Graphs and Noncommutative Koszul Algebras

Gregory N. Hartman

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Edward Green, Chair
Ezra Brown
Daniel Farkas
Peter Linnell
Charles Parry

April 23, 2002
Blacksburg, Virginia

Keywords: Directed graphs, Koszul algebras, Quivers, Representations
Copyright 2002, Gregory N. Hartman

Graphs and Noncommutative Koszul Algebras

Gregory N. Hartman

(ABSTRACT)

A new connection between combinatorics and noncommutative algebra is established by relating a certain class of directed graphs to noncommutative Koszul algebras. The directed graphs in this class are called full graphs and are defined by a set of criteria on the edges. The structural properties of full graphs are studied as they relate to the edge criteria.

A method is introduced for generating a Koszul algebra Λ from a full graph G. The properties of Λ are examined as they relate to the structure of G, with special attention being given to the construction of a projective resolution of certain semisimple Λ-modules based on the structural properties of G. The characteristics of the Koszul algebra $\hat{\Lambda}$ that is derived from the product of two full graphs G and G' are studied as they relate to the properties of the Koszul algebras Λ and Λ' derived from G and G'.

Acknowledgements

Above all, I thank God my Father for life, joy, peace, and guidance, especially over the past five years. May my life and this thesis bring Him glory. “To the only God our Savior be glory, majesty, power and authority, through Jesus Christ our Lord, before all ages, now and forevermore! Amen.” Jude 25.

To Michelle: I am so thankful of your love and patience as all too often my response to your question of “What are you thinking about?” has been “Math.” I especially appreciate your extra effort in these recent days as I have pushed to finalize my work. I love you.

To my parents, Harvey and Kristine Hartman, and Michelle’s parents, Max and Sandy Bird: Thank you for your support, encouragement, and generosity. You all have been such a blessing, and I thank God for you.

To Dr. Green: Thank you for your guidance and encouragement as I have made my first endeavor into the world of mathematical research. I appreciate the way you have both pushed me to discover and let me peer over your shoulder as you did the same. Your excitement and desire to learn is inspiring.

To Eileen Shugart: Thank you for the many opportunities you have afforded me to develop professionally. I appreciate the advice and guidance that you have given me as a teacher and as an SGTA. Thank you for believing in me enough to see that I received the recognition from others that you thought I deserved and praising me in terms that I am not worthy of. Most of all, I thank you for your friendship.

To my committee: Thank you for your help and instruction during my five years here. Each of you amazes me with your understanding of mathematics.

To my fellow graduate students: Thank you for making life at Tech so much better than I first thought it would be. I have enjoyed my time with you while learning in the classroom, studying in the Commons Room, chatting in the hallways, and playing basketball in the gym. I especially thank Dan and Dustin for being memorable and enjoyable officemates.

Contents

List of Figures

Chapter 1

Introduction

1.1 Background

The main result of this dissertation is the establishment of a new connection between combinatorics and noncommutative algebra. This is done by linking a certain class of directed graphs, called full graphs, to quotients of path algebras that are Koszul algebras.

There already exists several examples of connections between commutative algebra and other mathematical fields. Stanley-Reisner rings arise from a study of simplicial complexes in topology; a special case of this is the study of quotients of polynomial rings derived from undirected graphs, where graph theoretical properties can influence the structure of the associated ring. Toric ideals were derived from the study of toric varieties in algebraic geometry; a very basic property of toric ideals is that they are generated by binomial elements. In both examples, the study of one field helped produce new results in the other related field. We are seeking new results about Koszul algebras by relating them to full graphs.

Koszul algebras were first explicitly defined by Priddy in 1970 [16]. Since then the theory has grown to be rich and full with connections arising with numerous fields, such as algebraic geometry (including the study of toric ideals), Stanley-Reisner rings, and Cohen-Macauley domains [5, 13, 14, 15]. Fröberg [6] provides a useful survey of many of the interesting properties of Koszul algebras along with an extensive bibliography. While worthy of study in and of themselves, Koszul algebras also have important applications to commutative algebra, algebraic topology, Lie theory and quantum groups [2, 3, 12, 16, 17].

Before discussing the results of this dissertation, we will first review some background information. Throughout this dissertation, K will denote a field.

1.2 Graded and Koszul Algebras

A *graded K-algebra* is a family $\{R_i, \phi_{i,j}\}_{i,j=0}^{\infty}$ of K-vector spaces and maps satisfying:

1. R_0 is a K-algebra

2. each R_i is a R_0-R_0-bimodule

3. $\phi_{i,j}\colon R_i \otimes_{R_0} R_j \to R_{i+j}$

4. the following diagram commutes:

$$\begin{array}{ccc} R_i \otimes_{R_0} R_j \otimes_{R_0} R_k & \xrightarrow{1\otimes\phi_{j,k}} & R_i \otimes_{R_0} R_{j+k} \\ {\scriptstyle \phi_{i,j}\otimes 1}\downarrow & & \downarrow{\scriptstyle \phi_{i,j+k}} \\ R_{i+j} \otimes_{R_0} R_k & \xrightarrow{\phi_{i+j,k}} & R_{i+j+k} \end{array}$$

For simplicity of notation, a graded K-algebra $\{R_i, \phi_{i,j}\}$ will be written as $R = \coprod_{i\geq 0} R_i$, and the image of $\phi_{i,j}$ will be written as R_iR_j. Finally, we say that a graded K-algebra R is *generated in degrees 0,1* if, for all $i, j \geq 0$, the maps $\phi_{i,j}$ are surjective.

Graded modules are defined similarly. A *graded left R-module M* is a family of left R_0-modules and maps $\{M_j, \psi_{i,j}\}_{i=0,j=-\infty}^{\infty}$ satisfying:

1. $\psi_{i,j}\colon R_i \otimes_{R_0} M_j \to M_{i+j}$

2. the following diagram commutes:

$$\begin{array}{ccc} R_i \otimes_{R_0} R_j \otimes_{R_0} \otimes M_k & \xrightarrow{1\otimes\psi_{j,k}} & R_i \otimes_{R_0} M_{j+k} \\ {\scriptstyle \phi_{i,j}\otimes 1}\downarrow & & \downarrow{\scriptstyle \psi_{i,j+k}} \\ R_{i+j} \otimes_{R_0} M_k & \xrightarrow{\psi_{i+j,k}} & M_{i+j+k} \end{array}$$

Again, for simplicity of notation, a graded left R-module $M = \{M_i, \psi_{i,j}\}$ will be written as $M = \coprod_{i=-\infty}^{\infty} M_i$, and the image of $\psi_{i,j}$ will be written as R_iM_j. The elements of M_j are said to be *homogeneous* elements of degree j. M is *generated in degree k* if $M = RM_k$. A *graded left submodule* $N \subseteq M$ is a family of left R_0-submodules $N_j \subseteq M_j$ such that $R_iN_j \subseteq N_{i+j}$. Similar definitions hold for graded right submodules.

A *graded left (right) ideal* I of R is a graded left (right) R-submodule of R. A *graded ideal* I of R is both a left and right graded ideal of R. I is a *homogeneous* graded ideal if it can be generated by homogeneous elements. If I is a homogeneous ideal, then the grading on R induces a grading on the quotient ring R/I. We are most interested in *quadratic ideals*, that is, ideals I generated in degree 2 (so $I = RI_2$, which we denote by $I = \langle I_2\rangle$).

A *quadratic algebra* is a graded algebra R where R_0 is semisimple and R is generated in degrees 0,1 by relations of degree 2. To be more specific, let $T = T_{R_0}(R_1) = R_0 \oplus R_1 \oplus (R_1 \otimes_{R_0} R_1) \oplus \cdots = \coprod_{i\geq 0} R_1^{\otimes i}$ be the tensor algebra of the R_0-R_0-bimodule R_1. The canonical map $\pi\colon T \to R$ is surjective with kernel P. Setting $P_2 = P \cap (R_1 \otimes_{R_0} R_1)$, we say that $R \cong T/P$ is quadratic if P is quadratic; i.e., if $P = \langle P_2\rangle$.

Set $L = \coprod_{i\geq 1} R_i$. If R_0 is a semisimple artinian ring, then L is the *graded Jacobson radical* of R, with $R_0 \cong R/L$. Set $\Lambda = R/I$ for some homogeneous ideal $I \subseteq L$, and give Λ the induced grading. Set $J = \coprod_{i\geq 1} \Lambda_i$. If R_0 is semisimple artinian, then Λ_0 will be semisimple artinian as well, since $\Lambda_0 \cong \Lambda/J \cong (R/I)/(L/I) \cong R/L \cong R_0$. Thus J is the graded Jacobson radical of Λ.

In this dissertation we will restrict our study to a special class of graded algebras. A graded K-algebra Λ is *split basic finitely 0,1-generated* if the following hold:

1. $\Lambda_0 \cong \prod_{i=1}^{n} K$
2. Λ_1 has finite K-dimension
3. Λ is generated in degrees 0,1

If Λ is split basic finitely 0,1-generated, and as a graded right Λ-module Λ_0 admits a graded projective resolution

$$\cdots P_2 \to P_1 \to P_0 \to \Lambda_0 \to 0$$

such that each P_i is generated in degree i, then Λ is a *Koszul algebra*. Such a resolution is said to be *linear*. It is known that all Koszul algebras are quadratic algebras [2, 6, 11].

There are numerous equivalent definitions for Λ being Koszul [6]; we will give only one other definition. Given a graded algebra Λ, the Yoneda algebra, $\mathrm{E}(\Lambda)$, is $\coprod_{n\geq 0} \mathrm{Ext}_{\Lambda}^{n}(\Lambda_0,\Lambda_0)$, given a multiplicative structure via the Yoneda product. If Λ is split basic finitely generated, then $\mathrm{E}(\Lambda)$ is also a split basic finitely generated graded K-algebra, where $\mathrm{E}(\Lambda)_i = \mathrm{Ext}_{\Lambda}^{i}(\Lambda_0,\Lambda_0)$. Λ is a Koszul algebra if and only if $\mathrm{E}(\Lambda)$ is generated in degrees 0,1.

1.3 Path Algebras

Let Γ be a finite *quiver* (a finite directed graph). The *path algebra* $K\Gamma$ is the K-algebra with K-basis consisting of all directed paths. Multiplication is defined on the basis by the concatenation of paths when possible; it is zero otherwise. All of the graded algebras that will be referenced here will be quotients of path algebras.

As a brief introduction to the theory of path algebras, we give two basic facts here about these algebras. First, it is easy to see that $K\Gamma$ is finite dimensional if and only if Γ contains no oriented cycles. Also note that the free associative K-algebra on n noncommuting variables is isomorphic to the path algebra $K\Gamma$ where Γ consists of one vertex and n loops. This fact is of special interest in this dissertation. Path algebras are of interest for many reasons [1, 9, 11]; one of these reasons is that the class of algebras that can be represented in the form of $K\Gamma/I$ for some quiver Γ and ideal I includes all finitely generated associative K-algebras [10].

One can give an alternative view to path algebras. Given a path algebra $R{=}K\Gamma$, denote by R_0 the subalgebra generated by the vertices. The vector space M which is generated by the arrows of R is a R_0-R_0-bimodule. Then R is isomorphic to the tensor algebra $T_{R_0}(M) = R_0 \oplus M \oplus (M \otimes_{R_0} M) \oplus (\otimes_{R_0}^{3} M) \oplus \cdots$ [7]. A path algebra $K\Gamma$ can be given a natural grading, called the length grading, where each path is given degree equal to its length. Thus $K\Gamma = \coprod_{i\geq 0}(K\Gamma)_i$, where $(K\Gamma)_i$ is the K-vector space generated by all directed paths in Γ of length i (vertices are given length 0). It is easy to see that $K\Gamma$ is generated in degrees 0,1. If I is a homogeneous ideal of $K\Gamma$, then $K\Gamma/I$ inherits this length grading and is also generated in degrees 0,1. Let L denote the ideal of $K\Gamma$ generated by all the arrows of Γ (so $L = \langle (K\Gamma)_1 \rangle = \sum_{i\geq 1}(K\Gamma)_i$). L is the graded Jacobson radical of $K\Gamma$ with the length grading.

All the algebras studied in this dissertation will be quadratic algebras of the form $\Lambda = K\Gamma/I$. Clearly $\Lambda_0 = (K\Gamma/I)/(L/I)$ is semisimple; more specifically, $\Lambda_0 = \prod_{i=1}^{n} K$ where n is the number of vertices in Γ. $J = L/I$ is the Jacobson radical of Λ .

If I is a homogeneous ideal of $K\Gamma$, then $K\Gamma/I$ is also a split basic, finitely 0,1 generated K-algebra. In fact, we can make a stronger statement. If Λ is any split basic finitely 0,1 generated graded K-algebra then there exists a finite quiver Γ and graded ideal $I \subseteq \sum_{i\geq 2}(K\Gamma)_i$ such that $\Lambda \cong K\Gamma/I$ as graded K-algebras [11]. Because of this description of split basic finitely 0,1 generated graded K-algebras, all algebras in this dissertation will be viewed as quotients of path algebras.

Again, it is known that for $\Lambda = K\Gamma/I$ to be Koszul, I must be generated by quadratic elements. Quadratic generators of I alone are not enough to guarantee that Λ is Koszul; however, the following result is well known [2, 11].

Theorem 1.3.1 *Let Γ be a finite quiver, K be a field, and I an ideal of $K\Gamma$. If I has a quadratic Gröbner basis then $K\Gamma/I$ is Koszul.*

1.4 Overview of Results

The directed graphs studied in this dissertation are called *full graphs*, and since the definition of full graphs is rather technical, we will give only a brief description and save the complete definition for chapter two. In a full graph each edge is assigned a positive integer value and each vertex is given a label consisting of an ordered tuple of letters according to certain set of rules. If n is the largest value assigned to an edge of a full graph G, then we say that G is *n-full*, and the label of each vertex of G will be an $n+1$-tuple. The set of all letters of an n-full graph G will be denoted X_G. Each n-full graph will produce a set $\widetilde{\mathcal{G}}$ whose elements are of the form $((\mathbf{x_i}, \mathbf{x_j}), (\mathbf{x_k}, \mathbf{x_l})) \in X_G^4$ which are ordered pairs of ordered pairs of letters in X_G.

We now define the ring environment we will work in. A quiver Γ is an *associated quiver* to G if the following hold: 1) The elements of X_G can be put into a one to one correspondence with the arrows of Γ (where we will identify $\mathbf{a} \in X_G$ with arrow $\overrightarrow{\mathbf{a}}$ in Γ), 2) if ($\mathbf{a}$,$\mathbf{b}$) is one of the ordered pairs of an element of $\widetilde{\mathcal{G}}$, then the path $\overrightarrow{\mathbf{a}}\overrightarrow{\mathbf{b}}$ exists in Γ, 3) for every element (($\mathbf{a}$,$\mathbf{b}$),($\mathbf{c}$,$\mathbf{d}$)) $\in \widetilde{\mathcal{G}}$, $o(\overrightarrow{\mathbf{a}}) = o(\overrightarrow{\mathbf{c}})$ and $t(\overrightarrow{\mathbf{b}}) = t(\overrightarrow{\mathbf{d}})$ (where o and t refer to the origin and terminus of an arrow, respectively), and 4) for every vertex $\mathbf{v}$ in Γ, there exists an arrow $\overrightarrow{\mathbf{a}}$ such that $o(\overrightarrow{\mathbf{a}}) = \mathbf{v}$ or $t(\overrightarrow{\mathbf{a}}) = \mathbf{v}$.

An *associated triple* (K, G, Γ) is a field K, an n-full graph G, and a quiver Γ associated to G. Given an associated triple (K, G, Γ), we construct the path algebra $K\Gamma$. Next we define a set of elements of $K\Gamma$, $\mathcal{G} = \{\overrightarrow{\mathbf{a}}\,\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{c}}\,\overrightarrow{\mathbf{d}}\ |((\mathbf{a},\mathbf{b}),(\mathbf{c},\mathbf{d}))\in \widetilde{\mathcal{G}}\}$. In future references we will supress the superscript arrows and simply write the elements of $\mathcal{G}$ in the form **ab-cd**. Set $I = \langle \mathcal{G} \rangle$, and define I to be the *associated ideal* to (K, G, Γ).

Given an associated triple (K, G, Γ), all of our work will be done in the context of the K-algebra $\Lambda = K\Gamma/I$, where I is formed as above. We call Λ the *associated algebra* to (K, G, Γ). By definition I will be generated by quadratic, binomial elements; viewing $K\Gamma$ as a tensor algebra we see that Λ is a quadratic algebra.

Given a field K and an n-full graph G where $|X_G| = m$, the quiver consisting of one vertex and m loops will be an associated quiver to G. Thus, one choice for Λ could be a quotient of the free associative K-algebra on m noncommuting variables. In general, other choices for Γ exist and, in general, there are nonisomorphic K-algebras Λ associated to a given full graph G. However, most of the work done in this dissertation is independant of the choice of associated quiver, and therefore readers unfamiliar with path algebras can usually think in terms of quotients of free algebras.

The first goal of this dissertation is to relate full graphs to Koszul algebras; i.e., to show that given an associated triple (K, G, Γ), the associated algebra Λ is a Koszul algebra. Then we will seek connections between the combinatorial properties of G and the algebraic properties of Λ. We use these connections in the construction of a minimal projective resolution of Λ_0. In light of this motivation, we will prove the following results:

Theorem 1.4.1 *If (K, G, Γ) is an associated triple, then the associated ideal I has a quadratic Gröbner basis, and hence the associated algebra Λ is a Koszul algebra.*

Theorem 1.4.2 *Let an associated triple (K, G, Γ) be given with associated algebra Λ. If G is n-full, then gl.dim(Λ) $\leq n + 1$. Furthermore, this bound is sharp for each n.*

Theorem 1.4.3 *Let an associated triple (K, G, Γ) be given with associated algebra Λ. Then in a minimal projective resolution of Λ_0*

$$\cdots \to P_{n+1} \to \cdots \to P_2 \to P_1 rightarrow P_0 \to \Lambda_0 \to 0$$

with maps $d^i \colon P^i \to P^{i-1}$, the maps d^1, d^2, and d^3 can be determined directly from the construction of G.

Other main results include methods of constructing full graphs. One method will show how to create an $(n + 1)$-full graph from an n-full graph. Another construction we will describe will create an $(n + m)$-full graph from a pair of full graphs which are n-full and m-full.

We will study the K-algebras associated to graphs derived from the above constructions. We show that if G is an n-full graph and G' is the $(n+1)$-full graph produced by the first method above, then we can construct a minimal projective resolution of Λ'_0 from a minimal projective resolution of Λ_0 (where Λ' and Λ are associated algebras to some associated triples (K, G', Γ') and (K, G, Γ), respectively). A similiar result will be shown for $(n+m)$-full graphs produced by the second method described above.

Finally, given a full graph G we will define an associated quiver Γ_C called the *canonical quiver*. If G is n-full and G' is $(n+1)$-full as described above, let Λ and Λ' be the associated algebras to the associated triples (K, G, Γ_C) and (K, G', Γ'_C). We will show how the combinatorial method of creating the new full graph G' from G can be interpreted as a ring construction creating Λ' from Λ involving one-point extensions.

Chapter 2

Full Graphs

A full graph is a directed graph whose edges are assigned a positive integer value and whose vertices are given a label consisting of an ordered tuple of letters, all according to a certain set of conditions. Before giving the conditions, we will establish some notation.

Let G be a directed graph; we will use $\overline{G}$ to denote the underlying undirected graph. $V(G)$ and $E(G)$ will denote the vertex and edge sets of G, respectively; similar notation is used for $\overline{G}$. The elements of $V(G)$ will be denoted by $\mathbf{v}$. A directed edge in $E(G)$ going from vertex $\mathbf{v}$ to vertex $\mathbf{w}$ will be denoted $\overrightarrow{e_{\mathbf{vw}}}$; an edge between $\mathbf{v}$ and $\mathbf{w}$ in $\overline{G}$ will be denoted $\overline{e_{\mathbf{vw}}}$. A directed path in G from vertex $\mathbf{v}$ to vertex $\mathbf{w}$ will be denoted $\overrightarrow{p_{\mathbf{vw}}}$; likewise, a path in $\overline{G}$ from $\mathbf{v}$ to $\mathbf{w}$ will be denoted $\overline{p_{\mathbf{vw}}}$ (when e is not used for an edge or p is not used for a path, the context will make clear whether a path or an edge is being referred to).

We will now provide the conditions that lead to the definition of a full graph.

2.1 Numbering Conditions

Let G be a directed graph, and assign to each edge a positive integer, complying with the following rules:

1. G cannot contain either of the subgraphs in Figure 2.1.

2. If G contains a vertex $\mathbf{u}$ and edges $\overrightarrow{e_{\mathbf{uv}}}$ and $\overrightarrow{e_{\mathbf{uw}}}$ numbered i, j respectively, with $|i - j| = 1$, then there exists a vertex $\mathbf{x}$ and paths $\overrightarrow{p_{\mathbf{ux}}}$ and $\overrightarrow{q_{\mathbf{ux}}}$ with the following properties:

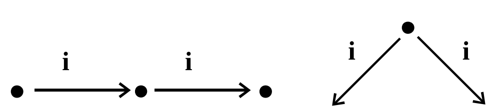


Figure 2.1: Forbidden subgraphs

(a) The first edge of $\overrightarrow{p_{\mathbf{ux}}}$ is $\overrightarrow{e_{\mathbf{uv}}}$; the first edge of $\overrightarrow{q_{\mathbf{ux}}}$ is $\overrightarrow{e_{\mathbf{uw}}}$.

(b) The edges of both paths are numbered with alternating i's and j's.

(c) The only vertices in both paths are $\mathbf{u}$ and $\mathbf{x}$.

The paths $\overrightarrow{p_{\mathbf{ux}}}$ and $\overrightarrow{q_{\mathbf{ux}}}$ form a cycle in $\overline{G}$; this cycle is called an *i,j cycle*; $\mathbf{x}$ is the *sink of the cycle.*

A sample i, j cycle is found in Figure 2.2:

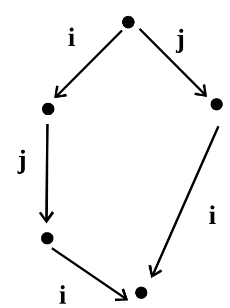


Figure 2.2: Sample i, j cycle

3. If G contains either of the subgraphs in Figure 2.3(a) or (b) where $|i - j| \geq 2$, then G also contains the graph shown in (c).

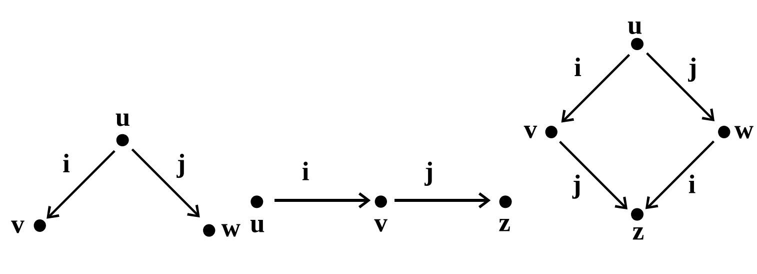


Figure 2.3: Establishing condition 3

4. Given any integer i assigned to an edge, $\overline{G}$ cannot contain any simple cycles C where all edges numbered i are oriented in the same direction when viewing C in G.

5. If G contains an edge $\overrightarrow{e_{\mathbf{uv}}}$ numbered i and an edge $\overrightarrow{e_{\mathbf{u'v'}}}$ numbered $i-1$ and a path $\overline{p_{\mathbf{uu'}}}$ such that no edge of $\overline{p_{\mathbf{uu'}}}$ is numbered $i-1, i$, then there exists a vertex $\mathbf{x}$ and

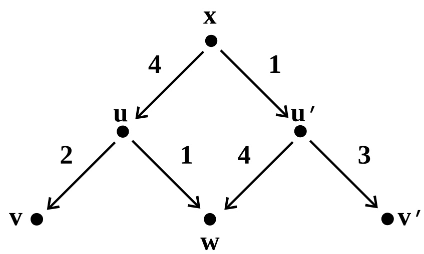


Figure 2.4: Establishing condition 5

undirected paths $\overline{p_{\mathbf{ux}}}$ and $\overline{p_{\mathbf{u'x}}}$ such that no edge in $\overline{p_{\mathbf{ux}}}$ is numbered $i-1, i$, or $i+1$ and no edge in $\overline{p_{\mathbf{u'x}}}$ is numbered $i-2, i-1$, or i.

Consider the example in Figure 2.4. The path **u w u′** does not contain any edges numbered 2 or 3. Vertex **x** exists and the path $\overline{p_{\mathbf{ux}}}$ (which is just the edge $\overrightarrow{e_{\mathbf{xu}}}$) does not have any edges numbered 1, 2, or 3; the path $\overline{p_{\mathbf{u'x}}}$ (which is the edge $\overrightarrow{e_{\mathbf{xu'}}}$) does not have any edges numbered 2, 3, or 4.

6. If there exists edges $\overrightarrow{e_{\mathbf{uv}}}$ and $\overrightarrow{e_{\mathbf{uv'}}}$ numbered i and undirected paths $\overline{p_{\mathbf{uu'}}}$ and $\overline{q_{\mathbf{uu'}}}$ such that:

   (a) $\overline{p_{\mathbf{uu'}}}$ does not contain any edges numbered $i-1$ or i

   (b) $\overline{q_{\mathbf{uu'}}}$ does not contain any edges numbered i or $i+1$

   then there exists an undirected path $\overline{r_{\mathbf{uu'}}}$ that does not contain any edges numbered $i-1$,i, or $i+1$.

Definition 2.1.1 *A directed graph with each edge assigned a number is said to be* numbered. *A numbered graph whose numbering adheres to the above rules is said to be* well numbered.

We say that numbered graphs are isomorphic if they are isomorphic as directed graphs and isomorphic edges have the same edge numbers. Unless specified otherwise, we will assume that all well numbered graphs have an edge numbered 1 in them; thus when we refer to a well numbered graph "whose largest edge number is n," we are referring to a graph with some edges numbered 1, some edges numbered n, and all other edges numbered with integers between 1 and n. A well numbered graph with edges numbered only 1, 3 and 4 could be described as a graph whose "largest edge number is 4." If G is a well numbered graph with smallest edge number n_1, largest edge number n_2, and for each $n_1 \leq j \leq n_2$, there is an edge in G numbered j, then we say that G has *consecutive edge numbers.*

Given a well numbered graph G whose largest edge number is n, we define $G[i]$ to be the well numbered graph whose edges are numbered $i+1 \cdots n+i$, where each edge number of

G is increased by some positive integer i. We can extend this notation to include negative integers for [i]; one must be careful so that the lowest edge number does not drop below 1. If G is a well numbered graph with edge numbers ranging from 3 to 7, G[-2] is well defined and is a well numbered graph; G[-3] is not well numbered. By default $G = G[0]$.

We make another comment about terminology. Let G be a well numbered graph, and let $\overline{p_{\mathbf{uv}}}$ be a path in G . If there exists an edge $\overrightarrow{e_{\mathbf{vw}}}$ numbered j where for all edge numbers i in $\overline{p_{\mathbf{uv}}}$, $|i-j| \geq 2$, then for every vertex $\mathbf{x}$ in $\overline{p_{\mathbf{uv}}}$, there exists an edge numbered j coming out of $\mathbf{x}$, as a result of condition # 3. We say such a path $\overline{p_{\mathbf{uv}}}$ is "j-moveable," and that the edge $\overrightarrow{e_{\mathbf{vw}}}$ "moves across" $\overline{p_{\mathbf{uv}}}$. The full importance of j-moveable paths will be seen in Chapter 3.

We end this section with a result about the structure of well numbered graphs that follows from our six conditions.

Lemma 2.1.1 *Let G be a connected well numbered graph. G has a source and a unique sink.*

Proof:

Let G be a connected well numbered graph. G contains no directed cycles, for having a directed cycle would violate condition #4. Since G is a finite directed graph with no directed cycles, it follows from elementary graph theory that G has at least one source and at least one sink.

Now assume that G has multiple sinks $\mathbf{s}_k$. We define a partial order $\leq_p$ on the vertices of G where $\mathbf{u} \leq_p \mathbf{v}$ if and only if there exists a directed path from $\mathbf{v}$ to $\mathbf{u}$.

Note that since G is connected, there exists at least one source from which directed paths lead to different sinks. Let $\mathbf{x}$ be a vertex of G minimal with respect to all vertices that have directed paths coming from them going into different sinks. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be sinks such that there exists directed paths $\overrightarrow{p_{\mathbf{xs_1}}}$ and $\overrightarrow{p_{\mathbf{xs_2}}}$, where the first edge of $\overrightarrow{p_{\mathbf{xs_1}}}$ is $\overrightarrow{e_{\mathbf{xv_1}}}$ and the first edge of $\overrightarrow{p_{\mathbf{xs_2}}}$ is $\overrightarrow{e_{\mathbf{xv_2}}}$, numbered i, j respectively.

By construction, the edges $\overrightarrow{e_{\mathbf{xv_1}}}$ and $\overrightarrow{e_{\mathbf{xv_2}}}$ begin an i, j cycle with sink $\mathbf{u}$. There exists a directed path from $\mathbf{u}$ to some sink $\mathbf{s}_a$ of G; if $\mathbf{s}_a \neq \mathbf{s}_1$ or $\mathbf{s}_a \neq \mathbf{s}_2$, then $\mathbf{v_1}$ or $\mathbf{v_2}$ (respectively) is a vertex which is $\leq_p \mathbf{x}$ with directed paths going to two different sinks, contradicting the minimality of $\mathbf{x}$.

Thus G has a unique sink. □

2.2 Constructions of Well Numbered Graphs

This section will describe methods for making new well numbered graphs from old ones.

Let G be any well numbered graph. Set $G' = G$ as numbered graphs and let $m \in \mathbb{Z}^+$ be given. We can make a new numbered graph H by connecting G to G' via edges $\overrightarrow{e_{\mathbf{v}_i\mathbf{v'}_i}}$ for all $\mathbf{v}_i \in V(G)$ (where we identify vertex $\mathbf{v}_i$ with its isomorphic vertex $\mathbf{v'}_i$ in $V(G')$), with each new edge receiving the number m.

Definition 2.2.1 *Let H be a graph formed from a well numbered graph G as described above. H is said to be the* complete extension of G by m. *If the largest edge number of G is n and $m = n + 1$, we simply say that H is the* complete extension of G. *The edges that connect G to G' are called the* connecting edges.

We now give a generalized version of the above construction. Given two directed graphs G and H, the *(Cartesian) product* of G and H, $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ with edges $\overrightarrow{e_{(\mathbf{v},\mathbf{u}),(\mathbf{v}',\mathbf{u}')}}$ if and only if 1) $\mathbf{v} = \mathbf{v}'$ and $\overrightarrow{e_{\mathbf{u}\mathbf{u}'}} \in E(H)$ or 2) $\mathbf{u} = \mathbf{u}'$ and $\overrightarrow{e_{\mathbf{v}\mathbf{v}'}} \in E(G)$. We extend this definition to numbered graphs.

