# Schur-class of finitely connected planar domains: the test-function approach

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Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

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April 18, 2011
Blacksburg, Virginia

Keywords: completely positive kernel, test functions, Schur class, extreme points.

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(ABSTRACT)

We study the structure of the set of extreme points of the compact convex set of matrix-valued holomorphic functions with positive real part on a finitely-connected planar domain $\mathcal{R}$ normalized to have value equal to the identity matrix at some prescribed point $t_0 \in \mathcal{R}$. This leads to an integral representation for such functions more general than what would be expected from the result for the scalar-valued case. After Cayley transformation, this leads to a integral Agler decomposition for the matrix Schur class over $\mathcal{R}$ (holomorphic contractive matrix-valued functions over $\mathcal{R}$). Application of a general theory of abstract Schur-class generated by a collection of test functions leads to a transfer-function realization for the matrix Schur-class over $\mathcal{R}$, extending results known up to now only for the scalar case. We also explain how these results provide a new perspective for the dilation theory for Hilbert space operators having $\mathcal{R}$ as a spectral set.

# Dedication

To my family, who encourages and supports me every step of the way in the pursuing of my dreams.

# Acknowledgements

I am very thankful to Dr. Joseph A. Ball for his endless patience and important clarifying ideas he provided while advising during the development of this thesis. Special thanks to Edgar Saenz Maldonado for his friendship and his important remarks done in our informal discussions of several topics of mathematics. Finally I would like to acknowledge the contribution for Professor David Sherman of the University of Virginia who pointed out the relevance of 1969 paper of William Arveson for the extreme point problems studied in this dissertation.

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# Chapter 1

# Introduction

Given two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, we let $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ denote the *Schur class* of holomorphic functions on the unit disk $\mathbb{D}$ with values in the closed unit ball of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ of bounded linear operators between the Hilbert space $\mathcal{U}$ and $\mathcal{Y}$. Then it is well known that the following three statements are equivalent:

1. $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$.
2. The de Branges-Rovnyak kernel

$$K_S(z,w) = \frac{I - S(z)S(w)^*}{1 - z\overline{w}}$$

is a positive kernel on $\mathbb{D}$, i.e.

$$\sum_{i,j=1}^{N} \langle K_S(z_i, z_j) y_j, y_i \rangle_{\mathcal{Y}} \geq 0 \tag{1.1}$$

for all $z_1, \dots, z_N \in \mathbb{D}$, $y_1, \dots, y_N \in \mathcal{Y}, N = 1, 2, \dots$

3. $S$ has a contractive transfer function realization, i.e. there is a Hilbert space $\mathcal{X}$ and a contractive operator matrix

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

$\left( \text{so } \|y\|^2 + \|x'\|^2 \leq \|u\|^2 + \|x\|^2 \text{ if } U \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x' \\ y \end{bmatrix} \right)$ so that

$$S(z) = D + zC(I - zA)^{-1}B, \text{ for } z \in \mathbb{D}.$$

Jim Agler generalized the previous equivalences for the case of the Schur class of holomorphic functions on $\mathbb{D}^2$ with values in the closed unit ball of $\mathcal{L}(\mathcal{U},\mathcal{Y})$ of bounded linear operators between the Hilbert space $\mathcal{U}$ and $\mathcal{Y}$ denoted by $\mathcal{S}_{\mathbb{D}^2}(\mathcal{U},\mathcal{Y})$. That is,

1. $S \in \mathcal{S}_{\mathbb{D}^2}(\mathcal{U},\mathcal{Y})$.
2. There exist two positive positive kernels $K_1(z,w)$ and $K_2(z,w)$ on $\mathbb{D}^2$ so that
$$I - S(z)S(w)^* = (1 - z_1\overline{w_1})K_1(z,w) + (1 - z_2\overline{w_2})K_2(z,w).$$
3. $S$ has a contractive $2D$ transfer function realization, i.e. there is a contractive operator matrix of the form
$$\mathbf{U} = \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{bmatrix} : \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \\ \mathcal{Y} \end{bmatrix}$$
so that
$$\begin{aligned} S(z_1,z_2) &= D + \begin{bmatrix} C_1 & C_2 \end{bmatrix} \left[ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} z_1 I_{\mathcal{X}_1} & 0 \\ 0 & z_2 I_{\mathcal{X}_2} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right]^{-1} \\ &\times \begin{bmatrix} z_1 I_{\mathcal{X}_1} & 0 \\ 0 & z_2 I_{\mathcal{X}_2} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \end{aligned}$$

The same equivalences cannot be generalized for the case of $\mathcal{S}_{\mathbb{D}^d}(\mathcal{U},\mathcal{Y})$ $(d > 2)$. If we want similar equivalences like the ones above it is necessary to define a new family of functions, namely the *Schur-Agler class*, that is

$$\begin{aligned} \mathcal{SA}_{\mathbb{D}^d}(\mathcal{U},\mathcal{Y}) &= \Big\{ S : \mathbb{D}^d \underset{\text{holo}}{\longrightarrow} \mathcal{L}(\mathcal{U},\mathcal{Y}) : \|S(T_1,\cdots,T_d)\| \leq 1 \text{ for all commutative} \\ & \quad T_1,\cdots,T_d \in \mathcal{BL}(\mathcal{K}) \text{ where } S(T_1,\cdots,T_d) = \sum_{n\in\mathbb{Z}_+^d} S_n \otimes T^n \text{ belongs to} \\ & \quad \mathcal{L}(\mathcal{U}\otimes\mathcal{K},\mathcal{Y}\otimes\mathcal{K}) \text{ if } S(z_1,\cdots,z_d) = \sum_{n\in\mathbb{Z}_+^d} S_n z^n \Big\}. \end{aligned}$$

Remarkably, using the von Neumann inequality or the Sz-Nagy dilation theorem ([3], Chapter 10, Section 2), it can be proven that $\mathcal{SA}_{\mathbb{D}}(\mathcal{U},\mathcal{Y}) = \mathcal{S}(\mathcal{U},\mathcal{Y})$. Also for the case of $\mathbb{D}^2$ it can be shown (by Ando dilation theorem) that $\mathcal{SA}_{\mathbb{D}^2}(\mathcal{U},\mathcal{Y}) = \mathcal{S}_{\mathbb{D}^2}(\mathcal{U},\mathcal{Y})$. But for the case $d > 2$ it is only true that $\mathcal{SA}_{\mathbb{D}^d}(\mathcal{U},\mathcal{Y}) \subsetneq \mathcal{S}_{\mathbb{D}^d}(\mathcal{U},\mathcal{Y})$ (see Paulsen [20] for an overview).

For this new setting of Schur-Agler class we have the following equivalences (due to [1]):

1. $S \in \mathcal{SA}_{\mathbb{D}^d}(\mathcal{U},\mathcal{Y})$.

2. There exist positive kernels $K_1, \cdots, K_d$ on $\mathbb{D}^d$ so that
$$I - S(z)S(w)^* = \sum_{k=1}^{d}(1 - z_k\overline{w_k})K_k(z,w). \tag{1.2}$$

3. There exists a contractive
$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$
so that
$$S(z) = D + C(I - Z(z)A)^{-1}Z(z)B,$$
where $\mathcal{X} = \begin{bmatrix} \mathcal{X}_1 \\ \vdots \\ \mathcal{X}_d \end{bmatrix}$ and $Z(z) = \begin{bmatrix} z_1 I_{\mathcal{X}_1} & & \\ & \ddots & \\ & & z_d I_{\mathcal{X}_d} \end{bmatrix}$.

If we use the alternative characterization of positive kernels, i.e. condition (1.1) (with a general domain $\Omega$ in place of the unit disk $\mathbb{D}$ on a function $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{Y})$) is equivalent to the condition:

*There exists a Hilbert space $\mathcal{X}$ and a function $H : \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$ so that $K(z,w)$ has a Kolmogorov decomposition*
$$K(z,w) = H(z)H(w)^*,$$
then we see that (1.2) can be rewritten in the form: *there exists Hilbert spaces $\mathcal{X}_1, \ldots, \mathcal{X}_d$ and functions $H_i : \mathbb{D}^d \to \mathcal{L}(\mathcal{X}_i, \mathcal{Y})$ so that*
$$I - S(z)S(w)^* = \sum_{i=1}^{d} H_i(z)(I - z_i\overline{w_i})I_{\mathcal{X}_i}H_i(w)^*. \tag{1.3}$$

Dritschel and McCullough have developed a different approach for the Schur-Agler class, namely, the test function approach. This approach consists in considering a collection $\Psi$ of holomorphic functions defined over a set $\Omega \subset \mathbb{C}$. Also, let us consider *admissible kernels* $K : \Omega \times \Omega \to \mathbb{C}$, that is $(1 - \psi(z)\overline{\psi(w)})K(z,w)$ positive kernel for each $\psi \in \Psi$. And then define the Schur-Agler class associated to the family $\Psi$, namely

$\mathcal{SA}_\Psi = \{S : \Omega \to \mathbb{C} : (I - S(z)\overline{S(w)})K(z,w)$ positive kernel for all admissible kernels $K\}$.

It has been proved that if we take $\Omega = \mathbb{D}^d$ and $\Psi = \{\psi_1, \ldots, \psi_d\}$ with $\psi_k(z) = z_k$ then $\mathcal{SA}_\Psi = \mathcal{SA}_{\mathbb{D}^d}(\mathbb{C}, \mathbb{C})$.

Under the assumptions that the family $\Psi$ separates point and $\sup_{\psi\in\Psi} |\psi(z)| < 1$ for each $z \in \Omega$, one has the following equivalences (given by Dritschel-Marcantognini-McCullough

[16] and closely related to earlier work of Ambrozie [4]). To state the result one recall the notion of completely positive kernel:

$$\Gamma : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})$$

is completely positive kernel if

$$\sum_{i,j=1}^{M} b_i^* \Gamma(z_i, z_j)[a_i^* a_j] b_j \geq 0 \text{ (as an element of } \mathcal{B}),$$

for all $z_1, \ldots, z_M \in \Omega, a_1, \ldots, a_M \in \mathcal{A}, b_1, \ldots, b_M \in \mathcal{B}$ and $M = 1, 2, \ldots$ (see Section 2.2.3 below).

We introduce the notation $\mathbb{E}(z) \in \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{U}_T, \mathcal{Y}_T)) : \mathbb{E}(z)(\psi) = \psi(z)$ (see Chapter 3, equation (3.2)).

1. $S \in \mathcal{SA}_\Psi$,
2. $S$ has Agler decomposition, that is, there exists a completely positive kernel:

   $$\Gamma : \Omega \times \Omega \to \mathcal{L}(\mathcal{C}_b(\Psi), \mathbb{C}),$$

   so that:

   $$I - S(z)\overline{S(w)} = \Gamma(z, w)(I - \mathbb{E}(z)\overline{\mathbb{E}(w)}),$$

3. $S$ has a contractive transfer function realization, that is there is a colligation:

   $$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathbb{C} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathbb{C}, \end{bmatrix}$$

   where $\mathcal{X}$ =Hilbert space equipped with a $*$-representation $\rho : \mathcal{C}_b(\Psi) \to \mathcal{L}(\mathcal{X})$ so that

   $$S(z) = D + C(I - \rho(\mathbb{E}(z))A)^{-1} \rho(\mathbb{E}(z))B. \tag{1.4}$$

For the case $\Omega = \mathcal{R}$ is a planar domain with $m+1$ holes with boundary $\partial\mathcal{R}$ consisting of $m+1$ connected components $\partial_0\mathcal{R}, \ldots, \partial_m\mathcal{R}$ (with $\partial_0\mathcal{R}$ the boundary of the unbounded components of $\mathbb{C} \setminus \mathcal{R}$), Dritschel-McCullough [17] (when $m = 2$) and Pickering [22] (general $m$) prove that there is a collection of test functions indexed by the $\mathcal{R}$-torus $\mathbb{T}_\mathcal{R} = \partial_0\mathcal{R} \times \cdots \times \partial_m\mathcal{R}$ $\Psi = \{\psi_\alpha : \alpha \in \mathbb{T}_\mathcal{R} = \partial_0\mathcal{R} \times \cdots \partial_m\mathcal{R}\}$ so that $\mathcal{S}_\mathcal{R}(\mathbb{C}, \mathbb{C}) = \mathcal{SA}_\Psi$, and the Agler decomposition ((2) in the latter equivalences) is given by an integral representation, more specifically, there is a completely positive kernel $\Gamma : \mathcal{R} \times \mathcal{R} \to \mathcal{L}(\mathcal{C}(\mathbb{T}_\mathcal{R}), \mathbb{C})$ so that

$$I - S(z)\overline{S(w)} = \Gamma(z, w)[I - \mathbb{E}(z)\overline{\mathbb{E}(w)}] = \int_{\mathbb{T}_\mathcal{R}} H(z, \alpha)(1 - \psi_\alpha(z)\overline{\psi_\alpha(w)})H(w, \alpha)^* \, d\mu(\alpha). \tag{1.5}$$

Note that (1.3) has the form (1.5) (with $\mathbb{D}^d$ in place of $\mathcal{R}$) if $\Psi$ is taken to be the coordinate functions $\Psi = \{\psi_i(z) = z_i : i = 1, \ldots, d\}$, $\{1, 2, \ldots, d\}$ in place of $\mathbb{T}_{\mathcal{R}}$ and $\mu$ is taken to be the discrete measure consisting of a unit point mass at each $i \in \{1, \ldots, d\}$.

The work of [17, 22] left open what is the analogue of (1.5) where $S$ is in the operator-valued Schur-class $\mathcal{S}_{\mathcal{R}}(\mathcal{U}, \mathcal{Y})$ over $\mathcal{R}$, i.e. the set of operator-valued functions over $\mathcal{R}$ with norm less or equal than 1. It turns out that analysis of the form of such an Agler decomposition (1.5) for the square matrix-valued Schur class $\mathcal{S}_{\mathcal{R}}(\mathcal{U}, \mathcal{U})$ is instrumental for one approach to the spectral set question over $\mathcal{R}$, which we now describe.

**Spectral set question** Let $\mathcal{R}$ denote a domain in $\mathbb{C}$ with boundary $\partial\mathcal{R}$. We say that an operator $T$ on a complex Hilbert space $\mathcal{X}$ has $\overline{\mathcal{R}}$ (the closure of $\mathcal{R}$) as a spectral set if $\sigma(T) \subset \overline{\mathcal{R}}$ and

$$\|s(T)\| \leq \|s\|_{\mathcal{R}} = \sup\{|s(z)| : z \in \mathcal{R}\}$$

for every rational function $s$ with poles off $\overline{\mathcal{R}}$.

The operator $T$ on $\mathcal{X}$ has a $\partial\mathcal{R}$-normal dilation if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{X}$ and a normal operator $N$ on $\mathcal{K}$ so that

$$s(T) = P_{\mathcal{X}} s(N)|_{\mathcal{X}},$$

for every rational function $f$ with poles off $X$, where $P_{\mathcal{X}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{X}$. It is easy to show that $\overline{\mathcal{R}}$ is a spectral set for $T$ if $T$ has a $\partial\mathcal{R}$-normal dilation

The **spectral set question** is the converse: if $T$ has $X$ as a spectral set then does it follow that $T$ has a $\partial\mathcal{R}$-normal dilation?

Arveson reformulated the problem as follows: $T$ has a $\partial\mathcal{R}$-normal dilation if and only if the representation $\varphi(s) = s(T)$ is a unital completely contractive algebra homomorphism from $\mathcal{S}_{\mathcal{R}}$ into $\mathcal{L}(\mathcal{X})$. So in Arveson language the spectral question is equivalent to asking whether or not every contractive unital representation $\varphi(s) = s(T)$ is still contractive for $S$ in the matrix-valued Schur class? In other words, given an operator $T$ for which $s(T)$ has norm at most 1 for all scalar-valued Schur class functions $s$, does it follow that $S(T)$ has norm at most 1 for all matrix-valued Schur class functions $S$?

If it can be shown that, if equation (1.5) holds for matrix-valued Schur class in the following form: *given $S$ in the matrix-valued Schur class there exists $\mu$ (still scalar measure), $H(z, \alpha)$ matrix-valued and $\psi_\alpha$ still scalar so that equation (1.5) still holds* then the Arveson reformulation of the spectral set question holds, and hence the spectral set question would have a positive solution.

It was the contribution of Dritschel-McCullough [17] to actually construct a $2 \times 2$ matrix Schur-class function $S$ on a triply connected domain having additional symmetries where the representation (1.5) fails; this was a key step in verifying that the spectral set question itself has a negative answer. We note that Agler-Harland-Raphael [2] also obtained a counterexample to the spectral set question over $\mathcal{R}$ using an alternative computational approach.

Our contribution is to obtain a more general representation which holds for matrix-valued Schur class in general (matrix-valued Schur class has had a important role on operator theory see e.g. [5],[13]). This leads to a realization formula (1.4) for the matrix-valued Schur class over multiply-connected domain $\mathcal{R}$. We actually present general theory for Schur-class with respect to matrix-valued collection of test functions. Just as in the scalar case we arrive at the representation (2) by a Cayley transformation of the corresponding representation function of the Herglotz class

$$\mathcal{H} = \{f \in Hol(\mathcal{R}) : Re\ f \geq 0\}$$

normalized to have $f(t_0) = I$. The integral representation of the Herglotz class in turn, follows from Choquet theory applied to this normalized convex set.

It turns out that an extreme point of this set fails, in general, to be expressible as a matrix-convex linear combination of scalar extreme points. This gives an alternate explanation for the failure rational dilation on multiply connected domains.

This dissertation is organized as follows. After a preliminary chapter reviewing needed basic material concerning reproducing kernel Hilbert spaces, $C^*$-algebras of matrix-valued continuous functions and some convexity analysis, Chapter 3 presents the main result of this thesis, that is, the analysis of operator-valued test-function approach which leads us to equivalent statements like the ones obtained for the Schur class for the unit disk (and the bidisk) and the Schur-Agler class of $\mathbb{D}^d$. The fourth chapter presents the Herglotz class of finitely connected planar domains and the analysis of its extreme points. Finally the last chapter uses the theory developed in Chapter 4 to provide an alternative perspective on the spectral set question for the case of a finitely connected planar domain.

# Chapter 2

# Preliminaries

## 2.1 $(\mathcal{A}, \mathcal{B})$-correspondences

We introduce some basic concepts of Hilbert $C^*$-Modules and correspondences following [19, 23].

Let $\mathcal{B}$ a $C^*$-algebra and $E$ a linear space. We say that $E$ is a (right) *pre-Hilbert $C^*$ module* over $\mathcal{B}$ if $E$ is a right module over $\mathcal{B}$ and is endowed with a $\mathcal{B}$-valued inner product $\langle \cdot, \cdot \rangle_E$ satisfying the following axioms for any $\lambda, \mu \in \mathbb{C}, e, f, g \in E$ and $b \in \mathcal{B}$:

1. $\langle \lambda e + \mu f, g \rangle_E = \lambda \langle e, g \rangle_E + \mu \langle f, g \rangle_E$;
2. $\langle e \cdot b, f \rangle_E = \langle e, f \rangle_E b$;
3. $\langle e, f \rangle_E^* = \langle f, e \rangle_E$;
4. $\langle e, e \rangle_E \geq 0$ (as an element of $\mathcal{B}$);
5. $\langle e, e \rangle = 0$ implies $e = 0$;
6. $(\lambda e) \cdot b = e \cdot (\lambda b)$.

If $E$ is a pre-Hilbert space module over $\mathcal{B}$, then $E$ is a normed linear space with norm given by

$$\|e\|_E = \|\langle e, e \rangle^{1/2}\|_{\mathcal{B}}.$$

Taking the completion of $E$ with this norm we get what we shall call a *Hilbert $C^*$-module over $\mathcal{B}$.*

Now we introduce the notion of an $(\mathcal{A}, \mathcal{B})$-correspondence. If $E$ is a right Hilbert $C^*$-module over $\mathcal{B}$ and $\mathcal{A}$ is another $C^*$-algebra, we say that $E$ is a $(\mathcal{A},\mathcal{B})$-*correspondence* if $E$ is also a left module over $\mathcal{A}$ which makes $E$ an $(\mathcal{A},\mathcal{B})$-bimodule:

$$(a \cdot e) \cdot b = a \cdot (e \cdot b) \text{ for all } a \in \mathcal{A}, e \in E \text{ and } b \in \mathcal{B}$$

$$(\lambda a) \cdot e = a \cdot (\lambda e)$$

with the additional compatibility condition

$$\langle a \cdot e, f\rangle_E = \langle e, a^* \cdot f\rangle_E.$$

We shall need the following definition.

Suppose that we are given three $C^*$-algebras $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ together with an $(\mathcal{A},\mathcal{B})$-correspondence $E$ and a $(\mathcal{B},\mathcal{C})$-correspondence $F$. Then we define the tensor product correspondence $E \otimes_{\mathcal{B}} F$ (usually abbreviated to $E \otimes F$) to be the completion of the linear span of all tensors $e \otimes f$ (with $e \in E$ and $f \in F$) subject to the identification

$$(e \cdot b) \otimes f = e \otimes (b \cdot f),$$

with left $\mathcal{A}$-action given by

$$a \cdot (e \otimes f) = (a \cdot e) \otimes f,$$

with right $\mathcal{C}$-action given by

$$(e \otimes f) \cdot c = e \otimes (f \cdot c),$$

and with $\mathcal{C}$-valued inner product $\langle \cdot, \cdot \rangle_{E\otimes F}$ given by

$$\langle e \otimes f, e' \otimes f'\rangle_{E\otimes F} = \langle \langle e, e'\rangle_E \cdot f, f'\rangle_F.$$

**Theorem 2.1.1.** *The inner product $\langle \cdot, \cdot \rangle_{E\otimes F}$ is positive semidefinite.*

*Proof.* For $e_1, \dots, e_N \in E, f_1, \dots, f_N \in F$ and $N = 1, 2, \dots$

$$\begin{aligned}
\sum_{i,j=1}^{N} \langle e_j \otimes f_j, e_i \otimes f_i\rangle_{E\otimes F} &= \sum_{i,j=1}^{N} \langle \langle e_j, e_i\rangle f_j, f_i\rangle_F \\
&= \sum_{i,j=1}^{N} \langle M_{ij} f_j, f_i\rangle_F, \,(\, M_{ij} = \langle e_j, e_i\rangle) \\
&= \langle M \cdot \underline{f}, \underline{f}\rangle_{F^N}, \left( M = (M_{ij})_{N\times N}, \underline{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} \right),
\end{aligned}$$

so

$$\sum_{i,j=1}^{N} \langle e_j \otimes f_j, e_i \otimes f_i\rangle_{E\otimes F} = \langle M \cdot \underline{f}, \underline{f}\rangle_{F^N}. \tag{2.1}$$

We have $M_{ij}$ can be seen as $M_{ij} = \langle e_j, e_i \rangle = e_i^* e_j$ (multiplication on the left by $e_i^*$, where $e_i^* : E \to \mathcal{B}$ is given by $e_i^* = \langle \cdot, e_i \rangle$). Thus for $e_1, \ldots, e_r, b_1, \ldots, b_r$ and $r = 1, 2, \ldots$ we have

$$\sum_{i,j=1}^{r} b_i^* e_i^* e_j b_j = \left\langle \sum_{j=1}^{r} e_j b_j, \sum_{i=1}^{r} e_i b_i \right\rangle_E \geq 0 \text{ (as an element of } \mathcal{B}),$$

which proves that $M$ is positive as an element of $\mathcal{B}^{N \times N}$. Therefore by general $C^*$-algebra theory applied to the $C^*$-algebra $\mathcal{B}^{N \times N}$, $M$ has a positive semidefinite square root $M^{1/2}$, and (2.1) simplifies to

$$\sum_{i,j=1}^{N} \langle e_j \otimes f_j, e_i \otimes f_i \rangle_{E \otimes F} = \langle M \cdot \underline{f}, \underline{f} \rangle_{F^N} = \langle M^{1/2} \cdot \underline{f}, M^{1/2} \cdot \underline{f} \rangle_{F^N} \geq 0.$$

□

*Remark* 2.1.1. Given a Hilbert space $\mathcal{H}$ we can see $\mathcal{H}$ as a right Hilbert $C^*$-module by making $\mathcal{B} = \mathbb{C}$ (complex numbers) and defining the right $\mathcal{B}$-action by

$$e \cdot b = be \text{ (scalar multiplication of } b \in \mathbb{C} \text{ and } e \in \mathcal{H})$$

and

$$\langle e, f \rangle_E^* = \overline{\langle e, f \rangle_E} \text{ (complex conjugate of } \langle f, e \rangle_E).$$

Moreover, $\mathcal{H}$ can be seen as $(\mathcal{A}, \mathcal{B})$-correspondence by taking $\mathcal{A} = \mathbb{C}$ and $\mathcal{B} = \mathbb{C}$, and defining

$$(a \cdot e) \cdot b = a \cdot (e \cdot b) = (ab)e \text{ (scalar multiplication), for all } a, b \in \mathbb{C} \text{ and } e \in \mathcal{H}.$$

Thus, given two Hilbert spaces $E$ and $F$ we can also define the *Hilbert space tensor product* $E \otimes F$ as the tensor product of the $(\mathbb{C}, \mathbb{C})$-correspondence $E$ and $(\mathbb{C}, \mathbb{C})$-correspondence $F$ subject to the identification

$$(be) \otimes f = e \otimes (bf),$$

with left $\mathbb{C}$-action given by

$$a(e \otimes f) = (ae) \otimes f,$$

with right $\mathbb{C}$-action given by

$$c(e \otimes f) = e \otimes (cf),$$

where $a, b, c \in \mathbb{C}$ and $e \in E, f \in F$,
and with $\mathbb{C}$-valued inner product $\langle \cdot, \cdot \rangle_{E \otimes F}$ given by

$$\langle e \otimes f, e' \otimes f' \rangle_{E \otimes F} = \langle \langle e, e' \rangle_E f, f' \rangle_F = \langle e, e' \rangle_E \langle f, f' \rangle_E,$$

for $e, e' \in E$ and $f, f' \in F$.

