are supermodular and by referring to a theorem provided by Topkis [32], they show the existence of a unique Nash equilibrium. Since this paper models an infinite-level inventory problem where replenishment is allowed, it does not apply to revenue management of perishable products such as airline seats and hotel rooms.

Many pricing games have been studied in the economics literature. Most of these studies focus on the socially optimal policies, degree of product variety, market structure, and leadership. Anderson and de Palma [9] study an oligopoly model of single-product firms. They determine a closed form solution for the optimal number of variants and price. Caplin and Nalebuff [6] provide sufficient conditions for the existence and uniqueness of pure strategy Nash equilibrium in a pricing game. The difference between the price competitions, also known as *strategic complement*, and quantity competitions, also known as *strategic substitutes*, has been addressed in the literature. Tirole [31] provides sufficient conditions for existence of Nash equilibrium under these two competition settings. In the next chapter, we introduce a competitive dynamic pricing model between many retailers.

1.3 Overview and Outline

This dissertation consists of two topics concerning dynamic pricing. In the first topic, we develop a game-theoretic model to describe dynamic-price competition. We consider a market with several retailers that offer substitutable products. For example, several airlines may offer the same itinerary with a slight difference in departure times. A passenger may be willing to fly at a less desirable time in order to save money. Consequently, the demand for each airline depends not only on its own ticket price, but also on those of its competitors. We develop a game-theoretic model to describe this phenomenon and study the Nash equilibrium of this price competition. We present this topic in Chapter 2.

In the second part of this dissertation, we work with Amtrak closely to study their revenue management practice and to improve it. Specifically, we study a distinctive service provided by Amtrak—called Auto Train—where the passengers bring their own vehicles on the train. Although Amtrak does change the price of their service over time, the judgment of a price change relies on experiences of their managers, and the Amtrak lacks a scientific approach to optimize their revenue. We first analyze the sales data provided by Amtrak with various statistical tools. After constructing a mathematical model to describe Amtrak's sales process, we use dynamic programming to formulate the optimal pricing strategy, and develop a computer program to compute the optimal pricing strategy. We present this topic in Chapter 3.

Chapter 2

Competitive Dynamic Pricing

As we described in Chapter 1.2, most studies in dynamic pricing assume that the retailer enjoys a monopoly. An important assumption of such monopolistic models is that the distribution of the random demand depends only on the price set by the monopolist. Although this assumption is quite realistic in the past when it was difficult for a customer to compare prices, it can be problematic nowadays because of the Internet. With a few minutes of surfing on the Internet, a customer can easily extract the real-time price of an airline ticket and can compare the prices offered by different airlines. For example, on March 6, 2004, an online travel agent returned the prices of round-trip tickets from New York City to San Francisco offered by United Airlines and American Airlines, summarized in Table 2.1. As seen in Table 2.1, United Airlines and American Airlines offered very similar products with different prices. Although there are loyal customers (memberships, personal preferences, or other reasons) who will always choose a particular airline, many customers have schedules flexible enough to take either itinerary, and will take into account the price when purchasing the tickets.

Consequently, the demand for a product depends not only on its price, but also on the price of a similar product from other suppliers. This observation motivates our work in this paper.

Table 2.1: Price competition between two airlines.

Airlines	Flight	Itinerary	Date	Departure Time	Arrival Time	Price on 4/6/04	Price on 4/9/04
American	59	JFK-SFO	6/1/04	7:15 am	10:34 am	$361.69	$291.70
Airlines	44	SFO-JFK	6/3/04	7:45 am	4:06 pm		
United	3	JFK-SFO	6/1/04	7:10 am	10:17 am	$301.70	$301.70
Airlines	16	SFO-JFK	6/3/04	8:00 am	4:19 pm		

However, there are some other studies in the context of supply chain management where the inventory level is the decision variable when replenishment is allowed. In such studies, common criteria for retailers are to maximize expected profit in a single-period model, or to maximize the long-run average profit on an infinite-time horizon. However, conditional on the total realized demand, the demand for each retailer is often formulated as a deterministic function of prices. For example, see Lippman and McCardle [21], Cachon and Zipkin [5], Mahajan and van Ryzin [22], Netessine and Rudi [24], Kirman and Sobel [16], and the references therein.

In this study, we develop a competitive dynamic pricing model, in which each retailer is given an initial stock to sell and the product price is the only decision variable. In Section 2.1, we develop the model and describe the underlying assumptions. In Section 2.3, we consider a single stage problem and determine its structural properties. Section 2.4 introduces a multistage game. In Section 2.5, we propose a practical policy when a retailer is not capable of keeping track of other retailers' real-time inventory levels.

2.1 Model and Assumptions

Consider n retailers that sell perishable products over a finite-time horizon. Each retailer offers a single type of product and products offered by different retailers are substitutable. Retailers sell the products over T time periods, where we count the time periods in the reverse chronological order so that period 1 is the last period. Inventory replenishment is not allowed and unsold products have no salvage value at the end of period 1. Customers arrive from period T to period 1 according to a Bernoulli process such that in each period the probability that a customer arrives is λ and the probability that no customer arrives is $1 - \lambda$. This assumption of costumer arrival is an approximation of a continuous-time homogenous Poisson process.[1] This type of demand approximation has been used in Subramanian [29].

Let C_i denote the initial inventory level of retailer i. At the beginning of each period, each retailer sets its product price in anticipation of a new customer that may arrive in that time period. From a retailer's standpoint, the sale ends when it sells out his stock, or at the end of period 1. The goal of each retailer is to set its product price at the beginning of each period in order to maximize its own expected total revenue.

When a customer arrives, he compares the prices of all retailers and then decides whether to buy one unit of his most preferable product or not to buy at all. In this dissertation, we use the multinomial logit (MNL) model to describe a customer's discrete choice. The MNL

[1] Our model can be extended to a time-dependent arrival process by properly scaling the time horizon.

model is widely used in the marketing science literature, and it assumes that each customer acts independently to maximize its own utility. In the MNL model, the utility of retailer i's product for a customer can be represented by a random variable

$$U_i = \alpha_i - \beta p_i + Z_i, \qquad i = 1, \ldots, n,$$

while the utility of no purchase is

$$U_0 = Z_0.$$

Parameter $\alpha_i, i = 1, \ldots, n$ models the quality, or brand image, of product i; p_i denotes the price of product i and we define $\boldsymbol{p} = (p_1, \ldots, p_n)$ as the price vector. Parameter β is the price response coefficient. The random variables Z_i, $i = 0, \ldots, n$, describe the idiosyncratic preference of each customer, and are independent and identically distributed Gumbel random variables with the following distribution function:

$$F_{Z_i}(z) = \exp(-(e^{-\frac{z}{\mu}+\gamma})), \qquad i = 0, \ldots, n,$$

where μ is a scale parameter and γ is Euler's constant $(= 0.5772\ldots)$. Each customer will independently make the decision that yields the highest utility.

According to the MNL model, a customer buys one unit from retailer i with probability: (see Ben-Akiva and Lerman [1] for a derivation)

$$q_i(\boldsymbol{p}) = P\Big(U_i = \max_{j=0,\ldots,n} U_j\Big) = \frac{e^{\alpha_i - \beta p_i}}{1 + \sum_{j=1}^{n} e^{\alpha_j - \beta p_j}}, \qquad i = 1, \ldots, n, \tag{2.1}$$

and will leave empty-handed with probability

$$q_0(\boldsymbol{p}) = \frac{1}{1 + \sum_{j=1}^{n} e^{\alpha_j - \beta p_j}}. \tag{2.2}$$

To calculate $q_i(\boldsymbol{p})$, $i = 0, \ldots, n$, we let $\mu = 1$ without loss of generality, because the scale parameter μ can be absorbed into the constants β and α_i, $i = 1, \ldots, n$. The first and second derivative of q_i are

$$\frac{\partial q_i(\boldsymbol{p})}{\partial p_i} = -\beta q_i(\boldsymbol{p})(1 - q_i(\boldsymbol{p})),\ \frac{\partial q_i(\boldsymbol{p})}{\partial p_j} = \beta q_i(\boldsymbol{p}) q_j(\boldsymbol{p}),\ \frac{\partial^2 q_i(\boldsymbol{p})}{\partial p_i \partial p_j} = -\beta^2 q_i(\boldsymbol{p}) q_j(\boldsymbol{p})(1 - 2q_i(\boldsymbol{p})).$$

$$\frac{\partial^2 q_i(\boldsymbol{p})}{\partial p_i^2} = \beta^2 q_i(1 - q_i(\boldsymbol{p}))(1 - 2q_i(\boldsymbol{p})),\ \frac{\partial^2 q_i(\boldsymbol{p})}{\partial p_j^2} = -\beta^2 q_i(\boldsymbol{p}) q_j(1 - 2q_j), \text{ for all } i \text{ and } j \neq i.$$

Note that $q_i(\boldsymbol{p}), i = 0, \ldots, n$ are continuous in p_j, $j = 0, \ldots, n$.

From Equation (2.1), we see that $q_i(\boldsymbol{p})$—the probability retailer i sells one item—is monotonically decreasing in its price p_i, and that

$$\lim_{p_i \to \infty} q_i(\boldsymbol{p}) = \lim_{p_i \to \infty} p_i q_i(\boldsymbol{p}) = 0, \qquad i = 1, \ldots, n.$$

Therefore, if retailer i sells out its stock, it can set $p_i = \infty$ so that its demand and its expected revenue is equal to zero. The probability of a sale during period t is $\lambda \sum_{j=1}^{n} q_j(\boldsymbol{p})$ and the probability of no sale in period t is $1 - \lambda + \lambda q_0(\boldsymbol{p})$.

We now review the concept of a supermodular function that will be helpful in addressing the characteristics of the equilibrium in later sections. We say $g(x, y)$ is supermodular if $g(x_1, y_1) + g(x_2, y_2) \geq g(x_1, y_2) + g(x_2, y_1)$ for all $x_1 > x_2$ and $y_1 > y_2$. A non-negative function $h : X \equiv (x_1, \ldots, x_n) \to R$ is log-supermodular (log-submodular) if and only if for all $x, y \in X$, $h(x \vee y)h(x \wedge y) \geq (\leq) h(x)h(y)$. Equivalently, if h is a twice differentiable function, then it is log-supermodular, if and only if $\partial^2 \log h / \partial x_i \partial x_j \geq (\leq) 0$ for all i and j, $j \neq i$. Now, we introduce a lemma extracted from Vives [34] to address structural properties of the maximizer of a modular function. Suppose we want to maximize $g(x, y)$ by choosing y for a given x and define

$$y(x) \equiv \min \arg\max_{y} g(x, y).$$

Lemma 2.1.1 *If $g(x, y)$ is supermodular, or if $g(x, y)$ is log-supermodular, then $y(x)$ is non-decreasing in x.*

Proof: First, consider the case when $g(x, y)$ is supermodular. Suppose that $x_1 > x_2$ and $y(x_1) = \bar{y}$. Since $g(x, y)$ is supermodular, for any $y > \bar{y}$, we have

$$g(x_1, y) + g(x_2, \bar{y}) \geq g(x_1, \bar{y}) + g(x_2, y),$$

or equivalently we have the following string of inequalities

$$g(x_2, \bar{y}) - g(x_2, y) \geq g(x_1, \bar{y}) - g(x_1, y) \geq 0,$$

where the second inequality follows because $y(x_1) = \bar{y}$. On the other hand, since for any $y > \bar{y}$, we have $g(x_2, y) \leq g(x_2, \bar{y})$, it follows that $y(x_2) \leq \bar{y}$.

Now consider that $g(x, y)$ is log-supermodular and suppose $x_1 > x_2$ and $y(x_1) = \bar{y}$. For any $y > \bar{y}$, since $g(x, y)$ is log-supermodular, we have

$$g(x_1, y)g(x_2, \bar{y}) \geq g(x_1, \bar{y})g(x_2, y).$$

Therefore,

$$\frac{g(x_2, \bar{y})}{g(x_2, y)} \geq \frac{g(x_1, \bar{y})}{g(x_1, y)} \geq 1,$$

where the second inequality follows because $y(x_1) = \bar{y}$. Consequently, since for any $y > \bar{y}$, $g(x_2, y) \leq g(x_2, \bar{y})$, it follows that $y(x_2) \leq \bar{y}$. $\square$

2.2 The Effect of Price Competition

In this section, we address the consequences if a retailer ignores the competition from other retailers and uses a pricing policy as if it enjoyed monopoly in the market.

First consider the case where retailer 1 enjoys a monopoly. Because there is only one retailer in the market, each customer will either purchase one unit of retailer 1's product at price p_1 with probability

$$q_1(p_1) = \frac{e^{\alpha_1 - \beta p_1}}{1 + e^{\alpha_1 - \beta p_1}},$$

or leaves empty-handed with probability $1 - q_1(p_1)$, according to our MNL choice model in Equation (2.1). To maximize the expected total revenue for retailer 1, let $U_1(s_1, t)$ denote the expected additional revenue if it still has s_1 items in inventory with t remaining time periods. Retailer 1 can find its optimal dynamic pricing strategy by solving the following dynamic program.

$$\begin{aligned} U_1(s_1, t) &= \lambda\Big(\max_{p_1} q_1(p_1)(p_1 + U_1(s_1 - 1, t - 1)) \\ &\quad + (1 - q_1(p_1))U_1(s_1, t - 1)\Big) + (1 - \lambda)U_1(s_1, t - 1), \end{aligned} \tag{2.3}$$

with boundary conditions $U_1(\cdot, 0) = U_1(0, \cdot) = 0$. Let $p^*(s_1, t)$ denote the maximizer of the preceding dynamic program. In other words, $p^*(s_1, t)$ is retailer 1's optimal price when it has s_1 units of product with t remaining time periods. In the rest of this study, we refer to this optimal strategy of the monopolist as the *monopolistic strategy*. From retailer 1's standpoint, the monopolistic strategy is easy to compute, as well as its revenue function $U_1(s_1, t)$. For example, if we let $\alpha_1 = 4$, $\beta = 0.1$, $\lambda = 0.1$, then using Equation (2.3) iteratively we can compute $U_1(20, 600) = 895.59$.

