On Nearly Euclidean Thurston Maps

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(ABSTRACT)

Nearly Euclidean Thurston maps are simple generalizations of rational Lattès maps. A Thurston map is called *nearly Euclidean* if its local degree at each critical point is 2 and it has exactly four postcritical points. We investigate when such a map has the property that the associated pullback map on Teichmüller space is constant. We also show that no Thurston map of degree 2 has constant pullback map.

Dedication

To my son Christopher.

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Contents

List of Figures

Chapter 1

Introduction

Given a Gromov hyperbolic group G whose space at infinity is the 2-sphere, does G act properly discontinuously, cocompactly, and isometrically on hyperbolic 3-space? J. Cannon has proposed an ingenious and powerful general approach to this conjecture. Suppose that G is a Gromov hyperbolic group whose space at infinity is a 2-sphere. Let Γ be a locally finite Cayley graph for G, and let o be a vertex of Γ. Then, given a geodesic ray $F : [0, \infty) \to \Gamma$ with $F(0) = o$ and a positive integer n, one can define a disk at infinity $D(F, n)$ corresponding to the half-space of points closer to the tail of the ray than to the initial segment of the ray. More precisely,

$$D(F, n) = \{[S] \in \partial\Gamma : \lim_{r\to\infty} d(S(r), \Gamma \setminus H(F, n)) = \infty\}$$

where

$$H(F, n) = \{x \in \Gamma : d(x, F([n, \infty))) \leq d(x, F([0, n]))\}.$$

For each positive integer n, the collection

$$\mathcal{D}(n) = \{D(F, n) : F \text{ is a geodesic ray in } \Gamma \text{ with } F(0) = o\}$$

is a finite cover of the space at infinity of G. One can then try to understand the internal combinatorics of the way these families of disks ("shingles") cover the 2-sphere. In [5], J. Cannon and E. Swenson prove that G acts properly discontinuously, cocompactly, and isometrically on hyperbolic 3-space if and only if the sequence $\{\mathcal{D}(n)\}_{n=1}^{\infty}$ of disk covers satisfies the conformality axioms formulated by J. Cannon in [4]. The authors also show in [5] that for every integer $n \geq 2$ the elements of $\mathcal{D}(n)$ can be obtained from the elements of $\mathcal{D}(n-1)$ by a finite recursion.

Finite subdivision rules were developed to give models for the above sequences of disk covers. A *finite subdivision rule* $\mathcal{R}$ consists of a finite CW complex $S_{\mathcal{R}}$ which is the union of its closed 2-cells, a subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$, and a continuous map $\phi_{\mathcal{R}} : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$ whose restriction to every open cell is a homeomorphism onto an open cell. Furthermore,

$S_{\mathcal{R}}$ must have the property that for each closed 2-cell $\tilde{t}$ of $S_{\mathcal{R}}$, there is a cell structure t on a 2-disk such that t has at least three vertices, all vertices and edges of t are in ∂t, and the characteristic map $\psi_t : t \to S_{\mathcal{R}}$ takes each open cell homeomorphically onto an open cell.

Finite subdivision rules are strongly related to postcritically finite branched covering maps on a 2-sphere. The reason is the following. If $S_{\mathcal{R}}$ is a closed surface, then $\phi_{\mathcal{R}}$ is a branched covering map. Due to the Riemann-Hurwitz formula, if $S_{\mathcal{R}}$ is connected and orientable, then either (i) $\phi_{\mathcal{R}}$ is a homeomorphism, (ii) $S_{\mathcal{R}}$ is a torus and $\phi_{\mathcal{R}}$ is a covering map, or (iii) $S_{\mathcal{R}}$ is a 2-sphere. If $\mathcal{R}$ properly subdivides any tile type, then case (i) cannot occur. In [8], Example 1.3.2 shows that case (ii) can occur. In [7], Example 4.3 shows that case (iii) can occur. This example is known as the binary square subdivision rule with two tile types $\mathcal{L}$ on the 2-sphere. In this example, the rational functions that realize $\phi_{\mathcal{L}}$ include a classical example of a rational map whose Julia set is the 2-sphere. Our attention is focused on the case (iii). We wish to understand for that case when $\phi_{\mathcal{R}}$ can be realized by a rational map.

In a CBMS Conference in 1983, William Thurston showed a necessary and sufficient condition for a postcritically finite branched covering map f on a 2-sphere to be equivalent to a rational map, where $f \sim g$ if there exist homeomorphisms $h_0 : (S^2, P_f) \to (S^2, P_g)$ and $h_1 : (S^2, P_f) \to (S^2, P_g)$ for which $h_0 \circ f = g \circ h_1$ and h_0 is isotopic, rel P_f, to h_1. Thurston did not publish his proofs of the theorems, but fortunately for us proofs were given later by Adrian Douady and John Hubbard in [10]. His main theorems were 1) that a Thurston map f is equivalent to a rational map iff Σ_f (the Teichmüller map induced by f) has a fixed point, and 2) if (S^2, ν_f) is hyperbolic, then Σ_f has a fixed point iff there are no Thurston obstructions(for precise definitions, see Chapter 2). The following phenomenon is an example of an obstruction. Suppose f is a rational Thurston map and $A \subset \overline{\mathbb{C}} \setminus P_f$ is an essential annulus containing, in each complementary component, at least two points of P_f. Suppose $f^{-1}(A)$ contains connected components A_i which are essential subannuli of A mapping by degree d_i onto A. The Grötzsch inequality asserts that $\sum_i \operatorname{mod}(A_i) \leq \operatorname{mod}(A)$ with equality iff the A_i are right Euclidean subcylinders in the canonical Euclidean metric on A. This implies that $\sum_i \frac{1}{d_i} \leq 1$ with equality if and only if f is a flexible Lattès map. This gives conditions on the degrees d_i which must be satisfied by a rational map. This condition can be phrased in terms of how homotopy classes of simple closed curves behave under backward iteration of a Thurston map. In principle, it is possible to check this criterion if the map is given by a subdivision rule. In practice, the sufficiency part is hard to verify, since it involves the behavior of the map on infinitely many homotopy classes of curves in the complement of the postcritical set.

Much about Thurston obstructions remains mysterious. While producing an obstruction for a specific example may not be difficult, no one has yet discovered an algorithm for determining the existence or non-existence of an obstructing multicurve in the general setting. The Berstein-Levy theorem for hyperbolic topological polynomials (see [15]) remains the best result for a non-existence of an obstruction: If the forward orbit of every critical point contains a critical point, then the topological polynomial is equivalent to a complex polynomial.

In the hyperbolic case, Thurston's pullback map is very complicated —it is transcendental— and in all known examples it is infinite-to-one. To better understand Thurston's pullback map and Thurston obstructions, Cannon-Floyd-Parry-Pilgrim [9] have introduced a nice class of Thurston maps called nearly Euclidean Thurston maps. A Thurston map f is nearly Euclidean if $|P_f| = 4$ and the local degree $\deg(f, x) \leq 2$ for all $x \in S^2$. These maps are simple generalizations of Lattès maps but yet nontrivial. In Chapter 3 we present definitions and basic facts concerning nearly Euclidean Thurston maps. In Chapter 3 we also present our main example. The subdivision map of the main example is denoted by f. In this example it is not immediately clear how many components there are in $f^{-1}(\gamma)$, or what the slope (see Chapter 4 for the definition) is of any essential nonperipheral component of $f^{-1}(\gamma)$ for any essential nonperipheral curve γ in $S^2 \setminus P_f$ (see Chapter 2 for definitions). A big reason for the difficulty in finding slopes of preimages is the "backtracking" that occurs near the two points that map to postcritical points but are not critical points or postcritical points.

Computing Thurston matrices involves degrees and numbers of essential components of pullbacks of invariant multicurves. These degrees and numbers of components are described for nearly Euclidean maps in Chapter 3. In Chapter 4 we prove that our main example is combinatorially equivalent to a rational map and we also provide the location of the fixed point of the Thurston pullback map in the Teichmüller space.

In Chapter 5 we investigate when a nearly Euclidean Thurston map has the property that the associated pullback map on Teichmüller space is constant. In [9], Cannon et al. provide a characterization of those nearly Euclidean Thurtson maps whose induced maps on Teichmüller space are constant. Theorem 10.2 of [9], reduces this characterization to a purely algebraic problem concerning finite Abelian groups generated by two elements. Our main result is focused on this algebraic problem. In Appendix B, we show that no Thurston map of degree 2 has constant pullback map. Part of this proof relies on the fact that there does not exist a nearly Euclidean Thurston map with degree 2 whose Teichmüller map is constant.

Proposition 5.1 of [1], essentially due to Curtis McMullen, shows that a Thurston map f has constant pullback map if it factors through a trivial Teichmüller space. The question at the end of Section 5.1 of [1] asks, in part, whether every Thurston map f whose Teichmüller map is constant is as in Proposition 5.1 of [1]. In Chapter 5 and Appendix D we construct examples where the method of Proposition 5.1 of [1] does not apply.

Chapter 2

Preliminaries

2.1 Background

We first establish some notation and standard definitions. All maps in this thesis will be orientation preserving. We use S^2 to denote an oriented topological 2-sphere; and specifically, we use $\overline{\mathbb{C}}$ to denote the Riemann sphere. Let $f : S^2 \to S^2$ be an orientation-preserving branched map of degree d, $d \geq 2$, and let Ω_f be its set of critical points. According to the Riemann-Hurwitz formula, f has $2d-2$ critical points, counted with multiplicity. We define the *postcritical set of f* to be

$$P_f := \bigcup_{n>0} f^{\circ n}(\Omega_f),$$

i.e., P_f is the union of all the forward orbits of the set of critical points under f. If P_f is finite, we call f a *Thurston map*. An obvious example would be any postcritically finite rational map on the Riemann sphere with degree at least two. Another source of examples comes from oriented finite subdivision rules with edge-pairings; they have been extensively studied since they provide insight into how metric conformal structures arise as limits of discrete structures; see [7],[8]. A third way to produce examples of Thurston maps is by mating two critically finite polynomials [16],[20],[14]. Having shown examples of Thurston maps, we define Thurston equivalence, invariant multicurve, and hyperbolic orbifold, and then state Thurston's theorem.