We note that it is standard in much of the literature to define a Hilbert $C^*$-module to have $\mathcal{B}$-valued inner product linear in the second variable and conjugate linear in the first variable (see e.g. [19]). We prefer to extend the convention for Hilbert spaces in the mathematical literature to the Hilbert-module context.

## 2.2 Reproducing Kernel Hilbert Spaces

### 2.2.1 Positive Kernels

Suppose we are given a Hilbert space $\mathcal{E}$ and a set $\Omega$. We let $\mathcal{L}(\mathcal{E})$ denote the space of bounded linear operators on $\mathcal{E}$. We say that:

$$K : \Omega \times \Omega \to \mathcal{L}(\mathcal{E})$$

is a *positive kernel* if:

$$\sum_{i,j=1}^{M} \langle K(w_i, w_j)e_j, e_i \rangle \geq 0,$$

for all $w_1, \dots, w_M \in \Omega$, $e_1, \dots, e_M \in \mathcal{E}$ and $M = 1, 2, 3, \dots$.

The following result is well known (see [3]), and is a standard extension of the main result of [7] to the case of operator-valued kernels.

**Theorem 2.2.1.** *Given $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{E})$. The following are equivalent:*

1. *$K$ is a positive kernel, i.e.*

$$\sum_{i,j=1}^{M} \langle K(w_i, w_j)e_j, e_i \rangle_{\mathcal{E}} \geq 0,$$

*for all $w_1, \dots, w_M \in \Omega$, $e_1, \dots, e_M \in \mathcal{E}$ and $M = 1, 2, 3, \dots$.*

2. *There is a Hilbert Space $\mathcal{H}$ of $\mathcal{E}$-valued functions on $\Omega$ such that*
   (a) *$K(\cdot, w)e \in \mathcal{H}$ for each $w \in \Omega$ and $e \in \mathcal{E}$, and,*
   (b) *$\langle f, K(\cdot, w)e \rangle_{\mathcal{H}} = \langle f(w), e \rangle_{\mathcal{E}}$, for each $w \in \Omega$ and $e \in \mathcal{E}$.*

3. *$K$ has a Kolmogorov decomposition, i.e.*

$$K(z, w) = H(z)H(w)^* \in \mathcal{L}(\mathcal{E}),$$

*for some $H : \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{E})$ where $\mathcal{X}$ is a Hilbert space.*

*Proof.* (1) $\Rightarrow$ (2) Define an inner product on the space $\mathcal{H}_0$ of finite linear combinations of kernel elements $k_w e$ ($w \in \Omega$, $e \in \mathcal{E}$) by

$$\left\langle \sum_{j=1}^{N} k_{w_j} e_j, \sum_{i=1}^{N} k_{w_i} e_i \right\rangle_{\mathcal{H}_0} = \sum_{i,j=1}^{N} \langle K(w_i, w_j)e_j, e_i \rangle_{\mathcal{E}}$$

(the positive kernel assumption guarantees that the inner product is positive semidefinite).

Let $\mathcal{N}$ be the set of all elements $\mathcal{H}_0$ having self inner product equal to 0. A consequence of the Schwarz inequality is that the inner product is well defined and still positive semidefinite on $\mathcal{H}_0/\mathcal{N}$. Let $\mathcal{H}$ be the Hilbert space completion of $\mathcal{H}_0/\mathcal{N}$. Elements $f$ of $\mathcal{H}$ can be identified as $\mathcal{E}$-valued functions on $\Omega$ via the formula

$$\langle f(w), e\rangle_{\mathcal{E}} = \langle f, k_w e + \mathcal{N}\rangle_{\mathcal{H}}.$$

When this is done, then an original kernel elements $k_w e + \mathcal{N}$ is identified with the function

$$(k_w e + \mathcal{N})(z) = K(z,w)e.$$

(2) $\Rightarrow$ (3) Take $\mathcal{X} = \mathcal{H}$. Define

$$H : \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{E})$$
$$H(z) : f \mapsto f(z), \text{ for each } z \in \Omega.$$

Then

$$H(w)^* : \mathcal{E} \to \mathcal{X}$$

is given by

$$e \mapsto K(\cdot, w)e.$$

Thus

$$H(z)H(w)^* e = H(z)(K(\cdot, w)e) = K(z,w)e. \text{ for all } e \in \mathcal{E},$$

therefore

$$K(z,w) = H(z)H(w)^*.$$

(3) $\Rightarrow$ (1) For $w_1, \ldots, w_M \in \Omega$ and $e_1, \ldots, e_M \in \mathcal{E}$ and $M = 1,2,3\ldots$ we have

$$\begin{aligned}
\sum_{i,j=1}^{M} \langle K(w_i, w_j)e_j, e_i\rangle_{\mathcal{E}} &= \sum_{i,j=1}^{M} \langle H(w_i)H(w_j)^* e_j, e_i\rangle_{\mathcal{E}} \\
&= \left\langle \sum_{j=1}^{M} H(w_j)^* e_j, \sum_{i=1}^{M} H(w_i)^* e_i \right\rangle_{\mathcal{X}} \\
&= \left\| \sum_{j=1}^{M} H(w_j)^* e_j \right\|^2 \geq 0,
\end{aligned}$$

and conclude that $K$ is a positive kernel. $\square$

*Remark* 2.2.1. If we are given a Hilbert space $\mathcal{H}$ whose elements are functions $f : \Omega \to \mathcal{E}$ such that for each $w \in \Omega$, $E_w : f \mapsto f(w)$ is bounded then it is possible to find a positive kernel $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{E})$ such that $\mathcal{H} = \mathcal{H}(K)$ and satisfying

1. $K(\cdot, w)e \in \mathcal{H}$, for each $w \in \Omega$ and $e \in \mathcal{E}$; and
2. $\langle f, K(\cdot, w)e\rangle_{\mathcal{H}} = \langle f(w), e\rangle_{\mathcal{E}}$ for each $w \in \Omega$ and $e \in \mathcal{E}$.

In fact, defining $K(z, w) = E_z E_w^*$ we get the required positive kernel satisfying conditions (1) and (2).

We now develop some results for matrix-valued kernels which we shall need which are close to known results but do not appear in the literature in the precise form which we need.

Define the space $\mathcal{H}(K) \otimes \mathcal{U}$ to be the Hilbert space tensor product of $\mathcal{H}(K)$ and $\mathcal{C}_2(\mathcal{U}, \mathcal{E})$ (where $\mathcal{C}_2(\mathcal{U}, \mathcal{E})$ is collection of Hilbert Schmidt operator from $\mathcal{U}$ to $\mathcal{E}$). Elements of $\mathcal{H}(K)\otimes\mathcal{U}$ can be seen as functions $f : \Omega \to \mathcal{C}_2(\mathcal{U}, \mathcal{E})$.

We have

$$K(\cdot, w)U \in \mathcal{H}(K) \otimes \mathcal{U}, \text{ for each } U \in \mathcal{C}_2(\mathcal{U}, \mathcal{E})$$

and

$$\langle f, K(\cdot, w)U\rangle_{H(K)\otimes\mathcal{U}} = tr(U^* f(w)).$$

We shall be particularly interested in right multiplication operators $R_\psi$ acting between two such tensor-product spaces defined as follows. For $\psi : \Omega \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$, define

$$R_\psi : \mathcal{H}(K) \otimes \mathcal{Y} \to \mathcal{H}(K) \otimes \mathcal{U}$$

$$f(z) \to f(z)\psi(z).$$

We assume that $R_\psi$ so defined maps $\mathcal{H}(K)\otimes\mathcal{Y}$ onto $\mathcal{H}(K)\otimes\mathcal{U}$. Let us compute $(R_\psi)^* K(\cdot, w)U$, for $U \in \mathcal{C}_2(\mathcal{U}, \mathcal{E})$.

$$\begin{aligned}
\langle f, (R_\psi)^* K(\cdot, w)U\rangle_{\mathcal{H}(K)\otimes\mathcal{Y}} &= \langle R_\psi f, K(\cdot, w)U\rangle_{\mathcal{H}(K)\otimes\mathcal{U}} \\
&= \langle (R_\psi f)(w), U\rangle_{\mathcal{C}_2(\mathcal{U},\mathcal{E})} \\
&= \langle f(w)\psi(w), U\rangle_{\mathcal{C}_2(\mathcal{U},\mathcal{E})} \\
&= tr[U^* f(w)\psi(w)] \\
&= tr[f(w)\psi(w)U^*] \\
&= tr[f(w)(U\psi(w)^*)^*] \\
&= tr[(U\psi(w)^*)^* f(w)] \\
&= \langle f(w), U\psi(w)^*\rangle_{\mathcal{C}_2(\mathcal{Y},\mathcal{E})} \\
&= \langle f, K(\cdot, w)U\psi(w)^*\rangle_{\mathcal{H}(K)\otimes\mathcal{Y}},
\end{aligned}$$

for all $f \in \mathcal{H}(K) \otimes \mathcal{Y}$, therefore

$$(R_\psi)^* K(\cdot, w)U = K(\cdot, w)U\psi(w)^*.$$

**Proposition 2.2.2.**

$$\|R_\psi\| \leq 1 \textit{ as an operator from } \mathcal{H}(K)\otimes\mathcal{Y} \to \mathcal{H}(K)\otimes\mathcal{U}$$

$\Leftrightarrow$

$$0 \leq \sum_{i,j=1}^{M} tr[X(w_i)K(w_i,w_j)X(w_j)^*] - tr[\psi(w_i)X(w_i)K(w_i,w_j)X(w_j)^*\psi(w_j)^*]$$

$$= \sum_{i,j=1}^{M} tr[X(w_j)^*(I-\psi(w_j)^*\psi(w_i))X(w_i)K(w_i,w_j)]$$

$$\textit{for all } w_1,\ldots,w_M \in \Omega,\ X:\Omega\to\mathcal{C}_2(\mathcal{E},\mathcal{U}) \textit{ and } M=1,2,\ldots$$

$\Leftrightarrow$

$$k_{X,\psi,K}(z,w) = tr[X(w)^*(I-\psi(w)^*\psi(z))X(z)K(z,w)],\ \textit{is a positive kernel from}$$

$$\Omega\times\Omega \textit{ to } \mathbb{C} \textit{ for all functions } X:\Omega\to\mathcal{C}_2(\mathcal{E},\mathcal{U}).$$

*Proof.* Note that $\|R_\psi\| \leq 1$ if and only if

$$\left\|\sum_{j=1}^{M} K(\cdot,w_j)X(w_j)^*\right\|^2 - \left\|(R_\psi)^*\sum_{j=1}^{M} K(\cdot,w_j)X(w_j)^*\right\|^2 \geq 0,$$

$$\text{for all } w_1,\cdots,w_M\in\Omega,\ X:\Omega\to\mathcal{C}_2(\mathcal{U},\mathcal{E}) \text{ and } M=1,2,\cdots.$$

$\Leftrightarrow$

$$\left\|\sum_{j=1}^{M} K(\cdot,w_j)X(w_j)^*\right\|^2 - \left\|\sum_{j=1}^{M} K(\cdot,w_j)X(w_j)^*\psi(w_j)^*\right\|^2 \geq 0,$$

$\Leftrightarrow$

$$\left\langle \sum_{j=1}^{M} K(\cdot,w_j)X(w_j)^*, \sum_{i=1}^{M} K(\cdot,w_i)X(w_i)^*\right\rangle_{\mathcal{H}(K)\otimes\mathcal{U}}$$

$$-\left\langle \sum_{j=1}^{M} K(\cdot,w_j)X(w_j)^*\psi(w_j)^*, \sum_{i=1}^{M} K(\cdot,w_i)X(w_i)^*\psi(w_i)^*\right\rangle_{\mathcal{H}(K)\otimes\mathcal{Y}} \geq 0,$$

$\Leftrightarrow$

$$\sum_{i,j=1}^{M}\left\langle K(w_i,w_j)X(w_j)^*, X(w_i)^*\right\rangle_{\mathcal{C}_2(\mathcal{U},\mathcal{E})}$$

$$-\sum_{i,j=1}^{M}\left\langle K(w_i,w_j)X(w_j)^*\psi(w_j)^*, X(w_i)^*\psi(w_i)^*\right\rangle_{\mathcal{C}_2(\mathcal{Y},\mathcal{E})} \geq 0,$$

$\Leftrightarrow$

$$\sum_{i,j=1}^{M} tr[X(w_i)K(w_i,w_j)X(w_j)^*] - \sum_{i,j=1}^{M} tr[\psi(w_i)X(w_i)K(w_i,w_j)X(w_j)^*\psi(w_j)^*] \geq 0,$$

$\Leftrightarrow$

$$\sum_{i,j=1}^{M} tr[X(w_j)^*X(w_i)K(w_i,w_j)] - \sum_{i,j=1}^{M} tr[X(w_j)^*\psi(w_j)^*\psi(w_i)X(w_i)K(w_i,w_j)] \geq 0,$$

$\Leftrightarrow$

$$\sum_{i,j=1}^{M} tr[X(w_j)^*(I-\psi(w_j)^*\psi(w_i))X(w_i)K(w_i,w_j)] \geq 0.$$

$\Leftrightarrow$

$$k_{X,\psi,K}(z,w) = tr[X(w)^*(I-\psi(w)^*\psi(z))X(z)K(z,w)]$$

is a positive kernel from $\Omega \times \Omega$ to $\mathbb{C}$ for each $\psi \in \Psi$, and $X : \Omega \to \mathcal{C}_2(\mathcal{E},\mathcal{U})$. $\square$

### 2.2.2 Dual kernel basis

Dual kernel basis for the scalar case appears in the work of Dristchel-Marcantognini-McCullough [16].

Assume $\Omega = \{z_1, \dots, z_N\}$ consists of finitely many points and $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{Y})$ is a strictly positive kernel on $\Omega$. Then we may form the *dual kernel basis* for $\mathcal{H}(\mathcal{K})$ consisting of functions

$$\{\delta_{z_i} \otimes y : i = 1, \dots, N; y \in \mathcal{Y}\}$$

where $\delta_{z_i} \otimes y$ is given by

$$(\delta_{z_i} \otimes y)(z) = \begin{cases} y, & z = z_i \\ 0, & z \neq z_i \end{cases}.$$

Define the matrix $[L(z_i,z_j)]_{i,j=1}^N$ by

$$\langle L(z_i,z_j)y, y'\rangle_{\mathcal{Y}} = \langle \delta_{z_j} \otimes y, \delta_{z_i} \otimes y'\rangle_{\mathcal{H}(\mathcal{K})}.$$

One can compute

$$[L(z_i,z_j)]_{i,j=1}^N = \left([K(z_i,z_j)]_{i,j=1}^N\right)^{-1}.$$

In fact, given any $f \in \mathcal{H}(\mathcal{K})$, we have $f = \sum_{j'=1}^{N} \delta_{z_{j'}} \otimes f(z_{j'})$, then

$$\begin{aligned}
\langle f(z_j), y\rangle_{\mathcal{Y}} &= \langle f, K(\cdot, z_j)y\rangle_{\mathcal{H}(\mathcal{K})} \\
&= \left\langle f, \sum_{l=1}^{N} \delta_{z_l} \otimes K(z_l, z_j)y \right\rangle_{\mathcal{H}(\mathcal{K})} \\
&= \left\langle \sum_{j'}^{N} \delta_{z_{j'}} \otimes f(z_{j'}), \sum_{l=1}^{N} \delta_{z_l} \otimes K(z_l, z_j)y \right\rangle_{\mathcal{H}(\mathcal{K})} \\
&= \sum_{j',l=1}^{N} \langle L(z_l, z_{j'})f(z_{j'}), K(z_l, z_j)y\rangle_{\mathcal{H}(\mathcal{K})} \\
&= \sum_{j',l=1}^{N} \langle K(z_j, z_l)L(z_l, z_{j'})f(z_{j'}), y\rangle_{\mathcal{Y}}, \text{ for all } y \in \mathcal{Y}.
\end{aligned}$$

We conclude

$$\sum_{j',l=1} K(z_j, z_l)L(z_l, z_{j'})f(z_{j'}) = f(z_j),$$

for all $f(z_1), \ldots, f(z_N) \in \mathcal{E}$ and therefore

$$[L(z_i, z_j)]_{i,j=1}^{N} = \left([K(z_i, z_j)]_{i,j=1}^{N}\right)^{-1}.$$

**Theorem 2.2.3.** *Given $F$ be an $\mathcal{L}(\mathcal{E}, \mathcal{E}_*)$-valued function on $\Omega$, the following are equivalent:*

1. *$R_F : \mathcal{H}(\mathcal{K}) \otimes \mathcal{E}_* \to \mathcal{H}(\mathcal{K}) \otimes \mathcal{E}$ is contractive.*
2. $$\sum_{i,j=1}^{N} tr[X(z_j)^*(I - F(z_j)^*F(z_i))X(z_i)K(z_i, z_j)] \geq 0,$$
   *for all choices of function $X : \Omega \to \mathcal{C}_2(\mathcal{Y}, \mathcal{E})$.*
3. $$\sum_{i,j=1}^{N} tr[L(z_i, z_j)f(z_j)(I - F(z_j)F(z_i)^*)f(z_i)^*] \geq 0,$$
   *for all choices of function $f : \Omega \to \mathcal{C}_2(\mathcal{E}_*, \mathcal{Y})$.*

*Proof.* (1) $\Leftrightarrow$ (2) amounts to Proposition 2.2.2.

(1) $\Leftrightarrow$ (3) Using dual kernels basis, we have that $\|R_F\| \leq 1$ if and only if

$$\begin{aligned} 0 \leq \|f\|^2 - \|R_F f\|^2 &= \sum_{i,j=1}^{N} \langle \delta_{z_j} \otimes f(z_j), \delta_{z_i} \otimes f(z_i) \rangle_{\mathcal{H}(\mathcal{K}) \otimes \mathcal{E}_*} \\ &\quad - \sum_{i,j=1}^{N} \langle \delta_{z_j} \otimes f(z_j) F(z_j), \delta_{z_i} \otimes f(z_i) F(z_i) \rangle_{\mathcal{H}(\mathcal{K}) \otimes \mathcal{E}} \\ &= \sum_{i,j=1}^{N} \langle L(z_i, z_j) f(z_j), f(z_i) \rangle_{\mathcal{C}_2(\mathcal{E}_*, \mathcal{Y})} \\ &\quad - \sum_{i,j=1}^{N} \langle L(z_i, z_j) f(z_j) F(z_j), f(z_i) F(z_i) \rangle_{\mathcal{C}_2(\mathcal{E}, \mathcal{Y})} \\ &= \sum_{i,j=1}^{N} tr[L(z_i, z_j) f(z_j)(I - F(z_j) F(z_i)^*) f(z_i)^*]. \end{aligned}$$

□

### 2.2.3 Completely Positive Kernels

Given two unital $\mathcal{C}^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ and a set of points $\Omega$, we say

$$\Gamma : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})$$

is a *completely positive kernel* if

$$\sum_{i,j=1}^{M} b_i^* \Gamma(z_i, z_j)[a_i^* a_j] b_j \geq 0 \text{ (as an element of } \mathcal{B}),$$

for all $z_1, \ldots, z_M \in \Omega, a_1, \ldots, a_M \in \mathcal{A}, b_1, \ldots, b_M \in \mathcal{B}$ and $M = 1, 2, \ldots$

In what follows, the symbol $\bigvee$ stands for the closed linear span. The following refines the results of [12] and [14].

**Theorem 2.2.4.** *Given $\Gamma : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})$, the following are equivalent:*

1. *$\Gamma$ is a completely positive kernel.*
2. *There is a $(\mathcal{A}, \mathcal{B})$-correspondence $\boldsymbol{\mathcal{H}}(\Gamma)$ whose elements are $\mathcal{B}$-valued functions*

$$(w, a) \mapsto f(w, a) \text{ on } \Omega \times \mathcal{A}$$

*which are conjugate linear in the $\mathcal{A}$-argument with left $\mathcal{A}$-action given by*

$$a_0 \cdot f(w,a) = f(w, a_0^* a)$$

*such that*

*(a)* $k_{w,a_0}$ *is given by*

$$k_{w,a_0}(z,a) = \Gamma(z,w)[a^* a_0]$$

*is in* $\boldsymbol{\mathcal{H}}(\Gamma)$ *for all* $w \in \Omega$ *and* $a_0 \in \mathcal{A}$*, and*

*(b)*

$$f(w,a_0) = \langle f, k_{w,a_0} \rangle_{\boldsymbol{\mathcal{H}}(\Gamma)}$$

*for all* $w \in \Omega,\ a_0 \in \mathcal{A}$.