Now, consider the more realistic case when retailer 2 introduces its product, which directly competes with retailer 1's product. To highlight the competition impact, we assume that retailer 1 ignores the competition and still uses the monopolistic strategy, and retailer 2 knows retailer 1's strategy and takes advantage of this ignorance to maximize its expected revenue. Therefore, from retailer 2's standpoint, a rational strategy is to maximize its own expected revenue given the strategy of retailer 1. Let $U_2(s_1, s_2, t)$ denote the additional expected revenue for retailer 2 with t time periods remaining, if retailers 1 and 2, respectively, have s_1 and s_2 units of product in inventory. Note that the information about the inventory levels is given to both retailers. We can find the optimal policy for retailer 2 by solving the following dynamic program.

$$\begin{aligned} U_2(s_1, s_2, t) \quad = \quad & \lambda\Big(\max_{p_2} q_1(p_1^*(s_1, t), p_2)U_2(s_1 - 1, s_2, t - 1)) + q_2(p_1^*(s_1, t), p_2) \\ & (p_2 + U_2(s_1, s_2 - 1, t - 1)) + q_0(p_1^*(s_1, t), p_2)U_2(s_1, s_2, t - 1)\Big) \\ & +(1 - \lambda)U_2(s_1, s_2, t - 1), \end{aligned} \tag{2.4}$$

with boundary conditions $U_2(\cdot, 0, \cdot) = U_2(\cdot, \cdot, 0) = 0$. Note that we do not need a boundary condition for $s_1 = 0$, because after retailer 1 sells out its stock, its product price—from retailer 2's and the customers' standpoints—is equal to ∞, which allows retailer 2 to recursively solve the preceding dynamic program.

The participation of retailer 2 can potentially bring significant effect on retailer 1's revenue, if retailer 1 ignores the competition. Consider the same example with $\alpha_1 = 4$, $\beta = 0.1$, $\lambda = 0.1$, and $U_1(20, 600) = 895.59$. Letting $\alpha_2 = 5$, Table 2.2 shows the expected revenue for both retailers, if retailer 1 uses the monopolistic strategy and retailer 2 uses the corresponding optimal policy defined by Equation (2.4).

Table 2.2: Expected revenue with the monopolistic strategy.

	Expected revenue	
c_2	retailer 1	retailer 2[a]
0	895.59	0
5	865.07	325.21
10	827.40	580.93
15	781.59	795.14
20	726.92	974.86
25	663.73	1121.73
30	594.49	1236.33
35	524.13	1319.32
40	461.18	1372.01

[a]This is $U_2(20, c_2, 600)$ defined by Equation (2.4).

As seen in Table 2.2, if the initial stock of retailer 2 is much smaller than that of retailer 1, then retailer 1 still dominates the market and therefore the monopolistic strategy still performs well for retailer 1. On the other hand, if the initial stock of retailer 2 is comparable or larger than that of retailer 1, then using the monopolistic strategy, retailer 1 tends to set the price too high, which will leave unwanted inventory at the end. Consequently, retailer 1's expected revenue with the monopolistic strategy becomes much lower than it would expect.

The fact that with the monopolistic strategy, retailer 1 will likely to have unwanted inventory at the end when retailer 2 enters the market suggests that a different strategy is necessary. Taking into account the competition from retailer 2, a rational retailer 1 would want to change its strategy to maximize its own expected revenue. After retailer 1 adopts a new strategy, retailer 2 would also want to make adjustments, and then retailer 1, and so on, until each retailer's strategy is the best response against the other retailer. Using these observations, in the next section, we define the Nash equilibrium of the market.

2.3 Single-Stage Game

In this section, we consider a scenario where the objective of each retailer is to maximize the current expected revenue from the arriving customer. Because each retailer wants to maximize the current expected revenue and does not take into account how the sale might affect its inventory and therefore its revenue in the future, we call this policy the *myopic policy.* This policy would be the optimal policy for each retailer if the arriving customer is the last customer, or if each retailer has unlimited inventory. Myopic optima and its structural properties facilitate the analysis of qualitative properties of our multistage game.

The single-stage game consists of (1) *the players,* which in our model are the retailers indexed by $i = 1, \ldots, n$; (2) *the strategies,* which in our model are the price of the products denoted by $p_i \in R^+$, $i = 0, \ldots, n$; (3) *the payoffs*, which in our model are the expected revenue $g_i(\boldsymbol{p})$ received by each player.

Note that $g_i(\boldsymbol{p})$ denote the immediate expected revenue of retailer i. We denote the price vector posted by the retailers by $\boldsymbol{p} = (p_1, \ldots, p_n)$ and denote $g_i(\boldsymbol{p})$ as follows.

$$g_i(\boldsymbol{p}) = \lambda p_i \cdot q_i(\boldsymbol{p}) = \frac{\lambda p_i e^{\alpha_i - \beta p_i}}{1 + \sum_{j=1}^{n} e^{\alpha_j - \beta p_j}}, \qquad i = 0, \ldots, n, \tag{2.5}$$

Retailer i's objective is to choose p_i to maximize its payoff function $g_i(\boldsymbol{p})$, $i = 0, \ldots, n$. Let p_{-i} denote the prices charged by retailers other than retailer i. For a given p_{-i}, we define

$$\phi_i(\boldsymbol{p}_{-i}) \equiv \min \arg \max_{p_i} g_i(\boldsymbol{p}). \tag{2.6}$$

as the retailer i's best response function. When there is a tie, we let the retailer choose the smallest price that maximizes its payoff. We will show that in this single-stage game, there is a unique maximizer p_i for $g_i(\boldsymbol{p})$, $i = 1, \ldots, n$. A Nash equilibrium arises if each retailer's price is the best response to the prices posted by other retailers, so that no retailer has incentives to change its price. In other words, a joint policy $\boldsymbol{p}^* = (p_1^*, \ldots, p_n^*)$ would be a Nash equilibrium if for $i = 1, \ldots, n$,

$$g_i(\boldsymbol{p}^*) = \max_{p_i} \; g_i(p_1^*, \ldots, p_{i-1}^*, p_i, p_{i+1}^*, \ldots, p_n^*). \tag{2.7}$$

In general, Nash equilibria in pure strategies in non-cooperative games are not guaranteed to exist. In addition, uniqueness of Nash equilibrium is quite useful in order to characterize the equilibrium and to study the properties of the equilibrium outcome with respect to the parameters of the game. To prove that there exists such a Nash equilibrium, we first show that the payoff function $g_i(\boldsymbol{p})$ is quasiconcave in p_i.

Lemma 2.3.1 *The expected current revenue $g_i(\boldsymbol{p})$ in Equation (2.5) is strongly quasiconcave in p_i.*

Proof: Taking the first derivative of $g_i(\boldsymbol{p})$ with respect to p_i yields

$$\frac{\partial g_i(\boldsymbol{p})}{\partial p_i} = q_i(\boldsymbol{p})[1 - \beta p_i(1 - q_i(\boldsymbol{p}))]. \tag{2.8}$$

Because $h(p_i) \equiv 1 - \beta p_i(1 - q_i(\boldsymbol{p}))$ is strictly decreasing in p_i, and that $h(0) = 1$ and $\lim_{p_i \to \infty} h(p_i) < 0$, there exists a unique solution $\hat{p}_i < \infty$ such that $h(\hat{p}_i) = 0$. It then follows that $g_i(\boldsymbol{p})$ is quasiconcave in p_i, because $\pi_i(\boldsymbol{p})$ increases when $p_i < \hat{p}_i$, reaches its maximum when $p_i = \hat{p}_i$, and then decreases when $p_i > \hat{p}_i$. □

From Lemma 2.3.1, for a given $\boldsymbol{p}_{-i}$, we can modify Equation (2.6) as follows.

$$\phi_i(\boldsymbol{p}_{-i}) \equiv \arg\max_{p_i} g_i(\boldsymbol{p}). \tag{2.9}$$

We next show some properties of the best response function (2.9).

Lemma 2.3.2 *The best response function $\phi_i(\boldsymbol{p}_{-i})$ is a non-decreasing function with respect to p_j, for $j \neq i$.*

Proof: The payoff function $g_i(\boldsymbol{p})$ is log-supermodular in p_i and p_j because

$$\frac{\partial^2 \log g_i(\boldsymbol{p})}{\partial p_i p_j} = \beta^2 q_i(\boldsymbol{p}) q_j(\boldsymbol{p}) \geq 0.$$

Applying Lemma 2.1.1, we conclude that the best response of retailer i, $\phi_i(\boldsymbol{p}_{-i})$ is increasing in its own price. □

Lemma 2.3.3 *The best response function $\phi_i(\boldsymbol{p}_{-i})$ is uniformly bounded for all $\boldsymbol{p}_{-i}$.*

Proof: Because

$$q_i(\boldsymbol{p}) = \frac{e^{\alpha_i - \beta p_i}}{1 + \sum_{j=1}^{n} e^{\alpha_j - \beta p_j}} \leq \frac{e^{\alpha_i - \beta p_i}}{1 + e^{\alpha_i - \beta p_i}} \leq \frac{e^{\alpha_i}}{1 + e^{\alpha_i}}, \tag{2.10}$$

it follows that

$$1 - \beta p_i(1 - q_i(\boldsymbol{p})) \leq 1 - \beta p_i\Big(\frac{1}{1 + e^{\alpha_i}}\Big).$$

Because the right-hand side of the preceding becomes negative if $p_i > (1 + e^{\alpha_i})/\beta$, from Equation (2.8), we can conclude that for $p_i > (1 + e^{\alpha_i})/\beta$,

$$\frac{\partial g_i(\boldsymbol{p})}{\partial p_i} < 0.$$

Therefore, retailer i's payoff function $g_i(\boldsymbol{p})$ decreases in p_i for $p_i > (1 + e^{\alpha_i})/\beta$, independent of the price levels of other retailers. Consequently, retailer i should never set a price greater than $(1 + e^{\alpha_i})/\beta$, so we have

$$\phi_i(\boldsymbol{p}_{-i}) \leq \frac{1 + e^{\alpha_i}}{\beta},$$

and the lemma is proved. □

There are many different methods to show the existence of Nash equilibrium, such as verifying the quasiconcavity or modularity of the payoff function. Since Nash equilibrium is a fixed point of the best response mapping, most of these methods use fixed point theorems to show the existence of an equilibrium. Compactness of strategy space is an assumption necessary for applying the fixed point theorem. Although in our model, the strategy space is not finite because we allow a retailer to set the price equal to infinity once it sells out its stock, we can use Lemma 2.3.3 to construct a equivalent game with a finite strategy space. We prove the uniqueness of a Nash equilibrium in the next theorem.

Theorem 2.3.1 *There exist a unique Nash equilibrium in pure strategies for the static game characterized by Equation (2.7).*

Proof: Based on Theorem 2.1 in Vives [34], a Nash equilibrium exists if the strategy sets are nonempty convex and compact, and the payoff of retailer i is continuous in the actions of all retailers and quasiconcave in its own action. The expected payoff function $g_i(\boldsymbol{p})$ is continuous in p_i, since $p_i q_i(\boldsymbol{p})$ is continuous in p_i and λ is independent of p_i. The continuity of the best response function is also straightforward, since it is the maximizer of a continuous function with respect to the actions of all retailers. In our single-period model, although each retailer can choose a price from $[0, \infty)$, which is not compact, we can construct an equivalent game by restricting retailer i to choose a price in $[0, (1 + e^{\alpha_i})/\beta]$—a nonempty convex and compact set. This claim is true because according to Lemma 2.3.3 retailer i does not need to consider any price greater than $(1 + e^{\alpha_i})/\beta$ when searching for its best response. In addition, by Lemma 2.3.1, the payoff function $g(\boldsymbol{p})$ is quasiconcave in p_i. Consequently, there exists a Nash equilibrium.

To verify the uniqueness of Nash equilibrium, Vives([34], page 152) provides a sufficient condition as follows:

$$\sum_{\forall j \neq i} \left| \frac{\partial^2 \log g_i(\boldsymbol{p})}{\partial p_i p_j} \right| \leq \left| \frac{\partial^2 \log g_i(\boldsymbol{p})}{\partial p_i^2} \right|.$$

Using (2.5), we have

$$\sum_{\forall j \neq i} \left| \frac{\partial^2 \log g_i(\boldsymbol{p})}{\partial p_i p_j} \right| = \beta^2 q_i(\boldsymbol{p}) \sum_{\forall j \neq i, 0} q_j,$$

and

$$\left| \frac{\partial^2 \log g_i(\boldsymbol{p})}{\partial p_i^2} \right| = 1/p^2 + \beta^2 q_i(\boldsymbol{p})(1 - q_i(\boldsymbol{p})),$$

which verifies the inequality, since $1 - q_i(\boldsymbol{p}) = q_0 + \sum_{\forall j \neq i, 0} q_j$ and that ϕ_i has a unique fixed point. $\square$

We denote $\boldsymbol{p}^*$ as the unique Nash equilibrium price vector in the single-period model, which is also the myopic price vector. Next we study the structural properties of this Nash equilibrium.