Two Thurston maps f and g are called *Thurston equivalent* iff there exist homeomorphisms $h_0 : (S^2, P_f) \to (S^2, P_g)$ and $h_1 : (S^2, P_f) \to (S^2, P_g)$ for which $h_0 \circ f = g \circ h_1$ and h_0 is isotopic, rel P_f, to h_1. In this case, if g is a rational map we also say that f is *realized* by g. In particular, we have the following commutative diagram:

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{h_1} & (S^2, P_g) \\ f\Big\downarrow & & \Big\downarrow g \\ (S^2, P_f) & \xrightarrow{h_0} & (S^2, P_g). \end{array}$$

Let f be a Thurston map with postcritical set P_f. A simple closed curve γ is *nonperipheral* if each connected component of $S^2 \setminus \gamma$ contains no fewer than two points of P_f. A simple closed curve γ in $S^2 \setminus P_f$ is *peripheral* if it can be shrunk to a point in P_f. An *f-invariant multicurve* is a finite set Γ of simple, closed, disjoint, essential, nonperipheral, non-homotopic curves in $S^2 \setminus P_f$ such that for each $\gamma \in \Gamma$, each component of $f^{-1}(\gamma)$ is either null-homotopic, peripheral or homotopic in $S^2 \setminus P_f$ to an element of Γ. (By essential we mean not null-homotopic). The Thurston matrix $A^\Gamma : \mathbb{R}^{|\Gamma|} \to \mathbb{R}^{|\Gamma|}$, for an invariant multicurve Γ, is defined in coordinates by

$$A^\Gamma_{\gamma\delta} = \sum_\alpha \frac{1}{\deg(f : \alpha \to \delta)}$$

where the sum is taken over components α of $f^{-1}(\delta)$ which are isotopic to γ in $S^2 \setminus P_f$. The matrix A^Γ has non-negative entries, so there is a leading eigenvalue which is real and positive by the Perron-Frobenius theorem. Let $\lambda(\Gamma)$ denote this leading eigenvalue. The orbifold $\mathcal{O}_f$ associated to f is the topological orbifold whose underlying space is S^2 and whose weight function $\nu_f(x)$ at x is given by $\nu_f(x) = \operatorname{lcm}\{\deg(f^{\circ n}, y) : f^{\circ n}(y) = x\}$; here $\deg(\cdot)$ denotes the local degree. The Euler characteristic of $\mathcal{O}_f$ is

$$\chi(\mathcal{O}_f) = 2 - \sum_{x \in P_f} \Big(1 - \frac{1}{\nu_f(x)}\Big).$$

From the definition of the weight function ν_f, $\nu_f(x) > 1$ if and only if $x \in P_f$. Also, note that the definition of the Euler characteristic of the orbifold is similar to the definition of the Euler characteristic of the underlying space except that the contribution of a vertex to the Euler characteristic is $1/\nu_f(x)$ instead of 1. The orbifold $\mathcal{O}_f$ is said to be *hyperbolic* if and only if $\chi(\mathcal{O}_f) < 0$. The following theorem is *Thurston's topological characterization of rational maps*; a proof can be found in [10].

Theorem 2.1.1. (Thurston). *A Thurston map f with hyperbolic orbifold is equivalent to a rational map if and only if for any f-invariant multicurve Γ, $\lambda(\Gamma) < 1$. In that case, the rational map is unique up to conformal conjugation.*

An f-invariant multicurve with $\lambda(\Gamma) \geq 1$ is called a Thurston obstruction. If the Thurston map f is not equivalent to a rational map, then f is said to be *obstructed*.

2.2 Thurston's pullback map

Let f be a Thurston map and let $\mathcal{T}(\mathcal{O}_f)$ be the Teichmüller space of $\mathcal{O}_f = (S^2, \nu_f)$. We may consider the space $\mathcal{T}(\mathcal{O}_f)$ as the space of complex structures on $\mathcal{O}_f$, up to the equivalence of isotopy fixing the elements of P_f. A complex structure on $\mathcal{O}_f$ pulls back under f to a complex structure on $(S^2, f^{-1}(\nu_f))$ and this extends to a complex structure on $\mathcal{O}_f$. In this way we get a map $\Sigma_f : \mathcal{T}(\mathcal{O}_f) \to \mathcal{T}(\mathcal{O}_f)$. Since it is obtained by lifting complex structures under f, we say that Σ_f is the *Thurston pullback map* induced by f. The map Σ_f is holomorphic (see Proposition 4.1 of [2]) and does not increase Teichmüller distances. In Proposition 3.3 of [10], Douady and Hubbard show that $\Sigma_f^{\circ 2}$ strictly decreases the distances between points in $\mathcal{T}(\mathcal{O}_f)$ if $\mathcal{O}_f$ is hyperbolic.

Theorem 2.2.1. (Thurston). *An orientation-preserving postcritically finite branched map is equivalent to a rational map if and only if Σ_f has a fixed point.*

In Section 9 of [10], Douady and Hubbard show that for any Thurston map f, $\chi(\mathcal{O}_f) \leq 0$. If O_f is not hyperbolic, then $\mathcal{T}(\mathcal{O}_f)$ is a single point unless the map f has exactly 4 postcritical points and $\nu_f(x) = 2$ for all $x \in P_f$. In that case, O_f is double-covered by a torus T_f, f lifts to a covering map of this torus, and there is a 2×2 matrix A_f which represents the induced map on homology $H_1(T_f, \mathbb{Z}) \to H_1(T_f, \mathbb{Z})$. In this case, Thurston's characterization is the following.

Theorem 2.2.2. (Thurston). *A Thurston map with orbifold $(2, 2, 2, 2)$ is equivalent to a rational map if and only if the eigenvalues of A_f are not real, or if A_f is a multiple of the identity.*

2.3 Teichmüller Theory

A Riemann surface is a connected oriented topological surface together with a *complex structure*: a maximal atlas of charts $\phi : U \to \mathbb{C}$ with holomorphic overlap maps. For a given oriented, compact topological surface X, let $\mathcal{C}(X)$ be the set of all complex structures on X. For any orientation-preserving homeomorphism $\psi : X \to X$ there is an induced map $\psi^* : \mathcal{C}(X) \to \mathcal{C}(X)$ defined by

$$\psi^*(c) = \{\phi \circ \psi : \psi^{-1}(U) \to \mathbb{C}\}$$

where c is the complex structure on X, $c = \{\phi : U \to \mathbb{C}\}$. It is furthermore true that any orientation-preserving branched covering map $f : X \to Y$ induces a map $f^* : \mathcal{C}(Y) \to \mathcal{C}(X)$, see Section 1 of [2].

Let $A \subset X$ be finite. The Teichmüller space of (X, A) is

$$\mathrm{Teich}(X, A) := \mathcal{C}(X) / \sim_A$$

where $c_1 \sim_A c_2$ if and only if $c_1 = \psi^*(c_2)$ for some orientation-preserving homeomorphism $\psi : X \to X$ which is the identity on A, and which is isotopic to the identity relative to A. Let $f : X \to Y$ be an orientation-preserving branched covering map. In view of the homotopy-lifting property, if

- $B \subset Y$ is finite and contains the critical value set V_f of f, and
- $A \subseteq f^{-1}(B)$,

then $f^* : \mathcal{C}(Y) \to \mathcal{C}(X)$ descends to a well-defined map σ_f between the corresponding Teichmüller spaces:

$$\begin{array}{ccc} \mathcal{C}(Y) & \xrightarrow{f^*} & \mathcal{C}(X) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Teich}(Y,B) & \xrightarrow{\sigma_f} & \mathrm{Teich}(X,A) \end{array}$$

This map is known as the *pullback map* induced by f. Note that this map depends on the sets A and B. For more details, see Section 1 of [2].

Let $f : X \to Y$ and $g : Y \to Z$ be orientation-preserving branched covering maps. If $A \subset X$ and $C \subset Z$ are such that

- $V_f \subseteq B$ and $V_g \subseteq C$,
- $A \subseteq f^{-1}(B)$ and $B \subseteq g^{-1}(C)$,

then C contains the critical values of $g \circ f$ and $A \subseteq (g \circ f)^{-1}(C)$. Thus

$$\sigma_{g\circ f} : \mathrm{Teich}(Z, C) \to \mathrm{Teich}(X, A)$$

can be decomposed as $\sigma_{g\circ f} = \sigma_f \circ \sigma_g$; where

$$\sigma_g : \mathrm{Teich}(Z, C) \to \mathrm{Teich}(Y, B)$$

and

$$\sigma_f : \mathrm{Teich}(Y, B) \to \mathrm{Teich}(X, A).$$

The Uniformization Theorem says that any simply connected Riemann surface is conformally equivalent to one of the three domains: the open unit disk, the complex plane or the Riemann sphere. Let P be a finite subset of S^2. Due to the Uniformization Theorem we may regard $\mathrm{Teich}(S^2, P)$ as the quotient space of the set of all orientation-preserving homeomorphisms $\phi : (S^2, \phi(P)) \to (\mathbb{P}^1, \phi(P))$, with $\phi_1 \sim \phi_2$ if there exists a Möbius transformation μ such that $\mu \circ \phi_1 = \phi_2$ on P, and $\mu \circ \phi_1$ is isotopic to ϕ_2 relative to P.