Definition 2.2.2 *If G' and G'' are numbered graphs, the* numbered product *of G' and G'', $G' \Box_N G''$, is the graph $G' \Box G''$ with edges $\overrightarrow{e_{(\mathbf{v},\mathbf{u})(\mathbf{v}',\mathbf{u})}}$ numbered j where the edges $\overrightarrow{e_{\mathbf{v}\mathbf{v}'}}$ are numbered j in G', and edges $\overrightarrow{e_{(\mathbf{v},\mathbf{u})(\mathbf{v},\mathbf{u}')}}$ numbered i where the edges $\overrightarrow{e_{\mathbf{u}\mathbf{u}'}}$ are numbered i in G''.*

The vertices $(\mathbf{v}_i, \mathbf{u}_j)$ of $G' \Box_N G''$ are denoted $\mathbf{v}_{i,j}$. By the construction of the product, for each vertex $\mathbf{u}_j$ of G'' there exists an isomorphic copy of G' which is denoted G'_j; we similarly define G''_i.

In general, if G' and G'' are well numbered graphs, $G' \Box_N G''$ will not be well numbered. Indeed, if G' and G'' both have edges numbered 1, then $G' \Box_N G''$ will have a vertex with two edges numbered 1 coming out, violating condition # 1.

One way to avoid this problem is to shift the numbering of one of the graphs. If G' and G'' are well numbered with largest edge numbers n and m respectively, then shifting G'' to $G''[n]$ will give a graph with edge numbers in $\{n+1, \cdots, n+m\}$. Thus $G' \Box_N G''[n]$ (where the $[\,\cdot\,]$ is performed first, then $\cdot \Box_N \cdot$) will not have any vertices with two edges coming out that are numbered the same. In fact, $G' \Box_N G''[n]$ is well numbered; this will be proved in Theorem 2.2.1. Because shifting the numbering of one graph produces a nice result in terms of preserving well numberedness, we give this construction a special name.

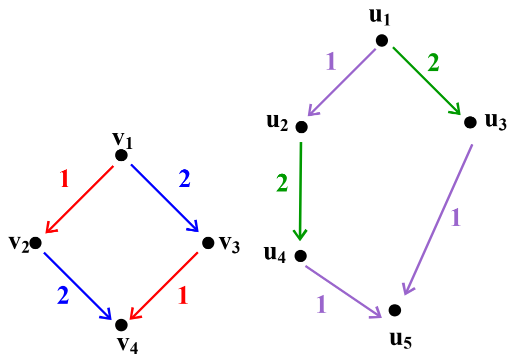
Definition 2.2.3 *Let G', G'' be numbered graphs where the largest edge number of G' is n, and let $G = G' \Box_N G''[n]$. G is the* complete extension of G' by G'', *which is denoted $G' \Box_C G''$.*

Again, an example may be useful.

Example 2.2.1 We will let G' and G'' be the well numbered graphs in Figure 2.5.

Setting $G = G' \Box_C G''$, we get the graph in Figure 2.6. We label only a few vertices and edges to avoid unnecessary clutter. □

Note: the construction in Definition 2.2.1 is just a special case of Definition 2.2.2. The complete extension of G by m is the same graph as $G \Box_N G''$ where G'' is the graph consisting of 2 vertices connected by an edge numbered m; the complete extension of G is the same graph as $G \Box_C G''$ where G'' is the graph with only 2 vertices connected by one edge which is numbered 1.



(a) G' (b) G''

Figure 2.5: Well numbered graphs G and G' in Example 2.2.1

As mentioned before, the numbered graph operation $[\,\cdot\,]$ is performed before the operation $\cdot\square_N\cdot$. Likewise, $[\,\cdot\,]$ is performed before $\cdot\square_C\cdot$. Thus $G'\square_C G''[i] = G'\square_N G''[n+i]$

A few more examples may be helpful. Let G' be a numbered graph with edges numbered from 1 to 3; let G'' be numbered with edge numbers from 1 to 5. $G = G'\square_C G''$ has edges numbered from 1 to 8; $G'\square_C G''[2]$ will have largest edge number 10 and no edges numbered 4 or 5. $G'[2]\square_C G''$ will have largest edge number 10, but no edges numbered 1 or 2 (copies of G'' will have edges numbered from 6 to 10). This is the same as $(G'\square_C G'')[2] = G'[2]\square_N G''[5]$. A final example makes use of negative integers within the [ ]; $G'[5]\square_C G''[-8]$ gives a graph wherein the copies of G' have edge numbers 6 through 8; the copies of G'' will have edge numbers from 1 to 5. In light of the structure of the product, we see that $G'[5]\square_C G''[-8]$ is isomorphic to $G''\square_C G'$ as numbered graphs.

The product of directed graphs is "commutative" and "associative"; $G\square H \cong H\square G$ and $(G\square H)\square I \cong G\square(H\square I)$. This follows directly from the definition of the product and that $A \times B \cong B \times A$ for sets A and B. We will usually refer to the graphs $G\square H$ and $H\square G$ as being the same, not just isomorphic. The numbered product is also "commutative" and "associative." While the complete extension is associative, it is not commutative. Given numbered graphs G' and G'' with largest edge numbers n and m, $G'\square_C G'' = G'\square_N G''[n] \neq G''\square_N G'[m] = G''\square_C G'$.

The following concept relating to these graphs will be useful in the subsequent proofs. Let $G = G'\square_N G''$ and let a path $p = \overrightarrow{p_{\mathbf{v}_{i,j}\mathbf{v}_{k,l}}}$ be given between two vertices $\mathbf{v}_{i,j}$ and $\mathbf{v}_{k,l}$ in G. We can "project" p into G'_s (or G''_t) by sending the edges in p that occur in any copy of G' to the corresponding edges in G'_s. The result will be a directed walk w in G'_s from $\mathbf{v}_{i,s}$ to $\mathbf{v}_{k,s}$; by elementary graph theory this contains a (not necessarily unique) directed path $p' = \overrightarrow{p_{\mathbf{v}_{i,s}\mathbf{v}_{k,s}}}$. We define a projection of p into G'_s to be any path $p' = \overrightarrow{p_{\mathbf{v}_{i,s}\mathbf{v}_{k,s}}}$ contained in the walk w.

Using Example 2.2.1, consider the path $p = \mathbf{v}_{1,1}\mathbf{v}_{3,1}\mathbf{v}_{3,3}\mathbf{v}_{3,5}\mathbf{v}_{1,5}\mathbf{v}_{2,5}$. Projecting onto

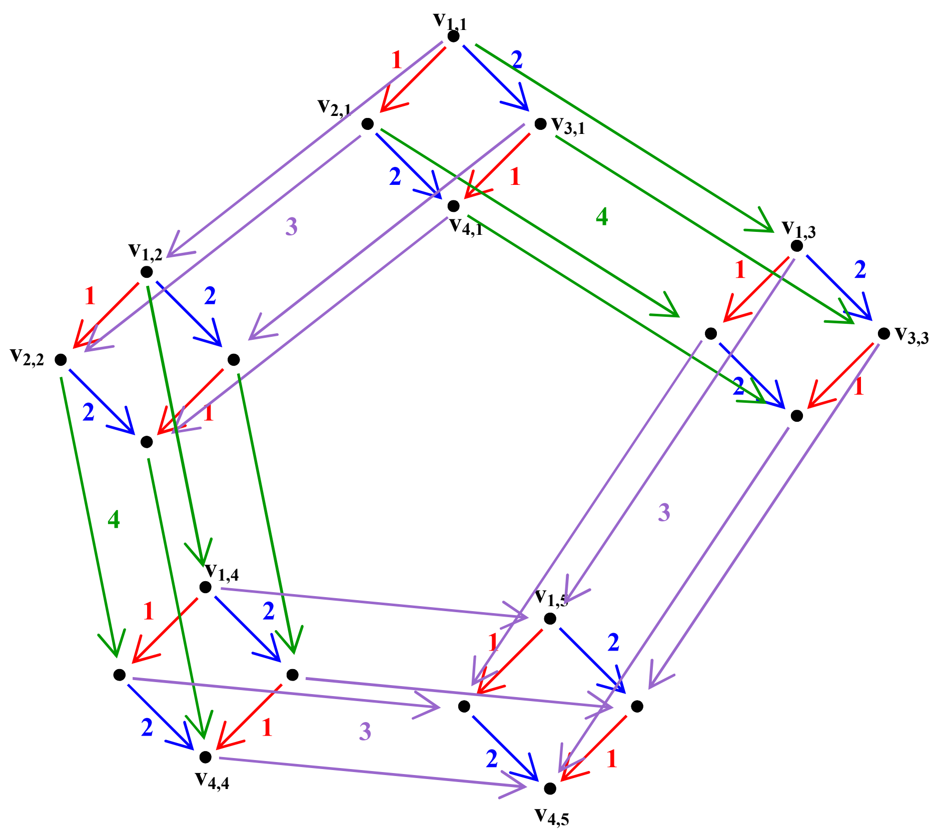


Figure 2.6: $G = G' \square_C G''$

G'_2, we initially get the directed walk $\mathbf{v}_{1,2}\mathbf{v}_{3,2}\mathbf{v}_{1,2}\mathbf{v}_{2,2}$, which then yields the directed path $\mathbf{v}_{1,2}\mathbf{v}_{2,2}$. Projecting into G''_2, we get the walk $\mathbf{v}_{2,1}\mathbf{v}_{2,1}\mathbf{v}_{2,3}\mathbf{v}_{2,5}\mathbf{v}_{2,5}\mathbf{v}_{2,5}$, which leads to the path $\mathbf{v}_{2,1}\mathbf{v}_{2,3}\mathbf{v}_{2,5}$.

We now prove a theorem that shows that if G' and G'' are well numbered, then the complete extension of G' by G'' is well numbered.

Theorem 2.2.1 *Let G' be a numbered graph with largest edge number n_1, and let G'' be numbered with smallest edge number n_2, $n_1 < n_2$. Set $G = G' \square_N G''$. G is well numbered if and only if G' and G'' are well numbered.*

Proof:
($\Leftarrow$) Assume G' and G'' are well numbered. We will verify that G satisfies the 6 conditions of being well numbered.

1. This follows directly from the construction of G and the fact that G' and G'' are well numbered and have no edge numbers in common.

2. We use the terminology initially used in defining well numbered. If $\overrightarrow{e_{\mathbf{uv}}}$ and $\overrightarrow{e_{\mathbf{uw}}}$ are both in a copy of G' or G'', then the condition is satisfied since G' and G'' are well numbered. Now consider when $\overrightarrow{e_{\mathbf{uv}}}$ is in a copy of G' and $\overrightarrow{e_{\mathbf{uw}}}$ is in a copy of G''. We then really have edges $\overrightarrow{e_{\mathbf{v}_{i,j}\mathbf{v}_{k,j}}}$ and $\overrightarrow{e_{\mathbf{v}_{i,j}\mathbf{v}_{i,l}}}$. By construction, there exists edges $\overrightarrow{e_{\mathbf{v}_{i,l}\mathbf{v}_{k,l}}}$ and $\overrightarrow{e_{\mathbf{v}_{k,j}\mathbf{v}_{k,l}}}$ in G, giving us a cycle and satisfying the condition.

3. This condition is satisfied in a manner similar to how condition #2 is satisfied.

4. Let a cycle C be given in $\overline{G}$. Again, if C is completely contained in a copy of G' or G'' then the condition is satisfied. Now assume that C is not completely contained in a copy of G' or G'', and there exists an edge number t such that all edges numbered t in C are oriented in the same direction in G. Without loss of generality, assume $1 \leq t \leq n$. Projecting C onto any copy of G' will give a cycle C' with all edges numbered t oriented in the same direction, contradicting the well numberedness of G'.

5. We again use the terminology used to define well numbered. Assume we have edges $\overrightarrow{e_{\mathbf{uv}}}$ and $\overrightarrow{e_{\mathbf{u'v'}}}$ numbered $i, i-1$ respectively and path $\overline{p_{\mathbf{uu'}}}$ with no edges numbered $i-1, i$. We consider two cases:

   (a) $i \leq n_1$ (a similar result holds for $i - 1 \geq n_2$): Thus we have edges $\overrightarrow{e_{\mathbf{v}_{j,k}\mathbf{v}_{l,k}}}$ and $\overrightarrow{e_{\mathbf{v}_{s,q}\mathbf{v}_{r,q}}}$ numbered $i, i-1$ respectively. We then have an edge $\overrightarrow{e_{\mathbf{v}_{s,k}\mathbf{v}_{r,k}}}$ in G'_k numbered $i-1$. Since G' is well numbered, there exists a vertex $\mathbf{v}_{x,k}$ and paths $\overline{p_{\mathbf{v}_{j,k}\mathbf{v}_{x,k}}}$, $\overline{p_{\mathbf{v}_{s,k}\mathbf{v}_{x,k}}}$ with no edges numbered $i-1, i, i+1$ and $i-2, i-1, i$, respectively. Project the second path into G'_q to obtain a path $\overline{p_{\mathbf{v}_{s,q}\mathbf{v}_{x,q}}}$ that does not contain any edges numbered $i-2, i-1, i$. Extend this path with edges in G''_x to the vertex $\mathbf{v}_{x,k}$; none of these edges are numbered $i-2, i-1, i$. Thus we have a vertex $\mathbf{x}=\mathbf{v}_{x,k}$ and paths $\overline{p_{\mathbf{v}_{j,k}\mathbf{v}_{x,k}}}$, $\overline{p_{\mathbf{v}_{s,q}\mathbf{v}_{x,k}}}$ that do not contain edges numbered $i-1, i, i+1$ and $i-2, i-1, i$, respectively, satisfying the condition.

   (b) $i - 1 = n_1, i = n_2$ (Hence this case only applies when $n_2 = n_1 + 1$): Thus we have an edges $\overrightarrow{e_{\mathbf{v}_{j,k}\mathbf{v}_{l,k}}}$ and $\overrightarrow{e_{\mathbf{v}_{s,r}\mathbf{v}_{s,q}}}$, and a path $\overline{p_{\mathbf{v}_{j,k}\mathbf{v}_{s,r}}}$ with no edges numbered $i-1, i$. Project this path into both G''_j and G'_r; this gives paths $\overline{p_{\mathbf{v}_{j,k}\mathbf{v}_{j,r}}}$ and $\overline{p_{\mathbf{v}_{j,r}\mathbf{v}_{s,r}}}$, respectively. We seek a vertex $\mathbf{x}$ and paths p, q from $\mathbf{v}_{j,k}$ and $\mathbf{v}_{s,r}$ to $\mathbf{x}$ with no edges numbered $i-2, i-1, i$ and $i-1, i, i+1$, respectively. We let $\mathbf{x} = \mathbf{v}_{j,r}$; set $p = \overline{p_{\mathbf{v}_{j,k}\mathbf{v}_{j,r}}}$ and $q = \overline{p_{\mathbf{v}_{s,r}\mathbf{v}_{j,r}}}$. p has no edges numbered $i-2$ since it is contained in a copy of G''; it also has no edges numbered $i-1, i$ since it is the projection of a path with no such edges. A similar statement is true for q. Thus the condition is satisfied.

6. This proof is similiar to that of #5. It is easily satisified if both edges are in the same copy of G' (or G''); so then consider when the edges are in different copies of G' (a similar proof holds for different copies of G'').

Thus we have edges $\overrightarrow{e_{\mathbf{v}_{j,k}\mathbf{v}_{l,k}}}$ and $\overrightarrow{e_{\mathbf{v}_{q,s}\mathbf{v}_{t,s}}}$ numbered i and paths $p = \overline{p_{\mathbf{v}_{j,k}\mathbf{v}_{q,s}}}$ with no edges numbered $i-1, i$ and $q = \overline{p_{\mathbf{v}_{j,k}\mathbf{v}_{q,s}}}$ with no edges numbered $i, i+1$. Project both paths into G'_k giving paths $p' = \overline{p_{\mathbf{v}_{j,k}\mathbf{v}_{q,k}}}$ and $q' = \overline{p_{\mathbf{v}_{j,k}\mathbf{v}_{q,k}}}$; since G' is well numbered there exists a path $r' = \overline{p_{\mathbf{v}_{j,k}\mathbf{v}_{q,k}}}$ in G'_k with no edges numbered $i-1, i, i+1$.

Project the paths p and q also into G''_q to get paths $p'' = \overline{p_{\mathbf{v}_{q,k}\mathbf{v}_{q,s}}}$ and $q'' = \overline{p_{\mathbf{v}_{q,k}\mathbf{v}_{q,s}}}$. Since G'' is well numbered, there exists a path $r'' = \overline{p_{\mathbf{v}_{q,k}\mathbf{v}_{q,s}}}$ in G''_q with no edges numbered $i-1, i, i+1$.

Set the path r to be the contatenation of the paths r' and r''; r is a path from $\mathbf{v}_{j,k}$ to $\mathbf{v}_{q,s}$ with no edges numbered $i-1, i, i+1$, satisfying the condition.

Since G satisfies the conditions, it is well numbered.

($\Rightarrow$) Now assume that G is well numbered. We wish to show that G' and G'' are well numbered. Again, we verify that G' and G'' satisfy the 6 conditions.

It is easy to see that if G satisfies conditions 1 through 4, then G' and G'' also satisfy these conditions. We really need only check conditions 5 and 6. Without loss of generality we consider only G'; for ease of notation, we will show that a particular copy of G' in G, namely G'_1, is well numbered.

5. Let edges $\overrightarrow{e_{\mathbf{v}_{k,1}\mathbf{v}_{l,1}}}$ and $\overrightarrow{e_{\mathbf{v}_{s,1}\mathbf{v}_{t,1}}}$ be given in G'_1 that are numbered $i-1$ and i respectively, as well as a path $\overline{p_{\mathbf{v}_{k,1}\mathbf{v}_{s,1}}}$ in G'_1 that does not contain any edges numbered $i-1$ nor i. Since G is well numbered, there exists a vertex $\mathbf{x}=\mathbf{v}_{x,y}$ in G and paths $\overline{p_{\mathbf{v}_{x,y}\mathbf{v}_{k,1}}}$ and $\overline{p_{\mathbf{v}_{x,y}\mathbf{v}_{s,1}}}$ that do not have any edges numbered $i-2, i-1, i$ and $i-1, -, i+1$, respectively. Project each path into G'_1; this gives paths $\overline{p_{\mathbf{v}_{x,1}\mathbf{v}_{k,1}}}$ and $\overline{p_{\mathbf{v}_{x,1}\mathbf{v}_{s,1}}}$ in G'_1 that do not contain any edges numbered $i-2, i-1, i$ and $i-1, i, i+1$, respectively. Thus the condition is satisfied.

6. The method of proving that G'_1 satisfies this condition is similar to that in 5. Let edges $\overrightarrow{e_{\mathbf{v}_{k,1}\mathbf{v}_{l,1}}}$ and $\overrightarrow{e_{\mathbf{v}_{s,1}\mathbf{v}_{t,1}}}$ be given in G'_1 that are numbered i, as well as paths $\overline{p_{\mathbf{v}_{k,1}\mathbf{v}_{s,1}}}$ and $\overline{q_{\mathbf{v}_{k,1}\mathbf{v}_{s,1}}}$ in G'_1 that do not contain edges numbered $i-1, i$ and $i, i+1$, respectively. Since G is well numbered, there exists a path $\overline{r_{\mathbf{v}_{k,1}\mathbf{v}_{s,1}}}$ in G that does not have any edges numbered $i-1, i$, nor $i+1$. Project this path into G'_1; this projected path satisfies the condition.

Since G'_1 satisfies the conditions, it is well numbered, and hence G' is well numbered. G'' is well numbered by similar arguments. $\Box$

Corollary 2.2.1 *Let G' and G'' be well numbered graphs. $G' \Box_C G''[i]$ is well numbered for all nonnegative integers i.*

Proof:

$G' \Box_C G''[i] = G' \Box_N G''[n+i]$ where n is the largest edge number of G'. Thus the smallest edge number of $G''[n+i]$ is greater than n. Apply the theorem. $\Box$

Another result follows. Let G' be a well numbered graph with smallest edge number n_1 and largest edge number n_2, and let G be the complete extension of G' by m, for some integer m. If $m \leq n_1 - 1$ or $m \geq n_2 + 1$, then G is well numbered by the theorem.

Now consider when G' does not have any edges numbered i for some $n_1 < i < n_2$ and G is the complete extension of G' by i. We cannot use Theorem 2.2.1 to prove that G is well numbered.

More generally, suppose there is another well numbered graph G'' and a set $I \subset \{n_1, n_1 + 1, \cdots, n_2\}$ such that no element of I is an edge number of G' and all edge numbers of G'' are elements of I. Is $G' \Box_N G''$ well numbered? Again, we cannot use Theorem 2.2.1, but by restricting G' and G'' to having a single source we can say something about its structure and then invoke this theorem. (We will show in Lemma 2.1.1 that well numbered graphs have at least one source; restricting to a single source is not unreasonable.)

Lemma 2.2.1 *Let G be a well numbered graph with a single source. If G has edge numbers in $\{n_1, \cdots, i-1, i+1, \cdots, n_2\}$ but no edges numbered i for some positive integers $n_1 < i < n_2$, then $G \cong G' \Box_N G''$ for some well numbered graphs G' and G''.*

Proof:

Let G be a numbered graph that satisfies the hypothesis, and let G_1 be the subgraph of G with all edges numbered $i+1 \cdots n_2$ removed. We claim that G_1 is not connected, and that all components are isomorphic to each other as numbered graphs.

We first show that G_1 is not connected. Assume that it is. Let $\overrightarrow{e_{\mathbf{uv}}}$ be any edge in G with an edge number j in $\{i+1, \cdots, n_2\}$. Since G_1 is connected, there exists a path $\overline{p_{\mathbf{uv}}}$ in G_1 wherein all edges have edge numbers in $\{n_1, \cdots, i-1\}$. This gives in G a cycle with only one edge numbered j, contradicting the well numberedness of G.

Thus G is not connected. Let $G'_{\mathbf{s}}$ denote the component of G_1 that contains the source, $\mathbf{s}$. We show that $\mathbf{s}$ has at least one edge coming from it in G numbered in $\{n_1 \cdots i-1\}$. Let $\overrightarrow{e_{\mathbf{uv}}}$ be any edge in G with edge number $j \leq i-1$ where $\mathbf{u}$ can be reached with a directed path $\overrightarrow{p_{\mathbf{su}}}$ with all edges in the path numbered $\geq i+1$. This path is then j-moveable, and hence we get an edge from $\mathbf{s}$ numbered $j \leq i-1$.

Let G'_x be a component of G_1 such that there exists a path $\overrightarrow{p_{\mathbf{uv}}}$ in G where $\mathbf{u}$ is a vertex of $G'_{\mathbf{s}}$ and $\mathbf{v}$ is a vertex of G'_x and all edges in $\overrightarrow{p_{\mathbf{uv}}}$ are numbered $\geq i+1$. We claim that $G'_x \cong G_1$.

Without loss of generality, let $\overrightarrow{e_{\mathbf{uu'}}}$ be an edge in $G'_{\mathbf{s}}$ numbered j. The path $\overrightarrow{p_{\mathbf{uv}}}$ is j-moveable, so there exists an edge $\overrightarrow{e_{\mathbf{vv'}}}$ in G'_x numbered j. By the properties of well numbered graphs, there also exists the path $\overrightarrow{p_{\mathbf{u'v'}}}$ in G with the same edge numbers, in the same order, as $\overrightarrow{p_{\mathbf{uv}}}$. Likewise, any edge coming into or out of vertices $\mathbf{u}$ or $\mathbf{u'}$ will "move" into G'_x. This process determines a one to one correspondence between the vertices and numbered edges of $G'_{\mathbf{s}}$ and G'_x, giving us an isomorphism of numbered graphs.

Since every component of G_1 can be "reached" from the source with a directed path in G with all edge numbers in the path $\geq i+1$, all components are hence isomorphic to $G'_{\mathbf{s}}$ as numbered graphs. Set $G' = G'_{\mathbf{s}}$.

Now consider that for every vertex $\mathbf{y}$ in $\overrightarrow{p_{\mathbf{uv}}}$ (a path from $G'_{\mathbf{s}}$ to G'_x), there exists a vertex $\mathbf{y}'$ in $\overrightarrow{p_{\mathbf{u'v'}}}$ and edge $\overrightarrow{e_{\mathbf{yy'}}}$ numbered j. Considering the previous paragraphs, for each vertex $\mathbf{y}$ in $\overrightarrow{p_{\mathbf{uv}}}$ we get another component of G_1 that is isomporphic to $G'_{\mathbf{s}}$. Extending this concept, we see that each component of G_1 is "separated" from some other component of G_1 by only one "edge"; i.e., if G'_y and G'_z are two components of G_1 and there exists an edge $\overrightarrow{e_{\mathbf{uv}}}$ numbered $j \geq i+1$ where $\mathbf{u}$ is in G'_y and $\mathbf{v}$ is in G'_z, then every vertex in G'_y is connected to its isomorphic vertex in G'_z by an edge numbered j in G. Thus the subgraph formed by G'_y, G'_z and the edges that connect them form a graph that isomorphic to the complete extension of $G'_{\mathbf{s}}$ by j.

Now consider G_2, the subgraph of G formed by removing all edges numbered $\leq i-1$. By arguments similar to that of the above, we find that G_2 is not connected and each component is isomorphic to each other as numbered graphs. Consider $G''_{\mathbf{s}}$, the component of G_2 that contains the source. Each vertex of $G''_{\mathbf{s}}$ is the "source" vertex of a component of G_1, for each vertex $\mathbf{v}$ of $G''_{\mathbf{s}}$ can be reached with a path $\overrightarrow{p_{\mathbf{sv}}}$ with all edges numbered $\geq i+1$. So each component of G_2 is made up of vertices that, when viewed in G_1, are isomorphic to each other in different components.

Set $G'' \cong G''_{\mathbf{s}}$. By the description of the structure of G as given above, we see that $G \cong G' \square_N G''$. Since G is well numbered, by Theorem 2.2.1 G' and G'' are well numbered. $\square$

Corollary 2.2.2 *Let G be a well numbered graph with a single source, smallest edge number n_1, and largest edge number n_2. Let $I \subset \{n_1 \cdots n_2\}$ such that the edge numbers of $G = \{n_1 \cdots n_2\} \backslash I$. Then there exists well numbered graphs $G_1, \cdots, G_k$ such that $G \cong G_1 \square_N G_2 \square_N \cdots \square_N G_k$ and each G_j has consecutive edge numbers.*

Proof:

We will proceed by induction on $|I| = j$.

$j = 0$ Trivial; $k = 1$

$j = 1$ Apply Lemma 2.2.1

$j-1 \to j$ Let $i = \min(I)$. By Lemma 2.2.1, there exists well numbered graphs G_1 and G'_2 such that G_1 has consecutive edge numbers $\{n_1, \cdots, i-1\}$, G'_2 has all edge numbers in $\{i+1, \cdots, n_2\}$, and $G \cong G_1 \square_N G'_2$. By the induction hypothesis, $G'_2 \cong G_2 \square_N \cdots \square_N G_k$ for some well numbered graphs $G_2, \cdots, G_k$ with consecutive edge numbers. Since $\cdot \square_N \cdot$ is associative, $G \cong G_1 \square_N G_2 \square_N \cdots \square_N G_k$. $\square$

Corollary 2.2.3 *Let G' and G'' be well numbered graphs, each with a single source. Let G' have smallest edge number n_1 and largest edge number n_2, and let $I \subset \{n_1 \cdots n_2\}$ such that no edge of G' is numbered with an element of I. Let all edge numbers of G'' be in I. $G' \square_N G''$ is well numbered.*

Proof:

By Corollary 2.2.2, there exists well numbered graphs $G_1, \cdots, G_k$ with consecutive edge numbers such that $G' \cong G_1 \Box_N G_2 \Box_N \cdots \Box_N G_j$ and $G'' \cong G_{j+1} \Box_N G_{j+2} \Box_N \cdots \Box_N G_k$. Thus $G' \Box_N G'' \cong G_1 \Box_N \cdots \Box_N G_j \Box_N G_{j+1} \Box_N \cdots \Box_N G_k$. Since $\cdot \Box_N \cdot$ is commutative, we can reorder and renumber these graphs so that G_1 has the lowest edge numbers of all the G_i's and if $s < t$, then the edge numbers of G_s are less than the edge numbers of G_t. By Theorem 2.2.1, each $G_i \Box_N G_{i+1}$ is well numbered; using the associative property and Theorem 2.2.1 inductively we get that $G_1 \Box_N \cdots \Box_N G_k$ is well numbered. Therefore $G' \Box_N G''$ is well numbered. $\Box$

We end this section with a special type of well numbered graphs. We define the *n-cube* to be the graph produced by the following: given 2^n vertices, give each vertex a unique n digit binary number (or, if you prefer, label each vertex with an unique binary n-tuple). Edges exist between vertices if and only if the label of the vertices differ by only 1 digit.

We will now show the n-cube can be made into a well numbered graph.

Lemma 2.2.2 *Let G be the graph produced by connecting two isomorphic n-cubes H and H' together where the only edges between H and H' are edges $\mathbf{uu'}$, where $\mathbf{u} \in H$, and $\mathbf{u'} \in H'$, $\mathbf{u} \sim \mathbf{u'}$. Then G is an n+1-cube.*

Proof:

Since H and H' are n-cubes, their vertices can be labelled with unique binary n-tuples, where adjacent vertices have labels that differ in only one position. Thus give each vertex of H and H' such a label, and make the isomorphism between the two graphs such that $\mathbf{u} \sim \mathbf{u'}$ if and only if $\mathbf{u}$ and $\mathbf{u'}$ have the same label. Now extend the label of all vertices by adding one position to the end of each label; for all labels in H, make this position a 0; for all labels in H', make this last position a 1. G is formed by connecting the isomorphic vertices of H to H' with edges. By construction, G has 2^{n+1} vertices, each with a n+1-tuple label, wherein vertices are adjacent if and only if their labels differ in only one position. Thus G is the $n+1$-cube. $\Box$

Corollary 2.2.4 *n-cubes can be well numbered.*

Proof:

We will proceed with induction on n. For $n = 2$, see Figure 2.7.

Now assume it is true for $n - 1$-cube. We will show it is true for the n-cube.

Let G be a $n - 1$-full $n - 1$-cube. By the proof of Lemma 2.2.2, we see that the n-cube is the complete extension of G by n. By Corollary 2.2.1, this is a well numbered graph.$\Box$

While the above proofs show that the n-cube can be formed by taking the complete extension of the $n - 1$ cube, it can also be shown that if G' is the n-cube and G'' is the m-cube, then $G_1 \Box_N G_2$ is the $n + m$ cube.

