Beurling-Lax Representations of Shift-Invariant Spaces, Zero-Pole Data Interpolation, and Dichotomous Transfer Function Realizations: Half-Plane/Continuous-Time Versions

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(ABSTRACT)

Given a full-range simply-invariant shift-invariant subspace $\mathcal{M}$ of the vector-valued L^2 space $L^2_{\mathcal{U}}(\mathbb{T})$ over the unit circle, the classical Beurling-Lax-Halmos (BLH) Theorem obtains a unitary operator-valued function W on $\mathbb{T}$ so that $\mathcal{M} = WH^2_{\mathcal{U}}$; in this case necessarily $\mathcal{M}^{\perp} = W\left(H^2_{\mathcal{U}}\right)^{\perp}$. The BLH Theorem of Ball-Helton [2] obtains such a representation for the case of a pair of shift-invariant subspaces $(\mathcal{M}, \mathcal{M}^{\times})$—with $\mathcal{M}$ forward full-range simply-invariant and $\mathcal{M}^{\times}$ backward full-range simply-invariant—forming a direct-sum decomposition of $L^2_{\mathcal{U}}(\mathbb{T})$ with a new almost everywhere invertible W on $\mathbb{T}$. For the case where $(\mathcal{M}, \mathcal{M}^{\times})$ is a finite-dimensional perturbation of the model pair $(H^2_{\mathcal{U}}(\mathbb{T}), H^2_{\mathcal{U}}(\mathbb{T})^{\perp})$, Ball-Gohberg-Rodman [1] obtained a transfer function realization formula for the representer W, parameterized from zero-pole data computed from $\mathcal{M}$ and $\mathcal{M}^{\times}$. Later work by Ball-Raney [4] extended this analysis to the nonrational case where the zero-pole data is taken in an appropriate infinite-dimensional operator-theoretic sense. Our current work obtains the analogue of these results for the case of a pair of subspaces $(\mathcal{M}, \mathcal{M}^{\times})$ of $L^2_{\mathcal{U}}(\mathbb{R})$ invariant under the forward and backward translation groups. These results rely on recent advances in the understanding of continuous-time infinite-dimensional input-state-output linear systems now codified in the book of Staffans [26].

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Chapter 1

Introduction

1.1 Overview; An Illustrative Diagram

Our purpose here is to present a number of theorems in the contexts of function theory and linear system theory. We will concentrate on three (related) types of results.

The first, found in Chapter 3, are representations of the type of the Beurling-Lax-Halmos theorems; these results provide connections between functions and spaces which are invariant under so-called shift operators. We state three such theorems: we give a classic version in Theorem 1.2.1, we re-state a version due to Ball-Raney in [4] as Theorem 3.5.4, and we give our own generalization thereof as Theorem 3.6.5.

The second type of result, which are considered in Chapter 4, are those that connect shift-invariant spaces to quintets of operators that encode zero-pole data; we generalize results of Gohberg-Kaashoek-Lerer-Rodman in [9]. These results are closely connected to the theory of linear systems, as these operator quintets have natural interpretations in terms of transfer functions of linear systems. These zero-pole data results are also closely connected to interpolation problems such as the Nevanlinna-Pick interpolation problem. Our zero-pole interpolation result is given as Theorem 4.2.10.

The final type of result, to be found in Chapter 5, relates to the question of finding transfer-function realizations: that is, given a function belonging to a certain class, we consider whether this function can be realized as the transfer function of a linear system. We give an operator-theoretic form of a realization formula in Theorem 5.2.1. Thereafter, we discuss two special cases where formulae may be explicitly computed; in these cases, the connection to finite-dimensional transfer function realizations becomes readily apparent.

Our particular purpose is to present continuous-time/half-plane versions of all of these theorems; that is, we consider functions in and subspaces of $L^2(\mathbb{R})$ and continuous-time linear systems.

The present document is efficiently summarized by Figure 1.1.

That is, we concern ourselves with three classes of objects: functions which are $L^2_{\mathcal{U}}(i\mathbb{R})$-regular, subspaces of $L^2_{\mathcal{U}}(i\mathbb{R})$ which are shift-invariant, and quintets of operators comprising so-called admissible Sylvester data sets. Precise definitions of these objects are given in Section 1.2 below. Our purpose here is to investigate the connections between these objects.

Remark 1.1.1 (Notation)**.** We identify standard notations which will be necessary in the rest of the introduction.

First, notation. We let $\mathcal{U}$ be a (possibly infinite-dimensional) Hilbert space, which

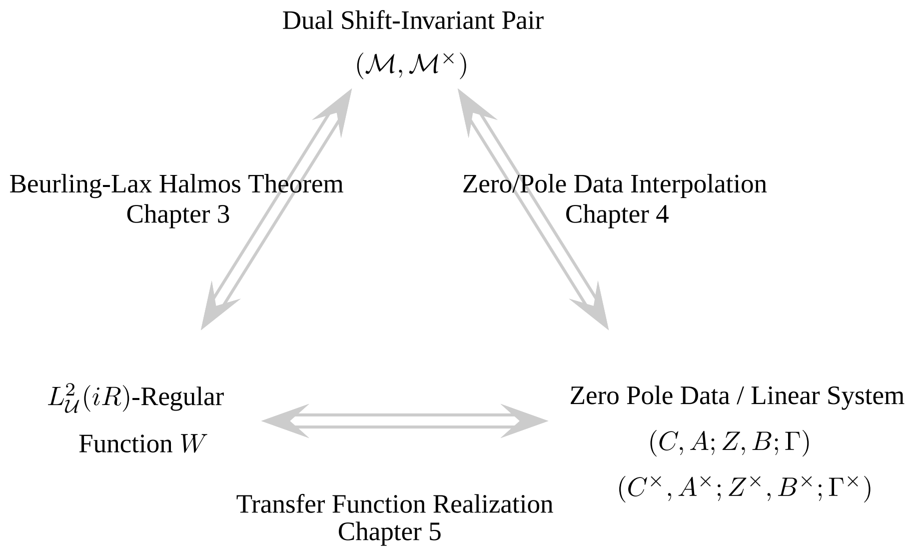

Figure 1.1: Guide to the Main Results

we will refer to as the coefficient space. L^2 is the usual space of (Lebesgue) square-integrable functions, and $L^2_{\mathcal{U}}$ is the space of such functions taking value pointwise in $\mathcal{U}$. We may specify the set for our $L^2_{\mathcal{U}}$ spaces: $L^2_{\mathcal{U}}(\mathbb{T})$ comprises functions on the unit circle in $\mathbb{C}$, and $L^2_{\mathcal{U}}(\mathbb{R})$ and $L^2_{\mathcal{U}}(i\mathbb{R})$ are functions on the real axis and imaginary axis, respectively.

We also have need of the Hardy space $H^2_{\mathcal{U}}(\mathbb{D})$. This the space of analytic functions on $\mathbb{D}$ with pointwise almost-everywhere nontangential limits on the circle $\mathbb{T}$, such that the limiting function is in $L^2_{\mathcal{U}}(\mathbb{T})$. This space can be identified with a subspace of $L^2_{\mathcal{U}}(\mathbb{T})$. Similarly, we have Hardy spaces associated with the right half plane Π_+ and the left half plane Π_-; these are $H^2_{\mathcal{U}}(\Pi_+)$ and $H^2_{\mathcal{U}}(\Pi_-)$ respectively.

We also use the standard notation M_θ to represent the operator of multiplication by the function θ. As a particular example, M_z on $L^2_{\mathcal{U}}(\mathbb{T})$ is given by

$$M_z f(z) = z f(z)$$

for $f \in L^2_{\mathcal{U}}(\mathbb{T})$.

1.2 On Beurling-Lax-Halmos Theorems

A cornerstone of modern function theory is the Beurling-Lax-Halmos Theorem (see [6], [20], and [12]), a version of which we state below. The essential nature of this theorem is to characterize the invariant subspaces of *shift operators*. In fact, this theorem completely characterizes all such invariant subspaces in terms of operator-valued functions which are pointwise-a.e. unitary—so-called *inner functions*.

Theorem 1.2.1. *The following are equivalent.*

1. *Let $\mathcal{M} \subset L^2_{\mathcal{U}}(\mathbb{T})$ have the properties that*

$$M_z\mathcal{M} \subset \mathcal{M}, \ \cap_{n\geq 0} M_z^n \mathcal{M} = 0, \ \textit{and} \ \cup_{n\leq 0} M_z^n\mathcal{M} \ \textit{is dense in} \ L^2_{\mathcal{U}}(\mathbb{T});$$

   where here M_z is the multiplication operator given by $M_z f(z) = zf(z)$ and $\mathcal{U}$ is a finite-dimensional Hilbert space.

2. *There exists a $\mathcal{L}(\mathcal{U})$-valued function $\theta(z)$, a.e. defined on $\mathbb{T}$ that is unitary for a.e. $z \in \mathbb{T}$ such that*

$$\mathcal{M} = M_\theta H^2_{\mathcal{U}}(\mathbb{D}).$$

   Here $H^2_{\mathcal{U}}(\mathbb{D})$ is the Hardy space of analytic functions on $\mathbb{D}$ that have a.e.-defined *nontangential limits in* $L^2_{\mathcal{U}}(\mathbb{T})$*.*

In the preceding Theorem 1.2.1, the operator M_z on $L^2_{\mathcal{U}}(\mathbb{T})$ is an example of a shift. As these operators are of central importance for us, we take a moment to discuss basic examples thereof. We consider the space ℓ^2 of bi-infinite square summable sequences:

$$\ell^2 = \left\{ \{a_i\} \mid \sum_{-\infty}^{\infty} |a_i|^2 < \infty \right\}$$

We define the suggestively-named *right shift operator* $\mathfrak{S}$ by

$$\mathfrak{S}\{a_i\} = \{a_{i-1}\}.$$

That is, $\mathfrak{S}$ shifts the sequence to the right by one. Clearly, $\mathfrak{S}$ is unitary. We may also define the *left shift operator*, which may be thought of as $\mathfrak{S}^{-1} = \mathfrak{S}^*$. Both $\mathfrak{S}$ and $\mathfrak{S}^*$ are examples of what we shall call *bilateral shifts*.

On the other hand, if we restrict $\mathfrak{S}$ to ℓ^2_+ (which we define as the set of elements $\{a_i\}$ of ℓ^2 such that $a_i = 0$ when $i < 0$), we see that $\mathfrak{S}\big|_{\ell^2_+}$ is merely isometric: while a left inverse exists, a right inverse does not. $\mathfrak{S}\big|_{\ell^2_+}$ is, however, a *pure isometry*, in that

$$\cap_{n \geq 0} \mathfrak{S}^n \ell^2_+ = \{0\}.$$

$\mathfrak{S}\big|_{\ell^2_+}$ is an example of what we shall call a *unilateral shift*.

In any of these examples, a family of operators may be generated by iteration. That is, for all $n \in \mathbb{N}$, $\mathfrak{S}^n$ is a (bilateral or unilateral, as appropriate) shift. Thus are these operators referred to as discrete shifts— that is, the family of operators is indexed by the integers. We will instead primarily concern ourselves with *continuous shifts*, which are operator semigroups and with real indices. These are discussed at length in section 2.1.

We take a moment to note the contributions of the authors credited in Theorem 1.2.1.

Original work on this problem appears to have been due to Beurling [6]. He considered the case wherein the coefficient space $\mathcal{U}$ was one dimensional: specifically, $\mathcal{U} = \mathbb{C}$. He was motivated by the problem of developing a spectral theory for the operator M_z on $L^2(\mathbb{T})$; the problem was of interest, of course, as M_z is neither self-adjoint nor normal. The essential idea was to analyze the operator's invariant subspaces as a first step toward developing the operator's spectral theory.

Further development on the theorem was due to Lax [20]; his contribution was to consider the invariant subspaces of the translation group acting on $L^2(\mathbb{R})$; this group is an example of a *continuous shift*.

Finally, the contribution of Halmos [12] was to extend to an arbitrary *discrete* shift operator acting on an abstract Hilbert space $\mathcal{X}$ with an arbitrary abstract coefficient Hilbert space $\mathcal{U}$; further the dimension of $\mathcal{U}$ was allowed to be infinite.

Another extension of Theorem 1.2.1 is of particular interest to us. But before discussing it, we first note that if $\mathcal{M} = M_\theta H^2_{\mathcal{U}}(\mathbb{D})$, then it follows—by the unitarity of θ—that $\mathcal{M}^\perp = M_\theta H^2_{\mathcal{U}}(\mathbb{D})^\perp$. A natural generalization of the Beurling-Lax-Halmos Theorem arises from considering under what circumstances a pair of spaces $(\mathcal{M}, \mathcal{M}^\times)$ can be simultaneously represented by a single function W according to the formulae

$$\mathcal{M} = M_W H^2_{\mathcal{U}}(\mathbb{D}), \quad \mathcal{M}^\times = M_W H^2_{\mathcal{U}}(\mathbb{D})^\perp.$$

Such representations have applications to computing Wiener-Hopf factorizations. In considering this question, we mark two more facts about $\mathcal{M}^\perp$. First, we note that $\mathcal{M}^\perp$ is invariant under $\mathfrak{S}^*$: $\mathfrak{S}^*\mathcal{M}^\perp \subset \mathcal{M}^\perp$; and second—and tautologically—we have that $\mathcal{M} \oplus \mathcal{M}^\perp = L^2_{\mathcal{U}}(\mathbb{T})$. In fact, a generalized Beurling-Lax-Halmos Theorem holds precisely when the pair $(\mathcal{M}, \mathcal{M}^\times)$ share properties analogous to those of the pair $(\mathcal{M}, \mathcal{M}^\perp)$, but omitting the orthogonality. This generalization was first proved by Ball-Helton for discrete shifts in [2]. A refined proof, wherein more careful attention was paid to the case where the coefficient space $\mathcal{U}$ is infinite-dimensional, due to Ball-Raney, is found in [4].

We may now succinctly state our purpose in chapter 3: we seek to mirror the work of Lax on the orthogonal case—i.e., when $\mathcal{M}^\times = \mathcal{M}^\perp$—by extending the result of Ball-Helton on the nonorthogonal case to continuous shifts.

Chapter 3 is organized along the following lines. Our essential idea is to analyze our shift group not in terms of its *generator*, which is an unbounded operator, but rather in terms of its *cogenerator*, which is bounded. We also make use of the Cayley transform as an isometry between $L^2_{\mathcal{U}}(i\mathbb{R})$ and $L^2_{\mathcal{U}}(\mathbb{T})$. These two tools allow us to reduce our problem to the previously solved case of discrete shifts on $L^2_{\mathcal{U}}(\mathbb{T})$ as found in [4]. We thus begin chapter 3 with a discussion of dual shift-invariant pairs; we continue on to a discussion of cogenerators of semigroups, and follow with an exposition on Cayley transforms. We then re-state the Beurling-Lax-Halmos Theorem as found in [4] and, finally state and prove our generalization thereof as Theorem 3.6.5.

1.3 On Linear Systems

Of great importance to both Chapters 4 and 5 are *linear systems*. We take a moment therefore to introduce them.

Linear systems are coupled vector differential and algebraic equations, traditionally written in the form

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}, \quad x(t_0) = x_0,$$

where A, B, C, and D are matrices of the appropriate sizes, $u(t)$ is a specified function, and $x(t_0) = x_0$ specifies initial data. Traditionally, $x(t)$ is called the state, and is pointwise an element of the state space $\mathcal{X}$; $u(t)$ is called the input and is pointwise an element of the input space $\mathcal{U}$; and $y(t)$ is called the output and is pointwise an element of the output space $\mathcal{Y}$. In solving a linear system, one seeks to compute the output function $y(t)$ in terms of a specified input function $u(t)$ and the initial state x_0.

An efficient method for solving linear systems is via Laplace transforms. If we assume for the moment that $x_0 = 0_{\mathcal{X}}$, the zero element of $\mathcal{X}$; denoting by $\widehat{\cdot}$ the bilateral Laplace transform; and letting $z \in i\mathbb{R}$ be the frequency variable; we have

$$\begin{cases} z\widehat{x}(z) &= A\widehat{x}(z) + B\widehat{u}(z) \\ \widehat{y}(z) &= C\widehat{x}(z) + D\widehat{u}(z) \end{cases}$$

These equations are readily solved for $\widehat{y}(z)$ in terms of $\widehat{u}(z)$ to find

$$\widehat{y}(z) = \big(C(zI - A)^{-1}B + D\big)\,\widehat{u}(z); \tag{1.1}$$

the $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function $G(z) := \widehat{C}(zI - \widehat{Z})^{-1}\widehat{B} + D$ is termed the *transfer function* of the system. The transfer function is thus a multiplicative operator mapping the (Laplace transform of) the input to the (Laplace transform of) the output. For any specified input function $u(t)$, then, the output function $y(t)$ may be computed via Laplace transforms and knowledge of G.

Equation (1.1) is called the *transfer-function* realization of $G(z)$.

Of course, various generalizations of linear systems are of widespread interest. Our concern will be to work with generalizations of linear systems where the state, input, and output spaces are all allowed to be infinite dimensional. Necessarily, this requires re-interpreting the system in terms of operator theory; we will generally allow all operators that arise from such considerations to be unbounded.

The formalism for studying such infinite-dimensional continuous-time linear systems has roots in the 1970s: cf. [8] and [13]. This formalism has only recently matured, however. In particular, we refer the reader to the books of Staffans [26] and of Weiss-Tucsnak [27].

1.4 On Zero/Pole Data Interpolation Theorems

In chapter 4, we consider so-called zero/pole data interpolation results. The nature of these results is to connect shift-invariant spaces with sets of operators. By the Beurling-Lax-Halmos Theorem 1.2.1 above, we know that any such spaces can be written as the image of a multiplication operator M_θ on analytic functions. These spaces necessarily therefore exhibit a rich zero/pole structure according to the zeroes and poles of θ. The role of the sets of operators will be to provide an alternate method of encoding this structure.

We begin with a short review of the history of these interpolation problems; this review should also serve to provide some intuition as to the nature of the results. Our discussion starts with the consideration of the zero/pole structure of functions and progresses to the inverse problem of finding an interpolating function which fits specified zero/pole data.

We begin with a consideration of the scalar case of *biproper* rational functions on $\mathbb{C}$; that is, functions of the form

$$r(z) = \frac{\Pi_i^M (z - z_i)^{\zeta_i}}{\Pi_j^N (z - p_j)^{\pi_j}},$$

where z_i is a zero with multiplicity ζ_i, p_j is a pole with multiplicity π_j, $z_i \neq \pi_j$, and $\sum \zeta_i = \sum \pi_j$. For such functions, given in this form, the locations and multiplicities of the zeroes and poles may be simply read off. And conversely, given a specified set of zeroes and poles with corresponding multiplicities (with equal sums), it is clearly possible to construct an interpolating biproper rational function. More interesting questions are to be found when one considers the case of rational matrix functions.

This topic seems to have been first pursued by Keldysch [19], who considered polynomial matrix functions. He concluded that zeroes of such functions are characterized not in terms of locations and multiplicities, but rather in terms of locations and *null chains*. These null chains determine analytic row-vector-valued functions which map the polynomial matrix functions to row-vector-valued power series. The null chain takes the form $\mathcal{N} = \{\begin{bmatrix} a_i & b_i \end{bmatrix}\}$ The corresponding analytic function is given by the obvious formula $W(z) = \sum_i \begin{bmatrix} a_i & b_i \end{bmatrix} z^i$. The number of nonzero terms in the null chain is the length of the null chain; this is also the order of the zero.

We consider a simple-but-illustrative example of these null chains. Consider the polynomial matrix

$$M(z) = \begin{bmatrix} z & 1 \\ 0 & z \end{bmatrix};$$

we claim that $M(z)$ has a zero of partial multiplicity 2 with a null chain of length 2 at the origin. We exhibit $W(z) = \begin{bmatrix} 0 & 1 \end{bmatrix} + z \begin{bmatrix} -1 & 0 \end{bmatrix}$ as a null chain of length 2. We compute $W(z)M(z)$ and re-order terms according to the power of z:

$$W(z)M(z) = \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} z + \mathcal{O}(z^2),$$

which vanishes at order $\mathcal{O}(z^2)$. This same information can be found by computing the Smith normal form for $M(z)$: we compute that

$$M(z) = \begin{bmatrix} 1 & 0 \\ z & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^2 \end{bmatrix} \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}$$

This work was extended by Gohberg-Sigal [11] to the case of rational matrix functions; this necessarily required the consideration of *pole chains* in addition to null chains. Another extension due to Gohberg-Rodman [10] considered analytic matrix functions.

Of great interest to us in the present work were the results of Bart-Gohberg-Kaashoek in [5] on rational matrix functions, wherein it was noticed that a function's pole chains are encoded in its transfer function realization. And, further, that the zero chains are encoded in the realization formula of the inverse. From this insight, the inverse problem seems to ask itself: given zeroes, null chains, poles, and pole chains, is it possible to construct an interpolating function?

This *zero/pole data interpolation* question was first considered in Gohberg-Kaashoek-Lerer-Rodman in [9]. They organized the zero data into a matrix B which encoded the null chains and a matrix Z which encoded the locations of the zeroes along with partial multiplicities; together, (Z, B) was termed a *null pair*. Similarly, the pole data was represented by the *pole pair* (C, A), which encoded the pole chains and locations and partial multiplicities of the poles, respectively. They were able to find an interpolating function W, but not uniquely.

Additional insight was added by Ball-Ran in [3], by considering a space $\mathcal{M}$ associated with W by $\mathcal{M} = W\mathcal{O}(\sigma)$, where $\sigma \subset \mathbb{C}$ containing the spectra of both A and Z, and $\mathcal{O}(\sigma)$ is the set of functions analytic on σ. To the null pair and pole pair, they added a fifth matrix Γ, called the coupling matrix, which satisfies the Sylvester equation $\Gamma A - Z\Gamma = BC$. The role of Γ is to couple the zero data with the pole data in such a way as to recover $\mathcal{M}$ according to the formula of Theorem 4.2.10. Thus the final interpolation problem is expressed in terms of a quintet of operators: the so-called Sylvester data set $(C, A; Z, B; \Gamma)$.

Finally, the work of Ball-Raney in [4] considers the case where the pole pair (C, A) and the null pair (Z, B) are in fact bounded operators on infinite-dimensional Hilbert spaces. The coupling operator Γ was in general unbounded. Further, they simultaneously considered a dual problem involving a dual data set $(C^\times, A^\times; Z^\times, B^\times; \Gamma^\times)$ with associated space $\mathcal{M}^\times$ subject to the additional constraint that $\mathcal{M} \dot{+} \mathcal{M}^\times = L^2_{\mathcal{U}}(\mathbb{T})$.

Our purpose in chapter 4 will be to extend the result of Ball-Raney to consider the case where the spaces $(\mathcal{M}, \mathcal{M}^\times)$ are subspaces of $L^2_{\mathcal{U}}(i\mathbb{R})$. As we shall see, this will necessitate allowing both Sylvester data sets

$$(C, A; Z, B; \Gamma) \quad \text{and} \quad (C^\times, A^\times; Z^\times, B^\times; \Gamma^\times)$$

to consist of unbounded operators.

Toward this end, chapter 4 is organized as follows. We first direct the reader's attention to section 2.3, in particular, which contains a "Graph Space Lemma" which will be of great use. We also wish to emphasize the importance of section 2.4 on linear systems, in terms of which our Sylvester data sets will have natural interpretation. Then chapter 4 begins with careful definitions of Sylvester data sets followed by an investigation of the role of the Sylvester equation. We then present a sequence of four Lemmas, which amount to special cases of the general data interpolation Theorem. Finally, we conclude with said Theorem.

1.5 On Transfer-Function Realizations

The last type of result with which we shall concern ourselves is that of transfer functions realizations, which we will consider in chapter 5. These realizations were mentioned in section 1.3 on linear systems.

The transfer function realization question is a simple version of an inverse problem: that is, given a function $W(z)$ with values pointwise in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$, we seek a quadruplet of operators $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ determining a linear system such that $W(z)$ is the transfer function. Finding this system gives an alternate formulation for $W(z)$:

$$W(z) = C(zI - A)^{-1}B + D.$$

Such a formulation for W is clearly advantageous: for example, under natural minimality conditions, the poles of W are given by the spectrum of A. Also, the behaviour as $z \to \infty$ is given by D.

We consider this realization theorem for a new class of functions in chapter 5. The essential insight is to make use of the structure developed in chapters 3 and 4. That is, given a function W which satisfies the hypotheses of the BLH type theorem of chapter 3, we construct shift-invariant subspaces according to said theorem. To these subspaces we apply the zero-pole data interpolation theorem as in chapter 4 in order to construct Sylvester data sets of operators. In contrast to the discrete-time case studied by Ball-Raney in [4], each operator in $(C, A; Z, B; \Gamma)$ may be unbounded. With some manipulation, these data sets can be seen to correspond to a continuous-time infinite-dimensional linear system of type studied by Staffans in [26] and Weiss-Tucsnak in [27].

The general realization theorem is found in section 5.2. Thereafter, we consider an illustrative special case: in section 5.3, we consider the case where W is a so-called *inner function.*

Chapter 2

Preliminary Definitions and Basic Results

2.1 Abstract (Semi)group Theory

2.1.1 Definition and basic properties

Definition 2.1.1. Let $\mathfrak{A}^s$ be a family of bounded operators on a Hilbert space $\mathcal{X}$, defined for all values $s \in \mathbb{R}$. This family is said to be a *Strongly Continuous Group* if the following hold:

1. $\mathfrak{A}^0$ is the identity operator on $\mathcal{X}$,

2. $\mathfrak{A}^s\mathfrak{A}^t = \mathfrak{A}^{s+t}$ for all s and t, and

3. $\lim_{s\to 0} \|\mathfrak{A}^s x - x\|_{\mathcal{X}} = 0$ for every $x \in \mathcal{X}$.

Alternately, if the family $\mathfrak{A}^s$ is defined for only nonnegative values of s, then $\mathfrak{A}^s$ is a *(forward) Strongly Continuous Semigroup* if

1. $\mathfrak{A}^0$ is the identity operator on $\mathcal{X}$,

2. $\mathfrak{A}^s\mathfrak{A}^t = \mathfrak{A}^{s+t}$ for all $s, t \geq 0$, and

3. $\lim_{s\downarrow 0} \|\mathfrak{A}^s x - x\|_{\mathcal{X}} = 0$ for every $x \in \mathcal{X}$.

If, instead, the family $\mathfrak{A}^s$ is defined only for nonpositive values of s, but $\mathfrak{A}^{-s}$ is a (forward) Strongly Continuous Semigroup, then we say $\mathfrak{A}^s$ is a *Backward Strongly Continuous Semigroup.*

We say that $\mathfrak{A}^s$ is a *(forward or backward) semigroup of isometries or contractions* if each $\mathfrak{A}^s$ is an isometry or contraction, respectively.

Remark 2.1.2. We introduce a notation that will often occur in formulae below. If $\mathfrak{T}^s$ is a strongly continuous forward semigroup, then we interpret $\mathfrak{T}^s$ with $s < 0$ to be the mapping $x \mapsto 0$. Similarly, if $\mathfrak{T}^s$ is a strongly continuous backward semigroup, then $\mathfrak{T}^s$ with $s > 0$ is the mapping $x \mapsto 0$.

Strongly-continuous semigroups have a norm growth restriction.

Definition 2.1.3. Given a strongly continuous (forward) semigroup $\mathfrak{T}$, the number

$$\omega_0(\mathfrak{T}) := \inf_{s>0} \frac{1}{s} \log \|\mathfrak{T}^s\|$$

is called the *(forward) growth bound of* $\mathfrak{T}$.

We similarly define the *(backward) growth bound* of a backward semigroup $\mathfrak{T}$ by

$$\omega_0(\mathfrak{T}) := \inf_{s<0} \frac{1}{s} \log \|\mathfrak{T}^s\| .$$

Theorem 2.1.4. *Let $\mathfrak{T}$ be a strongly continuous (forward) semigroup with growth bound ω_0. Then for any $\omega > \omega_0$, there exists a nonnegative constant $M_\omega < \infty$ such that*

$$\|\mathfrak{T}^s\| \leq M_\omega e^{\omega s}$$

for all s.

If, on the other hand, $\mathfrak{T}$ is a strongly continuous backward semigroup with growth bound ω_0, then for any $\omega < \omega_0$ there exists a nonnegative constant $M_\omega < \infty$ satisfying the same bound

$$\|\mathfrak{T}^s\| \leq M_\omega e^{\omega s}$$

for all s.

For the proof, we refer the reader to any of a number of standard references on semigroups, for example, cf. Theorem 2.2 of [22].

2.1.2 Infinitesimal Generator, Cogenerator

The first important property of strongly continuous semigroups is that they admit an infinitesimal representation in the form of an operator called the generator of the semigroup.

Definition 2.1.5. We define the *infinitesimal generator* A with domain $\mathcal{D}(A)$ of a strongly continuous group $\mathfrak{A}$ on a Hilbert space $\mathcal{X}$ to be the mapping

$$A : x \in \mathcal{D}(A) \mapsto \lim_{s \to 0} \frac{1}{s} \left(\mathfrak{A}^s x - x \right).$$

If, instead, $\mathfrak{A}$ is a strongly continuous (forward) semigroup on $\mathcal{X}$, then we define A

by

$$A : x \in \mathcal{D}(A) \mapsto \lim_{s \downarrow 0} \frac{1}{s} \left(\mathfrak{A}^s x - x \right).$$

And if $\mathfrak{A}$ is a strongly continuous backward semigroup, we define:

$$A : x \in \mathcal{D}(A) \mapsto \lim_{s \uparrow 0} \frac{1}{s} \left(\mathfrak{A}^s x - x \right).$$

In each case, the limit is evaluated with respect to the strong topology. The generator A is generally unbounded, and has domain $\mathcal{D}(A)$ defined to be the set of all x such that the defining limit exists.

For convenience, we include the following well-known result of Hille and Yosida which characterizes generators in terms of semigroups and vice-versa. We state the theorem as in [26], to which we also refer the reader for the proof.

Theorem 2.1.6 (Hille-Yosida)**.** *A linear operator $A : \mathcal{D}(A) \to \mathcal{X}$ is the infinitesimal generator of a strongly continuous semigroup $\mathfrak{A}$ on X satisfying $\|\mathfrak{A}^s\| \leq Me^{\omega t}$ if and only if the following conditions hold:*

1. *$\mathcal{D}(A)$ is dense in X;*

2. *every real $\lambda > \omega$ belongs to the resolvent of A, and*

$$\left\|(\lambda I - A)^{-n}\right\| \leq \frac{M}{(\lambda - \omega)^n} \text{for } \lambda > \omega \text{ and } n = 1, 2, 3, \ldots$$

It is worth noting the connection between generators of forward and backward semigroups.

Proposition 2.1.7. *Let $\mathfrak{A}^s$, with $s \leq 0$, be a backward strongly continuous semigroup with generator A and domain $\mathcal{D}(A)$. We define $\sigma = -s$ and construct a forward strongly continuous semigroup $\hat{\mathfrak{A}}^\sigma = \mathfrak{A}^s$ with corresponding generator $\hat{A}$ having domain $\mathcal{D}(\hat{A})$. Then $\mathcal{D}(A) = \mathcal{D}(\hat{A})$ and for all x in the domain, $Ax = -\hat{A}x$.*

Proof. We simply compute.

$$\begin{aligned} -\hat{A}x &= -\lim_{\sigma \downarrow 0} \frac{1}{\sigma} \left(\hat{\mathfrak{A}}^\sigma x - x \right) \\ &= \lim_{s \uparrow 0} \frac{1}{s} \left(\mathfrak{A}^s x - x \right) \\ &= Ax \end{aligned}$$

□

Remark 2.1.8. We note that the above result does not represent the relationship between the forward and backward semigroup *restrictions* of a group. Let $\mathfrak{T}$ be a strongly continuous group with generator T having domain $\mathcal{D}(T)$. For the moment, let $\mathfrak{T}_+^s$ denote $\mathfrak{T}^s\big|_{s \geq 0}$, the associated forward semigroup, and let $\mathfrak{T}_-^s$ similarly denote the backward semigroup. Then we have

$$\lim_{s \to 0} \frac{1}{s} \left(\mathfrak{T}^s x - x \right) = Tx$$

which implies that also

$$\lim_{s \downarrow 0} \frac{1}{s} \left(\mathfrak{T}_+^s x - x \right) = Tx$$

and

$$\lim_{s \uparrow 0} \frac{1}{s} \left(\mathfrak{T}_-^s x - x \right) = Tx.$$

That is, T is the generator of $\mathfrak{T}$, $\mathfrak{T}_+$, and $\mathfrak{T}_-$.

2.1.3 Laplace Transform Formulae

Given a strongly continuous semigroup, the Laplace transform provides an essential connection between the action of the semigroup and the action of the resolvent of the generator. We record the formula below; for the proof, we refer the reader to Proposition 2.3.1 of [27]. We remind the reader of our notation that for a forward semigroup $\mathfrak{T}$, we have that $\mathfrak{T}^s = 0$ when $s < 0$; cf. Note 2.1.2.