The *moduli space* $\mathrm{Mod}(S^2, P)$ is the space of all injections $\psi : P \hookrightarrow \mathbb{P}^1$ modulo postcomposition with Möbius transformations. If ϕ represents an element of $\mathrm{Teich}(S^2, P)$, the restriction $[\phi] \mapsto \phi|_P$ induces a universal covering $\pi : \mathrm{Teich}(S^2, P) \to \mathrm{Mod}(S^2, P)$ which is a local biholomorphism with respect to the complex structures on $\mathrm{Teich}(S^2, P)$ and $\mathrm{Mod}(S^2, P)$. In this case, $\dim(\mathrm{Teich}(S^2, P)) = \dim(\mathrm{Mod}(S^2, P)) = |P| - 3$.

For the special case $X = Y = S^2$ and $f : X \to Y$ a Thurston map, we have the following description. Assume that $B \subset S^2$ is a finite set containing at least three points with $f(\Omega_f) \subset B$. Then there is a Thurston pullback map $\zeta_f : \mathrm{Teich}(S^2, B) \to \mathrm{Teich}(S^2, f^{-1}(B))$ which may be defined as follows. Let $\tau \in \mathrm{Teich}(S^2, B)$ be represented by a homeomorphism $\phi : S^2 \to \mathbb{P}^1$. This homeomorphism ϕ defines a complex structure c on S^2 which can be pulled back via $f : S^2 \to S^2$ to a complex structure $f^*(c)$ on S^2 (one has to use the removable singularity theorem to define the complex structure near the critical points of f). The Uniformization Theorem guarantees the existence of an isomorphism $\psi : (S^2, f^*(c)) \to \mathbb{P}^1$. Then, $\zeta_f : \mathrm{Teich}(S^2, B) \to \mathrm{Teich}(S^2, f^{-1}(B))$ is defined by $\tau = [\phi] \mapsto \zeta(\tau) = [\psi]$. Note that $\phi \circ f \circ \psi^{-1}$ is analytic, thus a rational map R:

$$\begin{array}{ccc} S^2 & \xrightarrow{\psi} & \mathbb{P}^1 \\ f\big\downarrow & & \big\downarrow R \\ S^2 & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

It is well known that $\zeta_f : \mathrm{Teich}(S^2, B) \to \mathrm{Teich}(S^2, f^{-1}(B))$ is analytic. Assume now that $A \subseteq f^{-1}(B)$ contains at least three points. Then, there is an analytic submersion $\varpi : \mathrm{Teich}(S^2, f^{-1}(B)) \to \mathrm{Teich}(S^2, A)$ which consists of forgetting points in $f^{-1}(B) \setminus A$. The composition $\varpi \circ \zeta_f$ is the pullback map $\sigma_f : \mathrm{Teich}(S^2, B) \to \mathrm{Teich}(S^2, A)$ described above. We are particular interested in the case that $A = B = P_f$ and $|P_f| = 4$.

The following result, essentially due to McMullen, can be found in Proposition 5.1 of [1].

Theorem 2.3.1. *Let $s : \mathbb{P}^1 \to \mathbb{P}^1$ and $g : \mathbb{P}^1 \to \mathbb{P}^1$ be rational maps with critical value sets V_s and V_g. Let $A \subset \mathbb{P}^1$ be finite. Assume $V_s \subset A$ and $V_g \cup g(A) \subset s^{-1}(A)$. Then*

- *$f := g \circ s$ is a Thurston map,*
- *$V_g \cup g(V_s) \subseteq P_f \subseteq V_g \cup g(A)$ and*
- *the dimension of the image of $\sigma_f : \mathrm{Teich}(\mathbb{P}^1, P_f) \to \mathrm{Teich}(\mathbb{P}^1, P_f)$ is at most $|A| - 3$. In particular, if $|A| = 3$ the pullback map σ_f is constant.*

We refer to the assumptions of Theorem 2.3.1 as *McMullen's constant conditions*. In Chapter 5, we show that not every Thurston map whose Teichmüller map is constant is as in Theorem 2.3.1. More precisely, not every Thurston map whose Teichmüller map is constant satisfies McMullen's constant conditions.

2.4 Finite Subdivision Rules

A CW complex Y is a *subdivision* of a CW complex X if they have the same underlying space and every closed cell of Y is contained in a closed cell of X. A *finite subdivision rule* (abbreviated fsr) $\mathcal{R}$ consists of a finite 2-dimensional CW complex $S_{\mathcal{R}}$ which is the union of its closed 2-cells, a subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$, and a continuous cellular map $\phi_{\mathcal{R}} : \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$ whose restriction to every open cell is a homeomorphism onto an open cell. Furthermore, $S_{\mathcal{R}}$ must have the property that for each closed 2-cell $\tilde{t}$ of $S_{\mathcal{R}}$, there is a cell structure t on a 2-disk such that t has at least three vertices, all vertices and edges of t are in ∂t, and the characteristic map $\psi_t : t \to S_{\mathcal{R}}$ takes each open cell homeomorphically onto an open cell. The cell complex t is called a *tile type* of $\mathcal{R}$. Similarly, if $\tilde{e}$ is a closed 1-cell of $S_{\mathcal{R}}$, then a 1-disk e equipped with a characteristic map $\psi_e : e \to S_{\mathcal{R}}$ is called an *edge type* of $\mathcal{R}$. A fsr $\mathcal{R}$ is *orientation preserving* if there is an orientation on the union of the open 2-cells of $S_{\mathcal{R}}$ such that the restriction of $\phi_{\mathcal{R}}$ to each open 2-cell of $\mathcal{R}(S_{\mathcal{R}})$ is orientation preserving.

We assume throughout this thesis that the underlying space of $S_{\mathcal{R}}$ is homeomorphic to the two-sphere S^2 and that $\phi_{\mathcal{R}}$ is orientation-preserving. In this case, $\phi_{\mathcal{R}}$ is a postcritically finite branched covering of the sphere with the property that pulling back the tiles effects a recursive subdivision of the sphere. That is, for each $n \in \mathbb{N}$, there is a subdivision $\mathcal{R}^n(S_{\mathcal{R}})$ of the sphere such that f is a cellular map from the nth to the $(n-1)$st subdivisions. Thus, we may speak of tiles (which are closed 2-cells), faces (which are the interiors of tiles), edges and vertices at level n. It is important to note that formally, a finite subdivision rule is not a combinatorial object, since the map $\phi_{\mathcal{R}}$, which is part of the data, is assumed given. In other words: as a dynamical system on the sphere, the topological conjugacy class of $\phi_{\mathcal{R}}$ is well-defined.

Let $\mathcal{R}$ be a finite subdivision rule on the sphere such that $\phi_{\mathcal{R}}$ is orientation-preserving. The fsr $\mathcal{R}$ has *mesh going to zero* if for every open cover of $S_{\mathcal{R}}$, there is some integer n for which each tile at level n is contained in an element of the cover. A fsr $\mathcal{R}$ has *bounded valence* if there is an upper bound to the set of valences of vertices of $\mathcal{R}^n(S_{\mathcal{R}})$, the nth subdivision of $S_{\mathcal{R}}$, where n is any positive integer; that is, there is a uniform upper bound on the valence of any vertex at any level.

In [7] J.W. Cannon et al. studied a special class of postcritically finite branched coverings $F : S^2 \to S^2$, namely, those arising from finite subdivision rules $\mathcal{R}$ on the two-sphere satisfying the properties of having mesh going to zero and bounded valence. The former condition implies that F, as a topological dynamical system, is suitably expanding (see Section 2.5); the second, that there are no periodic branched points. They showed that if the subdivision rule $\mathcal{R}$ is conformal in Cannon's sense, then the subdivision map F is topologically conjugate to a rational map. More precisely,

Theorem 2.4.1. *Let $\mathcal{R}$ be a fsr such that $S_{\mathcal{R}}$ is a 2-sphere and the mesh of $\mathcal{R}$ approaches 0. If $\mathcal{R}$ is conformal in the sense of Cannon, then $\phi_{\mathcal{R}}$ is realizable by a rational map.*

2.5 Expanding Thurston maps

Let $f : S^2 \to S^2$ be a Thurston map and $\mathcal{C}$ be a Jordan curve in S^2 with $P_f \subset \mathcal{C}$. Fix a metric d on S^2 that induces the standard topology on S^2. For $n \in \mathbb{N}$ we denote by $\mathrm{mesh}(f, n, \mathcal{C})$ the supremum of the diameters of all connected components of the set $f^{-n}(S^2 \setminus \mathcal{C})$.

A Thurston map $f : S^2 \to S^2$ is called *expanding* if there exists a Jordan curve $\mathcal{C}$ in S^2 with $P_f \subset \mathcal{C}$ and

$$\lim_{n\to\infty} \mathrm{mesh}(f, n, \mathcal{C}) = 0 \tag{2.5.1}$$

According to Proposition 6.1(v) of [3], the set $f^{-n}(S^2\setminus\mathcal{C})$ has only finitely many components, so the supremum in the definition of $\mathrm{mesh}(f, n, \mathcal{C})$ is actually a maximum. According to Lemma 8.1 of [3], if the relation (2.5.1) is satisfied for one Jordan curve $\mathcal{C} \supset P_f$, then it actually holds for every such curve. This really is a topological property, as it is independent of the choice of the metric on S^2 if it induces the given topology on S^2. This notion of expansion for a Thurston map is equivalent to a similar concept of expansion introduced by P. Haïssinsky-Pilgrim (see Section 2.2 of [12] and Proposition 8.2 of [3]). If the Thurston map is a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ on the Riemann sphere $\overline{\mathbb{C}}$, then one can show that f is expanding if and only if f does not have periodic critical points if and only if its Julia set is equal to $\overline{\mathbb{C}}$. For more details, see Section 19 of [3].