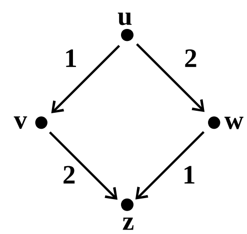


Figure 2.7: A well numbered 2-cube

2.3 Labelling a Numbered Graph

Let a numbered graph G be given wherein the largest number assigned to an edge is $n-1$. We will label each vertex $\mathbf{v}$ with an n-tuple of letters in $\prod_1^n \mathbf{X}$, where $\mathbf{X}$ is a countable set of letters. Set $\widehat{G} = G_0 \times \{1, \cdots, n\}$. An element $(\mathbf{v}, i) \in \widehat{G}$ will be written as $\mathbf{v}_i$. We will label the vertices of G as follows:

1. For all edges $\overrightarrow{e_{\mathbf{uw}}}$ in G, set $\mathbf{u}_i \sim \mathbf{w}_i$ for all $i \in \{1, \cdots, n\}, i \neq j, j+1$, where j is the edge number given to $\overrightarrow{e_{\mathbf{uw}}}$. Note that for any vertices $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{u}_i \sim \mathbf{w}_j$ only if $i = j$.

2. Let E be the equivalence relation generated by $\sim$. It follows from the note in 1) that $\mathbf{u}_i$ and $\mathbf{w}_j$ are in the same equivalence class only if $i = j$.

3. Let $\mathbf{X}$ be a countable set of letters. Associate each element of $\widehat{G}$ with a letter with a function $l\colon \widehat{G} \to \mathbf{X}$ where $l(\mathbf{u}_i) = l(\mathbf{v}_j)$ if and only if $\mathbf{u}_i$ and $\mathbf{v}_j$ are in the same equivalence class.

4. Assign to each vertex $\mathbf{v} \in G_0$ the n-tuple label $(l(\mathbf{v}_1), l(\mathbf{v}_2), \ldots, l(\mathbf{v}_n))$.

Let $X_G \subseteq \mathbf{X}$ denote the image of $\widehat{G}$ under l; $l(\widehat{G}) = X_G$. Thus X_G is the set of all letters used in the labels of the vertices of G. For simplicity of notation, we will usually suppress the function notation $l(\cdot)$ when it does not cause confusion and identify $l(\mathbf{v}_i)$ with $\mathbf{v}_i$. Also, we will usually write the labels of a vertex as $l(\mathbf{v}_1)l(\mathbf{v}_2)\ldots l(\mathbf{v}_n)$, the concatenation of the coordinates of its label.

Definition 2.3.1 *A numbered graph given a vertex labelling in accordance to the above procedure is said to be* saturated.

Definition 2.3.2 *A well numbered saturated graph is said to be* full. *A full graph whose largest edge number is n is said to be* n-full.

Some examples may be helpful.

Example 2.3.1 (See Figure 2.8)

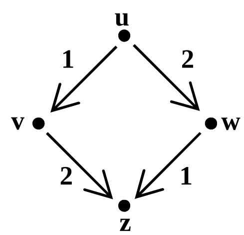


Figure 2.8: The well numbered graph in Example 2.3.1

The largest edge number is 2; thus $\widehat{G} = G_0 \times \{1,2,3\}$ and each vertex will be given a 3-tuple label. From $\overrightarrow{e_{\mathbf{uw}}}$ we have $\mathbf{u}_1 \sim \mathbf{w}_1$; from $\overrightarrow{e_{\mathbf{uv}}}$ we have $\mathbf{u}_3 \sim \mathbf{v}_3$; we also get that $\mathbf{w}_3 \sim \mathbf{z}_3$ and $\mathbf{v}_1 \sim \mathbf{z}_1$. The equivalence relation generated by these relations has the following 8 equivalence classes: $[\mathbf{u}_1]$, $[\mathbf{u}_2]$, $[\mathbf{u}_3]$, $[\mathbf{v}_1]$, $[\mathbf{v}_2]$, $[\mathbf{w}_2]$, $[\mathbf{w}_3]$, and $[\mathbf{z}_2]$. We'll let our set of letters be $\mathbf{X} = \{$a,b,c,$\cdots$,h$\}$. We define a function $l\colon \widehat{G} \to \mathbf{X}$ giving each vertex a label as shown in Figure 2.9. □

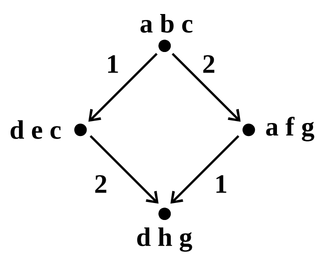


Figure 2.9: Labelling the graph in Example 2.3.1

Example 2.3.2 (See Figure 2.10)

The largest edge number here is 3; thus $\widehat{G} = G_0 \times \{1,2,3,4\}$ and each vertex will be given a 4-tuple label. Since $\overrightarrow{e_{\mathbf{st}}}$ is numbered 1 we have that $\mathbf{s}_3 \sim \mathbf{t}_3$ and $\mathbf{s}_4 \sim \mathbf{t}_4$; since $\overrightarrow{e_{\mathbf{su}}}$ is numbered 2 we have that $\mathbf{s}_1 \sim \mathbf{u}_1$ and $\mathbf{s}_4 \sim \mathbf{u}_4$; since $\overrightarrow{e_{\mathbf{sw}}}$ is numbered 3 we have that $\mathbf{s}_1 \sim \mathbf{w}_1$ and $\mathbf{s}_2 \sim \mathbf{w}_2$. The equivalence relation generated by all such relations has 12 equivalence classes: $[\mathbf{s}_1]$, $[\mathbf{s}_2]$, $[\mathbf{s}_3]$, $[\mathbf{s}_4]$, $[\mathbf{t}_1]$, $[\mathbf{t}_2]$, $[\mathbf{u}_2]$, $[\mathbf{u}_3]$, $[\mathbf{v}_2]$, $[\mathbf{w}_3]$, $[\mathbf{w}_4]$, and $[\mathbf{y}_3]$. We'll let our set of letters be $\mathbf{X} = \{$a,b,c,$\cdots$,l$\}$. Finally, a function $l\colon \widehat{G} \to \mathbf{X}$ gives each vertex a label as show in Figure 2.11. (The graph is drawn exactly as before; the edge numbers have been removed to help unclutter the image.) □

We give here some fundamental properties of the labels of full graphs.

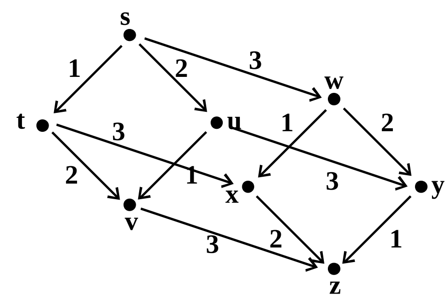


Figure 2.10: The well numbered graph in Example 2.3.2

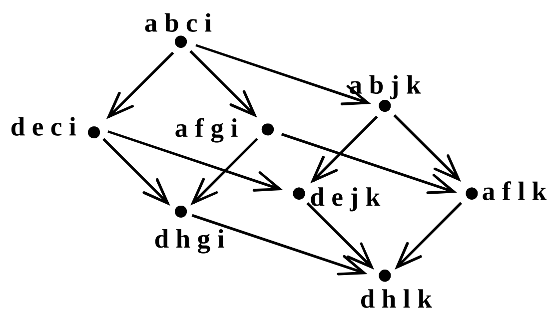


Figure 2.11: Labelling the graph in Example 2.3.2

Lemma 2.3.1 *Let vertices* $\mathbf{u}$ *and* $\mathbf{v}$ *of a full graph* G *be given. If* $l(\mathbf{u}_i) = l(\mathbf{v}_i)$*, then there exists a path* $\overline{p_{\mathbf{uv}}}$ *in which for all vertices* $\mathbf{x}$ *in* $\overline{p_{\mathbf{uv}}}$*,* $l(\mathbf{x}_i) = l(\mathbf{v}_i)$ *and no edge in* $\overline{p_{\mathbf{uv}}}$ *is numbered* $i-1, i$*.*

Proof:

Assume not. Let $\mathbf{Y} = \{\mathbf{x} \in G_0 \mid \exists$ an path $\overline{p_{\mathbf{ux}}}$ in which no edge is numbered $i, i-1\}$, and define the set $\mathbf{Z}$ similary for the vertex $\mathbf{v}$. Clearly either $\mathbf{Y} \cap \mathbf{Z} = \emptyset$ or $\mathbf{Y} = \mathbf{Z}$. By our assumption, $\mathbf{Y} \cap \mathbf{Z} = \emptyset$. This means that there is no edge not labelled i or $i-1$ in which one vertex of the edge is in $\mathbf{Y}$ and the other in $\mathbf{Z}$. Thus there is no generator of the equivalence relation that forces $l(\mathbf{u}_i) = l(\mathbf{v}_i)$, contradicting the fact that our graph had a saturated labelling.

Thus for all vertices $\mathbf{u}$ and $\mathbf{v}$, if $l(\mathbf{u}_i) = l(\mathbf{v}_i)$, then there exists an undirected path $\overline{p_{\mathbf{uv}}}$ in which no edges in the path are numbered $i, i-1$. Thus each edge $\overrightarrow{e_{\mathbf{xy}}}$ in the path generates the relation $\mathbf{x}_i \sim \mathbf{y}_i$, and so for all vertices $\mathbf{x}$ in $\overline{p_{\mathbf{uv}}}$, $\mathbf{x}_i \sim \mathbf{v}_i$, giving us $l(\mathbf{x}_i) = l(\mathbf{v}_i)$. $\square$

Corollary 2.3.1 *Let a full graph* G *be given. If* $\overrightarrow{e_{\mathbf{uv}}}$ *is an edge in* G *numbered* i*, then* $l(\mathbf{u}_i) \neq l(\mathbf{v}_i)$ *and* $l(\mathbf{u}_{i+1}) \neq l(\mathbf{v}_{i+1})$*.*

Proof:

Assume not. Then without loss of generality there exists an edge $\overrightarrow{e_{\mathbf{uv}}}$ in G numbered i such that $l(\mathbf{u}_i) = l(\mathbf{v}_i)$. By Lemma 2.3.1 we know that there exists a undirected path $\overline{p_{\mathbf{uv}}}$ such that for all vertices $\mathbf{x}$ in $\overline{p_{\mathbf{uv}}}$, $l(\mathbf{x}_i) = l(\mathbf{u}_i)$, and no edge in the path is numbered $i, i-1$. However, this then gives an undirected cycle in which only one edge is numbered i, violating condition #4 and contradicting the well numbering of Γ. Therefore $l(\mathbf{u}_i) \neq l(\mathbf{v}_i)$ and $l(\mathbf{u}_{i+1}) \neq l(\mathbf{v}_{i+1})$. $\square$

Chapter 3

Rings, Gröbner Bases and Full Graphs

3.1 Rings and Admissible Orders

Let R be a K-algebra and $\mathcal{B} = \{b_i\}_{i \in \mathcal{I}}$ be a multiplicative K-basis for R. That is, $\mathcal{B}$ is a K-basis for R as a vector space, and if $b, b' \in \mathcal{B}$, then $bb' \in \mathcal{B}$ or $bb' = 0$. We give two examples. 1) Let $R = K\langle x_1, x_2, \cdots, x_m \rangle$, the free associative algebra in m noncommuting variables. Let $\mathcal{B} = \{\text{monomials}\}$. 2) Let $R = K\Gamma$ for some quiver Γ, and let $\mathcal{B} =$\{finite directed paths\}. The length $l(p)$ of a path p is the number of arrows in p; vertices are included in $\mathcal{B}$ as paths of length 0. The multiplicative structure of $K\Gamma$ is given in chapter 1.

We desire a well order $>$ on $\mathcal{B}$ that is preserved, to a certain extent, by multiplication. Given $p, q, r \in \mathcal{B}$, we desire:

1. If $p < q$ then $pr < qr$ if $pr \neq 0 \neq qr$
2. If $p < q$ then $rp < rq$ if $rp \neq 0 \neq rq$
3. If $p = qr$, then $p \geq q$ and $p \geq r$.

Definition 3.1.1 *A well order $>$ on $\mathcal{B}$ is* admissible *if it satisfies the above conditions.*

We also say that R has an *ordered multiplicative basis* if R has a multiplicative basis $\mathcal{B}$ with an admissible ordering $>$ on $\mathcal{B}$. Path algebras are rather ubiquitous in many respects; we mention here that any ring with an ordered multiplicative basis is a quotient of some path algebra in a natural way [9]. With this in mind, we will give one example of an admissible order in the context of a path algebra.

The *left length-lexicographic order* is defined as follows. Given a path algebra $K\Gamma$, order the vertices and arrows arbitrarily, with the vertices all smaller than the arrows. If $p, q \in \mathcal{B}$, with $p = b_1 b_2 \cdots b_s$ and $q = b'_1 b'_2 \cdots b'_t$ (where $b_i, b'_j \in \mathcal{B}$), then $p > q$ if $s > t$, or if $s = t$, then for some $1 \leq i \leq s$, $b_j = b'_j$ for $j < i$, and $b_i > b'_i$.

Given some $r \in R\backslash\{0\}$, $r = \sum_{i\in\mathcal{I}} \alpha_i b_i$ with $\alpha_i \in K, b_i \in \mathcal{B}$. We define $\text{Tip}(r) = b_j$ where $b_j \geq b_i$ for all $i \in \mathcal{I}$ and $\alpha_j \neq 0$. The coefficient of Tip(r) is denoted by CTip(r); $\text{CTip}(r) = \alpha_j$ where $b_j =$Tip(r). Given some subset $X \subseteq R$, $\text{Tip}(X) = \{b \in \mathcal{B} \mid b = \text{Tip}(x)$ for some nonzero $x \in X\}$. $\text{NonTip}(X) = \mathcal{B}\backslash\text{Tip}(X)$.

Definition 3.1.2 *Let I be an ideal of a ring R with multiplicative basis $\mathcal{B}$ and admissible order $>$. A set $\mathcal{G} \subseteq I$ is a* Gröbner basis *for I with respect to $>$ if $<$Tip($\mathcal{G}$)$> = <$Tip(I)$>$. $\mathcal{G}$ is a* reduced Gröbner basis *for I if:*

1. *$\mathcal{G}$ is a Gröbner basis for I*
2. *For all $g \in \mathcal{G}$,* CTip(g) *= 1*
3. *For all $g \in \mathcal{G}$, no monomial of g -* Tip(g) *is in* Tip$(\mathcal{G})$

Gröbner basis theory has numerous applications in mathematics; it plays an important role in the implementation of many computer algebra systems. One difficulty in the application of Gröbner basis theory is the actual construction of a Gröbner basis given a finitely generated ideal I. However, other criteria exist to determine whether or not a particular set $X \subseteq R$ is a Gröbner basis for the ideal it generates. We will employ one such criterion in this dissertation; from a full graph G we will construct a set $\mathcal{G}$ and show that $\mathcal{G}$ is a Gröbner basis for $\langle\mathcal{G}\rangle$. To establish this criterion, we will need to define a few more terms.

As stated before, each element $r \in R\backslash\{0\}$ is a linear combination of monomials of the form $\sum_{i\in\mathcal{I}} \alpha_i b_i$ with $\alpha_i \in K, b_i \in \mathcal{B}$. If $\alpha_i \neq 0$, then the monomial $\alpha_i b_i$ is a *term* of r, and we say that b_i *occurs* in r. Given elements $b, b' \in \mathcal{B}$, we say b *divides* b' ($b \mid b'$) if there exists elements $c, d \in \mathcal{B}$ with $b' = cbd$. A set of elements $X \subseteq R$ is *tip reduced* if for all elements $x, y \in X$, Tip$(x)\nmid$ Tip(y). An element $r \in R$ is *uniform* if there exists elements $u, v \in \mathcal{B}$ such that $r = ur = rv$. In the context of path algebras, a linear combination of paths r is uniform if all paths begin at the same vertex u and all paths end at the same vertex v.

Two more definitions are needed to understand the important theorem that follows. First, let $f, g \in R$ and suppose there exists elements $b, c \in \mathcal{B}$ such that:

1. $\text{Tip}(f)c = b\text{Tip}(g)$.
2. $\text{Tip}(f) \nmid b$, $\text{Tip}(g) \nmid c$

Then the *overlap relation* of f and g by b, c, is: $o(f, g, b, c) = \dfrac{1}{\text{CTip}(f)}fc - \dfrac{1}{\text{CTip}(g)}bg.$

Secondly, let $X \subseteq R$, $f, g \in R$. Suppose there exists some element $x \in X$ such that Tip(x) divides some term $\alpha_j b_j$ of f; $\alpha_j b_j = \beta c\text{Tip}(x)d$ for some $c, d \in \mathcal{B}$ and $\beta \in K^*$. By adding $-\beta cxd + f = f'$, the coefficient of b_j is 0 in f'. We then say f *reduces to* f' by X, which is denoted by $f \Rightarrow_X f'$. We extend this definition to repeated reductions by X, so $f \Rightarrow_X g$ denotes that g can be obtained via repeated reductions of f by X.

Example 3.1.1 Let $R = K\langle x, y, z\rangle$, the noncommutative free algebra on three variables. R has a multiplicative basis $\mathcal{B} = \{\text{monomials}\}$, and order $\mathcal{B}$ with the left length-lexicographic ordering where $x > y > z$. Let $X = \{xy - yx, xz - zx, yz - zy\}$, and set $f = xyz - zyx$. We see that $\text{Tip}(xy - yx) \mid xyz$, a term of f; set $f' = f - (xy - yx)z = yxz - zyx$. Thus $f \Rightarrow_X f'$. We can continue by noting that $\text{Tip}(xz - zx) \mid yxz$, a term in f'; set $f'' = f' - y(xz - zx) = yzx - zyx$. Finally, $\text{Tip}(yz - zy) \mid yzx$, a term of f''; $f'' - (yz - zy)x = zyx - zyx = 0$. So we can say $f \Rightarrow_X 0$.

The following theorem is a variation of G. Bergman's Diamond Lemma [4, 8].

Theorem 3.1.1 *Let R be a K-algebra with a multiplicative basis $\mathcal{B}$ and admissible order $>$. Let $\mathcal{G}$ be a set of uniform, tip reduced elements of R. Suppose for every overlap relation with $g_1, g_2 \in \mathcal{G}$, $o(g_1, g_2, p, q) \Rightarrow_{\mathcal{G}} 0$. Then $\mathcal{G}$ is a Gröbner basis for $\langle\mathcal{G}\rangle$.* □

We will use the above theorem to show that a certain set $\mathcal{G}$ derived from a full graph G is a Gröbner basis for $\langle\mathcal{G}\rangle$. In order to do so, we must first define the ring environment that we will work in and give an admissible ordering.

3.2 Ring Environment

A full graph G will provide foundational structure from which a K-algebra of the form $K\Gamma/I$ (a quotient of a path algebra) will be constructed. The following will describe how G partially determines the structure of Γ and gives a generating set for I.

Set $\widetilde{\mathcal{G}} = \{((l(\mathbf{u}_i), l(\mathbf{u}_{i+1})), (l(\mathbf{w}_i), l(\mathbf{w}_{i+1}))) \mid \overrightarrow{e_{\mathbf{uw}}} \text{ is an edge in } G \text{ numbered } i\}$. Let Γ be a quiver such that:

1. The elements of X_G can be put into a one to one correspondence with the arrows of Γ (where we will identify $\mathbf{a} \in X_G$ with arrow $\overrightarrow{\mathbf{a}}$ in Γ)

2. If ($\mathbf{a}$,$\mathbf{b}$) is one of the ordered pairs of an element of $\widetilde{\mathcal{G}}$, then the path $\overrightarrow{\mathbf{a}}\overrightarrow{\mathbf{b}}$ exists in Γ

3. For every element (($\mathbf{a}$,$\mathbf{b}$),($\mathbf{c}$,$\mathbf{d}$)) $\in \widetilde{\mathcal{G}}$, $o(\overrightarrow{\mathbf{a}}) = o(\overrightarrow{\mathbf{c}})$ and $t(\overrightarrow{\mathbf{b}}) = t(\overrightarrow{\mathbf{d}})$ (where o and t refer to the origin and terminus of an arrow, respectively, in Γ).

4. If $\mathbf{v}$ is a vertex in Γ, then there exists an arrow $\overrightarrow{\mathbf{a}}$ such that $\mathbf{v} = o(\overrightarrow{\mathbf{a}})$ or $\mathbf{v} = t(\overrightarrow{\mathbf{a}})$.

Definition 3.2.1 *Given a full graph G and set $\widetilde{\mathcal{G}}$ as above, a quiver Γ is an* associated quiver to G *if the above conditions hold.*

Example 3.2.1 Consider the 2-full graph G in Figure 3.1.

$\widetilde{\mathcal{G}} = \{((\mathbf{a},\mathbf{b}),(\mathbf{d},\mathbf{e})), ((\mathbf{a},\mathbf{f}),(\mathbf{d},\mathbf{h})), ((\mathbf{b},\mathbf{c}),(\mathbf{f},\mathbf{g})), ((\mathbf{e},\mathbf{c}),(\mathbf{h},\mathbf{g}))\}$. Figure 3.2 gives three associated quivers. The first is natural in certain sense; every element of $\widetilde{\mathcal{G}}$ creates a "directed rectangle" in the quiver. Quiver (b) is in a sense a flattened version of (a). Finally,

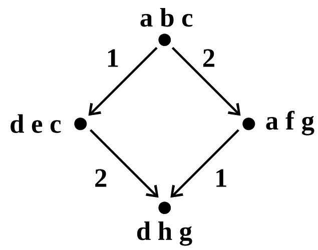


Figure 3.1: 2-full graph G in Example 3.2.1

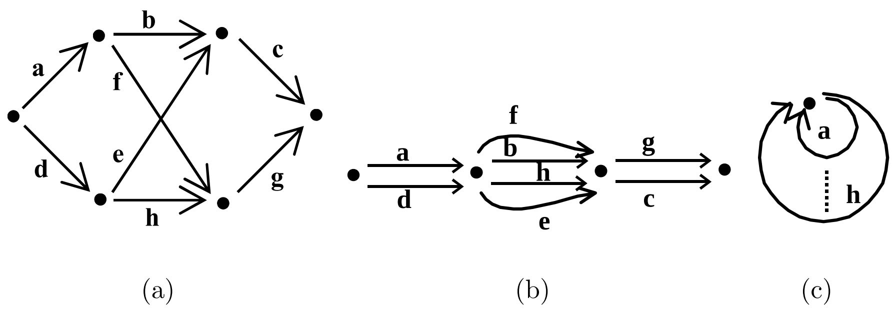


Figure 3.2: Associated quivers to G

(c) has one vertex with 8 loops (not all are shown). As mentioned before, this quiver forms an algebra isomorphic to the free algebra on 8 noncommuting variables. □

It is clear that these three quivers are "different;" we give next the definition of an isomorphism of associated quivers, which allows us to say that these associated quivers are pairwise nonisomorphic.

Definition 3.2.2 *Let G be a full graph and let Γ and Γ' be associated quivers to G, where the element $\boldsymbol{a} \in X_G$ is associated with the arrows $\overrightarrow{\mathbf{a}}$ in Γ and $\overrightarrow{\mathbf{a}}'$ in Γ'. Γ and Γ' are* isomorphic as associated quivers *if there exists a ring isomorphism $\phi\colon \Gamma \to \Gamma'$ such that $\phi(\overrightarrow{\mathbf{a}}) = \overrightarrow{\mathbf{a}}'$.*

In general, as shown in the example, there may be several nonisomorphic quivers which are associated to a full graph G. While all the work done in this dissertation will be done in the context of a quotient of a path algebra $K\Gamma$ for some field K and associated quiver Γ, unless specified otherwise, our results are developed independantly of the choice of Γ. Readers unfamiliar with path algebras can usually think in terms of quotients of free algbras, for as demonstrated by the example, if $|X_G| = m$, then the quiver with one vertex and m loops will be an associated quiver. Since this quiver is particulary nice, it is denoted Γ_F. For ease of notation, given a letter **a** in a label of G, we will also denote the arrow $\overrightarrow{\mathbf{a}} \in \Gamma$ as **a**.

Given a field K, a full graph G, and an quiver Γ associated to G, the *associated triple*

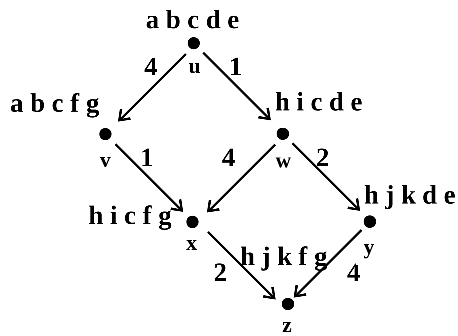


Figure 3.3: Demonstrating a 4-moveable path

(K, G, Γ) will determine an algebra $\Lambda = K\Gamma/I$, where I is an ideal of $K\Gamma$ whose generating set is determined by G as follows.

Set $\mathcal{G} = \{\mathbf{ab} - \mathbf{cd} \mid ((\mathbf{a}, \mathbf{b}), (\mathbf{c}, \mathbf{d})) \in \widetilde{\mathcal{G}}\}$. Define I to be the ideal generated by the elements of $\mathcal{G}$; $I = \langle \mathcal{G} \rangle$.

Definition 3.2.3 *Given an associated triple* (K, G, Γ) *and set* $\mathcal{G}$ *as described above,* $\mathcal{G}$ *is the* set of relations of G*,* I *is the* ideal associated to (K, G, Γ)*, and* $\Lambda = K\Gamma/I$ *is the* graded algebra associated to (K, G, Γ).

In Example 3.2.1, the set of relations of G is $\mathcal{G} = \{\mathbf{ab} - \mathbf{de}, \mathbf{af} - \mathbf{dh}, \mathbf{bc} - \mathbf{fg}, \mathbf{ec} - \mathbf{hg}\}$. The relation $\mathbf{ab} - \mathbf{de}$ came from the edge that is numbered 1 coming from the source. Now consider the full graph in Figure 3.3.

Edge $\overrightarrow{e_{\mathbf{uw}}}$ generates the relation **ab-hi**, as does edge $\overrightarrow{e_{\mathbf{vx}}}$; edge $\overrightarrow{e_{\mathbf{wy}}}$ generates the relation **ic-jk**, as does $\overrightarrow{e_{\mathbf{xz}}}$; $\overrightarrow{e_{\mathbf{uv}}}$ generates **de-fg**, as does $\overrightarrow{e_{\mathbf{wx}}}$ and $\overrightarrow{e_{\mathbf{yz}}}$. The path $\overrightarrow{p_{\mathbf{uy}}} = \overrightarrow{e_{\mathbf{uw}}}\overrightarrow{e_{\mathbf{wy}}}$ is 4-moveable. As discussed in Chapter 2, the vertex **u** having an edge numbered 4 coming from it necessitates that vertices **w** and **y** also have edges numbered 4 coming from it. But notice that not only does "an" edge numbered 4 "move" across the path, in some sense the "same" edge moves across in that each edge numbered 4 along this 4-moveable path generates the same relation. This fact plays an important role in showing that $\mathcal{G}$ is a Gröbner basis for $\langle \mathcal{G} \rangle$.

3.3 Well Ordering of X_G

Let G be a full graph, let X_G be the letters of G, and let (K, G, Γ) be an associated triple for some quiver Γ. We want to show that the set of relations $\mathcal{G}$ of G forms a Gröbner basis of the ideal it generates in $K\Gamma$. In order to do so, we must first give a well ordering on X_G (which are identified with the arrows of Γ) and then give an admissible ordering on the paths of Γ.

In general, we will define the letters that appear in the i^{th} coordinate of a label to be greater than letters in the $i+1^{st}$ coordinate. Now we will consider the i^{th} coordinate, and show a well order exists on the letters in that coordinate.

We begin by defining a binary relation $>_{i,e}$ on the letters in the i^{th} coordinate (where the subscript i refers to the i^{th} coordinate, and e refers to "edge"). Recall that the letters in the i^{th} coordinate are changed along edges numbered $i-1$ and i. For letters $\mathbf{a}$ and $\mathbf{b}$, we define $\mathbf{a} >_{i,e} \mathbf{b}$ if and only if there exists an edge $\overrightarrow{e_{\mathbf{uw}}}$ numbered $i-1$ or i with the i^{th} coordinates of **u,w** labelled **a**, **b** respectively. Let $>_{i,t}$ be the transitive closure of $>_{i,e}$. Let $\geq_{i,t}$, as usual, denote the relation $>_{i,t}$ with the reflexive property. $>_{i,t}$ can be extended to a total ordering $>_i$ if and only if $\geq_{i,t}$ is a partial order relation; we must show that $\geq_{i,t}$ is antisymmetric.

Assume that it is not; assume that there exists letters $\mathbf{a}$ and $\mathbf{b}$ appearing in the i^{th} coordinate such that $\mathbf{a} \neq \mathbf{b}$, $\mathbf{a} >_{i,t} \mathbf{b}$ and $\mathbf{b} >_{i,t} \mathbf{a}$. Without loss of generality, we'll assume that there is an edge $\overrightarrow{e}$ connecting a vertex labelled with **a** to a vertex labelled with **b**, and then assume that there is a sequence of edges $\overrightarrow{e_{\mathbf{1}}}, \overrightarrow{e_{\mathbf{2}}}, \cdots, \overrightarrow{e_{\mathbf{k}}}$ and letters $\mathbf{l_1}, \mathbf{l_2}, ..., \mathbf{l_{k-1}}$ such that we have $\mathbf{b} >_{i,e} \mathbf{l_1}$ from $\overrightarrow{e_{\mathbf{1}}}$, $\mathbf{l_1} >_{i,e} \mathbf{l_2}$ from $\overrightarrow{e_{\mathbf{2}}}$,..., and that $\mathbf{l_{k-1}} >_{i,e} \mathbf{a}$ from $\overrightarrow{e_{\mathbf{k}}}$.

By Lemma 2.3.1, there is an undirected path from the vertex of $\overrightarrow{e}$ labelled with **a** to the vertex of $\overrightarrow{e_{\mathbf{k}}}$ labelled with **a** in which no edge in the path is numbered $i-1$ nor i. This is true for all pairs of vertices with the same letter in the i^{th} position amongst the vertices of the edges $\overrightarrow{e_{\mathbf{1}}}, \overrightarrow{e_{\mathbf{2}}}, \cdots, \overrightarrow{e_{\mathbf{k}}}$.

By connecting each pair of vertices with the same letter with undirected paths of edges not numbered $i-1$ nor i, we have constructed an undirected cycle in which the only edges numbered $i-1$ and i are $\overrightarrow{e_{\mathbf{1}}}, \overrightarrow{e_{\mathbf{2}}}, \cdots, \overrightarrow{e_{\mathbf{k}}}$. By construction, these edges are all oriented in the same direction in this cycle, thus violating rule # 4. We then conclude that $\geq_{i,t}$ is antisymmetric, and hence $>_{i,t}$ can be extended to a total ordering $>_i$ on the letters in the i^{th} coordinate.