Definition 2.1.9. We let the notation $\mathcal{L} : L^2(\mathbb{R}) \to L^2(i\mathbb{R})$ denote the bilateral Laplace transform; thus

$$(\mathcal{L}f)(z) := \int_{-\infty}^{\infty} f(t)e^{-zt}dt.$$

Theorem 2.1.10. *Let $\mathfrak{A}$ be a forward strongly continuous semigroup with growth bound ω on a Banach space X, and let A be the generator of $\mathfrak{A}$. Choose $z \in \mathbb{C}$ with $\operatorname{Re} z > \omega$. Then $z \in \rho(A)$ and $\forall x \in X$*

$$(zI - A)^{-1}x = (\mathcal{L}\mathfrak{A}^t x)(z) = \int_0^{\infty} e^{-zt}\mathfrak{A}^t x dt.$$

Remark 2.1.11. We may always choose such a z: cf. Remark 5.4 of [22].

We also have cause to work with backward semigroups. The same formula, with slight modifications, holds for such semigroups. We record the formula and proof below.

Theorem 2.1.12. *Let $\mathfrak{A}$ be a backward strongly continuous semigroup with (backward) growth bound ω on a Banach space X, and let A be the generator of $\mathfrak{A}$. Choose*

$z \in \mathbb{C}$ *with* $\operatorname{Re} z < \omega$. *Then* $z \in \rho(A)$ *and* $\forall x \in X$

$$(A - zI)^{-1}x = (\mathcal{L}\mathfrak{A}^t x)(z) = \int_{-\infty}^{0} e^{-zt}\mathfrak{A}^t x dt.$$

Proof. For $t \geq 0$, we define a forward semigroup $\mathfrak{T}^t = \mathfrak{A}^{-t}$. Then $\mathfrak{T}^t$ is a forward strongly continuous semigroup with (forward) growth bound ω, and we let T be its infinitesimal generator. We use Theorem 2.1.10 to compute

$$\begin{aligned}(-T - zI)^{-1}x &= (\mathcal{L}\mathfrak{T}^t x)(-z) \\ &= \int_{0}^{\infty} e^{zt}\mathfrak{T}^t x dt,\end{aligned}$$

and by the change of variable $t \mapsto -t$ we get

$$\begin{aligned}&= -\int_{0}^{-\infty} e^{-zt}\mathfrak{A}^t x dt \\ &= \int_{-\infty}^{0} e^{-zt}\mathfrak{A}^t x dt.\end{aligned}$$

Now we may use Theorem 2.1.7 to connect the generator of $\mathfrak{T}^t$ to that of $\mathfrak{A}^t$ to get the claimed formula

$$(A - zI)^{-1}x = \int_{-\infty}^{0} e^{-zt}\mathfrak{A}^t x dt = (\mathcal{L}\mathfrak{A}^t x)(z).$$

□

2.1.4 Images of Semigroups under Bijective Isometries

Theorem 2.1.13. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. Let $\hat{}: \mathcal{X} \to \mathcal{Y}$ denote a linear bijective isometry with inverse denoted $\check{}: \mathcal{Y} \to \mathcal{X}$. To a generic linear operator A on $\mathcal{X}$ we associate a new operator $\hat{A}$ on $\mathcal{Y}$ according to the following formula:*

$$\hat{A}y := (A\check{y})\hat{}\,.$$

Let $\mathfrak{T}$ be a strongly continuous semigroup on $\mathcal{X}$ with generator T. Then the following hold:

1. *$\hat{\mathfrak{T}}$ is a strongly continuous semigroup on $\mathcal{Y}$.*
2. *$\hat{T}$ is the generator of $\hat{\mathfrak{T}}$ and $\mathcal{D}(\hat{T}) = (\mathcal{D}(T))\hat{}$.*

Proof. We begin with the statement (1). We evaluate

$$\hat{\mathfrak{T}}^0 y = \left(\mathfrak{T}^0 \check{y}\right)\hat{} = (\check{y})\hat{} = y$$

That is, $\hat{\mathfrak{T}}^0 = 1_{\mathcal{Y}}$, the identity on $\mathcal{Y}$. Next, letting $y = \hat{x}$ be an element of $\mathcal{Y}$, we check

$$\hat{\mathfrak{T}}^s \hat{\mathfrak{T}}^t y = \hat{\mathfrak{T}}^s \left(\mathfrak{T}^t x\right)\hat{} = \left(\mathfrak{T}^s \mathfrak{T}^t x\right)\hat{} = \left(\mathfrak{T}^{s+t} x\right)\hat{} = \hat{\mathfrak{T}}^{s+t} y.$$

Finally, we consider

$$\begin{aligned}\left\|\hat{\mathfrak{T}}^s y - y\right\|_{\mathcal{Y}}. &= \left\|\hat{\mathfrak{T}}^s \hat{x} - \hat{x}\right\|_{\mathcal{Y}} \\ &= \left\|(\mathfrak{T}^s x)\hat{} - \hat{x}\right\|_{\mathcal{Y}} \\ &= \left\|(\mathfrak{T}^s x - x)\hat{}\right\|_{\mathcal{Y}} \\ &= \left\|\mathfrak{T}^s x - x\right\|_{\mathcal{X}}.\end{aligned}$$

But by hypothesis, $\mathfrak{T}$ is strongly continuous on $\mathcal{X}$, so as $s \downarrow 0$, this norm goes to zero. Thus we see that $\hat{\mathfrak{T}}$ is strongly continuous on $\mathcal{Y}$.

Next we prove statement (2). Let $x \in \mathcal{D}(T)$ with image $\hat{x} =: y$. Further let $\hat{T}'$ be the generator of $\hat{\mathfrak{T}}$ and $\hat{T}$ to be the image of T under $\hat{}$. Then we consider

$$\begin{aligned}\hat{T}y - \hat{T}'y &= (Tx)\hat{} - \lim_{s\downarrow 0} \frac{1}{s}\left(\hat{\mathfrak{T}}^s \hat{x} - \hat{x}\right) \\ &= (Tx)\hat{} - \lim_{s\downarrow 0} \frac{1}{s}\left(\mathfrak{T}^s x - x\right)\hat{},\end{aligned}$$

but as $\hat{}$ is an isometry,

$$\begin{aligned}&= (Tx)\hat{} - \left(\lim_{s\downarrow 0} \frac{1}{s}\left(\mathfrak{T}^s x - x\right)\right)\hat{} \\ &= (Tx - Tx)\hat{} \\ &= (0_{\mathcal{X}})\hat{} = 0_{\mathcal{Y}}.\end{aligned}$$

We may conclude therefore that $\mathcal{D}(\hat{T}) \subset \mathcal{D}(\hat{T}')$ and that, on $\mathcal{D}(\hat{T}) \cap \mathcal{D}(\hat{T}')$, $\hat{T}x = \hat{T}'x$. A very similar argument, considering $Tx - \left(\hat{T}'\right)\check{} x$ gives the other containment, that $\mathcal{D}(\hat{T}') \subset \mathcal{D}(\hat{T})$. Thus $\hat{T} = \hat{T}'$. □

Remark 2.1.14. Perhaps the most important consequence of Theorem 2.1.13 comes from considering the Laplace transform as a linear bijective isometry between $L^2_{\mathcal{U}}(\mathbb{R})$ and $L^2_{\mathcal{U}}(i\mathbb{R})$. It allows us to conclude that "the Laplace transform of the generator is the generator of the Laplace transform".

2.1.5 General Shift (Semi)Groups

Classes of operators known as *Shifts* will be of great interest to us in the following work. These linear operators act on Hilbert spaces. We will work with two classes of shifts: discrete shifts and continuous shifts. We give general definitions of both, and then introduce the specific shifts in which we will almost exclusively be interested.

Definition 2.1.15 (Discrete Shifts)**.** Let $\mathcal{X} \subset \mathcal{Y}$ be Hilbert spaces.

An operator S_+ on $\mathcal{X}$ is a *(unilateral) discrete shift* if it is a pure isometry; that is, a discrete shift is an operator such that for all $x \in \mathcal{X}$, $\|S_+x\| = \|x\|$ and $\cap_{n\geq 0} S_+^n \mathcal{X} = \{0\}$.

If S is a unitary operator on $\mathcal{Y}$, then we call S a *bilateral discrete shift* if it is the minimal unitary extension of some discrete shift S_+ on $\mathcal{X}$. That is, we require that $Sx = S_+x$ for all $x \in \mathcal{X}$ and $\cup_{n\in\mathbb{N}} S^{-n}\mathcal{X}$ is dense in $\mathcal{Y}$.

We define continuous shifts in an entirely analogous way.

Definition 2.1.16. Again, let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces.

A semigroup of isometries $\mathfrak{A}_+$ acting on a Hilbert space $\mathcal{X}$ is a *(unilateral) continuous shift* if

$$\cap_{s\geq 0} \mathfrak{A}_+^s \mathcal{X} = \{0\}.$$

If $\mathfrak{A}$ is a unitary group acting on $\mathcal{Y}$, then we call $\mathfrak{A}$ a *bilateral continuous shift* if it is the minimal unitary extension of some unilateral continuous shift $\mathfrak{A}_+$ on $\mathcal{X}$. That is, we require that $\mathfrak{A}x = \mathfrak{A}_+x$ for all $x \in \mathcal{X}$ and $\cup_{s\leq 0}\mathfrak{A}^s\mathcal{X}$ is dense in $\mathcal{Y}$.

Lemma 2.1.17. *Let $\mathfrak{T}^s$ be a bilateral shift on a Hilbert space $\mathcal{X}$. Let $\mathcal{X}_+ \subset \mathcal{X}$ be invariant under $\mathfrak{T}^s$ when $s \geq 0$. Then the images of $\mathcal{X}_+$ under $\mathfrak{T}^s$ are nested in that whenever $s \geq t$ we have that $\mathfrak{T}^s\mathcal{X}_+ \subset \mathfrak{T}^t\mathcal{X}_+$.*

Proof. We consider the space $\mathfrak{T}^s\mathcal{X}_+$; since $\mathfrak{T}^s$ is unitary, we write

$$\mathfrak{T}^s\mathcal{X}_+ = \mathfrak{T}^t\mathfrak{T}^{-t}\mathfrak{T}^s\mathcal{X}_+ = \mathfrak{T}^t\mathfrak{T}^{s-t}\mathcal{X}_+ \subset \mathfrak{T}^t\mathcal{X}_+,$$

since $\mathfrak{T}^{s-t}\mathcal{X}_+ \subset \mathcal{X}_+$ for $s \geq t$. □

We present the following pair of Lemmas which describe adjoints of shifts.

Lemma 2.1.18. *Let $\mathcal{X}$ be a Hilbert space and let $\mathfrak{T}^s$ be a bilateral shift on $\mathcal{X}$. Let $\mathcal{X}_+$ be a subspace of $\mathcal{X}$ such that $\mathfrak{T}^s_+ := \mathfrak{T}^s_{|\mathcal{X}_+}$ is the unilateral shift of which $\mathfrak{T}^s$ is the minimal unitary extension. Then $\left(\mathfrak{T}^s\big|_{\mathcal{X}\ominus\mathcal{X}_+}\right)^*$ is a unilateral shift on $\mathcal{X} \ominus \mathcal{X}_+$.*

Proof. Let $\mathcal{X}_- := \mathcal{X} \ominus \mathcal{X}_+$ and define $\mathfrak{T}^s_- := (\mathfrak{T}^s)^*\big|_{\mathcal{X}_-}$. Incidentally, since $\mathfrak{T}^s$ is a unitary group, we know that $(\mathfrak{T}^s)^* = \mathfrak{T}^{-s}$.

As $\mathfrak{T}^s$ is a bilateral shift, both $\mathfrak{T}^s$ and $(\mathfrak{T}^s)^*$ are unitary; it follows that $\mathfrak{T}^s_-$ is a semigroup of isometries. Further, we have that $\mathfrak{T}^s_- : \mathcal{X}_- \to \mathcal{X}_-$. This statement holds because, if $x_- \in \mathcal{X}_-$ and $x_+ \in \mathcal{X}_+$,

$$((\mathfrak{T}^s)^*x_-, x_+) = (x_-, \mathfrak{T}^s x_+) = 0,$$

since $\mathfrak{T}^s : \mathcal{X}_+ \to \mathcal{X}_+$. That is, we have that $\mathfrak{T}_-^s$ is a semigroup of isometries on $\mathcal{X}_-$.

It remains to be seen that $\cap_{s\geq 0}\mathfrak{T}_-^s\mathcal{X}_- = \{0\}$. By Lemma 2.1.17, we know that $\cap_{0\leq s\leq T}\mathfrak{T}_-^s\mathcal{X}_- = \mathfrak{T}_-^T\mathcal{X}_+$. Then we consider

$$\mathfrak{T}_-^T\mathcal{X}_- = \mathfrak{T}^{-T}\left(\mathcal{X} \ominus \mathcal{X}_+\right) = \mathfrak{T}^{-T}\mathcal{X} \ominus \mathfrak{T}^{-T}\mathcal{X}_+ = \mathcal{X} \ominus \mathfrak{T}^{-T}\mathcal{X}_+.$$

But applying Lemma 2.1.17 again gives that $\mathfrak{T}^{-T}\mathcal{X}_+ = \cup_{-T\leq s\leq 0}\mathfrak{T}^{-T}\mathcal{X}_+$, so we have that

$$\cap_{0\leq s\leq T}\mathfrak{T}_-^s\mathcal{X}_- = \mathcal{X} \ominus \left(\cup_{0\leq s\leq T}\mathfrak{T}^{-T}\mathcal{X}_+\right).$$

But by hypothesis, $\mathfrak{T}^s$ is the minimal unitary extension of $\mathfrak{T}_+^s$, so taking the limit as $T \to \infty$ gives

$$\cap_{s\geq 0}T^s\mathcal{X}_- = \mathcal{X} \ominus \cup_{s\geq 0}T^{-s}\mathcal{X}_+ = \mathcal{X} \ominus \mathcal{X} = \{0\}.$$

□

Lemma 2.1.19. *Let $\mathcal{X}$ be a Hilbert space and let $\mathfrak{T}$ be a bilateral shift on $\mathcal{X}$. Then $\mathfrak{T}^*$ is also a bilateral shift on $\mathcal{X}$.*

Proof. As $\mathfrak{T}$ is a bilateral shift on $\mathcal{X}$, there must be a subspace $\mathcal{X}_+$ such that $\mathfrak{T}_+^s := \mathfrak{T}^s_{|\mathcal{X}_+}$ is a unilateral shift on $\mathcal{X}_+$ and such that $\mathfrak{T}^s$ is the minimal unitary extension of $\mathfrak{T}_+^s$.

Define $\mathcal{X}_- = \mathcal{X} \ominus \mathcal{X}_+$.

We consider $\mathfrak{T}_-^s := (\mathfrak{T}^s)^*_{|\mathcal{X}_-}$. By Lemma 2.1.18, $\mathfrak{T}_-^s$ is a unilateral shift. We show that $(\mathfrak{T}^s)^*$ is the minimal unitary extension of $\mathfrak{T}_-^s$.

For any $T > 0$, Lemma 2.1.17 gives that

$$\cup_{-T \leq s \leq 0} (\mathfrak{T}^s)^* \mathcal{X}_- = \left(\mathfrak{T}^{-T}\right)^* \mathcal{X}_-,$$

which may be rewritten as

$$\left(\mathfrak{T}^{-T}\right)^* (\mathcal{X} \ominus \mathcal{X}_+) = \mathfrak{T}^T \mathcal{X} \ominus \mathfrak{T}^T \mathcal{X}_+ = \mathcal{X} \ominus \mathfrak{T}^T \mathcal{X}_+,$$

and Lemma 2.1.17 again implies that this

$$= \mathcal{X} \ominus \cap_{0 \leq s \leq T} \mathfrak{T}^s \mathcal{X}_+.$$

In the limit as $T \to \infty$, we see that

$$\cup_{s \leq 0} (\mathfrak{T}^s)^* \mathcal{X}_- = \mathcal{X} \ominus \cap_{s \geq 0} \mathfrak{T}^s \mathcal{X}_+ = \mathcal{X} \ominus \{0\} = \mathcal{X}.$$

□

Remark 2.1.20. We note that if we restrict to considering only discrete shifts, then the preceding Lemma is a particular case of the more general notion of duals of subnormal operators; cf. [7].

Lemma 2.1.21. *Let $\mathfrak{T}$ be densely-defined operator on a Hilbert space $\mathcal{X}$ and let $\mathcal{M} \subset \mathcal{X}$ be a subspace. If $\mathcal{M}$ is invariant under $\mathfrak{T}$, then $\mathcal{M}^\perp = \mathcal{X} \ominus \mathcal{M}$ is invariant under $\mathfrak{T}^*$.*

Proof. Let $g \in \mathfrak{T}^* \mathcal{M}^\perp$. By definition, this means that there is some $f \in \mathcal{M}^\perp$ so that

$g = \mathfrak{T}^* f$. Then for any $m \in \mathcal{M}$,

$$(g, m) = (\mathfrak{T}^* f, m) = (f, \mathfrak{T} m) = 0$$

since $\mathfrak{T}\mathcal{M} \subset \mathcal{M}$ by hypothesis. Thus $\mathfrak{T}^*\mathcal{M}^\perp \subset \mathcal{M}^\perp$. □

2.2 Specific Strongly Continuous Groups and Semigroups: Hilbert Space Operators Acting on L^2 and H^2

Remark 2.2.1. We note that many of the results in this section are well known: Theorem 2.2.4 may be found as Theorem 9.5 of [25], while Theorems 2.2.7 and 2.2.10 may be found in [14], [27], and [26]. We present novel proofs in physicists' style, making use of Laplace transforms.

Also, in the following, we introduce the following notations. $\mathbb{R}$ is of course the real line; we set $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$. We identify $L^2_{\mathcal{U}}(\mathbb{R}_+)$ and $L^2_{\mathcal{U}}(\mathbb{R}_-)$ with subspaces of $L^2_{\mathcal{U}}(\mathbb{R})$. More generally, for any s, we view both $L^2_{\mathcal{U}}[-\infty, s]$ and $L^2_{\mathcal{U}}[s, \infty]$ as subspaces of $L^2_{\mathcal{U}}(\mathbb{R})$.

2.2.1 The Translation Groups and Semigroups

We will work almost exclusively with the following continuous shift acting on L^2.

Definition 2.2.2. For any $f \in L^2_{\mathcal{U}}(\mathbb{R})$ and for all $s \in \mathbb{R}$, we define the translation group by the following formula on $L^2_{\mathcal{U}}(\mathbb{R})$

$$\mathfrak{T}^s : f(t) \mapsto f(t-s).$$

If we restrict s to nonnegative values and f to $L^2_{\mathcal{U}}(\mathbb{R}_+)$, we get the *forward continuous shift* $\mathfrak{T}^s_+$ on $L^2_{\mathcal{U}}(\mathbb{R})$. If we restrict s to nonpositive values, we get the *backward continuous shift* $\mathfrak{T}^s_-$ on $L^2_{\mathcal{U}}(\mathbb{R})$.

We shall have ample cause to work with the Laplace transforms of the above shift operator.

Lemma 2.2.3. *Let* $f \in L^2(\mathbb{R})$*. Then*

$$(\mathfrak{T}^s f)\hat{}(z) = e^{-sz}\hat{f}(z).$$

Proof. $(\mathfrak{T}^s f)(t) = f(t-s)$, so

$$\begin{aligned}
(\mathfrak{T}^s f)\hat{}(z) &= \lim_{N\to\infty} \int_{-N}^{N} e^{-tz} f(t-s)dt \\
&= \lim_{N\to\infty} \int_{-N-s}^{N-s} e^{-(t'+s)z} f(t')dt' \\
&= e^{-sz} \lim_{N\to\infty} \int_{N-s}^{N-s} e^{-t'z} f(t')dt' \\
&= e^{-sz}\hat{f}(z),
\end{aligned}$$

where the limits are evaluated with respect to the $L^2_{\mathcal{U}}(i\mathbb{R})$ norm. □

Theorem 2.2.4. *The continuous shift* $\mathfrak{T}$ *is a strongly continuous group on* $L^2(\mathbb{R})$*.*

Proof of Theorem 2.2.4. By Lemma 2.2.3, we know that

$$(\mathfrak{T}^s f - f)\hat{}(x) = (e^{-sz} - 1)\hat{f}(z),$$

so we consider

$$\begin{aligned}\lim_{s\downarrow 0} \|\mathfrak{S}^s f - f\|_2^2 &= \lim_{s\downarrow 0} \left\|(e^{-sz} - 1)\hat{f}(z)\right\|_2^2 \\ &= \lim_{s\downarrow 0} \int_{-\infty}^{\infty} \left|e^{-sz} - 1\right|^2 \left|\hat{f}(z)\right|^2 dz,\end{aligned}$$

but $\left|e^{-sz} - 1\right|^2 \leq 4$, so by dominated convergence with $4\left|\hat{f}(z)\right|^2$ as the dominating function,

$$\begin{aligned}&= \int_{-\infty}^{\infty} \lim_{s\downarrow 0} \left|e^{-sz} - 1\right|^2 \left|\hat{f}(z)\right|^2 dz \\ &= 0\end{aligned}$$

Therefore $\mathfrak{T}^s$ is a strongly continuous semigroup on $L^2(\mathbb{R})$. □

We take a moment to justify calling the various T operator families shifts.

Theorem 2.2.5. *The forward continuous shift $\mathfrak{T}_+^s$ on $L^2_{\mathcal{U}}(\mathbb{R}_+)$ is a unilateral continuous shift. Similarly, the backward continuous shift $\mathfrak{T}_-^s$ on $L^2_{\mathcal{U}}(\mathbb{R}_-)$ is a unilateral continuous shift.*

The continuous shift $\mathfrak{T}^s$ is a bilateral continuous shift. Indeed, it is the minimal unitary extension $\mathfrak{T}_+^s$.

Proof. We note that $\mathfrak{T}_+^s L^2_{\mathcal{U}}(\mathbb{R}_+) = L^2_{\mathcal{U}}[s, \infty]$; therefore $\cap_{s\geq 0}\mathfrak{T}_+^s L^2_{\mathcal{U}}(\mathbb{R}_+) = \{0\}$. Thus

$\mathfrak{T}_+^s$ is a unilateral continuous shift.

That $\mathfrak{T}_-^s$ is also a unilateral continuous shift is similar.

Clearly, the definitions of $\mathfrak{T}^s$ and $\mathfrak{T}_+^s$ give, for all $f \in L^2_{\mathcal{U}}(\mathbb{R}_+)$, $\mathfrak{T}^s f(t) = \mathfrak{T}_+^s f(t)$. Further, note that $\mathfrak{T}^s L^2_{\mathcal{U}}(\mathbb{R}_+) = L^2_{\mathcal{U}}(s, \infty)$ for all s, so $\cup_{s \leq 0} \mathfrak{T}^s L^2_{\mathcal{U}}(\mathbb{R}_+)$ is dense in $L^2_{\mathcal{U}}(\mathbb{R})$; we conclude that $\mathfrak{T}^s$ on $L^2_{\mathcal{U}}(\mathbb{R})$ is a bilateral continuous shift. □

Remark 2.2.6. We introduce the space AC($\mathbb{R}$) of absolutely continuous functions on $\mathbb{R}$. See for example [25].

Theorem 2.2.7. *The infinitesimal generator of the continuous shift $\mathfrak{T}^s$ is given by the mapping*

$$T : f(t) \mapsto -f'(t)$$

with domain

$$\mathcal{D}(T) = \{f \in L^2(\mathbb{R}) \bigm| f \in AC(\mathbb{R}), f' \in L^2(\mathbb{R})\}.$$

Proof. This proof proceeds in two parts. First, we characterize functions f in $\mathcal{D}(T)$ as being such that their Laplace transforms satisfy $-z\hat{f}(z) \in L^2(i\mathbb{R})$. We then demonstrate that any function with such a Laplace transform is locally absolutely continuous and has derivative in $L^2(\mathbb{R})$. Combining these two parts gives the desired result.

We begin with the claim that $f \in \mathcal{D}(T)$ if and only if $-z\hat{f}(z) \in L^2(i\mathbb{R})$.

First we assume that $-z\hat{f}(z) \in L^2(i\mathbb{R})$. We show that the Laplace transform of the defining limit of the infinitesimal generator (cf. Definition 2.1.5) converges to

$-z\hat{f}(z)$. Consider

$$\left\|\frac{1}{s}\left(\mathfrak{T}^s f - f\right)\hat{} - (-z\hat{f})\right\|_2^2 = \left\|\frac{1}{s}\left(e^{-sz} - 1\right)\hat{f} + z\hat{f}\right\|_2^2$$

by Lemma 2.2.3. Then by the Mean Value Theorem, there exists a function $\sigma(s)$ with $0 < \sigma(s) < s$ and so that $\frac{1}{s}(e^{-sz} - 1) = -ze^{-\sigma(s)z}$, so

$$\begin{aligned} &= \left\|-ze^{-\sigma(s)z}\hat{f} + z\hat{f}\right\|_2^2 \\ &= \int_{-\infty}^{\infty} \left|e^{\sigma(s)z} - 1\right|^2 \left|-z\hat{f}(z)\right|^2 dx, \end{aligned}$$

which tends to zero as $s \downarrow 0$ by dominated convergence with $4\left|-z\hat{f}(z)\right|^2$, integrable since $-z\hat{f}(z) \in L^2$, as the dominating function. Since the limit exists, it follows that any f with $-z\hat{f}(z) \in L^2(\mathbb{R})$ is an element of $\mathcal{D}(T)$.

We now turn our attention to the converse: assume $f \in \mathcal{D}(T)$. This implies that there exists some $g \in L^2(\mathbb{R})$ with $\frac{1}{s}(\mathfrak{T}^s f - f) \to g$ in L^2. By the Plancherel theorem and Lemma 2.2.3 it follows that

$$\left\|\frac{1}{s}(e^{-sz} - 1)\hat{f} - \hat{g}\right\|_2^2 \to 0$$

as $s \downarrow 0$. By the Mean Value Theorem again, there exists some $\sigma(s)$ with $0 < \sigma(s) < s$ and $\frac{1}{s}(e^{-sz} - 1) = -ze^{-\sigma(s)z}$, so

$$\left\|-ze^{-\sigma(s)z}\hat{f}(z) - \hat{g}(z)\right\|_2^2 \to 0.$$

Equivalently, one may write

$$\left\| z\hat{f}(z) + e^{\sigma(s)z}\hat{g}(z) \right\|_2^2 \to 0.$$

But notice that

$$\begin{aligned} \left\| e^{\sigma(s)z}\hat{g}(z) - \hat{g}(z) \right\|_2^2 &= \left\| (e^{\sigma(s)z} - 1)\hat{g}(z) \right\|_2^2 \\ &= \int_{-i\infty}^{i\infty} \left| e^{\sigma(s)z} - 1 \right|^2 |\hat{g}(z)|^2 \, dz; \end{aligned}$$

this integral converges to 0 as $s \downarrow 0$ by dominated convergence with $|2\hat{g}|^2$ as the dominating function. We thus have that $e^{\sigma(s)z}\hat{g}(z) \to z\hat{f}(z)$ and also that $e^{\sigma(s)z}\hat{g}(z) \to \hat{g}(z)$, so $\hat{g}(z) = z\hat{f}(z)$ and therefore $z\hat{f}(z) \in L^2(i\mathbb{R})$.

We proceed to the second part of the proof: we show that $f \in L^2(\mathbb{R})$ is such that $z\hat{f}(z)$ is also in $L^2(\mathbb{R})$ if and only if f is locally absolutely continuous and has derivative f' in $L^2(\mathbb{R})$.

Let us first assume that $f \in L^2(\mathbb{R})$ is such that $z\hat{f}(z) \in L^2(i\mathbb{R})$, with the intention of showing that $f \in \mathrm{AC}(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$. Instead—and equivalently by the Plancherel Theorem—we show that if $f \in L^2(\mathbb{R})$ and $tf(t) \in L^2(\mathbb{R})$, then $\hat{f} \in L^2(i\mathbb{R})$, $\hat{f} \in \mathrm{AC}(i\mathbb{R})$, and $\hat{f}' \in L^2(i\mathbb{R})$.

We define some auxiliary functions: For $\tau \geq 0$, let

$$f_\tau(t) := \begin{cases} f(t) & \text{if } |t| \leq \tau \\ 0 & \text{otherwise} \end{cases};$$

let $g(t) := tf(t)$ and we define $g_\tau(t) := tf_\tau(t)$. The function $g(t) \in L^2(\mathbb{R})$ by hypothesis. Further, for every $\tau > 0$, both f_τ and g_τ are in $\mathrm{L}^1(-\tau, \tau)$.

Our first goal is to show that $\hat{f}$ is locally absolutely continuous. We first show that $\hat{f}_\tau$ is locally absolutely continuous: Consider that

$$\int_{ix_1}^{ix_2} \hat{g}_\tau(z)dz = \int_{ix_1}^{ix_2} \int_{-\tau}^{\tau} -tf_\tau(t)e^{-tz}dtdz,$$

and since $-tf_\tau$ is in $\mathrm{L}^1[-\tau,\tau]$ for every τ, Fubini-Tonelli implies

$$\begin{aligned} &= \int_{-\tau}^{\tau} \int_{ix_1}^{ix_2} -tf_\tau(t)e^{-tz}dzdt \\ &= \int_{-\tau}^{\tau} f_\tau(t)(e^{-itx_2} - e^{-itx_1})dt \\ &= \hat{f}_\tau(ix_2) - \hat{f}_\tau(ix_1). \end{aligned}$$

Thus $\hat{f}_\tau$ is locally absolutely continuous. To make a conclusion about $\hat{f}$, we take the limit as $\tau \to \infty$. The immediately preceding calculation shows that

$$\hat{f}_\tau(ix_2) = \hat{f}_\tau(ix_1) + \int_{ix_1}^{ix_2} \hat{g}_\tau(z)dz.$$

By construction, $L^2\text{-}\lim_{\tau\to\infty} f_\tau(t) = f(t)$; therefore we have that $L^2\text{-}\lim_{\tau\to\infty} \hat{f}_\tau(z) = \hat{f}(z)$. We also know that $\hat{g} \in L^2(i\mathbb{R})$, so then $\hat{g} \in \mathrm{L}^1(ix_1, ix_2)$ for any finite real x_1 and x_2; we then use dominated convergence to compute the limit $\int_{x_1}^{x_2} g_\tau ds$. Thus we have

$$\hat{f}(ix_2) = \hat{f}(ix_1) + \int_{ix_1}^{ix_2} \hat{g}(z)dz.$$

We conclude that $\hat{f}$ is locally absolutely continuous. Indeed, we also see that $\hat{f}' = \hat{g}$ which is in $L^2(i\mathbb{R})$ by hypothesis.

We turn our attention to the converse direction.

We assume that $f \in L^2(\mathbb{R})$ is locally absolutely continuous and that $f' \in L^2(\mathbb{R})$ and intend to show that $z\hat{f}(z) \in L^2(i\mathbb{R})$. Consider

$$\begin{aligned}(f')\hat{}(z) &= \lim_{\tau\to\infty} \int_{-\tau}^{\tau} e^{-zt} f'(t)dt \\ &= \lim_{\tau\to\infty} \left(e^{-zt} f(t)\big|_{-\tau}^{\tau} - \int_{-\tau}^{\tau} (-z)e^{-zt} f(t)dt \right) \\ &= \lim_{\tau\to\infty} \left(e^{-z\tau} f(\tau) - e^{z\tau} f(-\tau) + z\int_{-\tau}^{\tau} e^{-zt} f(t)dt \right);\end{aligned}$$

and, since we know that $\lim_{t\to\infty} f(t) = 0$, by Lemma 2.5.1 to come,

$$\begin{aligned}&= \lim_{\tau\to\infty} z\int_{-\tau}^{\tau} e^{-zt} f(t)dt \\ &= z\hat{f}(z).\end{aligned}$$

That is, $z\hat{f}(z) = (f'(t))\hat{} \in L^2(i\mathbb{R})$. □

We now have a great deal of information about our operator family $\mathfrak{T}^s$. It is a strongly continuous group on $L^2(\mathbb{R})$, a continuous bilateral shift, and we know both its generator and the domain of its generator. We now turn our attention to the shift semigroups which are derived from the shift group $\mathfrak{T}^s$.

The forward and backward shift semigroups have already been defined; see Definition 2.2.2. We take a moment to compute their adjoints.

Lemma 2.2.8. *Let $\pi_\pm : L^2_{\mathcal{U}}(\mathbb{R}) \to L^2_{\mathcal{U}}(\mathbb{R}_\pm)$ be orthogonal projections. Then for every s either nonnegative or nonpositive, as appropriate, $(\mathfrak{T}^s_\pm)^* = \pi_\pm(\mathfrak{T}^s)^*\big|_{L^2_{\mathcal{U}}(\mathbb{R}_\pm)} = \pi_\pm(\mathfrak{T}^{-s})\big|_{L^2_{\mathcal{U}}(\mathbb{R}_\pm)}$.*

Proof. Here, both $\mathfrak{T}_+^s$ and $\left(\mathfrak{T}_+^s\right)^*$ are bounded. Let $f, g \in L^2(\mathbb{R}_+)$, and embed both into $L^2(\mathbb{R})$ in the usual way. Then we may compute

$$\begin{aligned}&\left(f, \left(\mathfrak{T}_+^s\right)^* g\right) = (\mathfrak{T}_+^s f, g) = \int_0^\infty (\mathfrak{T}_+^s f)(\tau) g(\tau) d\tau = \int_s^\infty f(\tau - s) g(\tau) d\tau \\ &= \int_0^\infty \chi_{[0,\infty)} f(\tau') g(\tau' + s) d\tau' = \int_0^\infty f(\tau') \chi_{[0,\infty)} g(\tau' + s) d\tau' \\ &= (f, \chi_{[0,\infty)} \mathfrak{T}^{-s} g) = (f, \pi_+ \mathfrak{T}^{-s} g).\end{aligned}$$

The claimed result for $\left(\mathfrak{T}_-^s\right)^*$ follows by exchanging minuses for pluses. □

Definition 2.2.9. Let $\pi_\pm$ be the natural projection from $L^2_{\mathcal{U}}(\mathbb{R})$ onto $L^2_{\mathcal{U}}(\mathbb{R}_\pm)$ and let $\mathfrak{T}^s$ be the bilateral continuous shift on $L^2_{\mathcal{U}}(\mathbb{R})$.