3. $\Gamma$ *has a Kolmogorov decomposition, i.e., there exists an* $(\mathcal{A},\mathcal{B})$*-correspondence* $\boldsymbol{\mathcal{H}}$ *and a mapping* $w \mapsto k_w$ *from* $\Omega$ *to* $\boldsymbol{\mathcal{H}}$ *such that*

$$\Gamma(z,w)[a] = \langle a \cdot k_w, k_z \rangle_{\boldsymbol{\mathcal{H}}}.$$

*Proof.* (1) $\Longrightarrow$ (2) Take $\boldsymbol{\mathcal{H}}(\Gamma) = \bigvee\{a \cdot k_w b;\ a \in \mathcal{A}, b \in \mathcal{B} \text{ and } w \in \Omega\}$, with inner product

$$\left\langle \sum_{j=1}^{M} a_j \cdot k_{w_j} b_j, \sum_{i=1}^{M} a_i \cdot k_{w_i} b_i \right\rangle_{\boldsymbol{\mathcal{H}}(\Gamma)} = \sum_{i,j=1}^{M} b_i^* \Gamma(w_i,w_j)[a_i^* a_j] b_j.$$

(2) $\Longrightarrow$ (3) Take $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}(\Gamma)$ and define

$$J : \Omega \to \boldsymbol{\mathcal{H}}(\Gamma)$$
$$w \mapsto k_{w,\mathbb{1}}.$$

Then, clearly, by definition of the inner product $\langle \cdot, \cdot \rangle_{\boldsymbol{\mathcal{H}}(\Gamma)}$.

$$\langle a \cdot k_{w,\mathbb{1}}, k_{z,\mathbb{1}} \rangle_{\boldsymbol{\mathcal{H}}(\Gamma)} a \cdot k_{w,\mathbb{1}}(z,1) = \Gamma(z,w)[a].$$

(3) $\Longrightarrow$ (1) For all $z_1,\ldots,z_M \in \Omega, a_1,\ldots,a_M \in \mathcal{A}, b_1,\ldots,b_M \in \mathcal{B}$ and $M = 1,2,\ldots,$ we have

$$\begin{aligned}
\sum_{i,j=1}^{M} b_i^* \Gamma(z_i,z_j)[a_i^* a_j] b_j &= \sum_{i,j=1}^{M} b_i^* \langle a_i^* a_j k_{w_j}, k_{w_i} \rangle_{\boldsymbol{\mathcal{H}}} b_j \\
&= \sum_{i,j=1}^{M} \langle a_j \cdot k_{w_j} b_j, a_i \cdot k_{w_i} b_i \rangle_{\boldsymbol{\mathcal{H}}} \\
&= \left\langle \sum_{j=1}^{M} a_j \cdot k_{w_j} b_j, \sum_{i=1}^{M} a_i \cdot k_{w_i} b_i \right\rangle_{\boldsymbol{\mathcal{H}}} \geq 0
\end{aligned}$$

since the $\mathcal{B}$-valued $\mathcal{H}$-inner product is positive semidefinite. $\square$

Note that in general

$$(a_1 \cdot k_{w,a_0})(z,a) = k_{w,a_1 a_0}(z,a) = \Gamma(z,w)[a^* a_1 a_0] = k_{w,a_0}(z, a_1^* a).$$

### 2.2.4 Schur Products

Given two positive scalar kernels one way to construct another positive kernel is multiplying them pointwise (*Schur product*). For a non-commutative $\mathcal{C}^*$ algebra this operation is possible but not always will preserve positive definiteness. We will have to replace the pointwise multiplication by the pointwise composition of mappings, this clearly includes the usual scalar Schur product if we identify $z \in \mathbb{C}$ as $w \to zw$ in $\mathbb{C}$.

Let us now have two completely positive kernels $\Gamma : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{B})$ (with $\mathcal{A}$ a unital algebra) and $\Upsilon : \Omega \times \Omega \to \mathcal{L}(\mathcal{B}, \mathcal{C})$. Following [14] we define the *Schur product* of $\Gamma$ and $\Upsilon$ by setting

$$(\Upsilon \circ \Gamma)(z, w)[a] = \Upsilon(z, w)\Big[\Gamma(z, w)[a]\Big].$$

**Theorem 2.2.5.** *$\Upsilon \circ \Gamma : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{C})$ is a completely positive kernel.*

*Proof.* By Theorem 2.2.4 part (3):

$$\Upsilon(z, w)[l] = \langle l \cdot k_w^\Upsilon, k_z^\Upsilon \rangle_{\boldsymbol{\mathcal{H}}(\Upsilon)}$$

and

$$\Gamma(z, w)[a] = \langle a \cdot k_w^\Gamma, k_z^\Gamma \rangle_{\boldsymbol{\mathcal{H}}(\Gamma)}.$$

In the following computation we use the correspondence tensor product construction given by Theorem 2.1.1.

We compute

$$\begin{aligned} (\Upsilon \circ \Gamma)(z, w)[a] &= \Upsilon(z, w)\Big[\Gamma(z, w)[a]\Big] \\ &= \langle \Gamma(z, w)[a] \cdot k_w^\Upsilon, k_z^\Upsilon \rangle_{\boldsymbol{\mathcal{H}}(\Upsilon)} \\ &= \langle \langle a \cdot k_w^\Gamma, k_z^\Gamma \rangle_{\boldsymbol{\mathcal{H}}(\Gamma)} \cdot k_w^\Upsilon, \cdot k_z^\Upsilon \rangle_{\boldsymbol{\mathcal{H}}(\Upsilon)} \\ &= \langle (a \cdot k_w^\Gamma) \otimes k_w^\Upsilon, k_z^\Gamma \otimes k_z^\Upsilon \rangle_{\boldsymbol{\mathcal{H}}(\Gamma) \otimes \boldsymbol{\mathcal{H}}(\Upsilon)}, \\ &= \langle a \cdot (k_w^\Gamma \otimes k_w^\Upsilon), k_z^\Gamma \otimes k_z^\Upsilon \rangle_{\boldsymbol{\mathcal{H}}(\Gamma) \otimes \boldsymbol{\mathcal{H}}(\Upsilon)}, \end{aligned}$$

which by Theorem 2.2.4 implies $\Upsilon \circ \Gamma$ is a completely positive kernel. □

A immediate consequence of Theorem 2.2.5 is the following corollary.

**Corollary 2.2.6.**

$$\boldsymbol{\mathcal{H}}(\Upsilon \circ \Gamma) = \boldsymbol{\mathcal{H}}(\Gamma) \otimes \boldsymbol{\mathcal{H}}(\Upsilon).$$

*Remark.* For the particular case $\mathcal{B} = \mathcal{L}(\mathcal{E})$ ( for some Hilbert space $\mathcal{E}$), $\boldsymbol{\mathcal{H}}$ in Theorem 2.2.4 (part *(2)*) will take the form $\mathcal{H}(\Gamma) = \boldsymbol{\mathcal{H}}(\Gamma) \otimes \mathcal{E}$. Then the equivalences in Theorem 2.2.4 will be given by the following theorem.

**Theorem 2.2.7.** *Given $\Gamma : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{E}))$. The following are equivalent:*

1. *$\Gamma$ is a completely positive kernel.*
2. *There is a Hilbert space $\mathcal{H} = \mathcal{H}(\Gamma)$ of $\mathcal{E}$-valued functions on $\Omega \times \mathcal{A}$ conjugate-linear in $\mathcal{A}$-argument, equipped with a $*$-representation $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H}(\Gamma))$ such that:*
   *(a) $\Gamma(\cdot, w)[\cdot^* a]e \in \mathcal{H}(\Gamma)$ for each $w \in \Omega$, $a \in \mathcal{A}$ and $e \in \mathcal{E}$.*
   *(b) $\langle f, \Gamma(\cdot, w)[\cdot^* a]e\rangle_{\mathcal{H}(\Gamma)} = \langle (f(w, a), e\rangle_{\mathcal{E}}$.*
3. *$\Gamma$ has a Kolmorogov decomposition, i.e. there exists a Hilbert space $\mathcal{X}$ carrying a $*$-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{X})$ and $H : \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{E})$ so that*
$$\Gamma(z, w)[a] = H(z)\pi(a)H(w)^*.$$

## 2.3 Representation of $C^*$-algebras of matrix-valued continuous functions

We will need the following definitions:

1. Given a $C^*$-algebra, a Hilbert space $\mathcal{H}$, we will say the representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is *nondegenerate* if the $C^*$-algebra of operators $\pi(\mathcal{A})$ has trivial null space.

   Note that since $\pi$ is self-adjoint, then $\pi$ is nondegenerate is equivalent to the assertion that the closed linear span $[\pi(\mathcal{A})\mathcal{H}]$ of all vectors of the form $\pi(x)\xi$, $x \in \mathcal{A}$, $\xi \in \mathcal{H}$, is all of $\mathcal{H}$.
2. A representation $\pi$ of $\mathcal{A}$ is called *cyclic* if there is $\xi \in \mathcal{H}$ such that $[\pi(\mathcal{A})\xi] = \mathcal{H}$.
3. Let $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ a nondegenerate representation and let $\mathcal{H}_0$ be a subspace of $\mathcal{H}$ invariant under $\pi(\mathcal{A})$. Then $\pi_0(T) = \pi(T)|_{\mathcal{H}_0}$ defines a nondegenerate representation of $\mathcal{A}$ on $\mathcal{H}_0$. Such a $\pi_0$ is called a *subrepresentation* of $\pi$.
4. A representation $\pi$ of $\mathcal{A}$ is said to be *multiplicity-free* if $\pi$ does not have two nonzero orthogonal equivalent subrepresentations.

   Following the multiplicity theory on Arverson [8], we have $\pi$ is multiplicity-free if and only if $\pi(\mathcal{A})'$ is abelian (as a von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$).
5. Given $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and $\sigma : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ two representations of $\mathcal{A}$, we say $\pi$ is *equivalent* $\sigma$ (and write $\pi \cong \sigma$) if there is a unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $\sigma(x) = U\pi(x)U^*$ for all $x \in \mathcal{A}$.

6. A nonzero representation of $\mathcal{A}$ is called *irreducible* if $\pi(\mathcal{A})$ commutes with no nontrivial (self-adjoint) projections.

Let us consider a topological second countable completely regular Hausdorff space $\Omega$ and $\Omega_\beta$ its Stone-Cĕch compactification (then $\mathcal{C}_{\mathbb{C}}(\Omega_\beta)$ and $\mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$ are separable). We shall see that for the case $\mathcal{A} = \mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$ the representation theory is clearly tied up with the theory of measures on $\Omega_\beta$. Then we shall need the following elementary concepts of Borel measures.

Let $(X, \vartheta)$ a Borel measurable set. Given two Borel finite measures $\mu$ and $\nu$ defined on $(X, \vartheta)$, we say that:

1. $\mu$ is *equivalent* to $\nu$ if they have the same null sets.
2. $\mu$ and $\nu$ are *mutually singular* (and write $\mu \perp \nu$) if there are disjoint measurable sets $A$ and $B$ such that $X = A \cup B$ and $\nu(A) = \mu(B) = 0$.
3. $\nu$ is *absolute continuous* with respect to $\mu$ if $\nu(A) = 0$ for each $A$ for which $\mu(A) = 0$. $\left(\text{Then we have the Radon-Nikodym derivative of } \nu \text{ with respect to } \mu \left[\frac{d\nu}{d\mu}\right]\right)$.

**Theorem 2.3.1.** *If $\pi$ is a non degenerate cyclic $*$-representation of $\mathcal{A} = \mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$ on a separable Hilbert space $\mathcal{H}$ then there is a Borel measure $\mu$ on $\Omega_\beta$ such that*

$$\pi \cong \pi_\mu$$

*where $\pi_\mu$ is given by*

$$\pi_\mu(F) \cong M_F \text{ on } L^2_{\mathbb{C}^N}(\mu) \tag{2.2}$$

*for $F \in \mathcal{A}$.*

*Proof.* Since $\pi$ is cyclic then there exists $\xi \in \mathcal{H}$ so that $[\pi(\mathcal{A})\xi] = \mathcal{H}$. We consider the matrix units

$$E_{ij} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & 1 & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

with 1 in the $ij$ entry. We define the representations $\pi_i$ of $\mathcal{C}_{\mathbb{C}}(\Omega_\beta)$ via

$$\pi_i(f) = \pi(fE_{ii}).$$

We also define

$$\rho_i : \mathcal{C}_{\mathbb{C}}(\Omega_\beta) \to \mathbb{C}$$

by

$$\rho_i(f) = \langle \pi_i(f)\xi, \xi\rangle_{\mathcal{H}} = \langle \pi(fE_{ii})\xi, \xi\rangle_{\mathcal{H}}.$$

This $\rho_i$ is a positive linear functional on $\Omega$. In fact if $f \geq 0$ then $f = |g|^2$, $g \in \mathcal{C}_{\mathbb{C}}(\Omega_\beta)$ and therefore

$$\rho_i(f) = \langle \pi_i(\overline{g}g)\xi, \xi\rangle_{\mathcal{H}} = \langle \pi_i(g)\xi, \pi_i(g)\xi\rangle_{\mathcal{H}} \geq 0.$$

Thus by Riesz-Markov theorem there is a positive Borel measure $\mu_i$ on $\Omega$ so that

$$\rho_i(f) = \int_\Omega f d\mu_i.$$

If $f \in \mathcal{C}_{\mathbb{C}}(\Omega_\beta) \subset L^2(\mu_i)$, then $U_i : \mathcal{C}_{\mathbb{C}}(\Omega_\beta) \to \mathcal{H}$ given by $U_i(f) = \pi_i(f)\xi$ is isometric, since

$$\|f\|^2_{L^2(\mu_i)} = \int_\Omega |f|^2 d\mu_i = \rho_i(|f|^2) = \langle \pi_i(|f|^2)\xi, \xi\rangle_{\mathcal{H}} = \|\pi_i(f)\xi\|^2_{\mathcal{H}}.$$

Now, since $\mathcal{C}_{\mathbb{C}}(\Omega_\beta)$ is dense in $L^2(\mu_i)$ and the range of $U_i$ is dense in $\mathcal{H}$ then $U_i$ can be extended to $L^2(\mu_i)$ onto $\mathcal{H}$.

Using the fact that $\pi$ is cyclic we have

$$[\pi(E_{ii}AE_{ii})] = \pi(E_{ii})\mathcal{H},$$

and, since $\pi(E_{ii}) = \pi(E_{ii}^2) = \pi(E_{ii})^2 = \pi(E_{ii}^*) = \pi(E_{ii})^*$ then $\pi(E_{ii})$ is an orthogonal projection on $\mathcal{H}$.

If $f \in \mathcal{C}_{\mathbb{C}}(\Omega)$ (so $fI_{N\times N} \in \mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$) then $\pi(fI_{N\times N})\pi(E_{ii})\xi$ is dense in $\pi(E_{ii})\mathcal{H} = \mathcal{H}_i$ and

$$\|\pi(fI_{N\times N})\pi(E_{ii})\xi\|_{\mathcal{H}} = \|\pi(fE_{ii})\xi\|_{\mathcal{H}} = \|\pi_i(f)\xi\|_{\mathcal{H}} = \|f\|_{L^2(\mu_i)}.$$

Therefore $\pi(fI_{N\times N})$ can be extended to $\mathcal{H}_i = \pi(E_{ii})\mathcal{H}$ onto $L^2(\mu_i)$. Thus $\mathcal{H}_i \cong L^2(\mu_i)$.

And for $g \in L^2(\mu_i)$,

$$\begin{aligned} U_i(M_f g) = U_i(fg) &= \pi_i(fg)\xi \\ &= \pi(fgE_{ii})\xi \\ &= \pi(fI_{N\times N})\pi(gE_{ii})\xi = \pi(fI_{N\times N})\pi_i(g)\xi = \pi(fI_{N\times N})U_i(g), \end{aligned}$$

thus

$$U_i M_f = \pi(fI_{N\times N})U_i. \tag{2.3}$$

It can be shown

$$U_j^*\pi(fE_{jj}) = M_f U_j^*. \tag{2.4}$$

In fact, given $\xi' \in \mathcal{H}$ then $\xi' = U_j(g)$ for some $g \in L^2(\mu_i)$ (this is possible since $U_j$ is onto), then

$$M_f U_j^*(\xi') = M_f U_j^* U_j(g) = M_f g,$$

and

$$U_j^*\pi(fE_{jj})\xi' = U_j^*\pi_j(f)U_j(g) = U_j^*\pi_j(f)\pi_j(g)\xi = U_j^*\pi_j(fg)\xi = U_j^*U_j(fg) = fg = M_f g,$$

so

$$U_j^*\pi(fE_{jj}) = M_f U_j^*.$$

We claim that

$$U_j^*\pi(E_{ji})U_i : L^2(\mu_i) \to L^2(\mu_j)$$

is unitary. In fact for $f, g \in L^2(\mu_i)$

$$\begin{aligned} \langle U_j^*\pi(E_{ji})U_i(f), U_j^*\pi(E_{ji})U_i(g)\rangle_{L^2(\mu_j)} &= \langle \pi(E_{ji})U_i(f), \pi(E_{ji})U_i(g)\rangle_{\mathcal{H}} \ \text{(since } U_j^* \text{ is unitary)} \\ &= \langle \pi(E_{ij})\pi(E_{ji})U_i(f), U_i(g)\rangle_{\mathcal{H}} \\ &= \langle \pi(E_{ii})U_i(f), U_i(g)\rangle_{\mathcal{H}} \\ &= \langle U_i(f), U_i(g)\langle_{\mathcal{H}} = \langle f, g\rangle_{L^2(\mu_i)} \ \text{(since } U_i \text{ is unitary).} \end{aligned}$$

We also have

$$\begin{aligned} U_j^*\pi(E_{ji})U_iM_f &= U_j^*\pi(E_{ji})\pi(fI_{N\times N})U_i \ \text{(using equation (2.3))} \\ &= U_j^*\pi(fE_{ji})U_i \\ &= U_j^*\pi(fE_{ii})\pi(E_{ji})U_i \\ &= M_fU_j^*\pi(E_{ji})U_i \ \text{(using equation (2.4)),} \end{aligned}$$

which implies

$$\begin{aligned} U_j^*\pi(E_{ji})U_if &= U_j^*\pi(E_{ji})U_iM_f(1) \\ &= M_fU_j^*\pi(E_{ji})U_i(1) \\ &= M_f\varphi \ \text{(where } \varphi = U_j^*\pi(E_{ji})U_i(1)) \\ &= M_fM_\varphi 1 = M_\varphi M_f 1 = M_\varphi f, \end{aligned}$$

thus $U_j^*\pi(E_{ji})U_i = M_\varphi$.

By the claim stated above we have $M_\varphi$ is a unitary operator which implies $\mu_i$ and $\mu_j$ are absolutely continuous with respect to each other. We now introduce another isomorphism $V : L^2(\mu_j) \to L^2(\mu_i)$ given by

$$f \mapsto \sqrt{\left[\frac{d\mu_i}{d\mu_j}\right]}\, f,$$

then we have

$$\left\| \sqrt{\left[\frac{d\mu_i}{d\mu_j}\right]}\, f \right\|^2_{L^2(\mu_j)} = \int_\Omega |f|^2 \left[\frac{d\mu_i}{d\mu_j}\right] d\mu_j = \int_\Omega |f|^2 d\mu_i = \|f\|^2_{L^2(\mu_i)},$$

therefore, without loss of generality, we may assume $\mu_1 = \mu_2 = \cdots = \mu_N = \mu$.

Thus

$$\mathcal{H} \cong \bigoplus_{j=1}^{N} \mathcal{H}_j \cong L^2_{\mathbb{C}^N}(\mu).$$

So there is $U : \mathcal{H} \to L^2_{\mathbb{C}^N}(\mu)$ such that for $F \in \mathcal{A}$

$$U\pi(F) = M_F U.$$

□

*Remark* 2.3.1. It can be shown that if $\pi = \pi_\mu$ for a Borel measure $\mu$ then

$$\pi_\mu(\mathcal{A})' = \{M_f : f \in L^\infty(\mu)\}$$

and hence is abelian which implies that each $\pi_\mu$ is multiplicity-free.

We also need the following important result given by Lemma 2.2.3 [8].

If $\mathcal{A}$ is a $C^*$-algebra, and $\pi$ is a nondegenerate multiplicity-free representation of $\mathcal{A}$ on a separable Hilbert space then $\pi$ is a cyclic representation.

A direct consequence of Lemma 2.2.3 [8] and Theorem 2.3.1 is the following.

**Corollary 2.3.2.** *If $\pi$ is a nondegenerate multiplicity-free representation of $\mathcal{A} = \mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$ then there is a Borel measure $\mu$ on $\Omega_\beta$ with $\pi = \pi_\mu$.*

This is our version of Theorem 2.2.4 [8].

We also have the following important result given by a corollary of Theorem 2.2.4 [8] (see page 54 of [8]).

**Corollary 2.3.3.** *Let $\mathcal{H}$ be a separable Hilbert space and let $\pi$ be a representation of a separable abelian $C^*$-algebra on $\mathcal{H}$. Then $\pi$ is multiplicity-free if and only if $\pi$ is a cyclic representation.*

We state here another important result from [8], namely Theorem 2.2.2, which will be important for our work.

**Corollary 2.3.4.** *If $\mu$ and $\nu$ are two Borel finite measures on $X$, then $\pi_\mu$ is equivalent to $\pi_\nu$ if and only if $\mu$ is equivalent to $\nu$. $\pi_\mu$ is disjoint from $\pi_\nu$ if and only if $\mu \perp \nu$.*

**Corollary 2.3.5.** *The irreducible representations of $\mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$ are given by point evaluations: $\pi = \pi_{\delta_w}$, where $\delta_w$ is a unit point mass measure supported at the point $w \in \Omega_\beta$ so that the spectrum of $\mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$ (given by the equivalence classes of irreducible representations) can be identified with $\Omega_\beta$ (including the Borel structure).*

**Corollary 2.3.6.** *If $\pi$ is an irreducible representation of $\mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$ then $\pi(F)$ is compact (even finite dimensional with dimension equal $N$) for all $F \in \mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$.*

**Corollary 2.3.7.** *Any representation $\pi$ of $\mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$ on a separable Hilbert space is equivalent to one of the form*

$$\infty \cdot \pi_{\mu_\infty} \oplus 1 \cdot \pi_{\mu_1} \oplus 2 \cdot \pi_{\mu_2} \oplus \cdots$$

All this discussion is a mild generalization of sections $1.4-1.5$ in [8] where the representation theory for the scalar-valued case $\mathcal{A} = \mathcal{C}_{\mathbb{C}}(\Omega_\beta)$ is discussed. This derivation of the representation of a general representation of $\mathcal{A} = \mathcal{C}_{M_N(\mathbb{C})}(\Omega_\beta)$ as a direct integral of irreducible representations has the advantage of identifying explicitly the irreducible representations as evaluations, as opposed to applying the results in Chapter 4 of [8] for the case of a general GCR $C^*$-algebra $\mathcal{A}$.

## 2.4 Convexity Analysis

We will need the following definitions.

1. If $Z$ is a real Banach space and $\mathcal{W} \subseteq Z$, we say $\mathcal{W}$ is a *wedge* if $a+b \in \mathcal{W}$ and $ta \in \mathcal{W}$ whenever $a, b \in \mathcal{W}$ and $t \geq 0$.

2. $x \in \mathcal{W}$ is an *extreme point* of $\mathcal{W}$ if

$$x = (1-t)x_1 + tx_2,$$

   with $x_1, x_2 \in \mathcal{W}$ and $t \in (0,1)$, then $x_1 = x$ and $x_2 = x$.

3. $x \in \mathcal{W}$ is an *extreme direction* in $\mathcal{W}$ if $x \neq 0$ and if

$$x = x_1 + x_2,$$

   with $x_1, x_2 \in \mathcal{W}$, then $x_1 = t_1 x$ and $x_2 = t_2 x$ for $t_1, t_2 \geq 0$.

4. If $\mathcal{W}$ is a wedge, we say that the linear functional $\rho$ on $\mathcal{W}$ *strictly slices* $\mathcal{W}$ if $\rho(x) > 0$ whenever $x \in \mathcal{W}$ and $x \neq 0$.

   If a linear functional $\rho$ strictly slices $\mathcal{W}$, we define

   $$\mathcal{W}_\rho = \{x \in \mathcal{W} : \rho(x) = 1\},$$

   then it is clear that every nonzero element $x \in \mathcal{W}$ can be represented uniquely in the form $x = tx_1$ with $t > 0$ and $x_1 \in \mathcal{W}_\rho$.

   Let $X$ be a Hausdorff compact space.

5. We will denote by $M(X)$ the space of complex Borel measures and $M(X)^{N\times N}$ to the space of $N \times N$ matrix-valued complex Borel measures on $X$.

We consider the convex set

$$\mathcal{C} = \{\mu \in M(X)^{N\times N} : \mu(\Delta) \text{ positive (semidefinite) in } \mathbb{C}^{N\times N} \text{ for all Borel } \Delta,\ \mu(X) = I\}. \tag{2.5}$$

The following lemma gives a characterization of extreme points of $\mathcal{C}$ which is convenient to work with.

**Lemma 2.4.1.** *$\mu$ is an extreme point of $\mathcal{C}$ if and only if whenever $\nu \in M(X)^{N\times N}$ so that $\nu(X) = 0$ and $\mu \pm \nu \geq 0$ implies $\nu = 0$.*

*Proof.* ($\Rightarrow$) We can write

$$\mu = \frac{1}{2}(\mu + \nu) + \frac{1}{2}(\mu - \nu)$$

with $\mu \pm \nu \in \mathcal{C}$, which implies (since $\mu$ is an extreme point of $M(X)^{N\times N}$) $\mu + \nu = \mu = \mu - \nu$ (and therefore $\nu = 0$).