For rational Thurston maps we have the following statements.

Theorem 2.5.1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational Thurston map with Julia set $\mathcal{J}(f) = \overline{\mathbb{C}}$. Then for each sufficiently large $n \in \mathbb{N}$ there exists a quasicircle $\mathcal{C} \subset \overline{\mathbb{C}}$ with $P_f \subset \mathcal{C}$ and $f^n(\mathcal{C}) \subset \mathcal{C}$.*

Theorem 2.5.2. *Let $f : S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subset S^2$ be a Jordan curve with $P_f \subset \mathcal{C}$. Then for each sufficiently large $n \in \mathbb{N}$ there exists a Jordan curve $\tilde{\mathcal{C}}$ that is invariant for f^n and isotopic to C rel P_f.*

The main consequence of Theorem 2.5.2 is that we get such a two-tile subdivision rule for some iterate $F = f^n$ of every expanding Thurston map. This essentially allows one to describe the map F in terms of finite combinatorial data.

Corollary 2.5.3. *Let $f : S^2 \to S^2$ be an expanding Thurston map. Then for each sufficiently large n there exists a two-tile subdivision rule that is realized by $F = f^n$.*

2.6 Topological Coarse Conformal (cxc) Dynamics

A continuous, orientation-preserving, branched covering $f : S^2 \to S^2$ is called topologically cxc provided there exists a finite open cover $\mathcal{U}_0$ of S^2 by connected sets satisfying the following properties:

[**Expansion**] The mesh of the covering $\mathcal{U}_n$ tends to zero as $n \to \infty$, where $\mathcal{U}_n$ denotes the set of connected components of $f^{-n}(U)$ as U ranges over $\mathcal{U}_0$. That is, for any open cover $\mathcal{Y}$ of S^2 by open sets, there exists N such that for all $n \geq N$ and all $U \in \mathcal{U}_n$, there exists $Y \in \mathcal{Y}$ with $U \subset Y$.

[**Irreducibility**] The map f is locally eventually onto: for any $x \in S^2$, and any neighborhood W of x, there is some n with $f^n(W) = S^2$.

[**Degree**] The set of degrees of maps of the form $f^k|_{\tilde{U}} : \tilde{U} \to U$, where $U \in \mathcal{U}_n$, $\tilde{U} \in \mathcal{U}_{n+k}$, and n and k are arbitrary, has a finite maximum.

Because of Axiom [Degree], periodic or recurrent branch points cannot exist, i.e. branch points x for which the orbit $\mathcal{O}(x) = \{f^{\circ n}(x) : n \geq 0\}$ contains or accumulates on x.

A rational map f is called *semihyperbolic* if it has no parabolic cycles and no recurrent critical points in its Julia set. A rational map f which is chaotic on all $\overline{\mathbb{C}}$ is topologically cxc if and only if it is semihyperbolic, see e.g. Corollary 4..2.2 of [12].

The following theorem relates fsr and cxc. For more details see Section 4 of [12].

Theorem 2.6.1. *Suppose $\mathcal{R}$ is a finite subdivision rule for which $S_{\mathcal{R}}$ is the two-sphere and the subdivision map $\phi_{\mathcal{R}}$ is orientation preserving.*

- *If $\mathcal{R}$ has mesh going to zero, then there exists an open covering $\mathcal{U}_0$ such that $\phi_{\mathcal{R}}$ satisfies Axiom [Expansion] and [Irreducibility].*
- *If in addition $\mathcal{R}$ has bounded valence, then $\phi_{\mathcal{R}}$ satisfies Axiom [Degree], and so $\phi_{\mathcal{R}}$ is topologically cxc.*

Chapter 3

Nearly Euclidean Thurston Maps

Following Remark 3.5 of [16], a Lattès maps is a rational map from the Riemann sphere to itself such that each of its critical points is simple and it has exactly four postcritical points none of which is also critical. To better understand Thurston's pullback map, Thurston obstructions and the issue of conformality of finite subdivision rules, in [9] J. Cannon et al. introduced a nice class of Thurston maps. These are simple generalizations of Lattès maps.

3.1 Definitions and Preliminaries

To say that a Thurston map f is *Euclidean* we mean that its degree is at least 2, that its local degree at each of its critical points is 2, that it has at most four postcritical points and that none of its postcritical points is critical.

A Thurston map f is called *nearly Euclidean* (NET) if its local degree at each of its critical points is 2 and it has exactly four postcritical points.

The following lemma shows that a Euclidean Thurston map actually has exactly four postcritical points. As a consequence, if f is a Euclidean Thurston map, then the orbifold (S^2, ν_f) is Euclidean. On the other hand, if a nearly Euclidean Thurston map f is not Euclidean, then some postcritical point is a critical point and so the orbifold (S^2, ν_f) is hyperbolic.

Lemma 3.1.1. *Every Euclidean Thurston map has exactly four postcritical points, and so every Euclidean Thurston map is a NET map.*

Proof. Let f be a Euclidean Thurston map and let d be the degree of f. Since every point in S^2 has d preimages under f counting multiplicity, then $f^{-1}(P_f)$ has $d|P_f|$ points counting multiplicity. Applying the Riemann-Hurwitz formula and the fact that the local degree of f at each of its critical points is 2, then f has exactly $2d-2$ distinct critical points. Since f

is a Euclidean Thurston map,

$$\Omega_f \coprod P_f \subseteq f^{-1}(P_f)$$

and f maps each critical point to P_f with multiplicity 2. Hence $2|\Omega_f| + |P_f| \leq |f^{-1}(P_f)|$, i.e., $2(2d-2) + |P_f| \leq d|P_f|$. Thus, $0 \leq (|P_f| - 4)(d-1)$. Since $d > 1$, $|P_f|$ is at least 4. By definition of Euclidean Thurston map, $|P_f| \leq 4$. Therefore, $|P_f| = 4$ □

Lemma 3.1.2. *Let $f : S^2 \to S^2$ be a NET map with postcritical set P_f. Then $f^{-1}(P_f)$ contains exactly four points which are not critical points. The map is Euclidean if and only if these four points are the points of P_f.*

Proof. Let d be the degree of f. Then, $f^{-1}(P_f)$ has $d|P_f|$ points counting multiplicity, i.e., $f^{-1}(P_f)$ has $4d$ points counting multiplicity. Applying the Riemann-Hurwitz formula and the fact that the local degree of f at each of its critical points is 2, then f has exactly $2d-2$ distinct critical points. Hence, $2(2d-2) + |f^{-1}(P_f) \setminus \Omega_f| = 4d$. Thus, $|f^{-1}(P_f) \setminus \Omega_f| = 4$. If f is a Euclidean Thurston map, then $P_f \subset f^{-1}(P_f) \setminus \Omega_f$. Combining this fact and the previous lemma yields $4 = |P_f| \leq |f^{-1}(P_f) \setminus \Omega_f| = 4$. Thus $P_f = f^{-1}(P_f) \setminus \Omega_f$. Coversely, if $f^{-1}(P_f) \setminus \Omega_f = P_f$ then $|P_f| = 4$ and $P_f \cap \Omega_f = \emptyset$. So f is a Euclidean Thurston map. □

3.1.1 Standard commutative diagram

A Lattès map is a Euclidean Thurston map on the Riemann sphere, and so nearly Euclidean Thurston maps are closely related to Lattès maps. It is well known that a Lattès map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational map that is obtained from a finite quotient of a conformal torus endomorphism, i.e. the map f satisfies the following commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\overline{A}} & T \\ \Theta \big\downarrow & & \big\downarrow \Theta \\ \overline{\mathbb{C}} & \xrightarrow{f} & \overline{\mathbb{C}} \end{array}$$

where $\overline{A}$ is a map of a torus T that is a quotient of an affine map of the complex plane, and Θ is a two-to-one holomorphic map; see [17]. The next theorem shows that nearly Euclidean Thurston maps lift to maps of tori in a more general way. For the details of the proof, see Section 1 of [9].

Theorem 3.1.3. *Let f be a Thurston map. Then f is nearly Euclidean if and only if there exist branched covering maps $p_1 : T_1 \to S^2$ and $p_2 : T_2 \to S^2$ with degree two from the tori T_1 and T_2 to S^2 such that the set of branch points of p_2 is P_f and there exists a continuous map $\tilde{f} : T_1 \to T_2$ such that $p_2 \circ \tilde{f} = f \circ p_1$. If f is nearly Euclidean, then f is Euclidean if and only if the set of branched points of p_1 is P_f.*

Let $f: S^2 \to S^2$ be a nearly Euclidean Thurston map. Let $p_1: T_1 \to S^2$, $p_2: T_2 \to S^2$ and $\tilde{f}: T_1 \to T_2$ be as in Theorem 3.1.3 such that $p_2 \circ \tilde{f} = f \circ p_1$. For $j \in \{1,2\}$, let $P_j \subset S^2$ be the set of branched points of p_j and let $q_j: \mathbb{R}^2 \to T_j$ be a universal covering map. It is clear that $p_j \circ q_j: \mathbb{R}^2 \to S^2$ is a regular branched covering map whose local degree at every ramification point is 2. Let Γ_j and Λ_j be the set of deck transformations and the set of ramification points of $p_j \circ q_j$. We can choose q_j so that Γ_j is generated by the set of all Euclidean rotations of order 2 about the points of Λ_j. Now, given rotations $\psi(x) = 2\lambda - x$ and $\zeta(x) = 2\mu - x$ of order 2 about the points $\lambda, \mu \in \Lambda_j$, their composition $\psi \circ \zeta$ is the translation $\psi \circ \zeta(x) = x + 2(\lambda - \mu)$. We may, and do, normalize so that $0 \in \Lambda_j$. Thus, Λ_j is a lattice in $\mathbb{R}^2$ and the elements of Γ_j are the maps of the form $x \mapsto 2\lambda \pm x$ for some $\lambda \in \Lambda_j$.