By the term *Compressed Forward Shift* on $L^2_{\mathcal{U}}(\mathbb{R}_-)$, we mean the adjoint operator

$$\left(\mathfrak{T}_-^s\right)^* : f(t) \in L^2_{\mathcal{U}}(\mathbb{R}_-) \mapsto \pi_- f(t - s), \quad s \geq 0$$

and by *Compressed Backward Shift* on $L^2_{\mathcal{U}}(\mathbb{R}_+)$, we mean

$$\left(\mathfrak{T}_+^s\right)^* : f(t) \in L^2_{\mathcal{U}}(\mathbb{R}_+) \mapsto \pi_+ f(t - s), \quad s \leq 0.$$

We now have, in addition to the original bilateral continuous shift $\mathfrak{T}^s$, four unilateral continuous shifts: on $L^2_{\mathcal{U}}(\mathbb{R}_+)$, we have the forward shift $\mathfrak{T}_+^s$ and the compressed backward shift $\left(\mathfrak{T}_+^s\right)^*$; and on $L^2_{\mathcal{U}}(\mathbb{R}_-)$, we have the backward shift $\mathfrak{T}_-^s$ and the compressed forward shift $\left(\mathfrak{T}_-^s\right)^*$. The strong continuity of each follows directly from the strong continuity of $\mathfrak{T}^s$ itself.

We record the infinitesimal generators of the restricted semigroups as well as the domains of the generators.

Theorem 2.2.10. *Let $\Omega \subset \mathbb{R}$ and let $AC(\Omega)$ refer to absolutely continuous functions on Ω.*

The bilateral continuous shift group $\mathfrak{T}^s$ acting on $L^2_{\mathcal{U}}(\mathbb{R})$ has the generator

$$T : f(t) \mapsto -f'(t)$$

with domain

$$\mathcal{D}(T) = \{f \in L^2 \bigm| f \in AC(\mathbb{R}), f' \in L^2(\mathbb{R})\}.$$

The forward unilateral shift $\mathfrak{T}^s_+$ acting on $L^2_{\mathcal{U}}(\mathbb{R}_+)$ has generator

$$T_+ : f(t) \mapsto -f'(t)$$

with domain

$$\mathcal{D}(T_+) = \{f \in L^2(\mathbb{R}_+) \bigm| f \in AC(\mathbb{R}_+), f' \in L^2(\mathbb{R}_+), f(0) = 0\}.$$

The backward unilateral shift $\mathfrak{T}^s_-$ acting on $L^2_{\mathcal{U}}(\mathbb{R}_-)$ has generator

$$T_- : f(t) \mapsto f'(t)$$

with domain

$$\mathcal{D}(T_-) = \{f \in L^2(\mathbb{R}_-) \bigm| f \in AC(\mathbb{R}_-), f' \in L^2(\mathbb{R}_-), f(0) = 0\}.$$

The compressed forward unilateral shift $\left(\mathfrak{T}_-^s\right)^*$ *acting on* $L^2_{\mathcal{U}}(\mathbb{R}_-)$ *has generator*

$$T_-^* : f(t) \mapsto -f'(t)$$

with domain

$$\mathcal{D}(T_-^*) = \{f \in L^2(\mathbb{R}_-) \bigm| f \in AC(\mathbb{R}_-), f' \in L^2(\mathbb{R}_-)\}.$$

The compressed backward unilateral shift $\left(\mathfrak{T}_+^s\right)^*$ *acting on* $L^2_{\mathcal{U}}(\mathbb{R}_+)$ *has generator*

$$T_+^* : f(t) \mapsto f'(t)$$

with domain

$$\mathcal{D}(T_+^*) = \{f \in L^2(\mathbb{R}_+) \bigm| f \in AC(\mathbb{R}_+), f' \in L^2(\mathbb{R}_+)\}.$$

Proof. That the generator of the bilateral shift is the mapping $f \mapsto -f'$ is a clear corollary of Theorem 2.2.7, as is the domain of the generator of the forward shift when acting on $L^2(\mathbb{R})$. That the generator of the backward semigroups is the same has been discussed already in Note 2.1.8.

When we intersect $\mathcal{D}(T)$ with $L^2(\mathbb{R}_+)$, we get precisely $\mathcal{D}(T\big|_{L^2(\mathbb{R}_+)})$. We also immediately get $\mathcal{D}(-T\big|_{L^2(\mathbb{R}_-)})$ by symmetry.

By Lemma 2.2.8, we know that $(\mathfrak{T}_+^s\big|_{L^2(\mathbb{R}_+)})^* = \mathfrak{T}_-^s\big|_{L^2(\mathbb{R}_+)}$. Thus finding $\mathcal{D}(-T\big|_{L^2(\mathbb{R}_+)})$ reduces to determining the domain of the adjoint $T\big|_{L^2(\mathbb{R}_+)}$.

Define $\mathcal{D}^* := \{f \in L^2(\mathbb{R}_+) \bigm| f \in \mathrm{AC}(\mathbb{R}_+), f' \in L^2(\mathbb{R}_+)\}$.

We first demonstrate that $\mathcal{D}^* \subset \mathcal{D}(T_+^*)$. Let $f \in \mathcal{D}(T_+)$ and let $g \in \mathcal{D}^*$. Then we compute

$$(f, T_+^* g) = (T_+ f, g) = \int_+ f'\bar{g} = f\bar{g}\Big|_0^\infty - \int_+ f\bar{g}' = f\bar{g}\Big|_0^\infty + (f, -g').$$

By hypothesis, $f(0) = 0$. Since $f \in L^2(\mathbb{R}_+), f \in \mathrm{AC}(\mathbb{R}_+)$, and $f' \in L^2(\mathbb{R}_+)$, Lemma 2.5.1 gives $\lim_{t\to\infty} f(t) = 0$. The same is true of g. Thus we conclude that $\mathcal{D}^* \subset \mathcal{D}(T_+^*)$ and that when $g \in \mathcal{D}^*$, $T_+^* : g \mapsto -g'$.

Before we demonstrate the reverse inclusion, we first construct an auxiliary function $F \in \mathcal{D}(T_+)$. Fix $0 < T < \infty$. Choose some $f \in L^2(\mathbb{R}_+)$ with $\int_0^T f(s)ds = 0$ and $f(t) = 0$ when $t > T$. Note that, by construction, $f \perp 1$ on $L^2[0, T]$. Finally, let $F(t) := \int_0^t f(s)ds$; then $F \in \mathcal{D}(T_+)$, $T_+ F = -f$, and if $t > T$ then $F(t) = 0$.

Now assume that $g \in \mathcal{D}(T_+^*)$. Then there exists some $h \in L^2(\mathbb{R}_+)$ so that for any $x \in \mathcal{D}(T_+)$, the equality $(T_+ x, g) = (x, h)$ holds. (That is, $T_+^* g = h$.) In particular, this equality must hold for F. We compute:

$$\begin{aligned}
0 &= (F, h) - (T_+ F, g) \\
&= \int_0^\infty F(t)h(t)dt + \int_0^\infty f(t)g(t)dt \\
&= \int_0^T F(t)h(t)dt + \int_0^T f(t)g(t)dt
\end{aligned}$$

which, by integration by parts,

$$
\begin{aligned}
&= -\int_0^T f(t) \int_0^t h(s)dsdt + \int_0^T f(t)g(t)dt \\
&= \int_0^T f(t) \left(g(t) - \int_0^t h(s)ds \right) dt \\
&= \left(f(t), g(t) - \int_0^t h(s)ds \right)_{L^2(0,T)},
\end{aligned}
$$

where the interpretation as an inner product is justified since $\left(g(t) - \int_0^t h(s)ds\right) \in L^2[0,T]$ because $g, h \in L^2(\mathbb{R}_+)$. Thus $g(t) - \int_0^t h(s)ds$ is perpendicular to f, but f is an arbitrary function perpendicular to the constants in $L^2[0,T]$. It follows that $g(t) - \int_0^t h(s)ds$ itself equals a constant (which a priori may depend on T), which we call c_T. But since this same computation works for any arbitrary T, the constant cannot actually depend on T, so we conclude that there exists some constant c with

$$
g(t) - \int_0^t h(s)ds = c.
$$

Thus we see that $g \in \mathcal{D}(T_+)$ is locally absolutely continuous and $g' = h \in L^2(\mathbb{R}_+)$, which is the reverse inclusion that we sought.

We have now fully characterized $\mathcal{D}(T_+^*)$. □

2.2.2 Laplace Transforms of Shifts

There is one more fact about our shift (semi)groups that we need to record. We use the bilateral Laplace transform to induce a shift operator on $L^2_{\mathcal{U}}(i\mathbb{R})$. We give the definition and follow it immediately with an explicit formula.

Definition 2.2.11. We define the Laplace transform of the shift group $\widehat{\mathfrak{T}}^s$ on $L^2_{\mathcal{U}}(\mathbb{R})$ in the usual way that one defines such things; namely, by

$$\left(\widehat{\mathfrak{T}}^s f\right)(z) = \left(\mathfrak{T}^s \check{f}\right)^{\hat{}}(z)$$

Proposition 2.2.12. *We record the following formula:*

$$\left(\widehat{\mathfrak{T}}^s f\right)(z) = e^{-sz} f(z).$$

Proof. We need only compute. For $F \in L^2_{\mathcal{U}}(i\mathbb{R})$, there exists $f \in L^2_{\mathcal{U}}(\mathbb{R})$ with $F(z) = \hat{f}(z)$. Then

$$\begin{aligned}
\left(\widehat{\mathfrak{T}}^s F\right)(z) &= \left(\mathfrak{T}^s f\right)^{\hat{}}(z) \\
&= \int_{-\infty}^{\infty} \mathfrak{T}^s f(t) e^{-zt}\, dt \\
&= \int_{-\infty}^{\infty} f(t-s) e^{-zt}\, dt \\
&= e^{-sz} \hat{f}(z).
\end{aligned}$$

□

2.3 A Graph Space Lemma

If a Hilbert space $\mathcal{X}$ can be orthogonally decomposed into closed subspaces, then the following Lemma states that any closed subspace $\mathcal{M}$ of $\mathcal{X}$ is "almost" a graph space.

Lemma 2.3.1 (Almost Graph Space)**.** *Let $\mathcal{X}$ be a Hilbert space that has an orthogonal sum decomposition into (closed) subspaces $\mathcal{X}_-$ and $\mathcal{X}_+$; that is, $\mathcal{X} = \mathcal{X}_- \oplus \mathcal{X}_+$. Let $\mathcal{M} \subset \mathcal{X}$ be a closed subspace. Further let $P_{\mathcal{X}_-}$ be the orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}_-$.*

Define three additional subspaces related to $\mathcal{M}$ as follows: let $\mathcal{P} := \overline{P_{\mathcal{X}_-}\mathcal{M}}$, let $\mathcal{M}_0 = \mathcal{M} \cap \mathcal{X}_+$, and let $\mathcal{Z} = \mathcal{X}_+ \ominus \mathcal{M}_0$ with $P_{\mathcal{Z}}$ the orthogonal projection of $\mathcal{X}_+$ onto $\mathcal{Z}$.

Then there is a unique closed operator $\Gamma : \mathcal{D}(S_+) \subset \mathcal{P} \to \mathcal{Z}$, densely-defined, such that

$$\mathcal{M} = \{m_{\mathcal{P}} + m_{\mathcal{Z}} + m_0 \;\big|\; m_{\mathcal{P}} \in \mathcal{D}(\Gamma) \subset \mathcal{P},\ m_{\mathcal{Z}} \in \mathcal{Z},\ m_0 \in \mathcal{M}_0,\ \textit{and}\ \Gamma m_{\mathcal{P}} = m_{\mathcal{Z}}\}$$

Proof. For any $m_{\mathcal{P}} \in P_{\mathcal{X}_-}\mathcal{M}$, there exists at least one $m' \in \mathcal{X}_+$ so that $m_{\mathcal{P}} + m' \in \mathcal{M}$. Choose any such m' and define $\Gamma m_{\mathcal{P}} = P_{\mathcal{Z}} m'$. Note that this definition of $\Gamma m_{\mathcal{P}}$ is unique, because if $m'' \in \mathcal{X}_+$ so that $m_{\mathcal{P}} + m'' \in \mathcal{M}$, then

$$(m_{\mathcal{P}} + m') - (m_{\mathcal{P}} + m'') = m' - m''$$

must be in $\mathcal{M} \cap \mathcal{X}_+$; thus $P_{\mathcal{Z}}(m' - m'') = 0$. It also follows from this definition that $(m_{\mathcal{P}} + \Gamma m_{\mathcal{P}}) \in \mathcal{M}$; indeed, we let $m = m_{\mathcal{P}} + m'$ be the element of $\mathcal{M}$ associated with $m_{\mathcal{P}}$ so that

$$m = (m_{\mathcal{P}} + \Gamma m_{\mathcal{P}}) + (m' - \Gamma m_{\mathcal{P}}) = (m_{\mathcal{P}} + \Gamma m_{\mathcal{P}}) + (m' - P_{\mathcal{Z}} m'),$$

thus

$$m_{\mathcal{P}} + \Gamma m_{\mathcal{P}} = m - (m' - P_{\mathcal{Z}} m').$$

The latter quantity in parentheses is in $\mathcal{M}_0$ and thus in $\mathcal{M}$. Since $(m_{\mathcal{P}} + \Gamma m_{\mathcal{P}})$ is the difference of elements of $\mathcal{M}$, it is an element of $\mathcal{M}$.

Note that $\mathcal{M}_{\mathcal{P}}$ is the closure of $\mathcal{D}(\Gamma)$; Γ is thus trivially densely-defined.

Finally, to see that Γ is closed, assume the existence of a sequence $m_{\mathcal{P},n}$ in $\mathcal{D}(\Gamma)$ converging to $m_{\mathcal{P}} \in \mathcal{X}_-$ and such that $\Gamma m_{\mathcal{P},n}$ converges to $y \in \mathcal{X}_+$. In fact, since $\mathcal{M}$ is closed, $(m_{\mathcal{P}} + y) \in \mathcal{M}$. Clearly, $P_{\mathcal{X}_-}(m_{\mathcal{P}} + y) = m_{\mathcal{P}}$, so $m_{\mathcal{P}} \in \mathcal{D}(\Gamma)$. Further, since $\mathcal{M}$ and $\mathcal{X}_+$ are closed, so is $\mathcal{M}_0$. As is $\mathcal{Z}$, as an orthocomplement. Thus y is in $\mathcal{Z}$. It is the unique such element so that $(m_{\mathcal{P}} + y) \in \mathcal{M}$, so in fact $y = \Gamma m_{\mathcal{P}}$ and Γ is closed. $\square$

2.4 Continuous Time System Theory: Well-Posed Linear Systems

2.4.1 Introduction

Many of the results we are interested in are stated most naturally in the context of abstract systems. Systems wherein the underlying spaces are finite-dimensional have been well understood for some time. New, however, is the understanding of infinite-dimensional systems and, especially, the idea of L^2-admissible systems as laid out in [26]; preliminary versions of these ideas go back at least as far as [**?**], [13], among others.

2.4.2 Rigged Spaces

We introduce the so-called rigged spaces. In our context, these form a bi-infinite nested sequence of Hilbert spaces with each densely contained in the previous. These spaces are vital to the development of a theory of infinite-dimensional continuous-time linear systems. We take a moment to introduce these spaces now. For more thorough discussions, we refer the reader to [26] or [27].

Proposition 2.4.1. *Let $\mathcal{X}$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{X}}$ and let $A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{X}$ be a closed, densely-defined operator with nonempty resolvent on $\mathcal{X}$. Then for some $\alpha \in \rho(A)$ and $x \in \mathcal{D}(A)$, we equip $\mathcal{D}(A)$ with the norm*

$$\|x\|_1 := \|(\alpha I - A)x\|_{\mathcal{X}} .$$

Then with this norm, $\mathcal{D}(A)$ is a Hilbert space which we denote $\mathcal{X}_1$. Further, the norm $\|\cdot\|_1$ is independent of the choice of α, in that all such norms are equivalent. In particular, each norm is equivalent to the graph norm.

The preceding proposition is more or less a restatement of Proposition 2.10.1 in [27], to which we mostly refer the reader for the proof. We do, however, give an explicit proof of the equivalence of the rigged norms to the graph norm below in Lemma 2.4.4. We next define the rigged spaces with negative indices.

Proposition 2.4.2. *Let $\mathcal{X}$, A, and α be as in Proposition 2.4.1 above. For $x \in \mathcal{X}$, we define the norm*

$$\|x\|_{-1} := \left\|(\alpha I - A)^{-1}x\right\|_{\mathcal{X}} .$$

We define $\mathcal{X}_{-1}$ to be the completion of $\mathcal{X}$ with respect to the norm $\|\cdot\|_{-1}$. Then $\mathcal{X}_{-1}$ is a Hilbert space. Further, the norm $\|\cdot\|_{-1}$ is independent of choice of α.

As before, this proposition is a restatement of Proposition 2.10.2 from [27], to which we refer the reader for the proof.

Remark 2.4.3. We note that the construction of $\mathcal{X}_1$ may be iterated. That is, having defined $\mathcal{X}_1$, we may go on to define $\mathcal{X}_2$, $\mathcal{X}_3$, &c.

Similarly, having defined $\mathcal{X}_{-1}$, we may define $\mathcal{X}_{-2}$, $\mathcal{X}_{-3}$, &c.

Thus the rigged spaces form a bi-infinite densely nested sequence of Hilbert spaces with

$$\cdots \supset \mathcal{X}_{-2} \supset \mathcal{X}_{-1} \supset \mathcal{X} \supset \mathcal{X}_1 \supset \mathcal{X}_2 \supset \cdots$$

Further, by our construction, for all $k \in \mathbb{N}$, we have that $(\alpha I - A)^{-1} : \mathcal{X}_k \to \mathcal{X}_{k+1}$ is bijective.

We also note that, as $\mathcal{X}$ is a Hilbert space, one may also construct a bi-infinite sequence of nested spaces using A^* on $\mathcal{X}$ instead of A. In this case one recovers a sequence

$$\cdots \supset \mathcal{X}^*_{-2} \supset \mathcal{X}^*_{-1} \supset \mathcal{X} \supset \mathcal{X}^*_1 \supset \mathcal{X}^*_2 \supset \cdots$$

We clarify the relationship between the two sequences of rigged spaces in Lemma 2.4.6.

We now give a proof of an illustrative special case of the equivalence of the $\mathcal{X}_1$ norm and the graph norm on $\mathcal{D}(A)$.

Lemma 2.4.4. *Let $\mathcal{X}$ be a Hilbert space and let A be an operator that generates a rigged space structure on $\mathcal{X}$ such that 1 is in the resolvent of A. On the space $\mathcal{X}_1$,* *we define the following three norms: first, the graph norm*

$$\|x\|_G := \sqrt{\|x\|_{\mathcal{X}}^2 + \|Ax\|_{\mathcal{X}}^2}.$$

Second, another graph norm

$$\|x\|_M := \|x\|_{\mathcal{X}} + \|Ax\|_{\mathcal{X}}.$$

Finally, the usual rigged space norm

$$\|x\|_{\mathcal{X}_1} := \|(I - A)x\|_{\mathcal{X}}.$$

Then these norms are equivalent.

Proof. The equivalence of the two graph norms follow from the general result that norms on finite-dimensional spaces are equivalent. That is, there are constants $0 < m \leq M < \infty$ so that for any two nonnegative real numbers r_1 and r_2, we have that

$$m(r_1 + r_2) \leq \sqrt{r_1^2 + r_2^2} \leq M(r_1 + r_2).$$

Setting $r_1 = \|x\|_{\mathcal{X}}$ and $r_2 = \|Ax\|_{\mathcal{X}}$ gives the equivalence of the two graph norms.

We show that the Manhattan graph norm is equivalent to the rigged space norm. Clearly, we have that $\|x\|_{\mathcal{X}_1} = \|(I - A)x\|_{\mathcal{X}} \leq \|x\|_{\mathcal{X}} + \|Ax\|_{\mathcal{X}} = \|x\|_M$. To see the reverse inequality, consider

$$\begin{aligned}
\|x\|_M &= \|x\|_{\mathcal{X}} + \|Ax\|_{\mathcal{X}} \\
&= \left\|(I-A)^{-1}(I-A)x\right\|_{\mathcal{X}} + \|(Ix - Ix) + Ax\|_{\mathcal{X}} \\
&\leq \left\|(I-A)^{-1}\right\| \|(I-A)x\|_{\mathcal{X}} + \|x\|_{\mathcal{X}} + \|(I-A)x\|_{\mathcal{X}} \\
&\leq \left\|(I-A)^{-1}\right\| \|x\|_{\mathcal{X}_1} + \left\|(I-A)^{-1}(I-A)x\right\|_{\mathcal{X}} + \|x\|_{\mathcal{X}_1} \\
&= (2\left\|(I-A)^{-1}\right\| + 1)\|x\|_{\mathcal{X}_1}.
\end{aligned}$$

□

We note the following result which describes the pairing $(\cdot,\cdot)_{\mathcal{X}_1,\mathcal{X}_{-1}}$ in the case that the second argument is, in fact, an element of $\mathcal{X}$.

Lemma 2.4.5. *Let $\mathcal{X}$ be a Hilbert space and let Z be an operator that generates a rigged space structure on $\mathcal{X}$. Further let $x \in \mathcal{X}_1^*$ and let $y \in \mathcal{X}$, which we consider as a dense subset of $\mathcal{X}_{-1}$. Then*

$$(x, y)_{*1,-1} = (x, y)_{\mathcal{X}}.$$

Proof. As $x \in \mathcal{X}_1^*$, there exist an $\alpha \in \rho(Z)$ and an $x' \in \mathcal{X}$ so that we may write

$x = (\overline{\alpha}I - Z^*)^{-1}x'$. Similarly, there exists a $y' \in \mathcal{X}$ so that $y = (\alpha I - Z)y'$. Then

$$\begin{aligned}(x, y)_{(*1,-1)} &= \big((\overline{\alpha}I - Z^*)^{-1}x', (\alpha I - Z)y'\big)_{(*1,-1)} \\ &= (x', y')_{\mathcal{X}} \\ &= \big((\overline{\alpha}I - Z^*)^{-1}x', (\alpha I - Z)y'\big)_{\mathcal{X}} \\ &= (x, y)_{\mathcal{X}}\end{aligned}$$

□

Having defined the rigged spaces, we turn our attention to a result that serves to describe the connections between them.

Lemma 2.4.6. *Let $\mathcal{X}$ be a Hilbert space and let Z be an operator that generates the rigged spaces $\{\mathcal{X}_n\}$, $n = 0, \pm 1, \pm 2, \cdots$. Let the set of rigged spaces generated by Z^* be denoted $\{\mathcal{X}_{*,n}\}$. Then for $n = 0, 1, 2, \cdots$, $(\mathcal{X}_{-n})^* = \mathcal{X}_{*,n}$.*

Of special interest is the $n = 1$ case: $(\mathcal{X}_{-1})^* = \mathcal{X}_{*,1}$.

Proof. We proceed by induction. Clearly, $(\mathcal{X}_0)^* = \mathcal{X}^* = \mathcal{X} = \mathcal{X}_{*,0}$. We assume, then, that for some $n \geq 1$, $(\mathcal{X}_{-(n-1)})^* = \mathcal{X}_{*,(n-1)}$.

We set the parameter α from the definition of rigged spaces to 1.

First, we show that $\mathcal{X}_{*,n} \subset (\mathcal{X}_{-n})^*$. Let $x \in \mathcal{X}_{-n}$ and let $x^* \in \mathcal{X}_{*,n}$. Then we have that $x^*(x) := ((I - Z^*)^n x^*, (I - Z)^{-n}x)_{\mathcal{X}}$, which is clearly continuous.

To show the reverse inclusion, we let $l \in (\mathcal{X}_{-n})^*$. By construction of the rigged spaces, $(I - Z)^{-1} : \mathcal{X}_{-n} \to \mathcal{X}_{-(n-1)}$ is an isometry, so the composition $l \circ (I - Z) : \mathcal{X}_{-(n-1)} \to \mathcal{U}$ is continuous. By hypothesis, there exists some $y \in \mathcal{X}_{*,(n-1)}$ so that for every $x \in \mathcal{X}_{-(n-1)}$,

$$l \circ (I - Z) : x \mapsto (x, y)_{(\mathcal{X}_{-(n-1)}, \mathcal{X}_{*,(n-1)})}.$$

Recall that $\mathcal{X}_{-(n-1)}$ is dense in $\mathcal{X}_{-n}$; we choose some x' in their intersection and define $x \in \mathcal{X}_{-n}$ by $x := (I - Z)^{-1}x'$. Then we can compute

$$l(x') = \left((I - Z)^{-1}x', y)\right)_{(\mathcal{X}_{-(n-1)}, \mathcal{X}_{*,(n-1)})} = \left(x', (I - Z^*)^{-1}y\right)_{(\mathcal{X}_{-(n-1)}, \mathcal{X}_{*,(n-1)})}.$$

As $(I - Z^*)^{-1}y \in X_{*,n}$, this is nearly the representation we seek. So far, it is only defined on $\mathcal{X}_{-(n-1)}$; we seek to extend it to $\mathcal{X}_{-n}$.

For $x' \in \mathcal{X}_{-(n-1)} \subset \mathcal{X}_{-n}$, we have the estimate

$$\begin{aligned} |l(x')| &= \left((I - Z)^{-1}x', y)\right)_{(\mathcal{X}_{-(n-1)}, \mathcal{X}_{*,(n-1)})} \\ &\leq \left\|(I - Z)^{-1}x'\right\|_{\mathcal{X}_{-(n-1)}} \|y\|_{\mathcal{X}_{*,(n-1)}} = \|x'\|_{\mathcal{X}_{-n}} \|y\|_{\mathcal{X}_{*,(n-1)}}. \end{aligned}$$

We may thus extend l uniquely and continuously to $\mathcal{X}_{-n}$. If we define $y' := (I - Z^*)^{-1}y$, then we have the form

$$l : x' \in \mathcal{X}_{-n} \mapsto (x', y)_{(\mathcal{X}_{-n}, \mathcal{X}_{*,n})},$$

which demonstrates our reverse inclusion. □

2.4.3 Control and Observation Operators

We begin with a sequence of four definitions: we define Π_-- and Π_+-Admissible State/Output Pairs, and Π_-- and Π_+-Admissible Input/State Pairs.

Definition 2.4.7. Let $\mathcal{X}_{\mathcal{P}}^{\times}$ and $\mathcal{U}$ be Hilbert spaces, let $A^{\times} : \mathcal{X}_{\mathcal{P},1}^{\times} \to \mathcal{X}_{\mathcal{P}}^{\times}$ be a densely-defined linear operator, and let $C^{\times} : \mathcal{X}_{\mathcal{P},1}^{\times} \to \mathcal{U}$ be a linear operator. We call the pair $(C^{\times}, A^{\times})$ a Π_{-}-*Admissible State/Output Pair* if

1. $A^{\times}$ generates a strongly continuous forward semigroup $\mathfrak{A}^{\times}$ on $\mathcal{X}_{\mathcal{P}}^{\times}$,
2. $C^{\times}$ is bounded as an operator from $\mathcal{X}_{\mathcal{P},1}^{\times}$ to $\mathcal{U}$,
3. the mapping
$$\left(\mathcal{O}^{f}_{C^{\times},A^{\times}} x_0\right)(t) := C^{\times}\mathfrak{A}^{\times,t} x_0, t \geq 0$$
can be extended to a continuous map from $\mathcal{X}_{\mathcal{P}}^{\times}$ to $L^2_{\mathcal{U}}(0,\infty)$.

If, in addition, the operator $\mathcal{O}^{f}_{C^{\times},A^{\times}}$ is one-to-one, we call the pair $(C^{\times}, A^{\times})$ a Π_{-}-*Admissible Observable Pair*.

We name the (extended) operator $\mathcal{O}^{f}_{C^{\times},A^{\times}}$ on $\mathcal{X}_{\mathcal{P}}^{\times}$ the *forward Observation operator*.

Let $\mathcal{X}_{\mathcal{P}}$ and $\mathcal{U}$ be Hilbert spaces, let $A : \mathcal{X}_{\mathcal{P},1} \to \mathcal{X}_{\mathcal{P}}$ be a densely-defined linear operator, and let $C : \mathcal{X}_{\mathcal{P},1} \to \mathcal{U}$ be a linear operator. We call the pair (C, A) a Π_{+}-*Admissible State/Output Pair* if

1. A generates a strongly continuous backward semigroup $\mathfrak{A}$ on $\mathcal{X}_{\mathcal{P}}$,
2. C is bounded as an operator from $\mathcal{X}_{\mathcal{P},1}$ to $\mathcal{U}$, and
3. the mapping
$$\left(\mathcal{O}^{b}_{C,A} x_0\right)(t) := C\mathfrak{A}^{t} x_0, t \leq 0$$
can be extended to a continuous map from $\mathcal{X}_{\mathcal{P}}$ to $L^2_{\mathcal{U}}(-\infty,0)$.

If, in addition, the operator $\mathcal{O}^b_{C,A}$ is one-to-one, we call the pair (C, A) a Π_+-*Admissible Observable Pair.*

We define the (extended) operator $\mathcal{O}^b_{C,A}$ on $\mathcal{X}_\mathcal{P}$ to be the *backward Observation operator.*

Let $\mathcal{X}^\times_\mathcal{Z}$ and $\mathcal{U}$ be Hilbert spaces, let $Z^\times : \mathcal{X}^\times_{\mathcal{Z},1} \to \mathcal{X}^\times_\mathcal{Z}$ be a densely-defined linear operator, and let $B^\times : \mathcal{U} \to \mathcal{X}^\times_{\mathcal{Z},-1}$ be a linear operator. We call the pair $(Z^\times, B^\times)$ a Π_--*Admissible Input/State Pair* if

1. $Z^\times$ generates a strongly continuous forward semigroup $Z^\times$ on $\mathcal{X}_\mathcal{Z}$,
2. $B^\times$ is bounded as an operator from $\mathcal{U}$ to $\mathcal{X}^\times_{\mathcal{Z},-1}$, and
3. the mapping

$$\mathcal{C}^f_{Z^\times,B^\times} : u \in L^2(-\infty, 0) \mapsto \int_{-\infty}^0 \mathfrak{Z}^{\times,-s}_{|\mathcal{X}^\times_{\mathcal{Z},-1}} B^\times u(s) ds,$$

   a priori defined only for $u \in L^2_\mathcal{U}(0, T)$ for finite T with image in $\mathcal{X}^\times_{\mathcal{Z},-1}$, in fact has image in $\mathcal{X}^\times_\mathcal{Z}$ and extends to a bounded map from $L^2(-\infty, 0)$ into $\mathcal{X}^\times_\mathcal{Z}$.

If, in addition, the operator $\mathcal{C}^f_{Z^\times,B^\times}$ has dense range, then we call the pair $(Z^\times, B^\times)$ a Π_--*Admissible Controllable Pair*; we additionally call the operator $\mathcal{C}^f_{Z^\times,B^\times}$ on $L^2(-\infty, 0)$ the *backward Control operator.*

Finally, let $\mathcal{X}_\mathcal{Z}$ and $\mathcal{U}$ be Hilbert spaces, let $Z : \mathcal{X}_{\mathcal{Z},1} \to \mathcal{X}_\mathcal{Z}$ be a densely-defined linear operator, and let $B : \mathcal{U} \to \mathcal{X}_\mathcal{Z}$ be a linear operator. We call the pair (Z, B) a Π_+-*Admissible Input/State Pair* if

1. Z generates a strongly continuous backward semigroup $\mathfrak{Z}$ on $\mathcal{X}_\mathcal{Z}$,

2. B is bounded as an operator from $\mathcal{U}$ to $\mathcal{X}_{\mathcal{Z},-1}$, and

3. the mapping
$$\mathcal{C}^b_{Z,B} : u \in L^2(0,\infty) \mapsto -\int_0^\infty \mathfrak{Z}^{-s} Bu(s)ds,$$
a priori defined only for $u(s) \in L^2_{\mathcal{U}}(0,T)$ for finite T with image in $\mathcal{X}_{\mathcal{Z},-1}$, in fact has image in $\mathcal{X}_{\mathcal{Z}}$ and extends to a map from $L^2(0,\infty)$ into $\mathcal{X}_{\mathcal{Z}}$.

If, in addition, the operator $\mathcal{C}^b_{Z,B}$ has dense range, then we call the pair (Z,B) a Π_+-*Admissible Controllable Pair* and we call the operator $\mathcal{C}^b_{Z,B}$ from $L^2_{\mathcal{U}}(0,\infty)$ to $\mathcal{X}_{\mathcal{Z}}$ the *backward Control operator.*

Definition 2.4.8. A Π_--Admissible Observable pair $(C^\times, A^\times)$ is called Exactly Observable if the forward observation operator $\mathcal{O}^f_{C^\times,A^\times}$ is bounded below as a mapping from $\mathcal{X}^\times_{\mathcal{P}}$ to $L^2(0,\infty)$.