($\Leftarrow$) To show that $\mu$ is an extreme point we consider the contrapositive, that is

$\mu$ is not an extreme point implies there exists $\nu \neq 0$ in $M(X)^{N\times N}$ with $\nu(X) = 0$ and $\mu \pm \nu \geq 0$.

If $\mu$ is not an extreme point then

$$\mu = t\nu_1 + (1-t)\nu_2$$

for some $\nu_1, \nu_2 \in \mathcal{C}$ and $0 < t < 1$ with $\nu_1, \nu_2 \neq \mu$. Then we have

$$\mu = t\mu + (1-t)\mu = t\nu_1 + (1-t)\nu_2$$

and so

$$t(\mu - \nu_1) = (1 - t)(\nu_2 - \mu) = \nu.$$

Then $\nu$ is a measure so that $\nu(X) = 0$ and

$$\mu + \nu = \mu + (1 - t)(\nu_2 - \mu) = t\mu + (1 - t)\nu_2 \geq 0,$$

$$\mu - \nu = \mu - t(\mu - \nu_1) = (1 - t)\mu + t\nu_1 \geq 0$$

with $\nu \neq 0$ since $\mu \neq \nu_1, \nu_2$. □

We have an important result given by Lemma 1.3.4 [2], that relates the extreme point of $\mathcal{W}$ and the extreme directions of $\mathcal{W}_\rho$, which we present here.

**Lemma 2.4.2.** *Let $Z$ be a real vector space, let $\mathcal{W} \subseteq Z$ be a wedge and let $\rho$ be a linear functional on $Z$ that strictly slices $\mathcal{W}$. If $x$ is an extreme direction of $\mathcal{W}$, then $\rho(x)^{-1}x$ is an extreme point of $\mathcal{W}_\rho$. Conversely, if $x$ is an extreme point of $\mathcal{W}_\rho$ and $t > 0$, then $tx$ is an extreme direction of $\mathcal{W}$.*

Explicit characterization of extreme points leads to useful representation theorems in examples due to the following results given after the following two definitions.

1. Suppose $X$ is an nonempty compact subset of a locally convex space $E$, and that $\mu$ is a probability measure on $X$. (That is $\mu$ is a nonnegative regular Borel measure on $X$ with $\mu(X) = 1$). A point $x \in E$ is said to *represented* by $\mu$ if

$$f(x) = \int_X f d\mu \tag{2.6}$$

   for every continuous linear functional $f$ on $E$. A consequence of the Hahn-Banach theorem is that $\mu$ uniquely determines its representor $x$ whenever such $x$ exists.

2. If $\mu$ is a nonnegative regular Borel measure on the compact Hausdorff space $X$ and $S$ is a Borel subset of X, we say that $\mu$ is *supported* by $S$ if $\mu(X \setminus S) = 0$.

**Theorem 2.4.3.** *(Krein-Milman Theorem). If $X$ is a compact convex subset of a locally convex space, then $X$ is the closed convex hull of its extreme points.*

Choquet theory gives the following refinement of the Krein-Milman theorem.

**Theorem 2.4.4.** *Every point of a compact convex subset $X$ of a locally convex space is represented by a probability measure $\mu$ on $X$ which is supported by the closure of the extreme points of $X$.*

The proof of the equivalence of these two assertions is guaranteed by Proposition 1.2 [21].

*Remark* 2.4.1. One can view the formula (2.6) as asserting the existence of the Pettis integral

$$x = \int_X w d\mu(x) \tag{2.7}$$

(see e.g. Section II.3 of [15]). However the proof in [21] (see Proposition 1.1 there) that, for a given probability measure, there always exists an $x \in E$ satisfying (2.6) for all $f \in E^*$ uses no vector-valued integration theory, but rather uses duality and compactness arguments together with an exploitation of the special form of the integrand $F$ in (2.7) $F(w) = w$.

**Theorem 2.4.5.** *(Choquet). Suppose that $X$ is a metrizable compact convex subset of a locally convex space $E$, and that $x_0$ is an element of $X$. Then there is a probability measure $\mu$ on $X$ which represents $x_0$ and is supported by the extreme points of $X$.*

## 2.5 Extreme points of convex sets of measures

Following Arveson [9], we say that

1. Given an Hilbert space $\mathcal{H}$, an operator $T \in \mathcal{L}(\mathcal{H})$ *lives in* a closed subspace $\mathcal{M}$ of $\mathcal{H}$ if both $T$ and $T^*$ vanish on $\mathcal{M}^\perp$ (equivalently, $\mathcal{M}$ contains the range of both $T$ and $T^*$).
2. A finite collection $\{\mathcal{M}_1, \ldots, \mathcal{M}_M\}$ of subspaces of a Hilbert spaces $\mathcal{H}$ is called *weakly independent* if, whenever $T_i (\in \mathcal{L}(\mathcal{H}))$ lives in $\mathcal{M}_i$ and $T_1 + \cdots + T_M = 0$ then $T_1 = \cdots = T_M = 0$.

**Example 1.** Consider the wedge

$$\mathcal{W} = \{\mu : \mu \text{ is a positive scalar measure on a Hausdorff Borel space } X\},$$

then the set of extreme directions of $\mathcal{W}$ is

$$\{t\delta_x : t > 0,\ x \in X,\ \text{and } \delta_x \text{ is the point mass measure at } x\}.$$

To prove this assertion, let us fix $\mu$ an extreme direction of $\mathcal{W}$ and assume that its support is not a single point. Then there exist disjoint Borel sets $\Delta_1, \Delta_2 \subset X$ such that $\mu(\Delta_1), \mu(\Delta_2) > 0$.

If we define $\rho(\mu) = \mu(X)$ then $\rho$ is a linear functional that strictly slices $\mathcal{W}$ and then $0 < r = \rho(\mu)^{-1}\mu(\Delta_1) < 1$. We may define the following positive Borel measures

$$\mu_1(B) = r^{-1}\rho(\mu)^{-1}\mu(B \cap \Delta_1)$$

and

$$\mu_2(B) = (1-r)^{-1}\rho(\mu)^{-1}\mu(B \cap (X \setminus \Delta_1)),$$

thus we have

$$r\mu_1(B) + (1-r)\mu_2(B) = \rho(\mu)^{-1}\mu(B \cap \Delta_1) + \rho(\mu)^{-1}\mu(B \cap (X \setminus \Delta_1)) = \rho(\mu)^{-1}\mu(B)$$

for all Borel subset $B$ of $X$, therefore

$$r\mu_1 + (1-r)\mu_2 = \rho(\mu)^{-1}\mu.$$

But, by Lemma 2.4.2, $\rho(\mu)^{-1}\mu$ is an extreme point of $\mathcal{W}_\rho$ and so

$$\mu_1 = \mu_2 = \rho(\mu)^{-1}\mu,$$

and this implies

$$0 < r = \rho(\mu)^{-1}\mu(\Delta_1) = \mu_2(\Delta_1) = (1-r)^{-1}\rho(\mu)^{-1}\mu(\Delta_1 \cap (X \setminus \Delta_1)) = 0$$

and this is certainly a contradiction. So it must be true that the support of $\mu$ is a single point.

One can see that the other direction of the assertion is easily satisfied.

**Example 2.** If we consider the wedge

$$\mathcal{W} = \{\mu : \mu \text{ is an } N \times N \text{ positive operator measure on a Hausdorff Borel space } X\}$$

then the set of extreme directions of $\mathcal{W}$ is given by the set

$$\{tP\delta_x : t > 0, \delta_x \text{ is the point mass measure at } x (\in X) \text{ and } P \text{ is a rank-1 orthogonal projection}\}.$$

The proof is somewhat similar to the proof given in Example 1. Let us fix $\mu$ an extreme direction of $\mathcal{W}$.

**Step 1.** *supp* $\mu =$ single point.

If the support of $\mu$ were not a single point then there exist $\Delta_1, \Delta_2$ disjoint Borel sets with $\mu(\Delta_1), \mu(\Delta_2) \geq 0$ both not zero. Then $\mu$ can be written as

$$\mu = \mu_1 + \mu_2$$

where $\mu_i(E) = \mu(E \cap \Delta_i)$ for $i = 1, 2$. So $\mu_1(\Delta_2) = 0$ which implies $\mu_1 \neq t\mu$, which is a contradiction with the fact $\mu$ is an extreme direction of $\mathcal{W}$.

**Step 2.** Assume $\mu = P\delta_x$. We want to show $P$ is a rank-1 operator. If it were not so, then, by the Spectral theorem, $P = \sum_{k=1}^{N} t_k P_k$ where $P_k$'s are rank-1 pairwise orthogonal projections, and at least two $t_k$'s are strictly greater than zero. But then $P = Q_1 + Q_2$ with

$Q_j \geq 0$ and $Q_j \neq s_j P$ which, again, implies $\mu$ is not an extreme direction of $\mathcal{W}$. So it must be true $P$ is a rank-1 operator.

The strict slice is given, in this case, by $\rho(\mu) = tr\ \mu(X)$.

We also may define a slice (for $N > 1$) $\rho(\mu) = \mu(X)$ and then

$$\mathcal{W}_\rho = \{\mu \in \mathcal{W} : \mu(X) = I_{N\times N}\}.$$

As $\rho$ is matrix rather than real valued there is no way to view $\mathcal{W}_\rho$ is a strict slice of $\mathcal{W}$.

We now present an important result, given by Arveson [9] (Theorem 1.4.10), that characterizes the extreme points of the convex set $\mathcal{C}$ given by (2.5). We present here a proof of such result using elementary tools which is an alternative to the representation-theoretic proof given by Arveson.

**Theorem 2.5.1.** *$\mu \in M(X)^{N\times N}$ is an extreme point of $\mathcal{C}$ if and only if*

$$\mu = \sum_{k=1}^{n} \delta_{x_k} K_k$$

*with $n \geq 1$ where $K_1, \ldots, K_n$ are positive operators satisfying*

1. *$K_1 + \cdots + K_n = I$ and*
2. *$\{[K_1\mathbb{C}^N], \cdots, [K_n\mathbb{C}^N]\}$ is a weakly independent family of subspaces.*

*Proof.* ($\Leftarrow$) Suppose there exists $\nu \in M(X)^{N\times N}$ such that $\nu(X) = 0$ and $\mu \pm \nu \geq 0$. Let us take $\Delta$ to be a Borel set disjoint from $\{x_1, \ldots, x_n\}$, then

$$0 \leq (\mu \pm \nu)(\Delta) = \pm\nu(\Delta),$$

thus $\nu(\Delta) = 0$. By Jordan decomposition theory $\nu$ is supported on $\{x_1, \ldots, x_n\}$ and $\chi_\Delta \nu$ must be the zero measure. So

$$\nu = \sum_{j=1}^{n} \delta_{x_j} L_j.$$

Since $\nu(X) = 0$ then $\sum_{j=1}^{n} L_j = 0$. Since $\mu \pm \nu \geq 0$, in particular $(\mu \pm \nu)(\{x_j\}) \geq 0$ for each $j$. This implies $K_j \pm L_j \geq 0$ and so

$$-K_j \leq L_j \leq K_j.$$

And since $K_j = \sum_{k=1}^{N} t_k P_k$ with $P_1, \ldots, P_N$ spectral resolution for $\mathbb{C}^N$ with some $t_k$'s being zero, then

$$-\sum_{k, t_k \neq 0} P_k \leq -K_j \leq L_j \leq K_j \leq \sum_{k, t_k \neq 0} P_k.$$

If we let $t_{max} = \max_{k=1}^{N} t_k$, then

$$K_j \leq t_{max} P_{[K_j\mathbb{C}^N]} = \sum_{k, t_k \neq 0} t_{max} P_k.$$

Similarly

$$-t_{max} P_{[K_j\mathbb{C}^N]} \leq -K_j,$$

thus

$$-t_{max} P_{[K_j\mathbb{C}^N]} \leq L_j \leq t_{max} P_{[K_j\mathbb{C}^N]},$$

and so $L_j$ lives on $[K_j\mathbb{C}^N]$.

Since $\{[K_1\mathbb{C}^N], \ldots, [K_n\mathbb{C}^N]\}$ is a weakly independent family and $\sum_{j=1}^{N} L_j = 0$ then $L_j = 0$ for $j = 1, \ldots, n$. So $\nu = 0$ and therefore, by Lemma 2.4.1, $\mu$ is an extreme point of $\mathcal{C}$.

($\Rightarrow$) Suppose $\mu$ is an extreme point of $\mathcal{C}$.

**Step 1.** $\#(supp\ \mu) \leq N^2$.

*Proof.* Suppose there are $n$ $\big(n > N^2\big)$ disjoint Borel sets $\Delta_1, \ldots, \Delta_n$ with $\mu(\Delta_j) > 0$.

Define $\mu_j \in M(X)^{N \times N}$ by

$$\mu_j(E) = \mu(E \cap \Delta_j).$$

Then $\{\mu_1, \ldots, \mu_n\}$ is linearly independent in the real vector space $[M(X)^{N\times N}]_H$ of complex-Hermitian matrix-valued measures on $X$. Now define real linear functionals on $[M(X)^{N\times N}]_H$ by

$$L_i : \mu \mapsto \mu_{ii}(X),\ 1 \leq i \leq N,$$

$$L_{Re,ij} : \mu \mapsto Re\ \mu_{ij}(X),\ 1 \leq i < j \leq N,$$

$$L_{Im,ij} : \mu \mapsto Im\ \mu_{ij}(X),\ 1 \leq i < j \leq N.$$

So in total there are $N + \frac{N(N-1)}{2} + \frac{N(N-1)}{2} = N^2$ real linear functionals.

Note that for $1 \leq i \leq j \leq N$ we have

$$\mu_{ji}(X) = \mu_{ij}(X)^*. \tag{2.8}$$

Define now the real functional $L$ on $[M(X)^{N\times N}]_H$ by

$$L(\mu) = \begin{bmatrix} L_1(\mu) \\ \vdots \\ L_N(\mu) \\ L_{Re,ij}(\mu),\ 1 \leq i < j \leq N \\ L_{Im,ij}(\mu),\ 1 \leq i < j \leq N \end{bmatrix}.$$

Restricting $L$ to $\bigvee\{\mu_1, \ldots, \mu_n\}$ we realize that, by the null-kernel theorem, there exist real coefficients $c_1, \ldots, c_n$ such that $\nu = \sum_{l=1}^{n} c_l \mu_l \in \bigvee\{\mu_1, \ldots, \mu_n\}$ with $L(\nu) = 0$. This implies

$$\nu_{ii}(X) = 0$$

for $i = 1 \ldots N$;

$$\nu_{ij}(X) = 0$$

for $1 \leq i < j \leq N$; and using equation (2.9) then we have

$$\nu_{ij}(X) = 0$$

for $1 \leq i, j \leq N$. Therefore $\nu(X) = 0$.

Now, take $0 < \epsilon < \min\left\{\frac{1}{|c_j|} : j \text{ with } c_j \neq 0\right\}$, then

$$1 \pm \epsilon c_j \geq 0$$

for $j = 1, \ldots, n$. Then $\epsilon\nu \neq 0$ with $\epsilon\nu(X) = 0$, and for $E \subset X$ a Borel set we have

$$(\mu \pm \epsilon\nu)(E) = (1 \pm \epsilon c_j)\mu(E) \geq 0,$$

so, by Lemma 2.4.1, $\mu$ is not an extreme point of $\mathcal{C}$. □

**Step 2.** We may assume that

$$\mu = \sum_{j=1}^{n} \delta_{x_j} K_j$$

with $\sum_{j=1}^{n} K_j = I$ and $1 \leq n \leq N^2$.

We show now, $\{[K_1\mathbb{C}^N], \ldots, [K_n\mathbb{C}^N]\}$ is a weakly independent family. Suppose not, then there exist $L_1, \ldots, L_n$ so that $L_j$ lives in $[K_j\mathbb{C}^N]$, and $L_1 + \cdots + L_n = 0$ but not all $L_j = 0$.

We may assume $L_j$'s are selfadjoint, since $L_1 + \cdots + L_n = 0$ implies $L_1^* + \cdots + L_n^* = 0$ and so $(L_1 + L_1^*) + \cdots + (L_n + L_n^*) = 0$. So may take $L_j$ as $L_j + L_j^*$ if $L_j + L_j^* \neq 0$. And If all $L_j + L_j^*$ are 0, then at leat one of $\frac{L_j - L_j^*}{2i} \neq 0$ since not all $L_j$'s are zero, in this case we choose $L_j$ to be $\frac{L_j - L_j^*}{2i}$.

Define $\nu = \sum_{j=1}^{n} \delta_{x_j} L_j$, then $\nu(X) = 0$. We have $L_j$ lives in $[K_j\mathbb{C}^N]$ and so

$$-K_j \leq \epsilon L_j \leq K_j$$

for $\epsilon > 0$ small enough. Thus $\mu \pm \epsilon\nu \geq 0$ with $\nu \neq 0$ and $\nu(X) = 0$, so by Lemma 2.4.1 $\mu$ is not an extreme point of $\mathcal{C}$. □

**Corollary 2.5.2.** *If $\mu \in M(X)^{N\times N}$ is a spectral measure, i.e.*

$$\mu = \sum_{k=1}^{N} \delta_{x_k} P_k,$$

*where $\delta_{x_k}$ is the point mass measure at $x_k$ $(\in X)$ for $k = 1, \ldots, N$, and $P_1, \ldots, P_N$ are pairwise rank $-1$ orthogonal projections summing to $I$. Then $\mu$ is an extreme point of $\mathcal{C}$.*

*Proof.* We rewrite $\mu$ as $\mu = \sum_{k=1}^{M} \delta_{y_k} Q_k$ where $\{y_1, \ldots, y_M\}$ are distinct points in $X$ and $\{Q_1, \ldots, Q_M\}$ are orthogonal projections (not necessarily rank-1) with $Q_i Q_j = \delta_{ij} Q_i$ and $\sum_{i=1}^{M} Q_i = I$. Thus these orthogonal subspaces are certainly weakly independent. □

Let us show here an specific example of an extreme point of $\mathcal{C}$ which is not a spectral measure.

Consider $X = \{x_1, x_2, x_3\}$ and take $v_1, v_2, v_3$ to be three distinct unit vectors in $\mathbb{C}^2$ any two which are linearly independent in $\mathbb{C}^2$ satisfying an additional condition given below. Now consider the measure $\mu$ on $X$ define by

$$\mu(\{x_j\}) = c_j v_j v_j^*$$

where $c_j > 0$ for $j = 1, 2, 3$ with

$$c_1 v_1 v_1^* + c_2 v_2 v_2^* + c_3 v_3 v_3^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\left(\text{so } \mu(X) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$

Note that $\mu$ so defined is not an spectral measure.

Under these conditions we have the following lemma.

**Lemma 2.5.3.** *If $\nu \neq 0$ is a signed measure so that*

$$\nu(\{x_1, x_2, x_3\}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

*and $(\mu \pm \nu)(\{x_j\}) \geq 0$, then*

$$\nu(\{x_j\}) = w_j c_j v_j v_j^*$$

*where $-1 \leq w_j \leq 1$.*

*Proof.* If we choose the orthonormal basis $\{v_j, v_j^{\perp}\}$ and we write

$$\nu(\{x_j\}) = \begin{bmatrix} w_{11} & w_{12} \\ \overline{w_{12}} & w_{22} \end{bmatrix},$$

then

$$0 \leq P_{\{v_j\}^\perp}(\mu \pm \nu)(\{x_j\})P_{\{v_j\}^\perp} = \pm P_{\{v_j\}^\perp}\nu(\{x_j\})P_{\{v_j\}^\perp}.$$

This implies

$$P_{\{v_j\}^\perp}\nu(\{x_j\})P_{\{v_j\}^\perp} = 0$$

and so

$$\nu(\{x_j\}) = \begin{bmatrix} w_{11} & w_{12} \\ \overline{w_{12}} & 0 \end{bmatrix},$$

and because of our choice of the orthonormal basis we have

$$\mu(\{x_j\}) = \begin{bmatrix} c_j & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus

$$(\mu \pm \nu)(\{x_j\}) = \begin{bmatrix} c_j \pm w_{11} & \pm w_{12} \\ \pm\overline{w_{12}} & 0 \end{bmatrix} \geq 0$$

implies $w_{12} = 0$ $\left(\text{and so } \nu(\{x_j\}) = \begin{bmatrix} w_{11} & 0 \\ 0 & 0 \end{bmatrix}\right)$. Also the fact that

$$\nu(\{x_j\}) \leq \mu(\{x_j\}) \text{ forces } w_{11} \leq 1$$

and

$$-\nu(\{x_j\}) \leq \mu(\{x_j\}) \text{ forces } w_{11} \geq -1.$$

□

Now let us go back to the measure $\mu$ defined above by $\mu(\{x_j\}) = c_j v_j v_j^*$.

Assume $\mu$ is not an extreme point then by Lemma 2.4.1 there is a measure $\nu \neq 0$ on $X$ such that

$$\nu(X) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\mu \pm \nu \geq 0$.

By Lemma 2.5.3

$$\nu(\{x_j\}) = w_j c_j v_j v_j^*$$

where $-1 \leq w_j \leq 1$.

**Case 1.** Two $w's = 0$, say $w_1 = w_2 = 0$. Then $\nu(\{x_1\}) = \nu(\{x_2\}) = 0$ and so

$$0 = \nu(X) = \nu(\{x_1\}) + \nu(\{x_2\}) + \nu(\{x_3\}) = 0 + 0 + \nu(\{x_3\}) = \nu(\{x_3\}).$$

Thus $\nu = 0$, which a contradiction.

**Case 2.** One $w = 0$, say $w_3 = 0$. Then

$$w_1 c_1 v_1 v_1^* = -w_2 c_2 v_2 v_2^*$$

with $c_1, c_2 > 0$ and $\{v_1, v_2\}$ linearly independent. Then $w_1 = w_2 = 0$ and so $\nu = 0$ (again a contradiction).

**Case 3.** No $w = 0$. Then there are at least two $w's$ with the same sign. By replacing $\nu$ by $-\nu$, we may assume that the common sign is positive. By relabeling we may assume $w_1, w_2 > 0$. We have

$$0 \leq (\mu \pm \nu)(\{x_3\}) = c_3 v_3 v_3^* \pm w_3 c_3 v_3 v_3^* = c_3 v_3 v_3^* \mp (w_1 c_1 v_1 v_1^* + w_2 c_2 v_2 v_2^*)$$

because $\nu(\{x_1\}) + \nu(\{x_2\}) + \nu(\{x_3\}) = 0$. In particular this implies

$$0 \leq w_1 c_1 v_1 v_1^* + w_2 c_2 v_2 v_2^* \leq c_3 v_3 v_3^*$$

Note $w_1 c_1 v_1 v_1^* + w_2 c_2 v_2 v_2^*$ a rank-2 operator while $c_3 v_3 v_3^*$ a rank-1 operator.

Thus

$$0 \leq P_{\{v_3\}^\perp}(w_1 c_1 v_1 v_1^* + w_2 c_2 v_2 v_2^*) P_{\{v_3\}^\perp} \leq P_{\{v_3\}^\perp} c_3 v_3 v_3^* P_{\{v_3\}^\perp} = 0$$

and so

$$P_{\{v_3\}^\perp}(w_1 c_1 v_1 v_1^* + w_2 c_2 v_2 v_2^*) P_{\{v_3\}^\perp} = 0$$

which is a contradiction.

So our assumption must be incorrect and therefore any such $\mu$ defined as above is an extreme point.