Lemma 2.3.4 *For $i = 1, \ldots, n$ and $j \neq i$, $g_i(\boldsymbol{p})$ is log-supermodular in p_i, log-supermodular in (p_i, α_i), log-supermodular (p_i, λ), and log-submodular in (p_i, α_j).*

Proof. To show these properties, we calculate the first and second derivative of $\log g(.)$ whose signs show the results based on the definitions of the modularity properties of a function with respect to its parameters.

$$\log g_i(\boldsymbol{p}) = \log \lambda + \log p_i + \log q_i(\boldsymbol{p}).$$

$$\partial \log g_i(\boldsymbol{p})/\partial p_i = 1/p_i - \beta(1 - q_i(\boldsymbol{p})).$$

$$\partial^2 \log g_i(\boldsymbol{p})/\partial p_j \partial p_k = \beta^2 q_j q_k, \geq 0 \quad \forall j = 1, \ldots n, \text{ and } k \neq j.$$

$$\partial^2 \log g_i(\boldsymbol{p})/\partial p_i \partial \lambda = 0.$$

$$\partial^2 \log g_i(\boldsymbol{p})/\partial p_i \partial \alpha_i = \beta q_i(\boldsymbol{p})(1 - q_i(\boldsymbol{p})) \geq 0.$$

$$\partial^2 \log g_i(\boldsymbol{p})/\partial p_i \partial \alpha_j = -\beta q_i(\boldsymbol{p}) q_j \leq 0.$$

Proposition 2.3.1 *(1) The best response function $\phi_i(\boldsymbol{p}_{-i})$ is a non-decreasing function with respect to p_j, for all $j \neq i$. (2) Equilibrium price $p_i^* = \phi_i(\alpha_1, \ldots, \alpha_n, \beta, \lambda)$ is non-decreasing with respect to α_i and λ, and non-increasing with respect to α_j, $j \neq i$.*

Proof. Statement (1) follows from the log-supermodularity of the expected revenue with respect to (p_i, p_j). To show statement (2), we use the implicit function theorem. The implicit function of parameter a,$a = \alpha_i, \alpha_j, \lambda,$ and β, is $\partial\phi_i(.)/\partial a = -\frac{\partial^2 g_i(\boldsymbol{p})/\partial p_i \partial a}{\partial^2 g_i(\boldsymbol{p})/\partial p_i^2}$. The slope of the equilibrium price with respect to the parameters can be summarized as follows.

$$\frac{\partial\phi_i}{\partial\lambda} = -\frac{\partial^2 g_i(\boldsymbol{p})/\partial p_i \partial\lambda}{\partial^2 g_i(\boldsymbol{p})/\partial p_i^2}|_{\partial g_i(\boldsymbol{p})/\partial p_i = 0} = 0. \tag{2.11}$$

$$\frac{\partial\phi_i}{\partial\alpha_i} = -\frac{\partial^2 g_i(\boldsymbol{p})/\partial p_i \partial\alpha_i}{\partial^2 g_i(\boldsymbol{p})/\partial p_i^2}|_{\partial g_i(\boldsymbol{p})/\partial p_i = 0} = -\frac{-\lambda\beta\partial\phi_i/\partial\alpha_i q_i(\boldsymbol{p})(1 - q_i(\boldsymbol{p})) + \lambda q_i(\boldsymbol{p})^2}{-\lambda\beta q_i(\boldsymbol{p})}$$

$$\Rightarrow \frac{\partial\phi_i}{\partial\alpha_i} = \frac{q_i(\boldsymbol{p})}{\beta(2 - q_i(\boldsymbol{p}))} \geq 0. \tag{2.12}$$

$$\frac{\partial\phi_i}{\partial\alpha_j} = -\frac{\partial^2 g_i(\boldsymbol{p})/\partial p_i \partial\alpha_j}{\partial^2 g_i(\boldsymbol{p})/\partial p_i^2}|_{\partial g_i(\boldsymbol{p})/\partial p_i = 0} = -\frac{-\lambda\beta\partial\phi_i/\partial\alpha_j q_i(\boldsymbol{p})(1 - q_i(\boldsymbol{p})) - \lambda q_i(\boldsymbol{p})^2 q_j/(1 - q_i(\boldsymbol{p}))}{-\lambda\beta q_i(\boldsymbol{p})}$$

$$\Rightarrow \frac{\partial\phi_i}{\partial\alpha_j} = -\frac{q_i(\boldsymbol{p})q_j(\boldsymbol{p})}{\beta(1 - q_i(\boldsymbol{p}))(2 - q_i(\boldsymbol{p}))} \leq 0. \tag{2.13}$$

$$\frac{\partial^2 g_i(\boldsymbol{p})}{\partial p_i \partial\beta}|_{\partial g_i(\boldsymbol{p})/\partial p_i = 0} = \frac{q_i(\boldsymbol{p})}{\beta(-1 + \sum_{\forall j \neq i} q_j/(1 - q_j))} \neq 0. \tag{2.14}$$

which shows that the equilibrium price is non-decreasing in λ and α_1 and non-increasing in α_2. $\square$

Next, we study the monotonicity properties of equilibrium revenue with respect to the parameters.

Proposition 2.3.2 *Equilibrium revenue $g_i(\alpha_1, \ldots, \alpha_n, \beta, \lambda)$ is a non-decreasing function with respect to α_i and λ, and a non-increasing function with respect to α_j, $j \neq i$.*

Proof. The first derivative of (2.5) with respect to λ is

$$\frac{\partial g_i(\boldsymbol{p}^*)}{\partial \lambda} = p_i^* q_i(\boldsymbol{p}^*) \geq 0.$$

The first derivative of (2.5) with respect to α_i is

$$\frac{\partial g_i(\boldsymbol{p}^*)}{\partial \alpha_i} = \lambda q_i(\boldsymbol{p}) \frac{\partial p_i^*}{\partial \alpha_i} + \lambda p_i^* \sum_{j=1}^{n} (\frac{\partial q_i(\boldsymbol{p})}{\partial p_j} \frac{\partial p_j^*}{\partial \alpha_i} + \frac{\partial q_i(\boldsymbol{p})}{\partial \alpha_i}),$$

after substituting terms from our previous results, we have

$$\frac{\partial g_i(\boldsymbol{p}^*)}{\partial \alpha_i} = \frac{\lambda q_i(\boldsymbol{p})}{\beta(1 - \sum_{j \neq i} \frac{q_i(\boldsymbol{p}) q_j^2}{(1-q_i(\boldsymbol{p}))^2 (2-q_i(\boldsymbol{p}))})} \geq 0,$$

since $(2 - q_i(\boldsymbol{p})) > q_i(\boldsymbol{p})$ and $\sum_{j \neq i} q_j^2 < (\sum_{j \neq i} q_j)^2 < (1 - q_i(\boldsymbol{p}))^2$. Finally, the first derivative of (2.5) with respect to α_2 is

$$\frac{\partial g_i(\boldsymbol{p}^*)}{\partial \alpha_j} = \lambda q_i(\boldsymbol{p}) \frac{\partial p_i^*}{\partial \alpha_j} + \lambda p_i^* \sum_{k=1}^{n} (\frac{\partial q_i(\boldsymbol{p})}{\partial p_k} \frac{\partial p_k^*}{\partial \alpha_j} + \frac{\partial q_i(\boldsymbol{p})}{\partial \alpha_j}).$$

After rearrangements terms, we have

$$\frac{\partial g_i(\boldsymbol{p}^*)}{\partial \alpha_j} = \frac{\lambda q_i(\boldsymbol{p})}{\beta(1 - q_i(\boldsymbol{p}))} (\sum_{j \neq i} \frac{q_j^2}{(2 - q_j)} - 1),$$

which is non-positive for a duopoly setting. □

2.4 Multistage Game

In this section, we return to our dynamic game described in Section 2.1, where retailers compete in a pricing game over a finite-time horizon. In this game, the myopic policy is not necessarily the optimal policy for a retailer, because the retailer needs to take into account how the sale affects its inventory and therefore future revenue. To model this time dependency, we introduce a dynamic non-cooperative and non-zero-sum game between the retailers, hereafter *dynamic game.*

Suppose in period t a customer arrives and retailer i still has s_i items in inventory, $i = 1, \ldots, n$. If $t = 1$, then the Nash equilibrium is described by the myopic policy discussed in Section 2.3. In Nash equilibrium, retailer i's expected revenue is $\lambda g_i(\boldsymbol{p}^*)$.

To define a price equilibrium for the dynamic game when $t > 1$, we first define $V_i(\boldsymbol{s}, t)$ as retailer i's expected total revenue from period t to period 1 if all retailers use equilibrium strategies from period t to period 1. In period t, when a customer arrives, the game between the the retailers is for retailer i to choose p_i in order to maximize his payoff $J_i(\boldsymbol{p}, \boldsymbol{s}, t)$, $i = 1, \ldots, n$, where

$$\begin{aligned} J_i(\boldsymbol{p}, \boldsymbol{s}, t) \;\equiv\;& \lambda q_i(\boldsymbol{p})(p_i + V_i(\boldsymbol{s} - e_i, t-1)) \\ &+\lambda \sum_{j \in N - \{i\}} q_j(\boldsymbol{p}) V_i(\boldsymbol{s} - e_j, t-1) + (1 - \lambda + \lambda q_0(\boldsymbol{p})) V_i(\boldsymbol{s}, t-1). \end{aligned} \tag{2.15}$$

or equivalently,

$$\begin{aligned} J_i(\boldsymbol{p}, \boldsymbol{s}, t) \;\equiv\;& p_i + V_i(\boldsymbol{s} - e_i, t-1)) \\ &+ \sum_{j \in N - \{i\} + \{0\}} q_j(\boldsymbol{p})(V_i(\boldsymbol{s} - e_j, t-1) - V_i(\boldsymbol{s} - e_i, t-1) - p_i). \end{aligned} \tag{2.16}$$

where e_i is a $1 \times n$ vector that takes 1 for the ith element and 0 for the others, and $\boldsymbol{e}_0 \equiv (0, 0, \ldots, 0)$. If $\boldsymbol{p}^*(\boldsymbol{s}^*, t)$ denotes the equilibrium price characterized by Equation (2.16), then the equilibrium expected revenue function $V_i(\boldsymbol{s}, t)$ can be characterized as follows.

$$V_i(\boldsymbol{s}, t) = \lambda J_i(\boldsymbol{p}^*(\boldsymbol{s}, t), \boldsymbol{s}, t) + (1 - \lambda) V_i(\boldsymbol{s}, t-1).$$

We now introduce the following lemma.

Lemma 2.4.1 *$J_i(\boldsymbol{p}, \boldsymbol{s}, t)$ is strongly quasiconcave in p_i.*

Proof. Taking the first derivative of $g_i(\boldsymbol{p}, \boldsymbol{s}, t)$ with respect to p_i yields

$$\frac{\partial J_i(\boldsymbol{p}, \boldsymbol{s}, t)}{\partial p_i} = q_i(\boldsymbol{p})\Big(1 - \beta \sum_{0 \le j \le n, j \ne i} q_j(\boldsymbol{p})(p_i + V_i(\boldsymbol{s} - \boldsymbol{e}_i, t-1) - V_i(\boldsymbol{s} - \boldsymbol{e}_j, t-1))\Big), \quad (2.17)$$

Let $h(p_i) \equiv 1 - \beta \sum_{j=0, j\ne i}^{n} q_j(\boldsymbol{p})(p_i + V_i(\boldsymbol{s} - \boldsymbol{e}_i, t-1) - V_i(\boldsymbol{s} - \boldsymbol{e}_j, t-1))$. First, assume that $h(p_i) \ne 0$ for $p_i \ge 0$. In this case, since $h(p_i)$ is continues and $\lim_{p_i \to \infty} h(p_i) = -\infty$, it follows that $h(p_i) < 0$ for $p_i \ge 0$. This assures that $J_i(\boldsymbol{p}, \boldsymbol{s}, t)$ strictly decreases in p_i for all $p_i > 0$, and therefore is quasiconcave

Second, if there exists $\hat{p}_i$ such that $h(\hat{p}_i) = 0$, then the second derivative of $J_i(\boldsymbol{p}, \boldsymbol{s}, t)$ with respect to p_i evaluated at $\hat{p}_i$ is equal to

$$\left.\frac{\partial^2 J_i(\boldsymbol{p}, \boldsymbol{s}, t)}{\partial p_i^2}\right|_{p_i = \hat{p}_i} = -\beta q_i(\boldsymbol{p})|_{p_i = \hat{p}_i} < 0.$$

The preceding shows that $\hat{p}_i$ is a local maximum of $J_i(\boldsymbol{p}, \boldsymbol{s}, t)$ and that there does not exist an interior minimum for $p_i \in [0, \infty)$. It then follows that $\hat{p}_i$ is unique because otherwise there must exists an interior minimum for $J_i(\boldsymbol{p}, \boldsymbol{s}, t)$. Therefore $J_i(\boldsymbol{p}, \boldsymbol{s}, t)$ increases for $p_i \in [0, \hat{p}_i)$ and decreases for $p_i \in (\hat{p}_i, \infty)$, which assures that $J_i(\boldsymbol{p}, \boldsymbol{s}, t)$ is strongly quasiconcave in p_i. All together, we conclude that $J_i(\boldsymbol{p}, \boldsymbol{s}, t)$ is strongly quasiconcave in p_i. □

Using Lemma 2.4.1, we define the best response function when a customer arrives in period t and retailer i still has s_i units in inventory, $i = 1, \ldots, n$ as follows.

$$\psi_i(\boldsymbol{p}_{-i}, \boldsymbol{s}, t) \equiv \min \arg\max_{p_i} J_i(\boldsymbol{p}, \boldsymbol{s}, t). \quad (2.18)$$

Lemma 2.4.2 *For any price posted by retailer* j, $0 \le j \le n, j \neq i$, *the best response function of retailer* i *is bounded; that is,* $\sup_{p_{-i}} \psi_i(\boldsymbol{p}_{-i}, \boldsymbol{s}, t) < \infty$.