Since $\tilde{f}: T_1 \to T_2$ and $q_j: \mathbb{R}^2 \to T_j$, we have

$$\begin{array}{ccc} & & \mathbb{R}^2 \\ & & \big\downarrow{\scriptstyle q_2} \\ \mathbb{R}^2 & \xrightarrow[\tilde{f}\circ q_1]{} & T_2 \end{array}$$

By the standard lifting theorem from covering space theory, e.g., see Theorem 4.17 of [11], the unbranched covering map $\tilde{f} \circ q_1$ lifts to a continuous map $\hat{f}: \mathbb{R}^2 \to \mathbb{R}^2$ such that $q_2 \circ \hat{f} = \tilde{f} \circ q_1$. Since $\tilde{f} \circ q_1$ is a covering map, so is the map $\hat{f}$. Due to the fact that $\mathbb{R}^2$ is connected and simply connected, the map $\hat{f}$ must be a homeomorphism. Since $q_1 \circ \hat{f}^{-1}$ is also a covering map, we can replace q_1 by $q_1 \circ \hat{f}^{-1}$. In this case, $\tilde{f}$ lifts to the identity map. Thus, $\Lambda_1 \subseteq \Lambda_2$ and $\Gamma_1 \subseteq \Gamma_2$. So we obtain the standard commutative diagram

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{i_c} & \Lambda_2 \\ {\scriptstyle i_c}\big\downarrow & & \big\downarrow{\scriptstyle i_c} \\ \mathbb{R}^2 & \xrightarrow{id} & \mathbb{R}^2 \\ {\scriptstyle q_1}\big\downarrow & & \big\downarrow{\scriptstyle q_2} \\ T_1 & \xrightarrow{\tilde{f}} & T_2 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ S^2 & \xrightarrow{f} & S^2 \end{array}$$

where $id: \mathbb{R}^2 \to \mathbb{R}^2$ is the identity map and the maps from Λ_1 and Λ_2 are inclusion maps. The group Γ_j contains the group of deck transformations of q_j. It is the subgroup with index 2 consisting of translations of the form $x \mapsto 2\lambda + x$ with $\lambda \in \Lambda_j$. So we can identify T_j with $\mathbb{R}^2/2\Lambda_j$ and S^2 with $\mathbb{R}^2/\Gamma_j$. There is an implicit identification between $\mathbb{R}^2/\Gamma_1$ and $\mathbb{R}^2/\Gamma_2$ which is usually achieved by means of an isomorphism from Λ_2 to Λ_1. Since the map $\tilde{f}$ lifts to the identity map, in terms of group homomorphisms, we see that $\tilde{f}$ is the canonical group homomorphism from $\mathbb{R}^2/2\Lambda_1$ to $\mathbb{R}^2/2\Lambda_2$, i.e. $\tilde{f}(x+2\Lambda_1) = x + 2\Lambda_2$. Then the kernel of $\tilde{f} : \mathbb{R}^2/2\Lambda_1 \to \mathbb{R}^2/2\Lambda_2$ is the subgroup $\{x+2\Lambda_1 : x \in 2\Lambda_2\}$ which is precisely $2\Lambda_2/2\Lambda_1$. Since $\Lambda_1 \subseteq \Lambda_2$ and both are lattices, $2\Lambda_2/2\Lambda_1 \cong \Lambda_2/\Lambda_1$. Thus, $\deg(\tilde{f}) = |\Lambda_2/\Lambda_1|$. Finally, using the standard commutative diagram, we can conclude that $\deg(f) = \deg(\tilde{f}) = |\Lambda_2/\Lambda_1|$.

Applying the Riemann-Hurwitz formula twice, one sees that $\tilde{f}$ is unramified and that p_1 and p_2 are both ramified at exactly four points. Now, let $z \in P_1$. There exists $x \in T_1$ so that $p_1(x) = z$ and $\deg(p_1, x) = 2$. Since $p_2 \circ \tilde{f} = f \circ p_1$, it follows that $2\deg(f, p_1(x)) = \deg(p_2, \tilde{f}(x))\deg(\tilde{f}, x)$. For $i \in \{1, 2\}$, $\deg(p_i) = 2$ and $\tilde{f}$ is unramified; whence $\deg(f, z) = 1$ and $\deg(p_2, \tilde{f}(z)) = 2$. Thus, $f(P_1) \subset P_2 = P_f$ but the local degree of f at each element of P_1 is 1. By Lemma 3.1.2 $f^{-1}(P_2)$ contains four points that are not critical points. So the set P_1 is precisely the set whose elements are these four points.

Let $x \in P_2$ and $y \in T_1$ so that $p_1(y) = x$. Since P_2 is the postcritical set of f, then $p_2(\tilde{f}(y)) = f(p_1(y)) = f(x) \in P_2$. So $\tilde{f}(y)$ is one of the ramification points of p_2. Then, $p_1^{-1}(P_2)$ is contained in the set of $4\deg(f)$ points of T_1 which $\tilde{f}$ maps to a ramification point of p_2. This implies that $q^{-1}(p^{-1}(P_2)) \subset \Lambda_2$. More precisely, if $|P_2 \cap P_1| = m$ and $|P_2 \setminus P_1| = n$, so that $m + n = 4$, then $q^{-1}(p^{-1}(P_2)) \subset \Lambda_2$ consists of $m + 2n$ cosets of $2\Lambda_1$ in Λ_2.

3.1.2 Decomposition of Nearly Euclidean Thurston maps

Proposition 3.1.4. *Let f be a nearly Euclidean Thurston map with p_1, p_2, P_1 and P_2 as above. Let $h : S^2 \to S^2$ be any orientation-preserving homeomorphism so that $h(P_1) = P_2$ and define $g := h^{-1} \circ f$. Then g is a Euclidean Thurston map.*

Proof. Let Ω_f and Ω_g be the critical sets of f and g respectively. Since h is a homeomorphism, $\deg(g) = \deg(f) \geq 2$. It is clear that $\Omega_g = \Omega_f$ and that $\deg(g, x) = \deg(f, x) = 2$. Let $x \in P_1$. By the standard commutative diagram, there exists a point $\tilde{x} \in T_1$ so that $x = p_1(\tilde{x})$ and $p_2(\tilde{f}(\tilde{x})) = f(p_1(\tilde{x}))$. Then, $\deg(p_2, \tilde{f}(\tilde{x}))\deg(\tilde{f}, \tilde{x}) = \deg(f, p_1(\tilde{x}))\deg(p_1, \tilde{x})$. Since $\tilde{f}$ is an unbranched covering map, then $\deg(p_2, \tilde{f}(\tilde{x})) = \deg(f, p_1(\tilde{x}))\deg(p_1, \tilde{x})$. Due to the fact that $\tilde{x}$ is a ramification point of p_1, we have

$$2 \geq \deg(p_2, \tilde{f}(\tilde{x})) = 2\deg(f, p_1(\tilde{x})).$$

This forces $\deg(f, p_1(\tilde{x})) = 1$; i.e., $\deg(f, x) = 1$. Thus, P_1 contains no critical points of g and $\deg(p_2, \tilde{f}(\tilde{x})) = 2$. Hence, $f(x) = p_2(\tilde{f}(\tilde{x})) \in P_2$. Now, let $z \in \Omega_g$, then $g(z) = h^{-1}(f(z)) \in h^{-1}(P_2)$, i.e. $g(z) \in P_1$. Then $f(g(z)) \in P_2$ and so $g(g(z)) = h^{-1}(f(g(z))) \in h^{-1}(P_2)$. Thus $g^{\circ 2}(z) \in P_1$. Proceeding by induction, one sees that the forward orbit of the point z lies in the set P_1, a set of four elements containing no critical points of g. This means that g is a Euclidean Thurston map. $\square$

This tells us that every nearly Euclidean Thurston map can be expressed as a composition of a Euclidean Thurston map followed by an orientation-preserving homeomorphism from S^2 to itself. In the following lemma we consider the converse to the previous proposition.

Lemma 3.1.5. *Let $g : S^2 \to S^2$ be a NET map. Let p_1 and p_2 be the branched covering maps corresponding to g according to the standard commutative diagram and let P_1 and P_2 be their respective sets of branched points. Let $h : S^2 \to S^2$ be an orientation-preserving homeomorphism so that $h(P_2) \subset g^{-1}(P_2)$. Let $f := h \circ g$. If $\deg(g) \geq 5$ or $\deg(\mathrm{g}) = 3$ then f is a NET map.*

Proof. Since h is a homeomorphism, $\Omega_f = \Omega_g$. Let $x \in \Omega_f$. Since $g(x)$ is a critical value of g and P_2 is the set of postcrtitical points of g, we have $f(x) = h(g(x)) \in h(P_2)$. Then $f(x) \in g^{-1}(P_2)$ and so $g(f(x)) \in P_2$. Then $f(f(x)) = h(g(f(x))) \in h(P_2)$. By induction, one sees that $f^{\circ n}(x) \in h(P_2)$. Thus, $P_f \subseteq h(P_2)$. Because h is a homeomorphism and $|P_2| = 4$, we have $|P_f| \leq 4$. The local degree of f at each of its critical points is 2, therefore f is a Thurston map.

First case: $\deg(g) \geq 5$. This implies that $P_2 = g(\Omega_g)$. Otherwise there exists $x \in P_2 \setminus g(\Omega_g)$. Since $P_1 \subseteq g^{-1}(P_2)$ then $g^{-1}(x)$ contains at most 4 distinct points and so $\deg(g) \leq 4$, which is a contradiction. Now, let $y \in h(P_2)$, then there is $z \in P_2$ so that $y = h(z)$. Moreover, there exists $x \in \Omega_g = \Omega_f$ so that $z = g(x)$. Then $y = h(g(x)) = f(x)$ for some $x \in \Omega_f$. Therefore, $h(P_2) \subseteq P_f$ and so f is a NET map.