Repeat this to obtain a total ordering on the letters in each coordinate. Using the previously mentioned rule that the letters in the i^{th} coordinate are greater than the letters in the $i+1^{st}$ coordinate, we can now obtain a well ordering $>$ on all the letters. Given letters **a** and **b** appearing in the i^{th} and j^{th} coordinates, respectively, we define

$$\mathbf{a} > \mathbf{b} \Leftrightarrow \begin{cases} i < j & i \neq j \\ \mathbf{a} >_i \mathbf{b} & i = j \end{cases}.$$

Extending this to a left length-lexicographic order gives an admissible ordering on the paths of Γ.

This admissible ordering $>$ is not necessarily unique; it is dependant on how the partial ordering $>_{i,t}$ is extended to the total ordering $>_i$. What we are most concerned with, though, is finding the tip of the elements of $\mathcal{G}$. If an edge $\overrightarrow{e_{\mathbf{uv}}}$ of a full graph generates the relation **ab-cd**, then the origin of the edge $\overrightarrow{e_{\mathbf{uv}}}$ has **ab** in its label and the terminus of the edge has **cd** in its label. The general properties of $>$ are enough to show that **ab** is the tip of **ab-cd**.

3.4 Full Graphs and Gröbner Bases

We will now show that given an associated triple (K, G, Γ), $\mathcal{G}$ is a Gröbner basis for I, the ideal it generates. We will use Theorem 3.1.1; therefore we must show that $\mathcal{G}$ is a set of uniform, tip reduced elements in which all overlap relations reduce to 0. By construction of Γ, the elements of $\mathcal{G}$ are uniform.

Lemma 3.4.1 *Let (K, G, Γ) be given. The set of relations of G is tip reduced.*

Proof:

This follows directly from rule #6. Assume there are vertices **u**,**v** with edges numbered i coming out generating relations with the same tip, wlog **ab-cd** and **ab-ef**. By Lemma 2.3.1, there exists two undirected paths $\overline{p}$ and $\overline{q}$ from **u** to **v** such that $\overline{p}$ doesn't contain any edges numbered $i-1, i$ (hence fixing the **a**) and $\overline{q}$ doesn't contain any edges numbered $i, i+1$ (hence fixing the **b**). Then by rule #6 there exists an undirected path from **u** to **v** that does not contain any edges numbered $i-1, i$, nor $i+1$. Thus the relation moves across this undirected path, giving that the relations are in fact the same.

Therefore the set of relations of a full graph is tip reduced. □

Lemma 3.4.2 *Let an associated triple (K, G, Γ) be given, and let $\mathcal{G}$ be the set of relations. All overlap relations of $\mathcal{G}$ generated at a single vertex reduce to zero.*

Proof:

Since an overlap relation is generated only by consecutive numbers on edges, without loss of generality we reduce this problem to looking at an overlap generated by a 1 and 2, using only the first three coordinates of the labels of the graph.

Given any directed path whose edges are alternately numbered 1 and 2, and let **abc** and **xyz** denote the label of the first and last vertices of the path. We will first show that **abc** $\Rightarrow_{\mathcal{G}}$ **xyz**.

We will proceed by induction on the length n of the path. For $n = 1$, without loss of generality the edge will be numbered 1 and the labels will be **abc** and **xyc**. The relation is **ab-xy**; thus **abc - (ab-xy)c = xyc**, hence **abc** $\Rightarrow_{\mathcal{G}}$ **xyc**.

Now assume for some $k \geq 1$ that for paths of length k (where the first vertex is labelled **abc** and the last vertex is labelled **xyz**) that **abc** $\Rightarrow_{\mathcal{G}}$ **xyz**. Let $\overrightarrow{p}$ be a path of length $k+1$, and without loss of generality let the first vertex be labelled **abc**, the second **dec**, and the last **xyz**. Here we assume the first edge is numbered 1. By the base case, it is clear that **abc** $\Rightarrow_{\mathcal{G}}$ **dec**, and by our inductive hypothesis **dec** $\Rightarrow_{\mathcal{G}}$ **xyz**. Thus **abc** $\Rightarrow_{\mathcal{G}}$ **xyz**, and we have shown what we wanted to show.

Now look at an overlap relation generated at a vertex labelled **abc**. Thus there exists two paths $\overrightarrow{p}, \overrightarrow{q}$ from this vertex that meet at some other vertex labelled **xyz** such that each consists of edges numbered only 1 and 2. Without loss of generality $\overrightarrow{p}$ begins with a 1; thus $\overrightarrow{q}$ begins with a 2. The second vertex of each path will be labelled **dec** and **afg** respectively, giving us initial relations **ab-de** and **bc-fg**. Thus we have the overlap relation $o(\mathbf{ab}-\mathbf{de}, \mathbf{bc}-\mathbf{fg}, \mathbf{a}, \mathbf{c}) = (\mathbf{ab}-\mathbf{de})\mathbf{c} - \mathbf{a}(\mathbf{bc}-\mathbf{fg}) = \mathbf{afg}-\mathbf{dec}$. Each term is a label of the second vertex of each path. By our induction, $\mathbf{afg} \Rightarrow_{\mathcal{G}} \mathbf{xyz}$, and $\mathbf{dec} \Rightarrow_{\mathcal{G}} \mathbf{xyz}$. Thus $\mathbf{afg} - \mathbf{dec} \Rightarrow_{\mathcal{G}} \mathbf{xyz} - \mathbf{xyz}$, and hence $o(\mathbf{ab} - \mathbf{de}, \mathbf{bc} - \mathbf{fg}, \mathbf{a}, \mathbf{c}) \Rightarrow_{\mathcal{G}} 0$. □

Theorem 3.4.1 *Let (K, G, Γ) be an associated triple with set of relations $\mathcal{G}$. $\mathcal{G}$ is a Gröbner basis of $\langle \mathcal{G} \rangle$.*

Proof:

It has been shown that the elements of $\mathcal{G}$ are uniform, and from Lemma 3.4.1 we know that $\mathcal{G}$ is tip reduced. We now use Theorem 3.1.1 to show that $\mathcal{G}$ is a Gröbner basis of $\langle \mathcal{G} \rangle$ by showing that all overlap relations reduce to zero. We will do so by showing that all overlap relations are generated at a vertex, and hence reduce to zero by Lemma 3.4.2.

Let any overlap relation be given; that is, let two elements **ab-cd** and **be-fg** of $\mathcal{G}$ be given from edges numbered $i-1$ and i coming from vertices **v** and **u**, respectively. Since the letter **b** is shared in both relations, there exists by Lemma 2.3.1 a path between **v** and **u** that contains no edges numbered $i-1$ nor i. However, by rule #5 there exists a vertex **x** and undirected paths $\overline{p_{\mathbf{vx}}}$ (with no edges numbered $n-2, n-1$, nor n) and $\overline{p_{\mathbf{ux}}}$ (with no edges numbered $n-1, n$, and $n+1$). Thus the edge coming from **v** giving the relation **ab-cd** moves across the path to **x**, and the edge coming from **u** giving the relation **be-fg** also moves across to **x**. Thus this overlap relation is generated in the graph at a vertex.

Therefore $\mathcal{G}$ is a Gröbner basis for $\langle \mathcal{G} \rangle$. □

Corollary 3.4.1 *Let (K, G, Γ) be an associate triple with set of relations $\mathcal{G}$. The associated algebra $\Lambda = K\Gamma / \langle \mathcal{G} \rangle$ is a Koszul algebra.*

Proof:

Theorem 1.3.1 stated that given a path algebra $K\Gamma$ and an ideal I that has a quadratic Gröbner basis, then $K\Gamma/I$ is Koszul. Theorem 3.4.1 shows that $\mathcal{G}$ is a quadratic Gröbner basis for $\langle \mathcal{G} \rangle$; therefore $\Lambda = K\Gamma / \langle \mathcal{G} \rangle$ is Koszul. □

Chapter 4

Full Graphs and Projective Resolutions

4.1 Projective Resolutions of Right Λ-Modules

In the previous chapter we showed that given an associated triple (K, G, Γ), the associated algebra $\Lambda = K\Gamma/I$ is a Koszul algebra. By our definition in Chapter 1, this means that Λ_0 has a linear graded projective resolution.

This chapter will examine an algorithmic way to construct such a projective resolution of Λ_0. In [10] a method is described for finding a minimal projective resolution of a right Λ-module M. We refer the reader to this paper for a more detailed description of this process, including all proofs; we include here only the details needed for this dissertation.

For ease of notation, set $R = K\Gamma$. Thus $\Lambda = R/I$.

We find a projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

as follows:

Choose a set of elements $\{f_i^0\}_{i\in\mathcal{I}}$ in R such that the module $\coprod_i f_i^0 R / \coprod_i f_i^0 I$ maps onto M (where $\mathcal{I}$ is an indexing set; reference to such indexing sets is suppressed in the future). We normally take the set $\{f_i^0\}_i$ to consist of vertices of R with repetitions allowed. This gives the following short exact sequence:

$$0 \to \Omega_R^1(M) \to \coprod_i f_i^0 R \to M \to 0$$

Next choose sets $\{f_i^1\}_i$ and $\{f_i^{1*}\}_i$ of elements in $\coprod_i f_i^0 R$ such that each $f_i^1 \notin \coprod_i f_i^0 I$ and each $f_i^{1*} \in \coprod_i f_i^0 I$, where $\coprod_i f_i^1 R \oplus \coprod_i f_i^{1*} R = \Omega_R^1(M)$. (This is possible since R is hereditary. [9]

We establish a recursive definition for $n \geq 2$: choose elements $\{f_i^n\}_i$ and $\{f_i^{n*}\}_i$ where each $f_i^n \notin \coprod_i f_i^{n-1} I$ and each $f_i^{n*} \in \coprod_i f_i^{n-1} I$, where $\coprod_i f_i^n R \oplus \coprod_i f_i^{n*} R = \coprod_i f_i^{n-1} R \cap \coprod_i f_i^{n-2} I$. Continue until $\coprod_i f_i^{n-1} R \cap \coprod_i f_i^{n-2} I = 0$ or $\{f_i^n\}_i = \emptyset$.

Set $P_n = \coprod_i f_i^n R / \coprod_i f_i^n I$. It is well known that for all $x \in R$, xR is a projective right R-module; thus each P_n is a projective Λ-module. Let $d^n \colon P_n \to P_{n-1}$ be the boundary maps, which are homomorphisms induced by the inclusion $\coprod_i f_i^n R \subseteq \coprod_i f_i^{n-1} R$. This inclusion also implies that each f_j^n can be represented in $\coprod_i f_i^{n-1} R$ as $f_k^n = \sum_i f_i^{n-1} h_{i,k}^{n-1,n}$ for scalars $h_{i,k}^{n-1,n} \in R$. Thus the boundary maps d^n are determined by the matrix $(\overline{h}_{i,k}^{n-1,n})$, where $\overline{h}$ denotes the image of h in Λ.

The paper shows that

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

is indeed a projective resolution of M over Λ with the maps d^n. The resolution need not be minimal; however, it is shown that the $\{f_i^n\}_i$ can be chosen in such a way so that no proper K-linear combination of a subset of $\{f_i^n\}_i$ is in $\coprod_i f_i^{n-1} I + \coprod_i f_i^n L + \coprod_i f_i^{n*} L$, where L is the ideal of R generated by the arrows (hence L is the graded Jacobson radical of R). If the $\{f_i^n\}_i$ are chosen in this way, then the induced projective resolution is indeed minimal. We will make use of this fact to show that the projective resolution of Λ_0 we generate is minimal.

Before we actually show how to find sets $\{f_i^n\}_i$ for a module, we first give insight into how the recurrence relation was derived. We'll assume that we have completed the projective resolution up to the n^{th} step; thus we have the sets $\{f_i^{n-1}\}_i, \{f_i^n\}_i$, and $\{f_i^{n*}\}_i$, R-modules $\Omega_\Lambda^{n-1}(M), \Omega_\Lambda^n(M)$, and the first n projective Λ-modules in the projective resolution of M. Using this information, we show how to find sets $\{f_i^{n+1}\}_i$ and $\{f_i^{n+1*}\}_i$ and hence the projective Λ-module P_{n+1}. We establish the commutative diagram in Figure 4.1 with exact rows and columns. We assume that the sets $\{f_i^n\}_i$ and $\{f_i^{n*}\}_i$ were chosen so that

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \coprod f^{n-1} I & = & \coprod f^{n-1} I & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \coprod f^n R \oplus \coprod f^{n*} R & \longrightarrow & \coprod f^{n-1} R & \longrightarrow & \Omega_\Lambda^{n-1}(M) & \longrightarrow & 0 \\
 & & \psi \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_\Lambda^n(M) & \longrightarrow & P_{n-1} & \longrightarrow & \Omega_\Lambda^{n-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Figure 4.1: Establishing the recurrence relation I

the module $\coprod f^n R \oplus \coprod f^{n*} R$ would be the pullback completing the lower left hand square;

the fact that the kernel of ψ is $\coprod f^{n-1}I$ follows from the Snake Lemma.

Using the first column, we can establish the exact commutative diagram in Figure 4.2, where the module X is the pullback.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X & \longrightarrow & \coprod f^n R & \longrightarrow & \Omega^n_\Lambda(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \coprod f^{n-1} I & \longrightarrow & \coprod f^n R \oplus \coprod f^{n*} R & \longrightarrow & \Omega^n_\Lambda(M) & \longrightarrow & 0
\end{array}$$

Figure 4.2: Establishing the recurrence relation II

It is easily verified that X can be identified with $\coprod f^n R \cap \coprod f^{n-1} I$. We then seek elements $\{f_i^{n+1}\}_i$ and $\{f_i^{n+1*}\}_i$ such that $\coprod f^n R \cap \coprod f^{n-1} I = \coprod f^{n+1} R \oplus \coprod f^{n+1*} R$. Having done this, we can then take the top row of this diagram and create the new commutative diagram in Figure 4.3, which is the next iteration of the first commutative diagram, and

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \coprod f^n I & = & \coprod f^n I & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \coprod f^{n+1} R \oplus \coprod f^{n+1*} R & \longrightarrow & \coprod f^n R & \longrightarrow & \Omega^n_\Lambda(M) & \longrightarrow & 0 \\
 & & \psi \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^{n+1}_\Lambda(M) & \longrightarrow & P_n & \longrightarrow & \Omega^n_\Lambda(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Figure 4.3: Establishing the recurrence relation III

we can repeat the process.

4.2 Resolving Λ_0

We now apply the method described above to find a minimal projective resolution of the right Λ-module Λ_0.

We can set $\{f_i^0\}_i$={vertices of Γ, no repetitions}. The kernel of the map $\pi\colon \coprod_i f_i^0 R \to \Lambda_0$ equals $\coprod_i x_i R$ as the x_i range over the arrows of $K\Gamma$. Since I is generated by quadratic elements, by a simple length argument (which will be used repeatedly) we see that no $x_i \in \coprod_i f_i^1 I$. Therefore we set $\{f_i^1\}_i = \{$arrows of $K\Gamma\}$ and $\{f_i^{1*}\}_i = \emptyset$.

We next need sets $\{f_i^2\}_i$ and $\{f_i^{2*}\}_i$. By the recurrence relation, we desire $\coprod_i f_i^2 R \oplus \coprod_i f_i^{2*} R = \coprod_i f_i^1 R \cap \coprod_i f_i^0 I$. While we will show why later, we can choose $\{f_i^2\}_i = \mathcal{G}$; again, by a length argument we see that no element of $\mathcal{G}$ is an element of $\coprod_i f_i^1 I$. All we mention of $\{f_i^{2*}\}_i$ is that it is not necessarily empty, containing linear combinations of paths of length ≥ 3. Note that for $0 \leq n \leq 2$, the elements of $\{f_i^n\}_i$ are linear combinations of paths of length n. Once more, using a length argument, we see that no K-linear combinations of a subset of $\{f_i^n\}_i$ is in $\coprod_i f_i^{n-1} I + \coprod_i f_i^n L + \coprod_i f_i^{n*} L$, for all elements of the latter module are linear combinations of paths with length ≥ 3.

Finding $\{f_i^n\}_i$ and $\{f_i^{n*}\}_i$ for $n \geq 3$ is not a simple as it was for $n \leq 2$. In order to find these sets we turn to an algorithmic method provided in [Upcoming Green]. The basic principle of this algorithm is that all the elements of $\{f_i^n\}_i$ can be found through overlap relations between the tips of elements of $\{f_i^{n-1}\}_i$ and $\mathcal{G}$.

In order to talk about the tips of the elements of $\{f_i^{n-1}\}_i$, we must put an ordering on $\{f_i^{n-1}\}_i$. This is done recursively; we start by ordering the elements of $\{f_i^0\}_i$ arbitrarily and ordering the elements of $\{f_i^1\}_i = \{$arrows of $K\Gamma\}$ as described in Chapter 3. View $\{f_i^2\}_i$ as a subset of $\coprod_i f_i^1 R$. Thus the elements of $\{f_i^2\}_i$ are tuples of elements in R, whose coordinates are indexed by the elements of $\{f_i^1\}_i$ in decreasing order.

So for any $f_j^2 \in \{f_i^2\}_i$, $f_j^2 = (x_1, x_2, \cdots) \in \coprod_i f_i^1 R$. $\text{Tip}(\{f_i^2\}_i) = (0, \cdots, x_k, 0, \cdots)$, where $\text{Tip}(x_k)$=max$\{\text{Tip}(x_l)\}$, and if $\text{Tip}(x_k)$=$\text{Tip}(x_s)$, then $k > s$ in the ordering of $\{f_i^1\}_i$. Order $\{f_i^2\}_i$ by $f_j^2 > f_k^2$ if $\text{Tip}(f_j^2) > \text{Tip}(f_k^2)$ in the above order.

The tips of the elements of $\{f_i^{n-1}\}_i$ are defined recursively; let $f_j^{n-1} = (x_1, x_2, \cdots) \in \coprod_i f_i^{n-2} R$. $\text{Tip}(f_j^{n-1})$=$(0, \cdots, x_k, 0, \cdots)$ where $\text{Tip}(x_k)$=max$\{\text{Tip}(x_l)\}$, and if $\text{Tip}(x_k) = \text{Tip}(x_s)$, then $k > s$ in the ordering of $\{f_i^{n-2}\}_i$.

Example 4.2.1 Consider again the full graph G in Example 3.2.1, given again in Figure 4.4.

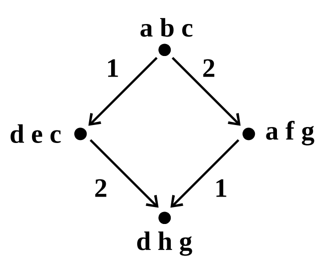


Figure 4.4: 2-full graph G in Example 4.2.1

Example 3.2.1 shows that the quiver Γ with 1 vertex v and 8 loops ({**a**,...,**h**}) is an associated quiver and $\mathcal{G}$ = {**ab-de**,**af-dh**,**bc-fg**,**ec-hg**}. Recall that $I = \langle\mathcal{G}\rangle$ and $\Lambda = K\Gamma/I$. In finding a projective resolution of Λ_0, we set $\{f_i^0\}_i = \{v\}$, $\{f_i^1\}_i$ = {**a**,...,**h**} and $\{f_i^2\}_i = \mathcal{G}$. As described in section 3.3, we can order $\{f_i^1\}_i$ by **a** > **d** > **b** > **e** > **f** > **h** > **c** > **g**. Viewing **ab-de** $\in \{f_i^2\}_i$ as a tuple in $\coprod_i f_i^1 R$, we see **ab-de**=(**b**,**-e**,0,0,0,0,0,0) (where the coordinates are indexed in decreasing order, not in any alphabetical order). Thus Tip(**ab-de**) = (**b**,0,0,0,0,0,0,0). Note that Tip(**ec-hg**) = (0,0,0,**c**,0,0,0,0) and Tip(**bc-fg**)=(0,0,**c**,0,0,0,0,0); so in ordering the elements of $\{f_i^2\}_i$ **bc-fg**>**ec-hg** since Tip(**bc-fg**)>Tip(**ec-hg**). In ordering all the elements of $\{f_i^2\}_i$ we have **ab-de**>**af-dh**>**bc-fg**>**ec-hg**. □

Now that we have defined the tips of the elements of $\{f_i^{n-1}\}_i$, we can discuss overlap relations between the tips of elements of $\{f_i^{n-1}\}_i$ and elements of $\mathcal{G}$. Let $f_1^{n-1} \in \{f_i^{n-1}\}_i$ with $\text{Tip}(f_1^{n-1}) = (0, \cdots, x_k, 0, \cdots)$, and let $g \in \mathcal{G}$. If there exists elements $p, q \in R$ such that $x_k p = q\text{Tip}(g)$ (where $\text{Tip}(g) \nmid p$ and $x_k \nmid q$), then an overlap relation is said to exist between the tip of f_1^{n-1} and g. We can use these overlaps to find the sets $\{f_i^n\}_i$; we first give an example and then formalize the algorithm.

Example 4.2.2 We refer back to the situation of Example 4.2.1 and find $\{f_i^n\}_i$ for $n \geq 3$. Consider the overlap generated between the elements **ab-de** $\in \{f_i^2\}_i$ and **bc-fg** $\in \mathcal{G}$. (**ab-de**)**c** - **a**(**bc-fg**) = **afg-dec** = (**af-dh**)**g** + **dhg** - **d**(**ec-hg**) - **dhg**. So (**ab-de**)**c** - **a**(**bc-fg**) = (**af-dh**)**g** - **d**(**ec-hg**), which gives (**ab-de**)**c** - (**af-dh**)**g** = **a**(**bc-fg**) - **d**(**ec-hg**). Note that the left hand side is in $\coprod_i f_i^2 R$ and the right hand side is in $\coprod_i f_i^1 I$. Thus (**ab-de**)**c** - (**af-dh**)**g** $\in \coprod_i f_i^2 R \cap \coprod_i f_i^1 I$, and we set f_1^3 =(**ab-de**)**c** - (**af-dh**)**g**. (Again, by a length argument, $f_1^3 \notin \coprod_i f_i^2 I$, so $f_1^3 \in \{f_i^3\}_i$ and $f_1^3 \notin \{f_i^{3*}\}_i$.) □

We claim, without proof, that $\{f_i^3\}_i = \{f_1^3\}$; we also note that since the elements of $\{f_i^{3*}\}_i$ are in $\coprod_i f_i^2 I$, each element of $\{f_i^{3*}\}_i$ has length ≥ 4. We next seek $\{f_i^4\}_i$ through overlaps with the tip of f_1^3 and elements of $\mathcal{G}$. $\text{Tip}(f_1^3)$ = (**c**,0,0,0), and clearly no overlap exists with elements in $\mathcal{G}$. Therefore we claim that $\{f_i^4\}_i = \emptyset$; indeed, it is not hard to see that $f_1^3 R \cap \coprod_i f_i^2 I = 0$.

It is also not hard to see (through length arguments) that $f_1^3 \notin \coprod_i f_i^2 I + f_1^3 L + \coprod_i f_i^{3*} L$; therefore we conclude that

$$0 \to f_1^3 R / f_1^3 I \to \coprod_i f_i^2 R / \coprod_i f_i^2 I \to \coprod_i f_i^1 R / \coprod_i f_i^1 I \to \coprod_i f_i^0 R / \coprod_i f_i^0 I \to \Lambda_0 \to 0$$

is a minimal projective resolution of Λ_0.

We now state a theorem that allows the finding of the $\{f_i^n\}_i$ through overlap relations.

Theorem 4.2.1 *Let (K,G,Γ) be an associated triple with associated algebra Λ. In the projective resolution*

$$\cdots \to \coprod_i f_i^j R / \coprod_i f_i^j I \to \cdots \coprod_i f_i^1 R / \coprod_i f_i^1 I \to \coprod_i f_i^0 R / \coprod_i f_i^0 I \to \Lambda_0 \to 0$$

let $\{f_i^0\}_i = \{$*vertices of* $K\Gamma\}$ *and* $\{f_i^1\}_i = \{$*arrows of* $K\Gamma\}$*. For* $n \geq 2$ *the sets* $\{f_i^n\}_i$ *can be determined through overlap relations between the tips of elements of* $\{f_i^{n-1}\}_i$ *and* $\mathcal{G}$*.*

Proof:

We proceed by induction on n. We omit the base case of $n = 2$ since it is similar to the induction step. Assume that we already have sets $\{f_i^0\}_i \cdots \{f_i^{n-1}\}_i$ and $\{f_i^{n-1*}\}_i$, and for $i = 0 \cdots n-1$, elements in $\{f_i^i\}_i$ are K-linear combinations of paths of length i.

Consider Figure 4.5, as given before.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \coprod f^{n-2}I & = & \coprod f^{n-2}I & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \coprod f^{n-1}R \oplus \coprod f^{n-1*}R & \longrightarrow & \coprod f^{n-2}R & \xrightarrow{\phi_{n-2}} & \Omega_\Lambda^{n-2}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_\Lambda^{n-1}(M) & \longrightarrow & P_{n-2} & \longrightarrow & \Omega_\Lambda^{n-2}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Figure 4.5: Establishing sets $\{f_i^n\}_i$ from overlaps

Let $f \in \{f_i^{n-1}\}_i$ and $g \in \mathcal{G}$ such that an overlap exists between $\text{Tip}(f)$ and $\text{Tip}(g)$. $\text{Tip}(f)$ lies in the k^{th} component of $\coprod_i f_i^{n-2}R$ for some k; since $f \in \{f_i^{n-1}\}_i$, it is the combination of paths of length $n-1$, whereas f_k^{n-2} is composed of paths of length $n-2$. Thus $\text{Tip}(f)$ is an arrow of $K\Gamma$. $\text{Tip}(g)$ is a quadratic element of the form $\mathbf{ab}$ (where $\mathbf{a}$, $\mathbf{b}$ are arrows). Thus $\text{Tip}(f)\mathbf{b} = \text{Tip}(g)$.

Set $h = f\mathbf{b} - f_k^{n-2}g$. $\text{Tip}(h) < \text{Tip}(f)$ since the tip of f is "killed off" in h. $h \in \coprod_i f_i^{n-2}R$; since $f\mathbf{b} \in \coprod_i f_i^{n-1}R$ and $f_k^{n-2}g \in I$, $h \in \text{Ker } \phi_{n-2}$ (see Figure 4.5). Thus $h \in \coprod_i f_i^{n-1}R \oplus \coprod_i f_i^{n-1*}R$, so h can be written uniquely as $\sum_i f_i^{n-1}r_i + \sum_j f_j^{n-1*}r'_j$, where the r_i and r'_j are in R. So $f\mathbf{b} - f_k^{n-2}g = \sum_i f_i^{n-1}r_i + \sum_j f_j^{n-1*}r'_j$, and $f\mathbf{b} - \sum_i f_i^{n-1}r_i = f_k^{n-2}g + \sum_j f_j^{n-1*}r'_j$. The left hand side is in $\coprod_i f_i^{n-1}R$; each $\{f_i^{n-1*}\}_i$ is in $\coprod_i f_i^{n-2}I$, so the right hand side is in $\coprod_i f_i^{n-2}I$. Thus $f\mathbf{b} - \sum_i f_i^{n-1}r_i \in \coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I$, and we choose it to be an element of $\{f_i^n\}_i$. Note that $f\mathbf{b} - \sum_i f_i^{n-1}r_i$ is a K-linear combination of paths of length n, and since $\text{Tip}(h) < \text{Tip}(f)$, $\text{Tip}(f\mathbf{b} - \sum_i f_i^{n-1}r_i) = \text{Tip}(f\mathbf{b}) = \mathbf{b}$. Repeat this process for all overlaps between the tips of elements of $\{f_i^{n-1}\}_i$ and $\mathcal{G}$.

We now show that a complete set $\{f_i^n\}_i$ is generated through this process. To do so we will show that $\coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I + \coprod_i f_i^{n-1}I = \coprod_i f_i^n R + \coprod_i f_i^{n-1}I$. (Since we have not given $\{f_i^{n*}\}_i$, we cannot directly show that $\coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I = \coprod_i f_i^n R \oplus \coprod_i f_i^{n*}R$. We instead use the fact that $\coprod_i f_i^{n*}R \subseteq \coprod_i f_i^{n-1}I$. If $\coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I + \coprod_i f_i^{n-1}I \neq \coprod_i f_i^n R + \coprod_i f_i^{n-1}I$, then something is "missing" from $\{f_i^n\}_i$.)

It should be clear that $\coprod_i f_i^n R + \coprod_i f_i^{n-1}I \subseteq \coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I + \coprod_i f_i^{n-1}I$. To show the other inclusion, assume that there exists elements in $\coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I$ that are not in $\coprod_i f_i^n R + \coprod_i f_i^{n-1}I$. Choose one of these elements x with a minimal tip.

Since $x \in \coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I$, it can be written as $\sum_i f_i^{n-1}r_i$, so $\text{Tip}(x) = \text{Tip}(f_j^{n-1})r_j$ for some $f_j^{n-1} \in \{f_i^{n-1}\}_i$. Also, x can be written as $\sum_i f_i^{n-2}m_i$, so $\text{Tip}(x) = f_k^{n-2}p\text{Tip}(g)s$, where $m_k = pgs$, $p, s \in R$, $g \in \mathcal{G}$, since $\mathcal{G}$ is a Gröbner basis for I. So $\text{Tip}(f_j^{n-1})r_j = p\text{Tip}(g)s$. If $\text{Tip}(f_j^{n-1})$ and $\text{Tip}(g)$ have an overlap, then this overlap produces an $f^n \in \{f_i^n\}_i$. Then $h = x - f^n u$, for some $u \in R$, has a smaller tip than x, and $h \in \coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I$, but $h \notin \coprod_i f_i^n R + \coprod_i f_i^{n-1}I$, contradicting the minimality of x. If $\text{Tip}(f_j^{n-1})$ and $\text{Tip}(g)$ do not have an overlap, then there exists some element $w \in R$ such that $\text{Tip}(x) = \text{Tip}(f_j^{n-1}wgs)$. $f_j^{n-1}wgs \in \coprod_i f_i^{n-1}I$; $h = x - f_j^{n-1}wgs$ has a smaller tip than x, and again $h \in \coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I$, but $h \notin \coprod_i f_i^n R + \coprod_i f_i^{n-1}I$, contradicting the minimality of x. Thus no such x can exist, and $\coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I + \coprod_i f_i^{n-1}I = \coprod_i f_i^n R + \coprod_i f_i^{n-1}I$.

We now note that $\coprod_i f_i^{n-1}I \subseteq \coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I$, so we can change our original claim that $\coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I + \coprod_i f_i^{n-1}I = \coprod_i f_i^n R + \coprod_i f_i^{n-1}I$ to $\coprod_i f_i^{n-1}R \cap \coprod_i f_i^{n-2}I = \coprod_i f_i^n R + \coprod_i f_i^{n-1}I$. It is clear that no element of $\{f_i^n\}_i$ is in $\coprod_i f_i^{n-1}I$, and no element of the form $f_i^{n-1}p_{i,j}g_j$ is in $\{f_i^n\}_i$ (where $f_i^{n-1} \in \{f_i^{n-1}\}_i, p_{i,j} \in R, g_j \in \mathcal{G}$). By length arguments we find that $p_{i,j} = 1$ for all i, j; thus we set $\{f_i^{n*}\}_i = \{f_i^{n-1}g_j\}_{i,j}$.