Proof: First denote $b_j \equiv -V_i(\boldsymbol{s} - \boldsymbol{e}_i, t-1) + V_i(\boldsymbol{s} - \boldsymbol{e}_j, t-1)$, and let $b \equiv \max_j b_j$, which is some constant. Rewrite Equation (2.17) as follows:

$$\begin{aligned}
\frac{\partial \Pi_i(\boldsymbol{p}, \boldsymbol{s}, t)}{\partial p_i} &= q_i(\boldsymbol{p})\Big(1 - \beta \sum_{j=0, j\neq i}^{n} q_j(\boldsymbol{p})(p_i - b_j)\Big) \\
&\le q_i(\boldsymbol{p})\Big(1 - \beta \sum_{j=0, j\neq i}^{n} q_j(\boldsymbol{p})(p_i - b)\Big) \\
&= q_i(\boldsymbol{p})\Big(1 - \beta(1 - q_i(\boldsymbol{p}))(p_i - b)\Big). \qquad (2.19)
\end{aligned}$$

Using Equation (2.10), we conclude that

$$1 - q_i(\boldsymbol{p}) \ge 1 - \frac{e^{\alpha_i}}{1 + e^{\alpha_i}} = \frac{1}{1 + e^{\alpha_i}}.$$

Therefore, Equation (2.19) becomes negative if

$$p_i > b + \frac{1 + e^{\alpha_i}}{\beta}.$$

In other words, $J_i(\boldsymbol{p}, \boldsymbol{s}, t)$ decreases in p_i for $p_i > b + (1 + e^{\alpha_i})/\beta$, independent of the prices posted by the other retailers. Therefore, retailer i does not need to consider any price that is greater than $b + (1 + e^{\alpha_i})/\beta$. Consequently, we conclude that $\sup_{\boldsymbol{p}_{-i}} \Phi_i(\boldsymbol{p}_{-i}, \boldsymbol{s}, t) \le b + (1 + e^{\alpha_i})/\beta$, which completes the proof. $\square$

Theorem 2.4.1 *There exist a Nash equilibrium in pure strategies for the multistage game.*

Proof: The proof of existence for the multistage game is similar to that of the single-stage game in Theorem 2.3.1. The expected payoff $J_i(\boldsymbol{p}, \mathbf{s}, t)$ is continuous in p_i, since $q_i(\boldsymbol{p})$ is continuous in p_i and $V_i(\boldsymbol{p}, \boldsymbol{s}, t)$ is independent of p_i. The expected revenue of the dynamic game, $J_i(\boldsymbol{p}, \mathbf{s}, t)$, is quasiconcave in p_i. On the other hand, according to Lemma 2.4.2, in a given state such as $(\boldsymbol{s}, t)$, since the best response functions are uniformly bounded, we can construct an equivalent game such that each retailer's strategy set is compact. The remaining of the proof follows from Theorem 2.1 (page 16) in Vives [34]. □

2.4.1 Structural Properties of the Nash Equilibrium

In this section, we study monotonicity properties of the equilibrium price and revenue of the multistage game. In a monopoly setting, there are many monotonicity properties that not only make intuitive sense but also can be proved in many mathematical models. Here are two examples:

1. Equilibrium prices, $p_i^*(\boldsymbol{s}, t), i = 1, \ldots, n$, are non-decreasing in t.

2. Equilibrium revenues, $V_i(\boldsymbol{s}, t)$, are non-decreasing in both t and s_i.

In our competition model, we find numerical examples in which these two monotonicity properties do not hold. First, in Table 2.3, we present a counterexample to the first property. In this case, we assume two retailers with $\alpha_1 = 10, \alpha_2 = 8, \beta = 0.1, \lambda = 0.01, s_1 = 20$, and $s_2 = 10$ and we determine the equilibrium price of retailer 1 for different numbers of periods. This example shows that the equilibrium price is not always increasing in time.

In Table 2.4, we use another example to show that the second property is also not always true. We set $\alpha_1 = 5, \alpha_2 = 4, \beta = 0.1, \lambda = 0.1, s_1 = 40, T = 600$, and determine the equilibrium revenue of retailer 2 for different s_2. This table shows that the equilibrium revenue of a retailer

Table 2.3: A counterexample for the equilibrium price to be increasing in time.

T	750	800	850	900	950	969	1000	1050
p_1^*	28.29	27.46	25.75	23.02	20.09	19.65	21.89	27.41

Table 2.4: A counterexample for the equilibrium revenue to be increasing in inventory.

s_2	10	15	20	25	26	27	30	32	35	40
V_2	405.43	526.09	612.11	664.52	669.80	671.46	500.7	465.71	466.32	466.28

is not always increasing in its own inventory level.

2.4.2 Nash Equilibrium Versus Monopolistic Strategy

In Section 2.2, we described how a retailer can lose revenue by ignoring the competition in the market when other retailers produce similar products. In this section, using numerical examples, we compare the expected revenue of the monopolistic strategy, discussed in Section 2.2, with the expected revenue in the Nash equilibrium.

We consider the same two-retailer example in Section 2.2, with $T = 600$, $\lambda = 0.1$, $\alpha_1 = 4$, $\alpha_2 = 5$, and $\beta = 0.1$. Table 2.6 reports the equilibrium revenues when we fix retailer 1's initial inventory level equal to 20, and change that of retailer 2 from 5 to 40. Columns 1 and 2 give the respective expected total revenues for retailer 1 and retailer 2 in Nash equilibrium. In columns 3 and 4, retailer 1 uses the monopolistic strategy—setting the price that would be optimal if retailer 1 enjoyed monopoly as described by Equation (2.3)—and retailer 2 maximizes its own expected revenue as described by Equation (2.4). The revenues in these two columns are expressed as percentage of their counterparts in Nash equilibrium.

Table 2.5: Expected revenues in different scenarios.

	Equilibrium revenue[a]		Monopolistic strategy[b]	
s_2	retailer 1	retailer 2	retailer 1[c]	retailer 2
5	866.73	324.14	99.81%	100.33%
10	833.55	576.50	99.26%	100.77%
15	794.89	783.53	98.33%	101.48%
20	749.40	947.97	97.00%	102.84%
25	696.69	1065.55	95.27%	105.27%
30	648.18	1131.27	91.72%	109.29%
35	623.32	1162.78	84.09%	113.46%
40	612.11	1174.53	75.34%	116.81%

[a]Both retailers use the price in equilibrium.

[b]retailer 1 uses the monopolistic strategy, and retailer 2 maximizes its own expected revenue.

[c]Simulation results with standard error no more than 0.0026%.

As we can see from the table, when s_2 is relatively small compared with s_1 so that retailer 1 dominates the market, retailer 1 can act as if it enjoyed monopoly without much harm. However, as s_2 increases, retailer 1 will lose more revenue if it does not adopt the game-theoretic model to set its product price. In addition, retailer 2 can increase its revenue by taking advantage of retailer 1's monopolistic pricing strategy.

2.5 A Practical Policy in the Case of Incomplete Information

In Section 2.4, we calculate the equilibrium prices given that the retailers have complete information. The complete information refers to the fact that each retailer knows the real–time inventory level of the other retailers and that the real-time inventory levels can be tracked without any additional cost. Although this assumption is theoretically appealing, in practice, having the real-time information about other retailers might be costly or sometimes impossible To get around this obstacle, in this section, we introduce a heuristic policy that does not require the real-time information about the inventory levels of other retailers. Using this policy, retailer 1 sets its price based on its own inventory level, the remaining time, and the initial inventory levels of the other retailers but not their real-time inventory levels.

Suppose that the initial inventory level of retailer i is $s_i, i = 1, \ldots, n$, when there are T remaining time periods. The idea behind the heuristic policy comes from two observations of equilibrium price and equilibrium revenue of the multistage game. In the first observation, if $\lambda T \gg \sum_{i=1}^{n} s_i$, then the equilibrium prices are such that the expected number of items that retailer i can sell is close to its initial inventory level. This observation suggests that the retailers should post prices such that they spread out their sales evenly across the time horizon. In the second observation, if $\lambda T \gg \sum_{i=1}^{n} s_i$, then the equilibrium price of retailer i is not that sensitive to the others' inventory levels, so knowing roughly the others' inventory levels can a retailer approximate equilibrium price to a good extent. We use these observations to develop our heuristic, hereafter *interpolation policy.* For simplicity, we first consider a market with two retailers and assume that retailer 1 does not know the real-time inventory level of retailer 2 and uses the interpolation heuristic. We next describe the interpolation heuristic.

Interpolation Heuristic Algorithm:

1. At the beginning of period T, denote the initial inventory levels by $\bar{s}_j, j = 1, 2$, and set the price equal to $p_1^*(\bar{s}_1, \bar{s}_2, T)$ described in the multistage game in Section 2.4

2. At the beginning of period t, observe the remaining items in inventory and compute $a_2 = \bar{s}_2 t/T$ and $d_2 = \lfloor a_2 \rfloor$.

3. Calculate the price based on only s_1 and t as follows:

$$\tilde{p}_1(s_1, t) = (d_2 + 1 - a_2)p_1^*(s_1, d_2, t) + (a_2 - d_2)p_1^*(s_1, d_2 + 1, t). \tag{2.20}$$

This heuristic policy linearly interpolates the Nash equilibrium prices. We generalize this policy to an n-retailer model by replacing Equation (2.20) with a summation of 2^{n-1} terms:

$$\begin{aligned} \tilde{p}(s_1, t) &= \sum_{s_2=d_2}^{d_2+1} \cdots \sum_{s_n=d_n}^{d_n+1} \\ &\quad \left[\left(\prod_{j=2}^{n} [\mathbf{1}(s_j = d_j)(d_j + 1 - a_j) + \mathbf{1}(s_j = d_j + 1)(a_j - d_j)] \right) p_1^*(s_1, s_2, \ldots, s_n, t) \right], \end{aligned}$$

where s_i is the initial inventory level of retailer i, $a_i = s_i t/T$, and $d_i = \lfloor a_i \rfloor$, $i = 2, \ldots, n$, and that the indicator function $\mathbf{1}(A)$ is equal to 1 if statement A is true, and is equal to 0 otherwise.

Next, we consider different policies used by retailer 2 when retailer 1 uses the interpolation policy, in order to demonstrate the performance of our policy.

To evaluate the interpolation heuristic, we first consider the scenario in which retailer 2 has complete information and is rational. Therefore, retailer 2 has the real-time inventory level of retailer 1 and wants to maximize its own expected revenue. To this end, retailer 2 calculates retailer 1's price using the interpolation algorithm and then calculates its own price. Let $V_2(s_1, s_2, t-1)$ be the maximized expected revenue of retailer 2 at the beginning of period $t-1$ if retailer i still have s_i items in inventory. The recursive equation for V_2 is

$$\begin{aligned} V_2(s_1, s_2, t) \equiv & \max_{p_2} \lambda q_2(\tilde{p}_1(s_1, t), p_2)(p_2 + V_2(s_1, s_2 - 1, t-1)) + \\ & \lambda q_1(\tilde{p}_1(s_1, t), p_2) V_2(s_1 - 1, s_2, t-1) + \\ & (1 - \lambda + \lambda q_0(\tilde{p}_1(s_1, t), p_2)) V_2(s_1, s_2, t-1). \end{aligned}$$

The boundary conditions are $U(\cdot, 0, \cdot) = U(\cdot, \cdot, 0) = 0$. We consider the same numerical example as in Section 2.4.2, and report the numerical results in Table 2.6. In this table, we include the old results from Table 2.6 to be able to compare the performances of different policies.

Finally, to derive a lower bound for the expected revenue for the heuristic policy, we assume retailer 2 has the real-time information about inventory levels and its goal is to minimize retailer 1's expected revenue. Let $U(s_1, s_2, t)$ denote retailer 1's minimized expected revenue at the beginning of period t if inventory level of retailer i is s_i, then the recursive equation can be written as

$$\begin{aligned} \min_{p_2} U(\tilde{p}_1(s_1, t), p_2, s_1, s_2, t) \equiv & \min_{p_2} \lambda q_1(\tilde{p}_1(s_1, t), p_2)(\tilde{p}_1(s_1, t) + U(s_1 - 1, s_2, t-1)) + \\ & \lambda q_2(\tilde{p}_1(s_1, t), p_2) U(s_1, s_2 - 1, t-1) + \\ & (1 - \lambda + \lambda q_0(\boldsymbol{p})(\tilde{p}_1(s_1, t), p_2) U(s_1, s_2, t-1), \end{aligned}$$

Table 2.6: Expected revenues in different scenarios.

	Equilibrium revenue[a]		Monopolistic strategy[b]		Interpolation heuristic[c]	
s_2	retailer 1	retailer 2	retailer 1[d]	retailer 2	retailer 1[e]	retailer 2
5	866.73	324.14	99.81%	100.33%	99.97%	100.17%
10	833.55	576.50	99.26%	100.77%	99.92%	100.26%
15	794.89	783.53	98.33%	101.48%	99.92%	100.41%
20	749.40	947.97	97.00%	102.84%	100.01%	100.69%
25	696.69	1065.55	95.27%	105.27%	100.25%	101.28%
30	648.18	1131.27	91.72%	109.29%	99.45%	101.96%
35	623.32	1162.78	84.09%	113.46%	97.54%	101.79%
40	612.11	1174.53	75.34%	116.81%	97.39%	101.38%

[a]Both retailers use the price in equilibrium.

[b]retailer 1 uses the monopolistic strategy, and retailer 2 maximizes its own expected revenue.

[c]retailer 1 uses the interpolation heuristic, and retailer 2 maximizes its own expected revenue.

[d]Simulation results with standard error no more than 0.0026%.