Second case: $\deg(g) = 3$. In this case g has 4 critical points. Since the preimage under g of every element of P_2 contains three points counting multiplicity, no such preimage contains two critical points. So g maps its four critical points bijectively to P_2. Hence, f maps its four critical points bijectively to $h(P_2)$. Therefore, $|P_f| = 4$ and so it is a NET map. □

The conclusion of the previous lemma may fail if either $\deg(g) = 2$ or $\deg(g) = 4$. For instance, considering the Lattès map

$$g(z) = \frac{(z-i)^2}{(z+i)^2},$$

one sees that $\deg(g) = 2$, $\Omega_g = \{\pm i\}$ and $P_g = P_2 = \{0, \pm 1, \infty\}$. Now let $h : S^2 \to S^2$ be any orientation-preserving homeomorphism so that $h(0) = i, h(\infty) = -i, h(1) = 1$ and $h(-1) = -1$. Then, $h(P_2) \subset g^{-1}(P_2) = \{\pm 1, \pm i, 0, \infty\}$. Let $f(z) = h(g(z))$. It is clear that $\Omega_f = \{\pm i\}$ and that f fixes its critical points. Then $P_f = \{\pm i\}$. Since $|P_f| = 2$, f it is not a nearly Euclidean Thurston map. For the case $\deg(g) = 4$ we may consider the nearly Euclidean rational map

$$g(z) = \frac{(z^2-i)^2}{(z^2+i)^2}.$$

It is easily verified that $\deg(g) = 4$, $\Omega_g = \{\pm(i)^{1/2}, \pm(-i)^{1/2}, 0, \infty\}$ and $P_g = P_2 = \{0, \pm 1, \infty\}$. Let $h : S^2 \to S^2$ be any orientation-preserving homeomorphism such that $h(0) = i$, $h(\infty) = -i$, $h(1) = 1$ and $h(-1) = -1$. Then $h(P_2) \subset g^{-1}(P_2)$. Letting $f := h \circ g$, one sees that $P_f = \{\pm i, -1\}$. Thus $|P_f| = 3$ and so f is not a nearly Euclidean Thurston map.

Under the assumptions of Lemma 3.1.5, we can say that $f = h \circ g$ is a nearly Euclidean Thurston map if it has at least four postcritical points. This provides a strategy for constructing new nearly Euclidean Thurston maps from old nearly Euclidean Thurston maps. The strategy is as follows. Suppose that $g = \phi_{\mathcal{Q}}$ is the finite subdivision map of a finite subdivision rule $\mathcal{Q}$ and let $h : S^2 \to S^2$ be an orientation-preserving homeomorphism that maps the 1-skeleton of S^2 into the 1-skeleton of its first subdivision $\mathcal{Q}(S^2)$, taking vertices of S^2 to vertices of $\mathcal{Q}(S^2)$. Then f is the subdivision map of a finite subdivision rule, $\mathcal{R}$. It is furthermore true that the subdivision complex of $\mathcal{R}$ is S^2 with cell structure the image under h of the cell structure of the original cell structure. The preceding discussion and the following lemma allow us to construct fsr whose subdivision maps are nearly Euclidean Thurston maps whose orbifolds are hyperbolic.

Lemma 3.1.6. *Let $g : S^2 \to S^2$ be a Euclidean Thurston map. Let $h : S^2 \to S^2$ be an orientation-preserving homeomorphism so that $h(P_g) \subset g^{-1}(P_g)$. Let $f := h \circ g$. Then f is a Thurston map. If $|P_f| = 4$ and h does not stabilize P_g, then f is a nearly Euclidean Thurston map such that $\chi(\mathcal{O}_f) < 0$.*

Proof. Since h is a homeomorphism, $\Omega_f = \Omega_g$. Let $x \in \Omega_f$. Since $g(x)$ is a critical value of g and P_g is the set of postcritical points of g, we have $f(x) = h(g(x)) \in h(P_g)$. Then $f(x) \in g^{-1}(P_g)$ and so $g(f(x)) \in P_g$. Then $f(f(x)) = h(g(f(x))) \in h(P_g)$. So $f^{\circ 2}(x) \in h(P_g)$. Proceeding by induction, one sees that $f^{\circ n}(x) \in h(P_g)$. Thus, $P_f \subseteq h(P_g)$. Because h is a homeomorphism and $|P_g| = 4$, we have $|P_f| \leq 4$. Therefore f is a Thurston map.

In this paragraph we assume that $|P_f| = 4$. In this case f is a nearly Euclidean Thurston map and $P_f = h(P_g)$. Since $h(P_g) \subseteq g^{-1}(P_g) = P_g \coprod \Omega_g$ and h does not stabilize P_g, there exists a point $y \in h(P_g) \cap \Omega_g$. Hence $P_f \cap \Omega_f \neq \emptyset$. Therefore, the orbifold $\mathcal{O}_f$ is hyperbolic. $\square$

3.2 Construction of the main example

We begin by setting $\Lambda_2 = \mathbb{Z}^2 = \langle (1,0), (0,1) \rangle$. Let F_2 be the closed rectangle $[0,2] \times [0,1]$ in the complex plane. This rectangle is a fundamental domain for the action of $\Gamma_2 = \{z \mapsto \pm z + 2\lambda : \lambda \in \Lambda_2\}$ on $\mathbb{R}^2$. We give F_2 a cell structure so that the boundary of F_2 is its 1-skeleton and its vertices are at $(0,0)$, $(1,0)$, $(2,0)$, $(2,1)$, $(1,1)$ and $(0,1)$. This rectangle should be viewed as a hexagon rather than a quadrilateral. Let $\mathcal{S}_2$ be the tiling of the plane by the images of F_2 under the elements of Γ_2.

Let $\Lambda_1 = \langle (2,-1), (0,3) \rangle$ be the sublattice of Λ_2 generated by $(2,-1)$ and $(0,3)$. A fundamental domain F_1 for the action of $\Gamma_1 = \{z \mapsto \pm z + 2\lambda : \lambda \in \Lambda_1\}$ on $\mathbb{R}^2$ is hatched in Figure 3.1. We give F_1 a cell structure so that the boundary of F_1 is its 1-skeleton and its vertices are at $(0,0)$, $(2,-1)$, $(4,-2)$, $(4,1)$, $(2,2)$ and $(0,3)$. We regard the hatched region in Figure 3.1 as a subdivision of F_1. Let $\mathcal{S}_1$ be the tiling of the plane by the images of F_1 under the elements of Γ_1.

Let $j \in \{1,2\}$. Let $T_j = \mathbb{R}^2/2\Lambda_j$, and let $q_j : \mathbb{R}^2 \to T_j$ be the canonical quotient map. Let $p_j : T_j \to \mathbb{R}^2/\Gamma_j$ be the canonical quotient map. We next define an identification map $\varphi : \mathbb{R}^2/\Gamma_2 \to \mathbb{R}^2/\Gamma_1$. The tiling $\mathcal{S}_j$ induces a tiling of $\mathbb{R}^2/\Gamma_j$ with one tile. The 1-skeleton of the cell complex $\mathbb{R}^2/\Gamma_j$ is a topological arc with four vertices and three edges. For our identification map $\varphi : \mathbb{R}^2/\Gamma_2 \to \mathbb{R}^2/\Gamma_1$ we choose an orientation-preserving cellular homeomorphism which maps the four points $p_2(q_2(1,0)), p_2(q_2(0,0)), p_2(q_2(0,1))$ and $p_2(q_2(1,1))$ to the four points $p_1(q_1(2,-1)), p_1(q_1(0,0)), p_1(q_1(0,3))$ and $p_1(q_1(2,2))$ in order. We use this homeomorphism to identify these two spaces and we identify the result with S^2. With this identification, the set of branch points of p_1 equals the set of branch points of p_2.

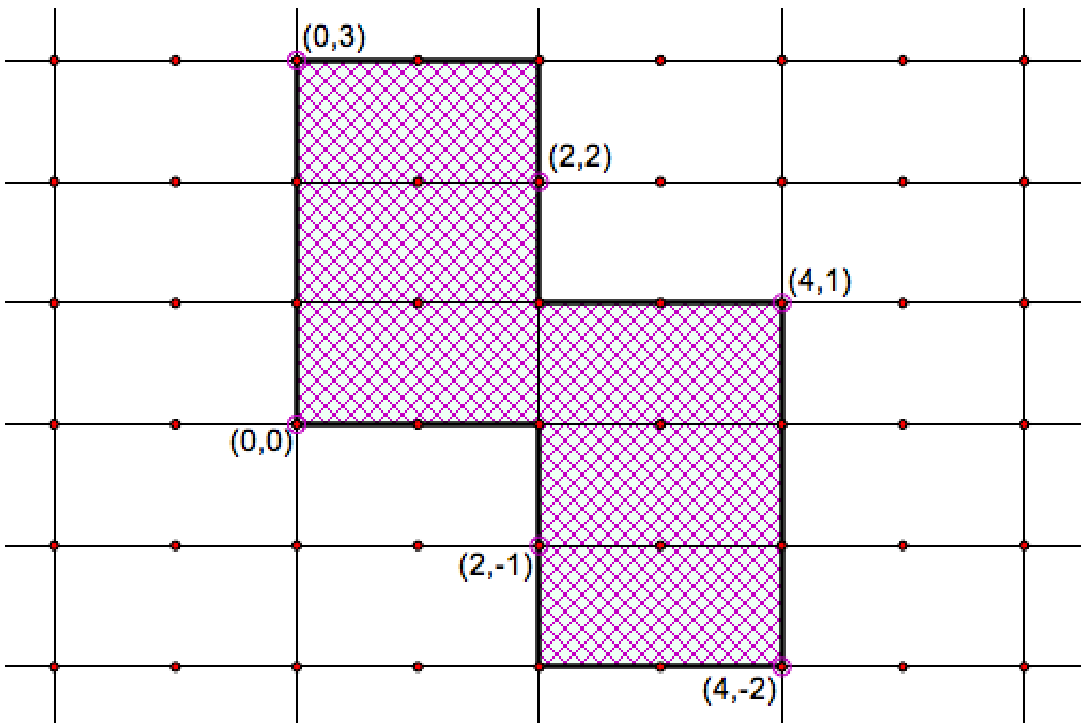


Figure 3.1: A fundamental domain F_1 for Γ_1.