Thus we can find a set $\{f_i^n\}_i$ using overlaps. □

An easy consequence of Theorem 4.2.1 is that when choosing the elements of $\{f_i^2\}_i$ through overlaps, we get that $\{f_i^2\}_i = \mathcal{G}$ as sets.

In finding $\{f_i^3\}_i$ through overlaps, we look at overlaps between the tips of $\{f_i^2\}_i$ and elements of $\mathcal{G}$. Thus we can find $\{f_i^3\}_i$ by looking at the overlap relations of elements of $\mathcal{G}$. We know from Theorem 3.4.1 that all overlaps of elements of $\mathcal{G}$ are generated at vertices of the graph G. The following theorem shows how $\{f_i^3\}_i$ can be determined completely by reading information off a full graph, without computation.

Theorem 4.2.2 *Let an associated triple (K, G, Γ) be given with associated algebra Λ. In a projective resolution of Λ_0, $\{f_i^3\}_i$ can be determined by i, j cycles in G. Therefore in a minimal projective resolution of Λ_0*

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \Lambda_0 \to 0$$

with maps $d^i\colon P_i \to P_{i-1}$, the maps d^1, d^2, and d^3 can be determined directly from the construction of G.

Proof:

Let any i, j cycle be given; without loss of generality let this cycle be a 1,2 cycle with source $\mathbf{v}$ and sink $\mathbf{u}$ (see Figure 4.6).

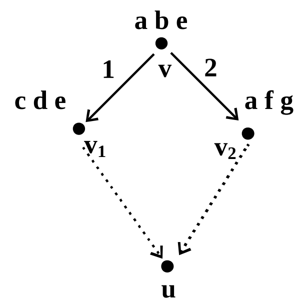


Figure 4.6: 1-2 cycle

Let the edge coming from **v** numbered 1 generate the relation **ab-cd**; let the edge numbered 2 generate the relation **be-fg**. Following the proof of Theorem 4.2.1, we set $h = (\mathbf{ab} - \mathbf{cd})\mathbf{e} - \mathbf{a}(\mathbf{be} - \mathbf{fg}) = \mathbf{afg} - \mathbf{cde}$. While we know this can be represented uniquely in $\coprod_i f_i^2 R \oplus \coprod_i f_i^{2*} R$, we show how to determine this from the graph. Note that **afg** and **cde** are part of the labels of $\mathbf{v}_2$ and $\mathbf{v}_1$, respectively. Since we are considering a 1,2 cycle, we know there exists an edge numbered 2 coming from $\mathbf{v}_2$ and an edge numbered 1 coming from $\mathbf{v}_1$ generating relations **af-hi** and **de-jk**, respectively. Note that Tip(**af-hi**)|**afg** and Tip(**de-jk**)|**cde**. Thus $h = (\mathbf{af} - \mathbf{hi})\mathbf{g} + \mathbf{hig} - \mathbf{c}(\mathbf{de} - \mathbf{jk}) - \mathbf{cjk}$. **hig** and **cjk** are labels of the next vertices in the cycle; we repeat the above reduction until the cubic monomials that are the remainders of the division cancel each other out (we know this will happen from Lemma 3.4.2). Note that all terms resulting from the "1" side of the 1,2 cycle have a negative coefficient and all terms coming from the "2" side have a positive coefficient. Note also that every time we divide by a relation coming from a 1 we get an element in $\coprod_i f_i^2 R$; dividing by relations coming from a 2 gives an element in $\coprod_i f_i^1 I$.

Thus $(\mathbf{ab} - \mathbf{cd})\mathbf{e} - \mathbf{a}(\mathbf{be} - \mathbf{fg}) = \sum_i f_i^2 r_i + \sum_j f_j^1 g_j$, where $r_i \in R$ and $g_j \in \mathcal{G}$. We know from Theorem 4.2.1 that $(\mathbf{ab} - \mathbf{cd})\mathbf{e} - \sum_i f_i^2 r_i$ can be taken as an element of $\{f_i^3\}_i$.

Note that there is a one to one correspondence between the edges numbered 1 in the 1,2 cycle and the terms of $(\mathbf{ab} - \mathbf{cd})\mathbf{e} - \sum_i f_i^2 r_i$; all $f_j^2 r_j$ that come from the "1" side now have a positive coefficient, and all $f_j^2 r_j$ coming from the "2" side have a negative coefficient. Also, for each term $f_j^2 r_j$, r_j is the letter in the third position of the label of the vertices of the edge numbered 1 that generated the f_j^2.

Thus to find a set $\{f_i^3\}_i$, consider all i, j cycles in G. Given a particular i, j cycle ($j = i+1$), the f^3 that is generated by it is $\sum_i f_i^2 r_i - \sum_j f_j^2 r_j$ where the first sum is taken over the edges numbered i coming from the "i" side of the cycle, and the second is taken over the edges numbered i from the "$i+1$" side, and the r_i and r_j are the letters in the $i+2$ place of the labels of the vertices of the edges.

The information to give $\{f_i^1\}_i$ comes directly from the labels of G; since $\{f_i^2\}_i = \mathcal{G}$ as sets we can immediately find $\{f_i^2\}_i$ by looking at the relations generated by G; finally, we can find $\{f_i^3\}_i$ by looking at all i, j cycles in G. Thus we can find P^1, P^2 and P^3 directly from G. The maps d^i are determined by the matrices $(\overline{h}_{i,k}^{n-1,n})$, where $h_{i,k}^{n-1,n}$ are scalars in

$f_k^n = \sum_i f_i^{n-1} h_{i,k}^{n-1,n}$ in R. Thus the maps are determined by $\{f_i^n\}_i$, $0 \leq n \leq 3$. □

Example 4.2.3 Consider Figure 4.7.

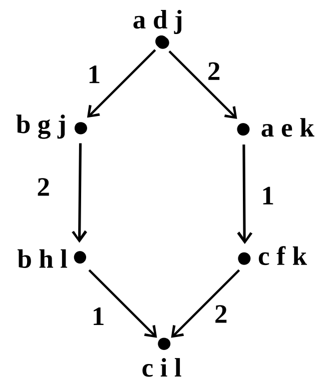


Figure 4.7: 2-full graph in Example 4.2.3

Set $\{f_i^1\}_i = \{$**a**,$\cdots$, **l**$\}$, $\{f_i^2\}_i = \{$**ad-hg**, **ae-cf**, **bh-ci**, **dj-ek**, **gj-hl**, **fk-il**$\}$. Without computation, we determine that $f^3 =$ (**ad-hg**)**j** + (**bh-ci**)**l** - (**ae-cf**)**k**. This can be verified by computing the overlap as in Theorem 4.2.1. □

Theorem 4.2.1 is proved in context of the Λ-module Λ_0. In [Upcoming Green] this method is generalized for all Λ-modules M.

So we have seen that the elements of $\{f_i^2\}_i$ come from single edges, and elements of $\{f_i^3\}_i$ come from vertices that are the source of two consecutively numbered edges. Elements of $\{f_i^4\}_i$ are found by overlaps between elements of $\{f_i^3\}_i$ and $\{f_i^2\}_i$; thus we are looking at an overlap between an element that is generated at a vertex which is the source of two edges numbered i and $i+1$, and an edge which must be numbered $i+2$ if an overlap is to be generated. In short, elements of $\{f_i^4\}_i$ are generated at vertices that are the source of 3 edges with consecutive numbers; in general, elements of $\{f_i^n\}_i$ are generated at vertices that are the source of $n-1$ edges with consecutive numbers coming out. Thus in order to find $\{f_i^{n+1}\}_i$, it is necessary that G be n-full; if G is n-full then we can find sets $\{f_i^j\}_i$ for j at most $n+1$. This puts a bound on the global dimension of Λ, which is stated as a corollary to Theorem 4.2.1.

Corollary 4.2.1 *Let an associated triple* (K, G, Γ) *be given, where* G *is n-full. Then the gl.dim(*Λ*)* $\leq n+1$*. Furthermore, this bound is sharp for each* n*.*

Proof:

If G has a vertex (namely, a source) with n edges numbered consecutively coming out, then we will be able to find a nonempty set $\{f_i^{n+1}\}_i$ through overlap relations. If G does not have a vertex with n edges coming out, then we can only find $\{f_i^i\}_i$ for $i \leq n$, giving a bound on the global dimension.

For each n, the n cube has a single source with n edges coming from the source. Therefore the associated algebra to the n cube will have global dimension $n+1$. Therefore the bound is sharp for each n. $\square$

As discussed in the paragraph before the corollary, we can find $\{f_i^{n+1}\}_i$ in a way that if $f \in \{f_i^{n+1}\}_i$, then there exists a vertex $\mathbf{v}$ with n edges coming out numbered consecutively $(r \cdots r+n-1)$ that generate f. Following the proof of Theorem 4.2.1, Tip$(f) = \mathbf{x}$, where $\mathbf{x}$ is the letter in the $r+n$ coordinate in the label of $\mathbf{v}$.

In section 2.2, it is shown that if G' is an n-full graph, then the complete extension of G by $n+1$ can be made $n+1$-full. Let G be the complete extension of G' by $n+1$; we will refer to the two isomorphic copies of G' in G as G_1 and G_2, where all edges numbered $n+1$ originate in G_1 and terminate in G_2.

The labels of the vertices of G_1 and G_2 will be identical except in the n^{th} and $n+1^{st}$ positions. Because of this all f's that come from edges numbered in $\{1, \cdots, n-2\}$ in G_1 will be the same as those f's that come from the corresponding edges in G_2. If $f_1^i = \sum_j f_j^{i-1} r_j$ comes from a vertex in G_1 with largest edge number $n-1$ coming out, then there will be a corresponding f_2^i in G_2, where $f_2^i = \sum_j f_j^{i-1} r_j'$. The f_j^{i-1} are the same in each summation; the r_j are letters in the n^{th} coordinate in the label of a vertex of G that gets changed to r_j' by an edge numbered $n+1$. An example will help illuminate this.

Example 4.2.4 Consider the 3-full graph G in Figure 4.8.

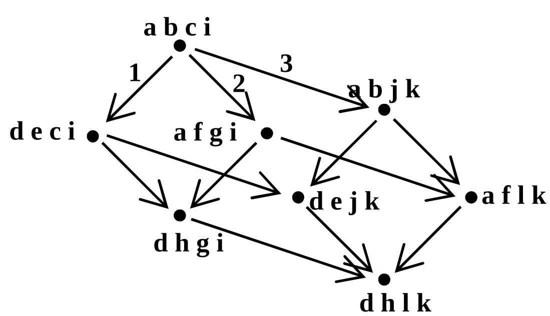


Figure 4.8: 3-full graph G in Example 4.2.4

Let Γ = quiver with one vertex and 12 loops. In resolving Λ_0, we set $\{f_i^0\}_i = \{\overline{v}\}$, $\{f_i^1\}_i$ = {**a** ... **l**}, $\{f_i^2\}_i = \mathcal{G}$ = {**ab** - **de**, **af** - **dh**, **bc** - **fg**, **ec** - **ig**, **bj** - **fl**, **ej** - **hl**, **ci** - **jk**, **gi** - **lk**}.

By Theorem 4.2.2 we find that $\{f_i^3\}_i$ = {(**ab** - **de**)**c** - (**af** - **di**)**g**, (**ab** - **de**)**j** - (**af** - **dh**)**l**, (**bc** - **fg**)**i** - (**bj** - **fl**)**k**, (**fc** - **hg**)**i** - (**ej** - **hl**)**k**}. Note that f_1^3 and f_2^3 are in the same components of $\coprod_i f_i^2 R$; only the coefficients of the $f^{2's}$ differ, as mentioned above.

We have an f^4 coming from the vertex where edges numbered 1,2, and 3 come out. We can find this f^4 using overlaps; there is an overlap between the tip of (**ab** - **df**)**c** - (**af** - **dh**)**g** in $\{f_i^3\}_i$ and **ci** - **jk** in $\mathcal{G}$. By following the procedure outlined in Theorem 4.2.1, we find that f_1^4 = ((**ab** - **de**)**c** - (**af** - **dh**)**g**)**i** - ((**ab** - **de**)**j** - (**af** - **dh**)**l**)**k** = f_1^3**i** - f_2^3**k**. This is the only element of $\{f_i^4\}_i$. □

Note that in taking the complete extension of the 2-cube, we got two f^3's coming from 1,2 cycles. The source then had edges numbered 1,2, and 3 coming from it, producing an f^4. This f^4 is the f^3 generated at the source by the 1,2 cycle multiplied by the last letter of the label of the source, minus the f^3 generated at the vertex in G_2 corresponding to the source multiplied by the last letter of the label of that vertex. This suggests a pattern, which is established in the following theorem.

Theorem 4.2.3 *Let G' be an n-full graph and let G be the complete extension of G', with isomorphic copies of G' in G labelled G_1 and G_2 as before. In finding a projective resolution of Λ_0, let all f's be found through overlap relations. Let $f \in \{f_i^j\}_i$ for some j be generated at a vertex* **v** *in G_1 by edges numbered consecutively $r \cdots s$, where $1 \le r \le s \le n$, and let f' be the corresponding element of $\{f_i^j\}_i$ that is generated at the corresponding vertex in G_2.*

1. *If $s \le n-1$, then $f = f'$*
2. *If $s = n$, then:*
   - (a) *$f = \sum_i f_i^{j-1}\mathbf{r}_i$ and $f' = \sum_i f_i^{j-1}\mathbf{r}'_i$, where the $\mathbf{r}'_i$ are the letters in the $n+1^{st}$ coordinate of labels of vertices in G_2 that correspond to $\mathbf{r}_i$, which are letters in the $n+1^{st}$ coordinate of vertices in G_1*
   - (b) *$f\mathbf{x} - f'\mathbf{y}$ is an element of $\{f_i^{j+1}\}_i$, where* **x** *is the letter in the $n+2^{nd}$ coordinate in the labels of all vertices in G_1 and* **y** *is the letter in the $n+2^{nd}$ coordinate in the labels of all vertices in G_2.*

Proof:

1 and 2(a) follow directly from the fact that G_1 is isomorphic to G_2 as numbered graphs and the fact that if **v** is a vertex in G_1, then its label differs from $\mathbf{v}'$ (the corresponding vertex in G_2) only in the $n+1$ and $n+2$ coordinates.

To prove 2(b), we show first that $f\mathbf{x} - f'\mathbf{y}$ is an element of $\coprod_i f_i^j R \cap \coprod_i f_i^{j-1} I$ (and thus can be taken to be an element of $\{f_i^{j+1}\}_i$), and then we show that it is the element that is generated by the overlaps created by edges numbered $r \cdots n+1$ coming from the vertex **v**.

By 2(a), $f\mathbf{x} - f'\mathbf{y} = (\sum_i f_i^{j-1}\mathbf{r}_i)\mathbf{x} - (\sum_i f_i^{j-1}\mathbf{r}'_i)\mathbf{y} = \sum_i f_i^{j-1}(\mathbf{r}_i\mathbf{x} - \mathbf{r}'_i\mathbf{y})$. $f\mathbf{x} - f'\mathbf{y}$ is in $\coprod_i f_i^j R$. Note that each element $\mathbf{r}_i\mathbf{x} - \mathbf{r}'_i\mathbf{y}$ in the sum $\sum_i f_i^{j-1}(\mathbf{r}_i\mathbf{x} - \mathbf{r}'_i\mathbf{y})$ is an element of $\mathcal{G}$ generated by an edge numbered $n+1$. Thus this sum is an element of $\coprod_i f_i^{j-1} I$, hence $f\mathbf{x} - f'\mathbf{y} \in \coprod_i f_i^j R \cap \coprod_i f_i^{j-1} I$.

We now show that this element is the element of $\{f_i^{j+1}\}_i$ that is generated by overlaps. $f = \sum_i f_i^{j-1}\mathbf{r}_i$; without loss of generality, renumber the f_i^{j-1} and $\mathbf{r}_i$ so that $\mathbf{r}_1$ is the letter in the $n+1^{st}$ coordinate of the vertex $\mathbf{v}$ (from Theorem 4.2.1 and the remarks following Corollary 4.2.1, $\mathbf{r}_1$ exists and $\text{Tip}(f) = \mathbf{r}_1$). By the construction of G, there exists an element $g = \mathbf{r}_1\mathbf{x} - \mathbf{r}'_1\mathbf{y}$ in $\mathcal{G}$, generated by an edge numbered $n+1$. Thus the tips of f and g generate an overlap; consider $h = f\mathbf{x} - f_1^{j-1}(\mathbf{r}_1\mathbf{x} - \mathbf{r}'_1\mathbf{y})$ as in the method of finding $\{f_i^{j+1}\}_i$ in Theorem 4.2.1.

$$\begin{aligned} h &= (\sum_{i\neq 1} f_i^{j-1}\mathbf{r}_i)\mathbf{x} + f_1^{j-1}\mathbf{r}_1\mathbf{x} - f_1^{j-1}\mathbf{r}_1\mathbf{x} + f_1^{j-1}\mathbf{r}'_1\mathbf{y} \\ &= \sum_{i\neq 1} f_i^{j-1}\mathbf{r}_i\mathbf{x} + f_1^{j-1}\mathbf{r}'_1\mathbf{y} \\ &= \sum_{i\neq 1} f_i^{j-1}(\mathbf{r}_i\mathbf{x}) - \sum_{i\neq 1} f_i^{j-1}(\mathbf{r}'_i\mathbf{y}) + \sum_{i\neq 1} f_i^{j-1}(\mathbf{r}'_i\mathbf{y}) + f_1^{j-1}\mathbf{r}'_1\mathbf{y} \\ &= \sum_{i\neq 1} f_i^{j-1}(\mathbf{r}_i\mathbf{x} - \mathbf{r}'_i\mathbf{y}) + f'\mathbf{y} \in \coprod_i f_i^{j-1}R \oplus \coprod_i f_i^{j-1*}R \end{aligned}$$

(We know from the proof of Theorem 4.2.1 that $h \in \coprod_i f_i^{j-1}R \oplus \coprod_i f_i^{j-1*}R$; we have now shown explicitly how it is represented as a sum in this set.)

Thus $f\mathbf{x} - f'\mathbf{y} = \sum_i f_i^{j-1}(\mathbf{r}_i\mathbf{x} - \mathbf{r}'_i\mathbf{y})$, and we can take $f\mathbf{x} - f'\mathbf{y}$ to be an element of $\{f_i^{j+1}\}_i$ that comes from an overlap. □

The benefit of this theorem is as follows. Let G' be an n-full graph with associated algebra Λ', and let all of the f's be known. Thus we have a minimal projective resolution

$$0 \to P'_{n+1} \to P'_n \to \cdots \to P'_1 \to P'_0 \to \Lambda'_0 \to 0.$$

Let G be the complete extension of G', and let Λ be an associated algebra. By Theorem 4.2.3 we may immediately find all the f's associated to a minimal projective resolution of Λ_0, and thus can immediately find all the projective modules in that resolution. An example follows.

Example 4.2.5 Consider the 3-full graph in Figure 4.9, previously shown in Figure 4.8.

This graph is the 3-cube; it is the complete extension of the 2-cube. Let G_1 be the 2-cube given by the edges numbered 1 and 2 at the source, and consider all the f's generated in a projective resolution of Λ'_0, for an associated algebra Λ'. $\{f_i'^0\}_i = \{v\}$ (where v are the vertices of some associated quiver Γ for G); $\{f_i'^1\}_i = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}\}$; $\{f_i'^2\}_i = \mathcal{G}' = \{\mathbf{ab}$ - $\mathbf{de}$, $\mathbf{af}$ - $\mathbf{dh}$, $\mathbf{bc}$ - $\mathbf{fg}$, $\mathbf{ec}$ - $\mathbf{hg}\}$; and $\{f_i'^3\}_i = \{(\mathbf{ab}$ - $\mathbf{de})\mathbf{c}$ - $(\mathbf{af}$ - $\mathbf{dh})\mathbf{g}\}$.

Now consider G_2, the other 2-cube in G with edges numbered 1 and 2. The labels of G_1 and G_2 are the same except in the third and fourth coordinates. In the third coordinate, the letter $\mathbf{c}$ in G_1 corresponds to the letter $\mathbf{j}$ in G_2, $\mathbf{g}$ corresponds to $\mathbf{l}$, and $\mathbf{d}$ corresponds to $\mathbf{k}$. We can immediately record the "new" elements of $\{f_i^1\}_i$ and $\{f_i^2\}_i$ by reading information off of the graph (here we will abuse notation a bit and regard $\{f_i'^1\}_i \subseteq \{f_i^1\}_i$, etc; $\{f_i^0\}_i$ will

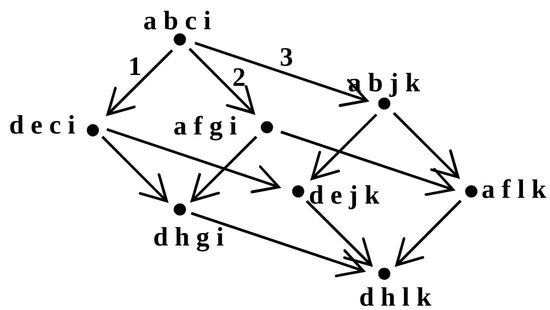


Figure 4.9: The 3-cube in Example 4.2.5

change based on the choice of the associated quiver Γ). We can find the new elements of $\{f_i^3\}_i$ in two ways; first, we can employ Theorem 4.2.2, which allows us to find the elements by reading from the graph without computation, or we can employ Theorem 4.2.3. We will use the latter method to demonstrate the usefullness of the theorem.

Consider $f_1^3 = (\mathbf{ab}$ - $\mathbf{de})\mathbf{c}$ - $(\mathbf{af}$ - $\mathbf{dh})\mathbf{g}$. Using 2a of Theorem 4.2.3, we replace **c** with **j** and **g** with **l**, giving $f_2^3 = (\mathbf{ab}$ - $\mathbf{de})\mathbf{j}$ - $(\mathbf{af}$ - $\mathbf{dh})\mathbf{l}$.

Consider **bc** - **fg** $\in \{f_i^2\}_i$; we can find its corresponding element in $\{f_i^2\}_i$ by making the same replacement, giving the element **bj** - **fl** $\in \{f_i^2\}_i$. Using 2b of Theorem 4.2.3, we see that (**bc** - **fg**)**i** - (**bj** - **fl**)**k** is an element of $\{f_i^3\}_i$. Finally, using this part of the theorem again, we find that $f_1^3\mathbf{d}$ - $f_2^3\mathbf{k}$ is the only element of $\{f_i^4\}_i$. These results match those found in Example 4.2.4, only with less computational effort. □

We have established the fact that when G is the complete extension of a full graph G', the projective resolution of Λ_0 is linked in a certain way to the projective resolution of Λ'_0. Recall from Chapter 2 that the complete extension of G' is just a special case of the general construction of a complete extension of G' by some full graph G''. This suggests that Theorem 4.2.3 is just a special case of a more general result. If $G = G'\Box_C G''$ for full graphs G' and G'', and the projective resolutions of Λ'_0 and Λ''_0 are known (i.e., all respective f's are known), can we find a full projective resolution of Λ_0? The answer is yes, and the following theorem shows how this can be accomplished.

We make one comment concerning the following proof. We have shown how to generate the elements $f \in \{f_i^k\}_i$ through overlap relations that correspond to a vertex in the graph having $k-1$ edges numbered consecutively coming out. Assume some f corresponds to a vertex with edges numbered $1 \cdots k-1$ coming out. We also know that $f = \sum_i f_i^{k-1}\mathbf{r}_i$ for some elements $f_i^{k-1} \in \{f_i^{k-1}\}_i$. It should seem plausible, and indeed it is true, that the elements f_i^{k-1} are generated at vertices with edges numbered $1 \cdots k-2$ coming out. The truth of this statement will be shown in Chapter 5, without dependance on this following theorem.

Theorem 4.2.4 *Let G' be an n-full graph, let G'' be an m-full graph, let $G = G'\Box_C G''$, and* *let* $\mathbf{v}$ *be a vertex in* G *which is the source of edges numbered consecutively* $h, \cdots, n, \cdots, n+k$ *for some* $k \geq 1$*. Denote by* $f_{\mathbf{v}}^{k+1}$ *the element of* $\{f_i^{k+1}\}_i$ *generated by the edges numbered* $n+1, \cdots, n+k$ *at* $\mathbf{v}$*;* $f_{\mathbf{v}}^{k+1} = \sum_{\mathbf{i}} f_{\mathbf{i} \in \mathcal{I}}^{k} \mathbf{r_i}$ *where each* $f_{\mathbf{i}}^{k}$ *is generated at vertex* $\mathbf{i}$ *by edges numbered* $n+1, \cdots, n+k-1$*. Let* $f_{\mathbf{i}}^{n+k-h+1}$ *denote the elements of* $\{f_i^{n+k-h+1}\}_i$ *generated at vertex* $\mathbf{i}$ *by edges numbered* $h, \cdots, n+k-1$ *for all* $\mathbf{i} \in \mathcal{I}$*, and let* $f_{\mathbf{v}}^{n+k-h+2}$ *denote the element of* $\{f_i^{n+k-h+2}\}_i$ *generated by the edges numbered* $h, \cdots, n+k$ *coming from vertex* $\mathbf{v}$*. The elements* $f_{\mathbf{v}}^{n+k-h+2} \in \{f_i^{n+k-h+2}\}_i$ *are determined by* $f_{\mathbf{v}}^{n+k-h+2} = \sum_{\mathbf{i} \in \mathcal{I}} f_{\mathbf{i}}^{n+k-h+1} \mathbf{r_i}$*.*

Proof:

We will proceed by induction on k; without loss of generality we set $h = 1$ to make the notation simpler. Assume vertex $\mathbf{v}$ is the source for edges numbered $1, \cdots, n+1$. We seek the element $f_{\mathbf{v}}^{n+2} \in \{f_i^{n+2}\}_i$ generated at this vertex by these edges. The edge numbered $n+1$, $\overrightarrow{e_{\mathbf{vw}}}$, generates an element $f_{\mathbf{v}}^2 = \mathbf{ab}\text{-}\mathbf{cd}$ of $\{f_i^2\}_i$. ($f_{\mathbf{v}}^2$ corresponds to $f_{\mathbf{v}}^{k+1}$ given in the statement of the theorem.) $f_{\mathbf{v}}^2 = \sum_{i=1}^{2} f_i^1 \mathbf{r}_i$, where $f_1^1 = \mathbf{a}$ and $f_2^1 = \mathbf{c}$ (also, $\mathbf{r}_1 = \mathbf{b}, \mathbf{r}_2 = -\mathbf{d}$; again these $f_{\mathbf{i}}^1$ correspond to the $f_{\mathbf{i}}^k$ as stated in the theorem).

By construction, the vertex $\mathbf{w}$ is the source of edges numbered $1 \cdots n$. Thus there exists the element $f_{\mathbf{w}}^{n+1} \in \{f_i^{n+1}\}_i$ generated by these edges. We also have an element $f_{\mathbf{v}}^{n+1}$ generated by the edges numbered $1 \cdots n$ at $\mathbf{v}$. (These two $f_{\mathbf{i}}^{n+1}$ correspond to the $f_{\mathbf{i}}^{n+k-h+1}$ in the statement of the theorem.) By Theorem 4.2.3, $f_{\mathbf{v}}^{n+1}\mathbf{b} - f_{\mathbf{w}}^{n+1}\mathbf{d}$ is the element of $\{f_i^{n+2}\}_i$ that is generated by the overlap relations at the vertex $\mathbf{v}$, establishing the base case.

Now assume the hypothesis is true for $k \geq 1$; that is, let vertex $\mathbf{v}$ be the source of edges numbered $1 \cdots n+k$ for some $k \geq 1$. Let the edges numbered $n+1 \cdots n+k$ generate $f_{\mathbf{v}}^{k+1} \in \{f_i^{k+1}\}_i$, where $f_{\mathbf{v}}^{k+1} = \sum_{\mathbf{i} \in \mathcal{I}} f_{\mathbf{i}}^k \mathbf{r_i}$, where each $f_{\mathbf{i}}^k$ is generated at vertex $\mathbf{i}$ by edges numbered $n+1 \cdots n+k-1$. By construction, each of these vertices $\mathbf{i}$ is also the source of edges numbered $1 \cdots n+k-1$, thus each $\mathbf{i}$ generates an element $f_{\mathbf{i}}^{n+k} \in \{f_i^{n+k}\}_i$ from these edges. We assume as the inductive hypothesis that the element $f_{\mathbf{v}}^{n+k+1} \in \{f_i^{n+k+1}\}_i$ generated by the edges numbered $1 \cdots n+k$ at $\mathbf{v}$ is equal to $\sum_{\mathbf{i} \in \mathcal{I}} f_{\mathbf{i}}^{n+k} \mathbf{r_i}$.

Now consider vertex $\mathbf{v}$ with edges numbered $1 \cdots n+k+1$ coming out. Let $f_{\mathbf{v}}^{k+2}$ denote the element in $\{f_i^{k+2}\}_i$ coming from the edges numbered $n+1 \cdots n+k+1$.

$$f_{\mathbf{v}}^{k+2} = \sum_{\mathbf{i} \in \mathcal{I}} f_{\mathbf{i}}^{k+1} \mathbf{r_i} = \sum_{\mathbf{i}} \sum_{\mathbf{j} \in \mathcal{J}} f_{\mathbf{j}}^k \mathbf{r_{i,j}} \mathbf{r_i} = \sum_{\mathbf{j}} f_{\mathbf{j}}^k \sum_{\mathbf{i}} \mathbf{r_{i,j}} \mathbf{r_i} = \sum_{\mathbf{j}} f_{\mathbf{j}}^k g_{\mathbf{j}}$$

where $\sum_{\mathbf{i}} \mathbf{r_{i,j}} \mathbf{r_i} = g_{\mathbf{j}} \in \mathcal{G}$. Each $f_{\mathbf{i}}^{k+1}$ is generated by edges numbered $n+1 \cdots n+k$; each $f_{\mathbf{j}}^k$ is generated by edges numbered $n+1 \cdots n+k-1$.

Let $f_{\mathbf{i}}^{n+k+1}$ denote the elements of $\{f_i^{n+k+1}\}_i$ generated by the edges numbered $1 \cdots n+k$ at the vertices $\mathbf{i} \in \mathcal{I}$. Set $f_{\mathbf{v}}^{n+k+2} = \sum_{\mathbf{i} \in \mathcal{I}} f_{\mathbf{i}}^{n+k+1} \mathbf{r_i}$. We wish to show that $f_{\mathbf{v}}^{n+k+2} \in \{f_i^{n+k+2}\}_i$. By the inductive hypothesis and the previous paragraph, we have

$$\sum_{\mathbf{i} \in \mathcal{I}} f_{\mathbf{i}}^{n+k+1} \mathbf{r_i} = \sum_{\mathbf{i}} \sum_{\mathbf{j} \in \mathcal{J}} f_{\mathbf{j}}^{n+k} \mathbf{r_{i,j}} \mathbf{r_i} = \sum_{\mathbf{j}} f_{\mathbf{j}}^{n+k} \sum_{\mathbf{i}} \mathbf{r_{i,j}} \mathbf{r_i} = \sum_{\mathbf{j}} f_{\mathbf{j}}^{n+k} g_{\mathbf{j}}.$$

Thus $f_{\mathbf{v}}^{n+k+2} \in \coprod_i f_i^{n+k+1} R \cap \coprod_i f_i^{n+k} I$, so it can be taken to be an element of $\{f_i^{n+k+2}\}_i$. We now show that it is the element that is generated at the vertex $\mathbf{v}$ via overlaps.

