The Discrete Hodge Star Operator and Poincaré Duality

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(ABSTRACT)

This dissertation is a unification of an analysis-based approach and the traditional topological-based approach to Poincaré duality. We examine the role of the discrete Hodge star operator in proving and in realizing the Poincaré duality isomorphism (between cohomology and homology in complementary degrees) in a cellular setting without reference to a dual cell complex. More specifically, we provide a proof of this version of Poincaré duality over $\mathbb{R}$ via the simplicial discrete Hodge star defined by Scott Wilson in [19] without referencing a dual cell complex. We also express the Poincaré duality isomorphism over both $\mathbb{R}$ and $\mathbb{Z}$ in terms of this discrete operator. Much of this work is dedicated to extending these results to a cubical setting, via the introduction of a cubical version of Whitney forms. A cubical setting provides a place for Robin Forman's complex of nontraditional differential forms, defined in [7], in the unification of analytic and topological perspectives discussed in this dissertation. In particular, we establish a ring isomorphism (on the cohomology level) between Forman's complex of differential forms with his exterior derivative and product and a complex of cubical cochains with the discrete coboundary operator and the standard cubical cup product.

Dedication

This dissertation is dedicated to my wonderful husband, Jonathan, and my beautiful daughter, Keziah. Throughout this process, they have each supported me in countless ways. Jonathan offered me encouragement that kept me going when I wanted to give up. He gave me confidence in my abilities and provided me with the perseverance to complete my Ph.D. He also selflessly took Keziah off my hands on days when I needed to work. Keziah, while less than two years of age, still managed to support me in ways that other people couldn't. After a long day of work on campus, everything was made worth it when I would open the front door and hear her shout with delight, "MOMMY!", and she would come running to give me a hug. Her sweet, innocent, easy-going personality has been a blessing in my life and has made being a mom and a Ph.D. student as easy as it can be. I am truly grateful for my little family.

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Chapter 1

Introduction

Poincaré Duality is arguably one of the most substantial results in algebraic topology. In its strongest form, it states that for a closed orientable n-dimensional manifold M, the homology and cohomology groups over $\mathbb{Z}$ of M are isomorphic in complementary degrees. Its proof relies on a notion of transversality. Traditionally, Poincaré Duality is recovered via the homology and cohomology groups of a cell complex $\mathcal{X}$ on M and its dual cell structure $\mathcal{X}^*$. This method requires M to admit a cell structure. For the more general manifold M satisfying the hypotheses of the theorem, the proof relies on the realization of the Poincaré Duality isomorphism as a cap product. The latter proof can be found in [10].

Throughout this paper, we focus on results in the context of cell complexes. Cell complexes have long been the foundation for intuition surrounding topological problems. Homology and cohomology groups are topological invariants that can be defined without reference to a cell complex. However, they may be calculated using a cell complex, and the results are independent of the cell structure chosen. Furthermore, numerical computation calls for cell complexes. Applications of the underlying theory of this paper can be seen in models of electromagnetism and other aspects of mathematical physics [4, 9, 15].

The traditional proof of Poincaré duality relies on a cell complex *and* its dual cell complex. This is a natural choice in that a cell complex in the presence of its dual makes transversality explicit through transverse intersections. Additionally, over $\mathbb{C}$, a dual complex allows for the recovery, in cellular terms, of the Hodge structure of the cohomology groups of a Riemann surface. As seen in [12], this is a key part of discrete complex analysis. However, bringing a cell complex's dual complex into the picture introduces twice as much information, an undesirable quality from a computational standpoint. Thus, we explore the extent to which the theory surrounding the Hodge star and Poincaré duality may be recovered in the absence of a dual complex. The theory developed by Scott Wilson in [19] is the motivation for the majority of the results that we present.

This paper serves as an alignment of an analysis-based perspective on Poincaré duality with the traditional topological perspective, expressed in the cellular setting without the usual reference to a dual cell complex. The discrete Hodge star operator defined by Wilson in [19] is the expression of the analysis-based perspective. Wilson demonstrates that his star may be used to prove a version of Poincaré duality like that expressed in de Rham cohomology. We prove that this result may be pushed further to recover Poincaré duality, as an isomorphism from cohomology to homology, over $\mathbb{R}$ on a cell complex without reference to its dual. We align this perspective with the topological perspective on Poincaré duality by interpreting the discrete Hodge star as the cap product with the fundamental class of M over $\mathbb{R}$. Although our proof of Poincaré duality does not extend over $\mathbb{Z}$, we define a new, analogous discrete Hodge star that agrees with the cap product with the fundamental class of M over $\mathbb{Z}$. In this way, we may realize the Poincaré duality map in its strongest form on a single cell complex.

Throughout this paper, various products play an important role in defining a discrete Hodge star operator on a cell complex without reference to its dual cell complex. These products are different on the (co)chain level, however, they agree with the standard products on (co)homology. Thus, the products offer different information from an analysis-based perspective, but their agreement on the (co)homology level establishes the alignment of this approach with that of the traditional algebraic topology perspective on Poincaré duality.

The above results are given in the traditional simplicial setting in Chapter 4. However, the bulk of this paper focuses on details surrounding Poincaré duality in a cubical setting. Cubical complexes are useful in many applications. For example, they can be used to model digital images [14]. It is also easier to work with cartesian products on a cubical complex than on a simplicial complex. The product of two cubes is again a cube; whereas the product of two simplices need not be a simplex.

Our cubical theory is heavily motivated by the work of Christian Mercat in [12] and Bobenko, Mercat, and Suris in [2]. In [12], Mercat defines a cup product of cochains on a 2-dimensional cubical complex. The study of this product is the foundation of the cubical cup product of arbitrary degree that we define in Section 3.2.1. Ultimately, our cubical theory creates a context into which further analysis-related results of [2, 12] may fit.

In Chapter 3, we give the following new results. We define $\mathcal{L}^2$ cubical Whitney forms and a cubical cup product that fits together with the wedge product of cubical Whitney forms. We define a cubical discrete Hodge star, analogous to Wilson's discrete star in [19], via this cubical cup product. We then prove Poincaré duality over $\mathbb{R}$ and show that star is the Poincaré duality map. We also define a discrete Hodge star over $\mathbb{Z}$ via the standard cellular cubical cup product, and we show that this star realizes the Poincaré duality map over $\mathbb{Z}$.

Because our theory is developed on a single cell complex without reference to its dual, we rely on a nondegenerate pairing on cohomology to recover transversality. The aforementioned cubical cup product, defined in Section 3.2.1, is our pairing, and the cubical Whitney forms are the avenue to proving that this pairing is nondegenerate on the cohomology level via the nondegenerate de Rham Poincaré duality pairing.

Our final contribution is the representation of Robin Forman's complex of nontraditional differential forms (defined on a simplicial complex in [7]) as a complex of cubical cochains. The desire to understand the place of Forman's work in a more traditional algebraic topology setting was the motivation for this result. The behavior of Forman's differential forms on a simplicial complex naturally defines a cell complex of kites that is associated with this simplicial complex. In Section 5.2, we define this associated kite complex. We then show that the complex of Forman's differential forms with his exterior derivative is isomorphic to the complex of cubical cochains defined on the associated kite complex together with the discrete coboundary operator. Hence, these complexes define isomorphic cohomology groups.

Note that in Chapter 3, we show that a kite is diffeomorphic to a cube, hence the use of the term "cubical" is appropriate here. Furthermore, the product of Forman's differential forms via composition suggests a cup product of cubical cochains on the associated kite complex, which we define in Section 5.4.2. In Section 5.4.3, we show that this cup product agrees with the cubical cup product we define in Section 3.2.1 on the cohomology level. Thus, in Chapter 5, we show that Forman's differential forms suggest a natural cubical structure that defines a complex of cubical cochains isomorphic to Forman's complex of differential forms. Together with the aforementioned product, we place Forman's work into the context of the cubical theory we give in Chapter 3.

Chapter 2

Background

This chapter provides a brief introduction to topics in algebraic topology pertaining to the results in this paper. We also give key definitions and establish notations that we use throughout the remaining chapters.

2.1 Cell Complexes

Throughout this paper, we develop theory on two specific cell complexes on a smooth manifold, namely simplicial and cubical. These complexes are *regular* cell complexes. The basic building blocks of any cell complex are topological spaces called k-cells, defined below.

Definition 2.1.1. A topological space c is called a k**-cell** if it is homeomorphic to a closed k-dimensional topological ball B^k. c is called a **open** k**-cell** if it is homeomorphic to $\operatorname{Int} B^k$.

Before we define a regular cell complex, we first define a cell complex, or CW complex.

Definition 2.1.2. A **CW complex** is a Hausdorff space $\mathcal{X}$ together with a collection of disjoint open cells C such that

1. For each open k-cell $c \in C$, there is a continuous map $f_c : B^k \to \mathcal{X}$ such that

   i. f_c maps $\operatorname{Int} B^k$ homeomorphically onto c, and

   ii. f_c maps $\operatorname{Bd} B^k$ into a finite union of open cells, each of dimension less than k.

2. A is closed in $\mathcal{X}$ if $A \cap \bar{c}$ is closed in $\bar{c}$ for all $c \in C$.

A regular cell complex places certain restrictions on the types of k-cells of which it is comprised.

Definition 2.1.3. A **regular cell complex** $\mathcal{X}$ is a Hausdorff space together with a collection of disjoint open cells C such that

1. For each open k-cell $c \in C$, there is a homeomorphism $f_c : B^k \to \mathcal{X}$ such that

   i. f_c maps $\operatorname{Int} B^k$ homeomorphically onto c, and

   ii. The image of $\operatorname{Bd} B^k$ under f_c equals the finite union of open cells, each of dimension less than k.

2. A is closed in X if $A \cap \bar{c}$ is closed in $\bar{c}$ for all $c \in C$.

Remark 2.1.4. A regular cell complex is a CW complex with the added conditions that the attachment maps f_c are homeomorphisms for all $c \in C$, and the image of the boundary of each cell under f_c is a subcomplex.

The condition that the attachment maps f_c are homeomorphisms guarantees that there are no identifications made on the boundary of a cell. For example, each edge has two distinct vertices in its boundary. See Figure 2.1 for an example of a CW complex that is not regular and an example of a regular cell complex. Note also that the intersection of any two n-cells in a regular cell complex is either nonempty or is the closure of a union of $(n-k)$-cells.

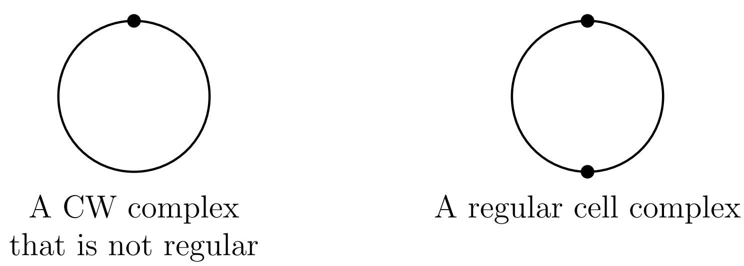


Figure 2.1: A comparison of a CW complex that is not regular with a regular cell complex.

As mentioned earlier, we will work with simplicial and cubical complexes. Thus, we define each explicitly in terms of its cells.

Definition 2.1.5. A set $\{a_0, \ldots, a_k\}$ of points of $\mathbb{R}^{k+1}$ is **geometrically independent** provided for any real numbers t_i, if

$$\sum_{i=0}^{k} t_i = 0 \quad \text{and} \quad \sum_{i=0}^{k} t_i a_i = \mathbf{0},$$

then $t_0 = t_1 = \cdots = t_n = 0$.

Definition 2.1.6. Let $\{a_0, \ldots, a_n\}$ be a geometrically independent set in $\mathbb{R}^n$. We define the n-simplex σ spanned by $a_0, \ldots, a_n$ to be the set of all points x of $\mathbb{R}^n$ such that

$$x = \sum_{i=0}^{n} t_i a_i, \quad \text{where each } t_i \text{ is nonnegative and} \quad \sum_{i=0}^{n} t_i = 1.$$

An n-simplex is denoted by its vertices $[a_0, \ldots, a_n]$. We denote a **standard n-simplex** by the vertices $[v_0, \ldots, v_n]$, where $v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the i^{th} position. $(t_0, \ldots, t_k)$ are called the **barycentric coordinates** of the point $\sum_i t_i v_i$ in $[v_0, \ldots, v_k]$. The **barycenter** of $[v_0, \ldots, v_k]$ is such that the barycentric coordinates are equal in each component. We denote the barycenter of a simplex σ by $\dot{\sigma}$.

Definition 2.1.7. A **face** of an n-simplex $\sigma = [a_0, \ldots, a_n]$ is a simplex spanned by a subset of $\{a_0, \ldots, a_n\}$. In other words, a face τ of σ is such that $\tau = [u_0, \ldots, u_k]$ where each $u_i = a_j$ for some j, $1 \leq j \leq n$, and $k \leq n$.

Definition 2.1.8. A **simplicial complex** X in $\mathbb{R}^n$ is a regular cell complex whose cells are simplices such that

1. Every face of a simplex of X is in X.
2. The intersection of any two simplices of X is a face of each of them.

Definition 2.1.9. The standard p-cube, e_p, is the subset of $\mathbb{R}^n$ consisting of points $(x_1, \ldots, x_n)$ such that $x_i = 0$ for $i > p$, and $0 \leq x_i \leq 1$ for $1 \leq i \leq p$.

In this way, the standard p-cube is a face of the n-cube. We will denote it by its nonzero variables $(x_1, \ldots, x_p)$.

Definition 2.1.10. A **cubical complex** K in $\mathbb{R}^n$ is a regular cell complex who cells are cubes such that

1. Every face of a cube of K is in K.

2. The intersection of any two cubes of K is a face of each of them.

Throughout this paper we will use the term “cubical structure” to refer to a cell complex in which each p-cell is the image of a diffeomorphism of the standard p-cube. So, each cell of a cubical structure is not necessarily a perfect cube. We define this term more carefully in Chapter 3.

In Chapter 5, we work with polyhedrons that are a subset of the standard n-simplex. We call these polyhedrons kites.

Definition 2.1.11. Let $\sigma = [v_0, \ldots, v_k]$ be a k-simplex and let $\tau = [u_0, \ldots, u_{k-p}]$ be a $(k-p)$-simplex in σ, $p \leq k$. Let $w_0, \ldots, w_{p-1}$ denote the vertices in σ that are not in τ. Let x_{v_i} denote the barycentric variable that is 1 at v_i. We define a ***p*-kite** in σ to be the polyhedron transverse to τ defined by the following subset of σ.

$$\{(x_0, \ldots, x_n) : x_{u_0} = \cdots = x_{u_{k-p}} \text{ and } 0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i,\, 0 \leq i \leq k\}.$$

Note that when $p = 2$, the above definition coincides with the more traditional definition of a kite as a quadrilateral with two distinct pairs of equal adjacent sides.

2.2 Homology and Cohomology Groups with $\mathbb{R}$ Coefficients

2.2.1 General Homology and Cohomology Groups Over $\mathbb{R}$

Definition 2.2.1. Let C_i denote an abelian group for all $i \geq 0$. Let $\partial_p : C_p \to C_{p-1}$ denote a homomorphism such that $\partial_p \circ \partial_{p+1} = 0$ for each $p \geq 0$ (we define $\partial_0 = 0$). Then the sequence

$$\cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

is called a **chain complex** and each C_i is called a **chain group**.

Because $\partial_p \circ \partial_{p+1} = 0$ for each p, $\operatorname{Im} \partial_{p+1} \subseteq \operatorname{Ker} \partial_p$. Thus, we have the following definition.

Definition 2.2.2. The $\boldsymbol{p^{th}}$ **homology group** of a chain complex is

$$H_p = \operatorname{Ker} \partial_p / \operatorname{Im} \partial_{p+1}.$$

From any chain complex, we may define its dual cochain complex as follows.

Definition 2.2.3. Define a **cochain group** by $C^p = C_p^* = \operatorname{Hom}(C_p, \mathbb{R})$. Let $\delta^p = \partial_{p+1}^* : C^p \to C^{p+1}$ be the map dual to ∂_p for all p. Because $\partial_p \circ \partial_{p+1} = 0$, it follows that $\delta^p \circ \delta^{p-1} = 0$. Thus, the sequence

$$\cdots \longrightarrow C^{p-1} \xrightarrow{\delta^{p-1}} C^p \xrightarrow{\delta^p} C_{p+1} \xrightarrow{\delta^{p+1}} \cdots$$

is called a **cochain complex**.

Note that for $\alpha \in C^p$, $\delta^p(\alpha) = \partial^*_{p+1}(\alpha) = \alpha \circ \partial_{p+1}$ by the definition of a dual homomorphism. Because $\delta^p \circ \delta^{p-1} = 0$, we have $\operatorname{Im} \delta^{p-1} \subseteq \operatorname{Ker} \delta^p$. Thus, we have the following definition.

Definition 2.2.4. The $\boldsymbol{p^{th}}$ **cohomology group** of a cochain complex is

$$H^p = \operatorname{Ker} \delta^p / \operatorname{Im} \delta^{p-1}.$$

We will work with (co)chain complexes defined on simplicial and cubical complexes.

2.2.2 Cellular (Co)Chains and (Co)Homology Groups

Definition 2.2.5. A **cellular *p*-chain** on a complex $\mathcal{X}$ is a linear combination of p-cells in $\mathcal{X}$. We denote the abelian group of cellular p-chains on $\mathcal{X}$ by $C_p(\mathcal{X})$.

A **cellular *p*-cochain** on $\mathcal{X}$ is a homomorphism that assigns a number to each p-chain in $C_p(\mathcal{X})$. We denote the abelian group of cellular p-cochains on $\mathcal{X}$ by $C^p(\mathcal{X}) = \operatorname{Hom}(C_p(\mathcal{X}), \mathbb{R})$.

Thus, given a homomorphism $\partial : C_p(\mathcal{X}) \to C_{p-1}(\mathcal{X})$ satisfying $\partial \circ \partial = 0$, we have the following chain complex.

$$\cdots \longrightarrow C_p(\mathcal{X}) \xrightarrow{\partial_p} C_{p-1}(\mathcal{X}) \xrightarrow{\partial_{p-1}} \cdots \longrightarrow C_1(\mathcal{X}) \xrightarrow{\partial_1} C_0(\mathcal{X}) \xrightarrow{\partial_0} 0.$$

And, if $\delta = \partial^* : C^p(\mathcal{X}) \to C^{p+1}(\mathcal{X})$ (hence $\delta \circ \delta = 0$), we have the following cochain complex.

$$\cdots \longrightarrow C^{p-1}(\mathcal{X}) \xrightarrow{\delta^{p-1}} C^p(\mathcal{X}) \xrightarrow{\delta^p} C_{p+1}(\mathcal{X}) \xrightarrow{\delta^{p+1}} \cdots.$$

Thus, we define cellular homology and cohomology associated with these complexes.

Definition 2.2.6. The $\boldsymbol{p^{th}}$ **cellular homology group of** $\boldsymbol{\mathcal{X}}$ is

$$H_p^{\mathcal{C}}(\mathcal{X}) = \operatorname{Ker} \partial_p / \operatorname{Im} \partial_{p+1}.$$

The $\boldsymbol{p^{th}}$ **cellular cohomology group of** $\boldsymbol{\mathcal{X}}$ is

$$H^p_{\mathcal{C}}(\mathcal{X}) = \operatorname{Ker} \delta_p / \operatorname{Im} \delta_{p-1}.$$

Throughout this paper, we will take $\mathcal{X} = X$ when referring to a simplicial complex and $\mathcal{X} = K$ for a cubical complex. In each case, we can explicitly define the connecting homomorphisms ∂ and δ. However, introducing these definitions requires defining an orientation on the cells of a simplicial complex and a cubical complex.

Definition 2.2.7. Let $\sigma = [v_0, \ldots, v_k]$ be a k-simplex for some $k \geq 0$. We define two orderings of its vertices to be equivalent if they differ by an even permutation. If $k \geq 1$, the orderings split into two equivalence classes. Each of these classes is called an **orientation** on σ. Note that a 0-simplex only has one orientation. An **oriented simplex** is a simplex σ together with an orientation on σ.

Throughout this paper, we take the **standard orientation of a** $\boldsymbol{k}$**-simplex** to be the orientation that agrees with the order of its vertices $v_0, \ldots, v_k$.

We may also define a **vector orientation of a** $\boldsymbol{k}$**-simplex** $\boldsymbol{\sigma}$ as follows.

$$v_0 - v_i, \ldots, \widehat{v_i - v_i}, \ldots, v_k - v_i.$$

Thus, we anchor the orientation at a vertex v_i in σ, and consider the vectors that emanate from v_i. The standard orientation of k-simplex and the vector orientation given by $v_1 -$

$v_0, \ldots, v_k - v_0$ are regarded as agreeing. The vector orientation anchored at v_i differs from the standard orientation by a sign of $(-1)^i$, where i is the number of moves to bring the anchor to the front of the list of vertices.

Definition 2.2.8. Let $\tau = (x_1, \ldots, x_p)$ be a p-cube for some $p \geq 1$. Regard the direction of increase of each variable as fixed in the direction from 0 to 1. We define two orderings of the variables of τ to be equivalent if they differ by an even permutation. For all $p \geq 2$, the orderings split into two equivalence classes. Each of these classes is called an **orientation** on τ. A 0-cube is defined to have a single orientation. A 1-cube is defined by a single variable x_1 and hence has a single orientation, also.

We will take the **standard orientation of a *p*-cube** to be the orientation that agrees with the ordering of its variables $x_1, \ldots, x_p$ for all $p \geq 1$. In the case where $p = 0$, we have noted that there is a single orientation. Thus, when we say "orient a p-cube, $p \geq 0$, by the order of its variables," it is implicit that we handle the case where $p = 0$ as mentioned.

Note that we may also give a **vector orientation of a *p*-cube** via the tangent directions in which the variables x_i increase. Changing the direction of k-vectors changes the orientation by a sign of $(-1)^k$. This allows for two orientations in the $p = 1$ case.

In Chapter 5, we relate the orientation of a k-simplex σ to the orientation of a p-kite contained in σ via a vector orientation of σ. This provides the motivation for our definition of an orientation on a p-kite.

Definition 2.2.9. Suppose $k, p \geq 0$ with $p \leq k$. We define the **orientation of a *p*-kite**

contained in a k-simplex σ as follows. Let $v_0, \dots, v_k$ be the vertices of σ. Suppose the orientation of σ is given by

$$v_0 - v_i, \dots, \widehat{v_i - v_i}, \dots, v_k - v_i.$$

Choose a subset of p vertices from $v_0, \dots, v_k$ that does not include the orientation anchor v_i. Call these vertices $w_0, \dots, w_{p-1}$, given in the order in which they appear in $v_0, \dots, v_k$. These vertices define a subset of the orientation vectors on σ.

$$w_0 - v_i, \dots, w_{p-1} - v_i.$$

We may associate each $w_i - v_i$ with the barycentric variable x_{w_i} that is 1 at w_i. This defines an orientation

$$x_{w_0}, \dots, x_{w_{p-1}}$$

on the p-kite in σ for which these serve as the free variables. Note that this is exactly the p-kite transverse to the $(k-p)$-simplex defined by the set of vertices in σ that is complementary to $w_0, \dots, w_{p-1}$.

We will see later that this definition is necessary for properly relating the orientation of a p-kite η to the orientation of the k-simplex σ that contains it. Note that the subset of orientation vectors of σ given by $w_0 - v_i, \dots, w_{p-1} - v_i$ does not make sense as an orientation of η because these vectors do not lie in the tangent space of η. Thus, by describing the orientation on η as $x_{w_0}, \dots, x_{w_{p-1}}$ we are in a sense "projecting" the orientation vectors of σ onto η.

For each complex, we now give the definitions of ∂. $\delta = \partial^*$ is then determined on a p-cochain α by

$$\delta(\alpha) = \alpha \circ \partial.$$

Definition 2.2.10. We define the **cellular simplicial boundary map** $\partial : C_p(X) \to C_{p-1}(X)$ for $p \geq 1$ by

$$\partial_p[v_0, \ldots, v_p] = \sum_{i=0}^{p} (-1)^i [v_0, \ldots, \widehat{v_i}, \ldots, v_p].$$

Definition 2.2.11. We define the **cellular cubical boundary map** $\partial : C_p(K) \to C_{p-1}(K)$ for $p \geq 1$ by

$$\partial_p \sigma = \sum_{i=1}^{p} (-1)^{i+1} (\sigma\big|_{x_i=1} - \sigma\big|_{x_i=0}).$$

By convention, $\partial_0 = 0$.

2.2.3 Singular (Co)Chains and (Co)Homology Groups on a Cell Complex

Definition 2.2.12. A **singular *p*-chain** on $\mathcal{X}$ is a continuous map $f : e_p \to \mathcal{X}$, where e_p denotes a standard cell in $\mathcal{X}$. We denote the abelian group of singular p-chains by $S_p(\mathcal{X})$.

So, a singular simplicial p-chain is a continuous map of the standard p-simplex into X; a singular cubical p-chain is a continuous map of the standard p-cube into K.

Definition 2.2.13. A **singular *p*-cochain** on $\mathcal{X}$ assigns a number to each singular p-chain in $S_p(\mathcal{X})$. We denote the abelian group of singular p-cochains by $S^p(\mathcal{X}) = \mathrm{Hom}(S_p(\mathcal{X}), \mathbb{R})$.

Thus, given a homomorphism $\partial : S_p(\mathcal{X}) \to S_{p-1}(\mathcal{X})$ satisfying $\partial \circ \partial = 0$, we have the following chain complex.

$$\cdots \longrightarrow S_p(\mathcal{X}) \xrightarrow{\partial_p} S_{p-1}(\mathcal{X}) \xrightarrow{\partial_{p-1}} \cdots \longrightarrow S_1(\mathcal{X}) \xrightarrow{\partial_1} S_0(\mathcal{X}) \xrightarrow{\partial_0} 0.$$

And, if $\delta = \partial^* : S^p(\mathcal{X}) \to S^{p+1}(\mathcal{X})$ (hence $\delta \circ \delta = 0$), we have the following cochain complex.

$$\cdots \longrightarrow S^{p-1}(\mathcal{X}) \xrightarrow{\delta^{p-1}} S^p(\mathcal{X}) \xrightarrow{\delta^p} S_{p+1}(\mathcal{X}) \xrightarrow{\delta^{p+1}} \cdots .$$

Thus, we define singular homology and cohomology associated with these complexes.

Definition 2.2.14. The $\boldsymbol{p^{th}}$ **singular homology group of $\boldsymbol{\mathcal{X}}$** is

$$H_p^{\mathcal{S}}(\mathcal{X}) = \operatorname{Ker} \partial_p / \operatorname{Im} \partial_{p+1}.$$

The $\boldsymbol{p^{th}}$ **singular cohomology group of $\boldsymbol{\mathcal{X}}$** is

$$H^p_{\mathcal{S}}(\mathcal{X}) = \operatorname{Ker} \delta_p / \operatorname{Im} \delta_{p-1}.$$

Below, we give the explicit map ∂ in the case where $\mathcal{X}$ is a simplicial complex and where $\mathcal{X}$ a cubical complex.

Definition 2.2.15. We define the **singular simplicial boundary map** $\partial : S_p(X) \to S_{p-1}(X)$ for $p \geq 1$ by

$$\partial_p \sigma = \sum_{i=0}^{p} (-1)^i \sigma\big|_{[v_0,\ldots,\hat{v}_i,\ldots,v_p]}.$$

Definition 2.2.16. We define the **singular cubical boundary map** $\partial : S_p(K) \to S_{p-1}(K)$ for $p \geq 1$ by

$$\partial_p f = \sum_{i=1}^{p} (-1)^{i+1} (f\big|_{x_i=1} - f\big|_{x_i=0}).$$

By definition, the singular simplicial boundary maps commutes with the cellular simplicial boundary map. The same is true in the cubical setting. In the simplicial case, the cellular chain $[v_0, \ldots, v_p]$ corresponds to the singular chain σ that is the identity map on $[v_0, \ldots, v_p]$. Thus, the image of $\sigma\big|_{[v_0,\ldots,\hat{v}_i,\ldots,v_p]}$ is $[v_0, \ldots, \hat{v}_i, \ldots, v_p]$. In particular, if ∂_c and ∂_s denote the cellular and singular boundary maps, respectively,

$$\partial_s \sigma = \sigma\big|_{\partial_c [v_0,\ldots,v_p]}.$$

Similarly, for the cubical setting.

2.2.4 De Rham Cohomology

We provide the following background on de Rham cohomology because it serves as motivation for both the theory in [19] and the theory that we develop in this paper. The details of this section are provided in [3].

Stokes' Theorem provides a natural dual relationship between differential forms on a smooth manifold and the singular chains of a cell complex $\mathcal{X}$ on M. Differential forms behave like cochains via integration along a chain.

Definition 2.2.17. The **de Rham complex** is the cochain complex of exterior differential forms on a smooth manifold M with the exterior derivative d in place of δ, as follows.

$$\cdots \longrightarrow \Omega^{p-1}(M) \xrightarrow{d^{p-1}} \Omega^p(M) \xrightarrow{d^p} \Omega^{p+1}(M) \xrightarrow{d^{p+1}} \cdots.$$

$\Omega^k(M)$ is the space of smooth k-forms on M.

Definition 2.2.18. The $\boldsymbol{p^{th}}$ **de Rham cohomology group** is given by

$$H^p_{dR}(M) = \operatorname{Ker} d_p / \operatorname{Im} d_{p-1}.$$

De Rham's theorem states that

$$H^p_{dR}(M) \cong H^p_{\mathcal{S}}(X; \mathbb{R})$$

for X a simplicial complex on a smooth manifold M.

Because differential forms are naturally related to cochains as described above, we frequently interchange the terms "differential form" and "cochain" throughout this paper. In doing so, we alert the reader that we are referring to a subset of differential forms in $\Omega(M)$ that take integral values only.

Isomorphisms on Cellular and Singular (Co)Homology

In this section, we provide the theorems that assert the following.

1. Cellular and singular simplicial (co)homology are isomorphic.

2. Cellular and singular cubical (co)homology are isomorphic.

3. Singular simplicial and singular cubical (co)homology are isomorphic.

4. Cellular simplicial and cellular cubical (co)homology are isomorphic.

The last result follows from the first three.

First, we recall a theorem from [10] that is useful in establishing the above isomorphisms on the cohomology level.

Theorem 2.2.19. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G.*

Proof. The proof is given in [10].

□

Theorem 2.2.20. *Let X be a simplicial complex on a smooth manifold M, and let $p \geq 0$ be arbitrary. Then,*

1. $H_p^{\mathcal{C}}(X) \cong H_p^{\mathcal{S}}(X)$.

2. $H^p_{\mathcal{C}}(X) \cong H^p_{\mathcal{S}}(X)$.

Proof. The proofs are given in [13] via a chain map between $C(X)$ and $S(|X|)$, where $|X|$ denotes the union of the simplices in X. This chain map is described in Remark 2.2.22.

□

Theorem 2.2.21. *Let K be a cubical complex on a smooth manifold M, and let $p \geq 0$ be arbitrary. Then,*

1. $H_p^{\mathcal{C}}(K) \cong H_p^{\mathcal{S}}(K)$.

2. $H^p_{\mathcal{C}}(K) \cong H^p_{\mathcal{S}}(K)$.

Proof. In [10], Hatcher gives the proof for an arbitrary CW complex. By definition, K is a regular CW complex. However, we specify the boundary in K. In an arbitrary CW complex, there are issues surrounding orientation of boundaries. In his proof, Hatcher gives a mechanism for addressing CW complexes and their topological boundaries. Thus, we must verify that the cellular boundary map defined in Definition 2.2.11 agrees with this mechanism.

Hatcher identifies $C_p(K)$ with $H_p(K^p, K^{p-1})$, where K^i denotes the i-skeleton of K. He handles the boundary map from $H_{p+1}(K^{p+1}, K^p)$ to $H_p(K^p, X^{p-1})$ via $H_p(K^p)$ as follows.

$$H_{p+1}(K^{p+1}, K^p) \xrightarrow{\partial_{p+1}} H_p(K^p) \xrightarrow{j_p} H_p(K^p, X^{p-1})$$

where ∂_{p+1} is the connecting homomorphism which is defined by the cubical singular boundary map and j_p is induced by the identity map. Because ∂_{p+1} commutes with the cubical cellular boundary map, Hatcher's mechanism agrees with the boundary convention defined in the cubical cellular setting. Thus, we may borrow the details of Hatcher's proof to obtain the desired result.

□

Remark 2.2.22. Let σ be a single p-cell in $C^p(\mathcal{X})$, and let f be the singular chain $f : e_p \to \mathcal{X}$ such that $\operatorname{Im} f = \sigma$. The above isomorphisms between singular and cellular homology are induced from the inclusion $i : C_p(\mathcal{X}) \to S_p(|\mathcal{X}|)$ given by $i(\sigma) = f$.

Let α be a singular p-cochain that takes value 1 on some singular p-chain $f : c_p \to \mathcal{X}$ and 0 otherwise. The image of f is a cellular p-chain, call it σ. Let β be the cellular p-cochain that takes value 1 on σ and 0 otherwise. Then the isomorphism between singular and cellular cohomology identifies α and β.

Theorem 2.2.23. *Let X and K be simplicial and cubical complexes, respectively, on a smooth manifold M. Let $p \geq 0$ be arbitrary. Then,*

1. $H_p^{\mathcal{S}}(X) \cong H_p^{\mathcal{S}}(K)$.

2. $H^p_{\mathcal{S}}(X) \cong H^p_{\mathcal{S}}(K)$.

Proof. In [6], Eilenberg and MacLane define a chain equivalence $f : S_p(X) \to S_p(K)$.

Recall that the standard n-simplex is denoted by its vertices $[v_0, \ldots, v_n]$ and the standard n-cube is denoted by its variables $(x_1, \ldots, x_n)$. Let T denote a singular simplicial cochain. Then the chain equivalence $f : S_p(X) \to S_p(K)$ is defined by

$$(fT)(x_1, \ldots, x_n) = T(v_0, \ldots, v_n)$$

where

$$\begin{aligned} v_0 &= 1 - x_1, \\ v_1 &= x_1(1 - x_2), \\ &\cdots \\ v_i &= x_1 \cdots x_i(1 - x_{i+1}), \qquad 0 < i < n, \end{aligned}$$

$$\cdots$$

$$v_n = x_1 \cdots x_n.$$

Because f is a chain equivalence, f induces the isomorphism in *1.* By Theorem 2.2.19, it follows that f also induces the isomorphism on cohomology in *2*.

□

Theorem 2.2.24. *Let X and K be simplicial and cubical complexes, respectively, on a smooth manifold M. Let $p \geq 0$ be arbitrary. Then,*

1. $H_p^{\mathcal{C}}(X) \cong H_p^{\mathcal{C}}(K)$.
2. $H_{\mathcal{C}}^p(X) \cong H_{\mathcal{C}}^p(K)$.