[e]Simulation results with standard error no more than 0.0052%.

where $\tilde{p}_1(s_1, t)$ is the price posted by retailer 1 based on s_1 and t. We use the same numerical study reported in Table 2.6 to evaluate the performance of the interpolation policy. We found that this policy makes more than 95% of the equilibrium revenue when the number of initial inventory of the retailers are almost the same. This percentage increases when the initial inventory of retailer 1, who uses the interpolation policy, is higher than that of retailer 2.

2.6 Conclusions

In this chapter, a dynamic pricing model has been addressed for a retailer in a competitive market. A mathematical model has been provided in order to analyze the competition between many retailers who can change the price of their products in real time. In this study, it has been shown that there exists a unique Nash equilibrium in the single-period case. In addition, some structural properties of the equilibrium price and the equilibrium revenue have also been shown. We have also generalized the problem by considering a multistage revenue management and have shown the existence of Nash equilibrium. Finally, a practical policy, called interpolation policy, is provided in the case of incomplete information, where the retailers do not have real-time inventory levels of other retailers.

In this study, the demand has been modelled so that during a single period at most one costumer can show up. As an extension, we can consider longer time periods, *i.e.* day, during which many customers can show up. This extension transfers the current problem into a periodic review revenue management which is, computationally, easier to apply to the real world problems.

In our interpolation policy, we assume that the retailers have no information about the real time inventory levels of other retailers. In reality, this information can be collected with an additional cost. To extend our model, we can assume that the inventory levels of other retailers can be collected at the beginning of each period with an additional cost and determine the optimal number of times that a retailer collect the real time information.

Finally, in our numerical study, we consider only 2 retailers in the market. In many markets, such as airlines, more than two retailers provide similar products. Considering 3, or more, retailer in the numerical study would be a useful extension of our study.

Chapter 3

Revenue Management of Auto Train at Amtrak

3.1 Overview and Motivation

Auto Train offered by Amtrak is a distinctive service in the United States that lets passengers bring their own vehicles on the train. This service is offered daily to shuttle between Lorton, VA and Sanford, FL. Amtrak starts selling the tickets about 335 days before the train's departure date. In this chapter, we study dynamic pricing strategies in order to maximize the expected revenue of this service.

Amtrak accepts different types of vehicle—including Automobile, VAN/SUV, and Motorcycle—to board the train. In addition, Amtrak also provides different types of accommodation for passengers, including Super Coach Seat, Superliner Lower level Coach Seat, Superliner Roomette, Superliner Accessible Bedroom, Superliner Bedroom Suite, Superliner Bedroom,

Table 3.1: Total capacity and price buckets of each accommodation.

Accommodation	Capacity	Bucket 0	Bucket 1	Bucket 2	Bucket 3	Bucket 4
Coach seats	200	$187	$150	$116	$91	$74
Automobile accommodation	240	$281	$247	$213	$182	$140
Sleeper	85	$264	$232	$201	$171	$125
Van accommodation	21	$450	$400	$346	$296	$228
Others (No RM)	53	N/A	N/A	N/A	N/A	N/A

and Family Bedroom. The capacities of these accommodations are fixed [1] and each accommodation has a base price, some of which have up to four discount levels. However, many of these accommodations have very low capacities, and Amtrak does not want to change the prices of these accommodations dynamically For our revenue management model, we only consider the accommodations for which revenue management is allowed by Amtrak. Therefore, for vehicle accommodations, we consider only automobile and VAN/SUV accommodations, and for passenger accommodations, we consider only Super Coach Seats and Superliner Roomette. For simplicity, they are called *automobile accommodations*, *van accommodations*, *coach seats*, and *sleepers*, respectively, throughout this study. The capacities and price buckets of these accommodations are summarized in Table 3.1.

For each party [2] of passengers to have a complete reservation, two types of ticket must be purchased, one for the vehicle and one for the passenger(s). The ticket type for the vehicle

[1] Adding capacity is possible but very costly, so we do not consider this possibility

[2] We call a group of passengers who share a vehicle as a "party" throughout this study.

depends on the type of vehicle the party owns, and the passengers can choose their own accommodations. They can either buy a coach seat for each passenger, or share a sleeper. Therefore, a complete reservation includes two separate costs: *base cost* and *upgrade cost*. The base cost is the cost of boarding the Auto Train, which includes the vehicle cost plus the cost of coach seats for all passengers in the party. If a party decides to upgrade to a sleeper, they also have to pay the upgrade cost. We will discuss these two types of cost in more detail in later sections.

Because of the nature of the Auto Train, where a successful reservation includes a ticket for the vehicle and a set of tickets for the passengers, our dynamic pricing model is new to the revenue management literature. In most revenue management and dynamic pricing studies, a retailer offers a perishable product in a limited time after which the product has no or little value. In our model, a retailer (Amtrak) has a set of components (vehicle accommodations and passenger accommodations) to produce a set of products (a complete set of accommodations for a party of passengers). In addition, different types of components (coach seats and sleepers, van and automobile accommodations) are available to make different types of final products.

The most related research to our study is Gallego and van Ryzin [15]. They study a multiproduct and continuous-time dynamic pricing model. They consider a retailer who has inventories of a set of components that are used to produce a set of products. They consider a finite-time horizon during which the products can be sold and after which the unsold products have no value. Using a stochastic point process demand, they provide a dynamic pricing model as an extension to their seminal work in Gallego and van Ryzin [14]. They, first, prove that the deterministic version of the problem, when the demand is a deterministic function of component prices and the time, provides an upper bound for the optimal expected revenue. Using the solution of the deterministic version, they provide two heuristics to solve the stochastic problem, namely *make-to-stock* and *make-to-order* policies. In the make-to-stock policy, the products are made in advance and an inventory of finished products are held. In the make-to-order policy, the products are priced in advance and the requests are satisfied in a first-come-first-serve manner. They simulate both the make-to-stock and the make-to-order heuristics using the results of the deterministic problem.

Since Amtrak requires a manager to authorize each price change, it is only possible to change the prices in a daily basis. So a discrete-time model is more practical and the continuous-time model provided by Gallego and van Ryzin [14] cannot address the Amtrak Auto Train problem. Interested readers are referred to You [37], Hersh and Ladany [17], and Talluri [30] for additional resources of multiproduct revenue management models.

In our model, due to the structure of the data set, we use a discrete time dynamic pricing model in which Amtrak can set its product prices at the beginning of each day. Therefore, instead of using an intensity control technique used by Gallego and van Ryzin [15], we use Markov Decision Process to determine the each product's optimal price at the beginning of each day.

The rest of this chapter is organized as follows. In Section 3.2, we use Amtrak's sales data to analyze the market. In Section 3.3, we construct a mathematical model to describe Amtrak's revenue management problem, and use dynamic programming to solve it.

3.2 Analysis of Sales Data

We analyze Amtrak's sales data in this section. In Section 3.2.1, we describe the available data set. In Section 3.2.2, we analyze the data set with various statistical tools. In Section 3.2.3, we construct the demand model and calibrate the demand parameters from the sales data.

3.2.1 Available Data Set

Sales data for Auto Train is provided by the Amtrak Revenue Management Department for more than 350 trains departing between October 2002 and September 2003. For each train the transaction data is available from 330 days before departure until the day of departure. Upon requesting a ticket for a vehicle, Amtrak assigns an identification number (also called the PNR-number) to the party. Since a complete reservation includes a ticket for the vehicle and a set of tickets for the passengers, there are at least 2 records associated with each PNR-number, one for the vehicle and one (or more) for the passenger(s). Each record represents a transaction and contains time and date of reservation, a flag to indicate whether the reservation is eventually cancelled or purchased, the price bucket of the reservation, number of passengers in the party, and base and upgrade costs. The base cost includes the cost for the vehicle and coach seats, and the upgrade cost represents the additional cost for additional accommodations such as the sleepers.

Amtrak does not record purchase inquiries that do not lead to eventual sales. It also does not record the price bucket of each accommodation if no transaction takes place on a specific day. In addition, some data are ambiguous so it is difficult to know the exact number of seats remaining, and whether a specific product is out of stock. Because of the lack of these data, we work with Amtrak Revenue Management Department to interpret the sales data and make a few assumptions. We next discuss these issues.

In the data set, Amtrak records the transaction price for each ticket sale, but does not record the price if no sales occur on a specific day. In other words, if on a given day, a few tickets for automobile accommodations and coach seats were sold, but there were no sales for van accommodations and sleepers, then we know the current prices of automobile accommodations and coach seats, and have no documented information about the price of van accommodations and sleepers on that day. The lack of each accommodation's daily price prevents us from establishing a correct demand model. To resolve this problem, we work with Amtrak's Revenue Management Department to reconstruct the data set with the following assumptions:

1. If a successful transaction (purchased or cancel) was made for an accommodation, we assign the transaction price as the price of the accommodation for that day.

2. If more than one price bucket was charged for the same accommodation on a given day, we assume that each price was posted with the same amount of time for that accommodation. For example, if two price buckets have been charged for the automobile accommodation during a day, we assume each of which was active for half of that day.

3. If no transaction takes place for an accommodation but the price of the last transaction and the next transaction are the same, we assign that same price for the given day.

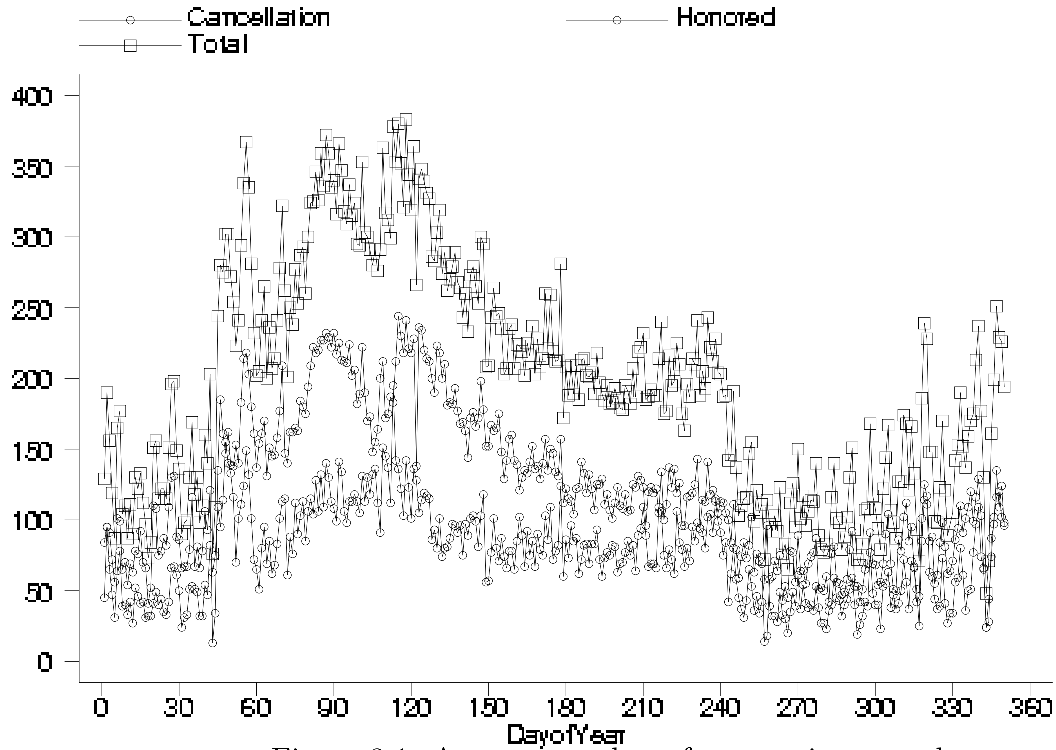

Figure 3.1: Average number of reservations per day.

4. If no transaction takes place for an accommodation but the price of the last transaction and the next transaction are different, we choose the price bucket of the transaction that took place closer to the given day.

3.2.2 Statistical Analysis of Data Set

In this section, we use the sales data to estimate the demand. In particular, we discuss five issues.

1. Seasonality

Seasonality is an important aspect of sales data especially in the travel industry that it should be taken into account in the data analysis. Seasonality can occur for trains departing in different months of a year. We take the daily average of total reservations, eventually purchased reservations, and eventually cancelled reservations over a year to address the seasonality. Figure 3.1 presents the results for trains departing in year 2003. This figure shows an obvious

Table 3.2: Average reservations for trains departing on different days in a week.

Day of departure	Sun.	Mon.	Tues.	Wed.	Thu.	Fri.	Sat.	Total
Ave. Res.	197.54	188.48	185.84	189.52	194.51	209.52	226.59	198.85
Ratio	0.99	0.95	0.93	0.95	0.98	1.05	1.14	1

seasonably across the year. For example, the average number of reservations during the winter and the early fall (January and February, and September 2003) is much lower than that during the spring (March and April 2003).

To take into account the seasonality, we use the sales data for trains that depart in summer 2003 (day 150 until day 240), because the demand over this period follows a consistent pattern. Our approach can also apply to sales data in the other seasons.

2. Trains are more popular on weekends.

Trains departing on different days during a week might have different levels of popularity. To determine whether the data is distinguishable on different departure days, we take the average number of reservations on trains departing on each day of a week as an index of popularity of the train with results shown in Table 3.2.

The first row in Table 3.2 lists the average reservations per day with the overall average in the last column. The results suggest that different days have different levels of popularity. We use the overall average as a base and will use the ratios listed in the second row to calculate a more accurate average reservations per day.

Table 3.3: Average reservations made on different days in a week.

Day of reservation	Sun.	Mon.	Tues.	Wed.	Thu.	Fri.	Sat.
Ave. Res.	146.97	262.84	231.85	228.31	204.11	192.59	161.01

Table 3.4: Correlation between the numbers of reservations one week and two weeks before departure.