Let $\tilde{g} : T_1 \to T_2$ be the canonical map, and let $g : S^2 \to S^2$ be the map which it induces. Then g is a Euclidean Thurston map. Its postcritical set P_g is the set of branch points of p_1 and p_2. It is also the subdivision map of a finite subdivision rule $\mathcal{Q}$. The subdivision complex of $\mathcal{Q}$ is S^2 with cell structure the push forward of $\mathcal{S}_1$ under $p_1 \circ q_1$. This is the same as the push forward of $\mathcal{S}_2$ under $p_2 \circ q_2$. Its first subdivision is the push forward of $\mathcal{S}_2$ under $p_1 \circ q_1$. Figure 3.2 indicates the action of g. The right portion of Figure 3.2 shows the push forward of $\mathcal{S}_1$ under $p_1 \circ q_1$ in S^2 and the left portion of Figure 3.2 shows the push forward of $\mathcal{S}_2$ under $p_1 \circ q_1$ in S^2. Most of the vertices in the left portion are labeled with preimages in $\mathbb{R}^2$.

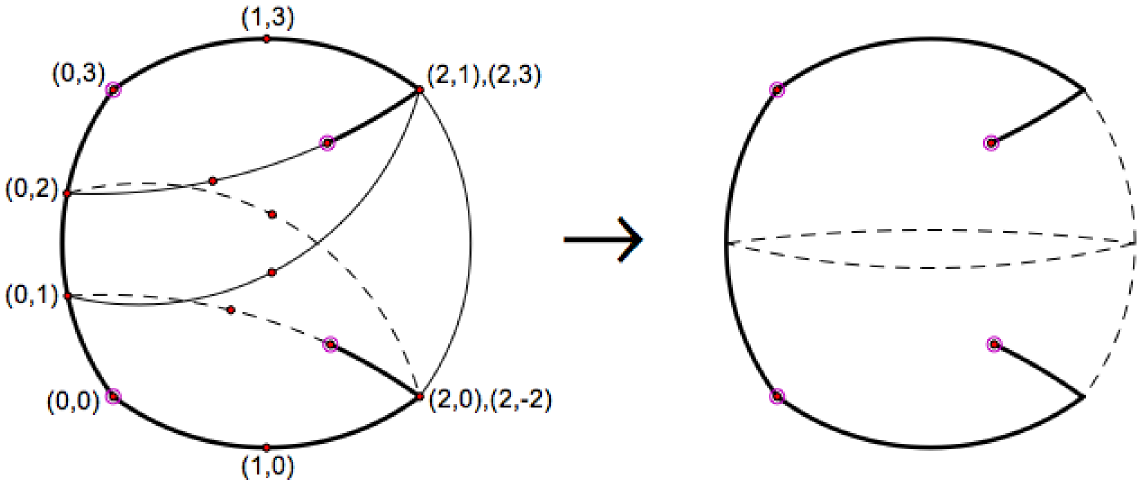


Figure 3.2: The map g.

There exists an orientation-preserving homeomorphism $h : S^2 \to S^2$ which takes the 1-skeleton of the push forward of $\mathcal{S}_1$ into the 1-skeleton of the push forward of $\mathcal{S}_2$ such that h fixes the images of $(1,0),(0,0),\ldots,(0,3),(1,3)$ and h maps the image of $(2,2)$ to the image of $(2,3)$ and the image of $(2,-1)$ to the image of $(2,0)$. Let $f = h \circ g$. The action of f is indicated in Figure 3.3. The map f preserves the edge labels which are given. We see that $h(P_g) \subset g^{-1}(P_g)$. By Lemma 3.1.6, f is a nearly Euclidean Thurston map and it is the subdivision map of a finite subdivision rule $\mathcal{R}$. The single tile type of $\mathcal{R}$ is a hexagon. The subdivision of the hexagon is shown in Figure 3.4. It is easy to check that $\mathcal{R}$ has bounded valence and that the mesh of $\mathcal{R}$ approaches 0 combinatorially.

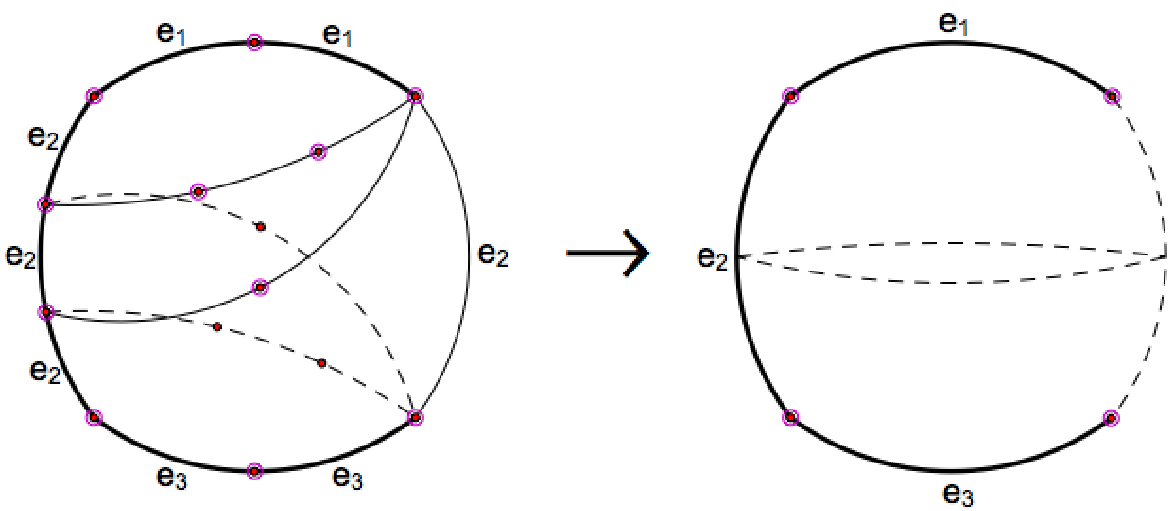


Figure 3.3: The map f.

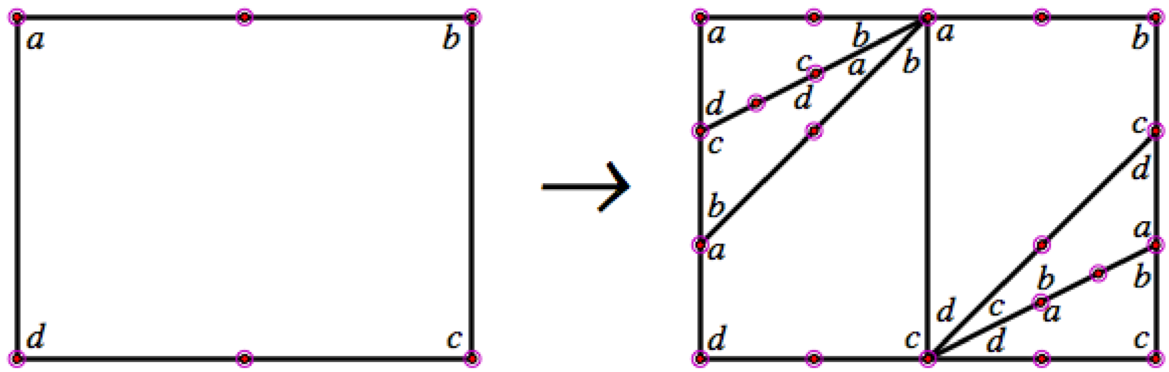


Figure 3.4: The subdivision of the tile type for the finite subdivision rule $\mathcal{R}$.

3.3 Slope of multicurves

Let $f : S^2 \to S^2$ be a nearly Euclidean Thurston map. We maintain the setting of Section 3.1. Note that $p_j \circ q_j : \mathbb{R}^2 \setminus \Lambda_j \to S^2 \setminus P_j$ is a regular covering with group of deck transformations Γ_j. In particular, every curve in $S^2 \setminus P_j$ has a lift to $\mathbb{R}^2 \setminus \Lambda_j$. Let (λ_j, μ_j) be an ordered basis of Λ_j. Let α be a multicurve in $S^2 \setminus P_j$, i.e., an essential nonperipheral simple closed curve in $S^2 \setminus P_j$. Suppose α has a lift to $\mathbb{R}^2$ which joins the points x and y.

- Since α is not null homotopic, then $x \neq y$.
- The deck transformations of $p_j \circ q_j$ are Euclidean isometries and this lift of α is also a lift of a closed curve in $T_j = \mathbb{R}^2/2\Lambda_j$. The $y = \gamma(x)$ for some translation $\gamma \in \Gamma_j$.

Hence the slope of the line through x and y is independent of the choice of lift of α to $\mathbb{R}^2$. The slope of such a line relative to the ordered basis (λ_j, μ_j) is the **slope of** α.