As a specific example we may take the case where

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$

$$c_1 = \frac{3}{2} - \frac{\sqrt{5}}{2}, \text{ and } c_2 = 2c_1.$$

So

$$c_1 v_1 v_1^* = \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix}, \ c_2 v_2 v_2^* = \begin{bmatrix} c_1 & c_1 \\ c_1 & c_1 \end{bmatrix}$$

and we demand from $c_3$ and $v_3$ to be defined so that

$$c_3 v_3 v_3^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - c_1 v_1 - c_2 v_2$$

is a rank-1 operator. And we define $\mu$ as above, that is

$$\mu(\{x_j\}) = c_j v_j v_j^*.$$

In Chapter 4 we shall have need to analyze more complicated convex sets of measures.

Let us now fix $m$ continuous functions in $C(X)$ $f_1, \ldots, f_m$ and consider the convex set

$$\begin{aligned}\tilde{\mathcal{C}} \;&=\; \{\mu \in M(X)^{N\times N} : \mu(\Delta) \text{ positive (semidefinite) in } \mathbb{C}^{N\times N} \text{ for all Borel } \Delta,\ \mu(X) = I,\\ &\quad\ \text{and } \mu(f_j) = 0,\ j = 1, \ldots, m\}.\end{aligned}$$

Then we may say the following about the extreme points of $\tilde{\mathcal{C}}$.

**Proposition 2.5.4.** *If $\mu \in M(X)^{N\times N}$ is an extreme point of $\tilde{\mathcal{C}}$ then*

$$\mu = \sum_{j=1}^{n} \delta_{x_j} \tilde{K}_j$$

*for $n \leq N^2(m+1)$.*

*Proof.* Suppose there exist $n$ ($n > N^2(m+1)$) disjoint Borel sets $\Delta_1, \ldots, \Delta_n$ with $\mu(\Delta_j) > 0$. Then $\{\mu_1, \ldots, \mu_n\}$ is linearly independent in the real vector space $[M(X)^{N\times N}]_H$ of complex-Hermitian matrix-valued measures on $X$. Now define real linear functionals on $[M(X)^{N\times N}]_H$ by

$$\begin{gathered}L_i : \mu \mapsto \mu_{ii}(X),\ 1 \leq i \leq N,\\ L_{Re,ij} : \mu \mapsto Re\ \mu_{ij}(X),\ 1 \leq i < j \leq N,\\ L_{Im,ij} : \mu \mapsto Im\ \mu_{ij}(X),\ 1 \leq i < j \leq N,\\ L_{i,r} : \mu \mapsto \mu_{ii}(f_r),\ 1 \leq i \leq N, 1 \leq r \leq m,\\ L_{Re,ij,r} : \mu \mapsto Re\ \mu_{ij}(f_r),\ 1 \leq i < j \leq N, 1 \leq r \leq m,\\ L_{Im,ij,r} : \mu \mapsto Im\ \mu_{ij}(f_r),\ 1 \leq i < j \leq N, 1 \leq r \leq m.\end{gathered}$$

So in total there are

$$N + \frac{N(N-1)}{2} + \frac{N(N-1)}{2} + Nm + \frac{N(N-1)}{2}m + \frac{N(N-1)}{2}m = N^2(m+1)$$

real linear functionals.

Note that for $1 \leq i \leq j \leq N$ and $1 \leq r \leq m$ we have

$$\mu_{ji}(X) = \mu_{ij}(X)^* \text{ and } \mu_{ji}(f_r) = \mu_{ij}(f_r)^*. \tag{2.9}$$

Define now the real functional $L$ on $[M(X)^{N\times N}]_H$ by

$$L(\mu) = \begin{bmatrix} L_1(\mu) \\ \vdots \\ L_N(\mu) \\ L_{Re,ij}(\mu),\ 1 \le i < j \le N \\ L_{Im,ij}(\mu),\ 1 \le i < j \le N \\ L_{1,r}(\mu),\ 1 \le r \le m \\ \vdots \\ L_{N,r}(\mu),\ 1 \le r \le m \\ L_{Re,ij,r}(\mu),\ 1 \le i < j \le N, 1 \le r \le m \\ L_{Im,ij,r}(\mu),\ 1 \le i < j \le N, 1 \le r \le m \end{bmatrix}.$$

Restricting $L$ to $\bigvee\{\mu_1, \ldots, \mu_n\}$ we realize that, by the null-kernel theorem, there exist real coefficients $c_1, \ldots, c_n$ such that $\nu = \sum_{l=1}^{n} c_l \mu_l \in \bigvee\{\mu_1, \ldots, \mu_n\}$ with $L(\nu) = 0$. This implies

$$\nu_{ii}(X) = 0$$

for $i = 1 \ldots N$;

$$\nu_{ij}(X) = 0$$

for $1 \le i < j \le N$; and using equation (2.9) then we have

$$\nu_{ij}(X) = 0$$

for $1 \le i, j \le N$. Thus $\nu(X) = 0$.

Also

$$\nu_{ii}(f_r) = 0$$

for $i = 1 \ldots N, 1 \le r \le m$;

$$\nu_{ij}(f_r) = 0$$

for $1 \le i < j \le N, 1 \le r \le m$; and using equation (2.9) then we have

$$\nu_{ij}(f_r) = 0$$

for $1 \le i, j \le N, 1 \le r \le m$. Thus $\nu(f_r) = 0$ for $1 \le r \le m$.

Similar reasoning to the one in Step 1 of Proposition 2.5.1 leads us to find $\epsilon > 0$ small enough so that $\mu \pm \epsilon\nu \ge 0$. So we have $\epsilon\nu \ne 0$, then by Lemma 2.4.1 $\mu$ is not an extreme point of $\tilde{\mathcal{C}}$. $\square$

# Chapter 3

# Test functions-Admissible kernels-Agler Decomposition

## 3.1 The Main Result

Let $\mathcal{E}$ a Hilbert space and $K : \Omega \times \Omega \to \mathcal{L}(\mathcal{E})$ a positive kernel (as defined in Chapter 2) and let $\Psi = \{\psi_\pi, \pi \in \mathcal{A}\}$, where each $\psi_\pi : \Omega \to \mathcal{L}(\mathcal{U}_T, \mathcal{Y}_T)$ ($\mathcal{U}_T$ and $\mathcal{Y}_T$ are Hilbert spaces). Elements of $\Psi$ will be called *test functions*. We assume that the collection $\Psi$ satisfies

$$\sup_{\pi \in \mathcal{A}} \|\psi_\pi(z)\| < 1 \text{ for each } z \in \Omega. \tag{3.1}$$

We say that $K$ is $\Psi$-*admissible* if $R_\psi : f(w) \to f(w)\psi(w)$ has operator norm $\leq 1$ as an operator from $\mathcal{H}(K) \otimes \mathcal{Y}_T$ to $\mathcal{H}(K) \otimes \mathcal{U}_T$ for all $\psi \in \Psi$. The collection of such $K$ will be denoted by $\mathcal{K}_\Psi(\mathcal{E})$.

Let $\mathcal{U}$ and $\mathcal{Y}$ Hilbert spaces and $S : \Omega \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$. We say that $S$ is in $\mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ if $R_S : \mathcal{H}(K) \otimes \mathcal{U} \to \mathcal{H}(K) \otimes \mathcal{Y}$ has $\|R_S\| \leq 1$ for all $K \in \mathcal{K}_\Psi(\mathcal{Y})$. By Proposition 2.2.2 we know equivalently $S \in \mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$ if and only if

$$k_{X,S,K} = tr[X(w)^*[I - S(w)^*S(z)]X(z)K(z,w)]$$

is a positive $\mathbb{C}$-valued kernel for all $K \in \mathcal{K}_\Psi(\mathcal{Y})$ and all choices of $X : \Omega \to \mathcal{C}_2(\mathcal{Y}, \mathcal{U})$.

For each $z \in \Omega$ we define

$$\mathbb{E}(z) : \Psi \to \mathcal{L}(\mathcal{U}_T, \mathcal{Y}_T)$$

by

$$\mathbb{E}(z)(\psi) = \psi(z). \tag{3.2}$$

We will denote by $\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T))$ the set of continuous bounded functions from $\Psi$ to $\mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)$. We may view $\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T))$ as a $(\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{U}_T)), \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T)))$-correspondence as follows. $\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T))$ is a right module over $\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ and a left module over $\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{U}_T))$. Then $\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X}$ is the tensor product of $(\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{U}_T)), \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T)))$-correspondence $\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T))$ with $(\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathbb{C})$-correspondence $\mathcal{X}$ (with left action of $\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ on $\mathcal{X}$ given by the representation $\rho$).

The following is the main result of this chapter.

**Theorem 3.1.1.** *Let $\Omega_0 \subset \Omega$. Given $S_0 : \Omega_0 \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ and dim $\mathcal{Y}_T < \infty$. The following are equivalent:*

1. *$S_0$ extends to $S \in \mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$,*
2. *$S_0$ has an Agler decomposition, i.e., for all $z, w \in \Omega_0$ we have*
$$I - S_0(z)S_0(w)^* = \Gamma(z, w)[I - \mathbb{E}(z)\mathbb{E}(w)^*],$$
*for some completely positive kernel $\Gamma : \Omega_0 \times \Omega_0 \to \mathcal{L}(\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathcal{L}(\mathcal{Y}))$.*
3. *$S_0$ has a weakly coisometric realization, i.e., there is a colligation*
$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$
*so $\mathbf{U}^*$ is isometric on*
$$\bigvee \left\{ \begin{bmatrix} L_{\mathbb{E}(w)^*}(I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* y_w \\ y_w \end{bmatrix} : w \in \Omega_0,\ y_w \in \mathcal{Y} \right\} \subset \begin{bmatrix} \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X} \\ \mathcal{Y} \end{bmatrix},$$
*where $\mathcal{X}$ is equipped with a unital $*$-representation $\rho : \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T)) \to \mathcal{L}(\mathcal{X})$, so that:*
$$S_0(z) = D + C(I - L^*_{\mathbb{E}(z)^*} A)^{-1} L^*_{\mathbb{E}(z)^*} B \text{ for all } z \in \Omega_0,$$
*where*
$$L_{\mathbb{E}(z)^*} : \mathcal{X} \to \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X}$$
*is given by*
$$L_{\mathbb{E}(z)^*} : x \mapsto \mathbb{E}(z)^* \otimes x,$$
*and*
$$L^*_{\mathbb{E}(z)^*} : \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X} \to \mathcal{X}$$
*is given on a pure tensor by*
$$L^*_{\mathbb{E}(z)^*} : g \otimes x \mapsto \rho(\mathbb{E}(z)g)x.$$

## 3.2 Ingredients

Assume as a first case that $\Omega_0 \subset \Omega$ is finite. We define

$$\begin{aligned}\mathcal{C} \;=\; & \{\Xi : \Omega_0 \times \Omega_0 \to \mathcal{L}(\mathcal{Y}); \Xi(z,w) = \Gamma(z,w)[I - \mathbb{E}(z)\mathbb{E}(w)^*], \text{ for some completely} \\ & \text{positive kernel } \Gamma : \Omega_0 \times \Omega_0 \to \mathcal{L}(\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathcal{L}(\mathcal{Y}))\} \subset \mathcal{V} = \{f : \Omega \times \Omega \to \mathcal{L}(\mathcal{Y})\}.\end{aligned}$$

Then $\mathcal{C}$ is a cone included in $\mathcal{V}$ inheriting the topology of point-wise weak-$^*$ convergence from $\mathcal{V}$.

We shall need the following lemmas.

**Lemma 3.2.1.** *$\mathcal{C}$ is closed.*

*Proof.* It will suffice to show that:

$$\text{given } \Xi_n \in \mathcal{C} \text{ with } \Xi_n \overset{pointwise}{\longrightarrow} \Xi \text{ then } \Xi \in \mathcal{C}$$

(i.e. $\Xi_n(z,w) = \Gamma_n(z,w)[I - \mathbb{E}(z)\mathbb{E}(w)^*]$ for some $\Gamma_n$ completely positive kernel).

**Step 1**. Let us fix a positive integer $n$, we have that for a fixed $z \in \Omega_0$,

$$\Xi_n(z,z) = \Gamma_n(z,z)[I - \mathbb{E}(z)\mathbb{E}(z)^*].$$

We see that $\|E(z)\|_\infty < 1$ for each $z \in \Omega$ (by equation (3.1)). Hence

$$\begin{aligned}\Gamma_n(z,z)[I] &= \Gamma_n(z,z)[(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{1/2}(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{-1}(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{1/2}] \\ &\leq \Gamma_n(z,z)[(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{1/2}\|(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{-1}\|(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{1/2}] \\ &\leq \Gamma_n(z,z)\Big[(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{1/2}\Big(\frac{1}{1 - \|\mathbb{E}(z)\|^2}\Big)(I - \mathbb{E}(z)\mathbb{E}(z)^*)^{1/2}\Big] \\ &= \Big(\frac{1}{1 - \|\mathbb{E}(z)\|^2}\Big)\Gamma_n(z,z)[(I - \mathbb{E}(z)\mathbb{E}(z)^*)] \\ &= \Big(\frac{1}{1 - \|\mathbb{E}(z)\|^2}\Big)\Xi_n(z,z),\end{aligned}$$

which implies

$$\|\Gamma_n(z,z)\| \leq M_z \|\Xi_n(z,z)\|, \text{ where } M_z = \frac{1}{1 - \|\mathbb{E}(z)\|^2}.$$

From the theory of Schur complement (e.g. 6 and 7, page 229 of [10]) we know

$$\|\Gamma_n(z,w)\| \leq M_z \|\Xi_n(z,z)\|^{1/2} M_w \|\Xi_n(w,w)\|^{1/2}.$$

**Step 2**. Since $\Omega_0$ is finite, it follows that

$$\|\Gamma_n(z,w)\| \leq M, \text{ for all } z, w \in \Omega_0.$$

Note that $\{\Gamma_n(z,w)\}$ is contained in the space $\mathcal{L}(\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T)),\mathcal{L}(\mathcal{Y}))$. It is well known (see e.g. Corollary 2 page 230 [15]) that $\mathcal{L}(\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T)),\mathcal{L}(\mathcal{Y}))$ is isometrically isomorphic as a Banach space to the dual space $(\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T))\hat{\otimes}\mathcal{C}_1(\mathcal{Y}))^*$ where $\mathcal{C}_1(\mathcal{Y})$ is the collection of trace class operator from $\mathcal{Y}$ to $\mathcal{Y}$, and $\hat{\otimes}$ indicates the completion of the algebraic tensor product in the projective tensor norm. In particular bounded subsets of $\mathcal{L}(\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T)),\mathcal{L}(\mathcal{Y}))$ are compact in the associated weak-* topology. Therefore there exists a converging subsequence $\{\Gamma_{n_k}\}$ to $\Gamma$, where

$$\Gamma_{n_k}(z,w) \underset{weak-*}{\longrightarrow} \Gamma(z,w)$$

means

$$tr\big(\Gamma_{n_k}(z,w)[f]X\big) \underset{k\to\infty}{\longrightarrow} tr\big(\Gamma(z,w)[f]X\big)$$

for each $f \in \mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T))$ and $X \in \mathcal{C}_1(\mathcal{Y})$.

Clearly $\Gamma$ is a completely positive kernel since for all $z_1,\dots,z_N \in \Omega_0$, $y_1,\dots,y_N \in \mathcal{Y}$ and $F_1,\dots,F_N \in \mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T))$, we have:

$$\begin{aligned}\sum_{i,j=1}^N \langle \Gamma(z_i,z_j)[F_i^*F_j]y_j,y_i\rangle_{\mathcal{Y}} &= \sum_{i,j=1}^N tr(\Gamma(z_i,z_j)[F_i^*F_j]y_jy_i^*)\ (y_jy_i^* \in \mathcal{C}_1(\mathcal{Y}) \\ &\quad \text{because it has rank 1}) \\ &= \lim_{k\to\infty}\sum_{i,j=1}^N \langle \Gamma_{n_k}(z_i,z_j)[F_i^*F_j]y_j,y_i\rangle_{\mathcal{Y}} \geq 0,\end{aligned}$$

because each $\Gamma_{n_k}$ is a completely positive kernel. Thus we have

$$\Xi_{n_k}(z,w) = \Gamma_{n_k}(z,w)[I-\mathbb{E}(z)\mathbb{E}(w)^*],$$

but

$$\Xi_{n_k}(z,w) \underset{k\to\infty}{\longrightarrow} \Xi(z,w)$$

(in the weak-$*$ topology of $\mathcal{L}(\mathcal{Y})$), and

$$\Gamma_{n_k}(z,w)[I-\mathbb{E}(z)\mathbb{E}(w)^*] \underset{k\to\infty}{\longrightarrow} \Gamma(z,w)[I-\mathbb{E}(z)\mathbb{E}(w)^*],$$

therefore

$$\Xi(z,w) = \Gamma(z,w)[I-\mathbb{E}(z)\mathbb{E}(w)^*],$$

and this concludes the proof. □

**Lemma 3.2.2.** *If for all $\Lambda \in \mathcal{V}^*$ such that $Re(\Lambda(G)) \geq 0$ for all $G \in \mathcal{C}$ then for $S \in \mathcal{SA}_\Psi(\mathcal{U},\mathcal{Y})$ we have $Re(\Lambda(I-S(z)S(w)^*)) \geq 0$, and $I-S(z)S(w)^* \in \mathcal{C}$*

*Proof.* Let $A = \{I - S(z)S(w)^*\}$ and $B = \mathcal{C}$. We have that $A$ is compact and, by Lemma 3.2.1, $B$ is closed. Also $\mathcal{V}$ is convex.

If $I - S(z)S(w)^* \notin \mathcal{C}$, then, by Theorem 3.4 (part (b)) in [24], there exist $\Lambda \in \mathcal{V}^*$ and $\sigma_1,\ \sigma_2 \in \mathbb{R}$ such that:

$$Re(\Lambda(I - S(z)S(w)^*)) < \sigma_1 < \sigma_2 < Re(\Lambda(G)),\ \text{for all}\ G \in \mathcal{C}.$$

Without loss of generality we can assume that $\sigma_1, \sigma_2 < 0$ since $\mathcal{C}$ is a cone, thus we have: If $I - S(z)S(w)^* \notin \mathcal{C}$, there exists $\Lambda \in \mathcal{V}^*$ such that $Re(\Lambda(I - S(z)S(w)^*)) < 0$ and $Re(\Lambda(G)) \geq 0$ for all $G \in \mathcal{C}$. Now taking the contrapositive of the statement above we get the desired result. □

**Lemma 3.2.3.** *If $\Xi(z,w) = H(z)H(w)^*$ is a positive kernel ($H : \Omega \to \mathcal{L}(\mathcal{X}, \mathcal{Y})$), then $\Xi \in \mathcal{C}$.*

*Proof.* Let $\psi_0$ be any particular element of $\Psi$. Then it suffices to find $G$ so that

$$\Xi(z,w) = G(z)(I - \psi_0(z)\psi_0(w)^*)G(w)^*. \tag{3.3}$$

Once we have such $G$ we can define $\Gamma$ by $\Gamma(z,w)(f) = G(z)\rho(f)G(w)^*$ where $\rho(f) = f(\psi_0)$ and the representation (3.3) can be rewritten as

$$\Xi(z,w) = \Gamma(z,w)[I - \mathbb{E}(z)\mathbb{E}(w)^*].$$

Let us pick any particular $\psi_0 \in \Psi$, so $\psi_0 : \Omega \to \mathcal{L}(\mathcal{U}_T, \mathcal{Y}_T)$ and choose $y_0 \in \mathcal{Y}_T$,. Let us set $P_0 = \langle \cdot, y_0 \rangle : \mathcal{Y}_T \to \mathbb{C}$. Then $P_0(I - \psi_0(z)\psi_0(w)^*)P_0^*$ is an scalar-valued operator and so

$$P_0(I - \psi_0(z)\psi_0(w)^*)P_0^* = 1 - P_0\psi_0(z)\psi_0(w)^*P_0^*.$$

Since $P_0\psi_0(z)$ is an operator from $\mathcal{U}_T$ to $\mathbb{C}$, then

$$P_0\psi_0(z) = [b_1(z)\ b_2(z) \cdots\ b_n(z)\ b_{n+1}(z) \cdots],$$

thus

$$\psi_0(w)^* : c \mapsto \begin{bmatrix} b_1(w)^* \\ b_2(w)^* \\ \vdots \end{bmatrix} c,$$

where $\begin{bmatrix} b_1(w)^* \\ b_2(w)^* \\ \vdots \end{bmatrix} \in \mathbb{B}^d,\ 1 \leq d \leq \infty.$

Then $\dfrac{1}{1 - P_0\psi_0(z)\psi_0(w)^*P_0^*}$ is a positive kernel since $P_0\psi_0(z)\psi_0(w)^*P_0^*$ is positive $\mathbb{C}$-valued kernel and

$$\frac{1}{1 - P_0\psi_0(z)\psi_0(w)^*P_0^*} = \sum_{n=0}^{\infty}(P_0\psi_0(z)\psi_0(w)^*P_0^*)^n,$$

where each term $(P_0\psi_0(z)\psi_0(w)^*P_0^*)^n$ is a positive $\mathbb{C}$-valued kernel (by Theorem 2.2.5 and the fact that the sum of positive kernels is a positive kernel), and hence has a Kolmogorov factorization

$$\frac{1}{1-P_0\psi_0(z)\psi_0(w)^*P_0^*} = H_0(z)H_0(w)^*.$$

Then

$$\begin{aligned}\Xi(z,w) &= H(z)H(w)^* \\ &= H(z)\Big(\frac{1}{1-P_0\psi_0(z)\psi_0(w)^*P_0^*}P_0(I-\psi_0(z)\psi_0(w)^*)P_0^*I_{\mathcal{X}}\Big)H(w)^* \\ &= H(z)(H_0(z)H_0(w)^*P_0(I-\psi_0(z)\psi_0(w)^*)P_0^*I_{\mathcal{X}})H(w)^*.\end{aligned}$$

Now, since $H_0(z) = [h_{01}(z)\ h_{02}(z)\ \cdots] : \mathcal{X}_0 \to \mathbb{C}$, then,

$$\begin{aligned}\Xi(z,w) &= H(z)H(w)^* \\ &= H(z)\Big(\sum_{n=1}^{\infty} h_{0n}(z)h_{0n}(w)^*P_0(I-\psi_0(z)\psi_0(w)^*)P_0^*\Big)I_{\mathcal{X}}H(w)^* \\ &= \sum_{n=1}^{\infty} H(z)\{h_{0n}(z)P_0(I-\psi_0(z)\psi_0(w)^*)P_0^*h_{0n}(w)^*I_{\mathcal{X}}\}H(w)^* \\ &= G(z)\begin{bmatrix} P_0(I-\psi_0(z)\psi_0(w))P_0^* & 0 & \cdots \\ 0 & P_0(I-\psi_0(z)\psi_0(w))P_0^* & \cdots \\ 0 & \ddots & \cdots \end{bmatrix} G(w)^*,\end{aligned}$$

where $G(z) = [H(z)h_{01}(z)\ H(z)h_{02}(z)\ \cdots]$. $\square$

**Lemma 3.2.4.** *Given $\rho : \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T)) \to \mathcal{L}(\mathcal{X})$ a unital $*$-representation. Define*

$$\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X} = \bigvee\{F \otimes x : F \in \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)), x \in \mathcal{X}\}$$

*with inner product*

$$\langle F \otimes x, G \otimes y\rangle_{\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X}} = \langle \rho(G^*F)x, y\rangle_{\mathcal{X}}.$$

*Then $\langle\cdot,\cdot\rangle_{\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X}}$ is positive semidefinite.*

*Proof.*

$\langle F\otimes x, G\otimes y\rangle_{\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X}} = \langle\langle F, G\rangle_{\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))}\cdot x, y\rangle_{\mathcal{X}} = \langle G^*F\cdot x, y\rangle_{\mathcal{X}} = \langle\rho(G^*F)x, y\rangle_{\mathcal{X}}$,

thus, by Theorem 2.1.1, $\langle\cdot,\cdot\rangle_{\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X}}$ is positive semidefinite. $\square$

**Lemma 3.2.5.** *Given $\rho$ as in Lemma 3.2.4 and $H \in \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T))$. Define*

$$L_H : \mathcal{X} \to \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{U}_T)) \otimes \mathcal{X}$$

$$x \mapsto H \otimes x$$

*then*

$$L_H^* : G \otimes x \mapsto \rho(H^* G)x.$$

*Proof.*

$$\begin{aligned}\langle L_H^*(G \otimes x), y\rangle_{\mathcal{X}} &= \langle G \otimes x, L_H y\rangle_{\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X}} \\ &= \langle G \otimes x, H \otimes y\rangle_{\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X}} \\ &= \langle \rho(H^* G)x, y\rangle_{\mathcal{X}},\end{aligned}$$

thus $L_H^*(G \otimes x) = \rho(H^* G)x$. □

**Lemma 3.2.6.** *Given $\rho$ as in Lemma 3.2.4, then*

$$\rho(\mathbb{E}(z)\mathbb{E}(w)^*) = L^*_{\mathbb{E}(z)^*} L_{\mathbb{E}(w)^*}$$

*Proof.* For any particular $x \in \mathcal{X}$,

$$\begin{aligned}L^*_{\mathbb{E}(z)^*} L_{\mathbb{E}(w)^*}(x) &= L^*_{\mathbb{E}(z)^*}(\mathbb{E}(w)^* \otimes x) \\ &= \rho(\mathbb{E}(z)\mathbb{E}(w)^*)x \text{ (by Lemma 3.2.5).}\end{aligned}$$

□

## 3.3 The proof of the Main Result

### 3.3.1 The case where $\Omega_0$ is finite

*Proof.* (1) $\Rightarrow$ (2). Let us consider first $\Omega_0 = \{z_1, \ldots, z_N\}$. We need to show $I - S_0(z)S_0(w)^* \in \mathcal{C}$. For such a purpose, using Lemma 3.2.2 it will be sufficient to prove the following.