	One week before departure		
Two weeks	Reservation	Paid	Cancelled
Reservation	0.2276	0.106	0.072
Paid	0.1437	0.111	0.543
Cancelled	0.025	-0.0051	-0.0302

3. More reservations are made on Mondays and Tuesdays.

Now, we determine whether the passengers make more reservations during weekdays or weekends. To do so, we take the average number of reservations on the day of reservation over all trains. Table 3.3 presents the results and suggests that more reservations are made at the beginning of the week, Mondays and Tuesdays, than other days of the week. Moreover, fewer reservations are made on Saturdays and Sundays.

Table 3.5: Correlation between the numbers of reservations one week and three weeks before departure.

	One week before departure		
Three weeks	Reservation	Paid	Cancelled
Reservation	0.1238	0.1436	0.0522
Paid	0.1414	0.1287	-0.0051
Cancelled	0.106	-0.0302	0.0721

4. Can we learn about future demand from past sales?

We are also interested in learning about future demand from the past sales. To this end, we determine the correlation between the numbers of reservations in the last two weeks before departure. Table 3.4 represents the correlation between different combinations of reservations, cancellations, and paid tickets between one week and two weeks before departure. The result in Table 3.4 shows no significant relationship between the numbers of sales a week before departure and two weeks before departure. Therefore, we conclude that we cannot learn about the future demand based on past sales. Table 3.5 presents a similar correlation analysis between demands one week and three weeks before departure.

5. Demand increases as the departure date approaches.

To estimate the demand, we calculate the weekly average number of reservations, cancellations, and paid tickets over the sales horizon. The results are presented in Figure 3.2. As expected,

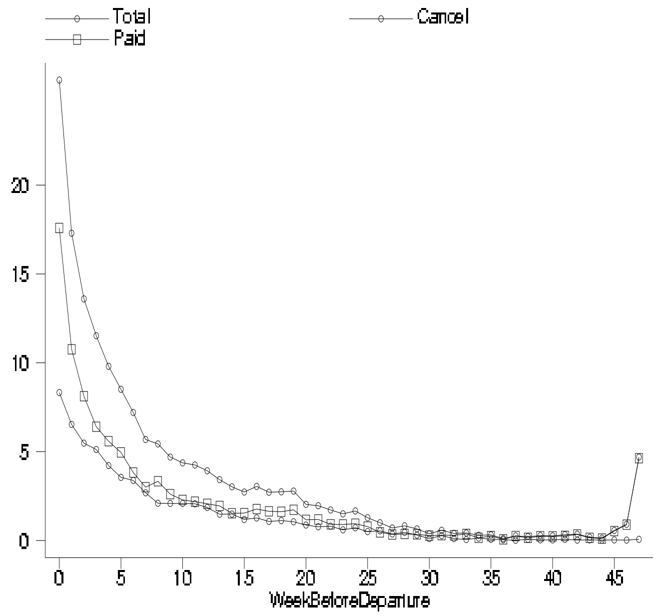

Figure 3.2: Average number of total, paid, and cancelled reservations per week.

the number of reservations increases toward the departure date. An early peak in the number of transactions on the opening day is followed by a duration of 20 weeks with very low numbers of reservations.

We also estimate the cancellation probability by taking the ratio between the number of canceled reservations and the total number of reservations. As seen in Figure 3.3, this the estimation of this cancellation probability becomes more stable toward the departure date.

3.2.3 Demand Model and Parameter Calibration from Sales Data

Accurate demand forecasting is a fundamental step in supply chain and revenue management. In this section, we use the knowledge about the data set to determine the demand distribution for different combinations of accommodations' prices.

From the sales data, the mean and the variance of the number of reservations per day are close to each other, which suggests that the demand follows a Poisson distribution. For example, the mean and the variance of the number of reservations for automobile accommodations

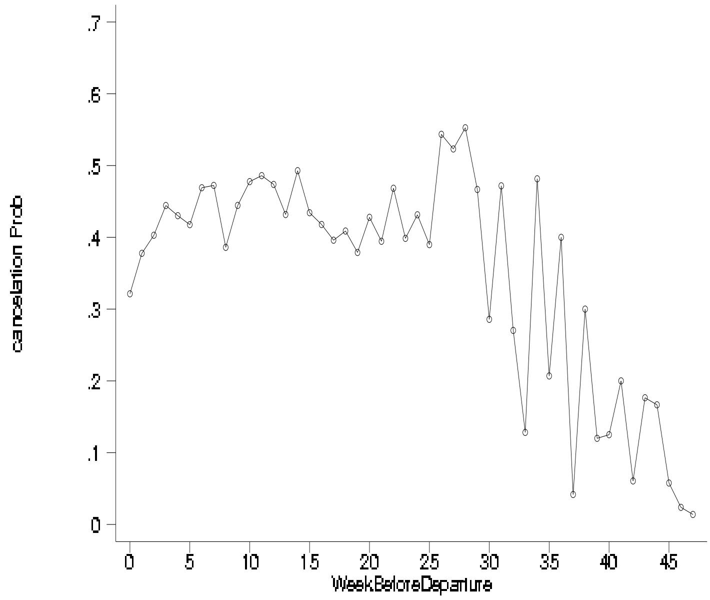

Figure 3.3: Cancellation probability in each week.

are 3.22 and 3.47, respectively, on the second to last day before departure with price bucket 1 for both automobile accommodations and coach seats. Table 3.6 presents the mean and the variance of different combinations of price buckets and days before departure.

In order to conclude the demand does follow a Poisson distribution, we run a goodness-of-fit, provided by Winston [36], test as follows.

- H_0: The total number of reservations for each type of vehicle on a given day and for a given price bucket follows a Poisson distribution.
- H_1: The total number of reservations for each type of vehicle on a given day and for a given price bucket does not follow a Poisson distribution.

There are 16375 [3] random variables that can be used for this hypothesis test. We use one example to explain how we run this hypothesis test. Consider the total number of reservations during the second to last day before departure when the price bucket is 1 for automobile accommodations and for coach seats. In our data set, we have 18 such instances and we

[3] 335(day) $\times$ 5(coach price bucket) $\times$ 5(vehicle price bucket) $\times$ 2(automobile and van) = 16375

Table 3.6: The mean and variance of the number of reservations per day.

Day	Auto.	Coach	Obs.	Mean	Var
0	1	1	15	3.84	3.61
1	1	1	18	5.17	4.87
2	1	1	15	3.27	4.35
12	1	1	12	1.97	1.36
13	1	1	13	2.01	2.91
14	1	1	12	2.33	2.42
0	1	2	5	2.81	2.69
1	1	2	3	4.67	4.33
2	1	2	6	3.01	2.83
12	2	2	5	2.41	2.29
13	2	2	6	2.33	2.27
14	2	2	8	3 .93	4.27

Table 3.7: Frequency of the number of reservations during two days before departure.

Number of reservations	2	3	4	5	6	7	8	9	10
Frequency	1	5	2	3	2	2	1	1	1

Table 3.8: Different statistics of the number of reservations during two days before departure.

Range of X	Observed	Expected	$(O-E)^2/E$
$X <= 3$	6	3.7302	1.381157
$3 < X \leq 5$	5	6.1981	0.231594
$5 < X \leq 7$	4	4.7143	0.108229
$X > 7$	3	2.7242	0.027922
Total	18	17.3668	1.7489

summarize the number of reservations for these 18 trains in Table 3.7.

The average number of reservations per day is $93/18 = 5.1667$. First, we assume the number of reservations follows a Poisson distribution with mean 5.1667 and compute its probability mass function. Second, we calculate the expected number of observations according to this probability mass function. For example, the probability of 3 reservations is $e^{(-5.1667)}(5.1667)^3/3! = 0.13$, and that it is expected to find 13% of the days with 3 reservation and therefore have 0.13*18=2.34 trains with 3 reservations. The results are summarized in Table 3.8. Note that in this table, we group the number of observations into groups because we only have 18 observations.

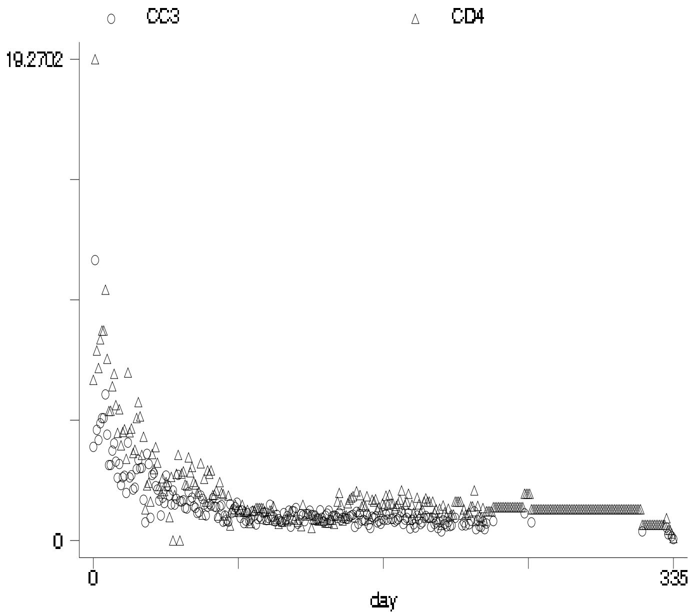

Figure 3.4: Average number of reservations for automobile accommodations.

Now, we use a chi-square test with 3 degrees of freedom and a test statistic of 1.7489. With $\alpha = 0.05$, the corresponding value from the chi-square table is 5.99, so we cannot reject H_0. Therefore, we conclude that the total number of reservations for automobile accommodations, when the price buckets of automobile accommodations and coach seats are both 1, on the second to last day before departure, is a Poisson random variable. Among all combinations of price buckets, we have chosen 15 combinations and found the same result by running similar tests. These results suggest that the total number of reservations during a day, for given price buckets, follows a Poisson distribution.

Now, we can determine the demand pattern over time. Plotting the average number of vehicle reservations, y, against time before departure, t, indicates that y is not a linear function with respect to t. As an example, in Figure 3.4, we plot the average number of reservations for automobile accommodations when the price buckets of automobile accommodations and coach seats are both 4.

The average number of reservations almost stays flat from the opening day through about thirty days before departure, and increases from thirty days before departure through the day of departure. Winston [36] (page 1290) provides a procedure to estimate the nonlinear relationships between two variables. In this procedure, first, the variables need to be plotted against each other to illustrate the nonlinear relationship. Second, this plot needs to be assigned to its best fit among the sample plots provided in the procedure steps. After finding the best fit, the procedure gives the directions to calculate the parameters of the model. Using this procedure and also after careful examinations on the data set for all price combinations, we conclude that the nonlinear relationship in the last thirty days before departure is of the form $y = \beta_0 \exp(\beta_1 t)$.

To estimate the parameters of the fitted curve for the last thirty days and for all combinations of price buckets, we use the following steps. First, we transform the observation values in order to linearize the relationship between y and t. To this end, we transform each value of y into $\ln y$ and each value of t into t. Second, we estimate the least square regression line for the transformed data. Let $\hat{\beta}_0$, $\hat{\beta}_1$, and s_e respectively denote the intercept of the least squares line, the slope of the least squares line, and the standard error of the regression estimate for the transformed data. Third, using these parameters, we estimate the relationship between the number of reservations and the time before departure for any price combination of automobile accommodations and coach seats as follows.

$$\hat{y} = \exp(\hat{\beta}_0 + \hat{\beta}_1 t + s_e^2/2).$$

On the other hand, for the remaining time horizon (30 days before departure through 335 days before departure), the total number of reservations for automobile accommodations appears to be linear and of the form

$$y = \beta_0(j_a, j_c) + \beta_1(j_a, j_c)t.$$

Using the sales data, we estimate the parameters of the demand function for each price combination. Table 3.9 summarizes the results. [4]

We also assume that the demand for van accommodations is a function of the van accommodations' price bucket, coach seats' price bucket, and time before departure. Using the same technique that we used to calculate the demand parameters for the automobile accommodations, we calculate the demand parameters of the van accommodations. The results are listed in Table 3.10. From this table, we conclude that the demand for the van accommodations is not significantly sensitive to the price or the time before departure. Between 90 trains that depart in the summer of 2003, there were only 5 trains that did not sell out its van accommodations.

Because of this observation and the current capacity of van accommodations, we recommend Amtrak to set its price in the highest price bucket for the entire sales horizon. Consequently, in Sections 3.3 and 3.4, we develop dynamic pricing models to manage the price of automobiles, coach seats, and sleepers, but not the price of van accommodations.

[4]Note that from our data analysis, there is a demand seasonality for trains departing on different days. For example, a train leaving on Sunday has a different demand distribution from a train leaving on Monday. To resolve this obstacle, we multiply the mean demand by the parameters listed in Table 3.2. To simplify our numerical analysis, we assume a Sunday train, whose parameter is very close to 1.