We know that $f_{\mathbf{v}}^{k+2} = \sum_{\mathbf{i}} f_{\mathbf{i}}^{k+1}\mathbf{r_i} = \sum_{\mathbf{j}} f_{\mathbf{j}}^{k} g_{\mathbf{j}}$. In finding $f_{\mathbf{v}}^{k+2}$ through overlaps, we considered the element $h = f_{\mathbf{v}}^{k+1}\mathbf{r_v} - f_{\mathbf{v}}^{k} g_{\mathbf{v}}$, which we determined was equal to $\sum_{\mathbf{i}\neq\mathbf{v}} f_{\mathbf{i}}^{k+1}\mathbf{r_i} + \sum_{\mathbf{j}\neq\mathbf{v}} f_{\mathbf{j}}^{k} g_{\mathbf{j}} \in \coprod_i f_i^{k+1}R \cap \coprod_i f_i^{k+1*}R$ by Theorem 4.2.1. Thus $h' = f_{\mathbf{v}}^{n+k+1}\mathbf{r_v} - f_{\mathbf{v}}^{n+k} g_{\mathbf{v}} = \sum_{\mathbf{i}\neq\mathbf{v}} f_{\mathbf{i}}^{n+k+1}\mathbf{r_i} + \sum_{\mathbf{j}\neq\mathbf{v}} f_{\mathbf{j}}^{n+k} g_{\mathbf{j}} \in \coprod_i f_i^{n+k+1}R \cap \coprod_i f_i^{n+k+1*}R$, and so we conclude that $f_{\mathbf{v}}^{n+k+2}$ is the element of $\{f_i^{k+2}\}_i$ generated by overlaps by edges numbered $1 \cdots n+k+1$ at the vertex $\mathbf{v}$. $\square$

Chapter 5

The Canonical Quiver

5.1 Definition of the Canonical Quiver

We defined in Chapter 3 the concept of an associated quiver to a full graph G. We noted there that several nonisomorphic quivers may be associated to G, and so far our results have been developed independant of the choice of the associated quiver.

Given a full graph G, there are two quivers that are of particular interest. The first, as described before, is the associated quiver Γ_{F} with one vertex and m loops, where $|X_G| = m$ (recall that X_G is the set of letters used in the labels of G). Γ_{F} is of interest since we can establish all the results of this dissertation up to this point independant of path algebras.

This quiver is, in a certain sense, the smallest among the associated quivers for G in that it has only one vertex (all associated quivers have the same number of arrows). It is characterized by all arrows having the same origin and terminus. We now look at the other extreme.

Consider an associated quiver Γ_{C} with the following property: if $\mathbf{a}$ and $\mathbf{b}$ are arrows in Γ_{C}, then $o(\mathbf{a}) \neq o(\mathbf{b})$, $o(\mathbf{a}) \neq t(\mathbf{b})$, $t(\mathbf{a}) \neq o(\mathbf{b})$, and $t(\mathbf{a}) \neq t(\mathbf{b})$ unless equality is required by $\mathcal{G}$ for Γ_{C} to be an associated quiver. (In this chapter we will continue to denote both the letter $\mathbf{a}$ in a label of G and the arrow $\overrightarrow{\mathbf{a}}$ in an associated quiver Γ as $\mathbf{a}$.)

The following lemma establishes some properties of such an associated quiver.

Lemma 5.1.1 *Let Γ_C be an associated quiver with the above property.*

1. *Γ_C is unique (up to isomorphism).*

2. *If Γ is an associated quiver $\ncong \Gamma_C$, then Γ_C has more vertices than Γ.*

Proof:

First, consider two quivers $\hat{\Gamma}$ and Γ' with the above property. If $o(\hat{\mathbf{a}}) = o(\hat{\mathbf{b}})$ in $\hat{\Gamma}$, then by our assumption the structure of $\mathcal{G}$ dictated this equality. Thus $o(\mathbf{a}') = o(\mathbf{b}')$ in Γ'. Likewise, if $o(\hat{\mathbf{a}}) \neq o(\hat{\mathbf{b}})$ in $\hat{\Gamma}$, then $o(\mathbf{a}') \neq o(\mathbf{b}')$ in Γ'. This shows that $o(\hat{\mathbf{a}}) = o(\hat{\mathbf{b}})$ if, and only if, $o(\mathbf{a}') = o(\mathbf{b}')$.

We can use the same line of reasoning to state the following: $o(\hat{\mathbf{a}}) = t(\hat{\mathbf{b}}) \Leftrightarrow o(\mathbf{a}') = t(\mathbf{b}')$, $t(\hat{\mathbf{a}}) = o(\hat{\mathbf{b}}) \Leftrightarrow t(\mathbf{a}') = o(\mathbf{b}')$, and $t(\hat{\mathbf{a}}) = t(\hat{\mathbf{b}}) \Leftrightarrow t(\mathbf{a}') = t(\mathbf{b}')$.

Define a ring map $\phi\colon \hat{\Gamma} \to \Gamma'$ by $\phi(\hat{\mathbf{p}}) = \mathbf{p}'$ for all paths $\hat{\mathbf{p}}$ in $\hat{\Gamma}$ (i.e., $\phi(\hat{\mathbf{p}}_1\hat{\mathbf{p}}_2 \cdots \hat{\mathbf{p}}_n) = \mathbf{p}'_1\mathbf{p}'_2 \cdots \mathbf{p}'_n$). By the previous paragraph, it is easy to see that this map is well defined and defines an isomorphism of associated quivers. Thus we have established, up to isomorphism, the uniqueness of Γ_{C}.

Now let $\hat{\Gamma}$ be a quiver with more vertices than Γ_{C}. Since $\hat{\Gamma}$ has more vertices, there exist arrows $\mathbf{a}$ and $\mathbf{b}$ such that, without loss of generality, $o(\mathbf{a}) = o(\mathbf{b})$ in Γ_{C} but $o(\hat{\mathbf{a}}) \neq o(\hat{\mathbf{b}})$ in $\hat{\Gamma}$. We know from our hypothesis that the relations of $\mathcal{G}$ forced $o(\mathbf{a}) = o(\mathbf{b})$. Since $o(\hat{\mathbf{a}}) \neq o(\hat{\mathbf{b}})$ in $\hat{\Gamma}$, $\hat{\Gamma}$ is not an associated quiver.

Consider a quiver $\hat{\Gamma}$ that has the same number of vertices as Γ_{C}, where $\hat{\Gamma} \ncong \Gamma_C$. Since $\hat{\Gamma} \ncong \Gamma_C$, we can again without loss of generality find arrows $\mathbf{a}$ and $\mathbf{b}$ such that $o(\mathbf{a}) = o(\mathbf{b})$ in Γ_{C} but $o(\hat{\mathbf{a}}) \neq o(\hat{\mathbf{b}})$ in $\hat{\Gamma}$. By the same argument as before, $\hat{\Gamma}$ is not an associated quiver.

Thus Γ_{C} is unique and has the most vertices of all associated quivers (up to isomorphism). □

Since this quiver Γ_{C} is unique, we give it a special name.

Definition 5.1.1 *Let G be a full graph. The associated quiver Γ_C with maximum number of vertices is the* canonical quiver*.*

We now give some more properties of the canonical quiver.

Lemma 5.1.2 *Let G be an n-full graph with edges numbered i for all i in $1 \cdots n$.*

1. *Γ_C has a unique source and sink.*
2. *Let $\mathbf{x}$ be a letter in the i^{th} coordinate in a vertex label of G. In all paths from the source of Γ_C that include the arrow $\mathbf{x}$, $\mathbf{x}$ is the i^{th} arrow of the path.*

Proof:

Consider the letters in the first coordinate in the labels of the vertices of G; they are changed only by edges numbered 1. This directly gives the fact that in Γ_{C} the arrows that correspond to these letters will have the same origin $\mathbf{v}$. Since in $\mathcal{G}$ no letter precedes these letters in any relation, there does not exist any arrow $\mathbf{a}$ with $t(\mathbf{a}) = \mathbf{v}$. Thus $\mathbf{v}$ is a source in Γ_{C}.

We now show that $\mathbf{v}$ is the only source in Γ_{C}. We will do this by showing that all arrows $\mathbf{a}_i$ in Γ_{C} corresponding to letters in the i^{th} coordinate of a label in G will have origin $o(\mathbf{a}_i) = \mathbf{v}_i$, where $\mathbf{v}_i$ is the terminus of some arrow $\mathbf{a}_{i-1}$ corresponding to a letter in the $i-1$ coordinate of a label, where $2 \leq i \leq n+1$.

Without loss of generality, assume there exists an arrow $\mathbf{b}$ in Γ_{C} where the letter $\mathbf{b}$ is in the second coordinate in a label in G, and that $\mathbf{w} = o(\mathbf{b})$ is not the terminus of any arrow that comes from a letter in the first coordinate. Since clearly no arrow coming from a letter in the i^{th} coordinate, $i \geq 2$, will have **w** as its terminus, we assume that **w** is another source in Γ_{C}.

Now, **b** cannot be in the label of any vertex that is the origin or terminus of an edge numbered 1, for then **w** would be the terminus of an arrow coming from the first coordinate. Thus **b** is in the label of vertices that are the sources or termini of edges numbered ≥ 2. Since edges numbered ≥ 3 do not affect the letters in the second coordinate, we consider only the edges numbered 2.

Certainly **b** is in the label of a vertex which is either the origin or terminus of an edge numbered 2. Without loss of generality **b** is in the label of the origin; this edge produces the element **bc-de** in $\mathcal{G}$. By the definition of an associated quiver, we know that **b** and **d** have the same origin in Γ_{C}.

Thus **d** cannot be in the label of any vertex which is the origin or terminus of any edge numbered 1. If **d** is in the origin (terminus) of any other edge numbered 2, then in Γ_{C} **d** will have the same origin as the arrow coming from the letter **d** changed to (changed from) across the edge numbered 2, which again is the same origin as **b**. Thus we can find a directed path from any vertex labelled with **b** in G to the sink of G wherein no edge is numbered 1. Let **y** denote the letter in the second coordinate of the label of the sink of G. By our previous statement, in Γ_{C}, $o(\mathbf{b}) = o(\mathbf{y})$.

Let **a** be the smallest letter found in the first coordinate in G. Thus **a** is in the label of some vertex **x** in G which is the terminus of an edge numbered 1; there exists a directed path in G from **x** to the sink in which no edges are numbered 1. **a** is in a relation **qr-as**; in Γ_C, $t(\mathbf{a}) = o(\mathbf{s})$. But since the path from **x** to the sink contained no edges numbered 1, we know as before that $t(\mathbf{a}) = o(\mathbf{s}) = o(\mathbf{y}) = o(\mathbf{b})$, contradicting our assumption that $o(\mathbf{b})$ was a source.

Thus our assumption is false, and for all arrows $\mathbf{a}_i$ coming from letters in the i^{th} coordinate, $o(\mathbf{a}_i) = t(\mathbf{a}_{i-1})$ for some arrow $\mathbf{a}_{i-1}$ coming from a letter in the $i-1$ coordinate in G, where $2 \leq i \leq n+1$. Thus **v** is the unique source. This also establishes the second statement of the lemma.

A similar argument can be used to now show the existence and uniqueness of the sink of Γ_{C}. □

We will use the term i^{th} *level* to describe the arrows of Γ_{C} that come from letters in the i^{th} coordinate of a label in G. This term will be usefull later; in the examples that follow one can get a visual perspective of why the word "level" is used. Let $(\Gamma_{\mathrm{C}})_0$ denote the set of vertices of Γ_{C}. Given i, the set $\{\mathbf{v} \in (\Gamma_{\mathrm{C}})_0 \mid \mathbf{v}$ is the origin of an arrow in the i^{th} level of $\Gamma_{\mathrm{C}}\}$ is the set of sources of the i^{th} level; we similarly define the set of sinks.

Example 5.1.1 We give an few examples of full graphs and their canonical quivers.

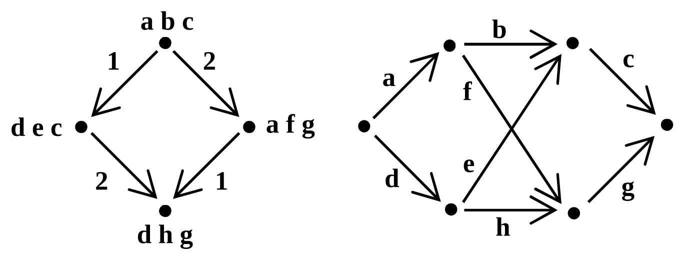


Figure 5.1: The 2-cube and its canonical quiver

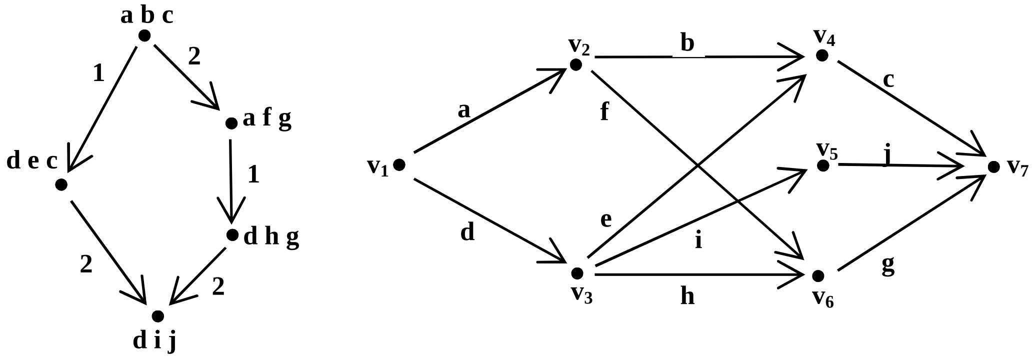


Figure 5.2: A 2-full graph and its canonical quiver

□

5.2 The Canonical Quiver and the Resolution of Simple Modules

Let an associated triple (K, G, Γ_C) be given and let Λ be the associated algebra. For each vertex $\mathbf{v} \in (\Gamma_C)_0$, there exists a simple Λ-module $S_{\mathbf{v}} = \overline{\mathbf{v}}\Lambda_0$ which is isomorphic to a copy of K, where $\overline{\mathbf{v}}$ denotes the image of $\mathbf{v}$ in Λ. $\Lambda_0 = \coprod_{\mathbf{v} \in (\Gamma_C)_0} \overline{\mathbf{v}}\Lambda_0$. In chapter 4 we investigated finding a minimal projective resolution of Λ_0; we can apply the same techniques to the problem of finding minimal projective resolutions of each of these vertex simple modules $S_{\mathbf{v}}$.

Consider the case where $\mathbf{v}$ is the unique source of Γ_C; we seek the minimal projective resolution

$$0 \to P_{n+1} \to P_n \to \cdots \to P_{\hat{\mathbf{v}}} \to P_0 \to S_{\mathbf{v}} \to 0$$

using the techniques developed earlier. Again let $R = K\Gamma_C$.

We set $\{f_i^0\}_i = \{\mathbf{v}\}$, so $P_0 = \mathbf{v}R/\mathbf{v}I \cong \overline{\mathbf{v}}\Lambda$. Consider the R-module homomorphism $\phi \colon \mathbf{v}R \to S_{\mathbf{v}}$ where $\phi(v) = 1$. $\mathrm{Ker}(\phi) = \coprod_i \mathbf{x}_i R$ as the $\mathbf{x}_i$ range over the arrows in Γ_C where $o(\mathbf{x}_i) = \mathbf{v}$. In other words, the $\mathbf{x}_i$ range over the arrows which come from letters in the first coordinate of the labels of G. Thus $\{f_i^1\}_i = \{$arrows with origin $\mathbf{v}\} = \{$arrows from letters in the first coordinate of labels of $G\}$.

We next seek $\{f_i^2\}_i$. We know that in resolving Λ_0 that $\{f_i^2\}_i = \mathcal{G}$; so in resolving a summand $S_\mathbf{v}$ of Λ_0 we expect our set $\{f_i^2\}_i$ to be a subset of $\mathcal{G}$. Considering the recursive defintion of $\{f_i^2\}_i$ and that $o(\mathbf{x}_i) = \mathbf{v}$ for all $\mathbf{x}_i \in \{f_i^1\}_i$, we conclude $\{f_i^2\}_i = \{g \in \mathcal{G} \mid \mathbf{v}g = g\}$. This correlates to precisely the relations of G that come from arrows numbered 1.

In finding $\{f_i^3\}_i$, we look at overlap relations between elements of $\{f_i^2\}_i$ and elements of $\mathcal{G}$; since $\{f_i^2\}_i$ are elements coming from edges numbered 1, in G we look for overlap relations generated by edges numbered 1 and 2 coming from a vertex. As we continue to find all the sets $\{f_i^n\}_i$, we see that each $f \in \{f_i^m\}_i$ will be generated at a vertex with edges numbered $1...m-1$ coming out.

This can be generalized for all vertices in Γ_C. Let $\mathbf{v}$ be a vertex which is a source in the j^{th} level of Γ_C. In finding a minimal projective resolution of $S_\mathbf{v}$, we set $\{f_i^0\}_i = \{\mathbf{v}\}$, $\{f_i^1\}_i = \{$arrows $\mathbf{x}$ in Γ_C with $o(\mathbf{x}) = \mathbf{v}\}$ and $\{f_i^2\}_i = \{g \in \mathcal{G} \mid \mathbf{v}g = g\}$. Note that all elements of $\{f_i^1\}_i$ will come from letters in the j^{th} coordinate of the labels of G, and hence all elements of $\{f_i^2\}_i$ will come from edges numbered j. The elements of $\{f_i^3\}_i$ will come from overlaps generated at vertices with edges numbered j and $j+1$ coming out; in general, the elements of $\{f_i^m\}_i$ will come from vertices with edges numbered $j \cdots j+m-2$ coming out.

This leads to a key observation first stated without explanation in chapter 4. In resolving Λ_0 using the techniques described in this dissertation, if $f^k = \sum_i f_i^{k-1}\mathbf{r}_i$ is generated at a vertex by edges numbered $i \cdots i+k-2$, then each f^{k-1} is generated at a vertex by edges numbered $i \cdots i+k-3$. The results of chapter 4 were found without regard to the specific structure of the associated quiver chosen; i.e., we developed techniques to find the sets $\{f_i^n\}_i$ without regard to what associated quiver was providing the underlying structure of our algebra. This key observation uses the specific structure of the canonical quiver to find information about the sets $\{f_i^n\}_i$; since these sets are the same regardless of the associated quiver used, this observation holds for all associated quivers.

Example 5.2.1 Consider the 3-full graph below and its corresponding canonical quiver.

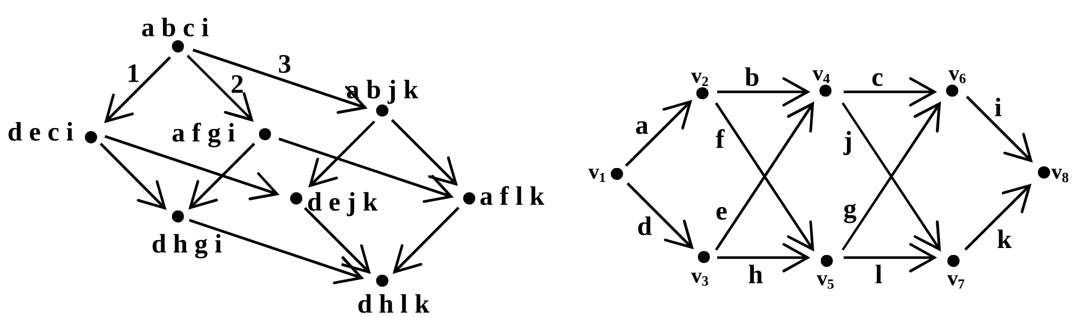


Figure 5.3: The 3-cube and its canonical quiver

In resolving $S_{\hat{\mathbf{v}}}$, $\{f_i^0\}_i = \{\mathbf{v}_1\}$, $\{f_i^1\}_i = \{\mathbf{a}, \mathbf{d}\}$, $\{f_i^2\}_i = \{\mathbf{ab} - \mathbf{de}, \mathbf{af} - \mathbf{dh}\}$, $\{f_i^3\}_i =$

$\{(\mathbf{ab}-\mathbf{de})\mathbf{c}-(\mathbf{af}-\mathbf{dh})\mathbf{g}, (\mathbf{ab}-\mathbf{de})\mathbf{j}-(\mathbf{af}-\mathbf{dh})\mathbf{l}\}$, $\{f_i^4\}_i = \{((\mathbf{ab}-\mathbf{de})\mathbf{c}-(\mathbf{af}-\mathbf{dh})\mathbf{g})\mathbf{i}-((\mathbf{ab}-\mathbf{de})\mathbf{j}-(\mathbf{af}-\mathbf{dh})\mathbf{l})\mathbf{k}\}$. In resolving $S_{\mathbf{v}'}$, we find $\{f_i^0\}_i = \{\mathbf{v_2}\}$, $\{f_i^1\}_i = \{\mathbf{b}, \mathbf{f}\}$, $\{f_i^2\}_i = \{\mathbf{bc}-\mathbf{fg}, \mathbf{bj}-\mathbf{fl}\}$, $\{f_i^3\}_i = \{(\mathbf{bc}-\mathbf{fg})\mathbf{i}-(\mathbf{bj}-\mathbf{fl})\mathbf{k}\}$. In resolving S_6, we have $\{f_i^0\}_i = \{\mathbf{v}_6\}$, $\{f_i^1\}_i = \{\mathbf{i}\}$ and in resolving S_8 all we have is $\{f_i^0\}_i = \{\mathbf{v}_8\}$.

5.3 Graph Constructions

Note the similarities in the construction of the canonical quivers of the 2-cube and 3-cube in Examples 5.1.1 and 5.2.1. By mere inspection, it seems that these quivers are somehow related; this section is devoted to studying this relationship.

Let G be an n-full graph with canonical quiver Γ_{C}, let G' be the complete extension of G with canonical quiver Γ'_{C}, and let Λ and Λ' be the respective associated algebras. As discussed earlier, we can view G' as having two isomorphic copies G_1 and G_2 of G inside connected by edges numbered $n+1$ from G_1 to G_2. G_1 can be given a vertex labelling in which the i^{th} letter in the label of a vertex $\mathbf{v}$ in G is the same as the i^{th} letter in the label of the corresponding vertex $\mathbf{v}'$ in G_1, for $i = 1 \cdots n+1$. The labels in G_2 differ from those in G_1 only in the $n+1$ and $n+2$ coordinates. Thus the relations of G are the same as those relations of G' generated in G_1; in some sense we can view the relations of G as a subset of the relations of G'. Also, there is a one to one correspondence between the relations of G and the relations of G' coming from G_2; the relations are identical for those edges numbered $\leq n-1$, and for a relation **ab-cd** generated by an edge numbered n in G, the relation $\mathbf{ab}'-\mathbf{cd}'$ will be generated in G_2 by the corresponding edge, where $\mathbf{b}'$ is the letter in the $n+1^{st}$ coordinate that $\mathbf{b}$ is changed to across an edge numbered $n+1$.

So from a purely combinatorial standpoint, Γ_{C} will be identical to Γ'_{C} up to the n^{th} level. At the $n+1$ level, Γ'_{C} will have two sinks. One sink $\mathbf{v}$ will be "identical" to the sink of Γ_{C} with "the same" arrows pointing to it. The other sink $\hat{\mathbf{v}}$ will be the sink of the new arrows created from the letters in the $n+1$ coordinate of the labels of vertices in G_2. The relation $\mathbf{ab}-\mathbf{cd}$ in G (i.e., in G_1) generated by an edge numbered n will again correspond to a relation $\mathbf{ab}'-\mathbf{cd}'$ generated by an edge numbered n in G_2; in Γ'_{C} $t(\mathbf{b}) = t(\mathbf{d}) = \mathbf{v}$; $t(\mathbf{b}') = t(\mathbf{d}') = \hat{\mathbf{v}}$; no relation will exist that will force $\mathbf{v} = \hat{\mathbf{v}}$, for no relation will exist that will force $t(\mathbf{b}) = t(\mathbf{b}')$, etc.

Thus in some sense the $n+1$ level of Γ'_{C} will contain two copies of the $n+1$ level of Γ_{C}. Finally, the edges numbered $n+1$ in G' will generate relations $\mathbf{bx}-\mathbf{b}'\mathbf{y}$ for each pair of letters $\mathbf{b}, \mathbf{b}'$ in the $n+1$ level of Γ'_{C}, where $\mathbf{x}$ and $\mathbf{y}$ are the letters in the $n+2$ coordinate of the labels of vertices in G_1 and G_2, respectively.

Example 5.3.1 Consider the 2-cube in Figure 5.1 and the 3-cube in Figure 5.3. □

The significance of all this is that we have established a connection between a construction on full graphs and a combinatorial construction on quivers. This is not entirely satisfying, since the corresponding path algebra is a ring and we have created a new ring by merely adding arrows and relations to the quiver in a systematic way.

We would like for this construction on full graphs to correspond to a more algebraic construction on the associated algebras; i.e., we seek an algebraic construction that transforms Λ to Λ'. We develop this construction in the context of matrix algebras.

Let $\Lambda = K\Gamma/\langle\rho\rangle$ for some quiver Γ and set of relations ρ. Let e be an idempotent of Λ; as a K-vector space Λ decomposes as $e\Lambda e \oplus e\Lambda(1-e) \oplus (1-e)\Lambda e \oplus (1-e)\Lambda(1-e)$, where $e\Lambda e$ and $(1-e)\Lambda(1-e)$ are also K-algebras, $(1-e)\Lambda e$ is a $(1-e)\Lambda(1-e)$-$e\Lambda e$-bimodule, and $e\Lambda(1-e)$ is a $e\Lambda e$-$(1-e)\Lambda(1-e)$-bimodule. We can recover the ring structure of Λ by considering the matrix ring $\begin{pmatrix}(1-e)\Lambda(1-e) & (1-e)\Lambda e\\ e\Lambda(1-e) & e\Lambda e\end{pmatrix}$. Elements of this matrix ring are 2×2 matrices of the form $\begin{pmatrix}\lambda & m\\ m' & \lambda'\end{pmatrix}$, where $\lambda \in (1-e)\Lambda(1-e)$, $m \in (1-e)\Lambda e$, $m' \in e\Lambda(1-e)$ and $\lambda' \in e\Lambda e$. Multiplication and addition are given by ordinary matrix operations. It is not hard to show that $\Lambda \cong \begin{pmatrix}(1-e)\Lambda(1-e) & (1-e)\Lambda e\\ e\Lambda(1-e) & e\Lambda e\end{pmatrix}$ as K-algebras.

Consider the special case where $e = \overline{\mathbf{v}}$, where $\mathbf{v}$ is a sink in Γ. Then $e\Lambda e \cong K$ and $e\Lambda(1-e) = 0$ for no paths begin at $\mathbf{v}$. The $(1-e)\Lambda(1-e)$-$e\Lambda e$-bimodule $(1-e)\Lambda e$ gives information about the images of paths that begin in $\Gamma_0 - \mathbf{v}$ (which corresponds to $(1-e)\Lambda(1-e)$) and end at $\mathbf{v}$.

We can also go in the other direction. Let $\Lambda = K\Gamma/\langle\rho\rangle$ and let M be a left Λ-module (hence M is a Λ-K-bimodule.) Consider the matrix ring $\Lambda' = \begin{pmatrix}\Lambda & M\\ 0 & K\end{pmatrix}$. Λ' can be viewed as the quotient of a path algebra $K\Gamma'/\langle\rho'\rangle$, where Γ' is a quiver constructed by adding a vertex, which is a sink, to Γ with arrows defined by M and $\rho' \supseteq \rho$ is a set of relations also determined by M. Λ' is said to be a *one-point extension* of Λ by the bimodule ${}_\Lambda M_K$, since the underlying quiver of Λ was "extended" by one "point." (Note: one-point extensions are explained in a more general context in [1]. There the terminology of a one-point extension is defined in terms of adding a source to a quiver; we have modified things here to suit the purposes of this dissertation.)

Now revisit the problem of extending G to get a new graph G' and analyzing the structure of Γ'_{C} from Γ_{C}. In light of combinatorial explanation of the construction of Γ'_{C}, we see that Λ' is constructed by two one-point extensions of Λ; $\hat{\Lambda} = \begin{pmatrix}\Lambda & M_1\\ 0 & K\end{pmatrix}$ is the algebra by adding one vertex to Γ_{C} with the corresponding relations determined by M_1, and $\Lambda' = \begin{pmatrix}\hat{\Lambda} & M_2\\ 0 & K\end{pmatrix}$ is the one-point extension of $\hat{\Lambda}$, with arrows and relations defined by M_2. The underlying quiver of $\hat{\Lambda}$ will be denoted $\hat{\Gamma}$; Γ'_{C} will still denote the underlying quiver of Λ'.

Knowing Λ', we can work backwards to find M_1 and M_2, but it is more useful to determine M_1 and M_2 knowing only Λ and the type of graph construction being applied to G.

In order to determine M_1 and M_2 we employ the theory of representations of quivers [1]. A representation (V, f) of a quiver Γ over a field K is a set of K-vector spaces $\{V_{\mathbf{i}} \mid \mathbf{i} \in \Gamma_0\}$ and K-linear maps $\{f_{\mathbf{a}} \colon V_{\mathbf{j}} \to V_{\mathbf{i}} \mid \mathbf{a} \colon \mathbf{i} \to \mathbf{j}$ is an arrow in Γ from vertex $\mathbf{i}$ to vertex $\mathbf{j}$ $\}$.

Assume each vector space is finite dimensional over K. The composition of these vector space maps is defined in the usual way. Let maps $f_{\mathbf{a}}\colon V_{\mathbf{j}} \to V_{\mathbf{i}}$ and $f_{\mathbf{b}}\colon V_{\mathbf{i}} \to V_{\mathbf{k}}$ be given; $f_{\mathbf{b}} \circ f_{\mathbf{a}}\colon V_{\mathbf{i}} \to V_{\mathbf{k}}$ is defined by $(f_{\mathbf{b}} \circ f_{\mathbf{a}})(v) = f_{\mathbf{b}}(f_{\mathbf{a}}(v)) = f_{\mathbf{ba}}(v)$, where **ba** is a path in Γ.