If $L \in \mathcal{V}^*$ with $Re(L(G)) \geq 0$ for all $G \in \mathcal{C}$ then $Re(\Lambda(I - S_0(z)S_0(w)^*)) \geq 0$, and $I - S_0(z)S_0(w)^* \in \mathcal{C}$.

Define

$$L_1(\Xi) = \frac{1}{2}(L(\Xi) + \overline{L(\Xi^\vee)}),$$

where we have set

$$\Xi^\vee(z, w) = \Xi(w, z)^*.$$

*Remark* 3.3.1. $L_1(\Xi) = Re(L(\Xi))$ in case $\Xi^\vee = \Xi$.

Let $\mathcal{H}_{L_1,\epsilon} = \{f : \Omega_0 \to \mathcal{Y}\}$ with inner product:

$$\Big\langle f, g \Big\rangle_{\mathcal{H}_{L_1,\epsilon}} = L_1(\Delta_{f,g}) + \epsilon^2 \sum_{j=1}^{N} tr(\Delta_{f,g}(z_j, z_j)),$$

where

$$\Delta_{f,g}(z, w) = f(z)g(w)^*.$$

$\langle \cdot, \cdot \rangle_{H_{L_1}}$ is clearly positive semidefinite since:

$$(\Delta_{f,f})^\vee(z, w) = (\Delta_{f,f}(w, z))^* = (f(w)f(z)^*)^* = f(z)f(w)^* = (\Delta_{f,f})(z, w)$$

implies (by Remark 3.3.1)

$$\langle f, f \rangle_{H_{L_1}} = L_1(\Delta_{f,f}) = Re(\Delta_{f,f}) \geq 0 \text{ (because } \Delta_{f,f} \in \mathcal{C} \text{ (by Lemma 3.2.3))}.$$

Then $\mathcal{H}_{L_1,\epsilon} \otimes \mathcal{U}_T$ can be identified with $\{f : \Omega_0 \to \mathcal{L}(\mathcal{U}_T, \mathcal{Y})\}$ with inner product

$$\Big\langle f, g \Big\rangle_{\mathcal{H}_{L_1,\epsilon}} = L_1(\Delta_{f,g}) + \epsilon^2 \sum_{j=1}^{N} tr(\Delta_{f,g}(z_j, z_j)).$$

Now, for $f \in \mathcal{H}_{L_1,\epsilon}$ the following two conditions are satisfied

1. $(\Delta_{f,f} - \Delta_{f\psi,f\psi})^\vee(z, w) = (\Delta_{f,f} - \Delta_{f\psi,f\psi})(z, w)$, and
2. $$\begin{aligned} (\Delta_{f,f} - \Delta_{f\psi,f\psi})(z, w) &= f(z)f(w)^* - f(z)\psi(z)\psi(w)^* f(w)^* \\ &= f(z)(I - \psi(z)\psi(w)^*)f(w)^* \\ &= H(z)(I - \psi(z)\psi(w)^*)H(w)^* \\ &= \Gamma(z, w)[I - \mathbb{E}(z)\mathbb{E}(w)^*], \end{aligned}$$

   where $\Gamma(z, w)[f] = H(z)\rho(f)H(w)^*,$ with $\rho(f) = f(\psi),$

which implies $\Delta_{f,f} - \Delta_{f\psi,f\psi} \in \mathcal{C}$.

Note that for each $w \in \Omega$ the map $f \mapsto f(w)$ is bounded since

$$\|f\|^2 = L_1(\Delta_{f,f}) + \epsilon^2 \sum_{j=1}^{N} tr(\Delta_{f,f}(z_j, z_j)) \geq \epsilon^2 \|f(w)\|^2.$$

Thus the map $E_w : f \mapsto f(w)$ is bounded and $\|E(w)\| \leq \dfrac{1}{\epsilon}$. Furthermore $R_\psi$ is contractive on $\mathcal{H}_{L_1,\epsilon}$, since

$$\begin{aligned}\|f\|^2 - \|R_\psi f\|^2 &= L_1(\Delta_{f,f} - \Delta_{f\psi,f\psi}) + \epsilon^2 \sum_{j=1}^{N} tr(f(z_j)(I - \psi(z_j)\psi(z_j)^*)f(z_j)^*) \\ &= Re(L(\Delta_{f,f} - \Delta_{f\psi,f\psi})) + \epsilon^2 \sum_{j=1}^{N} tr(f(z_j)(I - \psi(z_j)\psi(z_j)^*)f(z_j)^*) \geq 0\end{aligned}$$

because of the fact $\Delta_{f,f} - \Delta_{f\psi,f\psi} \in \mathcal{C}$ and equation (3.1).

Thus by Remark 2.2.1 there exists a positive kernel $K_\epsilon$ such that $\mathcal{H}_{L_1,\epsilon} = \mathcal{H}(K_\epsilon)$. And $\|R_\psi\| \leq 1$ on $\mathcal{H}_{L_1,\epsilon}$ for all $\epsilon$, hence $K_\epsilon$ is $\Psi$-admissible. Since $S_0 \in \mathcal{SA}_\Psi(\mathcal{U},\mathcal{Y})$, we conclude that $R_{S_0} : \mathcal{H}(K) \otimes \mathcal{Y} \to \mathcal{H}(K) \otimes \mathcal{U}$ is contractive as well.
Also

$$\begin{aligned}(\Delta_{f,f} - \Delta_{fS_0,fS_0})^\vee(z,w) &= ((\Delta_{f,f} - \Delta_{fS_0,fS_0})(w,z))^* \\ &= (f(w)f(z)^* - f(w)S_0(w)S_0(z)^*f(z)^*)^* \\ &= f(z)f(w)^* - f(z)S_0(z)S_0(w)^*f(w)^* \\ &= \Delta_{f,f}(z,w) - \Delta_{fS_0,fS_0}(z,w) \\ &= (\Delta_{f,f} - \Delta_{fS_0,fS_0})(z,w).\end{aligned}$$

So, by Remark 3.3.1,

$$L_1(\Delta_{f,f} - \Delta_{fS_0,fS_0}) = Re(L(\Delta_{f,f} - \Delta_{fS_0,fS_0})).$$

Then,

$$\begin{aligned}0 \leq \|f\|^2 - \|R_{S_0} f\|^2 &= L_1(\Delta_{f,f} - \Delta_{fS_0,fS_0}) + \epsilon^2 \sum_{j=1}^{N} tr(f(z_j)(I - S_0(z_j)S_0(z_j)^*)f(z_j)^*) \\ &= Re(L(\Delta_{f,f} - \Delta_{fS_0,fS_0})) \\ &\quad + \epsilon^2 \sum_{j=1}^{N} tr(f(z_j)(I - S_0(z_j)S_0(z_j)^*)f(z_j)^*) \\ &= Re(L(f(z)(I - S_0(z)S_0(w)^*)f(w)^*) \\ &\quad + \epsilon^2 \sum_{j=1}^{N} tr(f(z_j)(I - S_0(z_j)S_0(z_j)^*)f(z_j)^*) \text{ for all } \epsilon > 0.\end{aligned}$$

Letting $\epsilon$ tend to zero then gives

$$Re(L(f(z)(I - S_0(z)S_0(w)^*)f(w)^*)) \geq 0 \text{ for all } f.$$

For the special case $f(z) \equiv I$ we have:

$$Re(L(I - S_0(z)S_0(w)^*)) \geq 0.$$

Since $L$ was arbitrarily chosen, then by Lemma 3.2.2, $I - S_0(z)S_0(w)^* \in \mathcal{C}$.

### 3.3.2 Removal of the assumption that $\Omega_0$ is finite

Now we remove the assumption that $\Omega_0$ is finite.

We now assume that $\Omega_0$ is *any* subset of $\Omega$ (including possibly $\Omega_0 = \Omega$).

If we take $F$ to be a finite subset of $\Omega_0$, then we define

$$\begin{aligned} \Delta_F \;=\; & \{\Gamma : F \times F \to \mathcal{L}(\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathcal{L}(\mathcal{Y})) \text{ completely positive kernel :} \\ & I - S_0(z)S_0(w)^* = \Gamma(z,w)[1 - \mathbb{E}(z)\mathbb{E}(w)^*] \text{ for } z, w \in F\} \end{aligned}$$

From the discussion above we have that $\Delta_F$ is nonempty and by Lemma 3.9 [18] is compact. Now, for finite sets $F$ and $G$ with $F \subset G \subset \Omega_0$, we define

$$W_{G,F} : \Delta_G \to \Delta_F$$

$$W_{G,F}(\Gamma) = \Gamma|_{F \times F}.$$

Let $\mathcal{F}$ = collection of all finite subsets of $\Omega_0$ partially ordered by inclusion, the triple $(\Delta_F, W_{G,F}, \mathcal{F})$ is an inverse limit of nonempty compact spaces. Consequently, by Kurosh's Theorem (see [6] page 75) , for each $F \in \mathcal{F}$ there is $\Gamma_F \in \Delta_F$ so that whenever $F, G \in \mathcal{F}$ and $F \subset G$,

$$W_{G,F}(\Gamma_G) = \Gamma_F. \tag{3.4}$$

Now, define

$$\Gamma : \Omega_0 \times \Omega_0 \to \mathcal{L}(\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathcal{L}(\mathcal{Y}))$$

$$\Gamma(z,w) = \Gamma_F(z,w),$$

where $F \in \mathcal{F}$ and $z, w \in F$. Equation (3.4) guarantees that $\Gamma$ is well-defined.

Thus if $F$ is a finite subset, $f_z, f_w \in \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$ and $g_z, g_w \in \mathcal{L}(\mathcal{Y})$, then

$$\sum_{z,w \in F} g_z^* \Gamma(z,w)[f_z^* f_w] g_w = \sum_{z,w \in F} g_z^* \Gamma_F(z,w)[f_z^* f_w] g_w \geq 0$$

since $\Gamma_F$ is a completely positive kernel. Hence $\Gamma$ is positive, thus the proof of (1) $\Rightarrow$ (2) is complete.

(2) $\Rightarrow$ (3) Using Theorem 2.2.7 (part (3)) we have

$$\Gamma(z,w)[a] = H(z)\rho(a)H(w)^*,$$

therefore making $l = \langle (I - S_0(z)S_0(w)^*)y_w, y_z\rangle_{\mathcal{Y}}$, we have

$$\begin{array}{rcl} l & = & \langle \Gamma(z,w)[I - \mathbb{E}(z)\mathbb{E}(w)^*]y_w, y_z\rangle_{\mathcal{Y}} \\ & = & \langle H(z)\rho(I - \mathbb{E}(z)\mathbb{E}(w)^*)H(w)^*y_w, y_z\rangle_{\mathcal{Y}} \\ & = & \langle \rho(I - \mathbb{E}(z)\mathbb{E}(w)^*)H(w)^*y_w, H(z)^*y_z\rangle_{\mathcal{X}} \\ & = & \langle H(w)^*y_w, H(z)^*y_z\rangle_{\mathcal{X}} - \langle \rho(\mathbb{E}(z)\mathbb{E}(w)^*)H(w)^*y_w, H(z)^*y_z\rangle_{\mathcal{X}} \\ & = & \langle H(w)^*y_w, H(z)^*y_z\rangle_{\mathcal{X}} - \langle L^*_{\mathbb{E}(z)^*}L_{\mathbb{E}(w)^*}H(w)^*y_w\rangle_{\mathcal{X}} \text{ (by Lemma 3.2.6)} \\ & = & \langle H(w)^*y_w, H(z)^*y_z\rangle_{\mathcal{X}} \\ & - & \langle L_{\mathbb{E}(w)^*}H(w)^*y_w, L_{\mathbb{E}(z)^*}H(z)^*y_z\rangle_{\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X}}. \end{array}$$

So

$$\begin{array}{l} \langle L_{\mathbb{E}(w)^*}H(w)^*y_w, L_{\mathbb{E}(z)^*}H(z)^*y_z\rangle_{\mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X}} + \langle y_w, y_z\rangle_{\mathcal{Y}} \\ \quad = \langle H(w)^*y_w, H(z)^*y_z\rangle_{\mathcal{X}} + \langle S_0(w)^*y_w, S_0(z)^*y_z\rangle_{\mathcal{U}}. \end{array}$$

Thus

$$V : \begin{bmatrix} L_{\mathbb{E}(w)^*}H(w)^*y_w \\ y_w \end{bmatrix} \mapsto \begin{bmatrix} H(w)^*y_w \\ S_0(w)^*y_w \end{bmatrix}$$

is isometric from

$$\mathcal{D} = \bigvee \left\{ \begin{bmatrix} L_{\mathbb{E}(w)^*}H(w)^*y_w \\ y_w \end{bmatrix} \right\} \subset \begin{bmatrix} \mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

onto

$$\bigvee \left\{ \begin{bmatrix} H(w)^*y_w \\ S_0(w)^*y_w \end{bmatrix};\ w \in \Omega,\ y_w \in \mathcal{Y} \right\}.$$

We extend $V$ to all of $\begin{bmatrix} \mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ by $V|_{\mathcal{D}^\perp} = 0$. Thus $V$ is a contraction from $\begin{bmatrix} \mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ to $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$.

Write $V = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$ where $\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{C}_b(\Psi,\mathcal{L}(\mathcal{Y}_T,\mathcal{U}_T))\otimes\mathcal{X} \\ \mathcal{Y} \end{bmatrix}$, then

$$\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} L_{\mathbb{E}(w)^*}H(w)^*y_w \\ y_w \end{bmatrix} = \begin{bmatrix} H(w)^*y_w \\ S_0(w)^*y_w \end{bmatrix}. \tag{3.5}$$

From the first row of (3.5) we read off

$$A^* L_{\mathbb{E}(w)^*} H(w)^* y_w + C^* y_w = H(w)^* y_w.$$

Since $\sup_\psi\{\|\psi(w)\|\} < 1$ and $\|A^*\| \leq 1$, we see that $I - A^* L_{\mathbb{E}(w)^*}$ is invertible and

$$H(w)^* y_w = (I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* y_w,$$

then

$$L_{\mathbb{E}(z)^*}(I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* y_w = L_{\mathbb{E}(w)^*} H(w)^* y_w.$$

From the second row of (3.5) we read off

$$B^* L_{\mathbb{E}(w)^*} H(w)^* y_w + D^* y_w = S_0(w)^* y_w$$

then

$$B^* L_{\mathbb{E}(w)^*}(I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* + D^* = S_0(w)^*$$

which implies

$$S_0(z) = D + C(I - L^*_{\mathbb{E}(z)^*} A)^{-1} L^*_{\mathbb{E}(z)^*} B. \tag{3.6}$$

Making $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, we have $\mathbf{U}$ is weakly coisometric, i.e. $\mathbf{U}^*$ is isometric on

$$\bigvee \left\{ \begin{bmatrix} L_{\mathbb{E}(w)^*}(I - A^* L_{\mathbb{E}(w)^*})^{-1} C^* y_w \\ y_w \end{bmatrix};\ w \in \Omega,\ y_w \in \mathcal{Y} \right\}$$

$$= \bigvee \left\{ \begin{bmatrix} L_{\mathbb{E}(w)^*} H(w)^* y_w \\ y_w \end{bmatrix};\ w \in \Omega,\ y_w \in \mathcal{Y} \right\} = \mathcal{D} \text{ (since } \mathbf{U}^*|_{\mathcal{D}} = V \text{ is isometric).}$$

Combining with (3.6) we see that $\mathbf{U}$ is a weakly coisometric realization of $S_0$ and (3) follows.

(3) $\Rightarrow$ (2). We are given $S_0$ of the form (3.2) and must find $\Gamma$ so that (2) in the statement of the Main Theorem holds, i.e. $\Gamma$ gives an Agler decomposition. We actually are able to show that $S_0$ extends to a function $S$ defined on all of $\Omega$ for which (2) holds on all of $\Omega$.

Our candidate is:

$$\Gamma : \Omega \times \Omega \to \mathcal{L}(\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T)), \mathcal{L}(\mathcal{Y}))$$
$$\Gamma(z,w)[f] = C(I - L^*_{\mathbb{E}(z)^*} A)^{-1} \rho(f) (I - A^* L_{\mathbb{E}(w)^*})^{-1} C^*,$$

where $f \in \mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T))$.

This candidate is certainly a completely positive kernel since

$$\Gamma(z,w)[f] = H(z)\rho(f)H(w)^*, \text{ for } H(z) = C(I - L^*_{\mathbb{E}(z)^*} A)^{-1}.$$

It remains to show:

$$I - S_0(z)S_0(w)^* = C(I - L^*_{\mathbb{E}(z)^*}A)^{-1}\rho(I - \mathbb{E}(z)\mathbb{E}(w)^*)(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^*.$$

Since

$$\begin{aligned} A^*L_{\mathbb{E}(w)^*}(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* + C^* &= [A^*L_{\mathbb{E}(w)^*} + (I - A^*L_{\mathbb{E}(w)^*})](I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* \\ &= (I - A^*L_{\mathbb{E}(w)^*})^{-1}C^*, \end{aligned}$$

then

$$\begin{aligned} \mathbf{U}^* \begin{bmatrix} L_{\mathbb{E}(w)^*}(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* \\ I \end{bmatrix} &= \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} L_{\mathbb{E}(w)^*}(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* \\ I \end{bmatrix} \\ &= \begin{bmatrix} (I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* \\ D^* + B^*L_{\mathbb{E}(w)^*}(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* \end{bmatrix}. \\ &= \begin{bmatrix} (I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* \\ S(w)^* \end{bmatrix}. \end{aligned}$$

Because $\mathbf{U}$ is weakly coisometric, then:

$$\left\langle \mathbf{U}^* \begin{bmatrix} L_{\mathbb{E}(w)^*}(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* \\ I \end{bmatrix} y, \mathbf{U}^* \begin{bmatrix} L_{\mathbb{E}(z)^*}(I - A^*L_{\mathbb{E}(z)^*})^{-1}C^* \\ I \end{bmatrix} y' \right\rangle$$

$$= \left\langle \begin{bmatrix} L_{\mathbb{E}(w)^*}(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* \\ I \end{bmatrix} y, \begin{bmatrix} L_{\mathbb{E}(z)^*}(I - A^*L_{\mathbb{E}(z)^*})^{-1}C^* \\ I \end{bmatrix} y' \right\rangle,$$

so

$$\left\langle \begin{bmatrix} (I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* \\ S_0(w)^* \end{bmatrix} y, \begin{bmatrix} (I - A^*L_{\mathbb{E}(z)^*})^{-1}C^* \\ S_0(z)^* \end{bmatrix} y' \right\rangle$$

$$= \left\langle \begin{bmatrix} L_{\mathbb{E}(w)^*}(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^* \\ I \end{bmatrix} y, \begin{bmatrix} L_{\mathbb{E}(z)^*}(I - A^*L_{\mathbb{E}(z)^*})^{-1}C^* \\ I \end{bmatrix} y' \right\rangle,$$

$\Longrightarrow$

$$\langle C(I - L^*_{\mathbb{E}(z)^*}A)^{-1}(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^*y, y'\rangle + \langle S_0(z)S_0(w)^*y, y'\rangle$$

$$= \langle C(I - L^*_{\mathbb{E}(z)^*}A)^{-1}L^*_{\mathbb{E}(z)^*}L_{\mathbb{E}(w)^*}(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^*y, y'\rangle + \langle y, y'\rangle$$

$\Longrightarrow$

$$\begin{aligned} \langle (I - S_0(z)S_0(w)^*)y, y'\rangle &= \langle C(I - L^*_{\mathbb{E}(z)^*}A)^{-1}(I - L^*_{\mathbb{E}(z)^*}L_{\mathbb{E}(w)^*}) \\ &\times\ (I - A^*L_{\mathbb{E}(w)^*})^{-1}C^*y, y'\rangle \\ &= \langle C(I - L^*_{\mathbb{E}(z)^*}A)^{-1}\rho(I - \mathbb{E}(z)\mathbb{E}(w)^*) \\ &\times\ (I - A^*L_{\mathbb{E}(w)^*})^{-1}C^*y, y'\rangle. \end{aligned}$$

Therefore,

$$I - S_0(z)S_0(w)^* = C(I - L^*_{\mathbb{E}(z)^*}A)^{-1}\rho(I - \mathbb{E}(z)\mathbb{E}(w)^*)(I - A^*L_{\mathbb{E}(w)^*})^{-1}C^*.$$

Then $\Gamma$ given by

$$\Gamma(z,w)[f] = H(z)\rho(f)H(w)^*$$

with $H(z) = C(I - L^*_{\mathbb{E}(z)^*}A)^{-1}$ does the job.

(2) $\Rightarrow$ (1) Since (2) in the statement of the Main theorem holds, then (3) also holds, that is

$$S_0(z) = D + C(I - L^*_{\mathbb{E}(z)^*}A)^{-1}L^*_{\mathbb{E}(z)^*}B \text{ for all } z \in \Omega_0.$$

We realize that such a formula for $S_0$ still makes sense if $z \in \Omega$, therefore if we define

$$S(z) = D + C(I - L^*_{\mathbb{E}(z)^*}A)^{-1}L^*_{\mathbb{E}(z)^*}B \text{ for all } z \in \Omega,$$

then $S$ is an extension of $S_0$. So we have $S$ has a weakly coisometric realization, therefore $S$ has an Agler decomposition over all $\Omega$ by the proof of (3) $\Rightarrow$ (2) already done. Thus without loss of generality we may assume that (2) holds with $\Omega_0 = \Omega$. We then show that $S \in \mathcal{SA}_\Psi(\mathcal{U},\mathcal{Y})$, that is $R_S$ is contractive.