Table 3.9: The demand parameters for automobile accommodations.

j_a, j_c	β_0	β_1	s_e	α_0	α_1
0, 0	0.101	0.057	0.011	0.058	0.003
0,1	0.475	0.052	0.079	0.147	0.003
0,2	0.653	0.0522	0.079	0.251	0.003
0,3	1.023	0.0582	0.075	0.479	0.002
0,4	1.221	0.052	0.079	0.685	0.004
1,0	0.281	0.056	0.011	0.162	0.003
1,1	0.362	0.062	0.079	0.209	0.003
1,2	0.554	0.062	0.079	0.320	0.003
1,3	0.884	0.068	0.075	0.510	0.002
1,4	1.543	0.062	0.079	0.891	0.004
2,0	0.488	0.058	0.112	0.282	0.003
2,1	0.889	0.055	0.063	0.513	0.003
2,2	1.238	0.059	0.112	0.715	0.003
2,3	1.487	0.058	0.077	0.860	0.002
2,4	1.883	0.059	0.073	1.087	0.004

Table 3.9: Continue

j_a, j_c	β_0	β_1	s_e	α_0	α_1
3,0	0.651	0.057	0.066	0.378	0.003
3,1	1.102	0.066	0.098	0.636	0.004
3,2	1.372	0.067	0.073	0.792	0.005
3,3	1.693	0.068	0.073	0.978	0.005
3,4	2.026	0.079	0.073	1.171	0.008
4,0	0.943	0.061	0.122	0.545	0.003
4,1	1.352	0.071	0.119	0.781	0.005
4,2	1.625	0.072	0.119	0.938	0.0052
4,3	1.917	0.074	0.121	1.1073	0.007
4,4	2.449	0.091	0.121	1.415	0.005

Table 3.10: The parameters of the demand for van accommodations.

j_v, j_c	β_0	β_1	s_e	α_0	α_1
0, 0	0.101	0.057	0.011	0.058	0.003
0,1	0.475	0.052	0.079	0.147	0.003
0,2	0.653	0.0522	0.079	0.251	0.003
0,3	1.023	0.0582	0.075	0.479	0.002
0,4	1.221	0.052	0.079	0.685	0.004
1,0	0.047	0.067	0.147	0.178	0.001
1,1	0.051	0.045	0.147	0.192	0.001
1,2	0.056	0.045	0.147	0.209	0.001
1,3	0.069	0.045	0.147	0.258	0.001
1,4	0.089	0.045	0.147	0.333	0.001
2,0	0.049	0.071	0.262	0.184	0.001
2,1	0.054	0.045	0.147	0.201	0.001
2,2	0.059	0.071	0.147	0.220	0.001
2,3	0.077	0.051	0.147	0.288	0.001
2,4	0.096	0.058	0.147	0.361	0.001

Table 3.10: Continue

j_a, j_c	β_0	β_1	s_e	α_0	α_1
3,0	0.062	0.069	0.155	0.233	0.001
3,1	0.068	0.073	0.229	0.254	0.001
3,2	0.070	0.058	0.171	0.261	0.001
3,3	0.079	0.058	0.171	0.297	0.001
3,4	0.101	0.058	0.171	0.378	0.001
4,0	0.070	0.075	0.284	0.261	0.000
4,1	0.075	0.075	0.279	0.283	0.001
4,2	0.079	0.075	0.279	0.298	0.001
4,3	0.089	0.075	0.279	0.381	0.001
4,4	0.111	0.075	0.284	0.414	0.001

3.3 Revenue Management for Automobile and Coach Seats

In this section, we develop a mathematical model to manage the revenue for automobile accommodations and coach seats. In Section 3.4, we expand our model to incorporate sleepers as an alternative accommodation.

3.3.1 The Mathematical Model

Consider a multiproduct, discrete-time revenue management model for a single-leg Auto Train where automobile is the only accepted vehicle and coach seat is the only available accommodation for passengers. The initial inventories for both types of accommodation are fixed and cannot be replenished during the sales horizon. The unsold accommodations have no salvage value after the departure time. Cancellation and overbooking are not considered in this model.

The base price of all accommodations are denoted by $f_x(0), x \in \{a, c\}$, where a and c stand for automobile accommodations and coach seats, respectively, and 0 stands for the base price index. However, there are four discount classes that can be used to induce demand. We denote the price of accommodation x in discount class j by $f_x(j)$ for $j \in \{1, 2, 3, 4\}$, and $f_x(0) > f_x(1) > f_x(2) > f_x(3) > f_x(4)$; the base price is the most expensive fare class.

The time horizon is divided into days, and the price can be changed on a daily basis. In the beginning of each day, the decision of the revenue manager can be delineated by a vector of two elements, $J = (j_a, j_c)$, where j_x is the open bucket of accommodation x on that day. On each day, the revenue manager is free to choose any price bucket for each accommodation.

To complete a reservation, each party of passengers needs to purchase a transportation ticket, which includes the accommodation cost for both the vehicle and coach seats for all passengers. For example, if a party consists of two passengers, the transportation cost includes one ticket for the vehicle and two coach-seat tickets. Therefore, the passengers' decision to ride the train depends only on the price of transporting their vehicle and the price of a coach seat.

We denote the aggregate demand of automobile accommodations during day t by $D_a(j_a, j_c, t)$. We assume that D_a is a Poisson random variable with mean $\lambda(j_a, j_c, t)$. We denote the total number of passengers sharing a vehicle by M, which is a discrete-valued random variable that can take on values $1, 2, ..., \bar{M}(= 5)$. Therefore, the total number of passengers arriving on day t can be calculated by $\sum_{i=1}^{D_a} M_i$, where M_i, is the number of passengers sharing the ith automobile.

Suppose at the beginning of each day, a total of m automobile accommodations and n coach seats are available for sale. Given the remaining number of accommodations and time before departure, we adjust the bucket classes for the accommodations at the beginning of each day in order to maximize the total expected revenue. This problem can be formulated as a Markov Decision Process with states t, m, and n, and control variable J, which can be solved by dynamic programming.

Let $R(m, n, t)$ denote the maximum expected revenue from day t to day 0, the day of departure, if at the beginning of day t, we still have m automobile accommodations and n coach seats The decision at time t is which fare class to charge for the next day. Let p and q denote the total number of available accommodations at the beginning of the next day, namely day $t-1$. In other words, a total of $m-p$ automobile accommodations and a total of $n-q$ coach seats are sold during day t.

We can recursively calculate the value function $R(m,n,t)$, starting with $t=T$ as follows:

$$\begin{aligned} R(m,n,t) \quad &\equiv \quad \max_{j_a,j_c\in\{0,1,2,3,4\}} \sum_{p=0}^{m}\sum_{q=0}^{n}\Big(f_a(j_a)(m-p)+f_c(j_c)(n-q) \\ &\qquad +R(p,q,t-1)\Big)P_{j_a,j_c}(m,n,p,q). \end{aligned} \tag{3.1}$$

where $P_{j_a,j_c}(m,n,p,q)$ is the probability of selling $m-p$ automobile accommodations and $n-q$ coach seats during day t when the effective prices for automobile accommodations and coach seats are $f_a(j_a)$ and $f_c(j_c)$, respectively. In other words, P is the one-step (two-dimensional) transition probability matrix for the total number of remaining seats for given prices of automobile accommodations and coach seats. In the next section, we first introduce a procedure to calculate P, and then use it to compute the optimal policy.

3.3.2 Computation of the Optimal Policy

In this section, we provide an algorithm to calculate the transition probability matrix, P. As explained earlier, an arriving party must purchase one ticket for the automobile and one ticket for each passenger to complete a reservation. Conditioning on the number of reservations for automobile accommodation during a day, the transition probability can be determined as follows.

$$P_{j_a,j_c}(m,n,p,q) = \sum_{k=0}^{\infty} P'(m,n,p,q,k)Pr(D(j_a,j_c)=k). \tag{3.2}$$

where $P'(m,n,p,q,k)$ is the transition probability, given that the total number of reservations for automobile accommodation is k during a day. Note that $P'(m,n,p,q,k)$ is independent of t and $J = \{j_a, j_c\}$.

Next, we calculate the transition probability matrix P. Because each party has one vehicle, but may have more than one passenger, the total number of sales for automobile accommodation is less than or equal to that for the coach seat.

Using Equation (3.2), we calculate the transition probability $P'(m,n,p,q,k)$. Recall that the total number of passengers in a party is a discrete random variable that takes on values $\{1, 2, \ldots, \bar{M}\}$, independent of the arrival time and the prices.

We assume that if the total number of passengers in a party exceeds the total number of available seats, then the whole party leaves without riding the train. We call this assumption as *no-partial-fulfillment* (NPF) throughout this study. In Procedure 3.3.1, we calculate the transition probabilities $P(m,n,p,q,k)$, for $m \geq p$, $n \geq q$, and $k \geq 0$.

Procedure 3.3.1

1. If $\Big($ ($k = 0$ and either $m \neq p$ or $n \neq q$) or ($m > p$ and $n = q$) or ($m = p$ and $n > q$) or ($m - p > k$ and $n - q > \bar{M}k$) or ($m = p$ and $n = q$ and $n > \bar{M}$) $\Big)$, then $P(m,n,p,q,k) = 0$.

2. If $\Big($($m = p$ and $n = q$ and $k = 0$) or ($m = p$ and $n = q$ and $m = 0$) or ($m = p$ and $n = q$ and $n = 0$) $\Big)$, then $P(m,n,p,q,k) = 1$.

3. If $\Big($ $k = 1$ and $m = p + 1$ and $n > q$ $\Big)$, then $P(m,n,p,q,k) = P(M = (n - q))$.

4. If $\Big($ $m = p$ and $n = q$ and $n < \bar{M}$ $\Big)$, then $P(m,n,p,q,k) = (\sum_{i=n}^{\bar{M}-1} P(M = i + 1))^k$,

where $\bar{M}$ is the maximum capacity of an automobile.

5. If $\Bigg(m > 0$ and $n > 0$ and $k > 1 \Bigg)$. then

   - if $n \geq \bar{M}$, then $P(m,n,p,q,k) = \sum_{j=1}^{\bar{M}} P(m-1,n-j,p,q,k-1) \cdot P(M=j)$;

   - if $n < \bar{M}$, then $P(m,n,p,q,k) = \sum_{j=1}^{n} P(m-1,n-j,p,q,k-1) \cdot P(M=j) + \sum_{j=n+1}^{\bar{M}} P(m-1,n,p,q,k-1) \cdot P(M=j)$.

Suppose $\bar{N}$ is a large number such that the probability of having $\bar{N}$ or more parties in any day and for any price combination is negligible [5].

Step 1 lists the infeasible transitions and sets the transition probability to zero. For example, the transition probability is zero if the future state and the current state are the same, $m = p$ and $n = q$, while at least a request has been made, $k > 0$, and also the total number of coach seats is more than 5, because in this case the reservation should have been granted. Step 2 summarizes the cases for which the transition probability is trivially equal to 1. For example, if the total number of reservations is zero, then the probability the future state are equal to the present state is 1. Step 3 initializes the probability matrix for the last reservation, when only one party shows up. In this case, the transition probabilities are calculated based on the distribution of the number of passengers in the party. Step 4 accommodates the NPF assumption when there are fewer seats available than the passengers in a party. Step 5 gives the recursive formulation to calculate the transition probability in the general case when the total number of automobile accommodations is more than the maximum capacity, $\bar{M}$.

[5]Based on Amtrak's sales data, we choose $\bar{N}$= 30 in this study.

3.3.3 Numerical Results

A computer program based on Procedure 3.3.1 and Equation (3.2) has been developed to calculate the best price bucket for automobile accommodations and for coach seats. Since the demand distribution during the last few weeks before the departure date follows a different pattern from the demand in the rest of the sales horizon, we calculate the optimal price buckets for the last 35 days before departure. The associated difficulty with this choice is to determine the initial values of the remaining number of automobile accommodations and coach seats at the beginning of the 35th.

From the sales data, the total numbers of available automobile accommodations and coach seats, at the beginning of the 35th day before departure, are random variables with respective mean 177.79 and 117.41, and respective standard deviation 47.35 and 57.06.[6]. The ideal way to determine the expected revenue for this time period is to take the weighted average of the expected revenues among all observed numbers of the remaining automobile accommodations and coach seats, at the beginning of the 35th days before departure, and then calculate the optimal price buckets accordingly. However, the dispersion of the available accommodations is very large with numbers ranging from -6 to 232[7]. This level of dispersion makes the numerical study very tedious and time consuming. In our numerical study, subject to the limited capability of available software and hardware, we assume we start with 50 automobile accommodations and 50 coach seats, and run our program for 35 days. [8]

[6]The first and second moments of the available accommodations at the beginning of 35 days before departure among 91 trains departing during summer 2003.

[7]The negative number is due to overbooking.

[8]Alternatively, one can decrease the number of times that the price can be changed in order to be able

We list the optimal price buckets of automobile accommodations and coach seats for the last day before departure in Table 3.11. The columns present the number of available automobile accommodations and the rows represent the number of available coach seats. The first number in a cell corresponds to the optimal price bucket for automobile accommodations, and the second number corresponds to that for coach seats. Table 3.12 presents the optimal price buckets for 15 automobile accommodations and 10 coach seats. The first number in a cell gives the price bucket for automobile accommodations, and the second for coach seats.

The following structural properties have been observed from our numerical studies.

- For a given number of automobile accommodations and the remaining time, the optimal price of automobile accommodations is non-increasing in the number of coach seats. [9]

- For a given number of coach seats and the remaining time, the optimal price of automobile accommodations is non-decreasing with respect to the number of automobile accommodations.

- For a given number of automobile accommodations and coach seats, the optimal price of automobile accommodations and coach seats are non-decreasing with respect to the remaining time.

Because each party consists of one vehicle and at least one passenger and we do not allow

to calculate the optimal price throughout the entire sales horizon. For example the manger can review the inventory at the beginning of day 180, 90, 45, 35, 30, 29, 28, . . ., 1, and change the price only at the beginning of these days. The associated aggregate demand between the two review points can also be calculated.

[9] Note that a higher bucket number is associated with lower price level, for example price bucket 4 is the cheapest price level for all accommodations

Table 3.11: Optimal price buckets for different number of available accommodations.

car–coach	1	2	3	4	5	6	7	8	9	10
1	2, 4	2, 3	4, 1	4, 1	4, 1	4, 1	4, 1	4, 1	4, 1	4, 1
2	1, 4	2, 4	2, 3	2, 3	4, 1	4, 1	4, 1	4, 3	4, 3	4, 3
3	1, 4	1, 4	2, 4	2, 4	2, 3	2, 3	2, 3	4, 1	4, 1	4, 1
4	1, 4	1, 4	2, 4	2, 4	2, 4	2, 3	2, 3	2, 3	2, 3	4, 1
5	1, 4	1, 4	2, 4	2, 4	2, 4	2, 4	2, 3	2, 3	2, 3	2, 3
6	1, 4	1, 4	1, 4	2, 4	2, 4	2, 4	2, 4	2, 3	2, 3	2, 3
7	1, 4	1, 4	1, 4	2, 4	2, 4	2, 4	2, 4	2, 4	2, 3	2, 3
8	1, 4	1, 4	1, 4	2, 4	2, 4	2, 4	2, 4	2, 4	2, 3	2, 3
9	1, 4	1, 4	1, 4	2, 4	2, 4	2, 4	2, 4	2, 4	2, 4	2, 3
10	1, 4	1, 4	1, 4	1, 4	2, 4	2, 4	2, 4	2, 4	2, 4	2, 3

Table 3.12: Optimal price buckets for different days before departure.