A morphism $h\colon (V, f) \to (V', f')$ is a set of K-linear maps $h_{\mathbf{i}}\colon V_{\mathbf{i}} \to V'_{\mathbf{i}}$ such that for each arrow $\mathbf{a}\colon \mathbf{i} \to \mathbf{j}$ in Γ, the following diagram commutes:

$$\begin{array}{ccc} V_{\mathbf{j}} & \xrightarrow{h_{\mathbf{j}}} & V'_{\mathbf{j}} \\ {\scriptstyle f_{\mathbf{a}}}\big\downarrow & & \big\downarrow{\scriptstyle f'_{\mathbf{a}}} \\ V_{\mathbf{i}} & \xrightarrow[h_{\mathbf{i}}]{} & V'_{\mathbf{i}} \end{array}$$

The objects (V, f) together with the morphisms create the category of finite dimensional representations of Γ over K, which is denoted Rep Γ. Let f.d.$(K\Gamma)$ denote the category of $K\Gamma$-modules of finite K-dimension. We will describe M_1 and M_2 in terms of representations; to do so we will show these two categories are equivalent. We refer the reader to [1] for a more detailed treatment of this topic, including all proofs.

Define a functor T: Rep $\Gamma \to$ f.d.$(K\Gamma)$ as follows. Set $T(V, f) = \coprod_{\mathbf{i}\in\Gamma_0} V_{\mathbf{i}}$ as a K-vector space. For each $\mathbf{i} \in \Gamma_0$, let $\pi_{\mathbf{i}}\colon T(V, f) \to V_{\mathbf{i}}$ and $\iota_{\mathbf{i}}\colon V_{\mathbf{i}} \to T(V, f)$ denote the standard projection and injection of vector spaces.

For any path $\mathbf{p} = \overrightarrow{p_{\mathbf{ij}}}$ from a vertex $\mathbf{i}$ to a vertex $\mathbf{j}$, we have in (V, f) the map $f_{\mathbf{p}}\colon V_{\mathbf{j}} \to V_{\mathbf{i}}$. This induces the map $\overline{f}_{\mathbf{p}} = \iota_{\mathbf{i}} f_{\mathbf{p}} \pi_{\mathbf{j}}\colon T(V, f) \to T(V, f)$. It is easy to show that the map $\tilde{f}\colon K\Gamma \to \mathrm{End}(T(V, h))$ by $\tilde{f}(\mathbf{p}) = \overline{f}_{\mathbf{p}}$ is a K-algebra homomorphism, thus making $T(V, f)$ a $K\Gamma$-module.

We now check how T acts on morphisms in Rep Γ. Let $h\colon (V, f) \to (V', f')$ be a morphism in Rep Γ. We thus have maps $h_{\mathbf{i}}\colon V_{\mathbf{i}} \to V'_{\mathbf{i}}$ which then induce a vector space map $\tilde{h}\colon T(V, f) \to T(V', f')$. If $\mathbf{p}\colon \mathbf{i} \to \mathbf{j}$ is an arrow in Γ, we know $h_{\mathbf{i}} f_{\mathbf{p}} = f'_{\mathbf{p}} h_{\mathbf{j}}$. Thus $\tilde{h}\overline{f}_{\mathbf{p}} = \overline{f}'_{\mathbf{p}}\tilde{h}$, so $\tilde{h}\tilde{f}(\mathbf{p}) = \tilde{f}'(\mathbf{p})\tilde{h}$. In noting how these functions act on an element $v \in T(V, f)$, we see that $\tilde{h}\tilde{f}(\mathbf{p})(v) = \tilde{f}'(\mathbf{p})\tilde{h}(v)$; it writing $\tilde{f}(\mathbf{p})$ in the standard $K\Gamma$-module way, we see this says $\tilde{h}(\mathbf{p}v) = \mathbf{p}\tilde{h}(v)$, and so $\tilde{h}$ is a $K\Gamma$-module map. Thus define $T(h) = (\tilde{h})$, and hence we have defined T as a functor.

We now need a functor H: f.d.$(K\Gamma) \to$ Rep Γ. Let M be an object in f.d.$(K\Gamma)$. Let $\mathbf{v}_1 \cdots \mathbf{v}_m$ denote the vertices of $K\Gamma$; these are orthogonal idempotents where $1 = \mathbf{v}_1 + \cdots + \mathbf{v}_m$. Thus $M = \coprod_{i=1}^{m} \mathbf{v}_i M$. For every element $\mathbf{p} \in K\Gamma$, we have a map $\tilde{f}(\mathbf{p})\colon M \to M$ where $\tilde{f}(\mathbf{p})(m) = \mathbf{p}m$. For an arrow $\mathbf{a}\colon \mathbf{v}_i \to \mathbf{v}_j$ we have that $\tilde{f}(\mathbf{a})(M) = \mathbf{a}M = \mathbf{a}\mathbf{v}_j M = \mathbf{a}(\mathbf{v}_j M) = \tilde{f}(\mathbf{a})(\mathbf{v}_j M)$; also,$\tilde{f}(\mathbf{a})(M) = \mathbf{a}M = \mathbf{v}_i\mathbf{a}M \subseteq \mathbf{v}_i M$. So $\tilde{f}(\mathbf{a})$ restricts to a map $f_{\mathbf{a}}\colon \mathbf{v}_j M \to \mathbf{v}_i M$. We then define $H(M)$ to consist of the vector spaces $\mathbf{v}_i M$ for each $\mathbf{v}_i \in \Gamma_0$ and the linear maps $f_{\mathbf{a}}\colon \mathbf{v}_j M \to \mathbf{v}_i M$ for each arrow $\mathbf{a}\colon \mathbf{v}_i \to \mathbf{v}_j$ in Γ.

Now check how H acts on morphisms in f.d$(K\Gamma)$. Let $h\colon M \to N$ be a $K\Gamma$-module map. $h(\mathbf{v}_i M) = \mathbf{v}_i h(M) \subseteq \mathbf{v}_i N$, so by restriction we have the map $h_i\colon \mathbf{v}_i M \to \mathbf{v}_i N$. For an arrow $\mathbf{a}\colon \mathbf{v}_j \to \mathbf{v}_i$ and element $m \in M$, we have $\mathbf{a}h(m) = h(\mathbf{a}m)$, so $\mathbf{a}h_i(m) = h_j(\mathbf{a}m)$. This says that $f_{\mathbf{a}} h_i = h_j f'_{\mathbf{a}}$. Setting $H(h) = \{h_i\}$, we have $H(h)\colon H(M) \to H(N)$ is a morphism in Rep Γ. Thus H is a functor from f.d.$(K\Gamma)$ to Rep Γ.

Theorem 5.3.1 *Let K be a field and Γ a finite quiver. The functors T:* Rep $\Gamma \to$ f.d.$(K\Gamma)$ *and H:* f.d.$(K\Gamma)\to$ Rep Γ *are inverse equivalences of K-categories.* □

We need a bit more to study M_1 and M_2 as given before. M_1 is a Λ-module, where $\Lambda{=}K\Gamma/I$ for some quiver Γ and ideal I, and not a $K\Gamma$-module. However, the category of Λ-modules, mod(Λ), is a full subcategory of f.d.$(K\Gamma)$. We need to find an equivalent subcategory of Rep(Γ).

Given a quiver Γ, a *relation* σ is a K-linear combination of paths such that each path begins at the same vertex $\mathbf{v}_i$ and ends at the same vertex $\mathbf{v}_j$; $\sigma = \mathbf{v}_i\sigma\mathbf{v}_j$. Note that the definition of an associated quiver ensures that the elements of $\mathcal{G}$, the "relations of G", are relations according to this new definition. If ρ is a set of relations of Γ, we term the pair (Γ, ρ) a *quiver with relations*. Associated to this pair is the K-algebra $K(\Gamma, \rho) = K\Gamma/\langle\rho\rangle$. We will assume that each path in each relation of ρ is of length ≥ 2. Rep(Γ,ρ) is the full subcategory of Rep Γ whose objects are (V, f) where $f_\sigma = 0$ for each relation $\sigma \in \rho$.

Theorem 5.3.2 *Let K be a field and (Γ,ρ) be a quiver with relations.*

1. *The functor T:* Rep $\Gamma \to$ f.d.$(K\Gamma)$ *induces an equivalence of K-categories between* Rep (Γ,ρ) *and* f.d.$(K(\Gamma,\rho))$.

2. *An object (V, f) is projective in* Rep (Γ,ρ) *if and only if $T(V,f)$ is projective in* f.d.$(K(\Gamma,\rho))$.

3. *A sequence $(U,g) \to (V,f) \to (W, h)$ in* Rep (Γ,ρ) *is exact if and only if the sequence $T(U,g) \to T(V,f) \to T(W, h)$ is exact in* f.d.$(K(\Gamma,\rho))$.□

Given a vertex $\mathbf{v}_i \in \Gamma_0$, there exists a simple object (S_i, f) in Rep Γ where $V_i \cong K$, $V_j = 0$ for $i \neq j$, and all maps $f_{\mathbf{a}} = 0$. These vertex simples describe all the simple objects in Rep Γ; thus the analagous definition describes all the simple objects in Rep(Γ,ρ).

Setting $\Lambda = K(\Gamma, \rho) = K\Gamma/\langle\rho\rangle$, we have an idempotent $\overline{\mathbf{v}}_i$ for each vertex in Γ and thus we have projective Λ-modules $P_i = \Lambda\overline{\mathbf{v}}_i$. Consider $(V, f) = H(P_i)$. We have by the definition of H that in Rep(Γ, ρ), $V_j = \overline{\mathbf{v}}_jP_i$. By choosing as K-basis for P_i the elements $\overline{\mathbf{p}}$ where $\mathbf{p}$ is a path in Γ that ends at vertex $\mathbf{v}_i$, we get as a K-basis for V_j the elements $\overline{\mathbf{p}}$ where $\mathbf{p}$ is a path in Γ that ends at vertex $\mathbf{v}_i$ and starts at vertex $\mathbf{v}_j$. For an arrow $\mathbf{a}\colon \mathbf{v}_r \to \mathbf{v}_t$, we define the map $f_{\mathbf{a}}\colon V_t \to V_r$ by $f_{\mathbf{a}}(\overline{\mathbf{p}}) = \overline{\mathbf{ap}}$.

Example 5.3.2 Consider the quiver Γ given in Figure 5.4, the relation $\rho = \mathbf{ab} - \mathbf{cd}$ and the K-algebra $\Lambda = K\Gamma/\langle\rho\rangle$.

The representation of the vertex simple module S_4 associated with $\mathbf{v}_4$ is given by (U, g) in Figure 5.5(a); the K denotes the one dimensional K-vector space; each 0 corresponds to the zero vector space. Each arrow corresponds to a vector space map between vector spaces; here each map is the zero map.

The projective module $P_4 = \overline{\mathbf{v}}_4\Lambda$ has a representation given by (V, f) in Figure 5.5(b). Note that each vector space shown is one dimensional; this is because, modulo the relation,

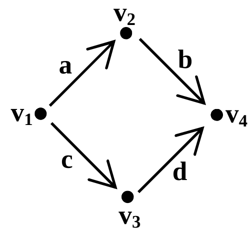
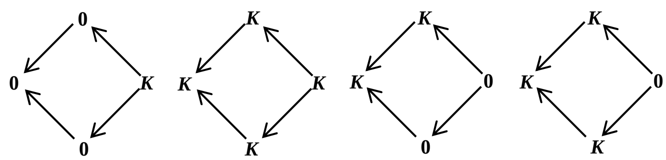


Figure 5.4: The quiver Γ in Example 5.3.2



(a) (U, g) (b) (V, f) (c) (W, h) (d) (X, k)

Figure 5.5: Representations of Λ-modules in Example 5.3.2

there is one path from each vertex to $\mathbf{v}_4$. Here the maps between the nonzero vector spaces are the identity maps.

The projective module $\overline{\mathbf{v}}_2\Lambda$ has a representation given by (W, h) in Figure 5.5(c). Note there is only one path in Γ from $\mathbf{v}_1$ to $\mathbf{v}_2$, hence the vector space W_1 is one dimensional. Since there are no paths from $\mathbf{v}_3$ or $\mathbf{v}_4$ to $\mathbf{v}_2$, $W_3 = W_4 = 0$. The arrow from W_2 to W_1 corresponds to the identity map; all other arrows correspond to the zero map.

In general, we should be more specific about the maps between vector spaces given by the arrows. In this dissertation, we are able to represent the maps with arrows without specifying the exact map represented, for we will only consider special cases of representations. We will be concerned with simple representations, projective representations, and the representations of kernels of maps from projectives onto simple representations.

For instance, consider $\Omega_\Lambda(S_4)$ in the short exact sequence $0 \to \Omega_\Lambda(S_4) \to P_4 \to S_4 \to 0$. By considering the representations of P_4 and S_4, we can easily see that the representation (X, k) of $\Omega_\Lambda(S_4)$ is given by Figure 5.5(d). Here all maps from the zero vector space are zero; all maps between nonzero vector spaces are the identity.

In light of this, we return to our comment about the need for specificity in our vector space maps. In the representations of simple modules, all maps are zero; in the representations of the projective modules that we consider in this dissertation, all nonzero vector spaces will be 1-dimensional and all maps between these spaces will be the identity. Thus in the representations of the kernels of the maps from projective modules onto simple modules, all maps between nonzero vector spaces will be the identity and all other maps will be the zero map. Thus, in this dissertation, we will not specify the specific maps represented by the arrows, and leave it to the reader to infer whether the zero or the identity map is being referred to. □

We are now prepared to describe M_1 and M_2. As a review of the problem we are considering, recall that G' is the complete extension of an n-full graph G. Γ'_{C} and Γ_{C} are the respective canonical quivers and $\Lambda' = K\Gamma'_{\mathrm{C}}/I'$ and $\Lambda{=}K\Gamma/I$ are the respective associated algebras. We have shown through a combinatorial method that Λ' is the result of two sucessive one-point extensions of Λ; $\Lambda'{=} \begin{pmatrix} \hat{\Lambda} & M_2 \\ 0 & K \end{pmatrix}$ where $\hat{\Lambda}{=} \begin{pmatrix} \Lambda & M_1 \\ 0 & K \end{pmatrix}$. The underlying quiver of $\hat{\Lambda}$ is denoted $\hat{\Gamma}$. We seek to describe the Λ-module M_1 and the $\hat{\Lambda}$-module M_2.

To do so, consider the simple $\hat{\Lambda}$-module $S_{\hat{\mathbf{v}}} = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$. We know $\hat{\Gamma}$ has one more vertex $\hat{\mathbf{v}}$ than Γ_{C}; $S_{\hat{\mathbf{v}}}$ is the vertex-simple module associated with this new vertex. Because of this additional vertex, $\hat{\Lambda}$ has an additional idempotent $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = e$. Set $P_{\hat{\mathbf{v}}} = \hat{\Lambda}e = \begin{pmatrix} \Lambda & M_1 \\ 0 & K \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & M_1 \\ 0 & K \end{pmatrix}$. $P_{\hat{\mathbf{v}}}$ is a projective $\hat{\Lambda}$-module, and is a projective cover of $S_{\hat{\mathbf{v}}}$. Consider the following short exact sequence:

$$0 \to \Omega_{\hat{\Lambda}}(S_{\hat{\mathbf{v}}}) \xrightarrow{\psi} P_{\hat{\mathbf{v}}} \xrightarrow{\phi} S_{\hat{\mathbf{v}}} \to 0$$

which is written alternatively

$$0 \to \begin{pmatrix} 0 & M_1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\psi} \begin{pmatrix} 0 & M_1 \\ 0 & K \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix} \to 0.$$

By describing $\Omega_{\hat{\Lambda}}(S_{\hat{\mathbf{v}}})$ we can describe M_1. To describe it, we consider its representation $(\hat{U}, \hat{g})$ in Rep($\hat{\Gamma}$,$\hat{\rho}$). We use Theorem 5.3.2 to know that the above short exact sequence gives the following short exact sequence

$$0 \to T(\Omega_{\hat{\Lambda}}(S_{\hat{\mathbf{v}}})) \xrightarrow{T(\psi)} T(P_{\hat{\mathbf{v}}}) \xrightarrow{T(\phi)} T(S_{\hat{\mathbf{v}}}) \to 0$$

which is also written as

$$0 \to (\hat{U}, \hat{g}) \xrightarrow{T(\psi)} (\hat{V}, \hat{f}) \xrightarrow{T(\phi)} (\hat{W}, \hat{h}) \to 0.$$

We claim we can describe $T(\Omega_{\hat{\Lambda}}(S_{\hat{\mathbf{v}}}))$ by describing the kernel of the map $T(P_{\hat{\mathbf{v}}}) \to T(S_{\hat{\mathbf{v}}})$.

Consider first $T(S_{\hat{\mathbf{v}}}) = (\hat{W}, \hat{h})$. This simple representation has one nonzero vector space which is a copy of K associated with the new vertex $\hat{\mathbf{v}}$, i.e., $\hat{W}_{\hat{\mathbf{v}}} \cong K, \hat{W}_i = 0$ for all other vertices $\mathbf{v}_i \neq \mathbf{v}$, and all maps $\hat{h}_{\mathbf{a}} = 0$.

The representation $T(P_{\hat{\mathbf{v}}}) = (\hat{V}, \hat{f})$ is given as follows. $P_{\hat{\mathbf{v}}} = \hat{\Lambda}\overline{v}$ where $\overline{v}$ is the idempotent of $\hat{\Lambda}$ associated to the new vertex $\hat{\mathbf{v}}$. Thus the vector spaces in this representation are $\hat{V}_i = \mathbf{v}_i P_{\hat{\mathbf{v}}}$, which again has as a K-basis the images $\overline{\mathbf{p}}$ in $\hat{\Lambda}$ of all paths $\mathbf{p}$ beginning at vertex $\mathbf{v}_i$ and ending at $\hat{\mathbf{v}}$. Make special note of $\hat{V}_{\hat{\mathbf{v}}} = \hat{\mathbf{v}}P_{\hat{\mathbf{v}}}$, which has a basis consisting of the images of paths beginning and ending at $\hat{\mathbf{v}}$. Since by construction $\hat{\mathbf{v}}$ is a sink, there is only one such path which is the trivial path $\hat{\mathbf{v}}$. Thus $\hat{V}_{\hat{\mathbf{v}}} \cong K$.

We know the rest of the vector spaces $\hat{V}_i$ by knowing the construction of G' from G. Let $\mathbf{v}$ be the sink of Γ_{C}; consider the representation (X, k) of the projective Λ-module $\Lambda\mathbf{v}$. Since we know Γ_{C} and the set of relations that determine Λ, we can explicitly define X_i for each vertex $\mathbf{v}_i$ in Γ_{C} and all maps $k_{\mathbf{a}}$ for each arrow $\mathbf{a}$. From the construction of G' (specifically, the relationship between G_1 and G_2) we know that the representations (X, k) and $(\hat{V}, \hat{f})$ will be "almost the same;" $\hat{V}_i \cong X_i$ for all $\mathbf{v}_i$ in Γ_{C}, $\mathbf{v}_i \neq \mathbf{v}$, $\hat{V}_{\mathbf{v}} = 0$, and $X_{\mathbf{v}} \cong \hat{V}_{\hat{\mathbf{v}}} \cong K$, where we view Γ_{C} as being a subquiver of $\hat{\Gamma}$. For all maps $k_{\mathbf{a}}\colon X_{\mathbf{v}} \to X_i$ there exists a map $f_{\mathbf{a}'}\colon \hat{V}_{\hat{\mathbf{v}}} \to \hat{V}_i$, where $\mathbf{a}'$ denotes the arrow coming from the letter in the label of G_2 corresponding to the letter (and hence, arrow) $\mathbf{a}$. All maps $f_{\mathbf{a}}\colon \hat{V}_{\mathbf{v}} \to \hat{V}_i$ are the zero map.

Again, we have the following short exact sequence $0 \to (\hat{U}, \hat{g}) \stackrel{T(\psi)}{\to} (\hat{V}, \hat{f}) \stackrel{T(\phi)}{\to} (\hat{W}, \hat{h}) \to 0$. We know from the action of ϕ that $T(\phi)$ maps $\hat{V}_{\hat{\mathbf{v}}}$ onto $\hat{W}_{\hat{\mathbf{v}}}$, and $T(\phi)$ takes all other vector spaces in $(\hat{V}, \hat{f})$ to zero. Thus $\hat{U}_i \cong \hat{V}_i$ for all $\mathbf{v}_i \neq \hat{\mathbf{v}}$, $\hat{U}_{\hat{\mathbf{v}}} = 0$ and for all maps $f_{\mathbf{a}}\colon \hat{V}_i \to \hat{V}_j$ where $\mathbf{v}_i \neq \hat{\mathbf{v}}$, $g_{\mathbf{a}} = f_{\mathbf{a}}$ and $g_{\mathbf{a}} = 0$ when $\mathbf{v}_i = \hat{\mathbf{v}}$.

We now have an explicit representation of the $\hat{\Lambda}$-module M_1. However, we really want a representation of M_1 as a Λ-module. Since we can view Λ as a subalgebra of $\hat{\Lambda}$, we can apply the forgetful functor $F\colon$ f.d.$(\hat{\Lambda}) \to$ f.d.(Λ), and abusing notation a bit, $F\colon \mathrm{Rep}(\hat{\Gamma},\hat{\rho}) \to \mathrm{Rep}(\Gamma_{\mathrm{C}},\mathcal{G})$. Since the action on M_1 in $\hat{\Lambda}$ is defined by the action of Λ on M_1 and $\hat{U}_{\hat{\mathbf{v}}} = 0$, $F(\hat{U}, \hat{g}) = (U, g)$ where $U_i = \hat{U}_i = F(\hat{U}_i)$ for all vertices $\mathbf{v}_i \neq \mathbf{v}$ in Γ_{C} and $g_{\mathbf{a}} = \hat{g}_{\mathbf{a}} = F(\hat{g}_{\mathbf{a}})$ for all arrows $\mathbf{a}$ in Γ_{C}.

We now have a representation of M_1 as a Λ-module; we can go one step further to show how this representation can be determined without first constructing $\hat{\Lambda}$. Consider the simple Λ-module S at the sink $\mathbf{v}$ of Γ_{C}, its projective cover $P = \Lambda\overline{\mathbf{v}}$, and the short exact sequence

$$0 \to \Omega_{\Lambda}(S) \to P \to S \to 0.$$

In repeating the work done before to determine the explicit representation of $(\hat{U}, \hat{g})$, we find that the representation of $\Omega_{\Lambda}(S)$ in $\mathrm{Rep}(\Gamma_{\mathrm{C}},\mathcal{G})$ is (U, g). So knowing only Λ and the connection between the full graphs G and G', we can construct $\hat{\Lambda}$.

We now turn to determining M_2; we will again use the approach of first considering its representation as a Λ'-module and then showing how to determine M_2 without a priori knowledge of Λ'.

Λ' is a one-point extension of $\hat{\Lambda}$, thus we know its underlying quiver Γ'_{C} will have one more vertex $\mathbf{v}'$, a sink, than $\hat{\Gamma}$. Consider the vertex simple module $S_{\mathbf{v}'}$ at this new vertex, its projective cover $P_{\mathbf{v}'} = \Lambda'\overline{\mathbf{v}}'$, and the short exact sequence

$$0 \to \Omega'_{\Lambda}(S_{\mathbf{v}'}) \stackrel{\psi}{\to} P_{\mathbf{v}'} \stackrel{\phi}{\to} S_{\mathbf{v}'} \to 0$$

which also gives a short exact sequence in $\mathrm{Rep}(\Gamma'_{\mathrm{C}},\rho')$

$$0 \to (U', g') \stackrel{T(\psi)}{\to} (V', f') \stackrel{T(\phi)}{\to} (W', h') \to 0.$$

In knowing some of the properties of the canonical quiver of a full graph, we know that Γ'_{C} will have one sink and therefore the new vertex $\mathbf{v}'$ is this sink; we also know that since there are only two letters in the $n+2$ coordinate of the labels of G' that there are only two arrows pointing to it, one each from $\hat{\mathbf{v}}$ and $\mathbf{v}$.

Finally, we can use information about the relations generated by edges numbered $n+1$ in G' to find information about the vector spaces and maps of (V', f'). There is only one path (an arrow) each from $\mathbf{v}$ and $\hat{\mathbf{v}}$ to $\mathbf{v}'$. Let $\mathbf{w}$ be any vertex in Γ_{C} that is a source in the nth level; there are t arrows each from $\mathbf{w}$ to $\mathbf{v}$ and $\hat{\mathbf{v}}$. By the relations in G' coming from edges numbered $n+1$, we know that there are thus t paths from $\mathbf{w}$ to $\mathbf{v}'$, modulo the relations. Thus for $\mathbf{v}_i, \mathbf{v}_j \neq \mathbf{v}, \mathbf{v}'$, the dimension of V'_i in (V', f') will be the same as the dimension of $\hat{V}_i$ in $(\hat{V}, \hat{f})$, the representation of $P_{\hat{\mathbf{v}}}$ as a $\hat{\Lambda}$-module, and for all arrows $\mathbf{a}\colon \mathbf{v}_i \to \mathbf{v}_j$, $f'_{\mathbf{a}} = \hat{f}_{\mathbf{a}}$. An analogous statement can be made using the representation (V, f) of $P_{\mathbf{v}}$ instead of $P_{\hat{\mathbf{v}}}$; this gives that $V'_{\mathbf{v}} = V_{\mathbf{v}}$ and for all arrows $\mathbf{a}\colon \mathbf{v}_i \to \mathbf{v}$, $f'_{\mathbf{a}} = f_{\mathbf{a}}$. Of course, $V'_{\mathbf{v}'} \cong K$.

Using the same techniques used before, we can describe (U', g'); $U'_{\mathbf{v}'} = 0$, $U'_i = V'_i$ for all $\mathbf{v}_i \neq \mathbf{v}'$, and $g'_{\mathbf{a}} = f'_{\mathbf{a}}$ for all arrows $\mathbf{a}\colon \mathbf{v}_i \to \mathbf{v}_j$ where $\mathbf{v}_j \neq \mathbf{v}'$; when $\mathbf{v}_j = \mathbf{v}'$, $g_{\mathbf{a}} = 0$.

This gives an explicit representation of M_2 as a Λ'-module using only knowledge of the construction of $\hat{\Lambda}$ and G'. To describe M_2 as a $\hat{\Lambda}$-module, we again apply the forgetful functor F to the representation (U', g'). Again, we do not truly "forget" any information, for all that is lost is the vector space $U'_{\mathbf{v}'}$, which is zero, and all maps from $U'_{\mathbf{v}'}$ which are also zero maps. Thus we can define Λ' using only knowlegde of the construction of G' and $\hat{\Lambda}$, enabling us to determine Λ' beginning with just knowledge of G' and Λ. Since the construction of G' from G is a combinatoric process, we cannot expect the construction of Λ' from Λ to be purely algebraic. The combinatoric overtones in the latter construction appear in the choices of the Λ and $\hat{\Lambda}$-modules by which the one-point extensions were created.

We now give an example to demonstrate the techniques discussed above.

Example 5.3.3 Consider the 2-full graph G and its canonical quiver Γ_{C} in Figure 5.2, the complete extension of this 2-full graph G' in Figure 5.6 and its canonical quiver Γ'_{C} in Figure 5.7.

We will attempt to give as much detail as possible in order to illuminate the discussion above. Let $\Lambda = K\Gamma_{\mathrm{C}}/I$, where $I = \langle \mathcal{G} \rangle$, $\mathcal{G} = \{\mathbf{ab}-\mathbf{de}, \mathbf{af}-\mathbf{dh}, \mathbf{bc}-\mathbf{fg}, \mathbf{ec}-\mathbf{ij}, \mathbf{hg}-\mathbf{ij}\}$ and let $\Lambda' = K\Gamma'_{\mathrm{C}}/I'$, where $I' = \langle \mathcal{G}' \rangle$, $\mathcal{G}' = \mathcal{G} \cup \{\mathbf{bl}-\mathbf{fn}, \mathbf{el}-\mathbf{io}, \mathbf{hn}-\mathbf{io}, \mathbf{ck}-\mathbf{lm}, \mathbf{jk}-\mathbf{om}, \mathbf{gk}-\mathbf{nm}\}$.

We want to describe Λ' as the matrix algebra $\begin{pmatrix} \hat{\Lambda} & M_2 \\ 0 & K \end{pmatrix}$, where $\hat{\Lambda} = \begin{pmatrix} \Lambda & M_1 \\ 0 & K \end{pmatrix}$. To do so, we find the Λ-module M_1 and the $\hat{\Lambda}$-module M_2. First, consider the simple $\hat{\Lambda}$-module $S_{\hat{\mathbf{v}}} = \begin{pmatrix} 0 & 0 \\ 0 & K \end{pmatrix}$ and the projective $\hat{\Lambda}$-module $P_{\hat{\mathbf{v}}} = \begin{pmatrix} 0 & M_1 \\ 0 & K \end{pmatrix}$. $S_{\hat{\mathbf{v}}}$ has representation given by $(\hat{W}, \hat{h})$ in Figure 5.8(c). Here we assume we already know the quiver $\hat{\Gamma}$ underlying $\hat{\Lambda}$ from our a priori knowledge of Γ'_{C}.

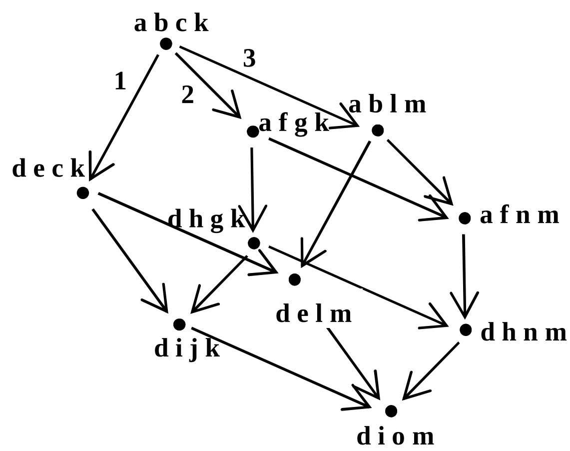


Figure 5.6: The 3-full graph G' in Example 5.3.3

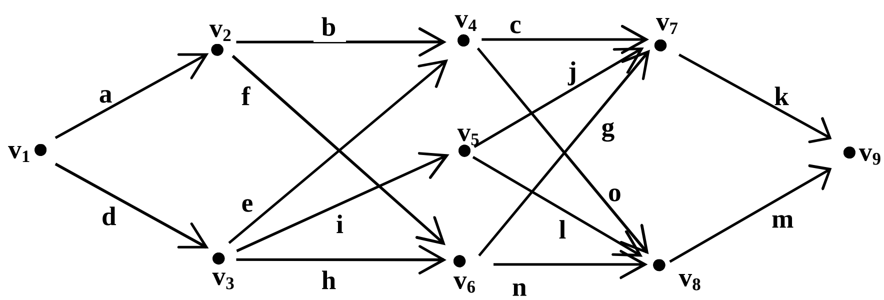
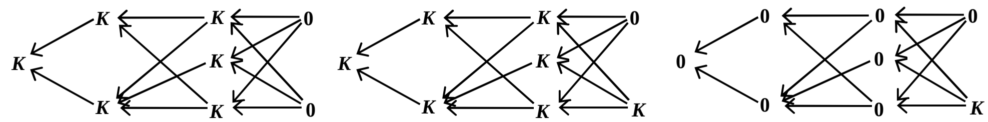


Figure 5.7: The canonical quiver of G' in Example 5.3.3

$P_{\hat{\mathbf{v}}}$ has representation $(\hat{V}, \hat{f})$ given in Figure 5.8(b). The dimension of each vector space $\hat{V}_i$ is given by the number of equivalence classes of paths from $\mathbf{v}_i$ to $\mathbf{v}_8$; on may check that, modulo the relations, there is only one path from each vertex $\mathbf{v}_i$ to $\mathbf{v}_8$, except for $\mathbf{v}_7$, where there is no such path. By the short exact sequence of $\hat{\Lambda}$-modules $0 \to M_1 \to P_{\hat{\mathbf{v}}} \to S_{\hat{\mathbf{v}}} \to 0$, it is visually easy to check that M_1 has representation $(\hat{U}, \hat{g})$ given in Figure 5.8(a).