A Π_+-Admissible Observable pair, (C,A), on the other hand, is called Exactly Observable if the backward observation operator $\mathcal{O}^b_{C,A}$ is bounded below as a mapping from $\mathcal{X}_{\mathcal{P}}$ to $L^2(-\infty,0)$.

A Π_--Admissible Controllable pair $(Z^\times, B^\times)$ is called Exactly Controllable if the forward control operator $\mathcal{C}^f_{Z^\times,B^\times}$ is also onto $\mathcal{X}^\times_{\mathcal{Z}}$.

A Π_+-Admissible Controllable pair (Z,B) is called Exactly Controllable if the backward control operator $\mathcal{C}^b_{Z,B}$ is also onto $\mathcal{X}_{\mathcal{Z}}$.

Having defined the Control and Observation operators, we present a pair of theorems exhibiting relationships between them under adjoints.

Theorem 2.4.9. *Let $\mathcal{X}$ be a Hilbert space. Let (C,A) be a Π_+-Admissible Exactly Observable pair on $\mathcal{X}$. Then $(-A^*, C^*)$ is a Π_--Admissible Exactly Controllable pair* *on $\mathcal{X}$, and*

$$\left(\mathcal{O}^b_{C,A}\right)^* = \mathcal{C}^f_{-A^*,C^*}.$$

Theorem 2.4.10. *Let $\mathcal{X}$ be a Hilbert space. Let (Z, B) be a Π_+-Admissible Exactly Controllable pair on $\mathcal{X}$. Then $(B^*, -Z^*)$ is a Π_--Admissible Exactly Observable pair on $\mathcal{X}$, and*

$$\left(\mathcal{C}^b_{Z,B}\right)^* = -\mathcal{O}^f_{B^*,-Z^*}.$$

Remark 2.4.11. We note that the preceding theorem is also true with all instances of the word "Exactly" deleted, as the proof shows.

Proof. First, we note that if Z generates a forward (resp. backward) strongly continuous semigroup, then $-Z^*$ generates a backward (resp. forward) strongly continuous semigroup. Similarly, if B is bounded, then so is B^*. To verify the other claimed properties of the pair (B^*, Z^*), we turn our attention to constructing the operator $\mathcal{O}^f_{B^*,-Z^*}$.

Fix some $T \in (0, \infty)$. We define

$$-\mathcal{C}^{b,T}_{Z,B} : f \in L^2_{\mathcal{U}}(0,T) \mapsto \int_0^T \mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}} B f(s) ds.$$

We first seek to compute $\left(\mathcal{C}^{b,T}_{Z,B}\right)^*$; later we will use this result to find $\left(\mathcal{C}^b_{Z,B}\right)^*$.

To this end, we will first regard $\mathcal{C}^b_{Z,B}$ and $\mathcal{C}^{b,T}_{Z,B}$ as mappings into $\mathcal{X}_{\mathcal{Z},-1}$. This is the natural assumption with which to begin the analysis, as their integrands are pointwise in $\mathcal{X}_{\mathcal{Z},-1}$. Later, we will apply the admissibility assumption on (Z, B) and treat both $\mathcal{C}^b_{Z,B}$ and $\mathcal{C}^{b,T}_{Z,B}$ as mappings onto $\mathcal{X}_{\mathcal{Z}}$.

Choose some $f \in L^2_{\mathcal{U}}(0,T)$ and $x^* \in (\mathcal{X}_{\mathcal{Z},-1})^* = \mathcal{X}_{Z^*,1}$. We compute

$$\begin{aligned} &(f, \left(\mathcal{C}^b_{Z,B}\right)^* x^*)_{L^2_{\mathcal{U}}(0,T)} \\ =&(\mathcal{C}^b_{Z,B} f, x^*)_{(-1,*1)} \\ =&(\int_0^T \mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}} B f(s) ds, x^*)_{(-1,*1)}, \end{aligned}$$

but by construction, the $(\cdot,\cdot)_{(-1,*1)}$ pairing is continuous, so this

$$\begin{aligned} =&\int_0^T (\mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}} B f(s), x^*)_{(-1,*1)} ds \\ =&\int_0^T ((f(s), B^* \left(\mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}}\right)^* x^*)_{\mathcal{U}} ds. \end{aligned}$$

But the quantity $B^*(\mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}})^* x^*$ is pointwise in $\mathcal{U}$ and is continuous in s by the continuity of both $\mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}}$ and B^*. As a function of s, therefore, it is in $L^2_{\mathcal{U}}(0,T)$. We therefore interpret the last integral above as

$$=(f, B^*(\mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}})^* x^*)_{L^2_{\mathcal{U}}(0,T)}.$$

As f and x^* were arbitrary, we conclude that

$$\left(\mathcal{C}^{b,T}_{Z,B}\right)^* x^*(s) = B^*(\mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}})^* x^*(s)$$

for $0 \le s \le T$.

Note that, by hypothesis, the pair (Z,B) is exactly controllable and so the operator $\mathcal{C}^b_{Z,B}$ is bounded from $L^2_{\mathcal{U}}(\mathbb{R}_+)$ onto $\mathcal{X}_{\mathcal{Z}}$. Clearly, the operator $\mathcal{C}^{b,T}_{Z,B}$ is also bounded with norm dominated by $\left\|\mathcal{C}^{b}_{Z,B}\right\|$. We write

$$\left\|\mathcal{C}^{b,T}_{Z,B}u\right\|_{\mathcal{X}} \leq M \left\|u\right\|_{L^2_{\mathcal{U}}(0,T)},$$

where $M = \left\|\mathcal{C}^{b}_{Z,B}\right\|$ is independent of T.

We construct the following bound. Consider

$$\begin{aligned}\left\|B^*(\mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}})^*x^*\right\|_{L^2_{\mathcal{U}}(0,T)} &= \sup_{u\in L^2_{\mathcal{U}}(0,T),\|u\|\leq 1} \left|(B^*(\mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}})^*x^*, u)_{L^2_{\mathcal{U}}(0,T)}\right| \\ &= \sup_{u\in L^2_{\mathcal{U}}(0,T),\|u\|\leq 1} \left|(x^*, -\mathcal{C}^{b,T}_{Z,B}u)_{(*1,-1)}\right|;\end{aligned}$$

but we know—by hypothesis—that $\mathcal{C}^{b,T}_{Z,B}u \in \mathcal{X}$; additionally, $x^* \in \mathcal{X}^*_{\mathcal{Z},1} \subset \mathcal{X}_{\mathcal{Z}}$, so we may apply Lemma 2.4.5 to get

$$= \sup_{u\in L^2_{\mathcal{U}}(0,T),\|u\|\leq 1} \left|(x^*, -\mathcal{C}^{b,T}_{Z,B}u)_{\mathcal{X}_{\mathcal{Z}}}\right|,$$

to which we can apply the estimate

$$\begin{aligned}&\leq \sup_{u\in L^2_{\mathcal{U}}(0,T),\|u\|\leq 1} \left\|x^*\right\|_{\mathcal{X}_{\mathcal{Z}}} M \left\|u\right\|_{L^2_{\mathcal{U}}(0,T)} \\ &\leq M \left\|x^*\right\|_{\mathcal{X}_{\mathcal{Z}}}.\end{aligned}$$

We denote the operator $x \mapsto B^*\left(\mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}}\right)^* x$ by $\mathcal{O}^{f,T}_{B^*,-Z^*}$. We may thus conclude that $\mathcal{O}^{f,T}_{B^*,-Z^*}$ extends continuously to map $\mathcal{X}_{\mathcal{Z}}$ into $L^2_{\mathcal{U}}([0,T])$.

Next we attempt to extend $\mathcal{O}^{f,T}_{B^*,-Z^*}$ to an operator into $L^2_{\mathcal{U}}(0,\infty)$. We consider

$L^2_{\mathcal{U}}(0,T)$ as embedded in $L^2_{\mathcal{U}}(0,\infty)$ and revisit the above inequality:

$$\left\| \mathcal{O}^{f,T}_{B^*,-Z^*} x^* \right\|_{L^2_{\mathcal{U}}(0,T)} \le M \left\| x^* \right\|_{\mathcal{X}_{\mathcal{Z}}}.$$

Taking limits as $T \to \infty$, we denote by $\mathcal{O}^{f}_{B^*,-Z^*}$ the limit $\lim_{T\to\infty} \mathcal{O}^{f,T}_{B^*,-Z^*}$, and conclude

$$\left\| \mathcal{O}^{f}_{B^*,-Z^*} x^* \right\|_{L^2_{\mathcal{U}}(0,\infty)} \le M \left\| x^* \right\|_{\mathcal{X}_{\mathcal{Z}}}.$$

So, we have seen that, if $\mathcal{C}^{b,T}_{Z,B}$ is taken to be a mapping from $L^2_{\mathcal{U}}(0,T)$ to $\mathcal{X}_{\mathcal{Z},-1}$, then $\mathcal{O}^{f,T}_{B^*,-Z^*}$ is a bounded mapping from $\mathcal{X}^*_{\mathcal{Z},1}$ to $L^2_{\mathcal{U}}(0,T)$; and the same can be extended to a mapping $\mathcal{O}^{f}_{B^*,-Z^*} : \mathcal{X}_{\mathcal{Z}} \to L^2_{\mathcal{U}}(0,\infty)$. If we now apply the hypothesis of exact controllability, then we know what $\mathcal{C}^{b,T}_{Z,B}$ is a mapping to $\mathcal{X}_{\mathcal{Z}}$. Under this additional assumption, and applying Lemma 2.4.5 again, we have

$$\begin{aligned}(-\mathcal{C}^{b,T}_{Z,B} u, x^*)_{(-1,*1)} &= (-\mathcal{C}^{b,T}_{Z,B} u, x^*)_{\mathcal{X}_{\mathcal{Z}}} \\ &= (u, \mathcal{O}^{f,T}_{B^*,-Z^*} x^*)_{L^2_{\mathcal{U}}(0,T)}\end{aligned}$$

for all $x \in \mathcal{X}^*_{\mathcal{Z},1}$. We know from above that $\mathcal{O}^{f,T}_{B^*,-Z^*}$ is bounded on $\mathcal{X}$; we further know by construction that $\mathcal{X}^*_{\mathcal{Z},1}$ is dense in $\mathcal{X}_{\mathcal{Z}}$; we may thus extend the above inner product continuously to all of $\mathcal{X}_{\mathcal{Z}}$ to get $(\mathcal{C}^{b,T}_{Z,B})^* x = -\mathcal{O}^{f,T}_{B^*,-Z^*} x$ for all $x \in \mathcal{X}_{\mathcal{Z}}$. In this relation we let $T \to \infty$ to get that $(\mathcal{C}^{b}_{Z,B})^* = -\mathcal{O}^{f}_{B^*,-Z^*}$.

Thus our operator $\mathcal{O}^{f}_{B^*,-Z^*}$ satisfies the extendability requirement for $(B^*, -Z^*)$ to be a Π_+-admissible pair.

Further, by the exact controllability hypothesis, we know that $\mathcal{C}^{b}_{Z,B}$ is onto. It follows that $\mathcal{O}^{f}_{B^*,-Z^*} = (-\mathcal{C}^{b}_{Z,B})^*$ is one-to-one. (We note that mere density of the range of

$\mathcal{C}^b_{Z,B}$ is enough to ensure that $\mathcal{O}^f_{B^*,-Z^*}$ is one-to-one.) Indeed, the stronger hypothesis that $\mathcal{C}^b_{Z,B}$ is onto guarantees that $\mathcal{O}^f_{B^*,-Z^*}$ is bounded below; we state this fact as the separate Lemma 2.4.12 below. □

Lemma 2.4.12. *Let X and Y be Banach spaces. Let $T : X \to Y$ be a continuous linear operator such that $T^* : Y^* \to X^*$ is onto. Then T is bounded below.*

Proof. T must be one-to-one, as T^* is onto. Further, as T^* is onto, it trivially has closed range; T must therefore also have closed range (Theorem 4.14 in [23]). Considering $\mathrm{Ran}(T)$ as a Banach space in its own right, then, we have that $T : X \to \mathrm{Ran}(T)$ is continuous, linear, one-to-one, and onto. We appeal to Corollary 2.12(c), also in [23], to conclude that T is bounded below. □

We will have many occasions where we will work with Laplace transforms of the Observation and Control operators rather than with the operators themselves directly. While we are content to leave the formula for the transform of the Control operator implicit, it will be to our benefit to record an explicit formula for the Laplace transform of the Observation operator. We do so below.

Lemma 2.4.13. *The bilateral Laplace transforms of the forward and backward observation and control operators are as follows.*

1. *The forward observation operator $\mathcal{O}^f_{C^\times,A^\times} : \mathcal{X}^\times_{\mathcal{Z}} \to L^2(\mathbb{R}_+)$ has transform $\hat{\mathcal{O}}^f_{C^\times,A^\times} : \hat{\mathcal{X}}^\times_{\mathcal{Z}} \to H^2(\Pi_+)$ given by*

$$\left(\hat{\mathcal{O}}^f_{C^\times,A^\times} x\right)(z) = C^\times (zI - A^\times)^{-1} x.$$

2. *The backward observation operator* $\mathcal{O}^b_{C,A} : \mathcal{X}_{\mathcal{Z}} \to L^2(\mathbb{R}_-)$ *has transform* $\hat{\mathcal{O}}^b_{C,A} : \hat{\mathcal{X}}_{\mathcal{Z}} \to H^2(\Pi_-)$ *given by*

$$\left(\hat{\mathcal{O}}^b_{C,A} x\right)(z) = -C(zI - A)^{-1}x.$$

Proof. Statement (1) is an immediate consequence of the well-known formula (cf., for example, [27] Proposition 2.3.1) for the Laplace transform of a semigroup in terms of the resolvent of its generator: that is, if $\mathfrak{S}^s$ is a forward semigroup with generator S and growth bound ω, then for all z with $\operatorname{Re} z > \omega$,

$$(zI - S)^{-1}x = \int_0^\infty e^{-zs}\mathfrak{S}^s x \; ds;$$

see Theorem 2.1.10.

In particular, our semigroups are contractive, so we may take $\omega = 0$. □

2.5 A Result on Functions in a Sobolev Space

The following result is well known in the context of Sturm-Liouville theorey; in particular, the following results appears (at least implicitly) on page 345 of [17]. We provide an independent proof.

Lemma 2.5.1. *If* $f \in L^2(\mathbb{R})$ *is locally absolutely continuous and* $f' \in L^2(\mathbb{R})$*, then* $\lim_{t\to\pm\infty} f(t) = 0$.

Proof. Assume that $\lim_{t\to\infty} f(t)$ does not exist. Then $\exists \varepsilon_0 > 0$ so that for any $M > 0$ there is an $m > M$ with $|f(t_m)| > 2\varepsilon_0$. Let $\{t_m\}$ be a sequence of such numbers going to infinity. We derive an inequality:

$$\begin{aligned}
|f(t) - f(t_m)| &= \left| \int_{t_m}^{t} f'(s) ds \right| \\
&\leq \int_{t_m}^{t} |f'(s)| \, ds \\
&\leq \|f'\|_2 \left\| 1_{(t_m,t)} \right\|_2 \\
&= \|f'\|_2 \sqrt{|(t - t_m)|}.
\end{aligned}$$

In particular, since $\|f'\|_2$ is fixed, if t is such that $\sqrt{|t - t_m|} \leq \frac{\varepsilon_0}{\|f'\|_2}$, we get

$$|f(t) - f(t_m)| \leq \varepsilon_0.$$

Now consider, for such a t,

$$\begin{aligned}
|f(t)| &= |f(t) - f(t_m) + f(t_m)| \\
&= |f(t_m) - (f(t_m) - f(t))| \\
&\geq ||f(t_m)| - |f(t_m) - f(t)|| \\
&\geq |f(t_m)| - |f(t_m) - f(t)| \\
&\geq 2\varepsilon_0 - \varepsilon_0 \\
&= \varepsilon_0.
\end{aligned}$$

That is, if t satisfies $|t - t_m| \leq \varepsilon^2 / \|f'\|_2^2$, then $|f(t)| \geq \varepsilon_0$. If we let $\{I_j\}$ be a set of disjoint intervals of length $\varepsilon^2 / \|f'\|_2^2$ such that there is at least one element of $\{t_m\}$

in each I_j, then we may estimate

$$\begin{aligned}
\|f\|_2^2 &= \int_{-\infty}^{\infty} |f(t)|^2 \, dt \\
&\geq \int_{\cup I_j} |f(t)|^2 \, dt \\
&\geq \sum_{j=1}^{\infty} \varepsilon_0^2 \frac{\varepsilon_0^2}{\|f'\|_2^2} \\
&= \infty,
\end{aligned}$$

which contradicts the hypothesis that $f \in L^2(\mathbb{R})$.

Clearly, an identical argument yields that $\lim_{t \to -\infty} f(t) = 0$ as well. □

Chapter 3

Generalized BLH Theorems

We give a version of the Buerling-Lax-Halmos Theorem appropriate to the continuous-time—alternately, right half-plane—case. We base our result on the version of the BLH Theorem given by Ball-Helton in [2]; however, for the proof thereof we refer the reader to [4], which is the cleaner proof.

3.1 Dual Shift-Invariant Pairs

Definition 3.1.1. If $\mathcal{M}$ is a closed subspace of $L^2_{\mathcal{U}}(\mathbb{R})$ and let $\mathfrak{T}$ be a shift on $L^2_{\mathcal{U}}(\mathbb{R})$. We say that $\mathcal{M}$ is *shift-invariant* if $\mathfrak{T}^s\mathcal{M} \subset \mathcal{M}$ for all $s \geq 0$.

A shift-invariant subspace $\mathcal{M}$ is *simply-invariant* if

$$\cap_{s\geq 0}\mathfrak{T}^s\mathcal{M} = \{0\}$$

and/or *full-range* if

$$\cup_{s\geq 0}\mathfrak{T}^{*s}\mathcal{M} \text{ is dense in } L^2_{\mathcal{U}}(\mathbb{R})$$

Alternately, if $\mathfrak{T}$ is a discrete shift, then the same definitions hold with the natural modifications: instead of demanding that the conditions hold for $s \geq 0$, we demand that they hold for $n \in \mathbb{N}$.

Definition 3.1.2. Let $\mathcal{M}$ and $\mathcal{M}^\times$ be a pair of closed subspaces of a Hilbert space $\mathcal{X}$. Let $\mathfrak{S}$ be a (continuous or discrete) bilateral shift on $\mathcal{X}$. We say that the pair $(\mathcal{M}, \mathcal{M}^\times)$ is a *dual shift-invariant pair* if the following hold:

1. $\mathcal{M}$ is full-range and simply invariant under $\mathfrak{S}$.

2. $\mathcal{M}^\times$ is full range and simply invariant under $\mathfrak{S}^*$.

3. $\mathcal{M}$ and $\mathcal{M}^\times$ together form a direct sum decomposition of $\mathcal{X}$:

$$\mathcal{M}\dot{+}\mathcal{M}^\times = \mathcal{X}.$$

Remark 3.1.3. We note that the conditions for a dual shift-invariant pair can be weakened. Let closed subspaces $(\mathcal{M}, \mathcal{M}^\times)$ form a direct-sum decomposition of $\mathcal{X}$. Then the following are equivalent:

1. $(\mathcal{M}, \mathcal{M}^\times)$ is a dual shift-invariant pair.

2. $\mathcal{M}$ is simply invariant under $\mathfrak{S}$ and $\mathcal{M}^\times$ is simply invariant under $\mathfrak{S}^*$.

3. $\mathcal{M}$ is full range under $\mathfrak{S}$ and $\mathcal{M}^\times$ is full range under $\mathfrak{S}^*$.

To see this, it suffices to consider the two decompositions

$$L^2_{\mathcal{U}}(i\mathbb{R}) = \mathfrak{S}^s L^2_{\mathcal{U}}(i\mathbb{R}) = \mathfrak{S}^s \mathcal{M} \dot{+} \mathfrak{S}^s \mathcal{M}^\times$$

and

$$L^2_{\mathcal{U}}(i\mathbb{R}) = \mathfrak{S}^{*s} L^2_{\mathcal{U}}(i\mathbb{R}) = \mathfrak{S}^{*s} \mathcal{M} \dot{+} \mathfrak{S}^{*s} \mathcal{M}^\times.$$

From the first decomposition, we see that $\mathcal{M}$ is simply invariant under $\mathfrak{S}$ if and only if $\mathcal{M}^\times$ is full-range invariant under $\mathfrak{S}^*$. From the second, we see that $\mathcal{M}$ is full-range invariant under $\mathfrak{S}$ if and only if $\mathcal{M}^\times$ is simply-invariant under $\mathfrak{S}^*$.

We exhibit a few properties of and connections between full-range and simply-invariant spaces.

Lemma 3.1.4. *Let $\mathfrak{S}$ be a (continuous or discrete) closed bilateral shift on a Hilbert space $\mathcal{X}$. Let $\mathcal{M}$ be a closed subspace of $\mathcal{X}$ such that $\mathfrak{S}\big|_{\mathcal{M}}$ is a unilateral shift and such that $\mathcal{M}$ is simply invariant under $\mathfrak{S}$. Then $\mathcal{M}^\perp$ is a Full-Range invariant space under $\mathfrak{S}^*$.*

Proof. Let s be either a nonnegative integer or a nonnegative real number, as appropriate to whether $\mathfrak{S}$ is a discrete or continuous shift, respectively. We note that being Full Range under $\mathfrak{S}^*$ means that

$$\mathcal{X} = \cup_{s\geq 0} (\mathfrak{S}^{*s})^* \mathcal{M}^\perp = \cup_{s\geq 0} \mathfrak{S}^s \mathcal{M}^\perp.$$

Then we consider that

$$\mathcal{X} = \mathfrak{S}^s \mathcal{X} = \mathfrak{S}^s \mathcal{M} \oplus \mathfrak{S}^s \mathcal{M}^\perp,$$

But by Lemma 2.1.17, we know that $\mathfrak{S}^s\mathcal{M} = \cap_{0\leq t\leq s}\mathfrak{S}^t\mathcal{M}$ and that $\mathfrak{S}^s\mathcal{M}^\perp = \cup_{0\leq t\leq s}\mathfrak{S}^t\mathcal{M}^\perp$, so that in fact

$$\mathcal{X} = \left(\cap_{0\leq t\leq s}\mathfrak{S}^t\mathcal{M}\right) \oplus \left(\cup_{0\leq t\leq s}\mathfrak{S}^t\mathcal{M}^\perp\right)$$

Taking the limit as $s \to \infty$ and applying the simple invariance of $\mathcal{M}$, we conclude that

$$\mathcal{X} = L^2\text{-clos}\,\cup_{t\geq 0}\mathfrak{S}^t\mathcal{M}^\perp.$$

□

3.2 Semigroups of Contractions

Definition 3.2.1. Let $\mathfrak{A}^s$ be a strongly continuous semigroup. Then we say that $\mathfrak{A}^s$ is a *semigroup of contractions* if

$$\|\mathfrak{A}^s\| \leq 1$$

for $s \geq 0$.

We can restate the preceding definition in terms of the growth bound of the semigroup (cf. Theorem 2.1.4). That is, a semigroup $\mathfrak{A}^s$ is a semigroup of contractions if it satisfies the bound $\|\mathfrak{A}^s\| \leq Me^{\omega s}$ with $M = 1$ and $\omega = 0$.

It is worthwhile to note that the Hille-Yosida Theorem 2.1.6 implies that if $\mathfrak{A}$ is a semigroup of contractions, then 1 is in the resolvent of its generator A. We make use of this fact to define another linear operator associated with $\mathfrak{A}$; namely, the cogenerator.

3.3 Cogenerators, FRSI under cogenerators

3.3.1 General Cogenerators and FRSI preservation properties

Definition 3.3.1. If $\mathfrak{A}^s$ is a semigroup of contractions with generator A, then we define the *cogenerator* of $\mathfrak{A}^s$ to be the operator

$$\mathbf{A} := (I + A)(I - A)^{-1}.$$

The cogenerator is already well characterized in the literature; see, for example, section 8 of chapter 3 of [21]. We re-state without further proof a few results which we will make particular use of.

Theorem 3.3.2 (Propositions 8.2, 8.3, and 9.2 of [21])**.** *Let $\mathfrak{A}$ be a strongly continuous semigroup of contractions and let $\mathbf{A}$ be its cogenerator. Then $\mathbf{A}$ is also a contraction. If additionally, for each $s \geq 0$, $\mathfrak{A}^s$ is normal, self-adjoint, unitary, completely nonunitary, or an isometry, then $\mathbf{A}$ exhibits the same property, and conversely.*

Remark 3.3.3. We should point out that an equivalent characterization of simple invariant subspaces is that the shift, restricted to the subspace, is completely nonunitary. Thus the preceding theorem implies that a subspace is simply invariant under a continuous shift if and only if it is simply invariant under the cogenerator of the continuous shift.

We append to Theorem 3.3.2 the following result about full-range invariance.

Lemma 3.3.4. *Let $\mathfrak{T}$ be a closed (continuous) bilateral shift with cogenerator* $\mathbf{T}$ *on a Hilbert space* $\mathcal{X}$*. Let* $\mathcal{M} \subset \mathcal{X}$ *be a closed subspace such that* $\mathfrak{T}\big|_{\mathcal{M}}$ *is a unilateral shift. Then* $\mathcal{M}$ *is full-range under* $\mathfrak{T}$ *if and only if it is full-range under* $\mathbf{T}$*.*

Proof. By Lemma 3.1.4, we know that $\mathcal{M}^{\perp}$ is simply invariant under $\mathfrak{T}^*$. By Theorem 3.3.2 (cf. Remark 3.3.3), we know that $\mathcal{M}^{\perp}$ is simply invariant under $\mathbf{T}^*$. But by Lemma 3.1.4 again, we conclude that $\mathcal{M}$ is full range under $\mathbf{T}$.

Clearly, this argument also works in reverse, as the cited Lemmas are in fact equivalent characterizations. □

We combine the previous results into the following Theorem regarding dual shift-invariant pairs. As we will be working extensively with these pairs of spaces, it is this Theorem that will be of the greatest practical use to us.

Theorem 3.3.5. *Let* $\mathfrak{T}$ *be a closed (continuous) bilateral shift on a Hilbert space* $\mathcal{X}$ *with cogenerator* $\mathbf{T}$*. Further let* $\mathcal{M}$ *and* $\mathcal{M}^{\times}$ *be closed subspaces of* $\mathcal{X}$*. Then the pair* $(\mathcal{M}, \mathcal{M}^{\times})$ *forms a dual shift-invariant pair for* $\mathfrak{T}$ *if and only if it also forms a dual shift-invariant pair for* $\mathbf{T}$*.*

3.3.2 Shift semigroup cogenerator

We have worked almost exclusively with the translation semigroup on $L^2_{\mathcal{U}}(\mathbb{R})$ and its Laplace transform, which is a shift on $L^2_{\mathcal{U}}(i\mathbb{R})$. We consider it more convenient to develop the following theory using the Laplace transform; that is, we act in the so-called frequency domain. We take a moment, therefore, to compute the cogenerator of the Laplace transformed shift semigroup.

Lemma 3.3.6. *Let $\hat{\mathfrak{T}}$ denote the semigroup on $L^2_{\mathcal{U}}(i\mathbb{R})$ given by*

$$\left(\hat{\mathfrak{T}}^s g\right)(z) = e^{-sz} g(z).$$

Then the cogenerator of $\hat{\mathfrak{T}}$ is $\widehat{\mathbf{T}}$ given by

$$\left(\widehat{\mathbf{T}} g\right)(z) = \frac{1-z}{1+z} g(z).$$

Proof. We recall that this semigroup has generator $\hat{T} = M_{-z}$; that is,

$$\left(\hat{T} g\right)(z) = -z g(z).$$

Applying the definition of the cogenerator $\widehat{\mathbf{T}}$ gives

$$\widehat{\mathbf{T}} g = (I + \hat{T})(I - \hat{T})^{-1} g = (I + M_{-z})(I - M_{-z})^{-1} g = \frac{1-z}{1+z} g(z)$$

□

For our purposes, we shall think of mapping from a semigroup of contractions to its cogenerator as a mapping from a continuous shift to a discrete shift; better, it is a mapping that preserves the property of dual shift-invariance. Next, we turn our attention to a mapping which transforms a discrete shift on $L^2_{\mathcal{U}}(i\mathbb{R})$ which is a contraction to a discrete shift on $L^2_{\mathcal{U}}(\mathbb{T})$.

3.4 Cayley transforms, FRSI under Cayley transform

We begin by exhibiting a bijective isometry between $L^2_{\mathcal{U}}(i\mathbb{R})$ and $L^2_{\mathcal{U}}(\mathbb{T})$. We follow the discussion beginning on page 105 of [15].

Theorem 3.4.1. *Let $f \in L^2_{\mathcal{U}}(i\mathbb{R})$ and let $\lambda \in \mathbb{T}$. We define*

$$\mathcal{C}f(\lambda) := \frac{\sqrt{2}}{1-\lambda} f\left(\frac{1+\lambda}{1-\lambda}\right).$$

Then $\mathcal{C}f \in L^2_{\mathcal{U}}(\mathbb{T})$ and $\mathcal{C}$ is a bijective isometry from $L^2_{\mathcal{U}}(i\mathbb{R})$ onto $L^2_{\mathcal{U}}(\mathbb{T})$ with inverse given by

$$\mathcal{C}^{-1}h(z) = \frac{\sqrt{2}}{z+1} h\left(\frac{z-1}{z+1}\right).$$

Proof. The transform $\mathcal{C}$ is precisely a weighted Cayley transform; specifically, it is derived from the transform that maps right half-plane to the unit disc. We leave it to the interested reader to verify that if $\lambda \in \mathbb{T} \setminus \{1\}$, then $\frac{1+\lambda}{1-\lambda} \in i\mathbb{R}$. It follows that $\mathcal{C}f(\lambda)$ is well defined on $\mathbb{T} \setminus \{1\}$. Now we compute

$$\|\mathcal{C}f(\lambda)\|^2_{L^2_{\mathcal{U}}(\mathbb{T})} = \oint_{\mathbb{T}} \left| \frac{\sqrt{2}}{1-\lambda} f\left(\frac{1+\lambda}{1-\lambda}\right) \right|^2 \, d\lambda,$$

and, upon changing variables to $z := \frac{1+\lambda}{1-\lambda}$, we find

$$= \int_{i\mathbb{R}} |f(z)|^2 \, dz,$$

which of course is $\|f\|^2_{L^2_{\mathcal{U}}(i\mathbb{R})}$. Thus $\mathcal{C}$ is an isometry.

A straightforward computation, which we leave to the reader, verifies that the claimed $\mathcal{C}^{-1}$ is in fact the inverse of $\mathcal{C}$. □

Given a linear operator on $L^2_{\mathcal{U}}(i\mathbb{R})$, we use the transform $\mathcal{C}$ to generate an operator on $L^2_{\mathcal{U}}(\mathbb{T})$ in the usual way. Specifically, we apply this construction to the cogenerator $\widehat{\mathbf{T}}$ of the shift semigroup $\hat{\mathfrak{T}}$ on $L^2_{\mathcal{U}}(i\mathbb{R})$.

Theorem 3.4.2. *Let $f \in L^2_{\mathcal{U}}(\mathbb{T})$ and let $\widehat{\mathbf{T}}$ be the cogenerator of the shift group $\hat{\mathfrak{T}}$ on $L^2_{\mathcal{U}}(i\mathbb{R})$. With $\widehat{\mathbf{T}}$, we associate a linear operator $\mathcal{C}(\widehat{\mathbf{T}})$ on $L^2_{\mathcal{U}}(\mathbb{T})$ by*

$$\mathcal{C}(\widehat{\mathbf{T}})f(\lambda) := \mathcal{C}\left(\widehat{\mathbf{T}}\mathcal{C}^{-1}f\right)(\lambda).$$

Then

$$\mathcal{C}(\widehat{\mathbf{T}})f(\lambda) = \lambda f(\lambda).$$

That is, the Cayley transform of the cogenerator $\widehat{\mathbf{T}}$, acting on $L^2_{\mathcal{U}}(\mathbb{T})$, is the operator of multiplication by λ. This operator is itself a (discrete) shift on $L^2_{\mathcal{U}}(\mathbb{T})$. For further discussion on this shift, see, e.g., [21] or [15].

Proof. We simply compute:

$$\begin{aligned}\mathcal{C}(\widehat{\mathbf{T}})f(\lambda) &= \mathcal{C}\left(\widehat{\mathbf{T}}\mathcal{C}^{-1}f\right)(\lambda)\\ &= \mathcal{C}\left(\frac{z-1}{z+1}\mathcal{C}^{-1}f\right)(\lambda)\\ &= \lambda f(\lambda).\end{aligned}$$

□

Thus our mapping $\mathcal{C}$ gives us a correspondance between the (discrete) shift $\widehat{\mathbf{T}}$ on $L^2_{\mathcal{U}}(i\mathbb{R})$ and the (discrete) shift $M_\lambda : f(\lambda) \in L^2_{\mathcal{U}}(\mathbb{T}) \mapsto \lambda f(\lambda)$. Further, as we show, $\mathcal{C}$ maps $\widehat{\mathbf{T}}$-invariant subspaces to M_λ-invariant subspaces; additionally, it preserves the properties of simple invariance and full range invariance.