Proof. The result follows from Theorem 2.2.20, Theorem 2.2.21, and Theorem 2.2.23.

□

Thus, we see that the (co)homology groups in play throughout this paper are all isomorphic. For this reason, we drop the notation identifying the group as cellular or singular. Where necessary, we make distinctions clear.

2.2.5 Topological Invariance of (Co)Homology Groups

Suppose M is a smooth manifold. Given two different simplicial complexes on M, or two different cubical complexes on M, the (co)homology groups defined by these complexes are the same. Thus, we may use $H_*(\mathcal{X})$ and $H_*(M)$ interchangeably. Similarly for $H^*(\mathcal{X})$ and $H^*(M)$.

Theorem 2.2.25. *Suppose X and Y are simplicial complexes on a manifold M. Then, for all p,*

1. $H_p(X) = H_p(Y)$.
2. $H^p(X) = H^p(Y)$.

Proof. The proof of *1* is given in [13]. *2* follows from Theorem 2.2.19. □

Lemma 2.2.26. *Any cubical complex K can be triangulated, i.e. subdivided into a simplicial complex.*

Proof. It suffices to show that the n-cube may be triangulated for all $n \geq 0$, and that these triangulations restrict in a consistent manner to boundaries. To see this, we construct a simplicial complex on K via induction on the dimension n of a top-dimensional cube in K.

Base Case: For $n = 0$ and $n = 1$, an n-cube is already a simplex. Nonetheless, we give the construction for $n = 1$ to make clear our inductive method. Place a vertex at the center of the 1-cube. The joins of the center vertex to the vertices in its boundary are 1-simplices. To construct a simplicial complex on the 2-cube σ, we begin with the 1-cubes contained in σ. Generate the 1-simplices associated with each 1-cube in the boundary of σ as above. Place a vertex in the center of the 2-cube. The joins of this vertex with the 1-simplices in the boundary are 2-simplices. In this way, we triangulate the 2-cube.

Inductive Hypothesis: Let $n \geq 0$ be arbitrary and assume the n-cube may be triangulated.

Inductive Step: Consider the $(n+1)$-cube. Triangulate its n-skeleton by the inductive hypothesis. Place a vertex at the center of the $(n+1)$-cube, namely the point whose coordinate entries are all $\frac{1}{2}$. The joins of the this vertex with the n-simplices in its n-skeleton are $(n+1)$-simplices. This yields a triangulation of the $(n+1)$-cube.

So, by induction, an n-cube may be triangulated.

Note that construction of the triangulation of the n-cube always builds on previous constructions on lower dimensional skeletons. Thus, we need not worry about the consistency of the construction at the boundary of each n-simplex in the triangulation of K.

□

Theorem 2.2.27. *Suppose K and L are cubical complexes on a manifold M. Then, for all p,*

1. $H_p(K) = H_p(L)$.
2. $H^p(K) = H^p(L)$.

Proof. By Lemma 2.2.26, K and L can each be triangulated, yielding two simplicial complexes X and Y, respectively. By Theorem 2.2.25, the (co)homology groups of X and Y are isomorphic. By Theorem 2.2.24, the (co)homology groups of K and L are isomorphic to the (co)homology groups of X and Y, respectively. The result follows by transitivity. □

2.3 Products on a Complex

In [18], Whitney defines products on an arbitrary complex $\mathcal{X}$ with integer coefficients. These definitions can be restated using real coefficients, instead. We make this change because we work with real coefficients throughout much of this dissertation.

A cap product is a product of a chain of degree $p+q$ and cochain of degree p that returns a chain of dimension q. A cup product is a product of cochains of arbitrary dimension p and q that returns a cochain of dimension $p+q$. One can show that the standard cup product on a complex $\mathcal{X}$ defines a ring structure on its cohomology groups.

In this section, and throughout the remainder of this paper, we will use the notation $\hat{\sigma}_k$ to denote a k-cochain that is 1 on the k-chain σ_k and 0 otherwise.

Whitney asserts that there are three defining properties of a cap product.

Definition 2.3.1. A **cap product** on an arbitrary cell complex $\mathcal{X}$ is a product $\cap : C_{p+q}(\mathcal{X}) \times C^p(\mathcal{X}) \to C_q(\mathcal{X})$ satisfying the following three conditions.

1. Suppose $\sigma_{p+q} \in C_{p+q}(\mathcal{X})$ and $\sigma_p(\mathcal{X}) \in C_p(\mathcal{X})$ are single cells. Then $\sigma_{p+q} \cap \hat{\sigma}_p$ is a q-chain in $\overline{\mathrm{St}(\sigma_p) \cdot \sigma_{p+q}}$.

2. $\partial(\sigma \cap \alpha) = (-1)^p(\partial\sigma \cap \alpha - \sigma \cap \delta\alpha)$ for $\sigma \in C_{p+q}(\mathcal{X})$ and $\alpha \in C^p(\mathcal{X})$.

3. For some real number $\gamma_\cap$, $I(\sigma_p \cap \hat{\sigma}_p) = \gamma_\cap$ for all $\sigma_p \in C_p(\mathcal{X})$. Note that I is the constant 0-cochain that takes value 1 on each 0-cell of $\mathcal{X}$.

$\mathrm{St}(\sigma_p)$ is the union of all cells in which σ_p is a face. $\overline{\mathrm{St}(\sigma_p)\cdot\sigma_{p+q}}$ is the closure of the union of $\mathrm{St}(\sigma_p)$ and σ_{p+q}. In general, $A \cdot B$ denotes the union of all cells in A and B.

Note that Whitney's second defining property of the cap product in [18] differs from the one we give above. We have altered this property so that the standard singular simplicial cap product, defined momentarily, meets the three properties given above. This alteration elicits results analogous to those given in [18] which can be proven similarly with this change, as we will see in detail below.

Whitney also asserts that there are three defining properties of a cup product of cochains.

Definition 2.3.2. A **cup product** on an arbitrary cell complex $\mathcal{X}$ is a product $\cup : C^p(\mathcal{X}) \times C^q(\mathcal{X}) \to C^{p+q}(\mathcal{X})$ satisfying the following three conditions.

1. Suppose $\sigma_p \in C_p(\mathcal{X})$ and $\sigma_q \in C_q(\mathcal{X})$. Then $\hat{\sigma}_p \cup \hat{\sigma}_q$ is a $(p+q)$-cochain in $\mathrm{St}(\sigma_p)\cdot\mathrm{St}(\sigma_q)$.

2. $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^p \alpha \cup \delta\beta$ for $\alpha \in C^p(\mathcal{X})$ and $\beta \in C^q(\mathcal{X})$.

3. For some real number $\gamma_\cup$, $I \cup \alpha = \gamma_\cup \alpha$ for all $\alpha \in C^p(\mathcal{X})$ and for all p, where I is the constant 0-cochain that takes value 1 on the 0-cells of $\mathcal{X}$.

Whitney states that a cup product and its corresponding cap product satisfy the following.

1. $(\alpha \cup \beta)(\sigma) = \beta(\sigma \cap \alpha)$ for all $\sigma \in C_{p+q}(\mathcal{X})$, $\alpha \in C^p(\mathcal{X})$, and $\beta \in C^q(\mathcal{X})$.

2. $\gamma_\cup = \gamma_\cap$.

We give the standard definitions of the singular products in both the simplicial and the cubical setting. See, e.g., [10]

Definition 2.3.3. The **singular simplicial cup product** $\cup : S^p(X) \times S^q(X) \to S^{p+q}(X)$ is defined on $\sigma \in S_{p+q}(X)$ by

$$(\alpha \cup \beta)(\sigma) = \alpha(\sigma\big|_{[v_0,\ldots,v_p]})\beta(\sigma\big|_{[v_p,\ldots,v_{p+q}]}).$$

Definition 2.3.4. The **singular simplicial cap product** $\cap : S_{p+q}(X) \times S^p(X) \to S_q(X)$ is defined by

$$\sigma \cap \varphi = \varphi(\sigma\big|_{[v_0,\ldots,v_p]})\sigma\big|_{[v_p,\ldots,v_{p+q}]}.$$

Remark 2.3.5. We have earlier stated that the domain of a singular chain is a standard simplex. Thus, we will take the notation $\sigma\big|_{[v_p,\ldots,v_{p+q}]}$ to implicitly mean $\sigma\big|_{[v_p,\ldots,v_{p+q}]}$ preceded by the orientation-preserving map $[v_0, \ldots, v_q] \mapsto [v_p, \ldots, v_{p+q}]$.

The cubical definitions first require the introduction of some notation. See, e.g., [11].

Let e_n be the standard n-cube and H an ordered subset $h_1, \ldots, h_p$ of the integers $1, \ldots, n$. Define $\lambda_H^\epsilon : e_p \to e_n$ ($\epsilon = 0$ or 1) by

$$\lambda_H^\epsilon(u_1, \ldots, u_p) = (v_1, \ldots, v_n)$$

where $v_i = \epsilon$ if $i \notin H$ and $v_{h_r} = u_r$ if $i = h_r$ for some r, $1 \leq r \leq p$. Thus, λ_H^0 is an isometry of e_p onto the p-face in e_n which contains the origin and lies in the subspace spanned by $u_{h_1}, \ldots, u_{h_p}$. λ_H^1 is an isometry of e_p onto the p-face in e_n which contains the point $(1, \ldots, 1)$ and is parallel to the subspace spanned by $u_{h_1}, \ldots, u_{h_p}$.

Definition 2.3.6. For singular cochains $\alpha \in S^p(K)$ and $\beta \in \mathcal{S}^q(K)$, the **singular cubical cup product** $\cup : S^p(K) \times S^q(K) \to S^{p+q}(K)$ is defined by

$$(\alpha \cup \beta)(\sigma) = \sum_H \rho_{HK}\ \alpha(\sigma \circ \lambda^0_H) \cdot \beta(\sigma \circ \lambda^1_K),$$

where $\sigma \in S_{p+q}(K)$ and $\rho_{HK} = \mathrm{sgn}(h_1, \ldots, h_p, k_1, \ldots, k_q)$.

Definition 2.3.7. For $\sigma \in S_{p+q}(K)$ and $\alpha \in \mathcal{S}^p(K)$, the **singular cubical cap product** $\cap : S_{p+q}(K) \times S^p(K) \to S_q(K)$ is defined by

$$\sigma \cap \alpha = \sum_H \rho_{HK}\ \alpha(\sigma \circ \lambda^0_H) \cdot \sigma \circ \lambda^1_K.$$

The above products can be defined in the cellular setting via Theorem 2.2.20, Theorem 2.2.21, and Remark 2.2.22.

The following theorems in [18] aid in proving that any two cap products, respectively cup products, agree up to multiplication by an integer on homology, respectively cohomology.

Theorem 2.3.8. *Let $\cap$ be any cap product with $\gamma_\cap = 0$. Then there is a bilinear operation $\wedge$ such that*

(1) $\sigma_{p+q} \wedge \hat{\sigma}_p$ *is a* $(q+1)$*-chain in* $\overline{St(\sigma_p) \cdot \sigma_{p+q}}$.

(2) $\sigma_p \cap \hat{\sigma}_p = \partial(\sigma_p \wedge \hat{\sigma}_p)$.

(3) $\sigma_{p+q} \cap \hat{\sigma}_p = \partial(\sigma_{p+q} \wedge \hat{\sigma}_p) + (-1)^p[\partial\sigma_{p+q} \wedge \hat{\sigma}_p - \sigma_{p+q} \wedge \delta\hat{\sigma}_p]$ *for* $q > 0$.

Proof. Let $p \geq 0$ be arbitrary. We will construct $\wedge$ by induction on q.

Base Case: Suppose $q = 0$. By assumption, $\sigma_p \cap \hat{\sigma}_p = \sum_{i=0}^{p} \Gamma_i v_i$ with $\sum_{i=0}^{p} \Gamma_i = 0$; v_i is a vertex in σ_p. Thus, we may construct $\sigma_q \wedge \hat{\sigma}_q$ so that $\partial(\sigma_q \wedge \hat{\sigma}_q) = \sum_{i=0}^{p} \Gamma_i v_i$. So, *(2)* holds, and *(1)* is trivially true.

Inductive Hypothesis: For arbitrary $p \geq 0$, suppose we can construct $\sigma_{p+r} \wedge \hat{\sigma}_p$ for all r, $0 \leq r < q$ satisfying the necessary conditions.

Inductive Step: Define $\sigma_{p+q} \wedge \hat{\sigma}_p$ to satisfy *(1)*, i.e. $\sigma_{p+q} \wedge \hat{\sigma}_p = 0$ if σ_p is not a face of σ_{p+q}. Suppose σ_p *is* a face of σ_{p+q}. We may define a q-form ζ as follows because $\partial\sigma_{p+q} \wedge \hat{\sigma}_p$ and $\sigma_{p+q} \wedge \delta\hat{\sigma}_p$ are previously constructed by the inductive hypothesis.

$$\zeta = \sigma_{p+q} \cap \hat{\sigma}_p - (-1)^p[\partial\sigma_{p+q} \wedge \hat{\sigma}_p - \sigma_{p+q} \wedge \delta\hat{\sigma}_p]$$

We will show that $\partial\zeta = 0$ on $\overline{\sigma}_{p+q}$, a contractible space. By the Poincaré Lemma, it will then follow that as $q > 0$, we may construct $\sigma_{p+q} \wedge \hat{\sigma}_p$ such that $\partial(\sigma_{p+q} \wedge \hat{\sigma}_p) = \zeta$.

By the inductive hypothesis, rerranging *(3)* yields the following two equalities.

$$\begin{aligned}\partial(\partial\sigma_{p+q} \wedge \hat{\sigma}_p) &= \partial\sigma_{p+q} \cap \hat{\sigma}_p - (-1)^p[\partial \circ \partial\sigma_{p+q} \wedge \hat{\sigma}_p - \partial\sigma_{p+q} \wedge \delta\hat{\sigma}_p] \\ &= \partial\sigma_{p+q} \cap \hat{\sigma}_p + (-1)^p \partial\sigma_{p+q} \wedge \delta\hat{\sigma}_p.\end{aligned}$$

$$\begin{aligned}\partial(\sigma_{p+q} \wedge \delta\hat{\sigma}_p) &= \sigma_{p+q} \cap \delta\hat{\sigma}_p - (-1)^{p+1}[\partial\sigma_{p+q} \wedge \delta\hat{\sigma}_p - \sigma_{p+q} \wedge \delta \circ \delta\hat{\sigma}_p] \\ &= \sigma_{p+q} \cap \delta\hat{\sigma}_p - (-1)^{p+1}\partial\sigma_{p+q} \wedge \delta\hat{\sigma}_p.\end{aligned}$$

By the definition of a cap product,

$$\partial(\sigma_{p+q} \cap \hat{\sigma}_p) = (-1)^p[\partial\sigma_{p+q} \cap \hat{\sigma}_p - \sigma_{p+q} \cap \delta\hat{\sigma}_p].$$

Thus,

$$\begin{aligned}
\partial\zeta &= \partial(\sigma_{p+q} \cap \hat{\sigma}_p) - (-1)^p \partial(\partial\sigma_{p+q} \wedge \hat{\sigma}_p) + (-1)^p \partial(\sigma_{p+q} \wedge \delta\hat{\sigma}_p) \\
&= (-1)^p[\partial\sigma_{p+q} \cap \hat{\sigma}_p - \sigma_{p+q} \cap \delta\hat{\sigma}_p] - (-1)^p[\partial\sigma_{p+q} \cap \hat{\sigma}_p + (-1)^p \partial\sigma_{p+q} \wedge \delta\hat{\sigma}_p] \\
&\quad +(-1)^p[\sigma_{p+q} \cap \delta\hat{\sigma}_p - (-1)^{p+1}\partial\sigma_{p+q} \wedge \delta\hat{\sigma}_p] \\
&= 0.
\end{aligned}$$

Hence, ζ is a cycle on $\overline{\sigma}_{p+q}$, and we may construct $\sigma_{p+q} \wedge \hat{\sigma}_p$ such that $\partial(\sigma_{p+q} \wedge \hat{\sigma}_p) = \zeta$. So,*(3)* holds.

Thus, by induction, $\wedge$ is a bilinear operation satisfying *(1), (2),* and *(3).* □

Theorem 2.3.9. *Let $\cup$ be any cup product with $\gamma_\cup = 0$. Then there is a bilinear operation $\vee$ such that*

(1) $\hat{\sigma}_p \vee \hat{\sigma}_0 = 0.$

(2) If $q < 0$, $\hat{\sigma}_p \vee \hat{\sigma}_q$ is a $(p + q - 1)$-chain in St(σ_p)·St(σ_q).

(3) $\hat{\sigma}_p \cup \hat{\sigma}_q = \delta(\hat{\sigma}_p \vee \hat{\sigma}_q) + \delta\hat{\sigma}_p \vee \hat{\sigma}q + (-1)^p \hat{\sigma}_p \vee \delta\hat{\sigma}_q.$

Proof. Because $\gamma_\cap = \gamma_\cup = 0$, we may construct $\wedge$ as in Theorem 2.3.8. Thus, $\vee$ can be constructed to correspond to $\wedge$ using the relationship

$$(\hat{\sigma}_p \vee \hat{\sigma}_q)(\sigma_{p+q}) = \hat{\sigma}_q(\sigma_{p+q} \wedge \hat{\sigma}_p).$$

□

The following two theorems assert agreement of two cap products, respectively cup products, on homology, respectively cohomology.

Theorem 2.3.10. *Let $\cap_a$ and $\cap_b$ be cap products on an arbitrary complex $\mathcal{X}$. Let $p, q \geq 0$ be arbitrary. Suppose also that $\sigma \in C_{p+q}(\mathcal{X})$ and $\alpha \in C^p(\mathcal{X})$ such that $\partial\sigma = \delta\alpha = 0$. Then,*

$$[\gamma_{\cap_b}(\sigma \cap_a \alpha)] = [\gamma_{\cap_a}(\sigma \cap_b \alpha)]$$

in $H_q(\mathcal{X})$.

Proof. Let $\cap' = \gamma_{\cap_b}\cap_a - \gamma_{\cap_a}\cap_b$. Note that $\cap'$ is a cap product on $\mathcal{X}$ by linearity. Furthermore, $\gamma_{\cap'} = \gamma_{\cap_b}\gamma_{\cap_a} - \gamma_{\cap_a}\gamma_{\cap_b} = 0$. Thus, by Theorem 2.3.8, there is a bilinear product $\wedge$ that satisfies (*1*), (*2*), and (*3*). Suppose $q = 0$. Then by (*2*),

$$\sigma \cap' \alpha = \partial(\sigma \wedge \alpha).$$

Suppose $q > 0$. Then, because σ is a cycle and α is a cocycle, (*3*) yields

$$\begin{aligned} \sigma \cap' \alpha &= \partial(\sigma \wedge \alpha) + (-1)^p[\partial\sigma \wedge \alpha - \sigma \wedge \delta\alpha] \\ &= \partial(\sigma \wedge \alpha). \end{aligned}$$

So, for all $q \geq 0$, $\sigma \cap' \alpha$ is homologous to 0. Because $\cap' = \cap_a - \cap_b$, $\sigma \cap_a \alpha$ is homologous to $\sigma \cap_b \alpha$, as desired.

□

Theorem 2.3.11. *Let $\cup_a$ and $\cup_b$ be cup products on an arbitrary complex $\mathcal{X}$. Let $p, q \geq 0$ be arbitrary. Suppose also that $\alpha \in C^p(\mathcal{X})$ and $\beta \in C^q(\mathcal{X})$ are cocycles. Then,*

$$[\gamma_{\cup_b}(\alpha \cup_a \beta)] = [\gamma_{\cup_a}(\alpha \cup_b \beta)]$$

in $H^{p+q}(\mathcal{X})$.

Proof. The proof is analogous to that of Theorem 2.3.10 using instead $\cup' = \gamma_{\cup_b} \cup_a - \gamma_{\cup_a} \cup_b$ and Theorem 2.3.9.

□

2.4 Poincaré Duality

In this section, we give the statement of Poincaré duality in its strongest form, i.e. with coefficients in $\mathbb{Z}$. We then provide the intuition behind its proof via a cell complex and its dual. We also discuss the Poincaré duality isomorphism as a cap product. Analogous results hold for coefficients in $\mathbb{R}$.

2.4.1 The Duality Theorem

Theorem 2.4.1. *(Poincaré Duality) Let M be an n-dimensional oriented closed manifold. Then for all p,*

$$H^p(M;\mathbb{Z}) \cong H_{n-p}(M;\mathbb{Z}).$$

Details of the proof can be found in [13]. Poincaré Duality requires a notion of transversality. The traditional proof of Poincaré Duality, in the simplicial cell complex setting, is given via a simplicial chain complex $C(X;\mathbb{Z})$ and its dual chain complex $D(X;\mathbb{Z})$. It is the presence of a dual complex that provides the transversality that is key in obtaining the result. For any p, $C^p(X;\mathbb{Z})$ is naturally isomorphic to $D_{n-p}(X;\mathbb{Z})$ as the p-chains of the complex are in 1-1 correspondence with the cochains of complementary degree in the dual complex.

The Poincaré Duality isomorphism is explicitly expressed as $D : H^p(M;\mathbb{Z}) \to H_{p-k}(M;\mathbb{Z})$ defined by $D(\alpha) = [M] \cap \alpha$. $[M]$ denotes the fundamental class of M and $\cap$ is the singular simplicial cap product. The fundamental class is a homology class of M that is most intuitively interpreted as the sum of the top dimensional simplices in X. So the cap product with the fundamental class realizes the Poincaré Duality isomorphism. Details surrounding this expression are in [10].

2.4.2 De Rham Poincaré Duality

The following background on de Rham Poincaré duality is in [3].

In de Rham theory, there is no associated homology theory. Thus, de Rham Poincaré Duality presents itself as an isomorphism on complementary de Rham cohomology groups.

Theorem 2.4.2. *Let M be a smooth oriented closed manifold. Then for all p,*

$$H^p_{dR}(M;\mathbb{R}) \cong H^{n-p}_{dR}(M;\mathbb{R}).$$

This isomorphism is a consequence of the existence of a nondegenerate pairing of differential forms via the smooth wedge product, $\alpha \times \beta \mapsto \int_M \alpha \wedge \beta$, where $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^{n-p}(M)$.

2.4.3 The Hodge Star Operator

In this section, we present theory surrounding the smooth Hodge star operator as given in [16], unless otherwise stated.

For a given smooth oriented Riemannian manifold M, the Hodge star operator $* : \Omega^p(M) \to \Omega^{n-p}(M)$ is a linear map defined on the space of differential forms $\Omega^*(M)$. We will see that $*$ gives a nondegenerate pairing of forms in complementary degrees and hence it provides an alternative way of recovering transversality. However, this transversality is strictly on the cochain level, rather than between chains and cochains. First, we give some background before introducing $*$.

Suppose M is a smooth oriented Riemannian manifold. A Riemannian manifold means that we have an inner product $\langle\ ,\ \rangle_x$ on the tangent spaces T_xM, $x \in M$, that varies smoothly in x. This structure defines an inner product on the cotangent spaces T^*_xM. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for T_xM, with orientation given by $v_1, \ldots, v_n$. Then $\{T^*_xM$ elements dual to the v_i's$\}$ is an orthonormal basis for T^*_xM.

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis for T^*_xM, then $\{e_{i_1} \wedge \cdots \wedge e_{i_p} : i_1, \ldots, i_p$ are distinct$\}$ can be used as an orthonormal basis for the exterior power $\Lambda^p T^*_xM$. Hence, it defines an inner product $\langle\ ,\ \rangle_{p,x}$ on $\Lambda^p T^*_xM$ for all p. A volume form ω is a differential n-form with value $e_1 \wedge \cdots \wedge e_n$ at T^*_xM.

Definition 2.4.3. For each p, the **Hodge star operator** $* : \Omega^p(M) \to \Omega^{n-p}(M)$ is defined by

$$(\alpha \wedge *\beta)(x) = \langle \alpha(x), \beta(x) \rangle_{p,x} \cdot \omega(x) \qquad \text{for all } \alpha, \beta \in \Omega^p(M).$$

Note that $*$ is linear in each fiber of $\Omega^*_x(M)$. One can check that

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \operatorname{sgn}\sigma \; e_{j_1} \wedge \cdots \wedge e_{j_{n-p}},$$

where $\{i_1, \ldots, i_p, j_1, \ldots, j_{n-p}\} = \{1, \ldots, n\}$ and $\operatorname{sgn}\sigma$ is the signature of the permutation $(i_1, \ldots, i_p, j_1, \ldots, j_{n-p})$ of $(1, \ldots, n)$.

Under this expression, we see that

$$**\,(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \operatorname{sgn}\sigma_1 \;\operatorname{sgn}\sigma_2 \; e_{i_1} \wedge \cdots \wedge e_{i_p},$$

where $\sigma_1 = (i_1, \ldots, i_p, j_1, \ldots, j_{n-p})$ and $\sigma_2 = (j_1, \ldots, j_{n-p}, i_1, \ldots, i_p)$. Note that $\operatorname{sgn}\sigma_1 = (-1)^{p(n-p)} \operatorname{sgn}\sigma_2$, because $p(n-p)$ swaps are necessary to rewrite σ_2 as σ_1. Thus, $** = \pm 1$ depending on dimension.

The Hodge star defines an $\mathcal{L}^2$ inner product via

$$\langle \alpha, \beta \rangle_{\mathcal{L}^2} = \int_M \alpha \wedge *\beta.$$

Thus, the nondegenerate de Rham Poincaré duality pairing of representatives of $H^p(M) \times H^{n-p}(M)$ can be expressed as follows.

$$\begin{aligned} \alpha \times \beta &= \int_M \alpha \wedge \beta \\ &= \pm \int_M \alpha \wedge **\,\beta \end{aligned}$$

$$= \pm\langle \alpha, *\beta \rangle_{\mathcal{L}^2}.$$

Note that the sign comes from $** = \pm 1$, as previously discussed.

The $\mathcal{L}^2$ inner product permits the identification of $H^p(M)$ with $\mathcal{H}^p$, the vector space of degree p harmonic differential forms. $\mathcal{H}^p$ is the intersection of the kernel of d with the kernel of d's adjoint. Because the adjoint takes the form $\pm * d*$ (see, e.g., [16]), $*$ maps $\mathcal{H}^p$ isomorphically to $\mathcal{H}^{n-p}$. Furthermore,

$$\begin{aligned} \alpha \times *\alpha &= \int_M \alpha \wedge *\beta \\ &= ||\alpha||^2_{\mathcal{L}^2} \end{aligned}$$

gives an explicit realization of the nondegeneracy called de Rham Poincaré duality in Section 2.4.2.

Chapter 3

The Discrete Hodge Star and Poincaré Duality on Cubical Structures

This chapter is an exposition of the ingredients leading to the proof of Poincaré duality over $\mathbb{R}$ on a cubical complex without reference to its dual complex. Its major results include the definition of a cubical cup product (Section 3.2.1), the definition of $\mathcal{L}^2$ cubical Whitney forms (Section 3.2.2), the definition of a cubical discrete Hodge star (Section 3.3.1), and the proof of Poincaré duality over $\mathbb{R}$ on a single cubical complex via this star. We also define a cubical discrete Hodge star over $\mathbb{Z}$, and we show that this star realizes the Poincaré duality map over $\mathbb{Z}$ on a single cubical complex (Section 3.3.4).

Throughout this chapter, M is a closed oriented n-dimensional manifold unless otherwise stated. Because we work with cubical structures, we first establish that any smooth manifold admits a cubical structure that arises from a cubical complex (Section 3.1). This justifies the usefulness of the cubical theory we develop.

Unless otherwise stated, K denotes a cubical structure on M, and all chain and cochain groups on K are taken to have real coefficients. $[M]$ denotes the fundamental class of M given by the sum of n-dimensional cubes in K.

3.1 Defining a Cubical Structure on a Manifold

Throughout this chapter, we develop theory pertaining to cubical complexes. However, as we will see, a smooth manifold M elicits a structure that is "cubical" in nature, but whose cells are not standard cubes. Fortunately, we can easily relate this structure to a cubical complex via a diffeomorphism. Before offering a proof of this relation, we give a formal definition of what is meant by a "cubical structure."

Definition 3.1.1. Let M be an n-dimensional manifold. A **cubical structure on $\boldsymbol{M}$** is a regular cell complex $\mathcal{K}$ on M that satisfies the following.

1. For each homeomorphism f_c in Definition 2.1.3, f_c is also a cellular map with the standard n-cube, e_n, as its domain.

2. M admits coordinate charts ψ with the following two properties.

i. $\psi \circ f_c$ extends to a diffeomorphism on an open neighborhood of e_n.

ii. $\psi \circ f_c(e_n)$ is a convex polyhedron whose natural polyhedral cell structure is identified by $\psi \circ f_c$ with the natural cubical cell structure of c.

Remark 3.1.2. A cellular map is a continuous map between cell complexes that takes k-skeletons to k-skeletons for all k. Thus, all of the lower-dimensional cells of $\mathcal{K}$ work out by the cell structure of the standard n-cube. Consequently, we may restrict our definition to top-dimensional cells.

Momentarily, we will show that any smooth manifold admits a cubical structure. First, let's consider two examples of cubical structures. In the first, we will define a cubical structure on the 1-torus. In the second, we will show how the cubical structure on the 1-torus can be extended and defined on the 2-torus. An analogous technique can be used to define a cubical structure on an n-torus. Note, when we say "n-torus" we mean the connected sum of n 2-dimensional tori.

Example 3.1.3. Suppose M is the 1-torus, a smooth 2-dimensional manifold. We will define a cubical structure of squares (or 2-cubes) on M. The complex follows nicely from the standard view of the 1-torus that identifies opposite edges of a square as seen in Figure 3.1.

Create a cubical structure K on M by partitioning the 1-torus with evenly spaced vertical and horizontal lines. The 0-cells, or vertices, of K are the intersections of these vertical lines. Note that the vertices at the four corners of the 1-torus representation are identified. The 1-cells, or edges, of K are the horizontal or vertical line segments between two adjacent vertices.

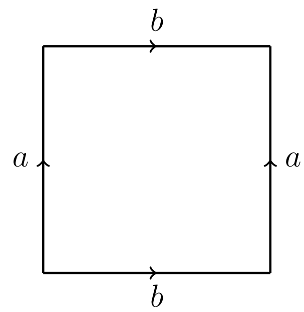


Figure 3.1: The 1-torus.

Note that the edges along the top and bottom are identified, as well as the edges along the left and right of M. One may assign an orientation to each of the edges as desired. The 2-cells, or faces, of K are the squares formed from the intersection of two adjacent vertical lines and two adjacent horizontal lines. The resultant cubical structure K is depicted in Figure 3.2.

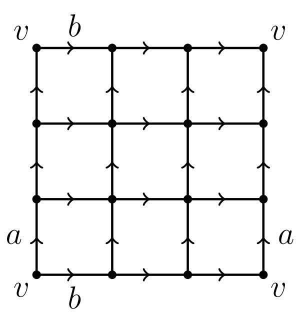


Figure 3.2: A cubical structure on the 1-torus.

Example 3.1.4. A technique similar to that used in Example 3.1.3 can be used to construct a cubical structure K on the 2-torus. Again, our complex will be of squares because the 2-torus is a 2-dimensional smooth manifold. We will represent the 2-torus by using 2 copies of the 1-torus square representation that are identified by cutting a small square out of each and attaching with identifications shown in Figure 3.3.

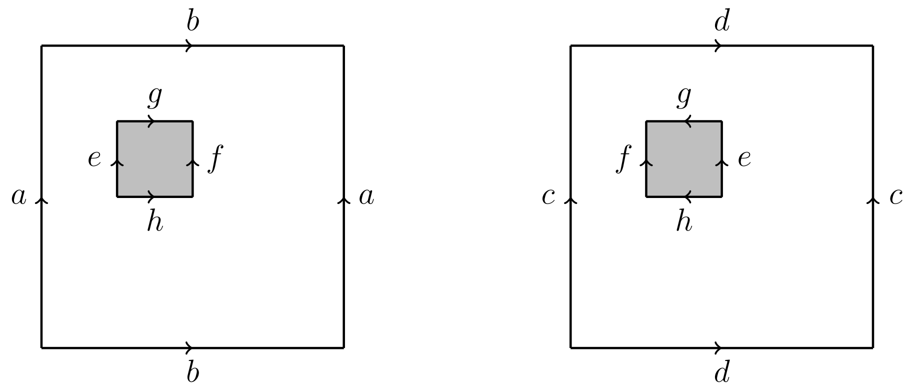


Figure 3.3: The 2-torus.

By using two layers of the 1-torus, we can easily lay down a square grid as discussed in Example 3.1.3 to create a cubical structure K. This complex is shown in Figure 3.4.

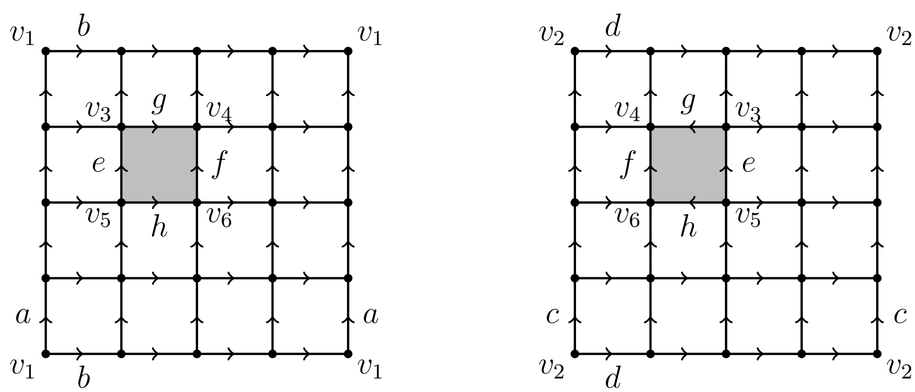


Figure 3.4: A cubical structure on the 2-torus.

A cubical structure on the n-torus can be constructed analogously by using n layers of the 1-torus and removing squares to attach successive layers.

We have now seen the existence of a cubical structure on the specific example of an n-torus. We can assert more generally that a cubical structure can always be defined on any smooth manifold M. The proof of this assertion hinges on the fact that an n-cube is diffeomorphic to a piece of an n-simplex. We can then borrow from the existence of a simplicial complex on any smooth manifold to construct our cubical structure K.

Theorem 3.1.5. *Any smooth manifold M admits a cubical structure.*

Proof. Let M be a smooth manifold. By the triangulation theorem in [17], there exists a smooth triangulation of M. The triangulation theorem also asserts that given the simplicial complex X of the triangulation, we have the following properties.

1. There is a homeomorphism φ of X onto M.

2. For n-simplex σ in X, there is a coordinate system ψ such that $\varphi(\sigma)$ remains on the interior of the coordinate neighborhood.

3. $\psi \circ \varphi$ is affine in σ.