Remaining time	0	1	2	3	4	5	6	7	8	9	10
Price buckets	4, 4	1, 4	2, 2	2, 1	2, 1	2, 1	1, 1	1, 1	1, 1	1, 1	1, 1

partial fulfillment, the unsold coach seats have no value after we sell out all automobile accommodations. On the other hand, since each party might request more than one coach seat, with more coach seats in hand, it is optimal to not to increase the price of automobile accommodations in order to induce the total demand. Therefore, the price of automobile accommodations is non-decreasing in the number of coach seats. On the other hand, when the number of automobile accommodations increases, since each party requests one automobile accommodation, having fewer automobile accommodations increases the chance of selling out all coach seats and leaving with unsold automobile accommodations at the end. Therefore, the price of automobile accommodations is non-decreasing in the number of available automobile accommodations. The last observation is consistent with the monotonic property addressed in the literature that more remaining time provides more chances to sell and that the optimal price are non-decreasing with respect to the remaining time.

3.4 Revenue Management for Automobile, Coach Seats, and Sleepers

In this section, we consider sleeper as an alternative accommodation type for passengers. As noted in Section 3.1, if a party decides to ride the train, they have a choice to purchase a sleeper. If a party adds a sleeper, in addition to the base cost and the cost of a sleeper, they have to pay a transportation cost equal to the minimum coach price bucket, currently \$74, for each passenger. After a careful study on the data set, we learned that this upgrading decision is a function of the coach seats' price and the sleepers' price. In addition, we learned that the upgrading decision does not depend on the price of vehicle accommodation and also the time of reservation.

3.4.1 The Mathematical Model

To manage the revenue of a single-leg Auto Train with sleeper upgrades, we use a multiproduct, discrete-time revenue management model. We consider automobile as the only accepted vehicle and coach seats and sleepers as the available accommodations for passengers.

The base price of accommodations are denoted by $f_x(0), x \in \{a, c, s\}$, where a, c, and s stand for automobile accommodations, coach seats, and sleepers, respectively, and 0 stands for the base price index. In each day, the state of open price buckets can be delineated by a vector of three elements $J = (j_a, j_c, j_s)$ where j_x is the open bucket of accommodation x in that day.

Each party of passengers needs to purchase a base ticket which includes the cost of vehicle accommodation plus the cost of coach seats for all passenger to complete a reservation. If a party decides to purchase a sleeper, then an additional upgrading cost would be added to their total cost.

The aggregate demand for the automobile accommodations during day t is a function of the remaining time, the automobile accommodations price, and the coach seats. This aggregate demand is denoted by $D_a(j_a, j_c, t)$. As in section 3.3, D_a is Poisson random variable with mean $\lambda(j_a, j_c, t)$. The probability of purchasing a sleeper is $u(j_c, j_s)$. Therefore, the total number of parties who choose a sleeper is a Poisson random variable with mean $\lambda(j_a, j_c, t)u(j_c, j_s)$ and the total number of parties who choose coach seats is a Poisson random variable with mean $\lambda(j_a, j_c, t)(1 - u(j_c, j_s))$. Based on the observations from the data set, we assume that all passengers in a party reserve only one sleeper, if they decide to upgrade their purchase. The total number of passengers in a party is denoted by M which is a discrete-valued random variable that can take on values $1, 2, ..., \bar{M}(= 5)$. Therefore, the total number of passengers arriving on day t can be calculated by $\sum_{i=1}^{D_c} M_i$, where M_i, is the number of passengers sharing the ith automobile.

Suppose at the beginning of each day, a total of m automobile accommodations, n coach seats, and s sleepers are available for sale. Given the remaining number of accommodations and the remaining time, we adjust the price buckets for the accommodations at the beginning of each day in order to maximize the total expected revenue. This problem can be formulated as a Markov Decision Process with states t, m, n, and s and control variable J, which can be solved by dynamic programming.

Let $R(m, n, s, t)$ denote the maximum expected revenue from day t to day 0, the day of departure, if at the beginning of day t, we still have have m automobile accommodations, n coach seats, and s sleepers. The decision at time t is which fare classes to charge for the next day. Let p, q, and r denote the total number of available accommodations at the beginning of the next day; namely day $t-1$. In other words, a total of $m-p$ automobile accommodations, $n - q$ coach seats, and $s - r$ sleepers are sold during day t.

We can calculate the value function $R(m, n, s, t)$, starting with $t = T$ as follows:

$$R(m, n, s, t) = \max_{j_a, j_c, j_s \in \{0,1,2,3,4\}} \sum_{p=0}^{m} \sum_{q=0}^{n} \sum_{r=0}^{s} \Big(f_a(j_a)(m - p) + f_c(j_c)(n - q) \qquad (3.3)$$

$$+f_s(j_s)(s-r) + R(p,q,r,t-1)\Big)\hat{P}_{j_a,j_c,j_s}(m,n,s,p,q,r)$$

where $\hat{P}_{j_a,j_c,j_s}(m,n,s,p,q,r)$ is the probability of selling $m-p$ automobile accommodations, $n-q$ coach seats, and $s-r$ sleepers during day t when the effective prices for automobile accommodations, coach seats, and sleepers are $f_a(j_a)$, $f_c(j_c)$, and $f_s(j_s)$, respectively. In other words, $\hat{P}$ is the one-step (three-dimensional) transition probability matrix for the total number of accommodations when the prices are given. In the next section, we first introduce a procedure to calculate $\hat{P}$, and then compute the optimal policy.

3.4.2 Computation of the Optimal Policy

In this section, Procedure 3.3.1 is extended in order to incorporate the sleepers into the model. As of Section 3.3.2, conditioning on the number of reservations for automobile accommodations during a day, the transition probability can be calculated as follows.

$$\hat{P}_{j_a,j_c,j_s}(m,n,s,p,q,r) = \sum_{k=0}^{\infty} P''(m,n,s,p,q,r,k) \cdot Pr(D(j_a,j_c)=k), \tag{3.4}$$

where $P''(m,n,s,p,q,r,k)$ is the transition probability, given that the total number of reservations for automobile accommodations during a day is k. Note that $P'(m,n,s,p,q,r,k)$ is independent of both t and $J=\{j_a,j_c,j_s\}$.

Now, we calculate the transition probability matrix, $\hat{P}(m,n,s,p,q,r,k)$, given that the total number of passengers in a party is a discrete random variable that takes on values $\{1,2,\ldots,\bar{M}\}$, M independent of the time and the prices.

We assume that if the total number of passengers in a party exceeds the total number of available coach seats, and the passengers decide not to buy a sleeper, then the whole party leaves without riding the train. We call this assumption as *no-partial-fulfillment* (NPF) throughout this study. In the following procedure, we calculate the probability transition $\hat{P}(m,n,s,p,q,r,k)$.

Procedure 3.4.1

(To calculate $\hat{P}(m,n,s,p,q,r,k),$ when $m \geq p, n \geq q,$ and $s \geq r$)

1. if $\Big((m-p>k)$ or $(n-q>\bar{M}k)$ or $(s-r>k)$ or $(m=p$ and $n=q$ and $n>\bar{M})\Big)$, then $\hat{P}(m,n,s,p,q,r,k)=0$

2. if $\Big((m=p$ and $n=q$ and $s=r$ and $k=0)$ or $(m=p$ and $n=q$ and $s=r$ and $m=0)\Big)$ then, $\hat{P}(m,n,s,p,q,r,k)=1$

3. if $(k=1)$ then,

   - if $(m=p+1$ and $n>q$, and $s=r)$ then $P(m,n,s,p,q,r,k)=P(M=n-q)(1-u(j_c,j_s))$.
   - if $(m=p+1$ and $n=q$, and $s=r+1)$ then $P(m,n,s,p,q,r,k)=u(j_c,j_s)$.
   - otherwise $P(m,n,s,p,q,r,k)=0$

4. if ($m=p$ and $n=q$ and $n<\bar{M}$ and $s=r$), then $P(m,n,s,p,q,r,k)=(\sum_{i=n}^{\bar{M}-1}P(M=i+1)\cdot(1-u(j_c,j_s)))^k$

5. if $(m>0$ and $n>0$ and $s>0$ and $k>1)$, then

   - if $n\geq\bar{M}$, then $P(m,n,s,p,q,r,k)=(1-u(j_c,j_s))\sum_{j=1}^{\bar{M}}\Big(P(m-1,n-j,s,p,q,r,k-1)\cdot P(M=j)\Big)+u(j_c,j_s)P(m-1,n,s-1,p,q,r,k-1)$

- if $n < \bar{M}$, then $P(m,n,s,p,q,r,k) = (1-u(j_c,j_s))\Big(\sum_{j=1}^{n} P(m-1,n-j,s,p,q,r,k-1)\cdot P(M=j)+\sum_{j=n+1}^{\bar{M}} P(m,n,s,p,q,r,k-1)pr(M=j)\Big)+u(j_c,j_s)P(m-1,n,s-1,p,q,r,k-1)$

Step 1 lists the infeasible transitions and sets the transitions to zero. Step 2 summarizes the cases for which the transition probability is trivially equal 1. For example, if the total number of reservations is zero, then the probability the future state is equal to the present state is 1. Step 3 calculates the upgrading probabilities. Step 4 accommodates the NPF assumption when there are fewer seats available than the passengers in a party and the passengers decide not to purchase a sleeper. Step 5 gives the recursive formulation for the general cases. To calculate the transition probabilities, using the historical sales data, we determine the probability of purchasing a sleeper as a function of both sleepers' price and coach seats' price as listed in Table 3.13.

3.4.3 Numerical Results

The best price buckets for automobile accommodations, coach seats, and sleepers have been calculated using Procedure 3.4.1 and Equation 3.4 for the remaining time and also the number of available accommodations. In addition to the relationships, in Section 3.3.3, the following relationships between the optimal price buckets and the number of unsold sleepers are observed.[10]

- For a given number of automobile accommodations, coach seats, and the remaining time,

[10] These statements are based on a limited numerical study, which might not be held in general.

Table 3.13: Probability of purchasing a sleeper.

Coach	Sleeper	Upgrade Probability
1	1	0.247
1	2	0.223
1	3	0.133
1	4	0.055
2	1	0.338
2	2	0.268
2	3	0.178
2	4	0.085
3	1	0.376
3	2	0.218
3	3	0.182
3	4	0.137
4	1	0.283
4	2	0.229
4	3	0.127

the optimal price of coach seats is non-increasing with respect to the number of sleepers.

- For a given number of automobile accommodations, coach seats and the remaining time, the optimal price of sleepers is non-increasing with respect to the number of sleepers.

- For a given number of coach seats, sleepers, and the remaining time, the optimal price of sleepers is non-increasing with respect to the number of automobile accommodations.

- For a given number of automobile accommodations, coach seats, and sleepers, the optimal price of sleepers is non-decreasing with respect to the remaining time.

Having more sleepers increases the chance of accommodating more parties for a given number of automobile accommodations and coach seats. Since the total demand depends on the price of coach seats, reducing the price of coach seats induces the demand and the total expected revenue. Therefore, the optimal price of coach seats is non-increasing with respect to the number of sleepers. The second observation is consistent with the monotonicity results addressed in the literature that the optimal price of an item does not decrease when the number of items increases. In the third observation, increasing the number of unsold automobile accommodations increases the total potential demand for coach seats and sleepers. To reserve the coach seats for parties, with the coach seats as their only preference, the price of sleepers decreases or remains the same in order to induce its demand and accommodate more parties. The last observation is consistent with the regular monotonic property that the optimal price is non-decreasing in the remaining time.

3.5 Conclusions

In this chapter, we have developed a revenue management model for Amtrak, the national railroad passenger corporation. Using the historical sales data, a statistical analysis has been provided in order to estimate the demand. A mathematical model has been introduced in order to maximize the expected revenue over the sales horizon. Finally, using a computer program, the optimal prices have been determined for all accommodations. This numerical study provides a manual for the Amtrak revenue managers to determine the optimal price buckets for each day and for each combination of remaining number of accommodations.

In our revenue management model, we have only calculated the lowest available bucket for a given day. In other words, we have assigned only one bucket to all customers who arrive in a given day. Therefore, we have not fully segmented different costumers, with different willingness to pay for the tickets, into different buckets. As an extension to this model, instead of calculating the lowest available bucket, we can determine the booking limits for each price bucket.

Revenue management is a new field of study in retail management that has been wildly studied in the literature and used in practice. Even though many published and unpublished papers have addressed interesting problems in this area, with different types of solutions, there are many remaining areas of research to pursue. For example, revenue management when customer act strategically and know that their purchases may affect future prices is a new venue of research. In this case, the price pattern that the seller uses is different than the models with myopic buyers.

Another open area of research in revenue management is the pricing of a multi-leg itinerary with different airlines providing different legs of the itinerary. The price level posted by each airline affects the demand for the itinerary. The pricing decisions for such itineraries are determined through a pre-determined codeshare. A cooperative game can be developed to address this pricing problem.

The author of this dissertation will pursue more in revenue management literature and hope to address the above problems.

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