Lemma 3.4.3. *Let $\widehat{\mathbf{T}}$ be a bilateral shift on $L^2_{\mathcal{U}}(i\mathbb{R})$ and let $\mathcal{M} \subset L^2_{\mathcal{U}}(i\mathbb{R})$ be a closed subspace such that $\widehat{\mathbf{T}}$ is a unilateral shift on $\mathcal{M}$. Let $\mathcal{CM}$ be the image of $\mathcal{M}$ under the mapping $\mathcal{C}$ and let $\mathcal{C}\widehat{\mathbf{T}}$ be the image of $\widehat{\mathbf{T}}$ under $\mathcal{C}$, each as defined above. Then $\mathcal{M}$ is $\widehat{\mathbf{T}}$-invariant, simply-invariant, or full-range invariant if and only if $\mathcal{CM}$ is $\mathcal{C}\widehat{\mathbf{T}}$-invariant, simply-invariant, or full-range invariant, respectively.*

Proof. We begin with the statement of equivalence of invariance. First assume that $\widehat{\mathbf{T}}\mathcal{M} \subset \mathcal{M}$. Then consider

$$\begin{aligned}
\left(\mathcal{C}(\widehat{\mathbf{T}})\right)(\mathcal{CM}) &= \{\left(\mathcal{C}(\widehat{\mathbf{T}})\right) g \mid g \in \mathcal{CM}\} \\
&= \mathcal{C}\{\left(\widehat{\mathbf{T}}\mathcal{C}^{-1} g\right) \mid g \in \mathcal{CM}\} \\
&= \mathcal{C}\{\widehat{\mathbf{T}} f \mid f = \mathcal{C}^{-1} g, g \in \mathcal{CM}\} \\
&= \mathcal{C}\left(\widehat{\mathbf{T}}\mathcal{M}\right) \\
&\subset \mathcal{CM}
\end{aligned}$$

The proof of the converse is essentially similar, except with the roles of $\mathcal{C}$ and $\mathcal{C}^{-1}$ reversed.

Next, we consider the equivalence of simple invariance. We assume that $\mathcal{M}$ is $\widehat{\mathbf{T}}$-simply invariant. A simple computation verifies that $\mathcal{C}(\widehat{\mathbf{T}}^N) = (\mathcal{C}(\widehat{\mathbf{T}}))^N$. In which case

$$\cap_{n=0}^{\infty}\mathcal{C}(\widehat{\mathbf{T}})^{n}\,(\mathcal{C}\mathcal{M}) = \mathcal{C}\left(\cap_{n=0}^{\infty}\left(\widehat{\mathbf{T}}^{n}\mathcal{M}\right)\right) = \mathcal{C}\{0\} = \{0\}.$$

Again, the converse statement is similar.

Finally, we consider the statement of equivalence of full-range invariance. We assume that $\cup_{n=0}^{\infty}\widehat{\mathbf{T}}^{-n}\mathcal{M} = L^2_{\mathcal{U}}(i\mathbb{R})$ and consider

$$\cup_{n=0}^{\infty}\mathcal{C}(\widehat{\mathbf{T}})^{-n}\,(\mathcal{C}\mathcal{M}) = \mathcal{C}\left(\cup_{n=0}^{\infty}\widehat{\mathbf{T}}^{-n}\mathcal{M}\right).$$

If we take the closure, then, we see that

$$\begin{aligned} L^2\text{-clos}\left(\cup_{n=0}^{\infty}\mathcal{C}(\widehat{\mathbf{T}})^{-n}\,(\mathcal{C}\mathcal{M})\right) &= L^2\text{-clos}\left(\mathcal{C}\left(\cup_{n=0}^{\infty}\widehat{\mathbf{T}}^{-n}\mathcal{M}\right)\right) \\ &= \mathcal{C}\left(L^2\text{-clos}\left(\cup_{n=0}^{\infty}\widehat{\mathbf{T}}^{-n}\mathcal{M}\right)\right) = \mathcal{C}L^2(i\mathbb{R}) = L^2(\mathbb{T}). \end{aligned}$$

Upon taking the limit $N \to \infty$, we get our desired statement of the equivalence of full-range invariance. The converse statement is again, of course, essentially similar, where one works with $\mathcal{C}^{-1}$ instead of $\mathcal{C}$. □

3.5 Generalized BLH Theorem due to Ball-Helton

For convenience, we restate—without proof, but with minor clarification of notation—the version of the Beurling-Lax-Halmos Theorem as found in [4]. This theorem was itself a restatement, complete with a more polished proof, of the result in [2]. First, however, we define some useful auxiliary spaces. These are the spaces of so-called trigonometric polynomials.

Definition 3.5.1. We define the following subspaces of $L^2_{\mathcal{U}}(\mathbb{T})$.

$$\mathcal{P}_{\mathcal{U}} := \left\{ f(\lambda) = \sum_{n\in\mathbb{Z}} f_n \lambda^n \;\middle|\; f_n \in \mathcal{U}, \text{ and } f_n = 0 \text{ for all but finitely many } n \right\}$$

$$\mathcal{P}_{\mathcal{U},+} := \left\{ f(\lambda) = \sum_{n\in\mathbb{Z}_+} f_n \lambda^n \;\middle|\; f_n \in \mathcal{U}, \text{ and } f_n = 0 \text{ for all but finitely many } n \right\}$$

$$\mathcal{P}_{\mathcal{U},-} := \left\{ f(\lambda) = \sum_{n\in\mathbb{Z}_-} f_n \lambda^n \;\middle|\; f_n \in \mathcal{U}, \text{ and } f_n = 0 \text{ for all but finitely many } n \right\}$$

Remark 3.5.2. Note that the space $\mathcal{P}_{\mathcal{U}}$ is dense in $L^2_{\mathcal{U}}(\mathbb{T})$. Further, $L^2\text{-clos}\,\mathcal{P}_{\mathcal{U},+} = H^2_{\mathcal{U}}(\mathbb{T})$ and $L^2\text{-clos}\,\mathcal{P}_{\mathcal{U},-} = H^2_{\mathcal{U}}(\mathbb{T})^{\perp}$. These latter two facts shed some light on statement (2) in the following Theorem 3.5.4.

Also before stating our main Theorem of the section, we introduce a new class of operator-valued functions.

Definition 3.5.3. We say that an a.e.-defined $\mathcal{L}(\mathcal{U})$-valued function W on $\mathbb{T}$ is $L^2(\mathbb{T})$*-regular* if

1. $W^{-1}(\lambda)$ exists for almost all $\lambda \in \mathbb{T}$ and both M_W and $M_{W^{-1}}$ are in $L^2_{\mathcal{L}(\mathcal{U})}(\mathbb{T})$; that is, multiplying by either $W(z)$ or $W^{-1}(z)$ maps $\mathcal{U}$ into $L^2_{\mathcal{U}}(\mathbb{T})$; and

2. the operator

$$M_W P_{H^2_{\mathcal{U}}} M_W^{-1} : \mathcal{P}_{\mathcal{U}} \to \mathrm{L}^1_{\mathcal{U}}(\mathbb{T})$$

has range in $L^2_{\mathcal{U}}(\mathbb{T})$ and extends to define a bounded operator from $L^2_{\mathcal{U}}(\mathbb{T})$ into itself.

Theorem 3.5.4 (3.4 of [4]). *We define a (discrete) shift M_λ on $L^2_{\mathcal{U}}(\mathbb{T})$ by $M_\lambda f(\lambda) = \lambda f(\lambda)$. Now suppose that $\mathcal{M}$ and $\mathcal{M}^\times$ are two subspaces of $L^2_{\mathcal{U}}(\mathbb{T})$. Then the following are equivalent*

1. *$(\mathcal{M}, \mathcal{M}^\times)$ is a dual shift-invariant pair with respect to M_λ.*

2. *There exists a L^2-regular $\mathcal{L}(\mathcal{U})$-valued function W so that*

$$\mathcal{M} = L^2\text{-clos}\, M_W P_{\mathcal{U},+} \text{ and } \mathcal{M}^\times = L^2\text{-clos}\, M_W P_{\mathcal{U},-},$$

   where M_W denotes the operator which multiplies by W.

Moreover, in this case the dual shift-invariant pair $(\mathcal{M}, \mathcal{M}^\times)$ uniquely determines W up to an invertible constant right factor; i.e., if W' is another $L^2_{\mathcal{U}}$-regular $\mathcal{L}(\mathcal{U})$-valued function as in part (2), then there is an invertible constant operator $X \in \mathcal{L}(\mathcal{U})$ so that $W'(\lambda) = W(\lambda)X$.

3.6 Generalized BLH Theorem

We may now state a generalized version of Theorem 3.5.4 which allows continuous shifts acting on $L^2_{\mathcal{U}}(i\mathbb{R})$. Before we do so, however, we will introduce two auxiliary subspaces of $L^2_{\mathcal{U}}(i\mathbb{R})$.

Definition 3.6.1. We define the following subspaces of $L^2_{\mathcal{U}}(i\mathbb{R})$:

$$\mathcal{Q}_{\mathcal{U}} := \left\{ f(z) = \sum_{n\in\mathbb{Z}} \frac{\sqrt{2}f_n}{z+1}\left(\frac{z-1}{z+1}\right)^n \mid f_n = 0 \text{ for all but finitely many } n \in \mathbb{Z} \right\},$$

$$\mathcal{Q}_{\mathcal{U},+} := \left\{ f(z) = \sum_{n\in\mathbb{Z}_+} \frac{\sqrt{2}f_n}{z+1}\left(\frac{z-1}{z+1}\right)^n \mid f_n = 0 \text{ for all but finitely many } n \in \mathbb{Z}_+ \right\},$$

and

$$\mathcal{Q}_{\mathcal{U},-} := \left\{ f(z) = \sum_{n\in\mathbb{Z}_-} \frac{\sqrt{2}f_n}{z+1}\left(\frac{z-1}{z+1}\right)^n \mid f_n = 0 \text{ for all but finitely many } n \in \mathbb{Z}_- \right\}.$$

Remark 3.6.2. As suggested by the notation, these spaces are intimately related to the $\mathcal{P}$-spaces defined as a part of Theorem 3.5.4. To wit: the $\mathcal{Q}$-spaces are the images under the inverse Cayley transform $\mathcal{C}^{-1}$ of the $\mathcal{P}$-spaces. Thus

$$\mathcal{Q}_{\mathcal{U}} = \mathcal{C}^{-1}\mathcal{P}_{\mathcal{U}},\ \mathcal{Q}_{\mathcal{U},+} = \mathcal{C}^{-1}\mathcal{P}_{\mathcal{U},+},\ \text{and } \mathcal{Q}_{\mathcal{U},-} = \mathcal{C}^{-1}\mathcal{P}_{\mathcal{U},-}.$$

It follows from this, as $\mathcal{C}$ is an isometry (cf. Theorem 3.4.1), that $\mathcal{Q}_{\mathcal{U}}$ is dense in $L^2_{\mathcal{U}}(i\mathbb{R})$, $L^2\text{-clos}\,\mathcal{Q}_{\mathcal{U},+} = H^2_{\mathcal{U}}(\Pi_+)$, and $L^2\text{-clos}\,\mathcal{Q}_{\mathcal{U},-} = H^2_{\mathcal{U}}(\Pi_-)$. These statements could also be proven directly, of course.

The structure of the $\mathcal{Q}$ spaces is such that $\mathcal{Q}_{\mathcal{U},+}$ is Full-Range Simply-Invariant under $\widehat{\mathbf{T}}$ and $\mathcal{Q}_{\mathcal{U},-}$ is Full-Range Simply-Invariant under $\widehat{\mathbf{T}}^*$, as exhibited in the following Lemma.

Lemma 3.6.3. *Let $\mathcal{Q}$, $\mathcal{Q}_{\mathcal{U},+}$, and $\mathcal{Q}_{\mathcal{U},-}$ be as above and let $\widehat{\mathbf{T}}$ be the multiplication* *operator on $L^2_{\mathcal{U}}(i\mathbb{R})$ defined by*

$$\widehat{\mathbf{T}}f(z) = \frac{z-1}{z+1}f(z).$$

Then $\mathcal{Q}_{\mathcal{U},+}$ is Full-Range Simply-Invariant under $\widehat{\mathbf{T}}$ and $\mathcal{Q}_{\mathcal{U},-}$ is Full-Range Simply-Invariant under $\widehat{\mathbf{T}}^*$.

As before, we introduce a new class of operator-valued functions on $i\mathbb{R}$.

Definition 3.6.4. We say that an a.e.-defined $\mathcal{L}(\mathcal{U})$-valued function W on $i\mathbb{R}$ is $L^2(i\mathbb{R})$*-regular* if

1. $W^{-1}(z)$ exists for almost all $z \in i\mathbb{R}$ and both $\frac{1}{z+1}M_W$ and $\frac{1}{z+1}M_{W^{-1}}$ are in $L^2_{\mathcal{L}(\mathcal{U})}(i\mathbb{R})$; that is, multiplying by either $W(z)$ or $W^{-1}(z)$ maps $\mathcal{U}$ into $L^2_{\mathcal{U}}(i\mathbb{R})$; and

2. the operator
$$M_W P_{H^2_{\mathcal{U}}} M_W^{-1} : \mathcal{Q}_{\mathcal{U}} \to (z+1)\mathrm{L}^1_{\mathcal{U}}(i\mathbb{R})$$
has range in $L^2_{\mathcal{U}}(i\mathbb{R})$ and extends to define a bounded operator from $L^2_{\mathcal{U}}(i\mathbb{R})$ into itself.

Note that if W is $L^2_{\mathcal{U}}(i\mathbb{R})$-regular, then multiplication by $\frac{1}{z+1}W(z)$ maps $\mathcal{U}$ into $L^2_{\mathcal{U}}(i\mathbb{R})$.

Theorem 3.6.5. *Let $\hat{\mathfrak{T}}$ on $L^2_{\mathcal{U}}(i\mathbb{R})$ be the Laplace transform of the translation group. Let $\mathcal{M}$ and $\mathcal{M}^\times$ be closed subspaces of $L^2_{\mathcal{U}}(i\mathbb{R})$ such that $\mathfrak{T}\big|_{\mathcal{M}}$ and $\mathfrak{T}^*\big|_{\mathcal{M}^\times}$ are unilateral shifts. Then the following are equivalent:*

1. *The pair $(\mathcal{M}, \mathcal{M}^\times)$ is a dual shift-invariant pair with respect to $\hat{\mathfrak{T}}$.*

2. *There exists a $L^2(i\mathbb{R})$-regular function W such that*

$$\mathcal{M} = L^2\text{-clos}\, W \mathcal{Q}_{\mathcal{U},+} \text{ and } \mathcal{M}^\times = L^2\text{-clos}\, W \mathcal{Q}_{\mathcal{U},-}.$$

Moreover, in this case, the dual shift-invariant pair $(\mathcal{M}, \mathcal{M}^\times)$ uniquely determines W up to an invertible constant right factor; that is, if W' is another function as in part (2), then there is an invertible constant operator $X \in \mathcal{L}(\mathcal{U})$ so that $W'(z) = W(z)X$.

Proof. We begin with the proof that (1) $\implies$ (2).

By Theorem 3.3.5, the pair $(\mathcal{M}, \mathcal{M}^\times)$ is dual shift-invariant under $\widehat{\mathbf{T}}$, the cogenerator of $\hat{\mathfrak{T}}$. But then by Theorem 3.4.3, the pair $(\mathcal{C}\mathcal{M}, \mathcal{C}\mathcal{M}^\times)$ is dual shift-invariant under $\mathcal{C}(\widehat{\mathbf{T}}) = M_\lambda$.

To the pair $(\mathcal{C}\mathcal{M}, \mathcal{C}\mathcal{M}^\times)$, then, we apply Theorem 3.5.4. Thus we have a $L^2_{\mathcal{U}}(\mathbb{T})$-regular function, which we call W', satisfying the following representations:

$$\mathcal{C}\mathcal{M} = L^2\text{-clos}\, W' \mathcal{P}_{\mathcal{U},+}, \text{ and } \mathcal{C}\mathcal{M}^\times = L^2\text{-clos}\, W' \mathcal{P}_{\mathcal{U},-}.$$

To these representations we apply $\mathcal{C}^{-1}$. As $\mathcal{C}^{-1}$ is an isometry from $L^2_{\mathcal{U}}(\mathbb{T})$ to $L^2_{\mathcal{U}}(i\mathbb{R})$, it preserves closures. Thus we get

$$\mathcal{M} = L^2\text{-clos}\, W'\left(\frac{z-1}{z+1}\right) \mathcal{Q}_{\mathcal{U},+}, \text{ and } \mathcal{M}^\times = L^2\text{-clos}\, W'\left(\frac{z-1}{z+1}\right) \mathcal{Q}_{\mathcal{U},-}.$$

If we identify $W(z) := W'\left(\frac{z-1}{z+1}\right)$, then we have the claimed representations of $\mathcal{M}$ and $\mathcal{M}^\times$.

Next we check that W is weighted $L^2(i\mathbb{R})$-regular.

We first note that

$$\frac{\sqrt{2}}{z+1}W(z) = \frac{\sqrt{2}}{z+1}W'\left(\frac{z-1}{z+1}\right) = \mathcal{C}^{-1}W'.$$

As $W' \in L^2_{\mathcal{U}}(\mathbb{T})$, so $\frac{1}{z+1}W(z) \in L^2_{\mathcal{U}}(i\mathbb{R})$, as claimed. Next, we clearly have that $W^{-1}(z) = (W')^{-1}\left(\frac{z-1}{z+1}\right)$; thus we similarly have that $\frac{1}{z+1}W^{-1}(z) = \frac{1}{\sqrt{2}}\mathcal{C}^{-1}(W')^{-1}$ must be in $L^2_{\mathcal{U}}(i\mathbb{R})$.

As W' is $L^2(\mathbb{T})$-regular, we know that

$$\mathbf{M}' := M_{W'}P_{H^2_{\mathcal{U}}(\mathbb{T})}M^{-1}_{W'}$$

extends to a bounded operator from $L^2_{\mathcal{U}}(\mathbb{T})$ into itself. But then the operator

$$\mathbf{M} := \mathcal{C}^{-1}(\mathbf{M}')$$

similarly extends to a bounded operator from $L^2_{\mathcal{U}}(i\mathbb{R})$ into itself.

From Theorem 3.5.4, we know that W' is unique up to an invertible constant right factor; but W' uniquely determines W, so it follows that W is uniquely determined up to a constant factor.

Now we show that (2) $\implies$ (1). We first show that $\mathcal{M}\dot{+}\mathcal{M}^\times = L^2_{\mathcal{U}}(i\mathbb{R})$. To do this, we show that the extended operator $M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}}$ is the projection from $L^2_{\mathcal{U}}(i\mathbb{R})$ onto $\mathcal{M}$ along $\mathcal{M}^\times$.

Let $f \in \mathcal{M}$. Then by hypothesis, there is a sequence $f_n \in \mathcal{Q}_{\mathcal{U},+}$ so that $Wf_n \to f$. Then, since $M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}}$ is bounded, $M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}} W f_n \to f$. But $M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}} W f_n = W f_n \in \mathcal{M}$, thus $M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}} f = f \in \mathcal{M}$. We may therefore conclude that $\mathcal{M} \subset \operatorname{Ran} M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}}$.

We now let $f \in L^2_{\mathcal{U}}(i\mathbb{R})$; as W is an (almost-everywhere) invertible mapping on $L^2_{\mathcal{U}}(i\mathbb{R})$, there exists some $g \in L^2_{\mathcal{U}}(i\mathbb{R})$ with $f = Wg$. As the space $\mathcal{Q}_{\mathcal{U}}$ is dense in $L^2_{\mathcal{U}}(i\mathbb{R})$, there exists a sequence $\{g_n\} \subset \mathcal{Q}\mathcal{U}$ so that $Wg_n \to f$. We may write $g_n = g_{n,+} + g_{n,-}$ where $g_{n,+} \in \mathcal{Q}_{\mathcal{U},+}$ and $g_{n,-} \in \mathcal{Q}_{\mathcal{U},-}$. We consider $M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}} f$:

$$\begin{aligned} M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}} f &= L^2\text{-lim}\, M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}} f_n \\ &= L^2\text{-lim}\, M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}} W g_n \\ &= L^2\text{-lim}\, M_W P_{H^2_{\mathcal{U}}(\Pi_+)} (g_{n,+} + g_{n,-}) \\ &= L^2\text{-lim}\, M_W g_{n,+}. \end{aligned}$$

But by our hypothesized representation of $\mathcal{M}$, we conclude that $L^2\text{-lim}\, W g_{n,+} \in \mathcal{M}$, and thus that $\operatorname{Ran} M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}} \subset \mathcal{M}$.

We conclude that $M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}}$ maps $L^2_{\mathcal{U}}(i\mathbb{R})$ onto $\mathcal{M}$.

Now we consider $f^\times \in \mathcal{M}^\times$. Then there is a sequence $g_{n,-} \in \mathcal{Q}_{\mathcal{U},-}$ so that $W g_{n,-} \to f^\times$. We compute

$$M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}} f^\times = L^2\text{-lim}\, M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}} W g_{n,-} = L^2\text{-lim}\, M_W P_{H^2_{\mathcal{U}}(\Pi_+)} g_{n,-} = \{0\}$$

Thus $M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}}$ is in fact the projection onto $\mathcal{M}$ along $\mathcal{M}^\times$. It follows that $L^2_{\mathcal{U}}(i\mathbb{R}) = \mathcal{M} \dot{+} \mathcal{M}^\times$.

We next show that $\mathcal{M}$ is invariant—indeed, simply invariant—under $\hat{\mathfrak{T}}$. By Theorem 3.3.5, it suffices to show that $\mathcal{M}$ has these properties under $\widehat{\mathbf{T}}$. We recall that $\widehat{\mathbf{T}}$ is an isometry on $L^2_{\mathcal{U}}(i\mathbb{R})$: this follows from Theorem 3.3.2, as $\hat{\mathfrak{T}}$ is an isometry. Then we have

$$\begin{aligned}\widehat{\mathbf{T}}\mathcal{M} &= \widehat{\mathbf{T}} L^2\text{-clos}\, W\mathcal{Q}_{\mathcal{U},+}\\ &= L^2\text{-clos}\,\widehat{\mathbf{T}}W\mathcal{Q}_{\mathcal{U},+},\end{aligned}$$

but as $\widehat{\mathbf{T}}$ is simply multiplication by $\frac{z-1}{z+1}$, cf. Lemma 3.3.6, $\widehat{\mathbf{T}}$ and W commute, so

$$\begin{aligned}&= L^2\text{-clos}\, W\widehat{\mathbf{T}}\mathcal{Q}_{\mathcal{U},+},\\ &\subset L^2\text{-clos}\, W\mathcal{Q}_{\mathcal{U},+}\\ &= \mathcal{M}\end{aligned}$$

Which is to say, $\mathcal{M}$ is $\widehat{\mathbf{T}}$-invariant. To see that $\mathcal{M}$ is actually simply invariant is fundamentally similar:

$$\cap_{n=1}^{\infty}\widehat{\mathbf{T}}^n\mathcal{M} = L^2\text{-clos}\, W\cap_{n=1}^{\infty}\widehat{\mathbf{T}}^n\mathcal{Q}_{\mathcal{U},+} = \{0\}.$$

The same argument, with the roles of $\widehat{\mathbf{T}}$ and $\mathcal{Q}_{\mathcal{U},+}$ exchanged for those of $\widehat{\mathbf{T}}^*$ and $\mathcal{Q}_{\mathcal{U},-}$, shows that $\mathcal{M}^\times$ is simply invariant under $\widehat{\mathbf{T}}^*$.

By Remark 3.1.3, it follows that the pair $(\mathcal{M}, \mathcal{M}^\times)$ is dual shift-invariant. □

Chapter 4

Data Representation Theorem

4.1 Sylvester Data Sets

We define both Π_+- and Π_--admissible Sylvester data sets at once. Here, Π stands for either Π_+ or Π_-.

Definition 4.1.1. We define an *infinitesimal* Π-*admissible Sylvester data set* to be a quintet of operators $(C, A; Z, B; \Gamma)$ acting on Hilbert spaces $\mathcal{X}_{\mathcal{P}}$, $\mathcal{X}_{\mathcal{Z}}$, and $\mathcal{U}$ as follows:

1. The pair $(C, A) : (\mathcal{D}(A), \mathcal{D}(A)) \subset (\mathcal{X}_{\mathcal{P}}, \mathcal{X}_{\mathcal{P}}) \to (\mathcal{U}, \mathcal{X}_{\mathcal{P}})$ is a Π exactly observable pair,

2. the pair $(Z, B) : (\mathcal{D}(Z) \subset \mathcal{X}_{\mathcal{Z}}, \mathcal{U}) \to (\mathcal{X}_{\mathcal{Z}}, \mathcal{X}_{\mathcal{Z},-1})$ is a Π exactly controllable pair,

3. the operator $\Gamma : \mathcal{D}(\Gamma) \subset \mathcal{X}_{\mathcal{P}} \to \mathcal{X}_{\mathcal{Z}}$ is closed and has dense domain,

and that Γ additionally satisfies the Sylvester equation requirement, that

4. the semigroup $\mathfrak{A}$ restricted to $\mathcal{D}(\Gamma)$ is strongly continuous in the graph norm on $\mathcal{D}(\Gamma)$, and if we denote the domain of the generator of $\mathfrak{A}_{|\mathcal{D}(\Gamma)}$ by $\mathcal{D}(\Gamma)_1$, then for all $x \in \mathcal{D}(\Gamma)_1$, the Sylvester equation $\Gamma Ax - Z\Gamma x = BCx$ holds.

We also introduce standard notation. We let

$$\mathfrak{S} := (C, A; Z, B; \Gamma)$$

denote a Π_+-admissible Sylvester data set. Additionally, we let

$$\mathfrak{S}^\times := (C^\times, A^\times; Z^\times, B^\times; \Gamma^\times)$$

denote a Π_--admissible Sylvester data set.

Definition 4.1.2. Given a Π_+-admissible Sylvester data set, we associate with it a subspace of $L^2_{\mathcal{U}}(\mathbb{R})$ by the following formula:

$$\mathcal{M}_{\mathfrak{S}} := \{\mathcal{O}^b_{C,A}x + f \mid x \in \mathcal{D}(\Gamma),\ f \in L^2_{\mathcal{U}}(\mathbb{R}_+),\ \text{ such that } \mathcal{C}^b_{Z,B}f = \Gamma x\}.$$

And given a Π_--admissible Sylvester data set, we associate with it the subspace of $L^2_{\mathcal{U}}(\mathbb{R})$ defined by

$$\mathcal{M}^\times_{\mathfrak{S}} := \{g + \mathcal{O}^f_{C^\times,A^\times}y \mid g \in L^2_{\mathcal{U}}(\mathbb{R}_-),\ y \in \mathcal{D}(\Gamma^\times),\ \text{such that } \mathcal{C}^f_{Z^\times,B^\times} = \Gamma^\times y\}$$

We take the Sylvester equation to be of the form $\Gamma Ax - Z\Gamma x = BCx$ for all appropriate x. This matches the Sylvester equation used in [4], wherein it arises naturally and is of immediate usefulness. In this current continuous-time context, we find that this form of the Sylvester equation is in fact the infinitesimal form of an equivalent equation involving the observation and control operators; we'll refer to this as the integrated Sylvester equation. Here we define the integrated Sylvester equation and demonstrate its equivalence to the infinitesimal version.

Definition 4.1.3. Let $(C, A; Z, B; \Gamma)$ be an infinitesimal Π_- Sylvester Data Set. Then for $x \in \mathcal{D}(\Gamma)$, we define the *integrated Sylvester equation* to be the equation

$$\Gamma\mathfrak{A}^{-s}x = -\int_0^s \mathfrak{Z}^{-t}_{|\mathcal{X}_{\mathcal{Z},-1}} BC\mathfrak{A}^{t-s}x dt - \mathfrak{Z}^{-s}\Gamma x. \tag{4.1}$$

Remark 4.1.4. We take a moment to note that the infinitesimal Sylvester equation in a discrete-time context has an integrated form, to which it is equivalent. Indeed, $(C, A; Z, B; \Gamma)$ be an Admissible Sylvester Data Set in the sense of [4]. As $A : \mathcal{D}(\Gamma) \to \mathcal{D}(\Gamma)$, we may use the Sylvester equation to compute $\Gamma A(Ax)$:

$$\begin{aligned}\Gamma A(Ax) &= BC(Ax) + Z\Gamma(Ax)\\ &= BCAx + Z(BCx + Z\Gamma x)\\ &= BCAx + ZBCx + Z^2\Gamma x\end{aligned}$$

Similarly, we may compute $\Gamma A(A^2x)$ and, in general, $\Gamma A(A^{n-1}x)$. A brief induction argument yields the (discrete-time) integrated Sylvester equation

$$\Gamma A^n x = \sum_{i=0}^{n-1} Z^i BCA^{n-1-i} + Z^n\Gamma x,$$

which is clearly a discrete version of Equation (4.1).

Of course, by setting $n = 1$ in the discrete-time integrated Sylvester equation, we recover the discrete-time infinitesimal Sylvester equation.

Theorem 4.1.5. *Let $(C, A; Z, B)$ be as in a Π_+ Sylvester Data Set, and let Γ be a closed (not necessarily bounded) operator with dense domain such that $\mathfrak{A}$ restricted to $\mathcal{D}(\Gamma)$ is strongly continuous in the graph norm. Then the infinitesimal form of the Sylvester equation*

$$\Gamma Ax - Z\Gamma x = BCx \tag{4.2}$$

holds for every $x \in \mathcal{D}(\Gamma)$ if and only if the integrated form of the Sylvester equation

$$\Gamma\mathfrak{A}^{-s}x + \mathfrak{Z}^{-s}\Gamma x = -\mathcal{C}^b_{Z,B}\mathfrak{T}^s\mathcal{O}^b_{C,A}x \tag{4.3}$$

holds for every $x \in \mathcal{D}(\Gamma)$.

Remark 4.1.6. We recall our notation that, if $\mathfrak{A}$ is a strongly continuous backward semigroup, then $\mathfrak{A}^s x = 0$ when $s > 0$. Thus

$$\mathcal{C}^b_{Z,B}\mathfrak{T}^s\mathcal{O}^b_{C,A}x = \int_0^\infty \mathfrak{Z}^{-t}_{|\mathcal{X}_{\mathcal{Z},-1}} B\mathfrak{T}^s C\mathfrak{A}^t x dt = \int_0^s \mathfrak{Z}^{-t}_{|\mathcal{X}_{\mathcal{Z},-1}} BC\mathfrak{A}^{t-s} x dt.$$

Proof. First we demonstrate that the infinitesimal form gives rise to the integrated form.

We choose an s contained in a right half-plane contained in $\rho(A) \cap \rho(Z)$—such an s is guaranteed to exist, as both A and Z generate strongly continuous semigroups. We add $s\Gamma x$ to both sides of the Sylvester equation to get

$$\Gamma(A + sI)x = (Z + sI)\Gamma x + BCx.$$

We define $x' := (A + sI)x$ and multiply both sides of our equation by $(Z + sI)^{-1}$ to get, after rearrangement of terms,

$$\Gamma(A + sI)^{-1}x' = (Z + sI)^{-1}\Gamma x' - (Z + sI)^{-1}BC(A + sI)^{-1}x'. \qquad (4.4)$$

Each term can now be recognized as a Laplace transform of a backward semigroup as per Theorem 2.1.12. Specifically, we have that

$$\Gamma(A + sI)^{-1}x' = \Gamma \int_{-\infty}^{\infty} e^{st}\mathfrak{A}^t x' dt$$

and

$$(Z + sI)^{-1}\Gamma x' = \int_{-\infty}^{\infty} e^{st}\mathfrak{Z}^t \Gamma x' dt.$$

The remaining term is a product of Laplace transforms; it therefore corresponds to the Laplace transform of a convolution:

$$(Z + sI)^{-1}BC(A + sI)^{-1}x' = \int_{-\infty}^{\infty} e^{st} \int_{-\infty}^{\infty} \mathfrak{Z}^\tau BC\mathfrak{A}^{\tau-s}x' d\tau dt$$

We may now apply an inverse Laplace transform to (4.4) to get

$$\Gamma\mathfrak{A}^{-s}x = -\int_0^s \mathfrak{Z}^{-t}_{|\mathcal{X}_{\mathcal{Z},-1}} BC\mathfrak{A}^{t-s}x dt - \mathfrak{Z}^{-s}\Gamma x.$$

Referring to our formulae for control and observation operators (cf. Definition 2.4.7) as well as Remark 4.1.6, we see this is precisely the claimed integrated form of the Sylvester equation.

Now we assume that the integrated form holds, and derive the infinitesimal form. The essence of the computation is to differentiate with respect to s in the $\mathcal{X}_{\mathcal{Z},-1}$ norm and evaluate at $s = 0$. We treat the left- and right-hand sides separately, beginning with the left-hand side.

In the following, we restrict to $x \in \mathcal{D}(\Gamma)_1$. The difference quotient of $\Gamma\mathfrak{A}^{-s}x$ is given by

$$\frac{1}{s}\left(\Gamma\mathfrak{A}^{-s}x - \Gamma x\right).$$

This difference quotient converges in $\|\cdot\|_{\mathcal{X}_\mathcal{Z}}$ to $-\Gamma Ax$ as $s \to 0$, since $\mathfrak{A}^{-s}$ is a strongly continuous semigroup on $\mathcal{D}(\Gamma)$ by condition (4) in Definition 4.1.1. That is, by definition, if $x \in \mathcal{D}(\Gamma)_1$, then $\frac{1}{s}(\mathfrak{A}^{-s}x - x)$ converges to $-Ax$ in the graph norm; equivalently,

$$\left\|\frac{1}{s}(\mathfrak{A}^{-s}x - x) + \mathfrak{A}^{-s}Ax)\right\|_{\mathcal{X}_\mathcal{P}}^2 + \left\|\Gamma\left(\frac{1}{s}(\mathfrak{A}^{-s}x - x) + \mathfrak{A}^{-s}Ax)\right)\right\|_{\mathcal{X}_\mathcal{Z}}^2$$

converges to zero. The second term here is of particular interest, as it implies the claimed convergence in $\mathcal{X}_\mathcal{Z}$.