We will show how to map a standard n-cube onto part of a standard n-simplex. Thus, the composition of this map with the map of a standard n-simplex into X followed by $\psi \circ \varphi$ will be a smooth map of a neighborhood of the cube to the coordinate chart of M in $\mathbb{R}^{n+1}$.

We will show details of the construction of a cube from a simplex for the case where $n = 2$. The general construction will follow analogously.

Consider the standard 2-simplex, F, as shown in Figure 3.5. Join the barycenter of each edge to the barycenter of F to create three 2-kites, one at each of the vertices in F as

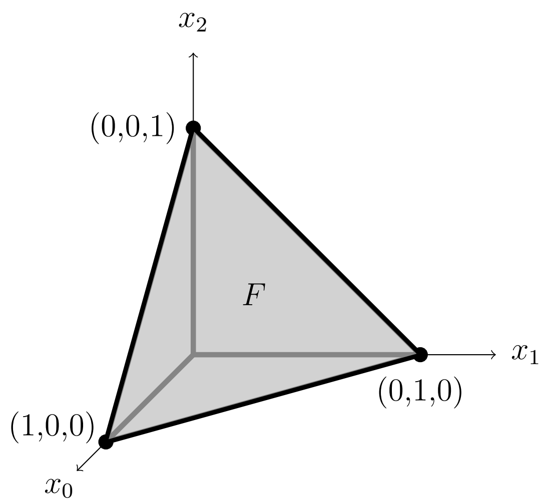


Figure 3.5: A standard 2-simplex, F.

pictured in Figure 3.6. We can smoothly map a cube to each of these kites. We will give the definition of the map of the standard 2-cube c onto the kite η nestled at (0,1,0), as highlighted in Figure 3.6. η has edges e_1, e_2, e_3, and e_4 formed from the intersection of the planes $x_2 = 0,\ x_0 = x_1,\ x_1 = x_2$, and $x_0 = 0$, respectively, with F.

Mapping c to η can be done by an orientation-preserving change of coordinates. This coordinate change, γ, associates the point (t_0, t_2) in c with the point (x_0, x_1, x_2) in η satisfying

$$x_0 = (1 - t_0)x_1, \qquad 0 \leq t_0 \leq 1$$

$$x_2 = t_2 x_1, \qquad 0 \leq t_2 \leq 1.$$

The relationship of c and η under γ is shown in Figure 3.7. For an explanation of why γ is orientation-preserving, see the general case below.

γ is a diffeomorphism between c and η by Lemma 3.1.7. Thus, γ followed by the composition

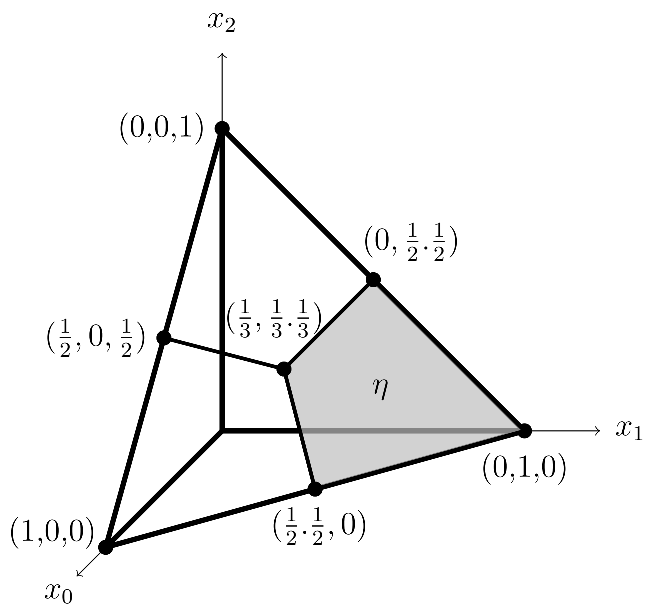


Figure 3.6: The kites of F.

of maps from the standard 2-simplex to its coordinate neighborhood in $\mathbb{R}^3$ is a smooth map defined on a neighborhood of c.

To generalize this construction, consider a standard n-simplex σ in $\mathbb{R}^{n+1}$. For some i, $0 \leq i \leq n$, let v_i be the vertex in σ where $x_i = 1$ and $x_j = 0$ for all $j \neq i, 0 \leq j \leq n$. We will define the coordinate change γ_i between the n-cube e_n and an n-kite η_i nestled at the vertex v_i.

η_i is determined by the intersection of the collection of equations $\{x_j = 0, x_j = x_i : 0 \leq j \leq n, j \neq i\}$ with σ. In particular, $\eta_i = \{(x_0, \ldots, x_n) : 0 \leq x_j \leq x_i$ for all j, $0 \leq j \leq n\}$. The coordinate change $\gamma_i : e_n \to \eta_i$ is orientation-preserving and associates the coordinates $(t_0, \ldots, \hat{t}_i, \ldots, t_n)$ in e_n with the coordinates $(x_0, \ldots, x_n)$ in η_i by

$$x_j = (1 - t_j)x_i, \quad 0 \leq t_j \leq 1, \qquad \text{if } j < i$$

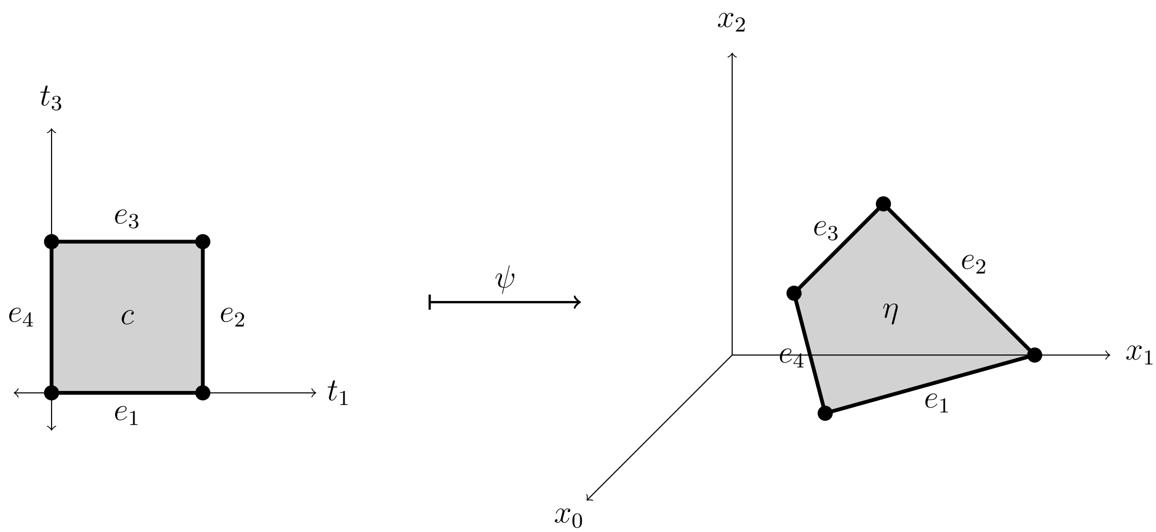


Figure 3.7: The relationship of c and η under γ.

$$x_j = t_j x_i, \quad 0 \le t_j \le 1, \qquad\qquad \text{if } j > i$$

for all $j, 0 \le j \le n, j \ne i$. Note: e_n lives in the $t_0 \cdots \hat{t}_i \cdots t_n$-hyperplane.

γ_i is an orientation-preserving diffeomorphism between e_n and η_i by Lemma 3.1.7 and Lemma 3.1.8. Hence, γ_i followed by the composition of the maps of the triangulation of M defines a diffeomorphism that extends to be defined on a neighborhood of the standard n-cube e_n within the hyperplane in which it sits, via an n-simplex. Thus, K satisfies property 2(i) of Definition 3.1.1.

To see that property 2(ii) is satisfied, we will show that the foundation of our construction, an n-kite η_i, is convex. The affine map from the n-simplex σ which contains η_i to its coordinate neighborhood will then preserve the convexity of η_i. Consequently, the image of the n-cube is a convex polyhedron in its coordinate neighborhood.

Choose an n-kite η_i. We will show that for any two points P and Q in η_i, with respective barycentric coordinates $(p_0, \ldots, p_n)$ and $(q_0, \ldots, q_n)$, the line segment $\overline{PQ}$ remains in η_i. Note, $\overline{PQ} = uP + (1-u)Q, \ 0 \leq u \leq 1$.

Recall $\eta_i = \{(x_0, \ldots, x_n) : 0 \leq x_j \leq x_i \text{ for all } j, \, 0 \leq j \leq n\}$. Thus, $p_j \leq p_i$ and $q_j \leq q_i$ for all j, $0 \leq j \leq n$. So,

$$up_j + (1-u)q_j \quad \leq \quad up_i + (1-u)q_i$$

for all j, $0 \leq j \leq n$, and $\overline{PQ}$ remains in η_i. Hence, η_i is convex and property 2 of Definition 3.1.1 is satisfied.

Thus, K is a cubical structure on M, as desired.

□

Remark 3.1.6. The construction of a cubical structure K on a smooth manifold M is local. We can, however, consider behavior on adjacent n-cubes because the diffeomorphism from an n-cube c to its coordinate neighborhood of M is defined on a neighborhood of c in the n-dimensional hyperplane in which c sits. In fact, we may use barycentric subdivision to widen our local view. Given an n-simplex σ, we can subdivide σ until we have the desired refinement. Then we can consider the n-cubes associated with the refined n-simplices. In this way we are able to guarantee we can view an n-cube and all of its surrounding, adjacent n-cubes in the same coordinate neighborhood. The importance of the behavior at adjacent n-cubes will be seen in Section 3.2.2 when we define cubical Whitney forms.

Lemma 3.1.7. *The map $\gamma_i : e_n \to \eta_i$ defined in Theorem 3.1.5 is a diffeomorphism for all i, $0 \le i \le n$.*

Proof. Because barycentric coordinates must sum to 1, γ_i can be reformulated as follows

$$\gamma_i\left((t_0, \ldots, \hat{t}_i, \ldots, t_n)\right) = \frac{1}{(1-t_0)+\cdots+(1-t_{i-1})+1+t_{i+1}+\cdots+t_n} \cdot (1-t_0, \ldots, 1-t_{i-1}, 1, t_{i+1}, \ldots, t_n).$$

To see that γ_i is injective, assume $\gamma_i\left((t_0, \ldots, \hat{t}_i, \ldots, t_n)\right) = \gamma_i\left((s_0, \ldots, \hat{s}_i, \ldots, s_n)\right)$. In the i^{th} component, we see that

$$\frac{1}{(1-t_0)+\cdots+(1-t_{i-1})+1+t_{i+1}+\cdots+t_n} = \frac{1}{(1-s_0)+\cdots(1-s_{i-1})+1+s_{i+1}+\cdots+s_n}.$$

Thus, the remaining component-wise equations yield

$$1-t_j = 1-s_j \qquad \text{if } j < i$$

$$t_j = s_j \qquad \text{if } j > i.$$

So, $(t_0, \ldots, \hat{t}_i, \ldots, t_n) = (s_0, \ldots, \hat{s}_i, \ldots, s_n)$ and γ_i is injective.

To see that γ_i is a surjection, consider an arbitrary point in η_i with barycentric coordinates $(x_0, \ldots, x_n)$. Note, $x_0 + \cdots + x_n = 1$. By definition of η_i, $0 < x_i \le 1$ and $0 \le x_j \le 1$ for each $j \ne i$. So, $\left(1-\frac{x_0}{x_i}, \ldots, 1-\frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right) \in e_n$. Thus,

$$\gamma_i\left((1-\frac{x_0}{x_i}, \ldots, 1-\frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})\right)$$

$$= \frac{1}{(1-(1-\frac{x_0}{x_i}))+\cdots+1-(1-\frac{x_{i-1}}{x_i}))+1+\frac{x_{i+1}}{x_i}+\cdots+\frac{x_n}{x_i}}$$

$$\cdot\left(1-(1-\frac{x_0}{x_i}),\ldots,1-(1-\frac{x_{i-1}}{x_i}),1,\frac{x_{i+1}}{x_i},\ldots,\frac{x_n}{x_i}\right)$$

$$=\frac{1}{\frac{x_0}{x_i}+\cdots+\frac{x_{i-1}}{x_i}+1+\frac{x_{i+1}}{x_i}+\cdots+\frac{x_n}{x_i}}\cdot\left(\frac{x_0}{x_i},\ldots,\frac{x_{i-1}}{x_i},1,\frac{x_{i+1}}{x_i},\ldots,\frac{x_n}{x_i}\right)$$

$$=\frac{1}{x_0+\cdots+x_{i-1}+x_i+x_{i+1}+x_n}\cdot(x_0,\ldots,x_{i-1},x_i,x_{i+1},\cdots,x_n)$$

$$=(x_0,\ldots,x_n),$$

and γ_i is surjective.

To see that γ_i is smooth, note that $0\leq t_j\leq 1$ for all $j\neq i$, $0\leq j\leq n$. So, we have

$$(1-t_0)+\cdots(1-t_{i-1})+1+t_{i+1}+\cdots+t_n\geq 1.$$

Thus, because γ_i is smooth in each component, it defines a smooth map from e_n to η_i.

Finally, to see that the inverse of γ_i is smooth, we will explicitly define it on a point in η_i with barycentric coordinates $(x_0,\ldots,x_n)$. Define $g:\eta_i\to e_n$ by

$$g\left((x_0,\ldots,x_n)\right)=\left(1-\frac{x_0}{x_i},\ldots,1-\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\ldots,\frac{x_n}{x_i}\right).$$

Note, because $(x_0,\ldots,x_n)\in\eta_i$, $x_i\neq 0$. To see that $g=\gamma_i^{-1}$, consider $g\circ\gamma_i$ and $\gamma_i\circ g$ given below.

$$g\circ\gamma_i\left((t_0,\ldots,\hat{t}_i,\ldots,t_n)\right)$$

$$=g\left(\frac{1}{(1-t_0)+\cdots(1-t_{i-1})+1+t_{i+1}+\cdots+t_n}\cdot(1-t_0,\ldots,1-t_{i-1},1,t_{i+1},\ldots,t_n)\right)$$

$$=(1-(1-t_0),\ldots,1-(1-t_{i-1}),t_{i+1},\ldots,t_n)$$

$$=(t_0,\ldots,\hat{t}_i,\ldots,t_n).$$

$\gamma_i \circ g\left((x_0, \ldots, x_n)\right)$

$$
\begin{aligned}
=&\gamma_i\left(\left(1-\frac{x_0}{x_i}, \ldots, 1-\frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)\right)\\
=&\frac{1}{(1-(1-\frac{x_0}{x_i})+\cdots+(1-(1-\frac{x_{i-1}}{x_i}))+1+\frac{x_{i+1}}{x_i}}\\
&\cdot\left(1-\big(1-\frac{x_0}{x_i}\big), \ldots, 1-\big(1-\frac{x_{i-1}}{x_i}\big), 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right)\\
=&\frac{1}{x_0+\cdots+x_{i-1}+x_i+x_{i+1}+x_n}\cdot(x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)\\
=&(x_0, \ldots, x_n).
\end{aligned}
$$

Thus, $g = \gamma_i^{-1}$. Furthermore, because $x_i \neq 0$, each component of γ^{-1} is smooth. Thus, γ_i^{-1} is also smooth.

Hence, γ_i is a diffeomorphism between an n-cube e_n and an n-kite η_i nestled at x_i. □

Lemma 3.1.8. *The map $\gamma_i : e_n \to \eta_i$ defined in Theorem 3.1.5 is orientation-preserving for all i, $0 \leq i \leq n$.*

Proof. By Lemma 3.1.7, γ_i is a diffeomorphism. In particular, the Jacobian determinant of γ_i is a continuous map from e_n to $\mathbb{R} - \{0\}$. To show that γ_i is orientation-preserving, fix a vertex v_i. We will show that the vector between v_i and any v_j, $i \neq j$, is oriented positively in the image of γ_i at v_i. Thus, the Jacobian determinant of γ_i will be positive at v_i. Because e_n is connected and the Jacobian determinant of γ_i is nonzero everywhere, it follows that γ_i is orientation-preserving at all points in e_n.

Consider an arbitrary coordinate t_j in e_n, $j \neq i$, $0 \leq t_j \leq 1$. Hold all other variables t_k constant at 1 if $k < i$ and 0 if $k > i$. We will use the formulation of γ_i given in Lemma 3.1.7 to argue our claim.

<u>Case 1</u>: t_j is such that $j < i$. Then $\gamma_i((1, \ldots, 1, t_j, 0, \ldots, 0))$ has nonzero entries only in the j^{th} and i^{th} spots. The j^{th} entry is $\dfrac{1-t_j}{2-t_j}$ and the i^{th} entry is $\dfrac{1}{2-t_j}$. Allowing t_j to run from 0 to 1 represents the motion of the vector between v_j and v_i. So we see that the j^{th} component decreases and the i^{th} component decreases, i.e. the motion is $v_i - v_j$.

<u>Case 2</u>: t_j is such that $j > i$. Again, $\gamma_i((1, \ldots, 1, t_j, 0, \ldots, 0))$ has nonzero entries only in the i^{th} and j^{th} spots. The i^{th} entry is $\dfrac{1}{1+t_j}$ and the j^{th} entry is $\dfrac{t_j}{1+t_j}$. Allowing t_j to run from 0 to 1 results in a decreasing i^{th} component and an increasing j^{th} component in the image of γ_i. This is the motion $v_j - v_i$.

In either case, the orientation vector between v_j and v_i points in the direction of increasing index value. Because j was arbitrary, the vectors in a standard oriented basis for the cube at $\gamma^{-1}(v_i)$ are mapped by the Jacobian of γ_i to the vectors in an ordered basis at v_i that agrees with the standard vector orientation of the simplex. Thus, we see that the Jacobian determinant, expressed via these bases, of γ_i is positive at the point v_i. Hence, it is positive at all points in e_n. In particular, γ_i is orientation-preserving on e_n.

$\square$

3.2 The Cubical Cup Product

The proof of Poincaré duality on a cell complex relies on a notion of transversality, typically realized through the transverse intersections of a cell complex with its dual. However, without reference to a dual complex, we rely on a nondegenerate pairing of cochains to provide our notion of transversality, namely a cup product.

In this section, we define the cubical cup product that we will ultimately use to define the cubical discrete Hodge star (Section 3.3.1) that is the Poincaré duality map over $\mathbb{R}$. The cup product we define is degenerate on the cochain level, but is nondegenerate on cohomology. The proof of nondegeneracy on cohomology, given in Section 3.2.3, relies on the borrowing of nondegeneracy from the smooth wedge product. Thus, we require a link between the cochains of K and smooth forms.

Cubical Whitney forms, defined in Section 3.2.2, provide this link. We show that the cup product of two cochains agrees with the wedge product of their Whitney forms. Because the cubical cup product is degenerate on the cochain level, its representation under the Whitney map W must also be degenerate. However, W provides a connection of cubical cohomology with de Rham cohomology. The proof of the de Rham version of Poincaré duality can be said to rely on transversality, which leads to nondegeneracy of the smooth wedge product at the level of all smooth differential forms (see [3]). Thus, on the cohomology level the pairing of Whitney forms is nondegenerate. Because we identify our cup product with this pairing, it follows that our cup product is also nondegenerate on the cohomology level.

Note that although Whitney forms are $\mathcal{L}^2$, we show that they are $\mathcal{L}^2$-differentiable, and hence the nondegeneracy of the "not quite" smooth wedge product may still be borrowed on the cohomology level.

One might wonder why we bother to define a new cup product when the singular cubical cup product defines a standard cellular cubical cup product. However, this cup product does not fit together with the cubical Whitney forms we define. Hence, we would seemingly have nothing to work with in proving its nondegeneracy. Although we define a new cup product that does not agree with the standard cup product on cochains, we show in Section 3.2.1 that our cup product agrees with the standard cellular cubical cup product on cohomology.

3.2.1 Defining a Cubical Cup Product

To define our cubical cup product, we must first introduce some notation.

Let K be a cubical structure on a closed, oriented n-dimensional manifold. Suppose $\sigma \in C_{p+q}(K)$ has standard orientation following the order of the variables $\{x_1, \ldots, x_{p+q}\}$. Let $\mathcal{F}_p = \{x_{i_1}, \ldots, x_{i_p}\}$ denote an arbitrary collection of p free basis variables from $\{x_1, \ldots, x_{p+q}\}$ ordered by ascending index values. Let $\{\mathcal{F}_p\}$ denote the collection of all possible collections of p free basis variables. Note: $\mathcal{F}_p^c = \{x_{i_{p+1}}, \ldots, x_{i_{p+q}}\}$ with variables in ascending index value order. Let $\mathcal{V} = \{v = (x_1, \ldots, x_{p+q}) : x_i \in \{0, 1\} \text{ for all } i\}$.

Let $v \in \mathcal{V}$. Suppose $\mathcal{F}_p = \{x_{i_1}, \ldots, x_{i_p}\}$ is given. Let $y_p(v)$ denote the p-face with free variables $x_{i_1}, \ldots, x_{i_p}$ and with remaining variables in $\mathcal{F}_p^c$ held constant according to their values at vertex v. Note: all variables and constants are assigned in their standard positions. Let $y_p^c(v)$ denote the q-face with free variables $x_{i_{p+1}}, \ldots, x_{i_{p+q}}$ and with remaining variables in $\mathcal{F}_p$ held constant with their values at vertex v. Again, all variables and constants are assigned in their standard positions.

Definition 3.2.1. Let $\alpha \in C^p(K), \beta \in C^q(K)$, and $\sigma \in C_{p+q}(K)$ with standard orientation $\{x_1, \ldots, x_{p+q}\}$. Then, the **cubical cup product** $\cup_c : C^p(K) \times C^q(K) \to C^{p+q}(K)$ is defined by

$$(\alpha \cup_c \beta)(\sigma) = \frac{1}{2^{p+q}} \sum_{\{x_{i_1},\ldots,x_{i_p}\} \in \{\mathcal{F}_p\}} \sum_{v \in \mathcal{V}} \operatorname{sgn}(x_{i_1}, \ldots, x_{i_{p+q}})\ \alpha(y_p(v))\ \beta(y_p^c(v)).$$

$\operatorname{sgn}(x_{i_1}, \ldots, x_{i_{p+q}})$ denotes the sign associated with rearranging $x_{i_1}, \ldots, x_{i_{p+q}}$ into the standard order $x_1, \ldots, x_{p+q}$.

Remark 3.2.2. If $p = 0$, the cup product simplifies to

$$(\alpha \cup_c \beta)(\sigma) = \frac{1}{2^{p+q}} \sum_{v \in \mathcal{V}} \alpha(y_p(v))\beta(y_p^c(v))$$

as there is only one way to "choose" zero free variables.

Theorem 3.2.3. $\cup_c$ *is a cup product on* K*, i.e. it satisfies the conditions of Definition 2.3.2.*

Proof. Property 1. Let σ_p and σ_q be basis elements of $C_p(K)$ and $C_q(K)$, respectively. Then, by the definition of $\cup_c$, $\hat{\sigma}_p \cup_c \hat{\sigma}_q$ will be nonzero only on $\sigma \in C_{p+q}(K)$ such that $y_p(v) = \sigma_p$ and $y_p^c(v) = \sigma_q$ for some choice of p-free variables $\mathcal{F}_p$ and for some vertex v in σ. Necessarily, $\sigma \subseteq \operatorname{St}(\sigma_p) \cdot \operatorname{St}(\sigma_q)$. Thus, $\hat{\sigma}_p \cup_c \hat{\sigma}_q$ is a $(p+q)$-form on $\operatorname{St}(\sigma_p) \cdot \operatorname{St}(\sigma_q)$, and Property 1 holds.

Property 2. Let $\sigma \in C_{p+q+1}(K)$. We will show

$$d(\alpha \cup_c \beta) = d\alpha \cup_c \beta + (-1)^p \alpha \cup_c d\beta$$

by calculating each term explicitly.

Throughout this argument, we will refer to a basis element $x_i \in \mathcal{F}_p^c$ or $x_i \in \mathcal{F}_{p+1}$. For the former case, we will assume that $x_i = x_{i_k}$ for some $k \in [p+1, \ldots, p+q+1]$. For the latter case, we will assume that $x_i = x_{i_{k'}}$ for some $k' \in [1, \ldots, p+1]$.

$$\begin{aligned}
d(\alpha \cup_c \beta)(\sigma) &= (\alpha \cup_c \beta)(\partial\sigma) \\
&= (\alpha \cup_c \beta)\Big(\sum_{x_i} (-1)^{i+1} (\sigma|_{x_i=1} - \sigma|_{x_i=0})\Big) \\
&= \frac{1}{2^{p+q}} \sum_{x_i} \sum_{\{\mathcal{F}_p : x_i \in \mathcal{F}_p^c\}} \\
&\quad \Big[\sum_{\mathcal{V}_{i,1}} \operatorname{sgn}(x_{i_1}, \ldots, \hat{x}_i, \ldots, x_{i_{p+q+1}})(-1)^{i+1} \alpha(y_p(v))\beta(y_p^c(v)\big|_{x_i=1}) \\
&\quad + \sum_{\mathcal{V}_{i,0}} \operatorname{sgn}(x_{i_1}, \ldots, \hat{x}_i, \ldots, x_{i_{p+q+1}})(-1)^{i} \alpha(y_p(v))\beta(y_p^c(v)\big|_{x_i=0})\Big].
\end{aligned}$$

Here $\mathcal{V}_{i,c}$ denotes the set $\{v \in \mathcal{V} : x_i = c\}$. Also, $\operatorname{sgn}(x_i, \ldots, \hat{x}_i, \ldots, x_{p+q+1})$ is the sign associated with rearranging $x_{i_1}, \ldots, \hat{x}_i, \ldots, x_{p+q+1}$ into the standard order $x_1, \ldots, \hat{x}_i, \ldots, x_{i_{p+q+1}}$.

$$\begin{aligned}
d\alpha \cup_c \beta(\sigma) &= \frac{1}{2^{p+q+1}} \sum_{\{\mathcal{F}_{p+1}\}} \sum_{v \in \mathcal{V}} \operatorname{sgn}(x_{i_1}, \ldots, x_{i_{p+q+1}}) d\alpha(y_{p+1}(v))\beta(y_{p+1}^c(v)) \\
&= \frac{1}{2^{p+q+1}} \sum_{x_i} \sum_{\{\mathcal{F}_{p+1} : x_i \in \mathcal{F}_{p+1}\}} \sum_{v \in \mathcal{V}} \operatorname{sgn}(x_{i_1}, \ldots, x_{i_{p+q+1}})(-1)^{k'+1} \\
&\quad \big[\alpha(y_{p+1}(v)\big|_{x_i=1}) - \alpha(y_{p+1}(v)\big|_{x_i=0})\big]\beta(y_{p+1}^c(v)) \\
&= \frac{1}{2^{p+q+1}} \sum_{x_i} \sum_{\{\mathcal{F}_{p+1} : x_i \in \mathcal{F}_{p+1}\}}
\end{aligned}$$

$$\left[\sum_{\mathcal{V}_{i,1}} \operatorname{sgn}(x_{i_1},\ldots,x_{i_{p+q+1}})(-1)^{k'+1}\alpha(y_{p+1}(v)\big|_{x_i=1})\beta(y^c_{p+1}(v)) \right. \quad (3.1)$$

$$\left. + \sum_{\mathcal{V}_{i,0}} \operatorname{sgn}(x_{i_1},\ldots,x_{i_{p+q+1}})(-1)^{k'}\alpha(y_{p+1}(v)\big|_{x_i=0})\beta(y^c_{p+1}(v))\right] \quad (3.2)$$

$$= \frac{1}{2^{p+q+1}} \sum_{x_i} \sum_{\{\mathcal{F}_p : x_i \in \mathcal{F}^c_p\}}$$

$$\left[\sum_{\mathcal{V}_{i,1}} \operatorname{sgn}(x_{i_1},\ldots,x_{i_{p+q+1}})(-1)^{k+1}\alpha(y_p(v))\beta(y^c_p(v)\big|_{x_i=1}) \right. \quad (3.3)$$

$$\left. + \sum_{\mathcal{V}_{i,0}} \operatorname{sgn}(x_{i_1},\ldots,x_{i_{p+q+1}})(-1)^{k}\alpha(y_p(v))\beta(y^c_p(v)\big|_{x_i=0})\right] \quad (3.4)$$

This last equality requires some explanation.

If $x_i \in \mathcal{F}_{p+1}$, recall $x_i = x_{i_{k'}}$ for some $k' \in [1,\ldots,p+1]$, and if $x_i \in \mathcal{F}^c_p$, then $x_i = x_{i_k}$ for some $k \in [p+1, p+q+1]$. So, in (3.1) and (3.2),

$$\begin{aligned} \operatorname{sgn}(x_{i_1},\ldots,x_{i_{p+q+1}}) &= \operatorname{sgn}(x_{i_1},\ldots,x_i,\ldots,x_{p+1},\ldots,x_{i_{p+q+1}}) \\ &= \operatorname{sgn}(x_{i_1},\ldots,\hat{x}_i,\ldots,x_{p+1},\ldots,x_{i_{p+q+1}})(-1)^{k'-i}. \end{aligned}$$

Further, in (3.3) and (3.4),

$$\begin{aligned} \operatorname{sgn}(x_{i_1},\ldots,x_{i_{p+q+1}}) &= \operatorname{sgn}(x_{i_1},\ldots,x_{p+1},\ldots,x_i,\ldots,x_{i_{p+q+1}}) \\ &= \operatorname{sgn}(x_{i_1},\ldots,x_{p+1},\ldots,\hat{x}_i,\ldots,x_{i_{p+q+1}})(-1)^{k-i}. \end{aligned}$$

Thus,

$$\operatorname{sgn}(x_{i_1},\ldots,x_i,\ldots,x_{p+1},\ldots,x_{i_{p+q+1}}) = \operatorname{sgn}(x_{i_1},\ldots,x_{p+1},\ldots,x_i,\ldots,x_{i_{p+q+1}})(-1)^{k-k'}$$

This explains the new expression of signs in the terms in (3.3) and (3.4) once we switch from a summand over $\{\mathcal{F}_{p+1} : x_i \in \mathcal{F}_{p+1}\}$ to a summand over $\{\mathcal{F}_p : x_i \in \mathcal{F}^c_p\}$.

Now, we will justify (3.1) = (3.3). A similar arguments establishes (3.2) = (3.4).

Choose an $\mathcal{F}_{p+1}$ and suppose $x_i \in \mathcal{F}_{p+1}$. Assume $v \in \mathcal{V}_{i,1}$. Then, $y_{p+1}(v)\big|_{x_i=1} = y_p(v)$ where y_p is determined from $\mathcal{F}_p = \mathcal{F}_{p+1} - \{x_i\}$. In particular, $x_i \in \mathcal{F}_p^c$. Similarly, $y_{p+1}^c(v) = y_p^c(v)\big|_{x_i=1}$. Thus, (3.1) = (3.3).

Continuing on with our last wedge product calculation,

$$
\begin{aligned}
(-1)^p \alpha \cup_c d\beta &= \frac{1}{2^{p+q+1}} \sum_{\mathcal{F}_p} \sum_{v \in \mathcal{V}} \operatorname{sgn}(x_{i_1}, \ldots, x_{i_{p+q+1}}) (-1)^p \alpha(y_p(v)) d\beta(y_p^c(v)) \\
&= \frac{1}{2^{p+q+1}} \sum_{x_i} \sum_{\{\mathcal{F}_p : x_i \in \mathcal{F}_p^c\}} \sum_{v \in \mathcal{V}} \operatorname{sgn}(x_{i_1}, \ldots, x_{i_{p+q+1}}) (-1)^p (-1)^{k-(p+1)} \\
&\quad \alpha(y_p(v)) \left[\beta(y_p^c(v)\big|_{x_i=1} - \beta(y_p^c(v)\big|_{x_i=0})\right] \\
&= \frac{1}{2^{p+q+1}} \sum_{x_i} \sum_{\{\mathcal{F}_p : x_i \in \mathcal{F}_p^c\}} \\
&\quad \left[\sum_{\mathcal{V}_{i,1}} \operatorname{sgn}(x_{i_1}, \ldots, x_{i_{p+q+1}}) (-1)^{k+1} \alpha(y_p(v)) \beta(y_p^c(v)\big|_{x_i=1}) \right. \qquad (3.5) \\
&\quad \left. + \sum_{\mathcal{V}_{i,0}} \operatorname{sgn}(x_{i_1}, \ldots, x_{i_{p+q+1}}) (-1)^{k} \alpha(y_p(v)) \beta(y_p^c(v)\big|_{x_i=0}) \right] \qquad (3.6)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d\alpha \cup_c \beta(\sigma) + (-1)^p \alpha \cup_c d\beta(\sigma) &= \frac{1}{2^{p+q+1}} \sum_{x_i} \sum_{\{\mathcal{F}_p : x_i \in \mathcal{F}_p^c\}} \\
&\quad \left[\sum_{\mathcal{V}_{i,1}} \operatorname{sgn}(x_{i_1}, \ldots, x_{i_{p+q}}) (-1)^{k+1} \alpha(y_p(v)) \beta(y_p^c(v)\big|_{x_i=1}) \right. \\
&\quad \left. + \sum_{\mathcal{V}_{i,0}} \operatorname{sgn}(x_{i_1}, \ldots, x_{i_{p+q+1}}) (-1)^{k} \alpha(y_p(v)) \beta(y_p^c(v)\big|_{x_i=0}) \right] \\
&= \frac{1}{2^{p+q}} \sum_{x_i} \sum_{\{\mathcal{F}_p : x_i \in \mathcal{F}_p^c\}} \left[\sum_{\mathcal{V}_{i,1}} \operatorname{sgn}(x_{i_1}, \ldots, \hat{x}_i, \ldots, x_{i_{p+q+1}}) \right. \\
&\quad (-1)^{i+1} \alpha(y_p(v)) \beta(y_p^c(v)\big|_{x_i=1})
\end{aligned}
$$

$$+\sum_{\mathcal{V}_{i,0}} \operatorname{sgn}(x_{i_1},\ldots,\hat{x}_i,\ldots,x_{i_{p+q+1}})$$
$$(-1)^i \alpha(y_p(v))\beta(y_p^c(v)\big|_{x_i=0})\Big]$$
$$= \quad d(\alpha \cup_c \beta)(\sigma), \text{ as desired.}$$

Note: The first equality above follows from adding (3.3) and (3.5) and adding (3.4) and (3.6). The middle equality above results from the following equality.

$$\operatorname{sgn}(x_{i_1},\ldots,x_{i_{p+q}}) = \operatorname{sgn}(x_{i_1},\ldots,\hat{x}_i,\ldots,x_{i_{p+q}})(-1)^{i-k} \tag{3.7}$$

Property 3. Let p be arbitrary and assume $\alpha \in C^p(K)$ is also arbitrary. Let σ be a single p-chain. Note that I is the constant 0-form that takes value 1 on all the vertices of K. Then,

$$\begin{aligned}
(I \cup_c \alpha)(\sigma) &= \frac{1}{2^p}\sum_{\mathcal{F}_0}\sum_{v\in V} \operatorname{sgn}(x_{i_1},\ldots,x_{i_p}) I(y_p(v))\alpha(y_p^c(v)) \\
&= \frac{1}{2^p}\sum_{v\in V} \operatorname{sgn}(x_1,\ldots,x_p) I(v)\alpha(x_1,\ldots,x_p) \\
&= \frac{1}{2^p}\sum_{v\in V} \alpha(x_1,\ldots,x_p) \\
&= \frac{1}{2^p}\left(2^p \alpha(x_1,\ldots,x_p)\right) \\
&= \alpha(x_1,\ldots,x_p).
\end{aligned}$$

Thus, $\gamma_{\cup_c} = 1$, and Property 3 holds.