Let $K \in \mathcal{K}_\Psi(\mathcal{E})$, $z_1,\dots,z_N \in \Omega$, $f : \{z_1,\dots,z_N\} \to \mathcal{C}_2(\mathcal{Y},\mathcal{E})$ and $N = 1,2,\dots$. Let us define

$$K_\epsilon(z_i,z_j) = K(z_i,z_j) + \epsilon^2\delta_{i,j}.$$

This $K_\epsilon$ is an (strictly) positive kernel. Then using the theory of dual basis developed in Section 2.2.2 we get $L_\epsilon$ so that

$$[L_\epsilon(z_i,z_j)]_{i,j=1}^N = \left([K_\epsilon(z_i,z_j)]_{i,j=1}^N\right)^{-1}.$$

Let us consider the kernel

$$P(z_i,z_j) = \sum_{i,j=1}^N tr[L_\epsilon(z_i,z_j)f(z_j)(I - S(z_j)S(z_i)^*)f(z_i)^*],$$

then we have

$$P = \sum_{i,j=1}^N tr[L_\epsilon(z_i,z_j)f(z_j)H(z_j)\rho(I - \mathbb{E}(z_j)\mathbb{E}(z_i)^*)H(z_i)^*f(z_i)^*].$$

By Corollary 2.3.7, we may assume that $\rho$ has the form

$$\rho = \infty \cdot \pi_{\mu_\infty} \oplus\ 1 \cdot \pi_{\mu_1} \oplus\ 2 \cdot \pi_{\mu_2} \oplus \cdots$$

with values equal to operators on $\mathcal{H} = (L^2_{\mathbb{C}^N}(\mu_\infty))^\infty \oplus \bigoplus_{r=1}^\infty (L^2_{\mathbb{C}^N}(\mu_r))^r$. With respect to this decomposition, the operator $H(z_i)^*$ then has column decomposition

$$H(z_i)^* = \begin{bmatrix} H_\infty(z_i)^* \\ col_{r=1}^\infty H_r(z_i)^* \end{bmatrix}.$$

As each $H_r(z_i)^* : \mathcal{Y} \to (L^2_{\mathbb{C}^N}(\mu_r))^r$ we have the still finer decomposition

$$H_r(z_i)^* = col_{m=1}^r H_{r,m}(z_i)^*$$

where each $H_{r,m}(z_i)^* : \mathcal{Y} \to L^2_{\mathbb{C}^N}(\mu_r)$ $(r = \infty, 1, 2, 3, \dots)$.

Finally we define $H_{r,m}(z_i, \psi)^* : \mathcal{Y} \to \mathbb{C}^N$ by

$$(H_{r,m}(z_i)^* y)(\psi) = H_{r,m}(z_i, \psi)^* y.$$

Then the adjoint of $H_{r,m}(z_i)^*$ is given by

$$(H_{r,m}(z_i)^*)^* f = H_{r,m}(z_i) f = \int_\Psi H_{r,m}(z_i, \psi) f(\psi) \mu_r(d\psi).$$

Then we get

$$P = \sum_{j,k=1}^\infty \int_\Psi \left[ \sum_{l=1}^k \sum_{i,j=1}^N tr[L_\epsilon(z_i, z_j) H_{k,l}(z_j, \psi)(I - \psi(z_j)\psi(z_i)^*) H_{k,l}(z_i, \psi)^* f(z_i)^*] \right] \mu_k(d\psi)$$

where

$$\sum_{i,j=1}^N tr[L_\epsilon(z_i, z_j) H_{k,l}(z_j, \psi)(I - \psi(z_j)\psi(z_i)^*) H_{k,l}(z_i, \psi)^* f(z_i)^*] \geq 0$$

for each $\psi$ because $R_\psi$ is contractive. Thus $P \geq 0$, and by Theorem 2.2.3 ((3) $\Rightarrow$ (2)) we have

$$\sum_{i,j=1}^N tr[X(z_j)^*(I - S(z_j)^* S(z_i)) X(z_i) K_\epsilon(z_i, z_j)] \geq 0,$$

for $X : \{z_1, \dots, z_N\} \to \mathcal{C}_2(\mathcal{E}, \mathcal{U})$. Since $\epsilon$ was any positive number then

$$\sum_{i,j=1}^N tr[X(z_j)^*(I - S(z_j)^* S(z_i)) X(z_i) K(z_i, z_j)] \geq 0,$$

therefore $R_S$ is contractive (by Theorem 2.2.3) and hence $S \in \mathcal{SA}_\Psi(\mathcal{U}, \mathcal{Y})$. □

*Remark* 3.3.2. If $\mathcal{U}_T = \mathcal{Y}_T$ then $\mathcal{C}_b(\Psi, \mathcal{L}(\mathcal{Y}_T, \mathcal{Y}_T)) \otimes \mathcal{X} \cong \mathcal{X}$ and $L^*_{\mathbb{E}(z)^*} = \rho(\mathbb{E}(z))$.

# Chapter 4

# Extreme Points

## 4.1 The Herglotz class of finitely connected planar domains

Let $\mathcal{R} \subset \mathbb{C}$ a finitely connected planar domain whose boundary $X = \partial\mathcal{R}$ consists of $m+1$ components $\partial_0\mathcal{R}, \ldots, \partial_m\mathcal{R}$.

Let us define

$$M^h(X) = \left\{ \mu \in M(X) : \int_X \phi_i d\mu = 0 \text{ for } i = 1, \ldots, m \right\}$$

where $\{\phi_1, \ldots, \phi_m\}$ is a fixed orthonormal basis for $L^{2,h}_{\mathbb{R}}(d\sigma)^\perp$ the subspace of measures $\mu \in M(X)$ such that its associated harmonic function $\hat{\mu}$ has a single-valued harmonic conjugate on $\mathcal{R}$.

If we consider

$$\mathcal{C}_1 = \{\tau \in M^h(X) : \tau \geq 0,\ \tau(X) = 1\},$$

then by [2], the extreme points of $\mathcal{C}_1$ consist of the set

$$\left\{ \sum_{r=0}^{m} w_r \delta_{x_r} : \boldsymbol{x} = (x_0, \ldots, x_m) \in \mathbb{T}_{\mathcal{R}} \right\}$$

where $\mathbb{T}_{\mathcal{R}} = \partial_0\mathcal{R} \times \cdots \times \partial_m\mathcal{R}$, $w_r > 0$ and $\sum_{r=0}^{m} w_r = 1$ and

$$B(\boldsymbol{x}) \begin{bmatrix} w_0 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

where

$$B(\boldsymbol{x}) = [\phi_i(x_j)]_{i=1,\ldots,n;\, j=0,\ldots,m}.$$

As in the classical case we may identify

$$\mathcal{H} = \{F \in Hol(\mathcal{R})^{N\times N}, Re\ F \geq 0,\ F(0) = I\} \subset Hol(\mathcal{R})^{N\times N},$$

with the convex set

$$\mathcal{C}^N = \{\mu \in M^h(X)^{N\times N} : \mu \geq 0,\ \mu(X) = I\} \subset M(X)^{N\times N}$$

with the locally convex topology of the uniform convergence on compact subsets on $\mathcal{H}$ corresponding to the weak-* topology on $\mathcal{C}^N$.

We will denote by $\Pi^N$ the set of extreme point of $\mathcal{C}^N$ and by $\widehat{\Pi^N} = \{F_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Pi^N\}$ the set of extreme points of $\mathcal{H}$.

Then Choquet theory implies

$$F(z) = \int_{\Pi^N} F_{\boldsymbol{\alpha}}(z) d\mu(\boldsymbol{\alpha}).$$

We will attempt to classify, as in the Section 2.5, the extreme points of $\mathcal{C}^N$. For such purpose we have the following result.

**Theorem 4.1.1.** *If*

$$\mu = \sum_{k=1}^{N} \mu_k^{(1)} P_k$$

*where $P_k$ are pairwise orthogonal projections summing to $I_N$ and $\mu_k^{(1)}$'s are scalar extreme points of $\mathcal{C}_1$ then $\mu$ is an extreme point of $\mathcal{C}^N$.*

*Proof.* In order to prove the theorem we will use Lemma 2.4.1. Consider then

$$\nu \in M^h(X)^{N\times N} \text{ so that } \nu(X) = 0 \text{ and } \mu \pm \nu \geq 0.$$

If we consider the orthonormal basis of $\mathbb{C}^N$ $\{e_1, \ldots, e_N\}$ we may write $P_j = e_j e_j^*$. Then

$$P_j(\mu + \nu)P_j = (\mu_j^{(1)} \pm \nu_{jj}^{(1)})P_j$$

where $\nu_{jj}^{(1)} = e_j^* \nu e_j$, then $\mu_j^{(1)} \pm \nu_{jj}^{(1)} \in M^h(X)^{N\times N}$ (for $N = 1$) with $\mu_j^{(1)} \pm \nu_{jj}^{(1)} \geq 0$ and $\nu_{jj}^{(1)}(X) = 0$. So, because $\mu_j^{(1)}$ is an extreme point of $\mathcal{C}_1$, $\nu_{jj}^{(1)} = 0$.

If we use the orthonormal basis $\{e_1, \ldots, e_N\}$ to represent all $N \times N$ matrices, we have

$$\mu = \begin{bmatrix} \mu_1^{(1)} & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mu_N^{(1)} \end{bmatrix} \text{ and } \nu = \begin{bmatrix} 0 & \nu_{12}^{(1)} & \cdots & \nu_{1N}^{(1)} \\ (\nu_{12}^{(1)})^* & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \nu_{N-1,N}^{(1)} \\ (\nu_{1N}^{(1)})^* & \cdots & (\nu_{N-1,N}^{(1)})^* & 0 \end{bmatrix}.$$

From $\nu(X) = 0$ we see that each of the scalar measures $\nu_{jj}^{(1)}$ $(1 \leq i, j \leq N)$ has $\nu_{jj}^{(1)}(X) = 0$.

Note

$$\mu \pm \nu = \begin{bmatrix} \mu_1^{(1)} & \pm\nu_{12}^{(1)} & \cdots & \pm\nu_{1N}^{(1)} \\ (\pm\nu_{12}^{(1)})^* & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \pm\nu_{N-1,N}^{(1)} \\ \pm(\nu_{1N}^{(1)})^* & \cdots & \pm(\nu_{N-1,N}^{(1)})^* & \mu_N^{(1)} \end{bmatrix}.$$

Hence, for each pair of indices $i, j$ $(i < j)$ and each Borel set $\Delta$

$$\begin{bmatrix} \mu_i^{(1)}(\Delta) & \nu_{ij}^{(1)}(\Delta) \\ \overline{\nu_{ij}^{(1)}(\Delta)} & \mu_j^{(1)}(\Delta) \end{bmatrix} \geq 0.$$

Taking the determinant gives

$$|\nu_{ij}^{(1)}(\Delta)|^2 \leq \mu_i^{(1)}(\Delta)\mu_j^{(1)}(\Delta).$$

In particular, $\nu_{ij}^{(1)}(\Delta) = 0$ whenever either $\mu_i^{(1)}(\Delta) = 0$ or $\mu_j^{(1)}(\Delta) = 0$. By assumption $\mu_i^{(1)}$ is an extreme point of $\mathcal{C}_1$, then, using Lemma 1.3.5 [2], we have

$$\mu_i^{(1)} = \sum_{r=0}^{m} w_r^{(i)} \delta_{x_r^{(i)}}$$

where $w_r^{(i)}$ is a positive number, $\delta_{x_r^{(i)}}$ is a unit point mass measure at $x_r^{(i)}$ (that belongs to the $r$-th component of $X$).

In particular $\nu_{ij}^{(1)}$ has the form

$$\nu_{ij}^{(1)} = \sum_{r=0}^{m} w_r^{(ij)} \delta_{x_r^{(i)}}$$

where $w_r^{(ij)}$ are real numbers such that $w_r^{(ij)} = 0$ whenever it is not the case that $x_r^{(i)} = x_r^{(j)}$.

The condition that $\nu_{ij}^{(1)}(X) = 0$ forces that

$$\sum_{r=0}^{m} w_r^{(ij)} = 0$$

(since $\nu_{ij}^{(1)}$ has the form $\sum_{r=0}^{m} w_r^{(ij)}\delta_{x_r^{(i)}}$ and $\nu_{ij}^{(1)} \in M^h(X)^{N\times N}$ (for $N=1$)). It follows (using the notation as in section 1.3 of [2])

$$\begin{bmatrix} w_0^{(ij)} \\ \vdots \\ w_m^{(ij)} \end{bmatrix} \in \ Ker\ A(\alpha^{(i)}).$$

But, by the analysis done in section 1.3 of [2], $dim\ Ker\ A(\alpha^{(i)}) = 1$ and $Ker\ A(\alpha^{(i)})$ has basis vector

$$\begin{bmatrix} w_0^{(i)} \\ \vdots \\ w_m^{(i)} \end{bmatrix} \text{ such that } w_s^{(i)} > 0$$

for all $s = 0, 1, \dots, m$. In particular if

$$\begin{bmatrix} w_0^{(i)} \\ \vdots \\ w_m^{(i)} \end{bmatrix}$$

is not the zero vector then all the components have constant sign which contradicts the condition

$$\sum_{r=0}^{m} w_r^{(ij)} = 0.$$

We conclude that $\nu_{ij}^{(1)}$ is the zero measure for all $i, j$ and hence $\nu = 0$. And hence $\mu$ is an extreme point of $\mathcal{C}^N$. $\square$

## 4.2 The Schur class

In this section we study the Schur class $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N)$ over $\mathcal{R}$, i.e. holomorphic functions on $\mathcal{R}$ whose values are contractive $N \times N$ matrices. The following give a precise equivalence between the strict Schur class ($S$ with values equal to strict contractions on $\mathcal{R}$) and the Herglotz class over $\mathcal{R}$.

**Theorem 4.2.1.** *Let $S : \mathcal{R} \to \mathbb{C}^{N\times N}$ is analytic with $\|S(z)\| < 1$ for $z \in \mathcal{R}$, then*

$$F = (\mathbb{1} + S)(\mathbb{1} - S)^{-1}$$

*has positive real part.*

*Proof.* Clearly $F$ is well defined since $\|S\| < 1$. Now, take $z \in \mathcal{R}$ and let

$$U(z) = [(\mathbb{1} + S)(\mathbb{1} - S)^{-1} + [(\mathbb{1} + S)(\mathbb{1} - S)^{-1}]^*](z).$$

Then

$$\begin{aligned}
U(z) &= [I+S(z)][I-S(z)]^{-1} + [[I-S(z)]^{-1}]^*[I+S(z)^*] \\
&= [[I-S(z)]^{-1}]^*\{[I-S(z)^*][I+S(z)] + [I+S(z)^*][I-S(z)]\}[I-S(z)]^{-1} \\
&= [[I-S(z)]^{-1}]^*\{2I - 2S(z)^*S(z)\}[I-S(z)]^{-1} \\
&= 2[[I-S(z)]^{-1}]^*\{I - S(z)^*S(z)\}[I-S(z)]^{-1} \geq 0.
\end{aligned}$$

□

For $F_{\boldsymbol{\alpha}} \in \widehat{\Pi^N}$ define $S_{\boldsymbol{\alpha}} = (F_{\boldsymbol{\alpha}} - \mathbb{1})(F_{\boldsymbol{\alpha}} + \mathbb{1})^{-1}$. Now we have the following important result.

**Theorem 4.2.2.** *If $S : \mathcal{R} \to \mathbb{C}^{N\times N}$ is analytic with $\|S(z)\| < 1$ for $z \in \mathcal{R}$ ($S \in \mathcal{S}^0_{\mathcal{R}}(\mathbb{C}^N, \mathbb{C}^N)$) then there is a positive measure $\mu$ on $\Pi^N$ and a measurable function $h$ whose values are functions $h(\cdot, \boldsymbol{\alpha})$ analytic in $\mathcal{R}$ so that*

$$I - S(z)S(w)^* = \int_{\Pi^N} h(z,\boldsymbol{\alpha})[I - S_{\boldsymbol{\alpha}}(z)S_{\boldsymbol{\alpha}}(w)^*]h(w,\boldsymbol{\alpha})^*\mu(d\boldsymbol{\alpha}).$$

*Proof.* **Case 1:** $S(0) = 0$. Then $\|S(x)\| < 1$ for $x \in \mathcal{R}$ by the Maximum Modulus Theorem for matrix-valued holomorphic functions.

Let

$$F = (\mathbb{1} + S)(\mathbb{1} - S)^{-1}, \tag{4.1}$$

we have:

$$F(0) = (\mathbb{1} + S)(\mathbb{1} - S)^{-1}(0) = [I + S(0)][I - S(0)]^{-1} = I.$$

Also by Theorem 4.2.1 $F$ is positive definite. Therefore $F$ belongs to $\mathcal{H}$.

On the other hand, from (4.1) we get $S = (F + \mathbb{1})^{-1}(F - \mathbb{1})$. Let

$$V = I - S(z)S(w)^*.$$

Then

$$\begin{aligned}
V &= I - [(F+\mathbb{1})^{-1}(F-\mathbb{1})](z)[(F+\mathbb{1})^{-1}(F-\mathbb{1})]^*(w) \\
&= I - [(F(z)+I)]^{-1}[F(z)-I][F(w)-I]^*[[F(w)+I]^{-1}]^*.
\end{aligned}$$

Making $B = [(F(z)+I)]^{-1}[F(z)-I][F(w)-I]^*[[F(w)+I]^{-1}]^*$, we have:

$$\begin{aligned}
V &= [F(z)+I]^{-1}[F(z)+I][F(w)+I]^*[[F(w)+I]^{-1}]^* - B \\
&= [F(z)+I]^{-1}\{[F(z)+I][F(w)+I]^* - [F(z)-I][F(w)-I]^*\}[F(w)+I]^{*-1} \\
&= [F(z)+I]^{-1}\{2[F(z)+F(w)^*]\}[F(w)+I]^{*-1}.
\end{aligned}$$

So

$$V = I - S(z)S(w)^* = 2[F(z)+I]^{-1}[F(z)+F(w)^*][[F(w)+I]^{-1}]^*. \tag{4.2}$$

Since $\Pi^N$ is compact, by Theorem 2.4.4, then there exists a (regular Borel) probability measure $\nu$ on $\Pi^N$ such that

$$L(F) = \int_{\Pi^N} L(F_{\boldsymbol{\alpha}})\nu(d\boldsymbol{\alpha}),$$

for every linear functional $L : \mathcal{H} \to \mathbb{C}$. In particular $L : F \to F(z)$ is such a linear functional and hence

$$[F(z)]_{ij} = \int_{\Pi^N} [F_{\boldsymbol{\alpha}}(z)]_{ij}\nu(d\boldsymbol{\alpha}).$$

Let us abbreviate this to the matrix identity

$$F(z) = \int_{\Pi^N} F_{\boldsymbol{\alpha}}(z)\nu(d\boldsymbol{\alpha})$$

for each $z \in \mathcal{R}$, where $\widehat{F}_{\boldsymbol{\alpha}} \in \Pi^N$. Then $F_{\boldsymbol{\alpha}} = (\mathbb{1} - S_{\boldsymbol{\alpha}})^{-1}(S_{\boldsymbol{\alpha}} + \mathbb{1})$.

Now, back in (4.2), we have:

$$\begin{aligned}
V &= 2[F(z)+I]^{-1}\Big\{\int_{\Pi^N} F_\alpha(z)\nu(d\boldsymbol{\alpha}) + \Big(\int_{\Pi^N} F_\alpha(w)\nu(d\boldsymbol{\alpha})\Big)^*\Big\}[[F(w)+I]^{-1}]^* \\
&= \int_{\Pi^N} \big\{2[F(z)+I]^{-1}[F_\alpha(z)+F_\alpha(w)^*][[F(w)+I]^{-1}]^*\big\}\nu(d\boldsymbol{\alpha}) \\
&= \int_{\Pi^N} \big\{2[F(z)+I]^{-1}\{[I-S_{\boldsymbol{\alpha}}(z)]^{-1}[S_{\boldsymbol{\alpha}}(z)+I] \\
&+ [S_{\boldsymbol{\alpha}}(w)+I]^*[[I-S_{\boldsymbol{\alpha}}(w)]^{-1}]^*\}[[F(w)+I]^{-1}]^*\big\}\nu(d\boldsymbol{\alpha}).
\end{aligned}$$

If we let

$$W = [I-S_{\boldsymbol{\alpha}}(z)]^{-1}[S_{\boldsymbol{\alpha}}(z)+I] + [S_{\boldsymbol{\alpha}}(w)+I]^*[[I-S_{\boldsymbol{\alpha}}(w)]^{-1}]^*,$$

then

$$\begin{aligned}
W &= [I-S_{\boldsymbol{\alpha}}(z)]^{-1}\{[S_{\boldsymbol{\alpha}}(z)+I][I-S_{\boldsymbol{\alpha}}(w)]^* \\
&+ [I-S_{\boldsymbol{\alpha}}(z)][S_{\boldsymbol{\alpha}}(w)+I]^*\}[[I-S_{\boldsymbol{\alpha}}(w)]^{-1}]^* \\
&= [I-S_{\boldsymbol{\alpha}}(z)]^{-1}\{2[I-S_{\boldsymbol{\alpha}}(z)S_{\boldsymbol{\alpha}}(w)^*]\}[[I-S_{\boldsymbol{\alpha}}(w)]^{-1}]^*.
\end{aligned}$$

Putting $W$ back in the equation for $V$, we get:

$$V = \int_{\Pi^N} \Big\{2[F(z)+I]^{-1}\{[I-S_{\boldsymbol{\alpha}}(z)]^{-1}\{2[I-S_{\boldsymbol{\alpha}}(z)S_{\boldsymbol{\alpha}}(w)^*]\}$$

$$\times[[I-S_{\boldsymbol{\alpha}}(w)]^{-1}]^*\}[[F(w)+I]^{-1}]^*\Big\}\nu(d\boldsymbol{\alpha}).$$

So

$$I - S(z)S(w)^* = \int_{\Pi^N} h(z, \boldsymbol{\alpha})[I - S_{\boldsymbol{\alpha}}(z)S_{\boldsymbol{\alpha}}(w)^*]h(w, \alpha)^* \nu(d\boldsymbol{\alpha})$$

where $h(\cdot, \boldsymbol{\alpha}) = 2[f(\cdot) + I]^{-1}[I - S_{\boldsymbol{\alpha}}(\cdot)]^{-1}$.

**Case 2:** $S(0) = A($ with $A \neq 0)$. Assume $\|A\| < 1$, so $I - A^*A$ is invertible.

Let $D_{A^*} = (I - AA^*)^{1/2}$ and $D_A = (I - A^*A)^{1/2}$. We have

$$A^*(I - AA^*) = (I - A^*A)A^* \text{ and } A(I - A^*A) = (I - AA^*)A.$$

Also the following identities are satisfied (see [25])

1. $A^*D_{A^*} = D_A A^*$;
2. $AD_A = D_{A^*}A$;
3. $A^*(D_{A^*})^{-1} = (D_A)^{-1}A^*$;
4. $A(D_A)^{-1} = (D_{A^*})^{-1}A$;

Let

$$S = \begin{bmatrix} (D_{A^*})^{-1} & -(D_{A^*})^{-1}A \\ -A^*(D_{A^*})^{-1} & D_A^{-1} \end{bmatrix}$$

and $T_S : \mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ be defined as follows:

$$T_S(Y) = [(D_{A^*})^{-1}Y - A(D_A)^{-1}][-A^*(D_{A^*})^{-1}Y + (D_A)^{-1}]^{-1},$$

whenever $-A^*(D_{A^*})^{-1}Y + (D_A)^{-1}$ is invertible. This holds for all $Y$ with $\|Y\| < 1$ since

$$S^*JS = J, \ J = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

We have

$$\begin{aligned} (T_S)^{-1}(Y) &= [(D_{A^*})^{-1} + YA^*(D_{A^*})^{-1}]^{-1}[A(D_A)^{-1} + Y(D_A)^{-1}] \\ &= D_{A^*}[I + YA^*]^{-1}(A + Y)(D_A)^{-1}. \end{aligned}$$

Now let $\varphi : \mathcal{R} \to \mathbb{C}^{N\times N}$ be defined as

$$\varphi(z) = T_S(\psi(z))$$

Then, clearly, $\varphi$ is analytic on $\mathcal{R}$ and $\|\varphi\| \leq 1$, and,

$$\varphi(0) = T_S(\psi(0)) = T_S(A) = 0 \text{ (using 4)},$$

then $\varphi$ satisfies the conclusion of Case 1.