Turning our attention to the right-hand side of (4.1), we claim that the derivative of the first term is given by BCx. This can be seen by computing the following (where all norms are in $\mathcal{X}_{\mathcal{Z},-1}$ unless otherwise specified):

$$\begin{aligned}
&\left\|\frac{1}{s}\int_0^s \mathfrak{Z}^{-t}BC\mathfrak{A}^{t-s}xdt - BCx\right\| \\
&= \left\|\frac{1}{s}\int_0^s \mathfrak{Z}^{-t}BC\mathfrak{A}^{t-s}xdt + \frac{1}{s}\int_0^s \mathfrak{Z}^{-t}BCxdt - \frac{1}{s}\int_0^s \mathfrak{Z}^{-t}BCxdt - BCx\right\| \\
&= \left\|\frac{1}{s}\int_0^s \mathfrak{Z}^{-t}BC(\mathfrak{A}^{t-s} - I)xdt + \frac{1}{s}\int_0^s (\mathfrak{Z}^{-t} - I)BCxdt\right\| \\
&\le \frac{1}{s}\int_0^s \left(\left\|\mathfrak{Z}^{-t}BC(\mathfrak{A}^{t-s} - I)x\right\| + \left\|(\mathfrak{Z}^{-t} - I)BCx\right\|\right) dt
\end{aligned}$$

As $\mathfrak{Z}^{-t}BC$ is norm-bounded for t in compact sets, the first integrand goes to zero as

$s \downarrow 0$ as a consequence of the strong continuity of $\mathfrak{A}$ on $\mathcal{D}(A) \subset \mathcal{X}_{\mathcal{P}}$. Similarly, in the second integrand, BCx is a fixed element of $\mathcal{X}_{\mathcal{Z},-1}$, so convergence of the integrand to zero is a consequence of the strong continuity of $\mathfrak{Z}^{-t}$ on $\mathcal{X}_{\mathcal{Z},-1}$.

Penultimately, the second term on the right-hand side of (4.1) is simplest. As $\mathcal{X}_{\mathcal{Z}}$ is precisely the domain of the generator of $\mathfrak{Z}^{-s}_{|\mathcal{X}_{\mathcal{Z},-1}}$, the derivative exists and equals $-Z\mathcal{C}^b_{Z,B}f = -Z\Gamma x$.

Finally, we may evaluate the derivative of (4.1) in the $\mathcal{X}_{\mathcal{Z},-1}$ norm to get $-\Gamma Ax = -BCx - Z\Gamma x$, or, rearranging,

$$\Gamma Ax - Z\Gamma x = BCx.$$

$\square$

We note that in Definition 4.1.1, the Sylvester equation requirement (4) is qualitatively different from requirements (1) through (3). The following Lemma sheds light on this requirement.

Lemma 4.1.7. *Let $\mathcal{M} \subset L^2_{\mathcal{U}}(\mathbb{R})$ be a subspace and let $(C, A; Z, B; \Gamma)$ be a quintet of operators satisfying conditions (1) through (3) in Definition 4.1.1 and such that $\mathcal{M}$ has the form*

$$\mathcal{M} = \{\mathcal{O}^b_{C,A}x + f \mid x \in \mathcal{D}(\Gamma),\ f \in L^2_{\mathcal{U}}(\mathbb{R}_+),\ \mathcal{C}^b_{Z,B}f = \Gamma x\}. \tag{4.5}$$

Then $\mathcal{M}$ is closed and, further, is shift-invariant if and only if Γ satisfies the Sylvester equation requirement (4) in Definition 4.1.1.

Remark 4.1.8. We wish to stress that the interpretation of Equation (4.5) is that, for any $m \in \mathcal{M}$,

$$m(t) = \begin{cases} \big(\mathcal{O}^b_{C,A}x\big)(t), & t < 0 \\ f(t), & t > 0 \end{cases}$$

Proof. We may see that $\mathcal{M}$ is closed by the following: Let m_n be a sequence in $\mathcal{M}$ which converges to m. By hypothesis, each m_n admits the representation $m_n = \mathcal{O}^b_{C,A}x_n + f_n$ for some $x_n \in \mathcal{D}(\Gamma)$ and $f \in L^2_{\mathcal{U}}(\mathbb{R}_+)$. As $\mathcal{O}^b_{C,A}x_n \in L^2_{\mathcal{U}}(\mathbb{R}_-)$ and $L^2_{\mathcal{U}}(\mathbb{R}_-)$ is closed, we conclude that $\mathcal{O}^b_{C,A}x_n \to f_- \in L^2_{\mathcal{U}}(\mathbb{R}_-)$. Similarly, the $f_n \to f_+ \in L^2_{\mathcal{U}}(\mathbb{R}_+)$. Thus $m = f_- + f_+$.

As the pair (C, A) is Π_+-admissible exactly observable, $\mathcal{O}^b_{C,A}$ is invertible; we may therefore conclude that $x_n \to x$, where $x = (\mathcal{O}^b_{C,A})^{-1}f_- \in \mathcal{X}_{\mathcal{P}}$. But we may say more: we also have by hypothesis that $\mathcal{C}^b_{Z,B}f_n = \Gamma x_n$. As the pair (Z, B) is exactly controllable, $\mathcal{C}^b_{Z,B}$ is bounded. We conclude that Γx_n converges to $\mathcal{C}^b_{Z,B}f_+$. But as Γ is closed, it must be that $x \in \mathcal{D}(\Gamma)$ and $\mathcal{C}^b_{Z,B}f_+ = \Gamma x$. We conclude that m takes the form $\mathcal{O}^b_{C,A}x + f_+$ such that $\mathcal{C}^b_{Z,B}f_+ = \Gamma x$; but this means that $m \in \mathcal{M}$ and that $\mathcal{M}$ is closed.

We first assume that the Sylvester equation requirement 4 in Definition 4.1.1 holds; this includes the assumption that the infinitesimal Sylvester equation holds. We appeal immediately to Lemma 4.1.5 and work instead with the integrated Sylvester equation.

We choose some element $m \in \mathcal{M}$ and assume it has a representation of the form (4.5). We examine what happens when we apply $\mathfrak{T}^s$:

$$\begin{aligned}(\mathfrak{T}^s m)(t) &= (\mathfrak{T}^s(\mathcal{O}^b_{C,A}x + f))(t) \\ &= C\mathfrak{A}^{t-s}x + f(t-s),\end{aligned}$$

which, in the light of Note 4.1.8 above, is to be interpreted as

$$= \begin{cases} C\mathfrak{A}^{t-s}x = C\mathfrak{A}^{t}\mathfrak{A}^{-s}x, & t < 0 \\ C\mathfrak{A}^{t-s}x, & 0 < t < s \\ f(t-s), & s < t \end{cases}$$

For fixed s, we define $x' := \mathfrak{A}^{-s}x$ and $f'(t) := C\mathfrak{A}^{t-s}x + f(t-s)$. By hypothesis, $\mathfrak{A} : \mathcal{D}(\Gamma) \to \mathcal{D}(\Gamma)$, so $x' \in \mathcal{D}(\Gamma)$. Also by hypothesis (C, A) comprises an exactly observable pair, so $C\mathfrak{A}^{t-s}x \in L^2_{\mathcal{U}}(\mathbb{R}_+)$; so, too, therefore is f'. Thus we have

$$\mathfrak{T}^s m = \mathcal{O}^b_{C,A}x' + f'(t).$$

for $x' \in \mathcal{D}(\Gamma)$ and $f' \in L^2_{\mathcal{U}}(\mathbb{R}_+)$. Since $\mathfrak{T}^s m$ has the right form to be in $\mathcal{M}$, all that need be checked is the coupling condition $\Gamma x' = \mathcal{C}^b_{Z,B}f'$.

Consider then

$$\begin{aligned} \mathcal{C}^b_{Z,B}f' &= \begin{cases} \mathcal{C}^b_{Z,B}C\mathfrak{A}^{t-s}x, & 0 < t < s \\ \mathcal{C}^b_{Z,B}f(t-s), & s < t \end{cases} \\ &= -\int_0^s \mathfrak{Z}^{-t}_{|\mathcal{X}_{\mathcal{Z},-1}} BC\mathfrak{A}^{t-s}x\,dt - \int_0^\infty \mathfrak{Z}^{-t-s}_{|\mathcal{X}_{\mathcal{Z},-1}} Bf(t)\,dt, \end{aligned}$$

which, by the integrated Sylvester equation,

$$= \Gamma\mathfrak{A}^{-s}x = \Gamma x'.$$

We conclude that $\mathcal{M}$ is shift-invariant.

Now we prove the converse direction; we assume that $\mathcal{M}$ is shift-invariant.

If $m \in \mathcal{M}$, then by (4.5),

$$\mathfrak{T}^s m = \mathfrak{T}^s(\mathcal{O}^b_{C,A}x + f) = C\mathfrak{A}^{t-s}x + f(t-s).$$

But as $\mathfrak{T}^s\mathcal{M} \subset \mathcal{M}$, we must have x' and f' so that

$$C\mathfrak{A}^{t-s}x + f(t-s) = C\mathfrak{A}^t x' + f'(t). \tag{4.6}$$

Evaluating (4.6) in different cases will give us the other results we claim.

When $t < 0$, (4.6) becomes

$$C\mathfrak{A}^{t-s}x = C\mathfrak{A}^t x'$$

so $\mathfrak{A}^{-s}x = x'$ since $C\mathfrak{A}^t$ has bounded left inverse. As both x and x' are in $\mathcal{D}(\Gamma)$, an immediate consequence of this is that $\mathfrak{A}^s : \mathcal{D}(\Gamma) \to \mathcal{D}(\Gamma)$, as claimed.

On the other hand, if $t > 0$, then (b) tells us

$$C\mathfrak{A}^{t-s}x\Big|_{0\le t<s} + f(t-s)\Big|_{t\ge s} = f'(t).$$

Applying $\mathcal{C}^b_{Z,B}$ to both sides, recalling that $\mathcal{C}^b_{Z,B}f' = \Gamma x' = \Gamma\mathfrak{A}^{-s}x$ gives

$$\Gamma\mathfrak{A}^{-s}x = \begin{cases} \mathcal{C}^b_{Z,B}C\mathfrak{A}^{t-s}x, & 0 < t < s \\ \mathcal{C}^b_{Z,B}f(t-s), & s < t \end{cases}.$$

We rewrite the right hand side, also expanding the control and observation operators, to get

$$\Gamma\mathfrak{A}^{-s}x = -\int_0^s \mathfrak{Z}^{-t}_{|\mathcal{X}_{\mathcal{Z},-1}} BC\mathfrak{A}^{t-s}x dt - \int_s^\infty \mathfrak{Z}^{-t}_{|\mathcal{X}_{\mathcal{Z},-1}} Bf(t-s)dt,$$

or, relabeling t in the right integral,

$$\Gamma\mathfrak{A}^{-s}x = -\int_0^s \mathfrak{Z}^{-t}_{|\mathcal{X}_{\mathcal{Z},-1}} BC\mathfrak{A}^{t-s}x dt - \int_0^\infty \mathfrak{Z}^{-t-s}_{|\mathcal{X}_{\mathcal{Z},-1}} Bf(t)dt.$$

But this last term can be identified with $\mathfrak{Z}^{-s}\mathcal{C}^b_{Z,B}f$, which can be further identified with $\mathfrak{Z}^{-s}\Gamma x$ by the coupling condition in the definition of $\mathcal{M}$; this gives, finally,

$$\Gamma\mathfrak{A}^{-s}x = -\int_0^s \mathfrak{Z}^{-t}_{|\mathcal{X}_{\mathcal{Z},-1}} BC\mathfrak{A}^{t-s}x dt - \mathfrak{Z}^{-s}\Gamma x \tag{4.7}$$

Equation (4.7) is nothing other than the integrated form of the Sylvester equation, cf. (4.1). Lemma 4.1.5 states that this is equivalent to the infinitesimal form of the Sylvester equation.

We will continue to work with the integrated Sylvester equation in order to show that $\mathfrak{A}^{-s}$ is strongly continuous on $\mathcal{D}(\Gamma)$ in the graph norm.

We now consider $\mathcal{D}(\Gamma)$ as a space in its own right, and we equip it with the Γ-graph norm. We seek to show that $\mathfrak{A}^{-s}$ is strongly continuous on $\mathcal{D}(\Gamma)$. We know by strong continuity of $\mathfrak{T}^s$ that for any $m \in \mathcal{M}$, we have $\mathfrak{T}^s m \to m$ strongly as $s \downarrow 0$. As $\mathcal{M}$ is shift-invariant, we have that $\mathfrak{T}^s m \in \mathcal{M}$; we may apply our representation to both m and $\mathfrak{T}^s m$, then, to get that an x and x^s in $\mathcal{X}$ and f and f^s in $L^2_{\mathcal{U}}(\mathbb{R}_+)$ so that

$$m = \mathcal{O}^b_{C,A}x + f, \quad \text{and} \quad \mathfrak{T}^s m = \mathcal{O}^b_{C,A}x^s + f^s,$$

and further that $x^s = \mathfrak{A}^{-s}x$. Strong continuity of $\mathfrak{T}^s$ guarantees therefore that

$$\mathcal{O}^b_{C,A}x^s + f^s \to \mathcal{O}^b_{C,A}x + f$$

strongly as $s \downarrow 0$; but we have that both $\mathcal{O}^b_{C,A} x^s$ and $\mathcal{O}^b_{C,A} x$ are in $L^2_{\mathcal{U}}(\mathbb{R}_-)$ and that both f^s and f are in $L^2_{\mathcal{U}}(\mathbb{R}_+)$, so it follows that $\mathcal{O}^b_{C,A} x^s \to \mathcal{O}^b_{C,A} x$ and that $f^s \to f$, where each convergence is strong.

As $\mathcal{O}^b_{C,A}$ is bounded by hypothesis, we conclude that $x^s \to x$ strongly.

We know that f^s converges strongly to f; but as $\mathcal{C}^b_{Z,B}$ is bounded, we may conclude that $\mathcal{C}^b_{Z,B} f^s \to \mathcal{C}^b_{Z,B} f$, also strongly. By our representation assumption on $\mathcal{M}$, we may conclude that $\Gamma x^s \to \Gamma x$ strongly.

We now have that both $x^s \to x$ and $\Gamma x^s \to \Gamma x$, implying precisely that $x^s \to x$ in the Γ graph norm. As $x^s = \mathfrak{A}^{-s} x$, we conclude that $\mathfrak{A}^{-s}$ is strongly continuous on $\mathcal{D}(\Gamma)$.

As $\mathfrak{A}^{-s}\big|_{\mathcal{D}(\Gamma)}$ is strongly continuous, it has a closed generator A' with a domain that we denote $\mathcal{D}(\Gamma)_1$, dense in $\mathcal{D}(\Gamma)$ in the graph norm. As A' is a restriction of A, we have that $\mathcal{D}(\Gamma)_1 \subset \big(\mathcal{D}(\Gamma) \cap \mathcal{D}(A)\big) \subset \mathcal{X}_{\mathcal{P}}$. But as, trivially,

$$\|\cdot\|_{\mathcal{X}_{\mathcal{P}}} \leq \|\cdot\|_{\mathcal{D}(\Gamma)},$$

$\mathcal{D}(\Gamma)_1$ is in fact dense in $\mathcal{D}(\Gamma)$ with respect to $\|\cdot\|_{\mathcal{X}_{\mathcal{P}}}$; it follows that $\mathcal{D}(\Gamma)_1$ is dense in $\mathcal{X}_{\mathcal{P}}$ with respect to $\|\cdot\|_{\mathcal{X}_{\mathcal{P}}}$.

□

4.2 The Data Representation Theorem

4.2.1 Four Lemmas on Data Representations in Special Cases

We present four Lemmas, which can be thought of as special cases of our data set representation Theorem. In fact, as we will show in Theorem 4.2.10 below, the general case can in a sense be reduced to these four special cases.

Lemma 4.2.1. *Let $\mathcal{P}^\times \subset L^2_{\mathcal{U}}(\mathbb{R}_+)$ be a closed subspace which is invariant under the compressed backward shift $\left(\mathfrak{T}^s_+\right)^*$. Then there exists a Hilbert space $\mathcal{X}^\times_{\mathcal{P}}$, a densely-defined operator $A^\times$ generating a rigged structure on $\mathcal{X}^\times_{\mathcal{P}}$, and a further operator $C^\times : \mathcal{X}^\times_{\mathcal{P},1} \to \mathcal{U}$ such that the pair $(C^\times, A^\times)$ constitutes a Π_--admissible exactly observable pair and that*

$$\mathcal{P}^\times = \operatorname{Ran} \mathcal{O}^f_{C^\times, A^\times}.$$

Remark 4.2.2. We note that an alternate proof may be found in [8] page 292.

Proof. We take inspiration from the analogous result for $\mathcal{M} \subset L^2(\mathbb{Z}_+)$ in which one may choose $\mathcal{X} = \mathcal{M}$, $A : \mathcal{X} \to \mathcal{X}$ to be the backward (discrete) shift, and $C : \mathcal{X} \to \mathcal{Y}$ to be

$$C : \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \mapsto x_0.$$

With these choices, it is easy to check that $\mathcal{O}_{C,A}\big|_{\mathcal{X}} = I_{\mathcal{X}}$ and that, trivially therefore, $\mathcal{M} = \mathcal{O}_{C,A}\mathcal{M}$.

Consequently, we choose $\mathcal{X} = \mathcal{M}$. We choose A to be the generator of $\left(\mathfrak{T}^s_+\right)^* \big|_{\mathcal{M}}$, the compressed backward (continuous) shift. By Proposition 2.2.7, $A : x \mapsto -x'$ with

$$\mathcal{D}(A) = \{x \in \mathcal{M} \subset L^2(\mathbb{R}_+) : x \in \mathrm{AC}(\mathbb{R}_+) \text{ and } x' \in L^2(\mathbb{R}_+)\}.$$

We choose $C : \mathcal{D}(A) \to \mathcal{U}$ to be the Dirac delta function at zero; that is, let $C : x(t) \in \mathcal{D}(A) \mapsto x(0)$. Then $C\left(\mathfrak{T}_+^s\right)^*$ is well defined as a mapping from $\mathcal{D}(A)$ to $\mathcal{U}$; therefore $\mathcal{O}_{C,A}^f : x \mapsto C\left(\mathfrak{T}_+^s\right)^* x$ is well defined as a mapping on $\mathcal{D}(A)$. For $x \in \mathcal{D}(A)$, $\mathcal{O}_{C,A}^f x = \{C\left(\mathfrak{T}_+^s\right)^* x(t)\}_{s\geq 0} = \{C\pi_+\mathfrak{T}^s x(t)\}_{s\geq 0} = \{Cx_+(t+s)\}_{s\geq 0} = \{x(s)\}_{s\geq 0}$, which can be identified with $x \in L^2(\mathbb{R}_+)$. In other words,

$$\mathcal{O}_{C,A}\big|_{\mathcal{D}(A)} = I\big|_{\mathcal{D}(A)}.$$

As $\mathcal{D}(A)$ is dense in $L^2(\mathbb{R}_+)$ and thus in $\mathcal{M}$, $\mathcal{O}_{C,A}$ has a unique continuous extension from $\mathcal{D}(A) \cap \mathcal{M}$ to $\mathcal{M}$: namely $I_{\mathcal{M}} =: \overline{\mathcal{O}_{C,A}}$. Thus with these choices of (C, A) and $\mathcal{X}$, we get $\mathcal{M} = \mathcal{O}_{C,A}\mathcal{M}$. What needs to be shown is that (C, A) form an exactly observable pair.

First we show that (C, A) is an L^2-admissible pair. A is, by definition, the generator of the semigroup $\left(\mathfrak{T}_+^s\right)^*$ which is strongly continuous by Theorem 2.2.4. To show that C has the required properties is, however, more delicate: it must be shown that there exists some $M > 0$ so that

$$\|Cx\|_{\mathcal{Y}} \leq M\left(\|x\|_2^2 + \|x'\|_2^2\right)^{\frac{1}{2}}$$

for every $x \in \mathcal{D}(A)$. Consider

$$\begin{aligned}
\|Cx\|_{\mathcal{U}} &= \|x(0)\|_{\mathcal{U}} \\
&= \int_0^1 \|x(0)\|_{\mathcal{U}}\, dt \\
&= \int_0^1 \left\| x(t) - \int_0^t x'(s) ds \right\| dt,
\end{aligned}$$

since $x \in \mathrm{AC}(\mathbb{R}_+)$, so

$$\begin{aligned}
&\leq \int_0^1 \|x(t)\|_{\mathcal{U}}\, dt + \int_0^1 \int_0^t \|x'(s)\|_{\mathcal{U}}\, dsdt \\
&\leq \int_0^1 \|x(t)\|_{\mathcal{U}}\, dt + \int_0^1 \int_0^1 \|x'(s)\|_{\mathcal{U}}\, dsdt, \\
&= \|x\|_{\mathrm{L}^1((0,1))} + \|x'\|_{\mathrm{L}^1((0,1))} \\
&\leq \|x\|_{L^2(0,1)} + \|x\|_{L^2(0,1)} \\
&\leq \|x\|_{L^2(\mathbb{R}_+)} + \|x\|_{L^2(\mathbb{R}_+)}\,.
\end{aligned}$$

But this last expression may be viewed as the Manhattan norm of $(\|x\|_{L^2(\mathbb{R}_+)}, \|x'\|_{L^2(\mathbb{R}_+)}) \in \mathbb{R}^2$, and it is well known that all norms on $\mathbb{R}^2$ are equivalent; therefore there exists some M so that

$$\|x\|_{L^2(\mathbb{R}_+)} + \|x\|_{L^2(\mathbb{R}_+)} \leq M \left(\|x\|_2^2 + \|x'\|_2^2 \right)^{\frac{1}{2}}.$$

Thus is the desired inequality shown. With this inequality, it is seen that C can be thought of as a bounded operator from $\mathcal{D}(A)$ to $\mathcal{U}$, if $\mathcal{D}(A)$ is equipped with the graph norm. Finally, the unique extension of $\mathcal{O}_{C,A}$ from $\mathcal{D}(A) \cap \mathcal{X}$ to $\mathcal{X}$ is, by the above discussion, $I_{\mathcal{X}}$. Trivially this extension satisfies the bound

$$\|\mathcal{O}_{C,A}x\|_{\mathcal{X}} = \|x\|_{\mathcal{X}} \leq 1 \cdot \|x\|_{\mathcal{X}}.$$

All conditions for (C, A) to be an L^2-admissible observable pair are thus satisfied.

But it is also claimed that (C, A) is exactly observable; one may verify this as follows: since $\mathcal{O}_{C,A} = I_{\mathcal{X}}$, clearly $\operatorname{Ker}\mathcal{O}_{C,A} = \{0\}$. By construction, $\operatorname{Ran}\mathcal{O}_{C,A} = \mathcal{M}$, which is closed by hypothesis. Since the kernel of $\mathcal{O}_{C,A}$ is trivial and its range is closed, the pair (C, A) is exactly observable.

That is, the pair (C, A) is an exactly observable pair of operators and $\mathcal{X} = \mathcal{M}$ is a Hilbert space so that $\mathcal{M} = \operatorname{Ran}\mathcal{O}_{C,A}$. □

Definition 4.2.3. We define the pair $(C^\times, A^\times)$ with state space $\mathcal{X}_{\mathcal{P}} = \mathcal{M}$ as constructed in Lemma 4.2.1 above to be the *model* Π_-*-admissible exactly observable pair* and use $(\mathbf{C}^\times, \mathbf{A}^\times)$ to denote it.

Lemma 4.2.4. *Let* $\mathcal{M}_0 \subset L^2_{\mathcal{U}}(\mathbb{R}_+)$ *be a closed subspace which is invariant under the forward shift* $\mathfrak{T}^s$*. Then there exists a Hilbert space* $\mathcal{X}_{\mathcal{Z}}$*, a densely-defined operator* Z *generating a rigged structure on* $\mathcal{X}_{\mathcal{Z}}$*, and a further operator* $B : \mathcal{U} \to \mathcal{X}_{\mathcal{Z},-1}$ *such that the pair* (Z, B) *constitutes a* Π_+*-admissible exactly controllable pair and that*

$$\mathcal{M}_0 = \operatorname{Ker}\mathcal{C}^b_{Z,B}.$$

Proof. We are given that $\mathcal{M}_0 \subset L^2_{\mathcal{U}}(\mathbb{R}_+)$ is forward shift invariant. But then by Lemma 2.1.21, we have that $\mathcal{P}^\times := L^2_{\mathcal{U}}(\mathbb{R}_+) \ominus \mathcal{M}$ is compressed backward shift invariant.

We apply Theorem 4.2.1 to $\mathcal{P}^\times$ to construct Hilbert space $\mathcal{X}^\times_{\mathcal{P}}$ and an exactly observable pair of operators $(C^\times, A^\times)$ such that $\mathcal{P}^\times = \mathcal{O}^b_{C^\times,A^\times}\mathcal{X}^\times_{\mathcal{P}}$.

By Theorem 2.4.9 we know that the pair $(-(A^\times)^*, (C^\times)^*)$ is an Π_--admissible exactly controllable pair. Further, we have that

$$\mathcal{M}_0 = (\mathcal{P}^\times)^\perp = \left(\operatorname{Ran}\mathcal{O}^b_{C^\times,A^\times}\right)^\perp = \operatorname{Ker}\left(\mathcal{O}^b_{C^\times,A^\times}\right)^* = \operatorname{Ker}\mathcal{C}^f_{-(A^\times)^*,(C^\times)^*}.$$

It therefore suffices to take $\mathcal{X}^\times_{\mathcal{Z}} = \mathcal{X}$, $Z = -(A^\times)^*$, and $B = (C^\times)^*$.

In particular, one choice is to set $(Z, B) = (-(\mathbf{A}^\times)^*, (\mathbf{C}^\times)^*)$ with $(\mathbf{C}^\times, \mathbf{A}^\times)$ as in Definition 4.2.7. $\square$

Definition 4.2.5. We define the pair (Z, B) constructed at the end of the proof of Lemma 4.2.4 above to be the *model* Π_+*-admissible exactly controllable pair* and use $(\mathbf{Z}, \mathbf{B})$ to denote it.

Lemma 4.2.6. *Let $\mathcal{P} \subset L^2_{\mathcal{U}}(\mathbb{R}_-)$ be a closed subspace which is invariant under the compressed forward shift $\left(\mathfrak{T}^s_-\right)^*$. Then there exists a Hilbert space $\mathcal{X}_{\mathcal{P}}$, a densely-defined operator A generating a rigged structure on $\mathcal{X}_{\mathcal{P}}$, and a further operator $C : \mathcal{X}_{\mathcal{P},1} \to \mathcal{U}$ such that the pair (C, A) constitutes a Π_+-admissible exactly observable pair and that*

$$\mathcal{P} = \operatorname{Ran}\mathcal{O}^b_{C,A}.$$

Proof. This proof closely parallels that of Lemma 4.2.1. It is, in fact, precisely a time-reversed version thereof.

We choose $\mathcal{X}_{\mathcal{P}} = \mathcal{P}$, A to be the generator of $\left(\mathfrak{T}^s_-\right)^*\big|_{\mathcal{P}}$, and C to be the point-evaluation operator at zero $C : f \in \mathcal{D}(A) \mapsto f(0)$. Having made these choices, we show that the pair (C, A) is Π_+-admissible exactly observable.

By definition, A is the generator of $\left(\mathfrak{T}_{-}^{s}\right)^{*}\big|_{\mathcal{P}}$, which is a strongly continuous forward semigroup by Theorem 2.2.4. We then define the rigged spaces $\mathcal{X}_{\mathcal{P},-1}$ and $\mathcal{X}_{\mathcal{P},1}$ with respect to A as per section 2.4.2.

We have fully characterized the generator of the full semigroup $\left(\mathfrak{T}_{-}^{s}\right)^{*}$ in Theorem 2.2.10. In particular, the domain $\mathcal{D}(T)$ consists of functions which are absolutely continuous. As $\mathcal{X}_{\mathcal{P},1} = \mathcal{D}(T)$, C is defined on $\mathcal{X}_{\mathcal{P},1}$.

We turn our attention to C in order to show that it is bounded as an operator from $\mathcal{X}_{\mathcal{P},1}$ to $\mathcal{U}$. This computation parallels a similar computation in the proof of Lemma 4.2.1. We consider, for $x \in \mathcal{X}_{\mathcal{P},1}$

$$\begin{aligned}
\|Cx\|_{\mathcal{U}} &= \|x(0)\|_{\mathcal{U}} \\
&= \int_{-1}^{0} \|x(0)\|_{\mathcal{U}}\, dt \\
&= \int_{-1}^{0} \left\| x(t) + \int_{t}^{0} x'(s)\, ds \right\|_{\mathcal{U}} dt \\
&\leq \int_{-1}^{0} \|x(t)\|_{\mathcal{U}}\, dt + \int_{-1}^{0}\int_{t}^{0} \|x'(s)\|_{\mathcal{U}}\, ds\, dt \\
&\leq \int_{-1}^{0} \|x(t)\|\, dt + \int_{-1}^{0}\int_{-1}^{0} \|x'(s)\|_{\mathcal{U}}\, ds \\
&= \|x\|_{\mathrm{L}^1(-1,0)} + \|x'\|_{\mathrm{L}^1(-1,0)} \\
&\leq \|x\|_{L^2(-1,0)} + \|x'\|_{L^2(-1,0)} \\
&\leq \|x\|_{L^2(\mathbb{R}_-)} + \|x'\|_{L^2(\mathbb{R}_-)},
\end{aligned}$$

but now we recognize that $x' = Ax$ and that $\mathcal{X}$ is a subspace of $L^2(\mathbb{R}_-)$, so we may write

$$= \|x\|_{\mathcal{X}} + \|Ax\|_{\mathcal{X}}\,.$$

We may now apply Lemma 2.4.4 to the last estimate to conclude the existence of some $M \geq 0$ such that

$$\|Cx\|_{\mathcal{U}} \leq M\,\|x\|_{\mathcal{X}_{\mathcal{P},1}}\,.$$

With these definitions, we have for $x \in \mathcal{X}_{\mathcal{P},1}$ that $\mathcal{O}^b_{C,A}x = \{C\mathfrak{A}^s x\}_{s\leq 0} = \{C\left(\mathfrak{T}^s_-\right)^* x\}_{s\leq 0} = \{Cx(t-s)\}_{s\geq 0} = \{x(-s)\}_{s\geq 0}$, which we may identify with $x \in L^2(\mathbb{R}_-)$. Thus on $\mathcal{X}_{\mathcal{P},1}$, we have $\mathcal{O}^b_{C,A} = I$, the identity operator. Clearly this extends continuously to all of $\mathcal{X}_{\mathcal{P}}$ and the extended operator (also called) $\mathcal{O}^b_{C,A}$ is also the identity. This extended $\mathcal{O}^b_{C,A}$ is trivially one-to-one.

So far we have that (C, A) is a Π_+-Admissible Observable Pair. But the extended operator $\mathcal{O}^b_{C,A}$ is also trivially onto, so we may conclude that (C, A) is in fact exactly observable.

Finally, also trivially, we have that $\mathcal{P} = \operatorname{Ran}\mathcal{O}^b_{C,A}$. □

Definition 4.2.7. We define the pair (C, A) constructed in the proof of Lemma 4.2.6 above to be the *model* Π_+*-admissible exactly observable pair* and use $(\mathbf{C}, \mathbf{A})$ to denote it.

Lemma 4.2.8. *Let $\mathcal{M}_0^\times \subset L^2(\mathbb{R}_-)$ be a closed subspace which is invariant under the backward shift $\mathfrak{T}^s_-$. Then there exists a Hilbert space $\mathcal{X}^\times_{\mathcal{Z}}$, a densely-defined operator $Z^\times$ generating a rigged structure on $\mathcal{X}^\times_{\mathcal{Z}}$, and a further operator $B^\times : \mathcal{U} \to \mathcal{X}^\times_{\mathcal{Z},-1}$ such that the pair $(Z^\times, B^\times)$ constitutes a Π_+-admissible exactly controllable pair and* *that*

$$\mathcal{M}_0^\times = \operatorname{Ker} \mathcal{C}^f_{Z^\times, B^\times}.$$

Proof. We consider $\mathcal{P} := L^2(\mathbb{R}_-) \ominus \mathcal{M}_0^\times$. By Lemma 2.1.21, we have that $\mathcal{P}$ is invariant under $\left(\mathfrak{T}^s_-\right)^*$, the compressed forward shift.

We may therefore apply Lemma 4.2.6 to $\mathcal{P}$ to get a Π_--admissible exactly observable pair (C, A) such that $\mathcal{P} = \operatorname{Ran} \mathcal{O}^b_{C,A}$.