We conclude that $\cup_c$ is a cup product, by definition.

□

As asserted by Whitney in [18], for any cup product, we may define an associated cap product via the relationship

$$(\alpha \cup \beta)(\sigma) = \beta(\sigma \cap \alpha).$$

Thus, we define the cubical cap product corresponding to $\cup_c$ as follows.

Definition 3.2.4. Let $\alpha \in C^p(K)$ and $\sigma \in C_{p+q}(K)$ with standard orientation $\{x_1, \ldots, x_{p+q}\}$. Then, the **cubical cap product** $\cap_c : C_{p+q}(K) \times C^p(K) \to C^q(K)$ is defined by

$$(\sigma \cap_c \alpha) = \frac{1}{2^{p+q}} \sum_{\{x_{i_1}, \ldots, x_{i_p}\} \in \{\mathcal{F}_p\}} \sum_{v \in \mathcal{V}} \text{sgn}(x_{i_1}, \ldots, x_{i_{p+q}})\ \alpha(y_p(v)) \cdot y_p^c(v).$$

One can easily check (by definition) that

$$(\alpha \cup_c \beta)(\sigma) \quad = \quad \beta(\sigma \cap_c \alpha) \tag{3.8}$$

for all $\alpha \in C^p(K)$, $\beta \in C^q(X)$, and $\sigma \in C_{p+q}(K)$.

$\cup_c$ and $\cap_c$ and the Standard Cellular Cubical Products

We have chosen our definition of the cubical cup product $\cup_c$ to fit nicely with the smooth wedge product of cubical Whitney forms defined momentarily in Section 3.2.2. As we will see, Whitney forms are defined over the entire n-cube. Thus, Whitney forms will not see a distinction between the vertices of a cube. Hence, a cubical product that is calculated over all of the vertices in the n-cube is necessary to fit together with Whitney forms. In this section, we show that even though $\cup_c$ evaluates over all vertices of the cube, it agrees with the standard cubical cup product (which evaluates on a subset of the vertices) on cohomology. This result is a consequence of Theorem 2.3.10, Theorem 2.3.11, and Theorem 2.2.21.

Recall the definition of the singular cubical cup product, with notations made clear in Section 2.3.

Definition 3.2.5. For singular cochains $\alpha \in S^p(K)$ and $\beta \in \mathcal{S}^q(K)$, the **singular cubical cup product** $\cup : S^p(K) \times S^q(K) \to S^{p+q}(K)$ is defined by

$$(\alpha \cup \beta)(\sigma) = \sum_H \rho_{HK}\ \alpha(\sigma \circ \lambda_H^0) \cdot \beta(\sigma \circ \lambda_K^1),$$

where $\sigma \in S_{p+q}(K)$ and $\rho_{HK} = \mathrm{sgn}(h_1, \ldots, h_p, k_1, \ldots, k_q)$.

Thus, via Theorem 2.2.21 and Remark 2.2.22, the standard cellular cubical cup product can be defined on a standard $(p+q)$-cube as follows.

Definition 3.2.6. For cellular cochains $\alpha \in C^p(K)$ and $\beta \in C^q(K)$, the **standard cellular cubical cup product** $\cup : C^p(K) \times C^q(K) \to C^{p+q}(K)$ is defined by

$$(\alpha \cup \beta)(x_1, \ldots, x_{p+q}) = \sum_H \rho_{HK}\ \alpha(\lambda_H^0(x_{h_1}, \ldots, x_{h_p})) \cdot \beta(\lambda_K^1(x_{k_1}, \ldots, x_{k_q})).$$

Note that the summation over H can be reinterpreted as the summation over choices of p-free variables, $\mathcal{F}_p$.

Via the relationship $(\alpha \cup \beta)(\sigma) = \beta(\sigma \cap \alpha)$, we may define the corresponding standard cubical cap product.

Definition 3.2.7. For $\alpha \in C^p(K)$ and $\sigma = (x_1, \ldots, x_{p+q}) \in C_{p+q}(K)$, the **standard cellular cubical cap product** $\cap : C_{p+q}(K) \times C^p(K) \to C_q(K)$ is defined by

$$(\sigma \cap \alpha) = \sum_H \rho_{HK}\ \alpha(\lambda_H^0(x_{h_1}, \ldots, x_{h_p})) \cdot \lambda_K^1(x_{k_1}, \ldots, x_{k_q}).$$

Theorem 3.2.8. *The standard cellular cubical cup product and the cubical cup product defined in Definition 3.2.1 agree on cohomology.*

Proof. By Theorem 2.3.11, it suffices to show that $\gamma_\cup = 1$. Let $p \geq 0$ be arbitrary and let $\alpha \in C^p(K)$. Then,

$$\begin{aligned} I \cup \alpha(x_1, \ldots, x_p) &= \operatorname{sgn}(1, \ldots, p)\ I(0, \ldots, 0) \cdot \alpha(x_1, \ldots, x_p) \\ &= \alpha(x_1, \ldots, x_p). \end{aligned}$$

Thus, $\gamma_\cup = 1$ and the result follows. □

Theorem 3.2.9. *The standard cellular cubical cap product the cubical cap product defined in Definition 3.2.4 agree on homology.*

Proof. $\gamma_\cap = \gamma_\cup = 1$. Thus, because $\gamma_{\cap_c} = 1$, Theorem 2.3.10 gives the desired result. □

3.2.2 Cubical Whitney Forms

We will see momentarily that $\cup_c$ is a nondegenerate pairing of $H^p(K)$ and $H^{n-p}(K)$. This result hinges on borrowing nondegeneracy from the smooth wedge product. To utilize this tool, we must first define cubical Whitney forms and establish the relationship of the cubical cup product with the smooth wedge product.

We will define the Whitney form of a basis element. The definition extends linearly. We will use the notation $\hat{\tau}_p \in C_p(K)$ for the p-form that is 1 on τ_p and 0 on all other p-chains.

Definition 3.2.10. Suppose τ_p is a basis element of $C^p(K)$. Then $\tau_p = y_p(v)$ for some n-cell σ, $\mathcal{F}_p$, and v a vertex in σ. Define the $\mathcal{L}^2$ **cubical Whitney form** of τ_p as follows

$$W\hat{\tau}_p = \left(\prod_{j=p+1}^{n} \left[1 - v(x_{i_j}) + (-1 + 2v(x_{i_j}))x_{i_j} \right] \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

The exterior product $dx_{i_1}, \ldots, dx_{i_p}$ given above is the restriction of the smooth exterior product on each n-cell. Below we discuss why a cubical Whitney form is $\mathcal{L}^2$.

$W\hat{\tau}_p$ will be zero on all n-cubes where τ_p is not a p-chain in that cube, and hence on these n-cubes' respective polyhedra on the manifold. This raises concern for the behavior of $W\hat{\tau}_p$ at the boundaries of adjacent polyhedra. Because the coordinates of each polyhedron are defined on a neighborhood containing it, a shared boundary will be seen in the respective coordinates of both polyhedra. For this reason, we can, and will, view polyhedra sharing a boundary as adjacent in their n-cube representations for the purposes of understanding the behavior of Whitney forms across boundaries.

Consider a cube, τ_n, that contains τ_p. We will show that the $\mathcal{L}^2$ exterior derivative of $W\hat{\tau}_p$ exists in all directions across the boundary of τ_n. To do so, let's first describe the behavior at the boundary between τ_n and adjacent cubes.

Recall that τ_p has free variables $x_{i_1}, \ldots, x_{i_p}$ and constant values assigned for $x_{i_{p+1}}, \ldots, x_{i_n}$. The behavior of $W\hat{\tau}_p$ as you move in the $+x_i$-direction can be described in two cases: (1) x_i is free in τ_p or (2) x_i is constant in τ_p. For both cases, we will move from τ_n into an adjacent cube τ_n' in the $+x_i$-direction. $\tau_n\big|_{x_i=1}$ is the $(n-1)$-face that τ_n and τ_n'.

<u>Case 1</u>: Suppose $x_i = x_{i_j}$ for some j where $1 \leq j \leq p$. Then, τ_p is not a p-chain in τ_n' since

x_i is free in τ_p. So, $W\hat{\tau}_p$ is 0 on τ_n', and thus has a jump discontinuity in this direction at $\tau_n\big|_{x_i=1}$. The graph of $W\hat{\tau}_p$ looks like a step function across $\tau_n\big|_{x_i=1}$, whose step value over τ_n decreases from 1 to 0 as you move away from τ_p (see Figure 3.8).

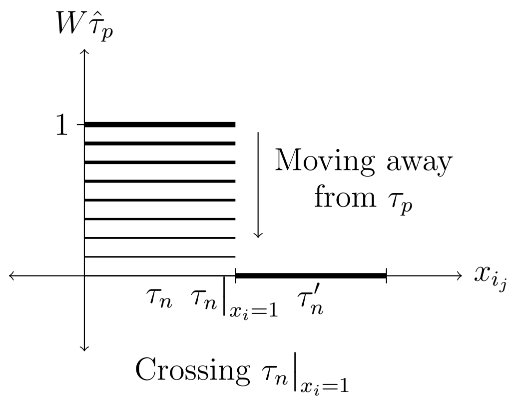


Figure 3.8: $W\hat{\tau}_p$ at adjacent cubes in the $+x_{i_j}$-direction, $1 \leq j \leq p$

Case 2: Suppose $x_i = x_{i_j}$ for some j where $p+1 \leq j \leq n$. Then τ_p is a p-chain in τ_n' since x_i is held constant in τ_p. Thus, $W\hat{\tau}_p$ is nonzero in τ_n'. In fact, the graph of $W\hat{\tau}_p$ across τ_n and τ_n' in the $+x_i$-direction is a peak, with its apex achieved on $\tau_n\big|_{x_i=1}$. It has height 1 when crossing $\tau_n\big|_{x_i=1}$ at τ_p and, if $\tau_p \neq \tau_n\big|_{x_i=1}$, its maximum height decreases to 0 as you move across $\tau_n\big|_{x_i=1}$ away from τ_p. Note that these peaks have equal and opposite signed slope on either side of its apex. (see Figure 3.9).

Thus, between any cube adjacent to τ_n, $W\hat{\tau}_p$ will either have a jump discontinuity or a peak. We are now ready to find $\dfrac{\partial W\hat{\tau}_p}{\partial x_i}$ for all i, $1 \leq i \leq n$.

Case 1: Suppose $x_i = x_{i_j}$ for some j, $1 \leq j \leq p$. Then x_i is free in τ_p and hence $W\hat{\tau}_p$ contains dx_i by definition of cubical Whitney forms. Because $dx_i \wedge dx_i = 0$, we have $\dfrac{\partial W\hat{\tau}_p}{\partial x_i} = 0$.

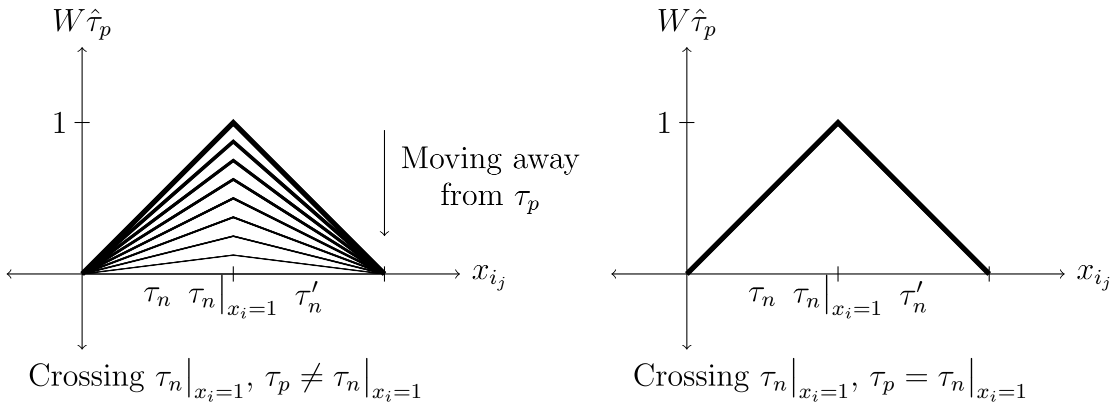


Figure 3.9: $W\hat{\tau}_p$ at adjacent cubes in the $+x_{i_j}$-direction, $p+1 \leq j \leq n$

Thus, $\dfrac{\partial W\hat{\tau}_p}{\partial x_i}$ does not appear in $dW\hat{\tau}_p$, and the issue of jump discontinuity in this direction is avoided.

Case 2: Suppose $x_i = x_{i_j}$ for some j, $p+1 \leq j \leq n$. Then x_i is constant in τ_p and hence $W\hat{\tau}_p$ peaks in the $+x_i$-direction. Because the slope of the peak is equal and opposite signed on either side of the apex, $\dfrac{\partial W\hat{\tau}_p}{\partial x_i}$ will be a step function, with steps at $\pm c$, where c is the slope of the peak. So for example, if we are differentiating across the shared boundary of the adjacent cubes at τ_p, the steps will have value ± 1 (see Figure 3.10). To show that this step function is the $\mathcal{L}^2$-derivative of $W\hat{\tau}_p$ in the $+x_i$-direction, we will show that it satisfies the weak definition of the $\mathcal{L}^2$-derivative (by Friedrichs, the weak derivative agrees with the strong derivative, see [8]).

We will show the calculation for the case where the apex has height 1. All other peaks have similar calculations (see Figure 3.11). Suppose $\varphi \in C_c^\infty$. By definition of the weak

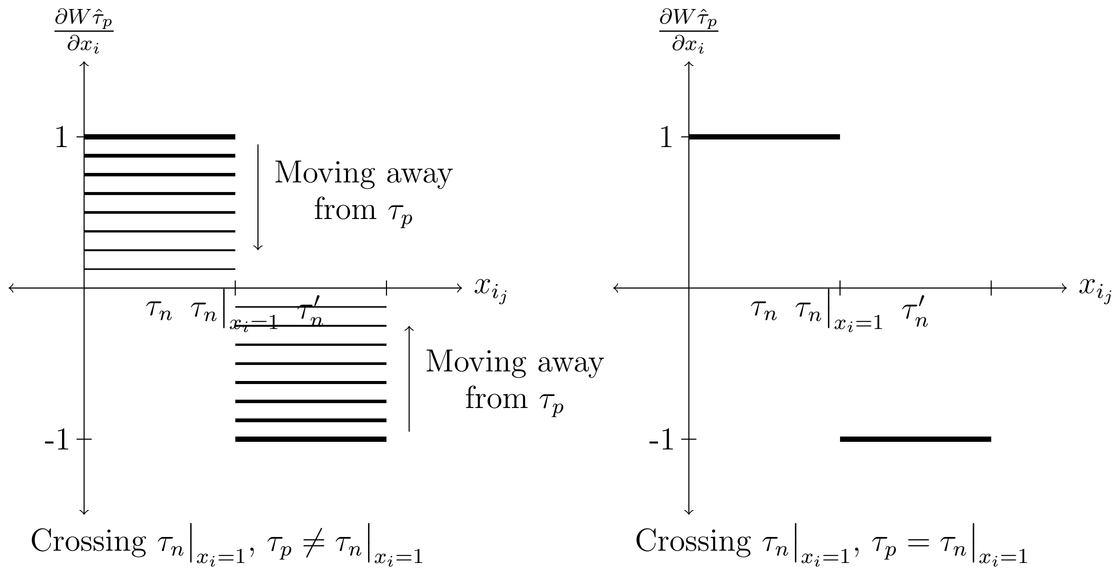


Figure 3.10: $W\hat{\tau}_p$ at adjacent cubes in the $+x_{i_j}$-direction, $1 \leq j \leq p$

$\mathcal{L}^2$-derivative,

$$\begin{aligned}
\int_0^2 \frac{\partial W\hat{\tau}_p}{\partial x_i} \cdot \varphi(x_i)\ dx_i &= -\int_0^2 W\hat{\tau}_p \cdot \varphi'(x_i)\ dx_i \\
&= -\int_0^1 x_i\varphi'(x_i)\ dx_i - \int_1^2 (2-x_i)\varphi'(x_i)\ dx_i \\
&= -\left[x_i\varphi'(x_i)\Big|_0^1 - \int_0^1 \varphi(x_i)\ dx_i \right] - \left[(2-x_i)\varphi(x_i)\Big|_1^2 + \int_1^2 \varphi(x_i)\ dx_i \right] \\
&= \left[\varphi(1) + \int_0^1 \varphi(x_i)\ dx_i \right] - \left[\varphi(1) + \int_1^2 \varphi(x_i)\ dx_i \right] \\
&= \int_0^1 \varphi(x_i)\ dx_i - \int_1^2 \varphi(x_i)\ dx_i \\
&= \int_0^2 g(x_i) \cdot \varphi(x_i)\ dx_i
\end{aligned}$$

where

$$g(x_i) = \begin{cases} 1, & \text{if } 0 \leq x_i \leq 1 \\ -1, & \text{if } 1 \leq x_i \leq 2 \end{cases}.$$

Thus, g is the $\mathcal{L}^2$-derivative of $W\hat{\tau}_p$ in the $+x_i$-direction. Note also that g is the step function depicted in Figure 3.11, i.e. $g = \dfrac{\partial W\hat{\tau}_p}{\partial x_i}$.

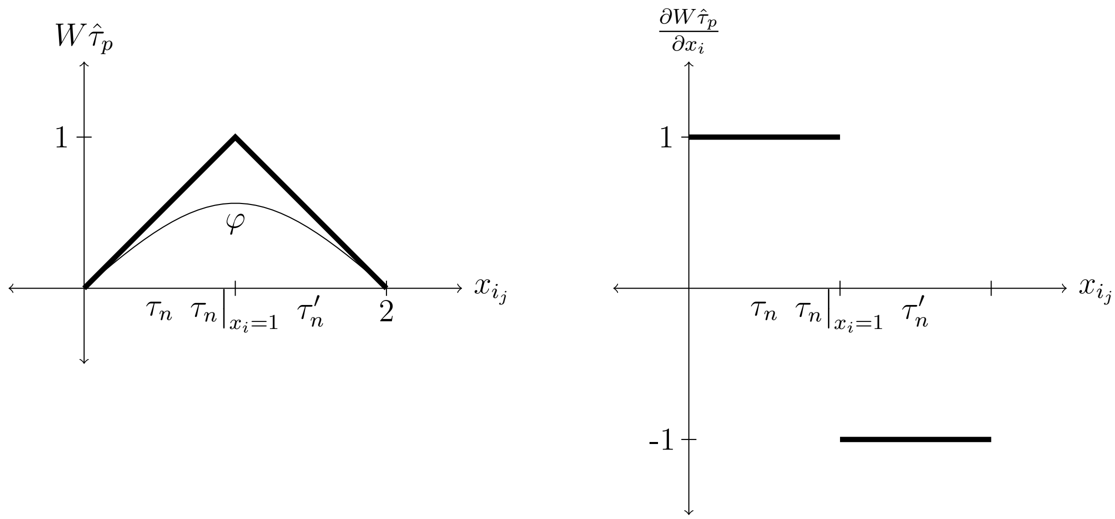


Figure 3.11: Crossing $\tau_n\big|_{x_i=1}$ at τ_p in the $+x_{i_j}$-direction, $1 \leq j \leq p$

A similar argument shows that regardless of each peak's apex height, $W\hat{\tau}_p$ is $\mathcal{L}^2$-differentiable in the $+x_i$-direction when x_i is constant in τ_p.

So, we see that for all x_i, $\dfrac{\partial W\hat{\tau}_p}{\partial x_i}$ exists and is the $\mathcal{L}^2$-derivative. Because τ_p was arbitrary, we conclude that cubical Whitney forms are both $\mathcal{L}^2$ and $\mathcal{L}^2$-differentiable, fitting their local definition nicely into the global cubical structure.

Theorem 3.2.11. *Let $p \geq 0$ be arbitrary. Suppose $\sigma \in C_p(K)$ and $\alpha \in C^p(K)$. Then,*

$$\alpha(\sigma) = \int_\sigma W\alpha.$$

Proof. By linearity, it suffices to prove the claim on a basis element $\hat{\tau} \in C^p(K)$ for some cell $\tau \in C_p(K)$. By definition, $\hat{\tau}(\tau) = 1$, and $\hat{\tau}$ takes the value 0 on all other p-cells. On any

n-cube containing τ,

$$W\hat{\tau} = \left(\prod_{j=p+1}^{n} \left[1 - v(x_{i_j}) + (-1 + 2v(x_{i_j}))x_{i_j} \right] \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

On τ, each x_{i_j} is held constant at its value at v for all j, $p+1 \leq j \leq n$. In other words, $x_{i_j} = v(x_{i_j})$, $p+1 \leq j \leq n$. So,

$$\begin{aligned} \int_\tau W\hat{\tau} &= \int_\tau \left(\prod_{j=p+1}^{n} \left[1 - v(x_{i_j}) + (-1 + 2v(x_{i_j}))v(x_{i_j}) \right] \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p} \\ &= \int_\tau \left(\prod_{j=p+1}^{n} \left[1 - 2v(x_{i_j}) + 2v(x_{i_j})^2 \right] \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p} \\ &= \int_\tau dx_{i_1} \wedge \cdots \wedge dx_{i_p} \\ &= 1. \end{aligned}$$

Note that $1 - 2v(x_{i_j}) + 2v(x_{i_j})^2 = 1$ for both $v(x_{i_j}) = 0$ and $v(x_{i_j}) = 1$. So, $\hat{\tau}$ and $W\hat{\tau}$ agree on τ.

By definition, $W\hat{\tau}$ is zero on all p-chains that are not in an n-cube for which τ is a face. To see that $\int_\sigma W\hat{\tau} = 0$ for $\sigma \in C_p(K)$, $\sigma \neq \tau$, when σ and τ appear in the same n-cube, we simply observe the following. If $\sigma \neq \tau$ is a p-chain in K, then x_{i_ℓ} must be constant in σ for some ℓ, $1 \leq \ell \leq p$. Thus, $dx_{i_\ell} = 0$ and $\int_\sigma W\hat{\tau} = 0$.

Hence,

$$\hat{\tau}(\sigma) = \int_\sigma W\hat{\tau}$$

for all $\sigma \in C_p(K)$.

□

Theorem 3.2.12. *Suppose $\sigma \in C_{p+q}(K)$, $\alpha \in C^p(K)$, and $\beta \in C^q(K)$. Then,*

$$(\alpha \cup_c \beta)(\sigma) = \int_\sigma W\alpha \wedge W\beta.$$

Remark 3.2.13. This theorem gives a direct relationship between the cubical cup product $\cup_c$ and the smooth wedge product $\wedge$. In particular, it further verifies that $\cup_c$ is in fact *the* cubical cup product.

Proof. It suffices to show the claim is true on basis elements. In particular, we will focus on basis elements that share a single vertex as these are the only candidates for a nonzero cubical cup product.

Choose p free variables and let v_0 be a vertex in σ. Define basis elements $\tau_p = y_p(v_0)$ and $\tau_q = y_p^c(v_0)$ from the chosen p variables. Then we have the following.

$$\begin{aligned}
(\hat{\tau}_p \cup_c \hat{\tau}_q)(\sigma) &= \frac{1}{2^{p+q}} \sum_{\mathcal{F}_p} \sum_{v \in \mathcal{V}} \operatorname{sgn}(x_{i_1}, \cdots, x_{i_{p+q}})\ \hat{\tau}_p(y_p(v))\ \hat{\tau}_q(y_p^c(v)) \\
&= \frac{1}{2^{p+q}} \operatorname{sgn}(x_{i_1}, \cdots, x_{i_{p+q}})\ \hat{\tau}_p(y_p(v_0))\ \hat{\tau}_q(y_p^c(v_0)) \\
&= \frac{1}{2^{p+q}} \operatorname{sgn}(x_{i_1}, \cdots, x_{i_{p+q}}).
\end{aligned}$$

On the other hand, we have the product of Whitney forms.

$$\begin{aligned}
\int_\sigma W\hat{\tau}_p \wedge W\hat{\tau}_q &= \int_\sigma \Bigg[\left(\prod_{j=p+1}^{p+q} \left[1 - v_0(x_{i_j}) + (-1 + 2v_0(x_{i_j}))x_{i_j}\right] \right) dx_{i_1} \wedge \cdots \wedge dx_{i_p} \\
&\quad \wedge \left(\prod_{j=1}^{p} \left[1 - v_0(x_{i_j}) + (-1 + 2v_0(x_{i_j}))x_{i_j}\right] \right) dx_{i_{p+1}} \wedge \cdots \wedge dx_{i_{p+q}} \Bigg] \\
&= \int_\sigma \operatorname{sgn}(x_{i_1}, \cdots, x_{i_{p+q}})
\end{aligned}$$

$$
\begin{aligned}
& \cdot \left(\prod_{j=1}^{p+q} \left[1 - v_0(x_{i_j}) + (-1 + 2v_0(x_{i_j}))x_{i_j} \right] \right) dx_1 \wedge \cdots \wedge dx_{p+q} \\
= \;& \operatorname{sgn}(x_{i_1}, \cdots, x_{i_{p+q}}) \\
& \cdot \int_0^1 \cdots \int_0^1 \left(\prod_{j=1}^{p+q} \left[1 - v_0(x_{i_j}) + (-1 + 2v_0(x_{i_j}))x_{i_j} \right] \right) dx_1 \cdots dx_{p+q} \\
= \;& \operatorname{sgn}(x_{i_1}, \cdots, x_{i_{p+q}}) \prod_{i=1}^{p+q} \left[\int_0^1 \left[1 - v_0(x_i) + (-1 + 2v_0(x_i))x_i \right] dx_i \right] \\
= \;& \operatorname{sgn}(x_{i_1}, \cdots, x_{i_{p+q}}) \prod_{i=1}^{p+q} \frac{1}{2} \\
= \;& \operatorname{sgn}(x_{i_1}, \cdots, x_{i_{p+q}}) \frac{1}{2^{p+q}}.
\end{aligned}
$$

Thus,

$$
(\hat{\tau}_p \cup_c \hat{\tau}_q)(\sigma) = \int_\sigma W\hat{\tau}_p \wedge W\hat{\tau}_q
$$

and the claim is proved.

□

3.2.3 Nondegeneracy of the Cubical Cup Product

As aforementioned, our notion of transversality without reference to a dual complex comes from a nondegenerate pairing. In this section, we show that the cubical cup product $\cup_c$ is this nondegenerate pairing on the cohomology level. The key ingredient to establishing this assertion is the borrowing of nondegeneracy from the smooth case. Our link to this nondegeneracy is cubical Whitney forms. As mentioned in the section introduction, W makes the connection of cubical cohomology with de Rham cohomology, where the de Rham version of Poincaré duality is nondegeneracy induced by the smooth wedge product. Theorem 3.2.12 identifies the smooth wedge product with our cubical cup product via W. Thus, in the event that the Whitney map is surjective, we may borrow nondegeneracy of the smooth wedge product on cohomology. We prove the surjectivity of W by showing that its composition with the de Rham map is the identity on cohomology. Although the proof of this assertion is not stated in detail by Wilson in [19], his comments provided the intuition for the proofs that we give in this section.

Theorem 3.2.14. *The smooth wedge product is a nondegenerate pairing of $\Omega^k(M)$ and $\Omega^{n-k}(M)$ for all k.*

Proof. [3] serves as a reference for the proof we give. Choose an arbitrary nonzero differential k-form α. Then there is a point p at which α is nonzero. Use the coordinates in a neighborhood p. We may write α as

$$\alpha = \sum_I f_I dx_I.$$

where I is a k-tuple given by $i_1, \dots, i_k$ and dx_I is the wedge product $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Because α is nonzero at p, $f_{I_0}(p) \neq 0$ for some k-tuple I_0. By continuity, there is a neighborhood U of p such that $f_{I_0}(x)$ has strictly the same sign as $f_{I_0}(p)$ for all $x \in U$.

Choose φ to be a nonnegative bump function such that $\varphi(p) > 0$ and φ is supported in U. Let I_0^c denote the $(n-k)$-tuple complementary to I_0. Then,

$$\alpha \wedge \varphi \; dx_{I_0^c} = f_{I_0} \varphi \; dx_{I_0} \wedge dx_{I_0^c}.$$

Note that for all k-tuples $J \neq I_0$, there is a j_ℓ in J that appears in I_0^c. Thus, $dx_J \wedge dx_{I_0^c} = 0$ for all $J \neq I_0$.

So,

$$\begin{aligned}\int_M \alpha \wedge \varphi \; dx_{I_0^c} &= \int_M f_{I_0}\varphi \; dx_{I_0} \wedge dx_{I_0^c} \\ &= \int_U f_{I_0}\varphi \; dx_{I_0} \wedge dx_{I_0^c} \qquad\qquad (\text{supp } \varphi \subseteq U) \\ &\neq 0.\end{aligned}$$

because $f_{I_0}(x)\varphi(x)$ has strictly the same sign as $f_{I_0}(p)$ for all $x \in U$. Thus, we have shown that every nonzero form in $\Omega^k(M)$ has a nonzero pairing with some form in $\Omega^{n-k}(M)$, i.e. $\wedge$ is nondegenerate. □

.

Definition 3.2.15. For any p, define the **de Rham map** $R : \Omega^p(M) \to C^p(K)$ by

$$(R\omega)(c) = \int_c \omega.$$

Lemma 3.2.16. *R and W are chain maps with respect to d_Ω and d_K.*

Proof. Suppose $\omega \in \Omega^p(M)$ and $c \in C_{p+1}(K)$. Then,

$$(d_K(R\omega))(c) = R\omega(\partial c) = \int_{\partial c} \omega$$

and

$$(R(d_\Omega\omega))(c) = \int_c d_\Omega\omega = \int_{\partial c} \omega$$

by Stokes' Theorem. Hence, $d_K \circ R = R \circ d_\Omega$.

Now, suppose $\alpha \in C^p(K)$ and again $c \in C_{p+1}(K)$. Then,

$$\int_c W(d_K\alpha) = (d_K\alpha)(c) = \alpha(\partial c).$$

Note the first equality is by Theorem 3.2.11. Also by Theorem 3.2.11,

$$\int_c d_\Omega \circ W\alpha = \int_{\partial c} W\alpha = \alpha(\partial c).$$

Thus, $W \circ d_K = d_\Omega \circ W$, and R and W are chain maps. □

Lemma 3.2.17. *R and W induce isomorphisms on their respective cohomology groups of all orders, $H^*_{dR}(M)$ and $H^*(K)$.*

Proof. For an arbitrary $p \geq 0$, it suffices to show $R \circ W$ and $W \circ R$ are the identity maps on $H^p(K)$ and $H^p_{dR}(M)$, respectively.

Suppose $\alpha \in C^p(K)$. Then,

$$(R \circ W)(\alpha)(c) = \int_c W\alpha = \alpha(c)$$

by Theorem 3.2.11.

Thus, $R \circ W$ is the identity map on $C^p(K)$. By Lemma 3.2.16, R and W are chain maps, and hence $R \circ W$ is well defined on $H^p(K)$. Thus, $R \circ W$ is the identity map on the cohomology level as well.

Now suppose $\omega_s \in \Omega^p(M)$ and $R\omega_s = \omega$ as an element of $C^p(K)$. Then,

$$(W \circ R)(\omega_s)(c) = \int_c W\omega$$

$$= \omega(c) \qquad \text{(Theorem 3.2.11)}$$

$$= \int_c \omega_s \qquad \text{(definition of } R). \tag{3.9}$$

Note: this is not quite the identity on $\Omega^p(M)$ because ω is only defined on cellular chains, whereas ω_s is a smooth form that lives both on and off cellular chains.

By Lemma 3.2.16, $W \circ R(\omega_s)$ is well-defined on cohomology. So, by (3.9), $W \circ R(\omega_s)$ and ω_s have the same values on every homology class. Thus, they represent the same cohomology class, i.e. $W \circ R$ is the identity map on $H^p_{dR}(M)$.

Thus, R and W induce isomorphisms between $(H^p_{dR}(M), d_\Omega)$ and $(H^p(K), d_K)$.

□

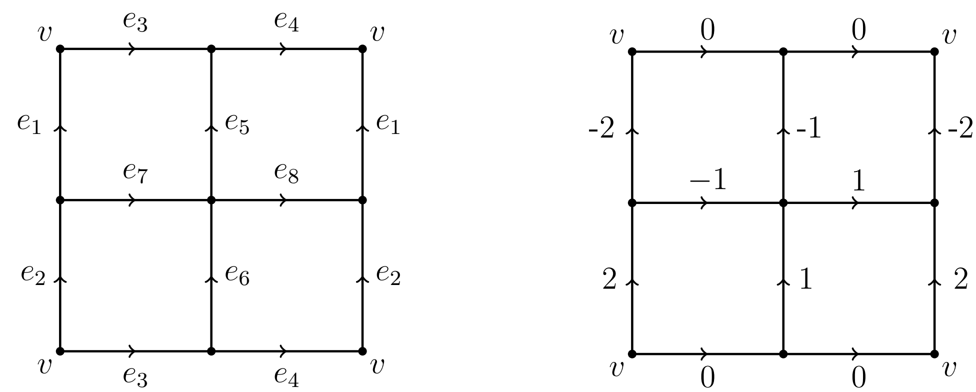
We have now seen that $W \circ R$ is not the identity map on $\Omega^p(K)$ when working on the cochain level. In fact, W cannot be a surjection of $C^p(K)$ by dimension considerations; the space of smooth differential forms (a subset of $\mathcal{L}^2$ differential forms) is infinite-dimensional, whereas $C^p(K)$ is finite-dimensional. Furthermore, W is injective on the cochain level by definition and by Theorem 3.2.11. So, if W was also surjective, it would transfer the nondegeneracy of the smooth wedge product to the cubical cup product on the cochain level via Theorem 3.2.12. However, we cannot say in general that the cubical cup product is nondegenerate on the cochain level. Consider the following example on the 2-dimensional 1-torus.