On the other hand, let

$$L = I - (T_S)^{-1}(\varphi(z))[(T_S)^{-1}(\varphi(w))]^*,$$

then:

$$\begin{aligned}
L &= I - D_{A^*}[I + \varphi(z)A^*]^{-1}[A + \varphi(z)](D_A)^{-1}(D_A)^{-1} \\
&\times\ [A^* + (\varphi(w))^*][I + A(\varphi(w))^*]^{-1}D_{A^*} \\
&= D_{A^*}[I + \varphi(z)A^*]^{-1}\Big\{[I + \varphi(z)A^*](D_{A^*})^{-2}[I + A\varphi(w)^*] \\
&-\ [A + \varphi(z)](D_{A^*})^{-2}[A^* + (\varphi(w))^*]\Big\}[I + A(\varphi(w))^*]^{-1}D_{A^*}.
\end{aligned}$$

Let us simplify the expression inside $\Big\{\cdots\Big\}$ in the previous equation,

$$\begin{aligned}
\Big\{\cdots\Big\} &= [(D_{A^*})^{-2} + \varphi(z)A^*(D_{A^*})^{-2}][I + A\varphi(w)^*] \\
&-\ [A(D_A)^{-2} + \varphi(z)(D_A)^{-2}][A^* + \varphi(w)^*] \\
&= (D_{A^*})^{-2} - A(D_A)^{-2}A^* + \varphi(z)[A^*(D_{A^*})^{-2}A - (D_A)^{-2}][\varphi(w)]^* \text{ (using 4)}, \\
&= (D_{A^*})^{-2}(I - AA^*) + \varphi(z)(A^*A - I)(D_A)^{-2}[\varphi(w)]^* \text{ (using 3, and 4)}, \\
&= I - \varphi(z)(\varphi(w))^* \text{ (using the definition of } D_{A^*}\text{and } D_A).
\end{aligned}$$

Back in the equation for $L$, we get:

$$L = D_{A^*}[I + \varphi(z)A^*]^{-1}\Big\{I - \varphi(z)(\varphi(w))^*\Big\}[I + A(\varphi(w))^*]^{-1}D_{A^*}.$$

Since $\varphi$ satisfies the conclusion of Case 1, we have

$$\begin{aligned}
I - S(z)S(w)^* &= L \\
&= D_{A^*}[I + \varphi(z)A^*]^{-1}\Big\{\int_{\Pi^N} h(z,\boldsymbol{\alpha})[I - S_{\boldsymbol{\alpha}}(z)S_{\boldsymbol{\alpha}}(w)^*]h(w,\boldsymbol{\alpha})^*\mu(d\boldsymbol{\alpha})\Big\} \\
&\times\ [I + A(\varphi(w))^*]^{-1}D_{A^*}.
\end{aligned}$$

So

$$I - S(z)S(w)^* = \int_{\Pi^N} H(z,\boldsymbol{\alpha})[I - S_{\boldsymbol{\alpha}}(z)S_{\boldsymbol{\alpha}}(w)^*]H(w,\boldsymbol{\alpha})^*\mu(d\boldsymbol{\alpha}),$$

where

$$H(\cdot,\boldsymbol{\alpha}) = D_{A^*}[I + \varphi(z)A^*]^{-1}h(\cdot,\boldsymbol{\alpha}).$$

$\square$

Application of the Main Theorem (2) $\Rightarrow$ (3) together with Remark 3.3.2 gives us the following.

**Corollary 4.2.3.** *(Transfer function realization). If* $S \in \mathcal{S}_{\mathcal{R}}(\mathbb{C}^N, \mathbb{C}^N)$ *then there is*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathbb{C}^N \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathbb{C}^N \end{bmatrix}$$

*where* $\mathcal{X}$ *is equipped with a unital* $*$*-representation* $\rho : \mathcal{C}_b(\Pi^N, \mathbb{C}^{N\times N}) \to \mathcal{L}(\mathcal{X})$*, so that:*

$$S(z) = D + C(I - \rho(\mathbb{E}(z))A)^{-1}\rho(\mathbb{E}(z))B \text{ for all } z \in \Omega.$$

# Chapter 5

# The spectral set question

Let us recall the spectral set question which we presented in the introduction.

Let $\mathcal{R}$ denote a domain in $\mathbb{C}$ with boundary $\partial\mathcal{R}$. We say that an operator $T$ on a complex Hilbert space $\mathcal{X}$ has $\overline{\mathcal{R}}$ (the closure of $\mathcal{R}$) as a spectral set if $\sigma(T) \subset \overline{\mathcal{R}}$ and

$$\|s(T)\| \leq \|s\|_{\mathcal{R}} = \sup\{|s(z)| : z \in \mathcal{R}\}$$

for every rational function $s$ with poles off $\overline{\mathcal{R}}$.

The operator $T$ on $\mathcal{X}$ has a $\partial\mathcal{R}$-normal dilation if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{X}$ and a normal operator $N$ on $\mathcal{K}$ with $\sigma(N) \subset \partial\mathcal{R}$ so that

$$s(T) = P_{\mathcal{X}} s(N)|_{\mathcal{X}},$$

for every rational function $f$ with poles off $X$, where $P_{\mathcal{X}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{X}$. It is easy to show that $\overline{\mathcal{R}}$ is a spectral set for $T$ if $T$ has a $\partial\mathcal{R}$-normal dilation

The **spectral set question** is the converse: if $T$ has $X$ as a spectral set then does it follow that $T$ has a $\partial\mathcal{R}$-normal dilation?

Let us also recall the Arveson reformulation for the spectral set question.

Given an operator $T$ for which $s(T)$ has norm at most 1 for all scalar-valued Schur class functions $s$, does it follow that $S(T)$ has norm at most 1 for all matrix-valued Schur class functions $S$?

Let us define

$$\mathcal{H}_1 = \{F \in Hol(\mathbb{T})^{N\times N}, Re\ F \geq 0,\ F(0) = I\} \subset Hol(\mathbb{T})^{N\times N}.$$

Using Herglotz representation theory (studied in [2]) we have that for $F \in \mathcal{H}_1$ with $F(0) = I$ $F$ can be represented as

$$F(z) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu(\xi) \tag{5.1}$$

where $\mu$ is a matrix-valued measure.

Also, by the results of Chapter 4, we know that elements $\boldsymbol{\alpha}$ of $\Pi^N$ have the form $\boldsymbol{\alpha} = (\xi_1, \ldots, \xi_n; K_1, \ldots, K_n)$ where $\xi_j \in \mathbb{T}$, $K_j \in \mathbb{C}^{N\times N}$, $K_j \geq 0$, $K_1 + \cdots + K_n = I$ and $\{[K_j\mathbb{C}^N] : j = 1, \ldots, n\}$ is weakly independent. So, given $\boldsymbol{\alpha} \in \Pi^N$ let us write $n = n(\boldsymbol{\alpha})$, $\xi = \xi(\boldsymbol{\alpha})$ and $K_j = K_j(\boldsymbol{\alpha})$ to indicate the dependence on $\boldsymbol{\alpha}$. Then Choquet theory analysis tells us that any $F \in \mathcal{H}_1$ with $F(0) = I$ can be represented as

$$F(z) = \int_{\Pi^N} \sum_{j=1}^{n(\boldsymbol{\alpha})} \frac{\xi_j(\boldsymbol{\alpha}) + z}{\xi_j(\boldsymbol{\alpha}) - z} K_j(\boldsymbol{\alpha}) d\mu(\boldsymbol{\alpha}), \tag{5.2}$$

where $K_j$ is a positive operator and $\mu$ is a scalar measure.

It is well known (see [20]) that the spectral set question over the unit disk has a positive answer. We show here how this follows easily from either representation (5.1) or (5.2); this gives a different proof from those presented in the standard texts (see e.g. [20]).

If $S$ is in $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N, \mathbb{C}^N)$ with $S(0) = 0$ then we can write

$$S(z) = (F(z) + I)^{-1}(F(z) - I)$$

where $F \in \mathcal{H}_1$, then using representation (5.1) we get

$$\begin{aligned}
I - S(z)S(w)^* &= (F(z) + I)^{-1}[(F(z) + I)(F(w)^* + I) \\
&\quad -(F(z) - I)(F(w)^* - I)](F(w)^* + I)^{-1} \\
&= (F(z) + I)^{-1}[2(F(z) + F(w)^*)](F(w)^* + I)^{-1} \\
&= 2(F(z) + I)^{-1}\left[\int_{\mathbb{T}} \left(\frac{\xi + z}{\xi - z} + \frac{\overline{\xi} + \overline{w}}{\overline{\xi} - \overline{w}}\right) d\mu(\xi)\right](F(w)^* + I)^{-1} \\
&= 2(F(z) + I)^{-1}\left[\int_{\mathbb{T}} \left(2\frac{1 - z\overline{w}}{(\xi - z)(\overline{\xi} - \overline{w})}\right) d\mu(\xi)\right](F(w)^* + I)^{-1} \\
&= 4(F(z) + I)^{-1}\left[\int_{\mathbb{T}} \left(\frac{1}{\xi - z}(1 - z\overline{w})\frac{1}{\overline{\xi} - \overline{w}}\right) d\mu(\xi)\right](F(w)^* + I)^{-1}.
\end{aligned}$$

And then

$$\begin{aligned}
I - S(T)S(T)^* &= 4(F(z)+I)^{-1}\int_{\mathbb{T}}(I_N\otimes(\xi I-T)^{-1})(I_N\otimes(I-TT^*))\\
&\quad\times(I_N\otimes(\overline{\xi}I-T^*)^{-1}))(d\mu(\xi)\otimes I_{\mathcal{X}})(F(w)^*+I)^{-1}\\
&= 4(F(z)+I)^{-1}\int_{\mathbb{T}}(I_N\otimes(\xi I-T)^{-1})(d\mu^{1/2}(\xi)\otimes I_{\mathcal{X}})\\
&\quad(I_N\otimes(I-TT^*))(I_N\otimes(\overline{\xi}I-T^*)^{-1}))(d\mu^{1/2}(\xi)\otimes I_{\mathcal{X}})\\
&\quad\times(F(w)^*+I)^{-1}.
\end{aligned}$$

So it follows from the last equality that if $T$ is a contraction then so it $S(T)$, thus the Arveson reformulation of the spectral set question for $\mathbb{D}$ has a positive answer.

We may obtain this same result using the second representation of $F$. If we go over the previous calculations for $I - S(z)S(w)^*$ and $I - S(T)S(T)^*$ we get

$$\begin{aligned}
I - S(z)S(w)^* &= 2(F(z)+I)^{-1}\left[\sum_{j=1}^{n(\boldsymbol{\alpha})}\int_{\Pi^N}\frac{2(1-z\overline{w})}{(\xi_j(\boldsymbol{\alpha})-z)(\overline{\xi_j(\boldsymbol{\alpha})}-\overline{w})}K_j(\boldsymbol{\alpha})d\mu(\boldsymbol{\alpha})\right]\\
&\quad\times(F(w)^*+I)^{-1}\\
&= 2(F(z)+I)^{-1}\left[\sum_{j=1}^{n(\boldsymbol{\alpha})}\int_{\Pi^N}K_j(\boldsymbol{\alpha})^{1/2}\frac{2(1-z\overline{w})}{(\xi_j(\boldsymbol{\alpha})-z)(\overline{\xi_j(\boldsymbol{\alpha})}-\overline{w})}\right.\\
&\quad\left.\times K_j(\boldsymbol{\alpha})^{1/2}d\mu(\boldsymbol{\alpha})\right](F(w)^*+I)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
I - S(T)S(T)^* &= 2(F(T)+I)^{-1}\left[\sum_{j=1}^{n(\boldsymbol{\alpha})}\int_{\Pi^N}(K_j(\boldsymbol{\alpha})^{1/2}\otimes I_{\mathcal{X}})(I_N\otimes 2(\xi_j(\boldsymbol{\alpha})I-T)^{-1})\right.\\
&\quad\left.\times(I_N\otimes(1-TT^*))(I_N\otimes(\overline{\xi_j(\boldsymbol{\alpha})}I-T^*))(K_j(\boldsymbol{\alpha})^{1/2}\otimes I_{\mathcal{X}})d\mu(\boldsymbol{\alpha})\right]\\
&\quad\times(F(T)^*+I)^{-1},
\end{aligned}$$

then, again, if $T$ is a contraction so is $S(T)$.

We would like now to study the spectral set question for the general finitely connected planer domain $\mathcal{R}$ with this same approach.

Theorem 1.1.21 of [2] tells us that for $f$ scalar-valued holomorphic with positive real part and $f(t_0)=1$ we have

$$f(z) = \int_{\mathbb{T}_{\mathcal{R}}} f_\alpha(z)d\mu(\alpha),$$

with $\mu$ a probability measure on $\mathbb{T}_{\mathcal{R}}$ and for $\alpha \in \mathbb{T}_{\mathcal{R}}$,

$$f_\alpha(z) = \int_{\partial\mathcal{R}} P_z(\lambda) d\mu_\alpha(\lambda)$$

where $\mu_\alpha$ is an extremal measure associated with $\alpha \in \mathbb{T}_{\mathcal{R}}$.

*Naive Conjecture 1*: If $F$ is $N \times N$ matrix-valued holomorphic on $\mathcal{R}$ with positive real part and $F(t_0) = I$ then

$$F(z) = \int_{\mathbb{T}_{\mathcal{R}}} f_\alpha(z) d\mu(\alpha)$$

with $\mu$ a positive matrix measure with $\mu(\mathbb{T}_{\mathcal{R}}) = I$.

If we suppose Naive Conjecture 1 is true then

$$\begin{aligned} I - S(T)S(T)^* &= 2(F(T)+I)^{-1} \int_{\mathbb{T}_{\mathcal{R}}} (f_\alpha(T) + f_\alpha(T)^*) d\mu(\alpha) (F(T)^* + I)^{-1} \\ &= 2(F(T)+I)^{-1} \int_{\mathbb{T}_{\mathcal{R}}} (Re\ f_\alpha(T)) d\mu(\alpha) (F(T)^* + I)^{-1}. \end{aligned}$$

Then if $Re\ f_\alpha(T) \geq 0$ for all $\alpha$ then $I - S(T)S(T)^* \geq 0$, and so we would have a positive answer for the spectral set question but this is a contradiction to Lemma 1.6 of [17] and Section 2.6 of [2].

What we do know for the multiply connected domain is that if $F$ is a $N \times N$ matrix-valued holomorphic on $\mathcal{R}$ with $Re\ F(z) \geq 0$ and $F(t_0) = I$ then

$$Re\ F(z) = \int_{\partial\mathcal{R}} P_z(\lambda) d\mu(\lambda).$$

We define $\mathcal{C}^N$ to be

$$\mathcal{C}^N = \left\{ \mu \in M(\partial\mathcal{R})^{N\times N} : \mu \text{ positive}, \mu(\partial\mathcal{R}) = I_N \text{ and } \int_{\partial\mathcal{R}} \phi_j(\lambda) d\mu(\lambda) = 0,\ j = 1, \ldots, m \right\},$$

where the $\phi_j$'s are the ones introduced in Section 4.1.

We use the second representation of $F$ to go after the extreme points of $\mathcal{C}^N$.

*Naive Conjecture 2*: The extreme points of $\mathcal{C}^N$ are of the form

$$\sum_{j=1}^{n} \mu_j^{(1)} K_j$$

where the $\mu_j^{(1)}$'s are extreme points of $\mathcal{C}_1$ (introduced in Section 4.1) and the $K_j$'s are positive operators satisfying

1. $K_1 + \cdots + K_n = I$ and
2. $\{[K_j\mathbb{C}^N] : j = 1, \ldots, n\}$ is weakly independent.

If Naive Conjecture 2 holds, then

$$F(z) = \int_{\Pi^N} \sum_{j=1}^{n(\boldsymbol{\alpha})} f^{(1)}_{x_j(\boldsymbol{\alpha})}(z) K_j(\boldsymbol{\alpha}) d\mu(\boldsymbol{\alpha}),$$

where $f^{(1)}_{x_j(\boldsymbol{\alpha})} = \widehat{\mu^{(1)}_j}$.

Then $S(z) = (F(z)+I)^{-1}(F(z)-I)$ belongs to $\mathcal{S}_{\mathcal{R}}(\mathbb{C}^N, \mathbb{C}^N)$ with $S(0) = 0$ and, making similar calculations to the ones done for the disk case, we have

$$\begin{aligned} I - S(z)S(w)^* &= 2(F(z)+I)^{-1} \sum_{j=1}^{n(\boldsymbol{\alpha})} \int_{\Pi^N} K_j(\boldsymbol{\alpha})^{1/2} \left( f^{(1)}_{x_j(\boldsymbol{\alpha})}(z) + \overline{f^{(1)}_{x_j(\boldsymbol{\alpha})}(w)} \right) \\ &\quad \times K_j(\boldsymbol{\alpha})^{1/2} d\mu(\boldsymbol{\alpha}) (F(w)^* + I)^{-1} \end{aligned}$$

and so

$$\begin{aligned} I - S(T)S(T)^* &= 2(F(T)+I)^{-1} \sum_{j=1}^{n(\boldsymbol{\alpha})} \int_{\Pi^N} (K_j(\boldsymbol{\alpha})^{1/2} \otimes I_{\mathcal{X}}) \Big( I_N \otimes \Big( f^{(1)}_{x_j(\boldsymbol{\alpha})}(T) \\ &\quad + f^{(1)}_{x_j(\boldsymbol{\alpha})}(T)^* \Big) \Big) (K_j(\boldsymbol{\alpha})^{1/2} \otimes I_{\mathcal{X}}) d\mu(\boldsymbol{\alpha}) (F(T)^* + I)^{-1}. \end{aligned}$$

Thus if $Re\ f^{(1)}_{x_j(\boldsymbol{\alpha})}(T) \geq 0$ for all $\boldsymbol{\alpha}$ then $I - S(T)S(T)^* \geq 0$, and so the spectral set question is answered affirmatively. Then again this is contradiction to Lemma 1.6 of [17] and Section 2.6 of [2].

We may try to reform the Naive Conjecture 2. To achieve that we take $\mu = \sum_{j=1}^n \mu^{(1)}_j K_j$ and since $\mu^{(1)}_j$ is an extreme point of $\mathcal{C}_1$ then

$$\mu^{(1)}_j = \sum_{r=0}^m w^{(j)}_r \delta_{x^{(j)}_r},$$

so

$$\mu = \sum_{j=1}^n \sum_{r=0}^m w^{(j)}_r \delta_{x^{(j)}_r} K_j.$$

Let $y^{(1)}_r, \ldots, y^{(n_r)}_r$ distinct elements in the list $x^{(1)}_r, \ldots, x^{(n)}_r$, then

$$\mu = \sum_{r=0}^m \sum_{k=1}^{n_r} \left( \sum_{j : x^{(j)}_r = y^{(k)}_r} w^{(j)}_r K_j \right) \delta_{y^{(k)}_r}.$$

Let us demand that

$$\widetilde{K}_{r,k} := \sum_{j:x_r^{(j)}=y_r^{(k)}} w_r^{(j)} K_j \geq 0$$

and $\{[\widetilde{K}_{r,k}\mathbb{C}^N] : r = 0, \ldots, m; k = 1, \ldots, n_r\}$ is weakly independent.

*Reformed Conjecture*: $\mu$ is an extreme point of $\mathcal{C}^N$ if and only if

$$\mu = \sum_{j=1}^{n} \mu_j^{(1)} K_j$$

where $\mu_j^{(1)}$ are distinct extreme points of $\mathcal{C}_1$. And the $K_j$'s are self-adjoint operators satisfying

1. $K_1 + \cdots + K_n = I$,
2. $\widetilde{K}_{r,k} \geq 0$, and
3. $\{[\widetilde{K}_{r,k}\mathbb{C}^N] : r = 0, \ldots, m; k = 1, \ldots, n_r\}$ is weakly independent.

Condition (1) in the Reformed Conjecture is equivalent to

$$\sum_{r=0}^{m} \sum_{k=1}^{n_r} \widetilde{K}_{r,k} = I.$$

With this Reformed formulation we may say the following.

**Theorem 5.0.4.** *If*

$$\mu = \sum_{j=1}^{n} \mu_j^{(1)} K_j$$

*where $\mu_j^{(1)}$ are distinct extreme points of $\mathcal{C}_1$. And the $K_j$'s are self-adjoint operators satisfying*

*1.* $K_1 + \cdots + K_n = I$,
*2.* $\widetilde{K}_{r,k} \geq 0$ *for* $r = 0, \ldots, m, k = 1, \ldots, n_r$, *and*
*3.* $\{[\widetilde{K}_{r,k}\mathbb{C}^N] : r = 0, \ldots, m; k = 1, \ldots, n_r\}$ *is weakly independent.*

*Then $\mu$ is an extreme point of $\mathcal{C}^N$.*

*Proof.* Suppose there exists $\nu \in M(X)^{N\times N}$ such that $\nu(X) = 0$ and $\mu \pm \nu \geq 0$. Let us take $\Delta$ to be a Borel set disjoint from $\{y_r^{(k)} : r = 0, \ldots, m, k = 1, \ldots, n_r\}$, then

$$0 \leq (\mu \pm \nu)(\Delta) = \pm\nu(\Delta),$$

thus $\nu(\Delta) = 0$. By Jordan decomposition theory $\nu$ is supported on $\{y_r^{(k)} : r = 0, \ldots, m, k = 1, \ldots, n_r\}$ and $\chi_\Delta \nu$ must be the zero measure. So

$$\nu = \sum_{r=0}^{m} \sum_{k=1}^{n_r} \delta_{y_r^{(k)}} \widetilde{L}_{r,k}.$$

Since $\nu(X) = 0$ then $\sum_{r=0}^{m} \sum_{k=1}^{n_r} \widetilde{L}_{r,k}$. Since $\mu \pm \nu \geq 0$, in particular $(\mu \pm \nu)(\{y_r^{(k)}\}) \geq 0$ for $r = 0, \ldots, m, k = 1, \ldots, n_r$. This implies $\widetilde{K}_{r,k} \pm \widetilde{L}_{r,k} \geq 0$ and so

$$-\widetilde{K}_{r,k} \leq \widetilde{L}_{r,k} \leq \widetilde{K}_{r,k}$$

And since $\widetilde{K}_{r,k} = \sum_{l=1}^{N} t_l P_l$ with $P_1, \ldots, P_N$ spectral resolution for $\mathbb{C}^N$ with some $t_l$'s being zero, then

$$-\sum_{l, t_l \neq 0} P_l \leq -\widetilde{K}_{r,k} \leq \widetilde{L}_{r,k} \leq \widetilde{K}_{r,k} \leq \sum_{l, t_l \neq 0} P_l.$$

If we let $t_{max} = \max_{l=1}^{N} t_l$, then

$$\widetilde{K}_{r,k} \leq t_{max} P_{[\widetilde{K}_{r,k}\mathbb{C}^N]} = \sum_{l, t_l \neq 0} t_{max} P_l.$$

Similarly

$$-t_{max} P_{[\widetilde{K}_{r,k}\mathbb{C}^N]} \leq -\widetilde{K}_{r,k},$$

thus

$$-t_{max} P_{[\widetilde{K}_{r,k}\mathbb{C}^N]} \leq \widetilde{L}_{r,k} \leq t_{max} P_{[\widetilde{K}_{r,k}\mathbb{C}^N]},$$

and so $\widetilde{L}_{r,k}$ lives on $[\widetilde{K}_{r,k}\mathbb{C}^N]$.

Since $\{[\widetilde{K}_{r,k}\mathbb{C}^N] : r = 0, \ldots, m; k = 1, \ldots, n_r\}$ is weakly independent and $\sum_{r=0}^{m} \sum_{k=1}^{n_r} \widetilde{L}_{r,k}$ then $L_{r,k} = 0$ for $r = 0, \ldots, m, k = 1, \ldots, n_r$. So $\nu = 0$ and therefore, by Lemma 2.4.1, $\mu$ is an extreme point of $\mathcal{C}$. □

We expect that there exist such extreme points $\mu = \sum_{j=1}^{n} \mu_j^{(1)} K_j$ for which not all $K_j$ are positive semidefinite.

The fact that the elements of $\Pi^N$ cannot be written as matrix-convex combinations of scalar extreme points explains how this approach to proving the spectral set question in the affirmative breaks down. With additional work this analysis of the structure of the extreme points should lead to an alternative negative solution of spectral set question.

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