By Theorem 2.4.9, though, we know that the pair $(-A^*, C^*)$ is a Π_+-admissible exactly controllable pair and that $(\mathcal{O}^b_{C,A})^* = \mathcal{C}^f_{-A^*,C^*}$. Further, we have that

$$\mathcal{M}_0^\times = (\mathcal{P})^\perp = (\operatorname{Ran} \mathcal{O}^b_{C,A})^\perp = \operatorname{Ker}(\mathcal{O}^b_{C,A})^* = \operatorname{Ker} \mathcal{C}^f_{A^*,C^*}.$$

It therefore suffices to take $\mathcal{X}^\times_{\mathcal{Z}} = \mathcal{M}_0^\times$, $Z^\times = -A^*$, $B^\times = C^*$.

One particular choice of $(Z^\times, B^\times)$ is $(-\mathbf{A}^*, \mathbf{C}^*)$, where $(\mathbf{C}, \mathbf{A})$ is the model Π_+-admissible exactly observable pair as in Definition 4.2.7. □

Definition 4.2.9. We define the pair $(Z^\times, B^\times)$ defined in Lemma 4.2.8 above to be the *model* Π_+*-admissible exactly controllable pair* and use $(\mathbf{Z}^\times, \mathbf{B}^\times)$ to denote it.

4.2.2 The General Data Representation Theorem

Theorem 4.2.10. *1. If $\mathcal{M}$ is a closed subspace of $L^2_{\mathcal{U}}(\mathbb{R})$, then it is forward shift-invariant if and only if there is a Π_+-admissible Sylvester data set $(C, A; Z, B; \Gamma)$*

so that

$$\mathcal{M} = \{\mathcal{O}^b_{C,A}x + f : x \in \mathcal{D}(\Gamma),\ f \in L^2_{\mathcal{U}}(\mathbb{R}_+),\ \mathcal{C}^b_{Z,B}f = \Gamma x\}.$$

2. *If $\mathcal{M}^\times$ is a closed subspace of $L^2_{\mathcal{U}}(\mathbb{R})$, then it is backward shift-invariant if and only if there is a Π_--admissible Sylvester data set $(C^\times, A^\times; Z^\times, B^\times; \Gamma^\times)$ so that*

$$\mathcal{M}^\times = \{g + \mathcal{O}^f_{C^\times,A^\times}y : y \in \mathcal{D}(\Gamma^\times),\ g \in L^2_{\mathcal{U}}(\mathbb{R}_-),\ \mathcal{C}^f_{Z^\times,B^\times}g = \Gamma^\times y\}.$$

3. *Let $(\mathcal{M}, \mathcal{M}^\times)$ be a pair of spaces which have the forms $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}$ and $\mathcal{M}^\times = \mathcal{M}^\times_{\mathfrak{S}^\times}$ for a Π_+-admissible Sylvester data set $\mathfrak{S}$ and a Π_--admissible Sylvester data set $\mathfrak{S}^\times$. Then the pair $(\mathcal{M}, \mathcal{M}^\times)$ forms a direct sum decomposition of $L^2_{\mathcal{U}}(\mathbb{R})$ if and only if the* coupling matrix

$$\Gamma_{\mathfrak{S},\mathfrak{S}^\times} := \begin{bmatrix} \mathcal{C}^f_{Z^\times,B^\times}\mathcal{O}^b_{C,A} & \Gamma^\times \\ \Gamma & \mathcal{C}^b_{Z,B}\mathcal{O}^f_{C^\times,A^\times} \end{bmatrix}$$

is invertible.

Proof of (1). We begin by assuming that $\mathcal{M}$ is shift-invariant and construct the Π_+-admissible Sylvester data set.

Let π_- be the orthogonal projection from $L^2_{\mathcal{U}}(\mathbb{R})$ onto $L^2_{\mathcal{U}}(\mathbb{R}_-)$. We define three auxiliary spaces $\mathcal{P} := \overline{\pi_-\mathcal{M}}$, $\mathcal{M}_0 := \mathcal{M} \cap L^2_{\mathcal{U}}(\mathbb{R}_+)$, and $\mathcal{Z} := L^2_{\mathcal{U}}(\mathbb{R}_+) \ominus \mathcal{M}_0$. Then we have that $\mathcal{P}$ is a compressed forward shift invariant subspace of $L^2_{\mathcal{U}}(\mathbb{R}_-)$ and $\mathcal{M}_0$ is a forward shift invariant subspace of $L^2_{\mathcal{U}}(\mathbb{R}_+)$.

We apply Lemma 4.2.6 to $\mathcal{P}$ and obtain a Hilbert space $\mathcal{X}_{\mathcal{P}}$ and the model Π_+-

admissible exactly observable pair $(\mathbf{C}, \mathbf{A})$ such that $\mathcal{P} = \mathcal{O}^b_{\mathbf{C},\mathbf{A}} \mathcal{X}_{\mathcal{P}}$.

Similarly, we may apply Lemma 4.2.4 to $\mathcal{M}_0$ to get a Hilbert space $\mathcal{X}_{\mathcal{Z}}$ and the model Π_+-admissible exactly controllable pair $(\mathbf{Z}, \mathbf{B})$ such that $\mathcal{M}_0 = \operatorname{Ker} \mathcal{C}^b_{\mathbf{Z},\mathbf{B}}$.

To construct the operator Γ, we apply Lemma 2.3.1 with $\mathcal{X} = L^2_{\mathcal{U}}(\mathbb{R})$, $\mathcal{X}_- = L^2_{\mathcal{U}}(\mathbb{R}_-)$ and $\mathcal{X}_+ = L^2_{\mathcal{U}}(\mathbb{R}_+)$. Our auxiliary spaces $\mathcal{P}$, $\mathcal{M}_0$, and $\mathcal{Z}$ correspond exactly to their Lemma 2.3.1 counterparts. Then we have a densely-defined closed operator $\Gamma : \mathcal{D}(\Gamma) \subset \mathcal{P} \to \mathcal{X}_+ \ominus \mathcal{M}_0 =: \mathcal{X}_{\mathcal{Z}}$ such that we may write

$$\mathcal{M} = \{m_p + f \mid m_p \in \mathcal{D}(\Gamma), f \in L^2_{\mathcal{U}}(\mathbb{R}_+), \Gamma m_p = P_{\mathcal{X}_{\mathcal{Z}}} f\}.$$

But as $\mathcal{O}^b_{\mathbf{C},\mathbf{A}} = I_{\mathcal{P}}$, the identity on $\mathcal{P}$, we may write

$$\mathcal{M} = \{\mathcal{O}^b_{\mathbf{C},\mathbf{A}} m_p + f \mid m_p \in \mathcal{D}(\Gamma), f \in L^2_{\mathcal{U}}(\mathbb{R}_+), \Gamma m_p = P_{\mathcal{X}_{\mathcal{Z}}} f\}.$$

Further, referring again to Lemma 4.2.4, we recognize that the operator $\mathcal{C}^b_{\mathbf{Z},\mathbf{B}}$ is given by $\mathcal{C}^b_{\mathbf{Z},\mathbf{B}} f = P_{\mathcal{X}_{\mathcal{Z}}} f$ for $f \in L^2_{\mathcal{U}}(\mathbb{R}_+)$. Thus we may write

$$\mathcal{M} = \{\mathcal{O}^b_{\mathbf{C},\mathbf{A}} m_p + f \mid m_p \in \mathcal{D}(\Gamma), f \in L^2_{\mathcal{U}}(\mathbb{R}_+), \Gamma m_p = \mathcal{C}^b_{\mathbf{Z},\mathbf{B}} f\}.$$

Thus $\mathcal{M}$ has the desired representation in terms of $(\mathbf{C}, \mathbf{A}; \mathbf{Z}, \mathbf{B}; \Gamma)$. All that remains to be seen that $(\mathbf{C}, \mathbf{A}; \mathbf{Z}, \mathbf{B}; \Gamma)$ satisfies the Sylvester equation. But in Lemma 4.1.7, we showed that satisfying the Sylvester equation is equivalent to $\mathcal{M}$ being shift-invariant.

We now assume that $(C, A; Z, B; \Gamma)$ is a Π_+-admissible Sylvester data set and define the space

$$\mathcal{M} = \{\mathcal{O}^b_{\mathbf{C},\mathbf{A}} m_p + f \;\big|\; m_p \in \mathcal{D}(\Gamma), f \in L^2_{\mathcal{U}}(\mathbb{R}_+), \Gamma m_p = \mathcal{C}^b_{\mathbf{Z},\mathbf{B}} f\}.$$

Then Lemma 4.1.7 immediately gives that $\mathcal{M}$ is both closed and shift-invariant. $\square$

Proof of (2). We assume that $\mathcal{M}^\times$ is backward shift-invariant and construct the Π_--admissible Sylvester data set. This proceeds in exact analogy to the proof of (1) above.

We let π_+ be the orthogonal projection from $L^2_{\mathcal{U}}(\mathbb{R})$ onto $L^2_{\mathcal{U}}(\mathbb{R}_+)$ and define three auxiliary spaces: $\mathcal{P}^\times := \overline{\pi_+\mathcal{M}^\times}$, $\mathcal{M}_0^\times := \mathcal{M} \cap L^2_{\mathcal{U}}(\mathbb{R}_-)$, and $\mathcal{Z}^\times := L^2_{\mathcal{U}}(\mathbb{R}_-) \ominus \mathcal{M}_0^\times$. We note that $\mathcal{P}^\times$ is a compressed backward shift invariant subspace of $L^2_{\mathcal{U}}(\mathbb{R}_+)$ and that $\mathcal{M}_0^\times$ is a backward shift invariant subspace of $L^2_{\mathcal{U}}(\mathbb{R}_-)$.

We apply Lemma 4.2.1 to $\mathcal{P}^\times$ to obtain a Hilbert space $\mathcal{X}^\times_{\mathcal{P}}$ and the model Π_--admissible exactly observable pair $(\mathbf{C}^\times, \mathbf{A}^\times)$ such that $\mathcal{P}^\times = \mathcal{O}^f_{\mathbf{C}^\times,\mathbf{A}^\times}\mathcal{X}_{\mathcal{P}}$.

We apply Lemma 4.2.8 to $\mathcal{M}_0^\times$ to obtain a Hilbert space $\mathcal{X}^\times_{\mathcal{Z}}$ and the model Π_--admissible exactly controllable pair $(\mathbf{Z}^\times, \mathbf{B}^\times)$ such that $\mathcal{M}_0^\times = \operatorname{Ker} \mathcal{C}^f_{\mathbf{Z},\mathbf{B}}$.

We construct the operator $\Gamma^\times$ by the application of Lemma 2.3.1 with $\mathcal{X} = L^2_{\mathcal{U}}(\mathbb{R})$, $\mathcal{X}_- = L^2_{\mathcal{U}}(\mathbb{R}_-)$, and $\mathcal{X}_+ = L^2_{\mathcal{U}}(\mathbb{R}_+)$; here our spaces $\mathcal{P}^\times$, $\mathcal{M}_0^\times$, and $\mathcal{Z}^\times$ correspond to their Lemma 2.3.1 non-cross counterparts. Then we have a closed, densely-defined operator $\Gamma^\times : \mathcal{D}(\Gamma^\times) \subset \mathcal{P}^\times \to \mathcal{Z}^\times$ such that

$$\mathcal{M}^\times = \{g + m_p \;\big|\; m_p \in \mathcal{D}(\Gamma^\times), g \in L^2_{\mathcal{U}}(\mathbb{R}_-), \Gamma^\times m_p = P_{\mathcal{X}^\times_{\mathcal{Z}}} g\}.$$

Similarly to before, we recognize that since $\mathcal{O}^f_{\mathbf{C}^\times,\mathbf{A}^\times} = I$ and that $\mathcal{C}^f_{\mathbf{Z},\mathbf{B}} = P_{\mathcal{X}^\times_{\mathcal{Z}}}$, we may rewrite this as

$$\mathcal{M}^\times = \{g + \mathcal{O}^f_{\mathbf{C}^\times,\mathbf{A}^\times} m_p \mid m_p \in \mathcal{D}(\Gamma^\times), g \in L^2_{\mathcal{U}}(\mathbb{R}_-), \Gamma^\times m_p = \mathcal{C}^f_{\mathbf{Z},\mathbf{B}} g\}.$$

Thus $\mathcal{M}^\times$ has the desired representation in terms of $(\mathbf{C}^\times, \mathbf{A}^\times; \mathbf{Z}^\times, \mathbf{B}^\times; \Gamma^\times)$. What remains to be seen is that $(\mathbf{C}^\times, \mathbf{A}^\times; \mathbf{Z}^\times, \mathbf{B}^\times; \Gamma^\times)$ satisfies the Sylvester equation. But again, we appeal to Lemma 4.1.7.

We now assume that $(C^\times, A^\times; Z^\times, B^\times; \Gamma^\times)$ is a Π_--admissible Sylvester data set and construct the space

$$\mathcal{M}^\times = \{g + \mathcal{O}^f_{\mathbf{C}^\times,\mathbf{A}^\times} m_p \mid m_p \in \mathcal{D}(\Gamma^\times), g \in L^2_{\mathcal{U}}(\mathbb{R}_-), \Gamma^\times m_p = \mathcal{C}^f_{\mathbf{Z},\mathbf{B}} g\}.$$

By Lemma 4.1.7, we see immediately that $\mathcal{M}^\times$ is both closed and backward shift-invariant. □

Proof of (3). We first characterize $\mathcal{M} \cap \mathcal{M}^\times$ in terms of the kernel of $\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$.

Let $h \in \mathcal{M} \cap \mathcal{M}^\times$, and we write $h = h_- + h_+$ with $h_- \in L^2_{\mathcal{U}}(\mathbb{R}_-)$ and $h_+ \in L^2_{\mathcal{U}}(\mathbb{R}_+)$. By hypothesis, h can be written as an element of $\mathcal{M}_{\mathfrak{S}}$, so

$$h_- + h_+ = \mathcal{O}^b_{C,A} x + f, \quad \text{with} \quad \mathcal{C}^b_{Z,B} f = \Gamma x.$$

But by orthogonality, we conclude that $h_- = \mathcal{O}^b_{C,A} x$ and $h_+ = f$; therefore also $\mathcal{C}^b_{Z,B} h_+ = \Gamma x$. As h is also an element of $\mathcal{M}^\times$, it has a representation as an element of $\mathcal{M}^\times_{\mathfrak{S}^\times}$:

$$h_- + h_+ = g + \mathcal{O}^f_{C^\times,A^\times} y, \quad \text{with} \quad \mathcal{C}^f_{Z^\times,B^\times} g = \Gamma^\times y.$$

As before, we may conclude that $h_+ = \mathcal{O}^f_{C^\times,A^\times}y$, $h_- = g$, and $\mathcal{C}^f_{Z^\times,B^\times}h_- = \Gamma^\times y$. Notice then that any $h \in \mathcal{M} \cap \mathcal{M}^\times$ has the form $h = \mathcal{O}^b_{C,A}x + \mathcal{O}^f_{C^\times,A^\times}y$ for a pair $(x, y) \in (\mathcal{D}(\Gamma), \mathcal{D}(\Gamma^\times))$ satisfying the additional constraints

$$\begin{cases} \mathcal{C}^b_{Z,B}\mathcal{O}^f_{C^\times,A^\times}y &= \Gamma x \\ \mathcal{C}^f_{Z^\times,B^\times}\mathcal{O}^b_{C,A}x &= \Gamma^\times y \end{cases}. \tag{4.8}$$

Now consider some pair $(x, y) \in (\mathcal{D}(\Gamma), \mathcal{D}(\Gamma^\times))$ satisfying the constraints (4.8) and construct $h = \mathcal{O}^b_{C,A}x + \mathcal{O}^f_{C^\times,A^\times}y$. Then $h \in \mathcal{M}$ as the coupling condition $\mathcal{C}^b_{Z,B}\mathcal{O}^f_{C^\times,A^\times}y = \Gamma x$ is satisfied. Similarly based on the other coupling condition we conclude that $h \in \mathcal{M}^\times$.

Putting the two previous paragraphs together, we conclude that $h \in \mathcal{M} \cap \mathcal{M}^\times$ if and only if it has the form $\mathcal{O}^b_{C,A}x+\mathcal{O}^f_{C^\times,A^\times}y$ satisfying the conditions (4.8) for a pair $(x, y) \in (\mathcal{D}(\Gamma), \mathcal{D}(\Gamma^\times))$. We also write

$$h = \begin{bmatrix} \mathcal{O}^b_{C,A} & -\mathcal{O}^f_{C^\times,A^\times} \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix}.$$

But as we may re-write the coupling conditions (4.8) as

$$\begin{bmatrix} \mathcal{C}^f_{Z^\times,B^\times}\mathcal{O}^b_{C,A} & \Gamma^\times \\ \Gamma & \mathcal{C}^b_{Z,B}\mathcal{O}^f_{C^\times,A^\times} \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we recognize that any pair $(x, -y)$ satisfying the coupling conditions is, in fact, in $\operatorname{Ker}\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$ and conversely.

Finally, then, we exhibit the mapping $\begin{bmatrix} \mathcal{O}^b_{C,A} & -\mathcal{O}^f_{C^\times,A^\times} \end{bmatrix}$ as a bijection from $\operatorname{Ker}\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$

to $\mathcal{M} \cap \mathcal{M}^\times$. In particular, we conclude that

$$\dim \operatorname{Ker} \Gamma_{\mathfrak{S},\mathfrak{S}^\times} = \dim \mathcal{M} \cap \mathcal{M}^\times. \tag{4.9}$$

Now we assume that the pair $(\mathcal{M}, \mathcal{M}^\times)$ forms a direct-sum decomposition of $L^2_{\mathcal{U}}(\mathbb{R})$. As this implies that $\mathcal{M} \cap \mathcal{M}^\times = \{0\}$, we immediately conclude by (4.9) that $\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$ is one-to-one. It therefore suffices to show that $\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$ is onto.

Choose some $h \in L^2_{\mathcal{U}}(\mathbb{R})$. By hypothesis, we may write $h = m^\times + m$ where $m^\times \in \mathcal{M}^\times$ and $m \in \mathcal{M}$; further

$$m^\times = g + \mathcal{O}^f_{C^\times,A^\times} y \quad \text{where } \mathcal{C}^f_{Z^\times,B^\times} g = \Gamma^\times y$$

and

$$m = \mathcal{O}^b_{C,A} x + f \quad \text{where } \mathcal{C}^b_{Z,B} f = \Gamma x.$$

As we may also use the decomposition $h = h_- + h_+$, we conclude

$$h_- = g + \mathcal{O}^b_{C,A} x \quad \text{and} \quad h_+ = \mathcal{O}^f_{C^\times,A^\times} y + f.$$

Applying control operators and re-writing in terms of $\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$, we have

$$\Gamma_{\mathfrak{S},\mathfrak{S}^\times} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{C}^f_{Z^\times,B^\times} & 0 \\ 0 & \mathcal{C}^b_{Z,B} \end{bmatrix} \begin{bmatrix} h_- \\ h_+ \end{bmatrix}. \tag{4.10}$$

As we chose h arbitrarily, we see that

$$\operatorname{Ran}\Gamma_{\mathfrak{S},\mathfrak{S}^\times} \supset \operatorname{Ran}\begin{bmatrix} \mathcal{C}^f_{Z^\times,B^\times} & 0 \\ 0 & \mathcal{C}^b_{Z,B} \end{bmatrix}$$

As the control operators are onto by hypothesis, we conclude that $\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$ is in fact onto.

We turn our attention to the converse; we assume that $\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$ is invertible. Thus $\dim\operatorname{Ker}\Gamma_{\mathfrak{S},\mathfrak{S}^\times} = 0$ and by (4.9) we conclude that $\mathcal{M}\cap\mathcal{M}^\times = \{0\}$. We now choose some $h = h_- + h_+ \in L^2_{\mathcal{U}}(\mathbb{R})$; referring to (4.10) and the preceeding analysis, we see that as $\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$ is invertible, we may solve for unique (x, y) such that

$$h = (\mathcal{O}^b_{C,A}x + f) + (g + \mathcal{O}^f_{C^\times,A^\times})$$

subject to $\mathcal{C}^b_{Z,B}f = \Gamma x$ and $\mathcal{C}^f_{Z^\times,B^\times}g = \Gamma^\times y$. Which is to say, h has the form $m^\times + m$ for $m^\times \in \mathcal{M}^\times$ and $m \in \mathcal{M}$. As h was arbitrary, we conclude that $\mathcal{M}+\mathcal{M}^\times = L^2_{\mathcal{U}}(\mathbb{R})$.

□

Chapter 5

Realization Theorem

We now consider the transfer-function realization question for $L^2_{\mathcal{U}}(i\mathbb{R})$-regular functions. That is, given such a function W, we seek a well-posed linear system which has W as its transfer function. We will make use of the preceding two chapters to do so: that is, we will first use W to generate a dual shift-invariant pair $(\mathcal{M}, \mathcal{M}^\times)$ and then find data sets $(C, A; Z, B; \Gamma)$ and $(C^\times, A^\times; Z^\times, B^\times; \Gamma^\times)$ which represent $(\mathcal{M}, \mathcal{M}^\times)$. We then use these operators to construct our linear system.

We introduce a standard notation for this chapter. Almost exclusively, we will be working with the Laplace transforms of the operators and spaces from the preceding chapters. That is, we will be working in the frequency domain, as opposed to the time domain. Normally we mark the difference with a "hat": for example, if A is an operator on $L^2_{\mathcal{U}}(\mathbb{R})$, we would write $\widehat{A}$ to denote the associated operator on $L^2_{\mathcal{U}}(i\mathbb{R})$ which is the Laplace transform of A. As we work only with the Laplace transform in this chapter, implementing the hat notation would result in an unthinkable proliferation of hats. To avoid this catastrophe, we dispense with the hats entirely.

Our realization formula is written in an unusual format. One reason for this is that we take advantage of the fact that the Hardy spaces $H^2_{\mathcal{U}}(\Pi_+)$ and $H^2_{\mathcal{U}}(\Pi_-)$ are reproducing kernel Hilbert spaces. We therefore discuss these spaces briefly before moving on to the main theorem.

5.1 Reproducing Kernel Hilbert Spaces

Definition 5.1.1. A *Reproducing Kernel Hilbert Space* is a Hilbert space $\mathcal{X}$ whose elements are $\mathcal{U}$-valued functions on a set Ω for which the operation of point evaluation is continuous for each $z \in \Omega$. That is, the mapping $\Lambda_z : \mathcal{X} \to \mathcal{U}$ given by $\Lambda_z f = f(z)$ is continuous.

By the Riesz representation theorem, it follows that there is a unique element $k_z(\cdot) \in \mathcal{X}$ such that, for all z, $\langle f, k_z \rangle_{\mathcal{X}} = f(z)$. We call this element k_z the *Reproducing Kernel* for $\mathcal{X}$.

In particular, we will focus on the Hardy spaces $H^2_{\mathcal{U}}(\Pi_+)$ and $H^2_{\mathcal{U}}(\Pi_-)$ as reproducing kernel Hilbert spaces. The reproducing kernels for these spaces are well known; we state the kernels below and refer the reader to [15] for the proof.

Theorem 5.1.2. *The Hardy space $H^2(\Pi_+)$ is a reproducing kernel Hilbert space with kernel*

$$k_z(\zeta) = \frac{1}{\zeta + \overline{z}},$$

where $z \in \Pi_+$.

Similarly, the Hardy space $H^2(\Pi_-)$ is a reproducing kernel Hilbert space with kernel

$$k_w^{\perp}(\zeta) = -\frac{1}{\zeta + \overline{w}},$$

where $w \in \Pi_-$.

5.2 The General Theorem

Theorem 5.2.1. *Let W be $L^2_{\mathcal{U}}(i\mathbb{R})$-regular (see Definition 3.6.4; define subspaces*

$$\mathcal{M} := L^2\text{-clos}\, M_W \mathcal{Q}_{\mathcal{U},+} \quad \textit{and} \quad \mathcal{M}^{\times} := L^2\text{-clos}\, M_W \mathcal{Q}_{\mathcal{U},-}.$$

Then the pair $(\mathcal{M}, \mathcal{M}^{\times})$ is a dual shift-invariant pair with respect to $\mathfrak{T}^t = M_{e^{st}}$ on $L^2_{\mathcal{U}}(i\mathbb{R})$ (see Definition 3.1.2).

Furthermore, with $\mathcal{M}$ we associate the Π_+-admissible Sylvester data set $\mathfrak{S} := (C, A; Z, B; \Gamma)$ and with $\mathcal{M}^{\times}$ we associate the Π_--admissible Sylvester data set $\mathfrak{S}^{\times} := (C^{\times}, A^{\times}; Z^{\times}, B^{\times}; \Gamma^{\times})$; we make both of these associations according to Theorem 4.2.10. We additionally let $\Gamma_{\mathfrak{S},\mathfrak{S}^{\times}}$ be the coupling matrix from the same Theorem. Then the operator $\mathcal{P} := I - M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1}$ acting on $L^2_{\mathcal{U}}(i\mathbb{R})$ is a (generally nonorthogonal) projection. Further, if we consider $\mathcal{P}$ as a block 2×2 operator on $H^2_{\mathcal{U}}(\Pi_-) \oplus H^2_{\mathcal{U}}(\Pi_+)$ according to the formula

$$\mathcal{P} = \begin{bmatrix} P_{H^2_{\mathcal{U}}(\Pi_-)} \\ P_{H^2_{\mathcal{U}}(\Pi_+)} \end{bmatrix} (I - M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1}) \begin{bmatrix} P_{H^2_{\mathcal{U}}(\Pi_-)} & P_{H^2_{\mathcal{U}}(\Pi_+)} \end{bmatrix}, \qquad (5.1)$$

then $\mathcal{P}$ admits the following realization formula: for all $w', z' \in \Pi_-$, $w, z \in \Pi_+$, and

$u, u', v, v' \in \mathcal{U}$

$$\left\langle \mathcal{P} \begin{bmatrix} k_{w'}^{\perp} u' \\ k_w u \end{bmatrix}, \begin{bmatrix} k_{z'}^{\perp} v' \\ k_z v \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} k_{w'}^{\perp}(z') & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u' \\ u \end{bmatrix}, \begin{bmatrix} v' \\ v \end{bmatrix} \right\rangle_{\mathcal{U} \times \mathcal{U}} - \left\langle \begin{bmatrix} C(z'I-A)^{-1} & 0 \\ 0 & C^{\times}(zI-A^{\times})^{-1} \end{bmatrix} \Gamma_{\mathfrak{S},\mathfrak{S}^{\times}}^{-1} \begin{bmatrix} (\bar{w}'I+Z^{\times})^{-1}B^{\times} & 0 \\ 0 & (\bar{w}I+Z)^{-1}B \end{bmatrix} \begin{bmatrix} u' \\ u \end{bmatrix}, \begin{bmatrix} v' \\ v \end{bmatrix} \right\rangle_{\mathcal{U} \times \mathcal{U}} \tag{5.2}$$

Proof. By our generalized Beurling-Lax-Halmos Theorem 3.6.5, we know that the pair $(\mathcal{M}, \mathcal{M}^{\times})$ is a dual shift-invariant pair and that $L^2_{\mathcal{U}}(i\mathbb{R}) = \mathcal{M} \dot{+} \mathcal{M}^{\times}$.

To this pair $(\mathcal{M}, \mathcal{M}^{\times})$, we may apply our general data representation Theorem 4.2.10—or rather, we apply the Laplace transform thereof. Dispensing with the usual hats to denote the Laplace transform, we thus have a Π_+-admissible Sylvester data set $\mathfrak{S} = (C, A; Z, B; \Gamma)$ such that

$$\mathcal{M} = \{\mathcal{O}^b_{C,A} x + f : x \in \mathcal{D}(\Gamma),\ f \in H^2_{\mathcal{U}}(\Pi_+),\ \mathcal{C}^b_{Z,B} f = \Gamma x\}. \tag{5.3}$$

a Π_--admissible Sylvester data set $\mathfrak{S}^{\times} = (C^{\times}, A^{\times}; Z^{\times}, B^{\times}; \Gamma^{\times})$ such that

$$\mathcal{M}^{\times} = \{g + \mathcal{O}^f_{C^{\times},A^{\times}} y : y \in \mathcal{D}(\Gamma^{\times}),\ g \in H^2_{\mathcal{U}}(\Pi_-),\ \mathcal{C}^f_{Z^{\times},B^{\times}} g = \Gamma^{\times} y\}. \tag{5.4}$$

and we know that the associated coupling matrix $\Gamma_{\mathfrak{S},\mathfrak{S}^{\times}}$ is invertible.

We identify $L^2_{\mathcal{U}}(i\mathbb{R}) = H^2_{\mathcal{U}}(\Pi_-) \oplus H^2_{\mathcal{U}}(\Pi_+)$ and concentrate on the operator $\mathcal{P}$ on $H^2_{\mathcal{U}}(\Pi_-) \oplus H^2_{\mathcal{U}}(\Pi_+)$, which we define to be the projection onto $\mathcal{M}^{\times}$ along $\mathcal{M}$. We exhibit two formulas for $\mathcal{P}$, whence will come the claimed realization formula.

From the proof of Theorem 3.6.5, we know that the operator $M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_{W^{-1}}$ acting on $L^2_{\mathcal{U}}(i\mathbb{R})$ is the projection onto $\mathcal{M}$ along $\mathcal{M}^\times$. It follows that

$$I - M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1}$$

is the projection onto $\mathcal{M}^\times$ along $\mathcal{M}$. We then construct the block 2×2 version of $\mathcal{P}$ according to (5.1). This is the first formula.

To exhibit the second formula, we consider an arbitrary $h \in L^2_{\mathcal{U}}(i\mathbb{R})$; we may of course write $h = h_- + h_+$ with $h_- \in H^2_{\mathcal{U}}(\Pi_-)$ and $h_+ \in H^2_{\mathcal{U}}(\Pi_+)$. Additionally, as $L^2_{\mathcal{U}}(i\mathbb{R}) = \mathcal{M}^\times \dot{+} \mathcal{M}$, there exist $m^\times$ and m so that $h = m^\times + m$. According to Eqs. 5.3 and 5.4, then, there exist $f \in H^2_{\mathcal{U}}(\Pi_+)$, $g \in H^2_{\mathcal{U}}(\Pi_-)$, $x \in \mathcal{X}_{\mathcal{P}}$, and $y \in \mathcal{X}^\times_{\mathcal{P}}$ so that

$$\begin{aligned} &m^\times = g + \mathcal{O}^f_{C^\times, A^\times} y, \text{ with } \mathcal{C}^f_{Z^\times, B^\times} g = \Gamma^\times y, \text{ and} \\ &m = \mathcal{O}^b_{C,A} x + f, \text{ with } \mathcal{C}^b_{Z,B} f = \Gamma x \end{aligned}$$

We thus have the representations $h_- = g + \mathcal{O}^b_{C,A} x$ and $h_+ = \mathcal{O}^f_{C^\times, A^\times} y + f$. For the second projection formula, we seek an operator $\mathcal{P}$ fulfilling the formula

$$\mathcal{P} \begin{bmatrix} h_- \\ h_+ \end{bmatrix} = \begin{bmatrix} g \\ \mathcal{O}^f_{C^\times, A^\times} y \end{bmatrix}.$$

Following the proof of Theorem 4.2.10, as $\Gamma_{\mathfrak{S}, \mathfrak{S}^\times}$ is invertible, we may solve for the pair (x, y) in terms of (h_-, h_+) thus:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \Gamma^{-1}_{\mathfrak{S}, \mathfrak{S}^\times} \begin{bmatrix} \mathcal{C}^f_{Z^\times, B^\times} & 0 \\ 0 & \mathcal{C}^b_{Z,B} \end{bmatrix} \begin{bmatrix} h_- \\ h_+ \end{bmatrix}$$

And from $g = h_- - \mathcal{O}^b_{C,A} x$, we have

$$\begin{bmatrix} g \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h_- \\ h_+ \end{bmatrix} - \begin{bmatrix} \mathcal{O}^b_{C,A} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus we may conclude that

$$\mathcal{P} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -\mathcal{O}^b_{C,A} & 0 \\ 0 & \mathcal{O}^f_{C^\times,A^\times} \end{bmatrix} \Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times} \begin{bmatrix} \mathcal{C}^f_{Z^\times,B^\times} & 0 \\ 0 & \mathcal{C}^b_{Z,B} \end{bmatrix}. \tag{5.5}$$

This is our second formula for $\mathcal{P}$.

We now note that computing the kernel Gramian of the first formula (5.1) leads to the left hand side of our claimed realization (5.2) in Theorem 5.2.1.