Example 3.2.18. Let K be the cubical structure on the 1-torus shown in Figure 3.12a, with edges e_i, $1 \leq i \leq 8$. Suppose $\alpha \in C^1(K)$ has values on the edges of K as shown in Figure 3.12b. Let $\beta \in C^1(K)$ be arbitrary such that $\beta(e_i) = b_i$ for each i. Then, taking $[M]$ to denote the fundamental class of M given by the sum of top-dimensional cubes of K,

$$\begin{aligned}
(\alpha \cup_c \beta)[M] &= \frac{1}{4}\sum_{\mathcal{F}_1}\sum_{v\in V} \operatorname{sgn}(x_{i_1}, x_{i_2})\alpha(y_1(v))\beta(y_1^c(v)) \\
&= \frac{1}{4}\sum_{v\in V}[\alpha(x_1, x_2(v))\beta(x_1(v), x_2) - \alpha(x_1(v), x_2)\beta(x_1, x_2(v))] \\
&= \frac{1}{4}\big[(-b_5 + b_7) + (b_3 - 0) + (0 + 2b_3) + (2b_7 - b_1) \\
&\quad +(b_1 + 2b_8) + (2b_4 - 0) + (0 + b_4) + (b_8 + b_5) \\
&\quad +(0 - b_3) + (-b_7 - b_6) + (-b_2 - 2b_7) + (-2b_3 - 0) \\
&\quad +(0 - 2b_4) + (-2b_8 + b_2) + (b_6 - b_8) + (-b_4 - 0)\big] \\
&= 0.
\end{aligned}$$

Thus, α pairs to zero with all $\beta \in C^1(K)$ and the cubical cup product is degenerate on the cochain level. Note, however, that α is exact and thus represents the zero class on the cohomology level.

The above example gives a specific counterexample to the nondegeneracy of the cubical cup product on cochains. It also confirms that if W is an injection, W cannot be a surjection on the cochain level. However, Lemma 3.2.17 confirms that W will be a surjection on cohomology. This gives us a way to recover a nondegenerate pairing on the cohomology level of our cubical structure. First, we must establish that the cubical cup product is well-defined. Nondegeneracy of the cubical cup product on the cohomology level will then follow easily from nondegeneracy of the smooth wedge product via W.



(a) The 1-torus with cubical structure K

(b) The values of $\alpha \in C^1(K)$.

Figure 3.12: The degeneracy of $\cup_c$ on the 1-torus

Theorem 3.2.19. *For any p, the cubical cup product is a well-defined map $H^p(K) \times H^{n-p}(K) \to \mathbb{R}$.*

Proof. It suffices to show that $(\alpha \cup_c d\beta)[M] = 0$ for all closed $\alpha \in C^p(K)$ and for all $\beta \in C^{n-p-1}(K)$. This follows directly from d being a derivation of $\cup_c$.

$$
\begin{aligned}
(\alpha \cup_c d\beta)[M] &= (-1)^p d(\alpha \cup_c \beta)[M] + (-1)^{p+1}(d\alpha \cup_c \beta)[M] \\
&= (-1)^p(\alpha \cup_c \beta)[\partial M] && \text{(Stokes' Theorem and } \alpha \text{ closed)} \\
&= 0 && (\text{by } \partial M = \emptyset).
\end{aligned}
$$

Hence, the cubical cup product is well-defined on cohomology. □

Theorem 3.2.20. *The cubical cup product is a nondegenerate pairing, $H^p(K) \times H^{n-p}(K) \to \mathbb{R}$.*

Proof. W is surjective on the cohomology level by Lemma 3.2.17. Also, W makes the connection of $H^*(K)$ with de Rham cohomology $H^*_{dR}(M)$ via Theorem 3.2.12, where the smooth wedge product is a nondegenerate pairing, see [3]. Thus, W transfers the nondegeneracy of the smooth wedge product to the cubical cup product on cohomology. □

3.3 The Discrete Hodge Star Operator

In the smooth case, the Hodge star realizes an isomorphism between de Rham cohomology groups of complementary degree. In a discrete setting, the Hodge star is traditionally defined via a cell complex and its dual (see, for e.g., [12]). Scott Wilson's innovation in [19] is the definition of a discrete Hodge star over $\mathbb{R}$ in a simplicial setting without reference to a dual cell complex. The choice in definition can be explained on an elementary level via the similarity of its representation to the smooth $*$ and $\langle\ ,\ \rangle_{\mathcal{L}^2}$ relationship, up to a sign. It also allows for a definition of a $*$ despite the degeneracy of the cup product on the cochain level. Wilson's star can be used to recover the duality isomorphism of cohomology groups in complementary degrees.

In this section, we define an analogous cubical discrete Hodge star over $\mathbb{R}$ (Section 3.3.1) and also a Hodge star over $\mathbb{Z}$ (Section 3.3.4). Over $\mathbb{R}$, we give the details surrounding the proof of star as an isomorphism between cohomology groups of complementary degree (Section 3.3.3). Our innovation is the proof of Poincaré duality over $\mathbb{R}$ via this same discrete star (Section 3.3.3), and the realization of star as the Poincaré duality map (over both $\mathbb{R}$

and over $\mathbb{Z}$) via the cubical cap product with the fundamental class of M (Section 3.3.3 and Section 3.3.4, respectively). This realization provides a more sophisticated justification for the choice of definition of the discrete Hodge star, including the omission of the sign present in the smooth case.

3.3.1 The Cubical Discrete Star Defined over $\mathbb{R}$

In Section 3.2.1, we defined the cubical cup product $\cup_c$, and in Section 3.2.3, we showed $\cup_c$ is nondegenerate on cohomology. We now define the cubical discrete Hodge star over $\mathbb{R}$ via $\cup_c$ analogously to Wilson's discrete Hodge star in [19].

Definition 3.3.1. We define the **cubical discrete Hodge star over** $\mathbb{R}$ $* : C^p(K) \to C^{n-p}(K)$ as follows

$$\langle *\alpha, \beta \rangle = (\alpha \cup_c \beta)[M]. \tag{3.10}$$

Here, $\langle\ ,\ \rangle$ is the discrete inner product defined on $\omega, \gamma \in C^p(K)$ by

$$\langle \omega, \gamma \rangle = \sum_{p\text{-faces, } c, \text{ in } K} \omega(c) \cdot \gamma(c).$$

The following property of $*$ regarding its relationship with the adjoint of d, d^*, will be important in proof of Poincaré duality in Section 3.3.3.

Lemma 3.3.2. *For each p,*

$$*d^p = (-1)^{p+1} (d^{n-p-1})^* * .$$

Proof. Let $\alpha \in C^p(K)$ and $\beta \in C^{n-p-1}(K)$. Then, because d is a derivation of $\cup_c$ and because $\partial M = \emptyset$,

$$\begin{aligned}
\langle *d^p\alpha, \beta\rangle &= (d^p\alpha \cup_c \beta)[M] && \text{(definition of } *) \\
&= (d^{n-1}(\alpha \cup_c \beta))[M] + (-1)^{p+1}(\alpha \cup_c d\beta)[M] && (d \text{ a derivation of } \cup_c) \\
&= (-1)^{p+1}(\alpha \cup_c d\beta)[M] && (\partial M = \emptyset) \\
&= (-1)^{p+1}\langle *\alpha, d^{n-p-1}\beta\rangle && \text{(definition of } *) \\
&= (-1)^{p+1}\langle (d^{n-p-1})^* * \alpha, \beta\rangle && \text{(definition of } d^*) \\
&= \langle (-1)^{p+1}(d^{n-p-1})^* * \alpha, \beta\rangle.
\end{aligned}$$

Because this is true for all $\beta \in C^{n-p-1}(K)$, $*d^p = (-1)^{p+1}(d^{n-p-1})^*$, and the claim is proved. □

3.3.2 The Discrete Hodge Decomposition

In this section, we prove the discrete Hodge decomposition using linear algebra and a dimension count of finite-dimensional vector spaces. The Hodge decomposition establishes a cochain group of arbitrary degree as isomorphic to the orthogonal direct sum of the image of d, the image of d^*, and the space of discrete harmonic forms (forms in both the kernel of d and the kernel of d^*). It is this decomposition that proves that each cohomology class has a unique harmonic representative, i.e. the p^{th} cohomology group is isomorphic to the space of harmonic differential p-forms.

The important result established in this section is Corollary 3.3.5. This is key in the proof of the isomorphism between cohomology groups of complementary degree, and the proof of Poincaré duality over $\mathbb{R}$.

Lemma 3.3.3. *For each p,* $\operatorname{Ker} d^{p*} = (\operatorname{Im} d^p)^{\perp}$.

Proof. Let $\alpha \in \operatorname{Ker}(d^p)^*$. Then for all $\beta \in C^p(K)$,

$$0 = \langle \beta, 0 \rangle = \langle \beta, d^{p*}\alpha \rangle = \langle d^p\beta, \alpha \rangle.$$

Thus, $\alpha \in (\operatorname{Im} d^p)^{\perp}$ and $\operatorname{Ker} d^{p*} \subseteq (\operatorname{Im} d^p)^{\perp}$.

Now suppose $\alpha \in (\operatorname{Im} d^p)^{\perp}$. Then for all $\beta \in C^p(K)$,

$$0 = \langle d^p\beta, \alpha \rangle = \langle \beta, (d^p)^*\alpha \rangle.$$

Because $d^{p*}\alpha$ is orthogonal to all $\beta \in C^p(K)$, $(d^p)^*\alpha = 0$. Thus, $\alpha \in \operatorname{Ker}(d^p)^*$ and $(\operatorname{Im} d^p)^{\perp} \subseteq \operatorname{Ker}(d^p)*$.

Hence, $\operatorname{Ker}(d^p)^* = (\operatorname{Im} d^p)^{\perp}$, as desired. □

Theorem 3.3.4. *For each p,*

$$C^p(K) = \operatorname{Im} d^{p-1} \oplus_{\perp} \operatorname{Im}(d^p)^* \oplus_{\perp} \left(\operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*\right).$$

Proof. Certainly, we have the following

$$C^p(K) = (\operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*)^{\perp} \oplus_{\perp} \left(\operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*\right).$$

We will show that $\operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^* = (\operatorname{Im} d^{p-1} \oplus_\perp \operatorname{Im}(d^p)^*)^\perp$, but first we must establish $\operatorname{Im} d^{p-1} \perp \operatorname{Im}(d^p)^*$. By definition of the adjoint operator,

$$\langle d^p\alpha, \beta\rangle = \langle \alpha, (d^p)^*\beta\rangle.$$

Assume $\alpha \in \operatorname{Im} d^{p-1}$ and $\beta \in C^{p+1}(K)$. Then $\alpha = d^{p-1}\gamma$ for some $\gamma \in C^{p-1}(K)$ and we have

$$\begin{aligned}
\langle \alpha, (d^p)^*\beta\rangle &= \langle d^p\alpha, \beta\rangle \\
&= \langle d^p \circ d^{p-1}\gamma, \beta\rangle \\
&= \langle 0, \beta\rangle = 0.
\end{aligned}$$

Thus $\alpha \perp (d^p)^*\beta$. Because α and β were arbitrary,

$$\operatorname{Im} d^{p-1} \perp \operatorname{Im} d^{p*}.$$

Now we will show $\operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^* = (\operatorname{Im} d^{p-1} \oplus_\perp \operatorname{Im}(d^p)^*)^\perp$ via containment in both directions.

($\subseteq$): Let $\omega \in \operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*$. By Lemma 3.3.3, $\operatorname{Ker}(d^{p-1})^* = (\operatorname{Im} d^{p-1})^\perp$. Similarly, because $d^{**} = d$, $\operatorname{Ker} d^p = (\operatorname{Im}(d^p)^*)^\perp$. Thus, $\omega \in (\operatorname{Im} d^{p-1})^\perp \cap (\operatorname{Im}(d^p)^*)^\perp$. In particular, $\omega \in (\operatorname{Im} d^{p-1} \oplus_\perp \operatorname{Im}(d^p)^*)^\perp$.

($\supseteq$): Let $\omega \in (\operatorname{Im} d^{p-1} \oplus_\perp \operatorname{Im}(d^p)^*)^\perp$. Choose arbitrary $c_1 d^{p-1}\alpha + c_2(d^p)^*\beta \in \operatorname{Im} d^{p-1} \oplus \operatorname{Im}(d^p)^*$. Then,

$$0 = \langle c_1 d^{p-1}\alpha + c_2(d^p)^*\beta, \omega\rangle$$

$$= \quad c_1\langle d^{p-1}\alpha, \omega\rangle + c_2\langle (d^p)^*\beta, \omega\rangle$$

$$= \quad c_1\langle \alpha, (d^{p-1})^*\omega\rangle + c_2\langle \beta, d^p\omega\rangle.$$

Suppose $c_1 = 1$ and $c_2 = 0$. Then, $0 = \langle \alpha, (d^{p-1})^*\omega\rangle$ for all $\alpha \in C^{p-1}(K)$. In particular, $0 = \langle (d^{p-1})^*\omega, (d^{p-1})^*\omega\rangle$, and because $\langle\ ,\ \rangle$ is positive semidefinite, $(d^{p-1})^*\omega = 0$. A similar argument with $c_1 = 0$ and $c_2 = 1$ estabishes $d^p\omega = 0$.

Thus, $\omega \in \operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*$.

This confirms $\operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^* = (\operatorname{Im} d^{p-1} \oplus_\perp \operatorname{Im}(d^p)^*)^\perp$, and hence

$$C^p(K) = \operatorname{Im} d^{p-1} \oplus_\perp \operatorname{Im}(d^p)^* \oplus_\perp \operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*.$$

□

Corollary 3.3.5. *For each p,*

$$H^p(K) \cong \left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^p.$$

Proof. By Lemma 3.3.3, $\operatorname{Im}(d^p)^* = (\operatorname{Ker} d^p)^\perp$. Combining this with the result of Theorem 3.3.4 yields

$$\operatorname{Ker} d^p = \operatorname{Im} d^{p-1} \oplus_\perp \left(\operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*\right).$$

Thus,

$$H^p(K) \quad = \quad \frac{\operatorname{Ker} d^p}{\operatorname{Im} d^{p-1}}$$

$$
\begin{aligned}
&= \frac{\operatorname{Im} d^{p-1} \oplus_{\perp} (\operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*)}{\operatorname{Im} d^{p-1}} \\
&\cong \operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*.
\end{aligned}
$$

Similarly, by Lemma 3.3.3, $\operatorname{Im} d^{p-1} = (\operatorname{Ker}(d^{p-1})^*)^{\perp}$. So, Theorem 3.3.4 yields

$$\operatorname{Ker}(d^{p-1})^* = \operatorname{Im}(d^p)^* \oplus_{\perp} \left(\operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*\right).$$

Hence, we have

$$
\begin{aligned}
\left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^p &= \frac{\operatorname{Ker}(d^{p-1})^*}{\operatorname{Im}(d^p)^*} \\
&= \frac{\operatorname{Im}(d^p)^* \oplus_{\perp} (\operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*)}{\operatorname{Im}(d^p)^*} \\
&\cong \operatorname{Ker} d^p \cap \operatorname{Ker}(d^{p-1})^*.
\end{aligned}
$$

Therefore,

$$H^p(K) \cong \left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^p, \text{ as desired.}$$

□

3.3.3 The Discrete Hodge Star as an Isomorphism

The discrete Hodge star can be viewed in two different ways. The is first as an operator that induces an isomorphism between cohomology groups in complementary degrees. The second, and more interesting viewpoint, is as the Poincaré Duality map between cohomology and complementary degree homology. Ultimately, we will establish our discrete Hodge star as the Poincaré duality map, i.e. the cubical cap product with the fundamental class of M, under the chain isomorphism defined in Definition 3.3.8. The first isomorphism is analogous to a result of Scott Wilson in [19]. The latter isomorphism and star as the Poincaré duality map are original contributions of this dissertation.

Duality of Cubical Cohomology Groups

In this section, we show that the discrete Hodge star induces an isomorphism between cohomology groups of complementary degree. Traditionally, the Hodge star realizes this isomorphism via the space of harmonic forms. However, our discrete Hodge star need not take harmonic forms to harmonic forms. Instead, we show that the image of an exact form under $*$ is coexact. This makes $*$ a well defined map between $H^p(K)$ and $\left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^{n-p}$. The nondegeneracy of $\cup_c$, proven in Theorem 3.2.20, establishes $*$ as an injection on $H^p(K)$, which sets the stage for proving $*$ is, in fact, an isomorphism between $H^p(K)$ and $\left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^{n-p}$. Combining this result with Corollary 3.3.5, we recover the duality isomorphism of cohomology groups via our star.

Theorem 3.3.6. *For each p, the discrete $*$ induces an isomorphism*

$$H^p(K) \cong \left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^{n-p}.$$

Proof. First, we must show that $* : H^p(K) \to \left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^{n-p}$ is well-defined. Suppose $\alpha \in \operatorname{Ker} d^p$. Then Lemma 3.3.2 yields

$$\begin{aligned} (d^*)^{n-p} * \alpha &= (d^{n-p-1})^* * \alpha \\ &= (-1)^{p+1} * d^p\alpha \end{aligned}$$

$$= (-1)^{p+1} * 0$$

$$= 0.$$

So, $*\alpha \in \ker(d^*)^{n-p}$. Suppose $\alpha = d^{p-1}\beta$ for some $\beta \in C^{p-1}(K)$. By Lemma 3.3.2,

$$\begin{aligned} *d^{p-1}\beta &= (-1)^p (d^{n-p})^* * \beta \\ &= (-1)^p (d^*)^{n-p+1} * \beta. \end{aligned}$$

Thus, if α is exact, then $*\alpha$ is coexact. This confirms that $* : H^p(K) \to \left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^{n-p}$ is well-defined.

Now, we will show that $*$ is an isomorphism. By Theorem 3.2.20, $\cup_c$ is a nondegenerate pairing of $H^p(K)$ and $H^{n-p}(K)$. Thus, (3.10) implies that, for $*$ defined on $H^p(K)$, $\operatorname{Im} *$ pairs nondegenerately with H^{n-p} via the discrete inner product. Hence,

$$\dim \operatorname{Im} * \geq \dim H^{n-p}(K)$$

By Corollary 3.3.5, $H^{n-p}(K) \cong \left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^{n-p}$. It follows that

$$\begin{aligned} \dim \operatorname{Im} * &\geq \dim H^{n-p}(K) \\ &\geq \dim \left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^{n-p}. \end{aligned}$$

On the other hand, $\left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^{n-p}$ is the codomain of $*$ and we have

$$\dim \operatorname{Im} * \leq \dim \left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^{n-p}.$$

Thus, $\dim \operatorname{Im} * = \dim \left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^{n-p}$ and $*$ is a surjection. Because $\cup_c$ is a nondegenerate pairing,

$$\dim H^p(K) = \dim H^{n-p}(K)$$

$$= \dim \operatorname{Im} *.$$

So, $*$ is a surjection on finite-dimensional vector spaces of equal dimension. Hence, $*$ is an injection, and consequently an isomorphism, as desired.

□

Corollary 3.3.7. *For each p,*

$$H^p(K) \cong H^{n-p}(K).$$

Proof. The result follows directly from Corollary 3.3.5 and Theorem 3.3.6.

□

Cubical Poincaré Duality

In this section, we give the original result that the discrete Hodge star induces an isomorphism between cohomology and complementary-degree homology with real coefficients, i.e. establishes Poincaré duality in its traditional form. In fact, we assert that star can be viewed as the cubical cap product with the fundamental class of M, and hence this isomorphism is the Poincaré duality map over $\mathbb{R}$. A major point of interest is that we will recover Poincaré duality on a single cell complex, in the absence of its dual complex.

The proof of Poincaré duality on a single cell complex hinges on a key observation that the discrete adjoint operator d^* intertwines with the discrete boundary operator ∂ under composition with the following map.

Definition 3.3.8. For each p, define $f_p : C_p(K) \to C^p(K)$ by $f_p(c) = \hat{c}$. Recall that for a single p-cell τ in $C_p(K)$, $\hat{\tau}(\tau) = 1$ and $\hat{\tau}$ is 0 on all other p-chains.

Remark 3.3.9. f_p is an isomorphism between $C_p(K)$ and $C^p(K)$ for all p.

To see the relationship of the adjoint and the boundary operator, we consider the behavior of d^* on basis elements.

Suppose $\tau_{p+1} \in C_{p+1}(K)$ is a basis element. By definition of d^*,

$$\langle d\hat{\tau}_p, \hat{\tau}_{p+1}\rangle = \langle \hat{\tau}_p, (d^p)^*\hat{\tau}_{p+1}\rangle,$$

for any basis element $\tau_p \in C_p(K)$.

Before calculating each of the above inner products explicitly, we will first narrow our choice of a basis element $\tau_p \in C_p(K)$ to only those in $\partial\tau_{p+1}$. Such a τ_p is the only candidate for a nonzero pairing $\langle d\hat{\tau}_p, \hat{\tau}_{p+1}\rangle$. The definition of the discrete inner product and the definition of d expose why the pairing is zero otherwise.

$$\begin{aligned}
\langle d\hat{\tau}_p, \hat{\tau}_{p+1}\rangle &= \sum_{(p+1)-\text{faces } c} d\hat{\tau}_p(c)\hat{\tau}_{p+1}(c) \\
&= \sum_{(p+1)-\text{faces } c} \hat{\tau}_p(\partial c)\hat{\tau}_{p+1}(c) \\
&= \hat{\tau}_p(\partial\tau_{p+1})\hat{\tau}_{p+1}(\tau_{p+1}) \\
&= \hat{\tau}_p(\partial\tau_{p+1}) \\
&= 0 \text{ if } \tau_p \notin \partial\tau_{p+1}.
\end{aligned}$$

Thus, we will consider only $\tau_p \in \partial\tau_{p+1}$.

Now we will calculate each inner product in the definition of d^*.

$$\begin{aligned}\langle d\hat{\tau}_p, \hat{\tau}_{p+1}\rangle &= \sum_{(p+1)-\text{faces } c} d\hat{\tau}_p(c)\hat{\tau}_{p+1}(c) \\ &= d\hat{\tau}_p(\tau_{p+1})\hat{\tau}_{p+1}(\tau_{p+1}) \\ &= \hat{\tau}_p(\partial\tau_{p+1}) \\ &= \pm 1. \end{aligned} \tag{3.11}$$

The sign that $\hat{\tau}_p(\partial\tau_{p+1})$ carries agrees with τ_p's orientation in $\partial\tau_{p+1}$.

On the other hand,

$$\begin{aligned}\langle \hat{\tau}_p, (d^p)^*\hat{\tau}_{p+1}\rangle &= \sum_{p-\text{faces } c} \hat{\tau}_p(c)(d^p)^*\hat{\tau}_{p+1}(c) \\ &= \hat{\tau}_p(\tau_p)(d^p)^*\hat{\tau}_{p+1}(\tau_p) \\ &= (d^p)^*\hat{\tau}_{p+1}(\tau_p). \end{aligned} \tag{3.12}$$

Thus, the definition of d^* yields (3.11) = (3.12) and $(d^p)^*\hat{\tau}_{p+1}(\tau_p) = \pm 1$, according to the orientation of $\tau_p \in \partial\tau_{p+1}$. So,

$$(d^p)^*\hat{\tau}_{p+1} = \widehat{\partial\tau_{p+1}}.$$

With this expression of d^*, we can now make the relationship of d^* and ∂ precise.

Theorem 3.3.10. *For each p, f_p is a chain map with respect to d^* and ∂, and hence induces an isomorphism between $\left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^p$ and $H_p(K)$.*

Proof. We will show $(d^p)^* \circ f_{p+1} = f_p \circ \partial_{p+1}$ on basis elements. Suppose τ_{p+1} is a $(p+1)$-face.

$$(d^p)^* \circ f_{p+1}(\tau_{p+1}) = (d^p)^*\hat{\tau}_{p+1}$$

$$= \widehat{\partial\tau_{p+1}}$$

$$= f_p(\partial\tau_{p+1})$$

$$= f_p \circ \partial(\tau_{p+1}).$$

Thus, f_p is a chain map. Because f_p is an isomorphism it induces the following isomorphism

$$\left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^p \cong H_p(K).$$

□

Corollary 3.3.11. (Cubical Poincaré Duality) *For each p,*

$$H^p(K) \cong H_{n-p}(K).$$

Proof. Transitivity of Corollary 3.3.5 and Theorem 3.3.10 proves the result. □

The discrete star's role in cubical Poincaré duality can be seen in Theorem 3.3.6. But, star's importance can be made more explicit by the following theorem.

Theorem 3.3.12. *Let $\alpha \in C^p(K)$ for any $p \geq 0$. Then,*

$$*\alpha = f_{n-p}([M] \cap_c \alpha).$$

Proof. Choose arbitrary $p \geq 0$ and let $\alpha \in C^p(K)$. Then,

$$*\alpha = \sum_{(n-p)\text{-cells } \sigma_i} k_i \hat{\sigma}_i,$$

where the k_i's are in $\mathbb{R}$. We abbreviate this sum $\sum k_i \hat{\sigma}_i$. By definition of $*$, for each i,

$$k_i = \langle *\alpha, \hat{\sigma}_i \rangle = (\alpha \cup_c \hat{\sigma}_i)[M].$$

For real numbers ℓ_i,

$$f_{n-p}([M] \cap_c \alpha) = \sum_{(n-p)\text{-cells } \sigma_i} \ell_i \hat{\sigma}_i.$$

We abbreviate this sum $\sum \ell_i \sigma_i$. Because $[M] \cap_c \alpha = \sum \ell_i \sigma_i$, for each i we have

$$\ell_i = \hat{\sigma}_i([M] \cap_c \alpha) = (\alpha \cup_c \hat{\sigma}_i)[M] = k_j.$$

The last equality holds because $\cap_c$ and $\cup_c$ are corresponding products. Thus,

$$*\alpha = f_{n-p}([M] \cap_c \alpha),$$

as desired.

□

Thus, the discrete star is the cubical cap product we have defined. More importantly, because the cubical cap product $\cap_c$ agrees with the standard cubical cap product on homology, the cubical discrete Hodge star is the Poincaré Duality map over $\mathbb{R}$ on a single cubical complex. Hence, star plays two important roles in duality in the discrete setting. As in the smooth case, star is the isomorphism between dual cohomology groups. However, the discrete Hodge star departs from the smooth case to also recover duality of cohomology and homology groups.

3.3.4 The Discrete Hodge Star and Poincaré Duality over $\mathbb{Z}$

The exposition of the details leading to the proof of Poincaré Duality reveals the need to work with coefficients in $\mathbb{R}$. Not only does $\cup_c$ have rational coefficients, but most of the isomorphisms we have uncovered rely on a dimension count in finite-dimensional vector spaces. These arguments fail because coefficients in $\mathbb{Z}$ allow for torsion. This section describes the original results that we can recover surrounding a discrete Hodge star and Poincaré Duality over $\mathbb{Z}$.

The Intertwining of d^* and ∂

We can recover Theorem 3.3.10 with integer coefficients. In other words, d^* and ∂ remain intertwined over $\mathbb{Z}$.

To see this, we first note that the discrete inner product $\langle\ ,\ \rangle$ is defined over $\mathbb{Z}$ because for $\mathbb{Z}$-valued cochains, it takes values only in $\mathbb{Z}$. Although the adjoint d^* usually refers to an inner product space (so, over $\mathbb{R}$, for example), we may instead define d^* with respect to a natural $\mathbb{Z}$-basis via the transpose of the matrix representing d. The basis of d is the collection of $\hat{c}$'s such that c is a single cell in K. Call the matrix representation of d M. Then, representing $\alpha \in C^p(K)$ and $\beta \in C^{p-1}(K)$ as column vectors under this basis, we have

$$\begin{aligned} \langle \alpha, d\beta \rangle &= \alpha^T(M\beta) \\ &= (\alpha^T M)\beta \\ &= (M^T\alpha)^T\beta \\ &= \langle d^*\alpha, \beta \rangle. \end{aligned}$$

In this way, we recover with $\mathbb{Z}$ coefficients a representation of d^* analogous to the usual inner product representation. Hence, we may apply a similar argument as in the case of real coefficients to obtain f_p a chain map with respect to d^* and ∂ over $\mathbb{Z}$ and

$$\left(\frac{\operatorname{Ker} d^*}{\operatorname{Im} d^*}\right)^p \cong H_p(K;\mathbb{Z}).$$

The Discrete Hodge Star as the Poincaré Duality map over $\mathbb{Z}$

The definition of the cubical discrete Hodge star in the case of real coefficients was defined by $\cup_c$. As we have noted, $\cup_c$ takes values in $\mathbb{Q}$ on cochains in $C^*(K;\mathbb{Z})$. Thus, we cannot use $\cup_c$ to define a cubical discrete Hodge star over $\mathbb{Z}$. We can, however, use the standard cubical cup product $\cup$ defined in Definition 3.2.6 because $\cup$ takes values in $\mathbb{Z}$ on $C^*(K;\mathbb{Z})$. Thus, we define a new cubical discrete Hodge star over $\mathbb{Z}$ as follows.

Definition 3.3.13. We define the **cubical discrete Hodge star over** $\mathbb{Z}$, $* : C^p(K;\mathbb{Z}) \to C^{n-p}(K;\mathbb{Z})$, via the discrete inner product $\langle\ ,\ \rangle$ and standard cubical cup product $\cup$ as follows.

$$\langle *\alpha, \beta\rangle = (\alpha \cup \beta)[M].$$

As aforementioned, we cannot recover Poincaré duality with this $*$ as in the real case. Our arguments over $\mathbb{R}$ relied on a dimension count in finite-dimensional vector spaces. However, defining $*$ in this way establishes it as the cap product with the fundamental class of M under the isomorphism f_{n-p} of Definition 3.3.8 on the cochain level.

Theorem 3.3.14. *Let $p \geq 0$ be arbitrary and let $\alpha \in C^p(K;\mathbb{Z})$. Then,*

$$*\alpha = f_{n-p}([M] \cap \alpha).$$

Proof. The proof is analogous to the proof of Theorem 3.3.12, replacing $\cup_c$ and $\cap_c$ with $\cup$ and $\cap$, respectively, and taking k_i and ℓ_i to be in $\mathbb{Z}$. □

Thus, we have defined a discrete Hodge star over $\mathbb{Z}$ on a single cubical complex that realizes the Poincaré duality map over $\mathbb{Z}$.

Chapter 4

The Discrete Hodge Star and Poincaré Duality in a Simplicial Setting

In Chapter 3, we proved the existence of Poincaré duality via a single cubical complex. The key ingredients for the proof were a nondegenerate pairing on the cohomology level of cubical cochains (obtained from Whitney forms and the deRham map) and the discrete Hodge star. In the simplicial setting, we have analogous ingredients. In this chapter, we report and provide additional details for results of Wilson in [19], and we extend Wilson's work to give a new proof of Poincaré duality over $\mathbb{R}$ on a simplicial cell complex without reference to its dual complex. We also show that Wilson's star is the standard cellular simplicial cap product with the fundamental class of M on homology, and hence is the Poincaré duality map over $\mathbb{R}$. Moreover, we define a new discrete Hodge star over $\mathbb{Z}$ on a single simplicial complex via the cellular simplicial cup product (Section 4.3.2). We prove that this star is the cellular cap product with the fundamental class of M on the cochain level. Because the cellular cap product intertwines with the singular cap product via Theorem 2.2.20, this Hodge star is the standard Poincaré duality map over $\mathbb{Z}$. Hence, the discrete Hodge star realizes Poincaré duality in its strongest form.

In [19], Scott Wilson provides a significant foundation for the original results given in this section. All new results in this chapter are stated as such, and unless otherwise noted, the remainder of the results are given in [19]. Wilson's definition of a discrete Hodge star over $\mathbb{R}$ is justified (beyond the simple observation that its definition mimicks the smooth $*$ and $\langle\ ,\ \rangle_{\mathcal{L}^2}$ relationship) through our assertion that star realizes the Poincaré duality map over $\mathbb{R}$ (Section 4.3.2). His choice is further understood through the definition of a new Hodge star that realizes Poincaré duality over $\mathbb{Z}$ (Section 4.3.2). Details of the foundation of this section are given in [19]. All other results are previously given in detail in the analogous cubical setting of Chapter 3. Therefore, we merely introduce the necessary definitions and results throughout this chapter without proof.

Throughout this chapter, unless otherwise stated, M is a closed oriented n-dimensional manifold that admits a simplicial complex X. $[M]$ denotes the fundamental class of M, which may be intuitively interpreted as the sum of n-simplices in X.

4.1 Simplicial Whitney Forms

Let τ be a p-simplex in $C^p(X)$ with barycentric coordinates $x_0, \ldots, x_p$. So, $0 \leq x_j \leq 1$ for all j and $\sum_{i=0}^{p} x_i = 1$.

Definition 4.1.1. For τ as above, we define the **simplicial Whitney form** of $\hat{\tau}$ by

$$W\hat{\tau} = p! \sum_{i=0}^{p} (-1)^i x_i \; dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_p.$$

Note that $W\hat{\tau}$ is defined to be nonzero on all n-simplices that contain τ. This definition extends linearly to define Whitney forms on all of $C^p(X)$. As in the cubical case, this is an embedding of simplicial cochains into the space of $\mathcal{L}^2$-forms.

W commutes with the exterior derivative d_Ω on $\Omega^*(M)$ and the coboundary map δ on $C^*(X)$, i.e. $d_\Omega \circ W = W \circ \delta$.

4.2 Wilson's Cup Product - A Nondegenerate Pairing of Differential Forms

As in the cubical setting, Wilson defines a cup product whose nondegeneracy is established via the nondegeneracy of the smooth wedge product. Recall that the validity of borrowing from the smooth nondegeneracy was established by showing the composition of the de Rham map with the Whitney map was an isomorphism on cohomology.

So, we define the **de Rham map** $R : \omega^p(M) \to C^p(X)$ by

$$R\omega(c) = \int_c \omega$$

for a differential form ω and a simplicial chain c.

Wilson defines his cup product via R and W directly.

Definition 4.2.1. Let X be a simplicial complex on a smooth manifold. We define **Wilson's cup product** $\cup_W : C^p(X) \times C^q(X) \to C^{p+q}(X)$ by

$$\alpha \cup_W \beta = R(W\alpha \wedge W\beta).$$

One can check that R is also a chain map with respect to d_Ω and δ. This leads to R and W inducing isomorphisms on their respective cohomology groups of all orders, as in the cubical setting.

In [19], Wilson refers briefly to his cup product agreeing with the standard cellular simplicial cup product on cohomology. We establish the details in Section 4.2.1. First, we show that $\cup_W$ is indeed a cup product as defined by Whitney in Defintion 2.3.2.

Theorem 4.2.2. *$\cup_W$ is a cup product.*

Proof. We show that $\cup_W$ satisfies the requirements of Definition 2.3.2

Property 1.