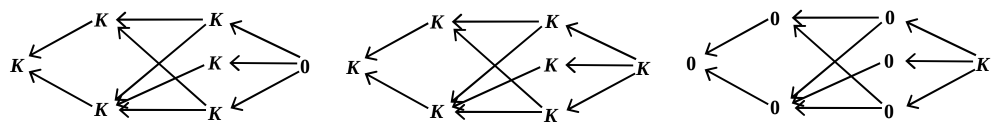
This representation of M_1 is given as a $\hat{\Lambda}$-module; we apply the forgetful functor F to get its representation as a Λ-module. In effect this "forgets" the action of $\hat{\Lambda}$ on the vector space $\hat{U}_8$; since this is 0, we have "not forgotten" anything. Thus the representation of M_1 as a Λ-module is given in Figure 5.9(a).

Since we know already Γ'_{C} and $\mathcal{G}'$, we know the underlying quiver $\hat{\Gamma}$ and relations $\mathcal{G}_1$ of $\hat{\Lambda}$, and have used this information already in determining the representations of $M_1, P_{\hat{\mathbf{v}}}$ and $S_{\hat{\mathbf{v}}}$. However, we have shown that the representation of M_1 as a Λ-module can be derived without knowledge of Γ'_{C} or $\mathcal{G}'$; we showed that M_1 can be given by the short exact sequence $0 \to M_1 \to P_{\mathbf{v}_7} \to S_{\mathbf{v}_7} \to 0$, where $S_{\mathbf{v}_7}$ is the vertex simple Λ-module associated with the sink $\mathbf{v}_7$ of Γ_{C} and $P_{\mathbf{v}_7} = \Lambda\overline{\mathbf{v}}_7$ is its projective cover.



(a) $(\hat{U}, \hat{g})$ (b) $(\hat{V}, \hat{f})$ (c) $(\hat{W}, \hat{h})$

Figure 5.8: Representations of $\hat{\Lambda}$-modules in Example 5.3.3



(a) (U, g) (b) (V, f) (c) (W, h)

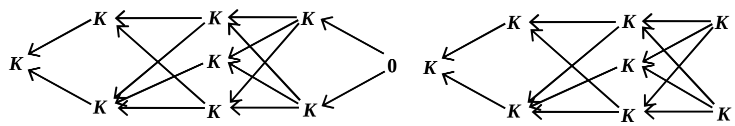
Figure 5.9: Representations of Λ-modules in Example 5.3.3

$S_{\mathbf{v}_7}$ has representation (W, h) given in Figure 5.9(c); $P_{\mathbf{v}_7}$ has representation (V, f) given in Figure 5.9(b), where each K represents the 1-dimensional vector space V_i, which has as a K-basis the equivalence classes of paths from $\mathbf{v}_i$ to $\mathbf{v}_7$. It is easy to check that modulo the relations, there is only one path from each vertex to $\mathbf{v}_7$.

Thus M_1 has representation given in Figure 5.9(a); while this is the same as what we determined above, this was done without a priori knowledge of Γ'_{C}.

Now consider $\hat{\Lambda}= \begin{pmatrix} \Lambda & M_1 \\ 0 & K \end{pmatrix}$. We will try to recover the underlying quiver $\hat{\Gamma}$ by studying the structure of $\hat{\Lambda}$; we presume to know nothing about Γ'_{C}. The underlying quiver has one more vertex $\mathbf{v}_8$ than Γ which is represented by the matrix element $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. It is easy to see that for any element $\lambda \in \Gamma_{\mathrm{C}}$, $\overline{\lambda} \cdot \overline{\mathbf{v}}_8 = \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$ (where we abuse notation and refer to Γ_{C} as a subquiver of $\hat{\Gamma}$). Also, for any $m \in M_1$, $1_\Lambda \cdot m \cdot \overline{\mathbf{v}}_8 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = m$, so now M_1 can be viewed as holding information concerning paths from vertices $\mathbf{v}_i$ to $\mathbf{v}_8$, for $i = 1 \cdots 7$.

How many arrows are there from $\mathbf{v}_4$ to $\mathbf{v}_8$? To answer this, we check the dimension of $\overline{\mathbf{v}}_4 M_1 \overline{\mathbf{v}}_8 = \begin{pmatrix} \overline{\mathbf{v}}_4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & M_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \overline{\mathbf{v}}_4 M_1$. In thinking of the representation of M_1, $\overline{\mathbf{v}}_4$ kills off all vector spaces except U_4, for given any path $\mathbf{p}$ from $\mathbf{v}_i$ to $\mathbf{v}_7, i \neq 4, \overline{\mathbf{v}_4 \mathbf{p}} = 0$ (here we abuse notation and refer to $\overline{\mathbf{v}}_4$ acting on the representation (U, g); technically we



(a) (U', g') (b) $F(U', g')$

Figure 5.10: Representations of Λ'-modules in Example 5.3.3

refer to $\overline{\mathbf{v}}_4$ acting on $H(U, g)$). $\overline{\mathbf{v}}_4 \cdot U_4 = U_4 \cong K$, so we have one arrow $\mathbf{v}_4 \to \mathbf{v}_8$. We label this arrow $\mathbf{l}$ in keeping with the lettering of G'. Repeating this shows there exists one arrow each from $\mathbf{v}_5$ and $\mathbf{v}_6$ to $\mathbf{v}_8$, which we label $\mathbf{o}$ and $\mathbf{n}$, respectively. We can also use this argument to show there are no arrows from $\mathbf{v}_7$ to $\mathbf{v}_8$; in considering $\overline{\mathbf{v}}_7 M_1$, $\overline{\mathbf{v}}_7$ kills off all vector spaces except U_7', but it is already 0. So $\dim(\overline{\mathbf{v}}_7 M_1 \overline{\mathbf{v}}_8) = 0$.

We can repeat this for all vertices in Γ_C; it will show there is only one path from each vertex in Γ_C to $\mathbf{v}_8$, modulo relations. For instance, this means that the paths $\mathbf{bl}$ and $\mathbf{fn}$ from $\mathbf{v}_2$ to $\mathbf{v}_8$ are "the same" in $\hat{\Lambda}$; thus we must have a relation $\mathbf{bl}$ - $\mathbf{fn}$. We can determine the other relations of $\mathcal{G}'$ that come from edges numbered 2 in this way. We know we have found all the arrows of $\hat{\Gamma}$ since we have used all the letters in the third coordinate of the labels of G'; thus with the relations $\hat{\mathcal{G}} = \mathcal{G} \cup \{\mathbf{bl} - \mathbf{fn}, \mathbf{el} - \mathbf{io}, \mathbf{hn} - \mathbf{io}\}$ (with the latter relations coming from our knowledge of G') we have completely described the algebra $\hat{\Lambda}$ in terms of a quiver and relations using only information from G, G', and Λ.

$\Lambda' = \begin{pmatrix} \hat{\Lambda} & M_2 \\ 0 & K \end{pmatrix}$. We can find the representation of M_2 as a Λ'-module using the same techniques involving short exact sequences used before (and here, now, we assume that we do know the structure of Γ_C' a priori). Doing so we find that M_2 has the following representation (U', g') given in Figure 5.10(a). Here each nonzero vector space U_i' represents the equivalence class of paths from $\mathbf{v}_i$ to $\mathbf{v}_9$. Applying the functor F gives the representation $F(U', g')$ given in Figure 5.10(b), where again we have effectively forgotten nothing.

Now let us determine M_2 without a priori knowledge of Γ_C'. Let (Y', y') and (X', x') denote the respective representations of the projective $\hat{\Lambda}$-modules $\hat{\Lambda}\overline{\mathbf{v}}_7$ and $\hat{\Lambda}\overline{\mathbf{v}}_8$, and let (U', g') denote the yet to be determined representation of the lammone-module M_2. From our work before this example, we noted that for $i = 1 \cdots 6$, $Y_i' = X_i' = U_i' \cong K$ and for all arrows $\mathbf{a}\colon \mathbf{v}_i \to \mathbf{v}_j$, $i, j = 1 \cdots 6$, $y_\mathbf{a}' = x_\mathbf{a}' = g_\mathbf{a}'$. $U_7' = Y_7', U_8' = X_8'$, and for arrows $\mathbf{a}, \mathbf{b}$ from $\mathbf{v}_7$ and $\mathbf{v}_8$, respectively, $g_\mathbf{a}' = y_\mathbf{a}'$ and $g_\mathbf{b}' = x_\mathbf{b}'$. This gives M_2 the representation 5.10(b).

Now we consider $\Lambda' = \begin{pmatrix} \hat{\Lambda} & M_2 \\ 0 & K \end{pmatrix}$ and construct its canonical quiver (i.e., we do not use a priori knowledge of Γ_C', but rather try to construct it from the information we have about Λ'). Γ_C' has one more vertex $\mathbf{v}_9$ than $\hat{\Gamma}$ which corresponds to the matrix element $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We again check the dimensions of $\overline{\mathbf{v}}_i M_2 \overline{\mathbf{v}}_9$ to find the number of paths, modulo relations, in

Γ'_{C} from $\mathbf{v}_i$ to $\mathbf{v}_9$. In doing so we find there is one arrow each from $\mathbf{v}_7$ and $\mathbf{v}_8$ to $\mathbf{v}_9$; we label these **k** and **m**, respectively, using our knowledge of G'. Dimension checking will also give us new relations; for instance, there is only one path from $\mathbf{v}_4$ to $\mathbf{v}_9$ giving us the relation **ck-lm**. The new arrows **k** and **m** utilize all letters from the labels of G'; the additional relations coming from the fact that there is only one path each from $\mathbf{v}_5$ and $\mathbf{v}_6$ to $\mathbf{v}_9$ give all the relations of $\mathcal{G}'$. Thus we stop adding arrows and relations, giving us the quiver Γ'_{C} and relations $\mathcal{G}'$ that we know we should get from knowing G'. $\square$

Chapter 6

Conclusions

The focus of this dissertation has been the establishment of a connection between combinatorics and noncommutative algebra. To do so, we defined a new class of directed graphs called full graphs that satisfied a certain set of combinatorial conditions on the edges. We then studied some of the properties of full graphs that followed as a result of the edge conditions.

Given a full graph G and field K, we defined the characteristics of an associated quiver Γ, and hence a path algebra $K\Gamma$. G also produced a set of relations $\mathcal{G}$ on Γ, which allowed us to form the algebra $K\Gamma/\langle\mathcal{G}\rangle$. We showed that $\mathcal{G}$ is a Gröbner basis for $\langle\mathcal{G}\rangle$, and hence $\Lambda = K\Gamma/\langle\mathcal{G}\rangle$ is a Koszul algebra.

We studied minimal projective resolutions of certain semisimple Λ-modules, giving special attention to how to generate certain projective modules and maps in the resolution using information from G. We showed how to generate the minimal projective resolution of the semisimple Λ-module Λ_0

$$0 \to P_{n+1} \stackrel{d_{n+1}}{\to} P_n \to \cdots \to P_2 \stackrel{d_2}{\to} P_1 \stackrel{d_1}{\to} P_0 \to \Lambda_0 \to 0$$

by considering overlaps of certain elements of $K\Gamma$, and showed how to find P_i and d_i for $1 \leq i \leq 3$ explicitly from G without computation. This also allowed us to connect the "size" of G to a "size" of Λ; if G is n-full then $\mathrm{gl.dim}(\Lambda) \leq n+1$.

We also defined a special associated quiver Γ_{C} called the canonical quiver. Given G' which is the complete extension of a full graph G, we studied how the canonical quiver Γ'_{C} of G' is related combinatorially to the canonical quiver Γ_{C} of G. We then showed how to algebra $K\Gamma'/\langle\mathcal{G}'\rangle$ could be derived from the algebra $K\Gamma/\langle\mathcal{G}\rangle$ using successive one point extensions.

We have shown how to generate a Koszul algebra from a full graph. We are interested in going the other direction: what Koszul algebras come from full graphs, and how can a full graph be recovered from such an algebra? We are also interested in finding all the projective modules in the minimal projective resolution of Λ_0 that we have constructed without computation, not just P_1, P_2 and P_3. Finally, we have produced sufficient combinatorial conditions on directed graphs that allow us to produce a Gröbner basis; using the same labelling and relation generating techniques, we are interested in finding necessary and sufficient conditions for producing a Gröbner basis from the graph. These are all topics that will motivate further research.

Appendix A

Examples

We give in this appendix some more insight into the numbering conditions given in Chapter 2. These conditions were established so that the resulting set of relations $\mathcal{G}$ would be a Gröbner basis for the ideal it generated in the constructed ring. We proved this was the case in three steps; we first showed that $\mathcal{G}$ was a tip reduced set, then that all overlap relations generated at a vertex in G reduce to zero, and finally that all overlap relations in $\mathcal{G}$ were produced at a vertex. Each of the conditions were determined to help prove at least one of the above steps. We give many examples below to show that each condition is independant of the others; no condition follows as a result of the other five.

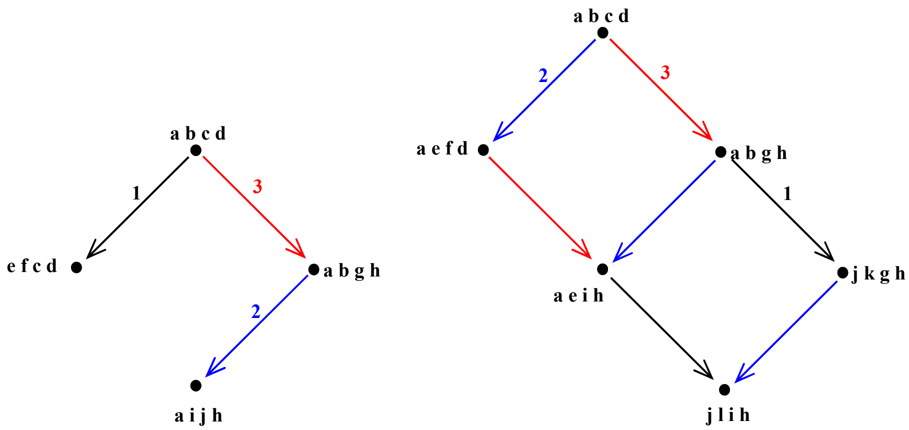
Condition 1 does not allow two directed edges that have the same source to have the same edge number. This helps ensure that $\mathcal{G}$ is tip reduced. This condition also does not allow the graph to contain a directed path of length 2 in which both edges are numbered the same, which allows for many graphs to produce reduced Gröbner bases. It is not yet clear whether or not all graphs that satisfy the conditions will produce a reduced Gröbner basis; certainly all examples given in this dissertation do.

Condition 2 clearly establishes that overlap relations which are generated at a vertex will reduce to zero. Condition 3 helps ensure that all overlaps are generated at a vertex and hence reduce. Figures A.1(a) and (b) give two examples of numbered graphs that fail to satisfy condition 3. In each case an overlap is generated that does not reduce to zero.

In Figure A.1(a), the edge numbered 1 and 2 create an overlap, which clearly does not reduce. Notice that since we do not have 1-3 cycle, we have two sinks; in such a situation we could not expect all overlaps to reduce. Figure A.1(b) does not contain any 1-3 cycles, but fails to satsify condition 3. This graph produces relations **ab-jk** and **bc-ef**, but (**ab-jk**)**c-a**(**bc-ef**) = **aef-jkc** $\not\Rightarrow_{\mathcal{G}} 0$.

In Figure A.2 we have a 1-3 cycle, but not one that satisfies condition 3. We then get relations **ab** - **ef** and **ef** - **kl**, so this graph clearly does not produce a reduced Gröbner basis. Worse yet, the overlap between relations **ef-kl** and **fc-mn** does not reduce, hence the relations do form a Gröbner basis.

Condition 4 gives "nice" structure in many ways. It aids in the systematic ordering of the elements of X_G; X_G must be given an admissable ordering before we can consider



(a) (b)

Figure A.1: Graphs that fail condition 3

generating Gröbner bases.

Consider Figure A.3(a). Note that in our system of ordering the elements, the edge numbered 1 coming from the source sets **a**>**d**. However, the other edge numbered 1 sets **d**>**a**, obviously causing problems in the manner in which we chose to order the elements. A solution around this problem is not straightforward, for even if we established some other ordering that set **a**>**d**, the second edge numbered 1 produces a the relation **-af**+**dh**; the coefficient of the tip is not 1. Thus the set $\mathcal{G}$ does not produce a reduced Gröbner basis; worse yet, the tip of this relation does not come from the label of the source of the directed edge that produced it. The fact that in a full graph the label of the origin of a directed edge supplied the tip of the relation is inherently used in several of the proofs in Chapter 3.

Also, without condition 4, Corollary 2.3.1 is not true. Consider Figure A.3(b). The edge numbered 2 creates the relation **bc-fc**; the **c** stays fixed. This corollary is also inherently used throughout this dissertation; after Chapter 2 we always assume that an edge produces a quadratic, binomial element in which all four letters are distinct.

Condition 5 forces all overlaps to be generated at a vertex. Consider the numbered graphs in Figures A.4 and A.5; neither graph satisfies condition 5 and each produces an overlap relation that does not come from a vertex.

In Figure A.4, the edge numbered 2 coming from the source at the top produces the relation **bc-hi**. The edge numbered 3 coming from the source near the middle of the graph produces the relation **cx-oz**. o(**bc-hi**,**cx-oz**,**b**,**x**) $\not\Rightarrow_{\mathcal{G}} 0$, so this graph does not produce a Gröbner basis.

The graph pictured in Figure A.5 also produces an overlap relation that does not come from a vertex; an overlap exists between the relation **bc-jk** generated by the edge num-

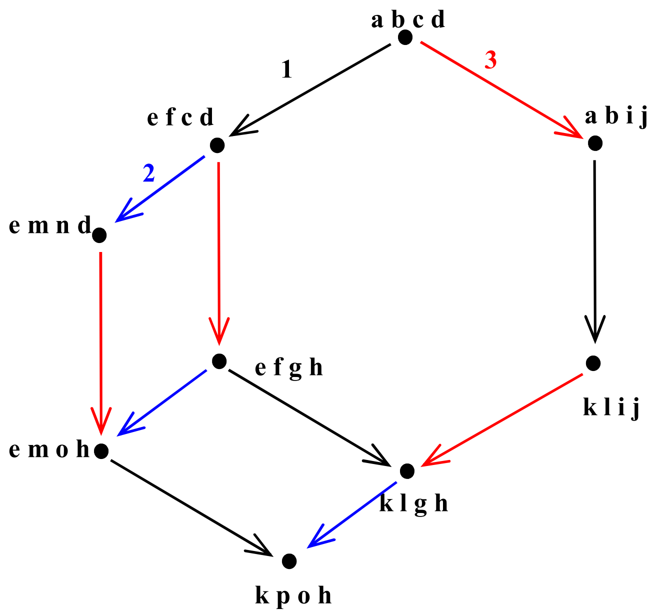


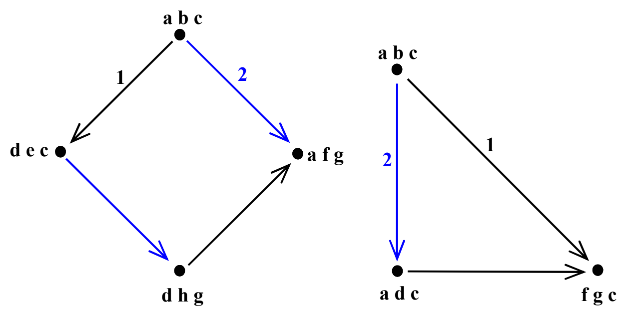
Figure A.2: Another graph that fails condition 3

bered 2 coming from the source on the left and the relation **ch-mn** generated by the edge numbered 3 coming from the source on the right. In this case, however, the relation does reduce to 0. This is probably due to the high level of symmetry involved in this graph.

Note that both of these examples utilized multiple sources; this may cause one to ask whether or not condition 5 is necessary in single source graphs. So far, the known answer to this question is a "weak" yes, this condition is necessary. An example exists in which the numbered graph does not satisfy condition 5, yet has a single source (i.e., when considering only single sourced graphs, condition 5 does not follow as a result of satisfying the remaining five conditions). However, in the examples so far created, all overlap relations do reduce, even though some are not generated at a vertex. All examples so far have edges numbered from 1 to 6, and hence the graphs are quite large and complicated; the smallest we have created has 31 vertices and over 80 edges. Due to the complexity of this graph, it is not included as an example.

We have shown so far two examples of graphs that do not satisfy all six of our conditions, yet still produce Gröbner bases. While we proved in Chapter 3 that our conditions are sufficient to produce a Gröbner basis, these examples show that they are not necessary. This opens the question of "What combinatoric conditions could be placed on a directed, numbered graph that are necessary and sufficient for producing a Gröbner basis?" Obviously, we do not know the answer to that question, else these conditions would have been included in this dissertation.

Finally, condition 6 establishes the fact that the set of relations produced is tip reduced. Without this condition, the numbered graph in Figure A.6 would be "full", although the



(a) (b)

Figure A.3: Graphs that fail condition 4

set of relations it generates is not tip reduced; the two edges numbered 2 coming from the sources produce the relations **bc-ij** and **bc-kl**. Again, the example we give is a graph with multiple sources; it is not yet clear whether or not condition 6 follows from the other five conditions when we restrict to single source graphs.

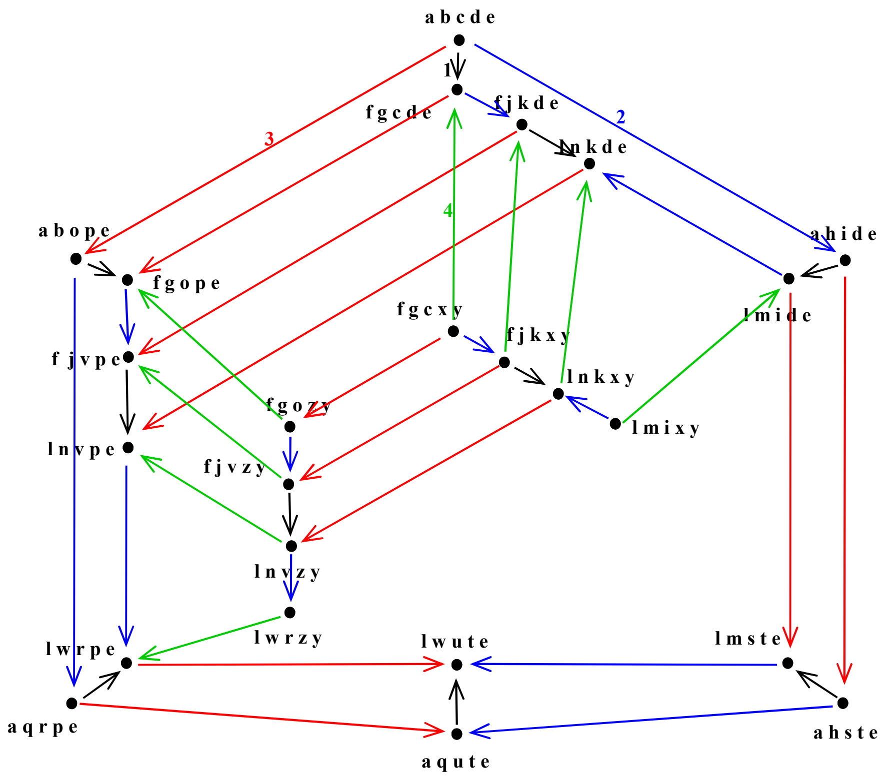


Figure A.4: A graph that fails condition 5

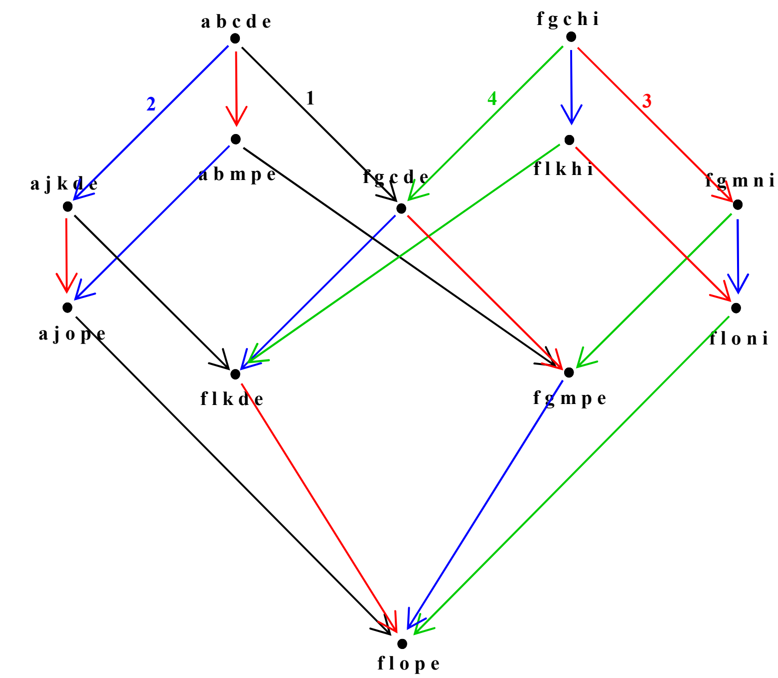


Figure A.5: Another graph that fails condition 5

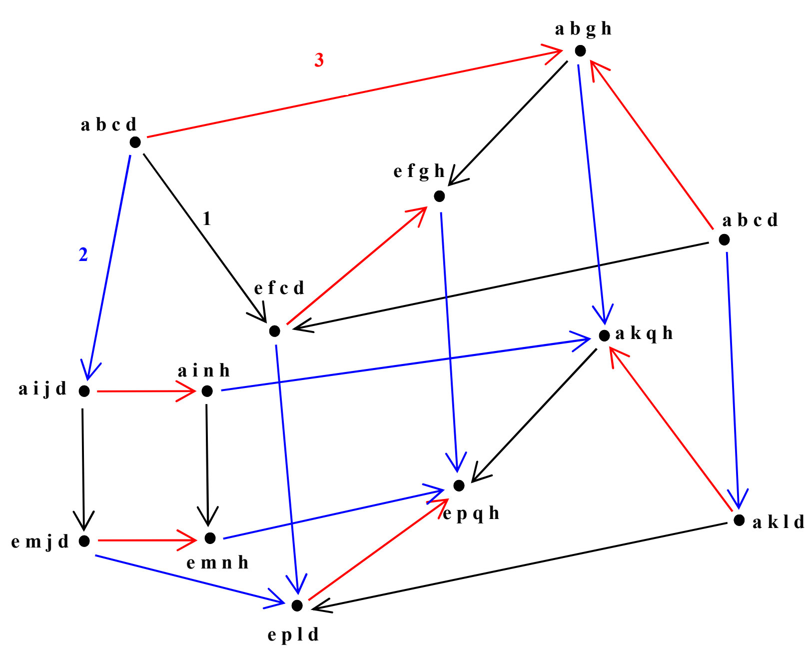


Figure A.6: A graph that fails condition 6

Bibliography

[1] M. Auslander, I. Reiten, and S. Smalø. *Representation Theory of Artin Algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, 1995.

[2] Jörgen Backelin and Ralf Fröberg. Koszul algebras, Veronese subrings and rings with linear resolutions. *Rev. Roumaine Math. Pures Appl.*, 30:85–97, 1985.

[3] A. Beilinson, V. Ginzburg, and W. Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9:473–527, 1996.

[4] George Bergman. The diamond lemma for ring theory. *Advances in Mathematics*, 29:178–218, 1978.

[5] Ralf Fröberg. On Stanley-Reisner rings. *Banach Center Publ.*, pages 57–70, 1990.

[6] Ralf Fröberg. Koszul algebras. *Advances in Commutative Ring Theory*, pages 337–350, 1997.

[7] Edward Green. Representation theory of tensor algebras. *J. Algebra*, 34:136–171, 1975.

[8] Edward Green. Noncommutative gröbner bases and projective resolutions. In *Proceedings of the Euroconference Computational Methods for Representations of Groups and Algebras, Essen*, volume 173 of *Progress in Mathematics*, pages 29–60. Basel, Birkhaüser Verlag, 1997.

[9] Edward Green. Multiplicative bases, Gröbner bases, and right Gröbner bases. *Journal of Symbolic Computation*, 29:601–623, 2000.

[10] Edward Green, Øyvind Solberg, and Daniel Zacharia. Minimal projective resolutions. *Transactions of the American Mathematical Society*, 353:2915–2939, 2001.

[11] Edward Green and Roberto Martínez Villa. Koszul and yoneda algebras. In *Representation Theory of Algebras*, volume 18, pages 247–297. Canadian Mathematical Society, Conference Proceedings, American Mathematical Society, 1994.

[12] Y.I. Manin. Some remarks on Koszul algebras and quantum groups. *Ann. Inst. Fourier*, 37:191–205, 1987.

[13] Hidefumi Ohsugi and Takayuki Hibi. Toric ideals generated by quadratic binomials. *Journal of Algebra*, 218:509–527, 1998.

[14] Hidefumi Ohsugi and Takayuki Hibi. Koszul bipartite graphs. *Advances in Applied Mathematics*, 22:25–28, 1999.

[15] Patrick Polo. On Cohen-Macauley posets, Koszul algebras and certain modules associated to Shubert varieties. *Bull. London Math. Soc.*, 27:425–434, 1995.

[16] Stewart B. Priddy. Koszul resolutions. *Transactions of the American Mathematical Society*, 152:39–60, November 1970.

[17] M. Rosso. Koszul resolutions and quantum groups. *Nuclear Phys. B. Proc. Suppl.*, 18:269–276, 1990.

Vita

Gregory N. Hartman

Education:

2002 Ph.D. in Mathematics
Virginia Tech, Blacksburg, VA

2000 M.S. in Mathematics
Virginia Tech, Blacksburg, VA

1991 B.S. in Mathematics
Liberty University, Lynchburg, VA

Work Experience:

1997-2002 Virginia Tech, Blacksburg, VA
Mathematics Department, Graduate Teaching Assistant

Awards:

GTA of the Year, 2001, Department of Mathematics, Virginia Tech

Virginia Tech Graduate Teaching Award, 2002