To complete our theorem, we use our second representation 5.5 to compute the so-called kernel Gramian. That is, for arbitrary $w', z' \in \Pi_-$, $w, z \in \Pi_+$, and $u, u', v, v' \in \mathcal{U}$, we compute

$$\left\langle \mathcal{P} \begin{bmatrix} k^\perp_{w'} u' \\ k_w u \end{bmatrix}, \begin{bmatrix} k^\perp_{z'} v' \\ k_z v \end{bmatrix} \right\rangle_{H^2(\Pi_-) \oplus H^2_{\mathcal{U}}(\Pi_+)} \tag{5.6}$$

As much due to a lack of space as for the purpose of clarity, we break the computation down into steps. The first term is trivial to compute:

$$\left\langle \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k^\perp_{w'} u' \\ k_w u \end{bmatrix}, \begin{bmatrix} k^\perp_{z'} v' \\ k_z v \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} k^\perp_{w'}(z') u' \\ 0 \end{bmatrix}, \begin{bmatrix} v' \\ v \end{bmatrix} \right\rangle_{\mathcal{U} \times \mathcal{U}}, \tag{5.7}$$

where, of course, the equality follows via the reproducing kernel property. The second term is the interesting one. We have

$$\begin{aligned}
&\left\langle \begin{bmatrix} -\mathcal{O}^b_{C,A} & 0 \\ 0 & \mathcal{O}^f_{C^\times,A^\times} \end{bmatrix} \Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times} \begin{bmatrix} \mathcal{C}^f_{Z^\times,B^\times} k^\perp_{w'} u' \\ \mathcal{C}^b_{Z,B} k_w u \end{bmatrix}, \begin{bmatrix} k^\perp_{z'} v' \\ k_z v \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} \mathcal{C}^f_{Z^\times,B^\times} k^\perp_{w'} u' \\ \mathcal{C}^b_{Z,B} k_w u \end{bmatrix}, \Gamma^{-1,*}_{\mathfrak{S},\mathfrak{S}^\times} \begin{bmatrix} -(\mathcal{O}^b_{C,A})^* & 0 \\ 0 & (\mathcal{O}^f_{C^\times,A^\times})^* \end{bmatrix} \begin{bmatrix} k^\perp_{z'} v' \\ k_z v \end{bmatrix} \right\rangle
\end{aligned} \tag{5.8}$$

We therefore consider inner products of the form $\left\langle \mathcal{C}^b_{Z,B} k_w u, x \right\rangle_{\mathcal{X}_\mathcal{Z}}$, where $x \in \mathcal{X}_\mathcal{Z}$ is arbitrary.

We compute

$$\left\langle \mathcal{C}^b_{Z,B} k_z u, x \right\rangle_{\mathcal{X}_\mathcal{Z}} = \left\langle \left(-\mathcal{O}^f_{B^*,-Z^*} \right)^* k_z u, x \right\rangle_{\mathcal{X}_\mathcal{Z}},$$

as control operators are adjoints of observation operators by Theorem 2.4.10; this then

$$= - \left\langle k_z u, \mathcal{O}^f_{B^*,-Z^*} x \right\rangle_{H^2_\mathcal{U}(\Pi_+)};$$

but we may take advantage of the reproducing kernel property, as well as our explicit formula for (Laplace transforms of) observation operators in Lemma 2.4.13, to see that this

$$\begin{aligned}
&= - \left\langle u, B^*(zI + Z^*)^{-1} x \right\rangle_\mathcal{U} \\
&= - \left\langle (\bar{z} I + Z)^{-1} B u, x \right\rangle_{\mathcal{X}_\mathcal{Z}}.
\end{aligned}$$

A similar computation gives that $\left\langle \mathcal{C}^f_{Z^\times,B^\times} k^\perp_{w'} u', x' \right\rangle = \langle -(\bar{w}' I + Z^\times)^{-1} B^\times u', x' \rangle$.

We may thus rewrite (5.8) as

$$\left\langle \begin{bmatrix} -\mathcal{O}^b_{C,A} & 0 \\ 0 & \mathcal{O}^f_{C^\times,A^\times} \end{bmatrix} \Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times} \begin{bmatrix} -(\bar{w}'I + Z^\times)^{-1}B^\times u' \\ -(\bar{w}I + Z)^{-1}Bu \end{bmatrix}, \begin{bmatrix} k^\perp_{z'}v' \\ k_z v \end{bmatrix} \right\rangle;$$

but in this form, we may explicitly evaluate the reproducing kernel inner products to get

$$\left\langle \begin{bmatrix} C(z'I - A)^{-1} & 0 \\ 0 & C^\times(zI - A^\times)^{-1} \end{bmatrix} \Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times} \begin{bmatrix} -(\bar{w}'I + Z^\times)^{-1}B^\times u' \\ -(\bar{w}I + Z)^{-1}Bu \end{bmatrix}, \begin{bmatrix} v' \\ v \end{bmatrix} \right\rangle; \tag{5.9}$$

Combining equations (5.7) and (5.9) gives the claimed result. □

5.3 The Inner Case

We now consider a special case wherein our L^2-regular function W is in fact *two-sided inner*. This is a very important special case for the reasons of being both illustrative and—perhaps more imporantly—motivating. In particular, as we work out the details of Theorem 5.2.1, we shall recover a much more common form of realization theorem; this motivates us in claiming that said Theorem is in fact a realization theorem. Before considering the details, we first give the definition of inner functions.

Definition 5.3.1. Let $W \in \mathcal{H}^\infty_{\mathcal{L}(\mathcal{U})}(\Pi_+)$ be contraction-valued. Considered as a function on $i\mathbb{R}$, we say that W is

1. *inner* if $W(z)$ is a.e. an isometry,

2. $*$-*inner* if $W(z)$ is a.e. a coisometry,

3. and *two-sided inner* if $W(z)$ a.e. unitary.

Note that W is two-sided inner if and only if it is both inner and $*$-inner.

One can similarly speak of an inner, $*$-inner, or two-sided inner function in $H^2_{\mathcal{L}(\mathcal{U})}(\Pi_-)$.

Remark 5.3.2. We note that if W is (Π_+) two-sided inner, then M_W is a mapping from $H^2_{\mathcal{U}}(\Pi_+)$ into itself.

By hypothesis, $W(z)$ is invertible on the line $i\mathbb{R}$ and equals $W^*(z)$. Noting that for $z \in i\mathbb{R}$ we have $z = -\bar{z}$, we may write

$$W^{-1}(z) = W^*(-\bar{z}). \tag{5.10}$$

This formula allows us to extend W^{-1} analytically to Π_-. Further, via this extension, we have that $W^{-1} \in \mathcal{H}^\infty_{\mathcal{L}(\mathcal{U})}(\Pi_-)$ and that M_W^{-1} maps $H^2_{\mathcal{U}}(\Pi_-)$ into itself.

Corollary 5.3.3. *Let $W \in \mathcal{H}^\infty_{\mathcal{L}(\mathcal{U})}(\Pi_+)$ be two-sided inner. In this case, the realization formula* (5.2) *of Theorem 5.2.1 takes the following form: for all $z, w \in \Pi_+$,*

$$\frac{I_{\mathcal{U}} - W(z)W(w)^*}{z + \bar{w}} = -B^*(zI + Z^*)^{-1}\Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times}(\bar{w}I + Z)^{-1}B. \tag{5.11}$$

Proof. We first note that inner functions are $L^2(i\mathbb{R})$-regular: as W is unitary, for any $u \in \mathcal{U}$, we have that

$$\left\| \frac{1}{z+1} M_W u \right\|_2^2 \leq \|u\|^2_{\mathcal{U}} \int_{i\mathbb{R}} \frac{1}{|z|^2 + 1} dz = \pi \|u\|^2_{\mathcal{U}}.$$

Also as W is unitary, the same estimate holds for $W^{-1} = W^*$. Further, we clearly have that, as an operator on $\mathcal{Q}_{\mathcal{U}}$, $\left\| M_W P_{H^2_{\mathcal{U}}} M_{W^{-1}} \right\| \le 1$; thus $M_W P_{H^2_{\mathcal{U}}} M_{W^{-1}}$ extends to a bounded operator on $L^2_{\mathcal{U}}(i\mathbb{R})$ with image in $L^2_{\mathcal{U}}(i\mathbb{R})$.

Now, due to the structure of $\mathcal{M}$ and $\mathcal{M}^\times$, we can in fact say more about the data sets $\mathfrak{S}$ and $\mathfrak{S}^\times$. That $\mathcal{M} \subset H^2_{\mathcal{U}}(\Pi_+)$ implies that $\mathcal{M}$ is fully represented by Lemma 4.2.4; thus

$$\mathfrak{S} = (0, 0, Z, B, 0)$$

with $\mathcal{X}_{\mathcal{P}} = \{0\}$ and $\mathcal{X}_{\mathcal{Z}} = H^2_{\mathcal{U}}(\Pi_+) \ominus \mathcal{M}$.

Somewhat similarly, we decompose $\mathcal{M}^\times$ as $\mathcal{M}^\times = H^2_{\mathcal{U}}(\Pi_-) \oplus P_{H^2_{\mathcal{U}}(\Pi_+)} \mathcal{M}^\times$; but then $P_{H^2_{\mathcal{U}}(\Pi_+)} \mathcal{M}^\times$, and thus all of $\mathcal{M}^\times$, is fully characterized by Lemma 4.2.1. It follows that

$$\mathfrak{S}^\times = (C^\times, A^\times, 0, 0, 0)$$

with $\mathcal{X}^\times_{\mathcal{P}} = P_{H^2_{\mathcal{U}}(\Pi_+)} \mathcal{M}^\times$ and $\mathcal{X}^\times_{\mathcal{Z}} = \{0\}$.

But we may say even more about $\mathfrak{S}^\times$. Recall that $\mathcal{M} = L^2\text{-clos}\, W \mathcal{Q}_{\mathcal{U},+}$ and that $\mathcal{M}^\times = L^2\text{-clos}\, \mathcal{Q}_{\mathcal{U},-}$. Thus for any $m \in \mathcal{M}$ and $m^\times \in \mathcal{M}^\times$, we have sequences $\{f_i\} \subset \mathcal{Q}_{\mathcal{U},+}$ and $\{g_i\} \subset \mathcal{Q}_{\mathcal{U},-}$ with $W f_i \to m$ and $W g_i \to m^\times$. We consider the $L^2_{\mathcal{U}}(i\mathbb{R})$ inner product $\langle W f_i, W g_j \rangle$, which of course equals $\langle f_i, g_j \rangle$ since W is unitary. By the continuity of the inner product, we may take limits as $i, j \to \infty$ to conclude that $\langle m, m^\times \rangle = 0$. That is, we conclude that $\mathcal{M}^\times = \mathcal{M}^\perp$. But then it follows that $\mathcal{X}^\times_{\mathcal{P}} = P_{H^2_{\mathcal{U}}(\Pi_+)} \mathcal{M}^\times = H^2_{\mathcal{U}}(\Pi_+) \ominus \mathcal{M} = \mathcal{X}_{\mathcal{Z}}$. And thus, by the proof of Lemma 4.2.4, we may conclude that in fact, we have that $C^\times = B^*$ and that $A^\times = -Z^*$, so that

$$\mathfrak{S}^\times = (B^*, -Z^*, 0, 0, 0)$$

Finally, we also thus know that

$$\Gamma_{\mathfrak{S},\mathfrak{S}^\times} = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{C}^b_{Z,B}\mathcal{O}^f_{B^*,-Z^*} \end{bmatrix}.$$

Recall that we also know that $\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$ is invertible; in this case this means that $\left(\mathcal{C}^b_{Z,B}\mathcal{O}^f_{B^*,-Z^*}\right)^{-1}$ exists—although the individual inverses do not—and that

$$\Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times} = \begin{bmatrix} 0 & 0 \\ 0 & \left(\mathcal{C}^b_{Z,B}\mathcal{O}^f_{B^*,-Z^*}\right)^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \left(\left(\mathcal{O}^f_{B^*,-Z^*}\right)^* \mathcal{O}^f_{B^*,-Z^*}\right) \end{bmatrix}.$$

Hereafter, we shall abuse notation slightly and write $\Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times} = -\left(\left(\mathcal{O}^f_{B^*,-Z^*}\right)^* \mathcal{O}^f_{B^*,-Z^*}\right)^{-1}$.

Now we may begin to specialize (5.2). We begin with the right hand side thereof. Applying our knowledge of data sets $\mathfrak{S}$ and $\mathfrak{S}^\times$, we have

$$\left\langle k^\perp_{w'}(z')u', v'\right\rangle_{\mathcal{U}} - \left\langle B^*(zI+Z^*)^{-1}\Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times}(\bar{w}I+Z)^{-1}Bu, v\right\rangle_{\mathcal{U}} \tag{5.12}$$

Considering the left hand side next, we see from (5.2) that $\mathcal{P}$ has, in fact, the block matrix form

$$(I - M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1}) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Writing temporarily $\mathbf{P} = I - M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1}$, we see thus that we have four terms to compute:

$$\underbrace{\left\langle \mathbf{P}k^\perp_{w'}u', k^\perp_{z'}v'\right\rangle}_{1} + \underbrace{\left\langle \mathbf{P}k_w u, k^\perp_{z'}v'\right\rangle}_{2} + \underbrace{\left\langle \mathbf{P}k^\perp_{w'}u', k_z v\right\rangle}_{3} + \underbrace{\left\langle \mathbf{P}k_w u, k_z v\right\rangle}_{4}$$

We take each term in turn.

Considering term 1, we have

$$\left\langle \mathbf{P}k_{w'}^{\perp}u', k_{z'}^{\perp}v' \right\rangle = \langle k_{w'}(z')u', v' \rangle - \left\langle M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1} k_{w'}^{\perp} u, k_{z'}^{\perp}v' \right\rangle .$$

But from Remark 5.3.2, we know that $M_W^{-1}k_{w'}^{\perp} \in H^2_{\mathcal{U}}(\Pi_-)$, so that $0 = P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1} k_{w'}^{\perp}$. Thus we may conclude that

$$\underbrace{\left\langle \mathbf{P}k_{w'}^{\perp}u', k_{z'}^{\perp}v' \right\rangle}_{1} = \langle k_{w'}(z')u', v' \rangle . \tag{5.13}$$

We turn to term 2. We have

$$\left\langle \mathbf{P}k_w u, k_{z'}^{\perp}v' \right\rangle = \left\langle k_w u, k_{z'}^{\perp}v' \right\rangle - \left\langle M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1} k_w u, k_{z'}^{\perp}v' \right\rangle .$$

The first inner product is clearly zero by orthogonality. The second inner product is also zero:

$$\left\langle M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1} k_w u, k_{z'}^{\perp}v' \right\rangle = \left\langle P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1} k_w u, M_W^{-1} k_{z'}^{\perp}v' \right\rangle .$$

Again from Remark 5.3.2, we know that $M_W^{-1}k_{z'}^{\perp} \in H^2_{\mathcal{U}}(\Pi_-)$, thus this inner product is also zero by orthogonality. We conclude

$$\underbrace{\left\langle \mathbf{P}k_w u, k_{z'}^{\perp}v' \right\rangle}_{2} = 0 \tag{5.14}$$

Term 3 comprises two inner products. The first inner product is zero by orthogonality, precisely as it was for term 2. The second inner product is also zero, as

$M_W^{-1}k_{w'}^{\perp} \in H^2_{\mathcal{U}}(\Pi_-)$ just as in term 1. Thus

$$\underbrace{\left\langle \mathbf{P}k_{w'}^{\perp}u', k_z v \right\rangle}_{3} = 0. \tag{5.15}$$

Finally, we compute term 4. We have

$$\begin{aligned} \langle \mathbf{P}k_w u, k_z v\rangle &= \langle k_w(z)u, v\rangle - \left\langle M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1} k_w u, k_z v \right\rangle \\ &= \langle k_w(z)u, v\rangle - \left\langle M_W^{-1} k_w u, P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1} k_z v \right\rangle \end{aligned} \tag{5.16}$$

In order to compute the second inner product, we consider terms of the form $\left\langle M_W^{-1}k_w u, f\right\rangle_{H^2_{\mathcal{U}}(\Pi_+)}$ for arbitrary $f \in H^2_{\mathcal{U}}(\Pi_+)$. By Remark 5.3.2, and inserting the dummy variable of integration ζ, we rewrite this as $\left\langle W^*(-\bar{\zeta})k_w(\zeta)u, f(\zeta)\right\rangle$. But we may rewrite this as $\left\langle k_w(\zeta)u, W(-\bar{\zeta})f(\zeta)\right\rangle$; in this last expression we may take advantage of the reproducing kernel property to get $\langle u, W(-\bar{w})f(w)\rangle_{\mathcal{U}}$. We may now move W back to the left side of the inner product to get $\langle W^{-1}(-\bar{w})u, f(w)\rangle_{\mathcal{U}}$. To this expression, we re-insert the reproducing kernel $k_w(\zeta)$ to conclude that

$$\left\langle M_W^{-1}k_w u, f\right\rangle_{H^2_{\mathcal{U}}(\Pi_+)} = \left\langle k_w(\zeta)W^{-1}(-\bar{w})u, f(\zeta)\right\rangle_{H^2_{\mathcal{U}}(\Pi_+)}.$$

We may thus rewrite (5.16) as

$$\langle \mathbf{P}k_w u, k_z v\rangle = \langle k_w(z)u, v\rangle - \left\langle M_W P_{H^2_{\mathcal{U}}(\Pi_+)} k_w W^{-1}(-\bar{w})u, k_z v \right\rangle$$

But as $k_w \in H^2_{\mathcal{U}}(\Pi_+)$, we may simply evaluate the inner product using the reproducing kernel on the right side to get

$$\langle k_w(z)u, v\rangle - \left\langle k_w(z)W(z)W^{-1}(-\bar{w})u, v\right\rangle;$$

we combine the two inner products to finally conclude that

$$\underbrace{\langle \mathbf{P}k_w u, k_z v\rangle}_{4} = \left\langle k_w(z)\left(I_{\mathcal{U}} - W(z)W^{-1}(-\bar{w})\right)u, v\right\rangle. \tag{5.17}$$

Thus the left hand side of (5.2) is computed by adding equations (5.13), (5.14), (5.15), and (5.17):

$$\left\langle k_{w'}^{\perp}(z')u', v'\right\rangle + \langle k_w(z)u, v\rangle - \left\langle k_w(z)W(z)W^{-1}(-\bar{w})u, v\right\rangle. \tag{5.18}$$

We have computed the right hand side of our equation as (5.12) and the left hand side as (5.18); setting these two expressions equal to each other gives—after cancellation of a common term—

$$\left\langle k_w(z)\left(I_{\mathcal{U}} - W(z)W^{-1}(-\bar{w})\right)u, v\right\rangle = -\left\langle B^*(zI + Z^*)^{-1}\Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times}(\bar{w}I + Z)^{-1}Bu, v\right\rangle.$$

Recalling the formula for $k_w(z)$ (cf. Theorem 5.1.2) and that $W^{-1}(-\bar{w}) = W^*(w)$, we conclude that

$$\frac{I_{\mathcal{U}} - W(z)W^*(w)}{z + \bar{w}} = -B^*(zI + Z^*)^{-1}\Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times}(\bar{w}I + Z)^{-1}B,$$

in exact agreement with the claimed equation (5.11).

□

Remark 5.3.4. We note that $\Gamma_{\mathfrak{G},\mathfrak{G}^\times}^{-1} = -\left(\left(\mathcal{O}^f_{B^*,-Z^*}\right)^* \mathcal{O}^f_{B^*,-Z^*}\right)^{-1} < 0$. Thus our factorization

$$\frac{I_{\mathcal{U}} - W(z)W^*(w)}{z+\bar{w}} = B^*(zI+Z^*)^{-1}(-\Gamma_{\mathfrak{G},\mathfrak{G}^\times}^{-1})(\bar{w}I+Z)^{-1}B$$

exhibits the deBranges-Rovnyak kernel

$$\frac{I_{\mathcal{U}} - W(z)W^*(w)}{z+\bar{w}}$$

as a positive kernel.

Remark 5.3.5. We note that our approach to the realization of a two-sided inner function complements that of Jacob-Zwart in [16].

5.4 The Wiener-Hopf Case

We consider a second illustrative special case wherein our representing function W has all zeroes and poles contained within Π_+. This case has applications to the construction of Wiener-Hopf factorizations; hence the name of this case.

With great similarity to the Inner Case, we will find that the Sylvester data sets are degenerate. This leads to a great simplification of our realization formula and allows for explicit computation of the kernel Gramians.

Corollary 5.4.1. *Let $W \in \mathcal{H}^\infty_{\mathcal{L}(\mathcal{U})}(\Pi_-)$ be invertible with $W^{-1} \in \mathcal{H}^\infty_{\mathcal{L}(\mathcal{U})}(\Pi_-)$ as well. Then the realization formula* (5.2) *of Theorem 5.2.1 takes the following form: for all*

$w \in \Pi_+$ and $z' \in \Pi_-$,

$$\frac{I_{\mathcal{U}} - W(z')W(-\bar{w})^{-1}}{z' + \bar{w}} = -C(z'I - A)^{-1}\Gamma^{-1}(\bar{w}I + Z)^{-1}B. \tag{5.19}$$

Proof. As both W and W^{-1} are bounded, it follows that W is $L^2(i\mathbb{R})$-regular. The argument is essentially the same as that in the first paragraph of the proof of Corollary 5.3.

Next, we note that $\mathcal{M}^\times = H^2_{\mathcal{U}}(\Pi_-)$. This fact follows immediately from the representation $\mathcal{M}^\times = L^2\text{-clos}\, W\mathcal{Q}_{\mathcal{U},-}$. As $W \in \mathcal{H}^\infty_{\mathcal{L}(\mathcal{U})}(\Pi_-)$, we may pass the closure past W to get $\mathcal{M}^\times = WH^2_{\mathcal{U}}(\Pi_-)$. But as $W^{-1} \in \mathcal{H}^\infty_{\mathcal{L}(\mathcal{U})}(\Pi_-)$ as well, we may say that $WH^2_{\mathcal{U}}(\Pi_-) = H^2_{\mathcal{U}}(\Pi_-)$.

Having identified $\mathcal{M}^\times$, we may construct the data set $(C^\times, A^\times; Z^\times, B^\times; \Gamma^\times)$. We first consider $P_{H^2_{\mathcal{U}}(\Pi_+)}\mathcal{M}^\times = \{0\}$. We then construct the pair $(C^\times, A^\times)$ by Lemma 4.2.1; we conclude that $C^\times = 0$. In this case, we may as well also take $A^\times = 0$. We construct the pair $(Z^\times, B^\times)$ by considering $\mathcal{M}^\times \cap H^2_{\mathcal{U}}(\Pi_-) = H^2_{\mathcal{U}}(\Pi_-)$. By Lemma 4.2.8, we have that $\operatorname{Ker} \mathcal{C}^f_{Z^\times,B^\times} = H^2_{\mathcal{U}}(\Pi_-)$; as this is the entire space, we conclude that $\mathcal{C}^f_{Z^\times,B^\times} = 0$. It follows that $B^\times = 0$, in which case we may take $Z^\times = 0$ as well. Finally, the choice $\Gamma^\times = 0$ suffices to complete our degenerate data set $(C^\times, A^\times; Z^\times, B^\times; \Gamma^\times) = (0, 0; 0, 0; 0)$.

We thus also know that

$$\Gamma_{\mathfrak{S},\mathfrak{S}^\times} = \begin{bmatrix} 0 & 0 \\ \Gamma & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{D}(\Gamma) \\ \{0\} \end{bmatrix} \to \begin{bmatrix} \{0\} \\ \mathcal{X}_{\mathcal{P}} \end{bmatrix}.$$

In this case, we interpret the invertibility of $\Gamma_{\mathfrak{S},\mathfrak{S}^\times}$ to mean that Γ^{-1} exists. We further identify $\Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times}$ as

$$\Gamma^{-1}_{\mathfrak{S},\mathfrak{S}^\times} = \begin{bmatrix} 0 & \Gamma^{-1} \\ 0 & 0 \end{bmatrix}$$

We may now specialize equation (5.2). We begin with the right-hand side, which we write as

$$\left\langle k^{\perp}_{w'}(z')u', v'\right\rangle - \left\langle C(z'I - A)^{-1}\Gamma^{-1}(\bar{w}I + Z)^{-1}Bu, v'\right\rangle \tag{5.20}$$

We turn our attention to the left-hand side of (5.2). As in the inner case, writing $\mathbf{P} = I - M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1}$, we recognize that we must compute four terms:

$$\underbrace{\left\langle \mathbf{P}k^{\perp}_{w'}u', k^{\perp}_{z'}v'\right\rangle}_{1} + \underbrace{\left\langle \mathbf{P}k_w u, k^{\perp}_{z'}v'\right\rangle}_{2} + \underbrace{\left\langle \mathbf{P}k^{\perp}_{w'}u', k_z v\right\rangle}_{3} + \underbrace{\left\langle \mathbf{P}k_w u, k_z v\right\rangle}_{4}$$

We begin with term 1.

$$\underbrace{\left\langle \mathbf{P}k^{\perp}_{w'}u', k^{\perp}_{z'}v'\right\rangle}_{1} = \left\langle k^{\perp}_{w'}(z')u', v'\right\rangle - \left\langle M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1}u', v'\right\rangle = \left\langle k^{\perp}_{w'}(z')u', v'\right\rangle, \tag{5.21}$$

as M_W^{-1} maps $H^2_{\mathcal{U}}(\Pi_-)$ onto $H^2_{\mathcal{U}}(\Pi_-)$.

Term 2 turns out to hold the most interest for us. We compute

$$\underbrace{\left\langle \mathbf{P}k_w u, k^{\perp}_{z'}v'\right\rangle}_{2} = \left\langle k_w u, k^{\perp}_{z'}v'\right\rangle - \left\langle M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1}k_w u, k_{z'}v'\right\rangle. \tag{5.22}$$

The first inner product is zero by orthogonality. To compute the second inner product, we first focus on terms of the form $\langle W^{-1}(\zeta)k_w(\zeta)u, f(\zeta)\rangle$ for arbitrary

$f \in H^2_{\mathcal{U}}(\Pi_+)$. Then

$$\begin{aligned}\left\langle W(\zeta)^{-1}k_w(\zeta)u, f(\zeta)\right\rangle &= \left\langle k_w(\zeta)u, \left(W(\zeta)^{-1}\right)^* f(\zeta)\right\rangle_{L^2_{\mathcal{U}}(i\mathbb{R})} \\ &= \left\langle k_w(\zeta)u, \left(W(-\bar{\zeta})^*\right) f(\zeta)\right\rangle_{L^2_{\mathcal{U}}(i\mathbb{R})}\end{aligned}$$

where we now recognize that $\left(W(-\bar{\zeta})^{-1}\right)^* \in \mathcal{H}^\infty_{\mathcal{L}(\mathcal{U})}(\Pi_+)$, so that

$$\begin{aligned}&= \left\langle k_w(\zeta)u, \left(W(-\bar{\zeta})^{-1}\right)^* f(\zeta)\right\rangle_{H^2_{\mathcal{U}}(\Pi_+)} \\ &= \left\langle u, \left(W(-\bar{w})^{-1}\right)^* f(w)\right\rangle_{\mathcal{U}} \\ &= \left\langle W(-\bar{w})^{-1}u, f(w)\right\rangle_{\mathcal{U}} \\ &= \left\langle k_w(\zeta)W(-\bar{w})^{-1}u, f(\zeta)\right\rangle_{H^2_{\mathcal{U}}(\Pi_+)}\end{aligned}$$

We now recognize that (5.22) may be rewritten as

$$-\left\langle W(\zeta)k_w(\zeta)W(-\bar{w})^{-1}u, k^\perp_{z'}v'\right\rangle. \tag{5.23}$$

We consider therefore the projection of $M_W k_w$ onto $H^2_{\mathcal{U}}(\Pi_-)$. We note that

$$M_W k_w u = \frac{W(\zeta)}{\zeta+\bar{w}}u = \frac{W(\zeta)-W(-\bar{w})}{\zeta+\bar{w}}u + \frac{W(-\bar{w})}{\zeta+\bar{w}}u.$$

The second term equals $k_w(\zeta)W(-\bar{w})u$ and is thus in $H^2_{\mathcal{U}}(\Pi_+)$. The first term is in $H^2_{\mathcal{U}}(\Pi_-)$, as the numerator is analytic in Π_- and zero at $\zeta = -\bar{w}$, canceling the pole.

We may finally thus rewrite (5.23) as

$$\begin{aligned}
&- \left\langle (W(\zeta) - W(-\bar{w}))k_w(\zeta)W(-\bar{w})^{-1}u, k_{z'}^{\perp}v' \right\rangle - \left\langle k_w(\zeta)W(-\bar{w})W(-\bar{w})^{-1}u, k_{z'}^{\perp}(\zeta) \right\rangle \\
&= - \left\langle (W(z') - W(-\bar{w}))k_w(z')W(-\bar{w})^{-1}u, v' \right\rangle_{\mathcal{U}} \\
&= \left\langle \left(I - W(z')W(-\bar{w})^{-1}\right) k_w(z')u, v' \right\rangle_{\mathcal{U}}
\end{aligned}$$

We finally conclude therefore that

$$\underbrace{\left\langle \mathbf{P}k_w u, k_{z'}^{\perp}v' \right\rangle}_{2} = \left\langle \left(I - W(z')W(-\bar{w})^{-1}\right) k_w(z')u, v' \right\rangle_{\mathcal{U}}. \tag{5.24}$$

We next turn our attention to term 3. We have that

$$\underbrace{\left\langle \mathbf{P}k_{w'}^{\perp}u', k_z v \right\rangle}_{3} = \left\langle k_{w'}^{\perp}u', k_z v \right\rangle - \left\langle M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1} k_{w'}^{\perp} u, k_z v \right\rangle = 0. \tag{5.25}$$

That the first inner product is zero is a consequence of the orthogonality of $H^2_{\mathcal{U}}(\Pi_-)$ and $H^2_{\mathcal{U}}(\Pi_+)$. The second inner product is zero follows from that fact that M_W^{-1} maps $H^2_{\mathcal{U}}(\Pi_-)$ onto $H^2_{\mathcal{U}}(\Pi_+)$.

Finally, we conclude with term 4. We have

$$\langle \mathbf{P}k_w u, k_z v \rangle = \langle k_w(z)u, v \rangle - \left\langle M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1} k_w u, k_z v, . \right\rangle$$

In order to compute this, we consider terms of the form $\left\langle M_W^{-1} k_w u, f \right\rangle$ for arbitrary

$f \in H^2_{\mathcal{U}}(\Pi_+)$. Then

$$\begin{aligned}\left\langle M_W^{-1} k_w u, f\right\rangle &= \left\langle W(\zeta)^{-1} k_w(\zeta) u, f(\zeta)\right\rangle \\ &= \left\langle k_w(\zeta) u, \left(W(\zeta)^{-1}\right)^* f(\zeta)\right\rangle_{L^2_{\mathcal{U}}(i\mathbb{R})} \\ &= \left\langle k_w(\zeta) u, \left(W(-\bar{\zeta})^{-1}\right)^* f(\zeta)\right\rangle_{L^2_{\mathcal{U}}(i\mathbb{R})};\end{aligned}$$

but we may identify this last inner product as actually taking place in $H^2_{\mathcal{U}}(\Pi_+)$ so that we may take advantage of the reproducing kernel property to get

$$\begin{aligned}&= \left\langle u, \left(W(-\bar{w})^{-1}\right)^* f(w)\right\rangle_{\mathcal{U}} \\ &= \left\langle W(-\bar{w})^{-1} u, f(w)\right\rangle_{\mathcal{U}} \\ &= \left\langle k_w(\zeta) W(-\bar{w})^{-1} u, f(\zeta)\right\rangle_{H^2_{\mathcal{U}}(\Pi_+)}.\end{aligned}$$

We may thus rewrite $\left\langle M_W P_{H^2_{\mathcal{U}}(\Pi_+)} M_W^{-1} k_w u, k_z v\right\rangle = \langle W(\zeta) k_w(\zeta) W(-\bar{w})^{-1} u, k_z(\zeta) v\rangle$. We now compute, using a similar argument,

$$\begin{aligned}\left\langle W(\zeta) k_w(\zeta) W(-\bar{w})^{-1} u, k_z(\zeta) v\right\rangle &= \left\langle k_w(\zeta) W(-\bar{w})^{-1} u, W(\zeta)^* k_z(\zeta) v\right\rangle_{L^2_{\mathcal{U}}(i\mathbb{R})} \\ &= \left\langle k_w(\zeta) W(-\bar{w})^{-1} u, W(-\bar{\zeta})^* k_z(\zeta)\right\rangle_{H^2_{\mathcal{U}}(\Pi_+)} \\ &= \left\langle W(-\bar{w})^{-1} u, W(-\bar{w})^* k_z(w)\right\rangle_{\mathcal{U}} \\ &= \langle u, k_z(w) v\rangle_{\mathcal{U}} \\ &= \langle k_w(z) u, v\rangle_{\mathcal{U}}.\end{aligned}$$

Finally, then, we may conclude that

$$\underbrace{\langle \mathbf{P} k_w u, k_z v\rangle}_{4} = 0. \tag{5.26}$$

Putting together equations (5.21), (5.24), (5.25), and (5.26), we conclude that the left-hand side of (5.2) equation equals

$$\left\langle k_{w'}^{\perp}(z')u', v'\right\rangle_{\mathcal{U}} + \left\langle \left(I - W(z')W(-\bar{w})^{-1}\right) k_w(z')u, v'\right\rangle_{\mathcal{U}} \tag{5.27}$$

Combining the right-hand side of (5.2) in (5.20) with the left-hand side in (5.27) gives our realization formula

$$\left\langle \left(I - W(z')W(-\bar{w})^{-1}\right) k_w(z')u, v'\right\rangle_{\mathcal{U}} = -\left\langle C(z'I - A)^{-1}\Gamma^{-1}(\bar{w}I + Z)^{-1}Bu, v'\right\rangle$$

□

Remark 5.4.2. We note that in the rational matrix case—i.e., where W is a rational matrix—a realization formula of the form

$$\frac{I - W(z')W(w)^{-1}}{z - w} = C(zI - A)^{-1}\Gamma^{-1}(wI - Z)^{-1}B,$$

i.e., formula (5.19) with w in place of $-\bar{w}$, and where $(C, A; Z, B; \Gamma)$ is a so-called global null-pole triple for W in the terminology of [1], was observed and developed by Katsnelson [18].

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