Property 2. R and W are chain maps with respect to d_Ω and d_X. Also, d_Ω is a derivation of the "smooth" wedge product. Because Whitney forms are not smooth, the wedge product is not quite the smooth wedge product. However, differentiation with respect to d_Ω still makes sense on Whitney forms and so we still have the desired derivation. Thus,

$$\begin{aligned} d_X(\alpha \cup \beta) &= d_X(R(W\alpha \wedge W\beta)) \\ &= R(d_\Omega(W\alpha \wedge W\beta)) \\ &= R(d_\Omega W\alpha \wedge W\beta + (-1)^p W\alpha \wedge d_\Omega W\beta) \\ &= R(d_\Omega W\alpha \wedge W\beta)) + (-1)^p R(W\alpha \wedge d_\Omega W\beta) \\ &= R(W(d_X\alpha) \wedge W\beta)) + (-1)^p R(W\alpha \wedge W(d_X\beta)) \\ &= (d_X\alpha) \cup_W \beta + (-1)^p \alpha \cup_W (d_X\beta). \end{aligned}$$

Property 3. We will explicitly show that $\gamma_{\cup_W} = 1$, i.e. for any p and for any $\alpha \in C^p(X)$, $I \cup_{W_c} \alpha = \alpha$. Recall that I is the constant 0-form that takes value 1 on each vertex in X. We give the proof on basis elements. The result follows by linearity.

If $p = 0$ the result is trivial. Therefore, we focus on $p \geq 1$.

Suppose $\sigma = [v_0, \ldots, v_p]$ is a p-simplex with barycentric coordinates $x_0, \ldots, x_p$. Then $\hat{\sigma}$ is the p-form that takes value 1 on σ and 0 on all other simplices. $I = \hat{v}_0 + \cdots \hat{v}_p$ on σ and

$$WI = x_0 + \cdots + x_p.$$

By definition of Whitney forms,

$$W\hat{\sigma} = p! \sum_{i=0}^{p} (-1)^i x_i \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_p.$$

So, by the definition of $\cup_W$,

$$(I \cup_W \hat{\sigma})(\sigma) \;=\; R(WI \wedge W\hat{\sigma})(\sigma)$$

$$\begin{aligned} &= \int_\sigma WI \wedge W\hat{\sigma} \\ &= p! \sum_{i=0}^{p} \int_\sigma (-1)^i x_i (x_0 + \cdots + x_p) dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_p. \end{aligned}$$

We may parametrize to reduce the calculation of each integral in the above sum to integration over the $x_0 \cdots \widehat{x_i} \cdots x_p$-hyperplane. The standard orientation of this coordinate hyperplane agrees with $(-1)^i$ times the standard orientation of σ. Because $\sum_{i=0}^{p} x_i = 1$, each integrand can be rewritten using $x_i = 1 - x_0 - \cdots - \widehat{x_i} - \cdots - x_p$.

$$\begin{aligned} x_i(x_0 + \cdots + x_p) &= (1 - x_0 - \cdots - \widehat{x_i} - \cdots - x_p)(x_0 + \cdots + x_{i-1} \\ &\quad +(1 - x_0 - \cdots - \widehat{x_i} - \cdots - x_p) + x_{i+1} + \cdots + x_p). \\ &= (1 - x_0 - \cdots - \widehat{x_i} - \cdots - x_p). \end{aligned}$$

For simplicity, we will use the notation

$$s_j = (1 - x_p - x_{p-1} - \cdots - \widehat{x_i} - \cdots - x_j).$$

So $x_i(x_0 + \cdots + x_p) = s_0$ and the integral over σ becomes the following for any $i \neq 0$.

$$\int_0^1 \int_0^{s_p} \int_0^{s_{p-1}} \cdots \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0 \; dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_p$$

If $i = 0$, we instead have

$$\int_0^1 \int_0^{s_p} \int_0^{s_{p-1}} \cdots \int_0^{s_2} s_1 \; dx_1 \wedge \cdots \wedge dx_p.$$

<u>Case 1</u>: $i \neq 0$. Suppose $i \neq p$. Then by Lemma 4.2.3,

$$\int_0^{s_p} \cdots \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0 \; dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{p-1} = \frac{1}{p!} {s_p}^p.$$

So,

$$\begin{aligned}\int_0^1\int_0^{s_p}\int_0^{s_{p-1}}\cdots\widehat{\int_0^{s_i}}\cdots\int_0^{s_1} s_0\ dx_0\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_p &= \int_0^1 \frac{1}{p!}{s_p}^p dx_p\\ &= -\frac{1}{(p+1)!}{s_p}^{p+1}\Big|_0^1\\ &= \frac{1}{(p+1)!}.\end{aligned}$$

Suppose $i\neq 0$ and $i=p$. Then, $p-1<i$ and by Lemma 4.2.3,

$$\begin{aligned}\int_0^1\int_0^{s_{p-1}}\cdots\int_0^{s_1} s_0\ dx_0\wedge\cdots\wedge dx_{p-1} &= \int_0^1 \frac{1}{(p!}{s_p}^p dx_p\\ &= -\frac{1}{(p+1)!}{s_p}^{p+1}\Big|_0^1\\ &= \frac{1}{(p+1)!}.\end{aligned}$$

<u>Case 2</u>: $i=0$. Suppose $p=1$. Then,

$$\begin{aligned}\int_0^1 s_1 dx_1 &= -\frac{1}{2}{s_1}^2\Big|_0^1\\ &= \frac{1}{2} = \frac{1}{(p+1)!}.\end{aligned}$$

Suppose $p\geq 2$. Then, $p>i$, so by Lemma 4.2.4,

$$\begin{aligned}\int_0^1\int_0^{s_p}\int_0^{s_{p-1}}\cdots\int_0^{s_2} s_1\ dx_1\wedge\cdots\wedge dx_p &= \int_0^1 \frac{1}{p!}{s_p}^p dx_p\\ &= -\frac{1}{(p+1)!}{s_p}^{p+1}\Big|_0^1\\ &= \frac{1}{(p+1)!}.\end{aligned}$$

By the above cases, we see that

$$(I\cup_W\hat{\sigma})(\sigma) = p!\sum_{i=0}^{p}\frac{1}{(p+1)!}$$

$$= p!\frac{p+1}{(p+1)!}$$
$$= 1 = \hat{\sigma}(\sigma).$$

$(I \cup_W \hat{\sigma})(\sigma') = 0$ on all other p-chains $\sigma' \neq \sigma$ by definition of $\hat{\sigma}$. Thus,

$$(I \cup_W \hat{\sigma}) = \hat{\sigma}$$

for all p-chains σ where p is arbitrary. Thus, by linearity, $\gamma_{\cup_W} = 1$.

Hence, $\cup_W$ satisfies the conditions of Definition 2.3.2 and $\cup_W$ is a cup product of simplicial differential forms.

□

Lemma 4.2.3. *Let $p \geq 1$ be an arbitrary integer. Suppose $1 \leq i \leq p$ for some integer i and j is an integer such that $j \neq i$ and $1 \leq j \leq p$.*

If $j < i$,

$$\int_0^{s_j} \cdots \int_0^{s_1} s_0 \, dx_0 \wedge \cdots \wedge dx_{j-1} = \frac{1}{(j+1)!} {s_j}^{j+1}.$$

If $j > i$,

$$\int_0^{s_j} \cdots \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0 \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{j-1} = \frac{1}{j!} {s_j}^{j}.$$

Proof. The proof is by induction on j and requires two base cases.

Base Case 1: Assume $i = 1$. We will show by induction on j that for all j, $2 \leq j \leq p$,

$$\int_0^{s_j} \cdots \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0 \, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{j-1} = \frac{1}{j!} {s_j}^{j}.$$

Suppose $j = 2$. Then,

$$\begin{aligned}\int_0^{s_2} s_0\ dx_0 &= -\frac{1}{2}{s_0}^2\Big|_0^{s_2}\\ &= \frac{1}{2}{s_2}^2\\ &= \frac{1}{j!}{s_j}^j.\end{aligned}$$

Now suppose arbitrary j is such that $2 \leq j \leq p-1$. Assume

$$\int_0^{s_j} \cdots \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0\ dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{j-1} = \frac{1}{j!}{s_j}^j.$$

Then,

$$\begin{aligned}\int_0^{s_{j+1}} \cdots \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0\ dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_j &= \int_0^{s_{j+1}} \frac{1}{j!}{s_j}^j\ dx_j.\\ &= -\frac{1}{(j+1)!}{s_j}^{j+1}\Big|_0^{s_{j+1}}\\ &= \frac{1}{(j+1)!}{s_{j+1}}^{j+1}.\end{aligned}$$

Thus, by induction, for $i = 1$ and for all $j,\ 2 \leq j \leq p$,

$$\int_0^{s_j} \cdots \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0\ dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{j-1} = \frac{1}{j!}{s_j}^j.$$

Base Case 2: Assume $j = 1$. Then for any $i > 1$, $i \neq 0$ so we have

$$\begin{aligned}\int_0^{s_1} s_0\ dx_0 &= -\frac{1}{2}{s_0}^2\Big|_0^{s_1}\\ &= \frac{1}{2}{s_1}^2\\ &= \frac{1}{(j+1)!}{s_j}^{j+1}.\end{aligned}$$

Inductive Hypothesis: Assume i is arbitrary, $2 \leq i \leq p$ and assume $j \neq i$, $1 \leq i \leq p$. If $j < i$, assume

$$\int_0^{s_j} \cdots \int_0^{s_1} s_0 \; dx_0 \wedge \cdots \wedge dx_{j-1} = \frac{1}{(j+1)!} {s_j}^{j+1}.$$

If $j > i$, assume

$$\int_0^{s_j} \cdots \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0 \; dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{j-1} = \frac{1}{j!} {s_j}^{j}.$$

Inductive Step: Consider $j+1$.

<u>Case 1</u>: $j+1 = i$. Consider $j+2$ instead. Note that $j+2 > i$. Then, by the inductive hypothesis for $j < i$,

$$\begin{aligned} \int_0^{s_{j+2}} \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0 \; dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge dx_{j+1} &= \int_0^{s_{j+2}} \frac{1}{(j+1)!} {s_j}^{j+1} dx_{j+1} \\ &= -\frac{1}{(j+2)! s_j}^{j+2} \Bigg|_0^{s_{j+2}} \\ &= \frac{1}{(j+2)!} {s_{j+2}}^{j+2}, \text{ as desired.} \end{aligned}$$

<u>Case 2</u>: $j+1 \neq i$ and $j < i$. Then, by the inductive hypothesis for $j < i$,

$$\begin{aligned} \int_0^{s_{j+1}} \cdots \int_0^{s_1} s_0 \; dx_0 \wedge \cdots \wedge dx_j &= \int_0^{s_{j+1}} \frac{1}{(j+1)!} {s_j}^{j+1} dx_j \\ &= -\frac{1}{(j+2)!} {s_j}^{j+2} \Bigg|_0^{s_{j+1}} \\ &= \frac{1}{(j+2)!} {s_{j+1}}^{j+2}, \text{ as desired.} \end{aligned}$$

<u>Case 3</u>: $j > i$. Then, by the inductive hypothesis for $j > i$,

$$\int_0^{s_{j+1}} \cdots \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0 \; dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_j = \int_0^{s_{j+1}} \frac{1}{j!} {s_j}^{j} \; dx_j.$$

$$\begin{aligned} &= -\frac{1}{(j+1)!}{s_j}^{j+1}\Big|_0^{s_{j+1}} \\ &= \frac{1}{(j+1)!}{s_{j+1}}^{j+1}, \text{ as desired.} \end{aligned}$$

In any case, the result holds for $j+1$.

By induction, for all $1 \leq i \leq p$ and for all $j \neq i$, $1 \leq j \leq p$,

if $j < i$,

$$\int_0^{s_j} \cdots \int_0^{s_1} s_0 \; dx_0 \wedge \cdots \wedge dx_{j-1} = \frac{1}{(j+1)!}{s_j}^{j+1}.$$

And, if $j > i$,

$$\int_0^{s_j} \cdots \widehat{\int_0^{s_i}} \cdots \int_0^{s_1} s_0 \; dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{j-1} = \frac{1}{j!}{s_j}^{j}.$$

Hence, the claim is proved. □

Lemma 4.2.4. *Let $p \geq 1$ be an arbitrary integer. Suppose $i = 0$ and $2 \leq j \leq p$. Then,*

$$\int_0^{s_j} \cdots \int_0^{s_2} s_1 \; dx_1 \wedge \cdots \wedge dx_{j-1} = \frac{1}{j!}{s_j}^{j}.$$

Proof. The proof is by induction on j.

Base Case: Suppose $j = 2$. Then,

$$\begin{aligned} \int_0^{s_2} s_1 \; dx_1 &= -\frac{1}{2}{s_1}^2\Big|_0^{s_2} \\ &= \frac{1}{2}{s_2}^2 \\ &= \frac{1}{j!}{s_j}^j, \end{aligned}$$

as desired.

Inductive Hypothesis: Now suppose for an arbitrary integer $j,\ 2 \leq j \leq p$,

$$\int_0^{s_j} \cdots \int_0^{s_2} s_1 \; dx_1 \wedge \cdots \wedge dx_{j-1} = \frac{1}{j!} {s_j}^j .$$

Inductive Step: By the inductive hypothesis,

$$\begin{aligned} \int_0^{s_{j+1}} \cdots \int_0^{s_2} s_1 \; dx_1 \wedge \cdots \wedge dx_j &= \int_0^{s_j} \frac{1}{j!} {s_j}^j \; dx_j \\ &= -\frac{1}{(j+1)!} {s_j}^{j+1} \Big|_0^{s_{j+1}} \\ &= \frac{1}{(j+1)!} {s_{j+1}}^{(j+1)}, \end{aligned}$$

and the result holds for $j + 1$.

Thus, the claim follows by induction. □

One may define the **Wilson cap product** with the fundamental class that corresponds to this cup product via the relationship

$$\beta([M] \cap_W \alpha) = (\alpha \cup_W \beta)[M].$$

Because R and W are isomorphisms, the smooth wedge product is a nondegenerate pairing of cohomology groups on X in complementary degree. A proof analogous to the proof of Theorem 3.2.20 recovers this result.

4.2.1 Wilson's Products and the Standard Simplicial Products

In this section, we show that Wilson's cap and cup products agree with the standard simplicial cap and cup products on homology and cohomology over $\mathbb{R}$. First, we recall the definition of the singular simplicial products.

Definition 4.2.5. The **singular simplicial cap product** $\cap : S_{p+q}(X) \times S^p(X) \to S_q(X)$ is defined by

$$\sigma \cap \varphi = \varphi(\sigma\big|_{[v_0,\ldots,v_p]})\sigma\big|_{[v_p,\ldots,v_{p+q}]}.$$

Definition 4.2.6. The **singular simplicial cup product** $\cup : S^p(X) \times S^q(X) \to S^{p+q}(X)$ is defined on $\sigma \in S_{p+q}(X)$ by

$$(\alpha \cup \beta)(\sigma) = \alpha(\sigma\big|_{[v_0,\ldots,v_p]})\beta(\sigma\big|_{[v_p,\ldots,v_{p+q}]}).$$

By Theorem 2.2.20 and Remark 2.2.22, we give the definition of the cellular singular products on a standard simplex.

Definition 4.2.7. The **cellular simplicial cap product** $\cap : C_{p+q}(X) \times C^p(X) \to C_q(X)$ is defined by

$$[v_0, \ldots, v_{p+q}] \cap \varphi = \varphi([v_0, \ldots, v_p])[v_p, \ldots, v_{p+q}].$$

Definition 4.2.8. The **cellular simplicial cup product** $\cup : C^p(X) \times C^q(X) \to C^{p+q}(X)$ is defined by

$$(\alpha \cup \beta)([v_0, \ldots, v_{p+q}]) = \alpha([v_0, \ldots, v_p])\beta([v_p, \ldots, v_{p+q}]).$$

These definitions satisfy the usual relationship

$$(\alpha \cup \beta)(\sigma) = \beta(\sigma \cap \alpha).$$

Theorem 4.2.9.

$$\gamma_\cap = \gamma_\cup = 1.$$

Proof. By definition of corresponding cap and cup products, $\gamma_\cap = \gamma_\cup$. We will show that $\gamma_\cup = 1$.

Let $p \geq 0$ be arbitrary and let $\alpha \in C^p(X)$. Then,

$$\begin{aligned}(I \cup \alpha)([v_0, \ldots, v_{p+q}]) &= I([v_0])\alpha([v_0, \ldots, v_p]) \\ &= \alpha([v_0, \ldots, v_p]).\end{aligned}$$

Thus, $\gamma_\cup = 1$ by definition. □

Theorem 4.2.10. *Wilson's cap product $\cap_W$ and the standard simplicial cap product $\cap$ agree on homology.*

Proof. We have previously shown in the proof of Theorem 4.2.2 that $\gamma_{\cup_W} = 1$. Thus, $\gamma_{\cap_W} = \gamma_\cap = 1$. The result follows from Theorem 2.3.10. □

Theorem 4.2.11. *Wilson's cup product $\cup_W$ and the standard simplicial cup product $\cup$ agree on cohomology.*

Proof. Because $\gamma_{\cup_W} = \gamma_\cup = 1$, the result follows from Theorem 2.3.11.

□

4.3 The Discrete Hodge Star on a Simplicial Complex

In this section, we define two discrete Hodge stars on a single simplicial complex: one over $\mathbb{R}$ and the other over $\mathbb{Z}$. These definitions of the discrete Hodge star arise from mimicking the smooth $*$ and $\langle\ ,\ \rangle_{\mathcal{L}^2}$ relationship (ignoring signs) in a way that permits them to serve as discrete Hodge stars even through cellular cup products that are degenerate on the cochain level. In either case, we show that the discrete Hodge star is the Poincaré duality map over its respective coefficient group, justifying the omission of the sign present in the smooth $*$ and $\langle\ ,\ \rangle_{\mathcal{L}^2}$ relationship. In the case of real coefficients, we also give an original proof of the traditional version of Poincaré duality on a single simplicial complex without reference to its dual via the discrete Hodge star.

4.3.1 The Discrete Hodge Star over $\mathbb{R}$

The discrete Hodge star defined in this section is a result given by Scott Wilson in [19].

Definition 4.3.1. Let $\langle\ ,\ \rangle$ be a non-degenerate positive definite inner product on $C^p(X;\mathbb{R})$ such that $C^i(X;\mathbb{R})$ is orthogonal to $C^j(X;\mathbb{R})$ for $i \neq j$. The **simplicial discrete Hodge star over** $\mathbb{R}$, $* : C^p(X;\mathbb{R}) \to C^{n-p}(X;\mathbb{R})$, is defined on $\sigma \in C^p(X;\mathbb{R})$ by

$$\langle *\sigma, \tau\rangle = (\sigma \cup_W \tau)[M].$$

where $[M]$ denotes the fundamental class of M.

As in the cubical setting, $*$ satisfies

$$*\delta^p = (-1)^{p+1}(\delta^{n-p-1})^* * .$$

Thus, for each p, $*$ induces an isomorphism

$$H^p(X;\mathbb{R}) \cong \left(\frac{\operatorname{Ker}\delta^*}{\operatorname{Im}\delta^*}\right)^{n-p}. \tag{4.1}$$

The Discrete Hodge Star as the Poincaré Duality Map

Via the arguments of Section 3.3.3, the isomorphism of (4.1) provides an avenue to recovering two major duality theorems. The first is stated by Wilson in [19]. The second, Poincaré duality (cohomology to homology) recovered on a single simplicial complex, is an original result.

Theorem 4.3.2. *For each p,*

$$H^p(X;\mathbb{R}) \cong H^{n-p}(X;\mathbb{R}).$$

As in the cubical case, δ^* and the simplicial boundary map ∂ commute via the isomorphism $f_p : C_p(X) \to C^p(X)$, $f_p(c) = \hat{c}$. This relationship is the foundation for the proof of Poincaré Duality on a single simplicial complex.

Theorem 4.3.3. *(Poincaré Duality) For each p,*

$$H^p(X;\mathbb{R}) \cong H_{n-p}(X;\mathbb{R}).$$

As in Theorem 3.3.12, we recover agreement of $*$ and the Wilson cap product with the fundamental class.

Theorem 4.3.4. *Let $p \geq 0$ and let $\alpha \in C^p(X;\mathbb{R})$. The simplicial discrete Hodge star over $\mathbb{R}$ is such that*

$$*\alpha = f_{n-p}([M] \cap_W \alpha).$$

Proof. The proof is analogous to the proof of Theorem 3.3.12, working over a simplicial complex with the products $\cap_W$ and $\cup_W$. □

By Theorem 4.2.10, $\cap_W$ agrees with the standard simplicial cap product on homology, and hence $*$ is the Poincaré Duality map over $\mathbb{R}$.

4.3.2 The Discrete Hodge Star over $\mathbb{Z}$

Recall the standard cellular simplicial cup product $\cup$ defined in Definition 4.2.8. By definition, $\alpha \cup \beta$ takes values in $\mathbb{Z}$ for $\alpha \in C^p(X;\mathbb{Z})$ and $\beta \in C^q(X;\mathbb{Z})$. Thus, we may use $\cup$ to define a discrete Hodge star over $\mathbb{Z}$ analogously to the discrete Hodge star over $\mathbb{R}$.

Definition 4.3.5. Let $\langle\ ,\ \rangle$ be the discrete inner product. The **simplicial discrete Hodge star over** $\mathbb{Z}$, $* : C^p(X;\mathbb{Z}) \to C^{n-p}(X;\mathbb{Z})$, is defined on $\sigma \in C^p(X;\mathbb{Z})$ by

$$\langle *\sigma, \tau \rangle = (\sigma \cup \tau)[M],$$

where [M] denotes the fundamental class of M, and $\cup$ is the standard cellular simplicial cup product defined in Definition 4.2.8.

Note that the discrete inner product is $\mathbb{Z}$-valued on cochains with coefficients in $\mathbb{Z}$ (refer to Section 3.3.4 for details).

Although we have defined $*$ over $\mathbb{Z}$ analogously to $*$ over $\mathbb{R}$, we cannot use $*$ to prove Poincaré duality over $\mathbb{Z}$ as in the real case. The arguments that establish Poincaré duality with real coefficients depend on finite-dimensional vector spaces. These arguments fail in the presence of torsion. However, the above definition of the discrete Hodge star identifies $*$ with the standard cellular simplicial cap product $\cap$ with the fundamental class of M on the cochain level under the map $f_p : C_p(X;\mathbb{Z}) \to C^p(X;\mathbb{Z})$ defined by $f(c) = \hat{c}$ for all $p \geq 0$.

Theorem 4.3.6. *Let $p \geq 0$ be arbitrary and let $\alpha \in C^p(X; \mathbb{Z})$. The simplicial discrete Hodge star over $\mathbb{Z}$ is such that*

$$*\alpha = f_{n-p}([M] \cap \alpha),$$

where $\cap$ denotes the cellular simplicial cap product of Definition 4.2.7.

Proof. The proof is analogous to the proof of Theorem 3.3.12, working over a simplicial complex with integer coefficients and using $\cap_W$ and $\cup_W$ instead. □

Thus, we have defined a discrete Hodge star on a single simplicial complex that agrees with the simplicial cap product with the fundamental class of M *on the cochain level*, by definition. Although we cannot use $*$ to prove Poincaré duality over $\mathbb{Z}$ as in the real case, the simplicial discrete Hodge star over $\mathbb{Z}$ is the Poincaré duality map.

Chapter 5

Forman's Complex of Differential Forms in a Cubical Setting

In [7], Robin Forman describes his nontraditional complex of differential forms in a simplicial setting. His differential forms are nontraditional in the sense that they act on a simplicial chain to return a chain, rather than a number. Despite this departure from the usual definition of forms, Forman asserts that the cohomology groups his complex of differential forms defines are isomorphic to the traditional cohomology groups of a complex of simplicial cochains.

This chapter provides an exposition of Forman's forms and their natural relationship to differential forms on a complex of kites. Because a complex of kites is by definition a cubical structure, we place Forman's complex of differential forms into the cubical setting, so that the theory of Chapter 3 applies to them.

Throughout this chapter, X denotes a simplicial cell complex on a smooth n-dimensional manifold M. K denotes the kite complex associated with X. We make no specific reference to a coefficient group, but the results follow over both $\mathbb{R}$ and $\mathbb{Z}$.

We highlight the original results in this chapter as follows. In Section 5.2, we define the kite complex K associated with the simplicial complex X via Forman's differential forms, and we establish an isomorphism of Forman's differential forms and the cubical cochains on K. In section 5.3, we define a signed version of Forman's differential operator (that defines a complex whose cohomology is the same as the cohomology of Forman's complex), and we prove that it is intertwined with the cubical discrete coboundary operator d. Hence, the cochain complexes defined by Forman's forms and the cubical forms on the kite complex are isomorphic. In this way, we offer a new proof of the isomorphism of the cohomology groups of Forman forms and the cubical cohomology groups. Last, we define a product of Forman's differential forms (Section 5.4.1), and we show that this cup product naturally defines a cubical cup product on the associated kite complex (Section 5.4.2). We conclude our results by showing that this cup product agrees with the cubical cup product we define in Section 3.2.1 on cohomology (Section 5.4.3). Thus, all of the results of Chapter 3 apply to Forman's complex of differential forms, e.g. the expressions of Poincaré duality.

5.1 Forman's Complex of Differential Forms on a Simplicial Complex

In this section, we give an overview of Forman's complex of differential forms found in [7]. This includes a brief overview of the properties of Forman's differential operator D and Forman's assertion that his complex of differential forms defines the same cohomology groups as the traditional cochain complex on a simplicial complex.

Definition 5.1.1. Suppose $p \geq 0$. We define the space of **Forman's differential *p*-forms** $\Omega^p_F(X)$ by

$$\Omega^p_F(X) = \bigoplus_{k \geq p} \{\text{local linear maps } \alpha : C_k(X) \to C_{k-p}(X)\}.$$

$\alpha \in \Omega^p_F(X)$ is locally linear in the sense that it takes a k-simplex σ to a linear combination of the $(k-p)$-cells within σ, i.e. its local $(k-p)$-cells. So, the image of each α is a subset of its pre-image.

Thus, we see that Forman's differential p-forms evaluate not only on simplicial p-chains, but also on k-chains for which $k \geq p$. Furthermore, the output of a Forman differential p-form on a k-chain is a $(k-p)$-chain, rather than a number. Consequently, Forman's forms are very different from traditional simplicial differential forms. In Section 5.2, we will explore just how closely Forman's differential forms can be related to a traditional view of discrete differential forms.

Now let us consider the behavior of a Forman differential p-form on a simplex of appropriate dimension. If $\alpha \in \Omega_F^p(X)$ and c is a k-simplex, $k \geq p$, then

$$\alpha(c) = \sum_{\substack{(k-p)\text{-simplices } b_i \\ b_i \subseteq c}} a_i \cdot b_i$$

for some collection of constants a_i. So a p-form takes a k-chain, $k \geq p$, to a linear combination of the $(k-p)$-cells that it contains.

Consider the following example to make this behavior more explicit.

Example 5.1.2. Consider a 2-simplex c with edges b_i and vertices b_i, $0 \leq i \leq 3$, as shown in Figure 5.1.

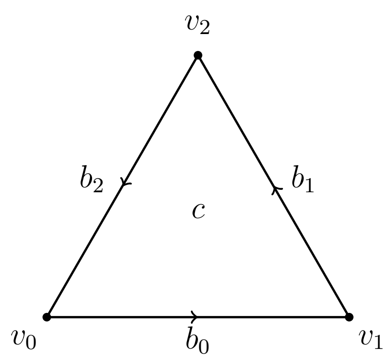


Figure 5.1: A 2-simplex c and its edges b_i and vertices v_i.

Let f be a 0-form, α a 1-form and ω a 2-form. Then we have the following.

$$f(c) = a \cdot c \qquad \alpha(b_j) = \sum_{v_i \in \partial b_j} s_{ji} \cdot v_i \qquad \omega(c) = \sum_{i=0}^{2} \ell_i \cdot v_i$$

$$f(b_i) = a_i \cdot b_i \qquad \alpha(c) = \sum_{i=0}^{2} t_i \cdot b_i$$

$$f(v_i) = A_i \cdot v_i$$

where a, a_i, A_i, s_{ji}, t_i, and ℓ_i are all constants.

Forman's forms are nontraditional. However, Forman defines a differential operator which leads to a differential complex whose cohomology agrees with the cohomology of the complex of traditional discrete differential forms.

Definition 5.1.3. We define **Forman's differential operator** $D : \Omega_F^p(X) \to \Omega_F^{p+1}(X)$ by

$$(D\omega)(c) = \partial(\omega(c)) - (-1)^p \omega(\partial c),$$

where c is a k-simplex, $k \geq p+1$. Thus, $(D\omega)(c) \in C_{k-(p+1)}(X)$.

Remark 5.1.4. Note that ∂ denotes the standard simplicial boundary map. Note also that D behaves similarly to the traditional simplicial coboundary operator δ in that D takes a p-form to a $(p+1)$-form, and $D \circ D = 0$.

We will use the notation

$$D\omega = \partial \circ \omega - (-1)^p \omega \circ \partial$$

without reference to a chain c interchangeably with the formal statement of D given in its definition.

Theorem 5.1.5.

$$D \circ D = 0.$$

Proof. The proof follows straight from the definition of D. Consider a p-form ω. Then, because $\partial \circ \partial = 0$,

$$D \circ D\omega \quad = \quad D(\partial \circ \omega - (-1)^p \omega \circ \partial)$$

$$= \partial(\partial \circ \omega - (-1)^p \omega \circ \partial) - (-1)^{p+1}(\partial \circ \omega - (-1)^p \omega \circ \partial) \circ \partial$$
$$= (-1)^{p+1}\partial \circ \omega \circ \partial - (-1)^{p+1}\partial \circ \omega \circ \partial$$
$$= 0.$$

□

By the above remark and Theorem 5.1.5, we have the following differential complex.

$$\Omega^*(X) : 0 \longrightarrow \Omega^0(X) \xrightarrow{D} \Omega^1(X) \xrightarrow{D} \cdots \xrightarrow{D} \Omega^n(X) \longrightarrow 0$$

Theorem 5.1.6. *The cohomology of Forman's complex of differential forms is precisely the cohomology of X with respect to δ. So,*

$$H^*(\Omega^*(X)) \cong H^*(X).$$

Proof. Forman gives the proof in his paper, [7]. This result is also established in an alternative manner by the exposition in the remainder of this chapter. □

5.2 The Associated Kite Complex Defined by Forman's Differential Forms

The behavior of Forman's differential forms on simplicial chains defines a complex whose cells are kites. Consequently, Forman's differential forms elicit a natural cubical structure. This section provides the details surrounding carefully defining the complex of kites associated with the simplicial complex via Forman's differential forms. We conclude this section by associating a Forman differential form with each cubical form on the associated kite complex. In this way, we establish an isomorphism between $\Omega^p_F(X)$ and $C^p(K)$.

To see the relationship between Forman's differential forms and the cubical cochains on an associated kite complex, we first consider an example. We will momentarily ignore orientations. Details surrounding choosing appropriate orientations will come later in this section.

Example 5.2.1. Consider a 2-simplex c and forms f, α and ω as in Example 5.1.2. We will show how each Forman form can be viewed as an evaluation on a kite chain of degree equal to the degree of the form.

First, we consider f, a 0-form. As we saw in Example 5.1.2, given an arbitrary chain in c, f returns a multiple of that chain. We will associate this calculation with a vertex in the center of each cell, namely its barycenter. We define an associated cubical 0-form f_c to return the appropriate coefficient on each chain. So, $f_c(\dot{c}) = a$, $f_c(\dot{b}_i) = a_i$, and $f_c(\dot{v}_i) = A_i$. Note that $\dot{\sigma}$ denotes the barycenter of σ.

Now consider α, a 1-form. We have previously seen that given c or an arbitrary edge b_i in c, α returns a linear combination of the b_i's or v_i's, respectively. α's behavior can be captured as an evaluation of a cubical 1-form α_c on an edge as follows. Denote the edge between $\dot{c}$ and some $\dot{b}_i$ by $\widehat{cb_i}$. Then define $\alpha_c(\widehat{cb_i}) = t_i$. Thus, $\alpha(c)$ can be viewed as assigning a number to each edge between $\dot{c}$ and $\dot{b}_i$. Similarly, $\alpha(b_i)$ can be viewed as assigning a number to each edge drawn between $\dot{b}_i$ and the vertices in its boundary. So, for example, $\alpha_c(\widehat{b_1 v_1}) = s_{11}$ and

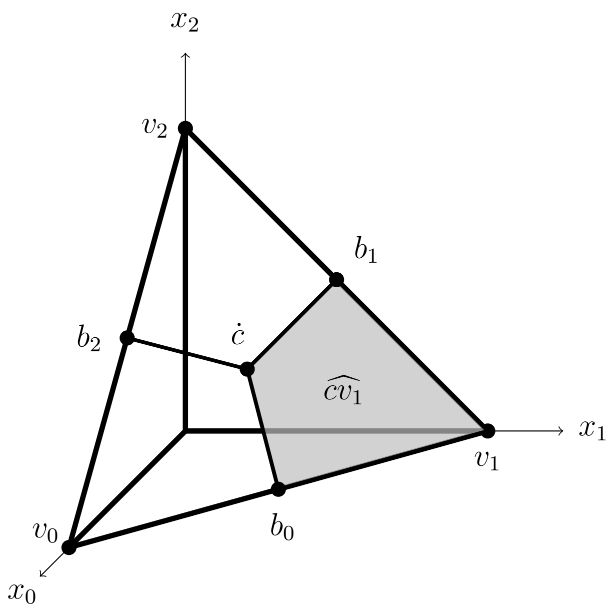


Figure 5.2: The 2-kite $\widehat{cv_1}$ between $\hat{c}$ and v_1.

$\alpha_c(\widehat{b_1 v_2}) = s_{12}$.

Lastly, consider ω, a 2-form. In Example 5.1.2, we saw that $\omega(c)$ returned a multiple ℓ_i of each vertex v_i in c. Consider the kite nestled at v_1 bounded by the planes $x_2 = x_1$, $x_2 = 0$, $x_0 = 0$, and $x_0 = x_1$, where x_i are the barycentric coordinates of c, as seen in Figure 5.2. We denote this kite $\widehat{cv_1}$. Define the cubical 2-form ω_c such that $\omega_c(\widehat{cv_1}) = \ell_1$. Similarly, $\omega_c(\widehat{cv_0}) = \ell_0$ and $\omega_c(\widehat{cv_2}) = \ell_2$ where $\widehat{cv_0}$ and $\widehat{cv_2}$ are the 2-kites nestled at v_0 and v_2, respectively.

In this way, Forman's differential forms naturally elicit an associated kite (and hence cubical) structure on c.

In general, if c is a k-simplex in X and α is a Forman p-form, the coefficient of a $(k-p)$-simplex $b \subseteq c$ in $\alpha(c)$ is associated with a cubical form evaluated on the p-kite that runs transverse to c and b, namely $\widehat{cb}$. We will use a system of equations and inequalities determined by barycentric coordinates to describe the associated kites explicitly.

Definition 5.2.2. Let $p \geq 0$ be arbitrary. Suppose a k-simplex c has vertices $v_0, \ldots, v_k$. Let $b \subseteq c$ be a $(k-p)$-simplex with vertices $u_0, \ldots, u_{k-p}$, a subset of $v_0, \ldots, v_k$. Let $w_0, \ldots, w_{p-1}$ be the vertices of c that are not in b. Let x_{v_i} denote the barycentric coordinate that is 1 at the vertex v_i. We define the ***p*-kite in *c* transverse to *b***, as follows.

$$\widehat{cb} = \{(x_0, \ldots, x_k) : x_{u_0} = \cdots = x_{u_{p-k}}, 0 \leq x_{w_i} \leq x_{u_0},\ \text{for all } i,\ 0 \leq i \leq p-1\}.$$

Note that this kite is analogous to the kites introduced in the proof of Theorem 3.1.5.

We must now settle the issue of assigning an orientation to $\widehat{cb}$.

Definition 5.2.3. Let c, b, and their vertices be as in Definition 5.2.2. Choose a vertex u_j in b as an anchor for our orientation. An **orientation of $\widehat{cb}$** is

$$x_{w_0}, \ldots, x_{w_{p-q}}$$

where $\{x_{w_i}\}$ are the free variables in $\widehat{cb}$ associated with the vectors

$$w_0 - u_j, \ldots, w_{p-1} - u_j. \tag{5.1}$$

Note that the anchor choice u_j for the vectors in (5.1) was arbitrary. Thus, we must establish consistency of the orientations associated with different anchors.

Theorem 5.2.4. *Let c be a k-simplex with vertices $v_0, \ldots, v_k$. Let $b \subseteq c$ be a $(k-p)$-simplex with vertices $u_0, \ldots, u_{k-p}$. Suppose $w_0, \ldots, w_{p-1}$ are the vertices in c that are not in b. Choose an arbitrary vertex u_j in b. Then the orientation of $\widehat{cb}$ given by*

$$w_0 - u_j, \ldots, w_{p-1} - u_j$$

is independent of our choice of u_j.

Proof. Suppose $k - p \geq 1$. To show the orientation of $\widehat{cb}$ will be the same regardless of our choice of anchor in b, we will show that the orientation $\widehat{cb}$ followed by a fixed orientation of b relative to the orientation of c has the same sign regardless of the anchor choice. To do so, we allow our anchor choice to vary continuously over all points in b, not just vertices. We will show that the orientation that arises from an arbitrary anchor is the sign of the determinant of the $k \times k$ matrix of the orientation vectors of c. By linear independence of the entries in this matrix, the determinant is nonzero. Thus, by the continuity of our anchor variation over the connected set b, the orientation of $\widehat{cb}$ is consistent regardless of our choice of anchoring vertex.

Orient b by the order of its vertices, $u_0, \ldots, u_{k-p}$. This orientation is the same as $u_1 - u_0, \ldots, u_{k-p} - u_0$, i.e. the orientation that results from anchoring at u_0. Let a denote the arbitrary point in b that is the anchor of the orientation of $\widehat{cb}$. Orient $\widehat{cb}$ by $w_0 - a, \ldots, w_{p-1} - a$. Each $w_i - a$ can be rewritten as the difference of two vectors anchored at u_0.

$$w_i - a = w_i - u_0 + u_0 - a = (w_i - u_0) - (a - u_0).$$

We will rewrite each of the above orientation vectors anchored at u_0 as column vectors whose entries follow the ordering of the vertices given by $w_0, \ldots, w_{p-1}, u_1, \ldots, u_{k-p}$. Write each $u_i - u_0$, $i \neq 0$, as a column vector. Note that each vector has a single entry of 1 in the u_i position. Write each $w_i - a$ as a column vector, also. Each of these column vectors contains a 1 in the w_i position and has entries in the $u_1, \ldots, u_{k-p}$ positions corresponding to the coefficient of $u_i - u_0$ in $a - u_0$. So, our matrix can be pictured as follows.

$$\left(\begin{array}{c|c} w_i - u_0 & \text{\Large 0} \\ \hline -(a - u_0) & u_i - u_0 \end{array}\right)$$

Note that the vectors $w_0 - u_0, \ldots, w_{p-1} - u_0$ are linearly independent as they are orientation vectors of vertices in c. Similarly the vectors $u_1 - u_0, \ldots, u_{k-p} - u_0$ are linearly independent. Thus, column operations can be used to arrive at the following matrix.

$$\left(\begin{array}{c|c} w_i - u_0 & \text{\Large 0} \\ \hline \text{\Large 0} & u_i - u_0 \end{array}\right)$$

The columns of this reduced matrix are the orientation vectors of c and are hence linearly independent. So, the matrix has nonzero determinant. As previously discussed, it follows that the orientation of the kite $\widehat{cb}$ followed by the orientation of b will be the same regardless of our choice of a. Thus, in particular, the orientation on $\widehat{cb}$ will be consistent regardless of the anchoring vertex u_j in b.

□

In defining a consistent orientation on a kite $\widehat{cb}$ in the associated kite complex, we considered the orientation of $\widehat{cb}$ followed by the orientation of b relative to the orientation of c. This combination of orientations was conveniently chosen to allow for the expression of the orientation of $\widehat{cb}$ as the determinant of a matrix. We will continue to work with this orientation mechanism throughout the remainder of this chapter and hence provide a notation describing it.

Definition 5.2.5. For arbitrary simplicial chains c and b, $\dim b \leq \dim c$, $\mathbf{sgn}(\widehat{\boldsymbol{cb}}, \boldsymbol{b}; \boldsymbol{c})$ denotes the sign associated with the orientation of the kite $\widehat{cb}$ followed by the orientation of b relative to the orientation of c.

Thus, $\operatorname{sgn}(\widehat{cb}, b; c) = 1$ when the orientation of $\widehat{cb}$ followed by the orientation of b agrees with the orientation of c, and $\operatorname{sgn}(\widehat{cb}, b; c) = -1$ when it disagrees.

To describe the boundary of $\widehat{cb}$, we must relate the barycentric coordinate variables of c to the cubical variables of $\widehat{cb}$. We may do this via the orientation of $\widehat{cb}$ associated with the vector orientation on c as follows.

Let

$$w_0 - u_j, \ldots, w_{p-1} - u_j$$

be as in Definition 5.2.3. Our claim is that the projection of these orientation vectors onto the tangent space of $\widehat{cb}$ gives a vector orientation of $\widehat{cb}$ emanating from the barycenter of b, $\dot{b}$.

To see this, recall that in the proof of Theorem 5.2.4, we showed that the above vector orientation is the same regardless of our anchor choice in b, even if the anchor is not a vertex of b. Consider the anchor to be $\dot{b}$. Note that $\dot{b}$ is a vertex of $\widehat{cb}$. So we have

$$w_0 - \dot{b}, \ldots w_{p-1} - \dot{b}.$$

The projection of these variables onto the tangent space of $\widehat{cb}$ gives a vector orientation of $\widehat{cb}$.

If we now take $\dot{b}$ to be the image of the origin under a diffeomorphism of the standard p-cube analogous to the diffeomorphism described in Theorem 3.1.5, then the orientation vectors of $\widehat{cb}$ emanate from the origin and increase to one. So, because $x_{u_0} = \cdots = x_{u_{k-p}}$ in $\widehat{cb}$, we may express the aforementioned diffeomorphism as

$$x_{w_i} = t_i x_{u_0} \qquad \text{for all } i,\ 0 \leq i \leq p-1,$$

where $t_0, \ldots, t_{p-1}$ are the cubical variables of the standard p-cube. In this way, we identify each barycentric variable x_{w_i} with a cubical variable t_i that determines $\widehat{cb}$. We may now define the boundary of $\widehat{cb}$.

Definition 5.2.6. The **boundary of $\widehat{cb}$** is given by the discrete cubical boundary map

$$\partial \widehat{cb} = \sum_{i=0}^{p-1} (-1)^i \left(\widehat{cb}\Big|_{t_i=1} - \widehat{cb}\Big|_{t_i=0} \right).$$

where each t_i is the cubical variable of $\widehat{cb}$ associated with the barycentric variable of c, x_{w_i}, as described above. Note that the x_{w_i}'s come in the order specified by the orientation of $\widehat{cb}$.

Remark 5.2.7. Each face in the boundary of $\widehat{cb}$ can be expressed in terms of the system of equations and inequalities that determines it. Recall our usual expression of the vertices in b, $u_0, \dots, u_{k-p}$, and the vertices of c that are not in b, $w_0, \dots, w_{p-1}$. Recall the diffeomorphism described above

$$x_{w_i} = t_i x_{u_0} \qquad \text{for all } i,\ 0 \leq i \leq p-1.$$

An evaluation of $t_\ell = 1$ for some ℓ gives the equation $x_{w_\ell} = x_{u_0}$. Thus, for an evaluation at 1, the kite in the boundary of $\widehat{cb}$ is described by the following system of equations and inequalities.

$$x_{u_0} = \cdots = x_{u_{k-p}} = x_{w_\ell}$$

$$0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i \neq \ell,\ 0 \leq i \leq p-1.$$

An evaluation of $t_\ell = 0$ for some ℓ gives the equation $x_{w_\ell} = 0$. Thus, for an evaluation at 0, the kite in the boundary of $\widehat{cb}$ is described by the following system of equations and inequalities.

$$x_{u_0} = \cdots = x_{u_{k-p}}$$

$$0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i \neq \ell,\ 0 \leq i \leq p-1.$$

$$x_{w_\ell} = 0.$$

These representations of the kites in the boundary of $\widehat{cb}$ will be utilized in the proofs of the lemmas used to prove Theorem 5.3.3 in Section 5.3.

Now that we have thoroughly discussed orientations and boundaries, we give the formal definition of the kite complex that is associated with the simplicial complex on which Forman's differential forms evaluate.

Definition 5.2.8. Suppose X is a simplicial complex on a smooth n-dimensional manifold M. Orient X standardly via the ordering of its vertices. Then the **kite complex** $\boldsymbol{K}$ **associated with** $\boldsymbol{X}$ is

$$K = \bigcup_{\left\{\substack{\text{all simplices} \\ c \text{ in } X}\right\}} \bigcup_{\left\{\substack{\text{all simplices} \\ b \subseteq c}\right\}} \widehat{cb},$$

where each $\widehat{cb}$ has standard orientation derived from the orientation on X as defined in Definition 5.2.3, and the topology of K is inherited from the topology of X.

In Example 5.2.1, we saw that there is a natural association between Forman differential forms and cubical cochains. Thus, we define the Forman form associated with each cubical form $\alpha \in C^*(K)$.

Definition 5.2.9. Let X be a simplicial complex on a smooth manifold M, and let K be its associated cubical structure. Given a cubical cochain $\alpha \in C^p(K)$, we define its **associated Forman differential form** $\boldsymbol{\alpha_F \in \Omega^p_F(K)}$ on a chain $c \in C_k(X)$, $k \geq p$, by

$$\alpha_F(c) = \sum_{\substack{b \in C_{k-p}(X), \\ b \subseteq c}} \operatorname{sgn}(\widehat{cb}, b; c)\ \alpha(\widehat{cb}) \cdot b.$$

The map taking α to α_F defines an isomorphism

$$C^p(K) \cong \Omega^p_F(X)$$

for all p. Furthermore, this definition is the building block for relating Forman's differential operator D and the discrete coboundary operator d.

5.3 Forman's Complex of Differential Forms and the Discrete Cochain Complex on the Associated Kite Complex

Our exposition of the kite complex associated with Forman's differential forms revealed the relationship between Forman's forms on a simplicial cell complex and cubical cochains on the associated kite complex. In this section, we expose the relationship between Forman's differential coboundary operator D and the cubical coboundary operator d (Theorem 5.3.3). It is this intertwining of a signed version of D with d that establishes the isomorphism between Forman's complex of differential forms and the cubical cochain complex on the associate kite complex. This relationship also allows us to define a cubical cup product based on a product of Forman's differential forms (Section 5.4.2).

We first define a signed version of Forman's differential operator D for purposes of fitting D together with the discrete coboundary operator d on the associated kite complex.

Definition 5.3.1. Define the **signed Forman coboundary operator** $D_F : \Omega_F^p(K) \to \Omega_F^{p+1}(K)$ by

$$D_F\alpha = (-1)^{p+1}D\alpha.$$

$D_F\alpha$ can be explicitly expressed as

$$D_F\alpha = (-1)^{p+1}\partial \circ \alpha + \alpha \circ \partial.$$

Thus, we see that D_F differs from D by moving the sign to sit with $\partial \circ \alpha$ instead. Note that $\partial \circ \alpha$ returns a chain whose dimension is lowered by $p+1$. So, $(-1)^{p+1}$ makes sense here.

Remark 5.3.2. The kernel, cokernel, and cohomology of D_F agree with the kernel, cokernel, and cohomology of D. It is important to note that the switch to D_F respects the invariants that D defines.

Theorem 5.3.3. *Given $\alpha \in C^p(K)$ and its associated Forman differential form $\alpha_F \in \Omega^p_F(X)$,*

$$D_F(\alpha_F) = (d\alpha)_F.$$

Proof. We give the proof for a basis element $\beta \in \Omega^p_F(X)$ that is associated with a cubical cochain that is nonzero on a single kite. First, we introduce the simplices and orientations relevant to our calculations.

Let c be a k-simplex and b a $(k-p)$-simplex in c. We will show the result is true for the basis element β such that $\beta(c) = b$ and $\beta(c') = 0$ for $c' \neq c$ a chain of dimension greater than or equal to k.

Let $v_0, \ldots, v_k$ be the vertices of c and let $u_0, \ldots, u_{k-p}$ be the subset of these vertices in b. Let $w_0, \ldots, w_{p-1}$ by the subset of vertices in c that are not in b. Note, $u_0, \ldots, u_{k-p}$ and $w_0, \ldots, w_{p-1}$ inherit their order from $v_0, \ldots, v_k$.

Our argument will include inspection of simplices in the boundary of b. We introduce an arbitrary $(k-p-1)$-simplex e in ∂b in advance. Let u_j be the vertex that is in b, but not in e. Let $u_a \neq u_j$ be a vertex in $\{u_0, \ldots, u_{k-p}\}$. u_a will serve as the anchor in defining an orientation on $\widehat{ce}$ and $\widehat{cb}$.

Orient c, b, and e standardly by the order of their vertices. By definition of the simplicial boundary operator, the sign of e in ∂b is $(-1)^j$. Note that j is the number of vertices that come before u_j in b.

Orient $\widehat{cb}$ by $w_0 - u_a, \ldots, w_{p-1} - u_a$. Orient $\widehat{ce}$ by $w_0 - u_a, \ldots, u_j - u_a, \ldots, w_{p-1} - u_a$, where u_j is inserted where it belongs in the ordering of the w_i's. In orienting $\widehat{ce}$, its orientation vectors are based at the barycenter of e, $\dot{e}$. Therefore, $\widehat{cb} = \widehat{ce}\big|_{t_j=1}$, where t_j is the cubical variable in $\widehat{ce}$ associated with x_{u_j}. So, the orientation of $\widehat{cb}$ in $\partial\widehat{ce}$ is $(-1)^\ell \cdot \widehat{cb}$, where ℓ is the number of w's that precede u_j.

Define a cubical p-form α by $\alpha(\widehat{cb}) = \text{sgn}(\widehat{cb}, b; c)$ and $\alpha(\sigma) = 0$ for all cubical p-cells $\sigma \neq \widehat{cb}$. Then,

$$
\begin{aligned}
\alpha_F(c) &= \sum_{\substack{(k-p)-\text{simplices} \\ \text{in } c,\, b'}} \text{sgn}(\widehat{cb'}, b'; c)\ \alpha(\widehat{cb'}) \cdot b' \\
&= \text{sgn}(\widehat{cb}, b; c)\ \alpha(\widehat{cb}) \cdot b \\
&= b.
\end{aligned}
$$

Thus, $\beta = \alpha_F$. For the remainder of the proof, we will use the notation α_F.

We are now ready to calculate $(D_F\alpha_F)$ and $(d\alpha)_F$ on an arbitrary simplex c' of dimension at least p.

$$D_F\alpha_F(c') \;=\; (-1)^{p+1}\partial\alpha_F(c') + \alpha_F(\partial c').$$

Because α_F takes a nonzero value on c only, $D_F\alpha_F(c') = 0$ except when $c' = c$ or $c \in \partial c'$.

By definition, $(d\alpha)_F(c') = 0$ if and only if $d\alpha(\widehat{c'f}) = 0$ for all kites $\widehat{c'f}$ where $\dim f = (\dim c') - p - 1$. The kites for which $d\alpha$ will be nonzero are those that contain $\widehat{cb}$ in their boundary. Lemma 5.3.5 and Lemma 5.3.6 show that the only kites that contain $\widehat{cb}$ in their boundary are $\widehat{c'b}$ or $\widehat{ce'}$, where c' is such that $c \in \partial c'$ or $c' = c$ and $e' \in \partial b$, respectively. Thus, both $D_F\alpha_F(c')$ and $(d\alpha)_F(c')$ are nonzero only when $c' = c$ or $c \in \partial c'$.

Consequently, our argument splits into two cases.

<u>Case 1</u>: $c' = c$.

$$\begin{aligned} D_F\alpha_F(c) &= (-1)^{p+1}\partial\alpha_F(c) + \alpha_F(\partial c) \\ &= (-1)^{p+1}\partial\alpha_F(c) \\ &= (-1)^{p+1}\partial b. \end{aligned}$$

By Definition 5.2.9 and by Lemma 5.3.5,

$$\begin{aligned} (d\alpha)_F(c) &= \sum_{\substack{(k-p-1)-\text{simplices} \\ \text{in } c,\ e'}} \operatorname{sgn}(\widehat{ce'}, e'; c)\ d\alpha(\widehat{ce'}) \cdot e' \\ &= \sum_{\substack{(k-p-1)-\text{simplices} \\ \text{in } \partial b,\ e'}} \operatorname{sgn}(\widehat{ce'}, e'; c)\ d\alpha(\widehat{ce'}) \cdot e' \end{aligned}$$

because Lemma 5.3.5 shows that $d\alpha(\widehat{ce'}) = 0$ for all $e' \notin \partial b$.

In order to show that the arbitrary element $e \in \partial b$ previously introduced has coefficient in $(d\alpha)_F(c)$ equal to the sign of its orientation in ∂b times $(-1)^{p+1}$, i.e. $(-1)^{j+p+1}$, we first calculate $d\alpha(\widehat{ce})$.

$$\begin{aligned} d\alpha(\widehat{ce}) &= \alpha(\partial\widehat{ce}) \\ &= \alpha((-1)^{\ell}\widehat{cb}) \\ &= (-1)^{\ell}\,\mathrm{sgn}(\widehat{cb}, b; c). \end{aligned}$$

Thus, the coefficient of e in $(d\alpha)_F(c)$ is

$$\begin{aligned} \mathrm{sgn}(\widehat{ce}, e; c)\ d\alpha(\widehat{ce}) &= [(-1)^{p+\ell+j+1}\,\mathrm{sgn}(\widehat{cb}, b; c)]\ d\alpha(\widehat{ce}) && \text{by Lemma 5.3.4 (1)} \\ &= (-1)^{p+\ell+j+1}\,\mathrm{sgn}(\widehat{cb}, b; c)[(-1)^{\ell}\,\mathrm{sgn}(\widehat{cb}, b; c)] \\ &= (-1)^{j+p+1}. \end{aligned}$$

Because e was arbitrary, this shows that $(d\alpha)_F(c)$ will be a linear combination of all $e' \in \partial b$, with the coefficient of each e' equal to its orientation sign in the boundary of b times $(-1)^{p+1}$. In other words, $d(\alpha)_F(c) = (-1)^{p+1}\partial b$.

Hence, $(D_F\alpha_F)(c) = (d\alpha)_F(c)$.

<u>Case 2</u>: $c \in \partial c'$.

c' is a $(k+1)$-simplex. Orient c' by its vertices $y_0, \ldots, y_{k+1}$. Let y_m be the vertex in c' that is not in c. Orient c by the order of its vertices, $y_1, \ldots, \hat{y}_m, \ldots, y_{k+1}$. Note that c will have orientation $(-1)^m \cdot c$ in $\partial c'$.

As above, orient b by its vertices $u_0, \ldots, u_{k-p}$. Let $w_0, \ldots, y_m, \ldots, w_{p-1}$ by the vertices in c'

that are not in b. Note that here y_m has been inserted into the place where it belongs in the list of w's.

Let u_a in b by the anchoring vertex as above. Orient $\widehat{cb}$ by $w_0 - u_a, \ldots, w_{p-1} - u_a$. Orient $\widehat{c'b}$ by $w_0 - u_a, \ldots, y_m - u_a, \ldots, w_{p-1} - u_a$. In orienting $\widehat{c'b}$, its orientation vectors are based at the barycenter of b, $\dot{b}$. Therefore, $\widehat{cb} = \widehat{c'b}\big|_{t_m=0}$, where t_m is the cubical variable in $\widehat{c'b}$ associated with x_{y_m}. So, the orientation of $\widehat{cb}$ in $\partial\widehat{c'b}$ is $(-1)^{s+1} \cdot \widehat{cb}$, where s is the number of w's that precede y_m.

Now, we calculate $D_F\alpha_F(c')$ and $(d\alpha)_F(c')$.

$$\begin{aligned} D_F\alpha_F(c') &= (-1)^{p+1}\partial\alpha_F(c') + \alpha_F(\partial c') \\ &= \alpha_F(\partial c') \\ &= \alpha_F((-1)^m c) \\ &= (-1)^m b. \end{aligned}$$

By Definition 5.2.9 and by Lemma 5.3.6,

$$\begin{aligned} (d\alpha)_F(c') &= \sum_{\substack{(k-p)-\text{simplices} \\ \text{in } c',\ f}} \operatorname{sgn}(\widehat{c'f}, f; c')\ d\alpha(\widehat{c'f}) \cdot f \\ &= \sum_{\substack{(k-p)-\text{simplices} \\ \text{in } c',\ f}} \operatorname{sgn}(\widehat{c'f}, f; c')\ \alpha(\partial\widehat{c'f}) \cdot f \\ &= \operatorname{sgn}(\widehat{c'b}, b; c')\ \alpha(\partial\widehat{c'b}) \cdot b \end{aligned}$$

because Lemma 5.3.6 shows that $\widehat{cb}$ appears in $\partial\widehat{c'b}$, and Lemma 5.3.7 shows that $\widehat{cb}$ does not appear in $\widehat{c'f}$ for all $f \neq b$. Thus, $d\alpha(\widehat{c'f}) = 0$ for all $f \neq b$. So, we need only calculate

$d\alpha(\widehat{c'b})$.

$$\begin{aligned} d\alpha(\widehat{c'b}) &= \alpha(\partial\widehat{c'b}) \\ &= \alpha((-1)^{s+1}\widehat{cb}) \\ &= (-1)^{s+1}\operatorname{sgn}(\widehat{cb}, b; c). \end{aligned}$$

Thus,

$$\begin{aligned} (d\alpha)_F(c') &= \operatorname{sgn}(\widehat{c'b}, b; c')\ d\alpha(\widehat{c'b}) \cdot b \\ &= [(-1)^{s+m+1}\operatorname{sgn}(\widehat{cb}, b; c)]\ d\alpha(\widehat{c'b}) \cdot b && \text{by Lemma 5.3.4 (2)} \\ &= (-1)^{s+m+1}\operatorname{sgn}(\widehat{cb}, b; c)\ \alpha((-1)^{s+1}\widehat{cb}) \cdot b \\ &= (-1)^{s+m+1}\operatorname{sgn}(\widehat{cb}, b; c)[(-1)^{s+1}\operatorname{sgn}(\widehat{cb}, b; c)] \cdot b \\ &= (-1)^m b. \end{aligned}$$

So, $D_F\alpha_F(c') = (d\alpha)_F(c')$ when $c \in \partial c'$.

We have now shown that for an arbitrary simplex c', $\dim c' \geq p+1$,

$$D_F\alpha_F(c') = (d\alpha)_F(c').$$

□

Lemma 5.3.4. *For c', c, b, and e as in the proof of Theorem 5.3.3 (including vertex labels),*

(1) $\operatorname{sgn}(\widehat{ce}, e; c) = (-1)^{p+\ell+j+1}\operatorname{sgn}(\widehat{cb}, b; c)$.

(2) $\operatorname{sgn}(\widehat{c'b}, b; c') = (-1)^{s+m+1}\operatorname{sgn}(\widehat{cb}, b; c)$.

Proof. (1). To show the desired relationship between the sign terms, we must first rewrite the orientations of c, b, and e in their vector representations. We will anchor our orientation at the vertex u_a in b. Note that $u_a = v_i$ for some i.

Recall that c is oriented by $v_0, \ldots, v_k$. So, the orientation on c can be rewritten as

$$(-1)^i(v_0 - u_a, \ldots, \widehat{v_i - u_a}, \ldots, v_k - u_a).$$

The orientation of $u_0, \ldots, u_{p-k}$ on b can be rewritten as

$$(-1)^a(u_0 - u_a, \ldots, \widehat{u_a - u_a}, \ldots, u_{k-p} - u_a).$$

The vector representation of the orientation of e depends on whether the anchor u_a comes before or after the variable u_j that is in b but not in e. Thus, the orientation of $u_0, \ldots, \hat{u}_j, \ldots, u_{k-p}$ on e can be rewritten as either

$$(-1)^a(u_0 - u_a, \ldots, \widehat{u_a - u_a}, \ldots, \widehat{u_j - u_a}, \ldots, u_{k-p} - u_a)$$

or

$$(-1)^{a+1}(u_0 - u_a, \ldots, \widehat{u_j - u_a}, \ldots, \widehat{u_a - u_a}, \ldots, u_{k-p} - u_a).$$

So, under this orientation convention,

$$\operatorname{sgn}(\widehat{cb}, b; c) = (-1)^{i+a} \operatorname{sgn}(w_0 - u_a, \ldots, w_{p-1} - u_a, u_0 - u_a, \ldots, \widehat{u_a - u_a}, \ldots, u_{k-p} - u_a)$$

and either $\operatorname{sgn}(\widehat{ce}, e; c) =$

$$(-1)^{i+a} \operatorname{sgn}(w_0 - u_a, \ldots, u_j - u_a, \ldots, w_{p-1} - u_a, u_0 - u_a, \ldots, \widehat{u_a - u_a}, \ldots, \widehat{u_j - u_a}, \ldots, u_{k-p} - u_a)$$

or $\operatorname{sgn}(\widehat{ce}, e; c) =$

$$(-1)^{i+a+1}\operatorname{sgn}(w_0 - u_a, ..., u_j - u_a, ..., w_{p-1} - u_a, u_0 - u_a, ..., \widehat{u_j - u_a}, ..., \widehat{u_a - u_a}, ..., u_{k-p} - u_a).$$

The sign comparison between $\operatorname{sgn}(\widehat{ce}, e; c)$ and $\operatorname{sgn}(\widehat{cb}, b; c)$ will thus be handled in two cases. Regardless of the case, the method is the same. We simply need to identify the number of moves necessary to move $u_j - u_a$ from its position in the orientation of $\widehat{ce}$ to its position in b.

Case 1: u_a comes before u_j.

Note that the above expression for $\operatorname{sgn}(\widehat{cb}, b; c)$ and the above expression for $\operatorname{sgn}(\widehat{ce}, e; c)$ each have coefficient $(-1)^{i+a}$. The sgnfactors in these expressions differ by moving $u_j - u_a$ as discussed. Moving $u_j - u_a$ from its position in $\widehat{ce}$ to the $w_{p-1} - u_a$ position requires $p - 1 - \ell$ moves. Recall that ℓ is the number of w's that come before u_j. The number of moves to then take $u_j - u_a$ to its correct spot in b is j since u_a comes before u_j. That is a total of $p - \ell + j - 1$ moves. In other words,

$$\operatorname{sgn}(\widehat{ce}, e; c) = (-1)^{p+\ell+j+1}\operatorname{sgn}(\widehat{cb}, b; c).$$

Case 2: u_a comes after u_j.

We handle this case analogously to Case 1. First, we note that the sign coefficients in $\operatorname{sgn}(\widehat{cb}, b; c)$ and $\operatorname{sgn}(\widehat{ce}, e; c)$ differ by a multiple of -1. This will be added to the sign associated with the number of swaps necessary to move $u_j - u_a$ from $\widehat{ce}$ to b. As in Case 1, $p - 1 - \ell$ moves are necessary to take $u_j - u_a$ to the $w_p - u_a$ position. This time, however,

$j+1$ moves are required to take $u_j - u_a$ to its correct position in b because u_a comes after u_j. This is a total of $p-\ell+j$ moves. Hence, we have

$$\begin{aligned}\operatorname{sgn}(\widehat{ce}, e; c) &= -(-1)^{p+\ell+j}\operatorname{sgn}(\widehat{cb}, b; c)\\ &= (-1)^{p+\ell+j+1}\operatorname{sgn}(\widehat{cb}, b; c).\end{aligned}$$

In either case, the result holds.

(2). To show the desired relationship, we must first rewrite the orientations of c', c, and b in their vector representations. We will anchor our orientation at the vertex u_a in b. Note that $u_a = y_t$ for some t.

Recall that c' is oriented by $y_0, \ldots, y_{k+1}$. So, the orientation on c' can be rewritten as

$$(-1)^t(y_0 - u_a, \ldots, \widehat{y_t - u_a}, \ldots, y_{k+1} - u_a).$$

The vector representation of the orientation of c depends on whether the anchor u_a comes before or after the variable y_m that is in c' but not in c. Thus, the orientation of $y_0, \ldots, \hat{y}_m, \ldots, y_{k+1}$ on c can be rewritten as either

$$(-1)^t(y_0 - u_a, \ldots, \widehat{y_t - u_a}, \ldots, \widehat{y_m - u_a}, \ldots, y_{k+1} - u_a)$$

or

$$(-1)^{t+1}(y_0 - u_a, \ldots, \widehat{y_m - u_a}, \ldots, \widehat{y_t - u_a}, \ldots, y_{k+1} - u_a).$$

The orientation of $u_0, \ldots, u_{p-k}$ on b can be rewritten as

$$(-1)^a(u_0 - u_a, \ldots, \widehat{u_a - u_a}, \ldots, u_{k-p} - u_a).$$

So, under this orientation convention, either $\operatorname{sgn}(\widehat{cb}, b; c) =$

$$(-1)^{a+t}\operatorname{sgn}(w_0 - u_a, ..., w_{p-1} - u_a, u_0 - u_a, ..., \widehat{y_t - u_a}, ..., \widehat{y_m - u_a}, ..., y_{k+1} - u_a).$$

or $\operatorname{sgn}(\widehat{cb}, b; c) =$

$$(-1)^{a+t+1}\operatorname{sgn}(w_0 - u_a, ..., w_{p-1} - u_a, u_0 - u_a, ..., \widehat{y_m - u_a}, ..., \widehat{y_t - u_a}, ..., y_{k+1} - u_a)$$

and

$$\operatorname{sgn}(\widehat{c'b}, b; c') = (-1)^{a+t}\operatorname{sgn}(w_0 - u_a, ..., y_m - u_a, ..., w_{p-1} - u_a, u_0 - u_a, ..., \widehat{u_a - u_a}, ..., u_{k-p} - u_a).$$

The sign comparison between $\operatorname{sgn}(\widehat{c'b}, b; c')$ and $\operatorname{sgn}(\widehat{cb}, b; c)$ will thus be handled in two cases. Regardless of the case, the method is the same. Because $y_m - u_a$ appears in $\operatorname{sgn}(\widehat{c'b}, b; c')$, but does not appear in $\operatorname{sgn}(\widehat{cb}, b; c)$, the sign difference is captured by identifying the number of moves necessary to manually move $y_m - u_a$ into place within $\operatorname{sgn}(\widehat{c'b}, b; c')$.

<u>Case 1</u>: u_a comes before y_m.

Note that each term has coefficient $(-1)^{t+a}$. The sgn terms differ by moving $y_m - u_a$ manually as discussed. Thus, we first move $y_m - u_a$ to the front of the list. The number of moves necessary is equal to the number of w's that come before y_m, namely s. After the other orientation vectors have been rearranged, we then move $y_m - u_a$ into its correct spot. This requires $m - 1$ moves because u_a comes before y_m, but u_a has been removed from the list. Thus, we have moved $y_m - u_a$ a total of $s + m - 1$ swaps. So,

$$\operatorname{sgn}(\widehat{c'b}, b; c') = (-1)^{s+m+1}\operatorname{sgn}(\widehat{cb}, b; c).$$

Case 2: u_a comes after y_m

Note that the sign coefficients in each differ by a multiple of -1. This will be added to the sign associated with the number of swaps necessary to move $y_m - u_a$ manually into its correct position in the list of orientation vectors. As in Case 1, we move $y_m - u_a$ to the front of the list using s moves. After reordering the remaining orientation vectors, we move $y_m - u_a$ into its correct spot. This requires the usual m moves because u_a comes after y_m. So, manually placing $y_m - u_a$ where it belongs requires $s + m$ moves. Thus,

$$\begin{aligned} \operatorname{sgn}(\widehat{c'b}, b; c') &= -(-1)^{s+m} \operatorname{sgn}(\widehat{cb}, b; c) \\ &= (-1)^{s+m+1} \operatorname{sgn}(\widehat{cb}, b; c). \end{aligned}$$

In either case, the result holds.

□

Lemma 5.3.5. *Given a k-simplex c, a $(k-p)$-simplex $b \subseteq c$, and a $(k-p-1)$-simplex e, $\widehat{cb} \in \partial \widehat{ce}$ if and only if $e \in \partial b$.*

Proof. Regardless of the direction of the implication, we will use the following notation. b has vertices $u_0, \ldots, u_{k-p}$. The vertices in c that are not in b are denoted by $w_0, \ldots, w_{p-1}$. If v_i is a vertex in c, x_{v_i} is its associated barycentric coordinate variable such that, at v_i, $x_{v_i} = 1$ and $x_{v_j} = 0$ for all $j \neq i$. The kite $\widehat{cb}$ can be described by the system of equations and inequalities that determines it (as discussed in the proof of Theorem 3.1.5) as follows.

$$x_{u_0} = \cdots = x_{u_j} = \cdots = x_{u_{k-p}}$$

$$0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i,\ 0 \leq i \leq p-1.$$

($\rightarrow$) The proof is by contrapositive. Suppose $e \notin \partial b$. Then $\{\text{vertices of } e\} \not\subseteq \{\text{vertices of } b\}$. Let $r_0, \ldots, r_{k-p-1}$ denote the vertices of e, and $s_0, \ldots, s_p$ denote the vertices in c that are not in e. At least one r_i is not in b.

Suppose r_e is a vertex in e that is not in b. The kite $\widehat{ce}$ is determined by the following system of equations and inequalities.

$$x_{r_0} = \cdots = x_{r_{k-p-1}}$$

$$0 \leq x_{s_i} \leq x_{r_0} \text{ for all } i,\ 0 \leq i \leq p.$$

The boundary of $\widehat{ce}$ can be described as discussed in Remark 5.2.7. Choose an arbitrary x_{s_ℓ}, $0 \leq \ell \leq p$. The boundary of $\widehat{ce}$ given by an evaluation equal to 0 of the cubical variable associated with x_{s_ℓ} is the kite

$$x_{r_0} = \cdots = x_{r_{k-p-1}}$$

$$0 \leq x_{s_i} \leq x_{r_0} \text{ for all } i \neq \ell,\ 0 \leq i \leq p$$

$$x_{s_\ell} = 0.$$

The boundary of $\widehat{ce}$ associated with an evaluation at 1 is the kite

$$x_{r_0} = \cdots = x_{r_{k-p-1}} = x_{s_\ell}$$

$$0 \leq x_{s_i} \leq x_{r_0} \text{ for all } i \neq \ell,\ 0 \leq i \leq p$$

Regardless of our choice of x_{s_ℓ} or the boundary evaluation value, the boundary of $\widehat{ce}$ that results will be such that x_{r_e} appears in the system of equations given by $x_{r_0} = \cdots = x_{r_{k-p-1}}$ and the system of equations given by $x_{r_0} = \cdots = x_{r_{k-p-1}} = x_{s_\ell}$. These equations cannot determine $\widehat{cb}$ because $r_e \notin b$, i.e. for all i, $r_e \neq u_i$. Thus, for all i, $x_{r_e} \neq x_{u_i}$ for some point in $\widehat{cb}$. So, $\widehat{cb} \notin \partial\widehat{ce}$.

($\leftarrow$) Suppose $e \in \partial b$. Then the vertices of e are $u_0, \ldots, \hat{u}_j, \ldots, u_{k-p}$ for some u_j. The vertices in c that are not in e are $w_0, \ldots, u_j, \ldots, w_{p-1}$. The kite $\widehat{ce}$ can be described by the system of equations and inequalities that determines it as follows.

$$x_{u_0} = \cdots = \widehat{x_{u_j}} = \cdots = x_{u_{k-p}}$$

$$0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i,\ 0 \leq i \leq p-1.$$

The boundary of $\widehat{ce}$ may be described as discussed in Remark 5.2.7. Thus, the boundary given by an evaluation equal to 1 of the cubical variable associated with x_{u_j} is determined by the following system of equations and inequalities.

$$x_{u_0} = \cdots = x_{u_j} = \cdots = x_{u_{k-p}}$$

$$0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i,\ 0 \leq i \leq p-1.$$

This is exactly $\widehat{cb}$. So, $\widehat{cb} \in \partial\widehat{ce}$. □

Lemma 5.3.6. *Given a k-simplex c, a $(k+1)$-simplex c', and a $(k-p)$-simplex b in both c and in c', $\widehat{cb} \in \partial\widehat{c'b}$ if and only if $c \in \partial c'$.*

Proof. Regardless of the direction of the implication, we will use the following notation. c' has vertices $y_0, \ldots, y_{k+1}$. b has vertices $u_0, \ldots, u_{k-p}$. The vertices in c' that are not in b are denoted by $w_0, \ldots, w_p$. If v_i is a vertex in c', x_{v_i} is its associated barycentric coordinate variable such that, at v_i, $x_{v_i} = 1$ and $x_{v_j} = 0$ for all $j \neq i$. The kite $\widehat{c'b}$ can be described by the system of equations and inequalities that determines it (as discussed in the proof of Theorem 3.1.5) as follows.

$$x_{u_0} = \cdots = x_{u_{k-p}}$$

$$0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i,\ 0 \leq i \leq p.$$

The boundary of $\widehat{c'b}$ may be described as in Remark 5.2.7. So an evaluation equal to 1 of the cubical variable associated to some x_{w_ℓ} gives the kite

$$x_{u_0} = \cdots = x_{u_{k-p}} = x_{w_\ell}$$

$$0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i \neq \ell,\ 0 \leq i \leq p.$$

An evaluation equal to 0 gives

$$x_{u_0} = \cdots = x_{u_{k-p}}$$

$$0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i \neq \ell,\ 0 \leq i \leq p$$

$$x_{w_i} = 0.$$

($\rightarrow$) The proof is by contrapositive. Assume $c \notin \partial c'$. Then $\{\text{vertices of } c\} \not\subseteq \{\text{vertices of } c'\}$. Let $r_0, \ldots, r_k$ denote the vertices of c. So, there is a vertex r_c in c that is not a vertex in c'.

Note that both c and c' must contain all the vertices of b, otherwise we cannot form $\widehat{cb}$ or $\widehat{c'b}$. Thus, r_c is not a vertex in b. So, if $s_0, \ldots, s_{p-1}$ denote the vertices in c that are not in b, then $r_c = s_\ell$ for some ℓ, $0 \leq \ell \leq p-1$.

By definition, $\widehat{cb}$ is determined by the following system of equations and inequalities.

$$x_{u_0} = \cdots = x_{u_{k-p}}$$

$$0 \leq x_{s_i} \leq x_{u_0} \text{ for all } i,\ 0 \leq i \leq p-1.$$

In particular, $0 \leq x_{r_c} \leq x_{u_0}$ is an inequality that determines $\widehat{cb}$. However, the equations and inequalities that determine the boundary of $\widehat{c'b}$ (as shown in the preamble of this proof) contain only those barycentric variables that correspond to vertices in c'. Because r_c is not in c', x_{r_c} does not appear in the system of equations and inequalities representations of the kites in $\partial\widehat{c'b}$. Thus, $\widehat{cb} \notin \partial\widehat{c'b}$.

($\leftarrow$) Assume $c \in \partial c'$. Then the vertices of c are $y_0, \ldots, \hat{y}_m, \ldots y_{k+1}$ for some y_m. Thus, $y_m = w_\ell$ for some ℓ, $0 \leq \ell \leq p$. In particular, $x_{y_m} = x_{w_\ell} = 0$ by the definition of $c \in \partial c'$. So, the kite $\widehat{cb}$ can be described by the system of equations and inequalities that determines it as follows.

$$x_{u_0} = \cdots = x_{u_{k-p}}$$

$$0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i \neq \ell,\ 0 \leq i \leq p$$

$$x_{w_\ell} = 0.$$

This is exactly the kite in the boundary of $\widehat{c'b}$ given by an evaluation equal to 0 of the cubical variable associated with x_{w_ℓ}. Thus, $\widehat{cb} \in \partial\widehat{c'b}$.

□

Lemma 5.3.7. *Suppose c' is a $(k+1)$-simplex, c is a k-simplex, and b and f are $(k-p)$-simplices. Suppose $b, f \subseteq c'$ and $b \subseteq c$. If $b \neq f$, then $\widehat{cb} \notin \partial\widehat{c'f}$.*

Proof. Suppose $b \neq f$. Let $\mu_0, \ldots, \mu_{k-p}$ be the vertices in f and $\nu_0, \ldots, \nu_p$ be the vertices in c' that are not in f. Let $u_0, \ldots, u_{k-p}$ be the vertices in b and $w_0, \ldots, w_{p-1}$ be the vertices in c that are not in b. Then $\widehat{c'f}$ is determined by the following system of equations and inequalities.

$$x_{\mu_0} = \cdots = x_{\mu_{k-p}}$$

$$0 \leq x_{\nu_i} \leq x_{\mu_0} \text{ for all } i,\ 0 \leq i \leq p.$$

$\widehat{cb}$ is determined by the following system of equations and inequalities.

$$x_{u_0} = \cdots = x_{u_{k-p}}$$

$$0 \leq x_{w_i} \leq x_{u_0} \text{ for all } i,\ 0 \leq i \leq p-1.$$

By Remark 5.2.7, an arbitrary kite in $\partial\widehat{cf}$ can be expressed as

$$x_{\mu_0} = \cdots = x_{\mu_{k-p}} = x_{\nu_\ell}$$

$$0 \leq x_{\nu_i} \leq x_{\mu_0} \text{ for all } i \neq \ell,\ 0 \leq i \leq p$$

or

$$x_{\mu_0} = \cdots = x_{\mu_{k-p}}$$

$$0 \leq x_{\nu_i} \leq x_{\mu_0} \text{ for all } i \neq \ell,\ 0 \leq i \leq p.$$

$$x_{\nu_\ell} = 0.$$

Because $b \neq f$, there is some vertex μ_r in f that is not in b. Note that x_{μ_r} appears in the system of equations in both boundary expressions given above. However, because μ_r is not in b, x_{μ_r} does not appear in the system of equations of $\widehat{cb}$. Thus, in $\widehat{cb}$, $x_{\mu r}$ is not constrained to equal any other variable. Therefore, neither of the expressions of $\partial\widehat{cf'}$ can describe $\widehat{cb}$, and $\widehat{cb} \notin \partial\widehat{c'f}$. □

We have previously established that $C^p(K) \cong \Omega^p(X)$ for all p. Theorem 5.3.3 intertwines the signed Forman exterior derivative D_F and the cubical exterior derivative d. so the chain complexes given by

$$\cdots \longrightarrow C^{p-1}(K) \xrightarrow{d} C^p(K) \xrightarrow{d} C^{p+1}(K) \xrightarrow{d} \cdots$$

and

$$\cdots \longrightarrow \Omega^{p-1}(X) \xrightarrow{D_F} \Omega^p(X) \xrightarrow{D_F} \Omega^{p+1}(X) \xrightarrow{D_F} \cdots$$

are isomorphic. Hence, they define isomorphic cohomology groups. Because D_F and D agree on cohomology, we have established

$$H^p(K) \cong H^p(\Omega(X))$$

for all p. Note that we have established Theorem 5.1.6 in an alternative manner via the associated kite complex.

5.4 Defining a Cup Product on the Associated Kite Complex

In this section, we define a product of Forman differential forms. This product, together with the identification of forman forms with cubical forms on the associated kite complex, defines a natural cubical cup product. We show that this cup product agrees with the cubical cup product of Chapter 3 that is used to define the discrete Hodge star over $\mathbb{R}$. This places Forman's theory into the context of the cubical theory that we develop in this paper.

5.4.1 A Product of Forman Differential Forms

We first define a product of Forman differential forms as suggested by Forman in [7]. We then show that this product is a derivation of the signed Forman coboundary operator D_F.

Definition 5.4.1. We define the **product of Forman differential forms** on a simplicial complex X of a smooth manifold M, denoted $\cup_F : \Omega^p_F(X) \times \Omega^q_F(X) \to \Omega^{p+q}_F(X)$, by

$$\alpha \cup_F \beta \quad = \quad \frac{1}{2}\left[\beta \circ \alpha + (-1)^{pq}\alpha \circ \beta\right]. \tag{5.2}$$

This definition is chosen so that $\cup_F$ satisfies the standard relationship of

$$\alpha \cup_F \beta = (-1)^{pq}(\beta \cup_F \alpha)$$

for all $\alpha \in \Omega^p_F(X)$ and $\beta \in \Omega^q_F(X)$. It also provides a derivation of D_F.

Theorem 5.4.2. *D_F is a derivation with respect to $\cup_F$.*

Proof. Let $\alpha \in \Omega^p_F(X)$ and $\beta \in \Omega^q_F(X)$. We will show that

$$D_F(\alpha \cup_F \beta) = D_F\alpha \cup_F \beta + (-1)^p \alpha \cup_F D_F\beta$$

by first calculating each term.

$$\begin{aligned}
D_F(\alpha \cup_F \beta) &= (-1)^{p+q+1}\partial(\alpha \cup_F \beta) + (\alpha \cup_F \beta) \circ \partial \\
&= \frac{1}{2}[(-1)^{p+q+1}(\partial\beta \circ \alpha + (-1)^{pq}\partial\alpha \circ \beta) + \beta \circ \alpha \circ \partial + (-1)^{pq}\alpha \circ \beta \circ \partial] \\
&= \frac{1}{2}[(-1)^{p+q+1}((-1)^{q+1}D_F\beta \circ \alpha - (-1)^{q+1}\beta \circ \partial \circ \alpha) \\
&\quad +(-1)^{p+q+1}((-1)^{pq+p+1}D_F\alpha \circ \beta - (-1)^{pq+p+1}\alpha \circ \partial \circ \beta) \\
&\quad +\beta \circ \alpha \circ \partial + (-1)^{pq}\alpha \circ \beta \circ \partial] \\
&= \frac{1}{2}[(-1)^p D_F\beta \circ \alpha + (-1)^{p+1}\beta \circ \partial \circ \alpha + (-1)^{pq+q}D_F\alpha \circ \beta \\
&\quad +(-1)^{pq+q+1}\alpha \circ \partial \circ \beta + \beta \circ \alpha \circ \partial + (-1)^{pq}\alpha \circ \beta \circ \partial].
\end{aligned}$$

$$\begin{aligned}
D_F\alpha \cup_F \beta &= \frac{1}{2}[\beta \circ D_F\alpha + (-1)^{(p+1)q}D_F\alpha \circ \beta] \\
&= \frac{1}{2}[\beta((-1)^{p+1}\partial \circ \alpha + \alpha \circ \partial) + (-1)^{pq+q}D_F\alpha \circ \beta] \\
&= \frac{1}{2}[(-1)^{p+1}\beta \circ \partial \circ \alpha + \beta \circ \alpha \circ \partial + (-1)^{pq+q}D_F\alpha \circ \beta]
\end{aligned}$$

$$\begin{aligned}(-1)^p\alpha \cup_F D_F\beta &= \frac{1}{2}(-1)^p\left[\beta\circ D_F\alpha(c)+(-1)^{p(q+1)}D_F\alpha\circ\beta(c)\right]\\ &= \frac{1}{2}[(-1)^p D_F\beta\circ\alpha+(-1)^{pq}\alpha((-1)^{q+1}\partial\circ\beta+\beta\circ\partial)]\\ &= \frac{1}{2}[(-1)^p D_F\beta\circ\alpha+(-1)^{pq+q+1}\alpha\circ\partial\circ\beta+(-1)^{pq}\alpha\circ\beta\circ\partial].\end{aligned}$$

Thus,

$$\begin{aligned}D_F\alpha\cup_F\beta+(-1)^p\alpha\cup_F D_F\beta &= \frac{1}{2}[(-1)^{p+1}\beta\circ\partial\circ\alpha+\beta\circ\alpha\circ\partial+(-1)^{pq+q}D_F\alpha\circ\beta\\ &\quad+(-1)^p D_F\beta\circ\alpha+(-1)^{pq+q+1}\alpha\circ\partial\circ\beta+(-1)^{pq}\alpha\circ\beta\circ\partial]\\ &= D_F(\alpha\cup_F\beta),\end{aligned}$$

as desired.

□

5.4.2 The Cubical Cup Product on the Associated Kite Complex

The identification of Forman differential forms and cubical cochains in Section 5.2 allows us to define a cubical cup product associated with the product of Forman forms $\cup_F$. Note that this results in a cup product with coefficients in $\mathbb{R}$. The intertwining of D_F and d then provides an avenue for proving that this cubical cup product is a derivation of d, and ultimately establishing that this product is a cup product as defined by Whitney in Definition 2.3.2.

Definition 5.4.3. We define the cubical interpretation of the product of Forman differential forms, namely the **Forman cubical cup product**, $\cup_{F_c} : C^p(K)\times C^q(K)\to C^{p+q}(K)$

implicitly. For $\alpha \in C^p(K)$ and $\beta \in C^q(K)$, $\alpha \cup_{F_c} \beta$ must satisfy

$$(\alpha \cup_{F_c} \beta)_F = \alpha_F \cup_F \beta_F.$$

Remark 5.4.4. We can reinterpret this definition explicitly as follows. Let $\alpha \in C^p(K)$ and $\beta \in C^p(K)$. Suppose σ is a k-simplex, $k \geq p+q$, in X and e is a $(k-p-q)$-simplex in σ. Then $(\alpha \cup_{F_c} \beta)(\widehat{\sigma e})$ is the coefficient of e in $(\alpha_F \cup_F \beta_F)(\sigma)$.

Theorem 5.4.5. *$\cup_{F_c}$ is a cup product on K, i.e. it satisfies the conditions of Definition 2.3.2.*

Proof. Property 1. Because $\cup_{F_c}$ is determined by $\cup_F$, it suffices to consider the behavior of the Forman cup product on basis elements of Forman forms. Let $\alpha \in \Omega^p_F(X)$ be nonzero on a single chain c of dimension $k \geq p$, with $\alpha(c) = b$ where b is a $(k-p)$-simplex. $\alpha \cup_F \beta$ will be nonzero only when paired with a basis element $\beta \in \Omega^p_F(X)$ of one of the following types.

$$\beta(c') = c \text{ and } 0 \text{ otherwise for some } (k+q)\text{-simplex } c'.$$

$$\text{Or, } \beta(b) = e \text{ and } 0 \text{ otherwise for some } (k-p-q)\text{-simplex } e.$$

Note that the latter requires $k \geq p+q$.

Consider β as in the first case. Then,

$$\begin{aligned}(\alpha \cup_F \beta)(c') &= \frac{1}{2}[\alpha \circ \beta(c') + (-1)^{pq}\beta \circ \alpha(c')] \\ &= \frac{1}{2}[\alpha(c) + 0] \\ &= \frac{1}{2}b\end{aligned}$$

and $\alpha \cup_F \beta$ is zero on all other chains. Thus, $\alpha \cup_F \beta$ is a $(p+q)$-form on $\mathrm{St}(c')$ and hence on $\mathrm{St}(c){\cdot}\mathrm{St}(c')$, as desired.

A similar argument shows that for β as in the second case, $\alpha \cup_F \beta$ is a $(p+q)$-form on $\mathrm{St}(c){\cdot}\mathrm{St}(b)$.

Note that $\alpha = \alpha'_F$ where $\alpha' \in C^p(K)$ such that $\alpha'(\widehat{cb}) = \mathrm{sgn}(\widehat{cb}, b; c)$ and is 0 otherwise. Also, in the first case, $\beta = \beta'_F$ where $\beta' \in C^q(K)$ such that $\beta'(\widehat{c'c}) = \mathrm{sgn}(\widehat{c'c}, c; c')$ and is 0 otherwise. Or, in the second case, $\beta = \beta''_F$ where $\beta'' \in C^q(K)$ such that $\beta''(\widehat{be}) = \mathrm{sgn}(\widehat{be}, e; b)$ and is 0 otherwise.

The above argument shows that $\alpha' \cup_{F_c} \beta'$ is a nonzero $(p+q)$-form only on $\mathrm{St}(\widehat{cb}){\cdot}\mathrm{St}(\widehat{c'c})$ and $\alpha' \cup_{F_c} \beta''$ is a nonzero $(p+q)$-form only on $\mathrm{St}(\widehat{cb}){\cdot}\mathrm{St}(\widehat{be})$. α' pairs to 0 with all other q-forms. Thus, Property 1 holds.

Property 2. The result follows from D_F a derivation with respect to $\cup_F$ because $D_F\omega_F = (d\omega)_F$ for any cubical form ω of arbitrary dimension. Let $\alpha \in C^p(K)$ and $\beta \in C^q(K)$. By definition, $d(\alpha \cup_{F_c} \beta)$ is determined by $(d(\alpha \cup_{F_c} \beta))_F$.

$$\begin{aligned}
(d(\alpha \cup_{F_c} \beta))_F &= D_F(\alpha \cup_{F_c} \beta)_F && \text{(Theorem 5.3.3)}\\
&= D_F(\alpha_F \cup_F \beta_F) && \text{(Definition 5.4.3)}\\
&= (D_F\alpha_F) \cup_F \beta_F + (-1)^p \alpha_F \cup_F (D_F\beta_F) && \text{(Theorem 5.4.2)}\\
&= (d\alpha \cup_{F_c} \beta)_F + (-1)^p(\alpha \cup_{F_c} d\beta)_F && \text{(Definition 5.4.3).}
\end{aligned}$$

Thus, $(d(\alpha \cup_{f_c} \beta))_F$ also determines $d\alpha \cup_{F_c} \beta + (-1)^p \alpha \cup_{F_c} d\beta$. So, as desired,

$$d(\alpha \cup_{F_c} \beta) = d\alpha \cup_{F_c} \beta + (-1)^p \alpha \cup_{F_c} d\beta.$$

Property 3. Let p be arbitrary. Consider an arbitrary $\alpha \in C^p(K)$. Let I be the constant 0-form that takes value 1 on each vertex in K. $I \cup_{F_c} \alpha$ is determined by $(I \cup_{F_c} \alpha)_F = (I_F \cup_F \alpha_F)$. Thus, $\gamma_{\cup_{F_c}} = \gamma_{\cup_F}$ where $I_F \cup_F \alpha_F = \gamma_{\cup_F} \alpha_F$. So, we will calculate $\gamma_{\cup_F}$.

First, we make I_F explicit. Given an arbitrary k-chain τ, for some $k \geq 0$,

$$\begin{aligned} I_F(\tau) &= (-1)^0 \operatorname{sgn}(\widehat{\tau\tau}, \tau; \tau) I(\widehat{\tau\tau}) \cdot \tau \\ &= \tau. \end{aligned}$$

So, we see that I_F is Forman's identity map. Thus, for an arbitrary simplex $\sigma \in C_\ell(X)$, $\ell \geq p$,

$$\begin{aligned} (I_F \cup_F \alpha_F)(\sigma) &= \frac{1}{2}[I_F \circ \alpha_F(\sigma) + (-1)^{(0)(p)} \alpha_F \circ I_F(\sigma)] \\ &= \frac{1}{2}[I_F(\alpha_F(\sigma)) + \alpha_F(\sigma)] \\ &= \frac{1}{2}[\alpha_F(\sigma) + \alpha_F(\sigma)] \\ &= \alpha_F(\sigma). \end{aligned}$$

This shows that $\gamma_{\cup_F} = 1$. Hence, $\gamma_{\cup_{F_c}} = 1$, and Property 3 holds.

Thus, $\cup_{F_c}$ is a cup product of cubical differential forms.

□

Because $\cup_{F_c}$ is a cup product, we may define a corresponding cap product $\cap_{F_c} : C_{p+q}(K) \times$

$C^p(K) \to C^q(K)$ via the relationship

$$\beta(\sigma \cap_{F_c} \alpha) = (\alpha \cup_{F_c} \beta)(\sigma).$$

Definition 5.4.6. Define the **Cubical Forman Cap Product** $\cap_{F_c} : C_{p+q}(K) \times C^p(K) \to C_q(K)$ implicitly by

$$\beta(\sigma \cap_{F_c} \alpha) = (\alpha \cup_{F_c} \beta)(\sigma)$$

where $\alpha \in C^p(K)$, $\beta \in C^q(K)$, and $\sigma \in C_{p+q}(K)$.

Theorem 5.4.7. $\cap_{F_c}$ *is a cap product on* K*.*

Proof. Let $\alpha \in C^p(K)$, $\beta \in C^q(K)$, and $\sigma \in C_{p+q}(K)$. $\cap_{F_c}$ satisfies the conditions of Definition 2.3.1 because $\cup_{F_c}$ is a cup product and

$$\beta(\sigma \cap_{F_c} \alpha) = (\alpha \cup_{F_c} \beta)(\sigma).$$

□

5.4.3 Agreement of the Forman Cubical Products and the Cubical Products

In this section, we show that the Forman cubical products and the cubical products defined in Section 3.2.1 agree on homology and cohomology. Thus, we provide a gateway between differential form theory developed by Forman and the theory of our cubical structures on a smooth manifold. To expose the connection between Forman and cubical structures, we rely on background pertaining to products developed by Whitney in [18].

Recall that in section 3.2.1, we defined the cubical cup product $\cup_c$ of differential forms on K and showed that it satisfied Whitney's cup product definition, Definition 2.3.1. From $\cup_c$, we defined the corresponding cubical cap product $\cap_c$ that satisfied Whitney's conditions in Definition 2.3.1.

The agreement of the Forman cubical products with the cubical products on homology and cohomology is a consequence of Theorem 2.3.10 and Theorem 2.3.11, respectively.

Theorem 5.4.8. *Let X be a simplicial complex on a smooth manifold M, and let K be its associated cubical structure. Let $p, q \geq 0$ be arbitrary and suppose $\sigma \in C^{p+q}(K)$ and $\alpha \in C^p(K)$. Then,*

$$[\sigma \cap_{F_c} \beta] = [\sigma \cap_c \beta]$$

in $(H_q(K), d)$.

Proof. We showed in Theorem 5.4.5 that $\gamma_{\cap_{F_c}} = \gamma_{\cup_{F_c}} = 1$. In Theorem 3.2.3, we showed that $\gamma_{\cap_c} = \gamma_{\cup_c} = 1$. Thus, the result follows from Theorem 2.3.10. □

Theorem 5.4.9. *Let X be a simplicial complex on a smooth manifold M, and let K be its associated cubical structure. Let $p, q \geq 0$ be arbitrary and suppose $\alpha \in C^p(K)$ and $\beta \in C^q(K)$. Then,*

$$[\alpha \cup_{F_c} \beta] = [\alpha \cup_c \beta]$$

in $(H^{p+q}(K), d)$.

Proof. Again, in Theorem 5.4.5 and Theorem 3.2.3, we showed that $\gamma_{\cup_{F_c}} = 1$ and $\gamma_{\cup_c} = 1$, respectively. Thus, the result follows from Theorem 2.3.11. □

We have previously shown that $\cap_c$ and $\cup_c$ agree with the standard cubical cap and cup products, respectively. Thus, the above theorems show that the Forman cubical products are also standard on homology and cohomology. In particular, the Forman cap product can also be viewed as the Poincaré duality map.

The Forman products are a simplification of $\cap_c$ and $\cup_c$. To see this, consider the comparison of the cup products $\cup_{F_c}$ and $\cup_c$. The statement for cap products will analogously follow.

$(\alpha \cup_c \beta)(\sigma)$ is calculated via all of the vertices in σ, i.e. 2^{p+q} vertices; whereas $(\alpha \cup_{F_c} \beta)(\sigma)$ uses only a subset of vertices, i.e. $2\binom{p+q}{p}$ vertices.

To make this subset explicit, consider a $(p+q)$-simplex c. Let v_i be a vertex in c. Recall from Section 5.4.2 that $(\alpha \cup_{F_c} \beta)(\widehat{cv_i})$ is the coefficient of v_i in $(\alpha_F \cup_F \beta_F)(c)$. Recall also that

$$(\alpha_F \cup_F \beta_F)(c) = \alpha_F \circ \beta_F(c) + (-1)^{pq} \beta_F \circ \alpha_F(c).$$

$\alpha_F \circ \beta_F(c)$ traverses each path from $\hat{c}$ to v_i via a p-simplex $b_j \subseteq c$. $\beta_F(c)$ is determined by $\beta(\widehat{cb_j})$ and $\alpha_F(\beta_F(c))$ is determined by $\alpha(\widehat{b_j v_i})$. So, each of these paths is determined by the p-simplex b_j through which it navigates. In particular, we may associate each path to the barycenter of b_j, namely $\hat{b}_j$, i.e. the shared vertex of $\widehat{cb_j}$ and $\widehat{b_j v_i}$. Because there $\binom{p+q}{q}$ paths from $\hat{c}$ to v_i via some b_j, Forman's cup product requires $\binom{p+q}{q}$ vertices to calculate $\alpha_F \circ \beta_F(c)$,

Similarly, $\beta_F \circ \alpha_F(c)$ traverses each path from $\hat{c}$ to v_i via a q-simplex $e_\ell \subseteq c$. As above, we may associate each path to the barycenter of e_ℓ, namely $\hat{e}_\ell$, the shared vertex of $\widehat{ce_\ell}$ and $\widehat{e_\ell v_i}$. Thus, because there are $\binom{p+q}{p}$ paths from $\hat{c}$ to v_i via some e_ℓ, Forman's cup product requires

$\binom{p+q}{p}$ vertices to calculate $\beta_F \circ \alpha_F(c)$.

Consequently, Forman's cup product requires a total of $\binom{p+q}{q} + \binom{p+q}{p} = 2\binom{p+q}{p}$ vertices in its calculation. To see that this number is less than 2^{p+q}, consider the binomial formula,

$$(x+y)^{p+q} = \sum_{k=0}^{p+q} \binom{p+q}{k} x^{p+q-k} y^k.$$

Letting x and y equal 1 yields,

$$2^{p+q} = \sum_{k=0}^{p+q} \binom{p+q}{k} \geq \binom{p+q}{p} + \binom{p+q}{q} = 2\binom{p+q}{p}.$$

Thus, Forman's cup product simplifies the computation of $\cup_c$ on cohomology. This is valuable from an applications standpoint.

Remark 5.4.10. We have chosen the definition of the Forman product $\cup_F$ to be such that $\cup_F$ is skew commutative on the cochain level. Suppose, we abandon this condition, and instead define a product $\cup'_F : \Omega^p_F(X) \times \Omega^q_F(X) \to \Omega^{p+q}_F(X)$ as follows.

$$\alpha \cup'_F \beta = \beta \circ \alpha.$$

Note that this is the product Forman suggests in [7]. To see that this product provides provides a derivation of D_F, let $\alpha \in \Omega^p_F(X)$ and $\beta \in \Omega^q_F(X)$. Then,

$$\begin{aligned} D_F(\alpha \cup'_F \beta) &= (-1)^{p+q-1}\partial \circ (\alpha \cup'_F \beta) + (\alpha \cup'_F \beta) \circ \partial \\ &= (-1)^{p+q-1}\partial \circ \beta \circ \alpha + \beta \circ \alpha \circ \partial, \\ D_F\alpha \cup'_F \beta &= \beta \circ D_F\alpha \\ &= \beta \circ ((-1)^{p-1}\partial \circ \alpha + \alpha \circ \partial) \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{p-1}\beta \circ \partial \circ \alpha + \beta \circ \alpha \circ \partial, \\
(-1)^p \alpha \cup_F' D_F\beta &= (-1)^p D_F\beta \circ \alpha \\
&= (-1)^p((-1)^{q-1}\partial \circ \beta + \beta \circ \partial) \circ \alpha \\
&= (-1)^{p+q-1}\partial \circ \beta \circ \alpha + (-1)^p \beta \circ \partial\alpha.
\end{aligned}
$$

Thus,

$$D_F(\alpha \cup_F' \beta) = D_F\alpha \cup_F' \beta + (-1)^p \alpha \cup_F' D_F\beta.$$

Furthermore, this product defines a cubical cup product $\cup' : C^p(K) \times C^1(K) \to C^{p+q}(K)$ by

$$(\alpha \cup' \beta)_F = \alpha_F \cup_F' \beta_F.$$

This cup product is actually the standard cubical cup product on the cochain level. To see this, we first reinterpret the standard cubical cup product, defined in Definition 3.2.6, to make sense in a kite setting.

Let $p, q \geq 0$ be arbitrary. Suppose c is an oriented k-simplex, $k \geq p+q$. Let b be an oriented $(k-p)$-simplex in b, and let e be an oriented $(k-p-q)$-simplex in b. If we assume that $\dot{c}$ and $\dot{e}$ map to the origin and $(1, \ldots, 1)$ under a diffeomorphism from $\widehat{ce}$ to the standard $(p+q)$-cube, respectively, then Definition 3.2.6 defines $(\alpha \cup \beta)(\widehat{ce})$ as follows.

$$(\alpha \cup \beta)(\widehat{ce}) = \operatorname{sgn}(\widehat{cb}, \widehat{be}; \widehat{ce})\ \alpha(\widehat{cb})\ \beta(\widehat{cb}).$$

So, the coefficient of e in $(\alpha \cup \beta)_F(c)$ is

$$
\begin{aligned}
\operatorname{sgn}(\widehat{ce}, e; c)(\alpha \cup \beta)(\widehat{ce}) &= \operatorname{sgn}(\widehat{ce}, e; c) \operatorname{sgn}(\widehat{cb}, \widehat{be}; \widehat{ce})\ \alpha(\widehat{cb})\ \beta(\widehat{cb}) \\
&= \operatorname{sgn}(\widehat{cb}, \widehat{be}, e; c)\ \alpha(\widehat{cb})\ \beta(\widehat{cb}).
\end{aligned}
$$

To see this last equality, suppose that $\mathrm{sgn}(\widehat{cb}, \widehat{be}; \widehat{ce}) = \epsilon$, where $\epsilon = \pm 1$. Then,

$$\begin{aligned} \mathrm{sgn}(\widehat{ce}, e; c) &= \mathrm{sgn}(\epsilon\ \widehat{cb}, \widehat{be}, e; c) \\ &= \epsilon\,\mathrm{sgn}(\widehat{cb}, \widehat{be}, e; c). \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{sgn}(\widehat{ce}, e; c)\,\mathrm{sgn}(\widehat{cb}, \widehat{be}; \widehat{ce}) &= (\epsilon\,\mathrm{sgn}(\widehat{cb}, \widehat{be}, e; c))(\epsilon) \\ &= \mathrm{sgn}(\widehat{cb}, \widehat{be}, e; c). \end{aligned}$$

Now, we consider the cubical cup product that $\cup'_F$ defines. By definition, $(\alpha \cup' \beta)(\widehat{ce})$ is the coefficient of e in $(\alpha_F \cup'_F \beta_F)(c)$. Because $(\alpha_F \cup'_F \beta_F)(c) = (\beta_F \circ \alpha_F)(c)$, we calculate the coefficient of e in $(\beta_F \circ \alpha_F)(c)$.

$$\begin{aligned} (\beta_F \circ \alpha_F)(c) &= \beta_F(\mathrm{sgn}(\widehat{cb}, b; c)\ \alpha(\widehat{cb}) \cdot b) \\ &= \mathrm{sgn}(\widehat{cb}, b; c)\ \alpha(\widehat{cb})\ \ \mathrm{sgn}(\widehat{be}, e; b)\ \beta(\widehat{be}) \cdot e \\ &= \mathrm{sgn}(\widehat{cb}, \widehat{be}, e; c)\ \alpha(\widehat{cb})\ \beta(\widehat{be}). \end{aligned}$$

This last equality holds by a similar argument, given above, for the sign term in $(\alpha \cup \beta)(\widehat{ce})$. Thus,

$$(\alpha \cup' \beta)(\widehat{ce}) = (\alpha \cup \beta)(\widehat{ce}).$$

Because c, b, and e were arbitrary, $\cup'$ and $\cup$ agree on the cochain level. So, the Forman product $\cup'_F$ corresponds to the standard cubical cup product.

In conclusion, we have shown that Forman's complex of differential forms in a simplicial setting, while nontraditional, can be conveniently related with the traditional cochain complex of its associated kite complex. The cubical products that arise naturally from Forman's complex agree with the standard cubical products. When we take the Forman product to be defined to allow for skew commutativity, this agreement is on the homology and cohomology level. If we instead ignore the commutativity condition, Forman's product defines the standard cubical cup product, and hence the standard cubical cap product by correspondence, on the *cochain* level. Thus, we have intertwined Forman's theory with the discrete cubical theory of Chapter 3.

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