Recall that for $j \in \{1, 2\}$, $|P_j| = 4$. If α is an essential, nonperipheral simple closed curve in $S^2 \setminus P_j$, then α separates two points, x and y, of P_j from the other two points of P_j. We call an arc in S^2 joining x and y which is disjoint from α a **core arc** for α. Giving a homotopy class of essential, nonperipheral simple closed curves in $S^2 \setminus P_j$ is equivalent to giving such a core arc.

3.4 Coset Numbers and corollaries

In [9], J. Cannon et al. prove that if f is a nearly Euclidean Thurston map and δ is a multicurve in $S^2 \setminus P_f$, then f maps every connected component of $f^{-1}(\delta)$ to δ with the same degree. Furthermore, the authors introduced the concept of coset numbers and characterized the number of essential components of $f^{-1}(\delta)$ in terms of coset numbers. The definition of coset numbers is as follows. Let A be a finite abelian group. Let H be a subset of A which is the disjoint union of four pairs $\{\pm h_1\}$, $\{\pm h_2\}$, $\{\pm h_3\}$, $\{\pm h_4\}$. Let B be a subgroup such that A/B is cyclic, and let a be an element of A so that $a+B$ generates A/B. Let n be the order of A/B. For each $k \in \{1,2,3,4\}$ there exists a unique $c \in \{0,\dots,n/2\}$ such that

$$(ca+B)\cap\{\pm h_k\} \neq \emptyset$$

Let c_1, c_2, c_3, c_4 be these four integers ordered so that $0 \le c_1 \le c_2 \le c_3 \le c_4$. These four numbers are called *coset numbers for H relative to B and a or relative to the generator $a+B$ and A/B*.

Now, let Λ_1 and Λ_2 be as in Section 3.1 and consider $A = \Lambda_2/2\Lambda_1$. If λ and μ form a basis of Λ_2, then $\lambda + 2\Lambda_1$ generates a cyclic subgroup B of A (this group may be the trivial group if $\lambda \in 2\Lambda_1$) and $\mu+2\Lambda_1$ is an element a whose image in A/B generates A/B, i.e., $(\mu+2\Lambda_1)+B$ generates A/B. From Section 3.1 we know that $q^{-1}(p_1^{-1}(P_2)) \subseteq \Lambda_2$, so $p^{-1}(P_2)$ is a subset of $\Lambda_2/2\Lambda_1$. Hence we may speak of coset numbers for the subset $H = p^{-1}(P_2)$ relative to B and a. In [9], J. Cannon et al. call these coset numbers the coset numbers for $q_1^{-1}(p_1^{-1}(P_2))$ relative to λ and μ. The coset number of $\eta \in q_1^{-1}(p_1^{-1}(P_2))$ is the smallest nonnegative integer c for which there exists an integer b such that $\pm\eta \in b\lambda + c\mu + 2\Lambda_1$. Based on these definitions and the settings of the previous section, the following theorem was proven. For the proof, see Theorem 4.1 of [9].

Theorem 3.4.1. *Let f be a nearly Euclidean Thurston map. Let δ be a multicurve in $S^2 \setminus P_f$ with slope $\frac{p}{q}$, where p and q are relatively prime integers not both 0. Let $\lambda = q\lambda_2 + p\mu_2 \in \Lambda_2$. Let d be the order of $\lambda + \Lambda_1$ in Λ_2/Λ_1. Let d' be the positive integer such that $dd' = |\Lambda_2/\Lambda_1| =$* $\deg(f)$. *Since p and q are relatively prime, there exists $\mu \in \Lambda_2$ such that λ and μ form a basis of Λ_2. Let c_1, c_2, c_3, c_4 be the coset numbers for the elements of $q_1^{-1}(p_1^{-1}(P_f))$ relative to λ and μ. Then the following statements hold.*

(1) Every connected component of $f^{-1}(\delta)$ maps to δ with degree d.

(2) The number of essential components of $f^{-1}(\delta)$ is $c_3 - c_2$.

(3) The number of peripheral components of $f^{-1}(\delta)$ is $c_2 - c_1 + c_4 - c_3$.

(4) The number of null homotopic components in $f^{-1}(\delta)$ is $c_1 - c_4 + d'$.

(5) The lines in $\mathbb{R}^2$ with slope p/q relative to the basis (λ_2, μ_2) of Λ_2 which map under $p_1 \circ q_1$ to essential simple closed curves in $S^2 \setminus P_f$ are exactly the Γ_1-translates of the lines with parametric forms $(x,y) = t\lambda + u\mu$ with parameter t and $c_2 < u < c_3$.

Combining Theorem 3.4.1 and Lemma A.1 of [19], we get the following two corollaries. In the next corollary assume $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the prime factorization of n; i.e. each p_i is prime and $p_1 < p_2 < \cdots < p_k$.

Corollary 3.4.2. *Let f be an expanding NET map with degree* $\deg(f) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. *Suppose that the number of essential components in the preimage of any simple closed curve γ in $S^2 \backslash P_f$ is less than p_1. If the orbifold $\mathcal{O}_f$ is hyperbolic then f is combinatorially equivalent to a rational map.*

Proof. Proceed by contradiction. Suppose there is an obstruction $\Gamma = \{\gamma\}$. Then, the set S defined by $S := \{\alpha : \alpha$ is a connected component of $f^{-1}(\gamma)$ isotopic to γ in $S^2 \setminus P_f\ \}$ is not empty and by assumption $|S| < p_1$. From 1 of Theorem 3.4.1, the function f maps every element of S to γ with the same degree, namely d, which is a divisor of $\deg(f)$. Since f is expanding, by Lemma A.1 of [19] there are no Levy cycles. Thus, $d \neq 1$. Then the multiplier $\lambda_\Gamma = \frac{|S|}{d} < \frac{p_1}{d} \leq 1$, which is a contradiction. □

Corollary 3.4.3. *Let f be an expanding NET map with degree* $\deg(f) = p$ *prime. If the orbifold $\mathcal{O}_f$ is hyperbolic then f is combinatorially equivalent to a rational map.*

Proof. Proceed by contradiction. Suppose there is an obstruction $\Gamma = \{\gamma\}$. Then, the set S defined by $S := \{\alpha : \alpha$ is a connected component of $f^{-1}(\gamma)$ isotopic to γ in $S^2 \setminus P_f\ \}$ is not empty. From 1 of Theorem 3.4.1, the function f maps every element of S to γ with the same degree, namely d, which is a divisor of p. So either $d = 1$ or $d = p$. Since f is expanding, by Lemma A.1 of [19] there are no Levy cycles. Thus, $d \neq 1$. Then the multiplier $\lambda_\Gamma = \frac{|S|}{d} = \frac{1}{p} < 1$, which is a contradiction. □

If the map f is not expanding, the last corollary may fail . For example, regard the sphere as the quotient space of two Euclidean quadrilaterals, "yellow" and "green", whose oriented boundaries are identified as shown below. We regard this as a CW-structure on the sphere.

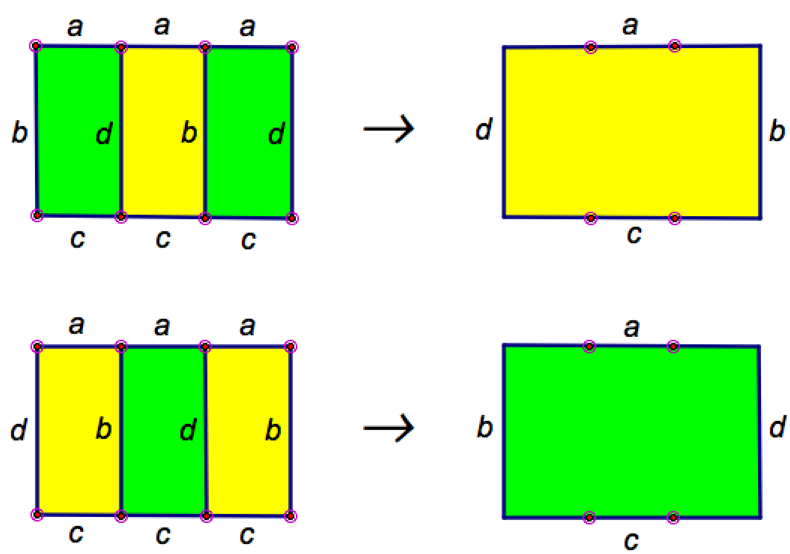


Figure 3.5: A non-expanding NET map of degree 3.

Denote this map by f. Note that f is a nearly Euclidean Thurston map of degree 3. Also, note that it has four simple critical points, each of which is fixed. If this nearly Euclidean Thurston map were combinatorially equivalent to a rational map R, then R would be a degree 3 rational map with four fixed critical points. Since a rational map of degree 3 has exactly four fixed points, then by the holomorphic index formula

$$\frac{1}{1-R'(z_1)}+\frac{1}{1-R'(z_2)}+\frac{1}{1-R'(z_3)}+\frac{1}{1-R'(z_4)}=1$$

where z_i are the fixed points of R. This yields $4=1$ which is impossible. Another way to show that this nearly Euclidean map is obstructed is by using Thurston's characterization theorem. Note that the essential vertical arc is a core arc for an f-invariant multicurve. The preimage of this multicurve has three connected components and f maps each of these connected components with degree 1. Only one of these connected components is essential and nonperipheral. Then the Thurston multiplier of this multicurve is 1. Therefore, the nearly Euclidean Thurston map given above is not combinatorially equivalent to a rational map.

Remark 3.4.1. The conclusion of Corollary 3.4.3 may fail if the orbifold is Euclidean. However, if the degree of f is either 2 or 3 the conclusion still holds. For more details and counterexamples see Appendix C.

3.4.1 Computation of coset numbers

The following lemma provides a way to compute the coset numbers c_1, c_2, c_3, c_4 in Theorem 3.4.1. For details of the proof, see Section 4 of [9].

Lemma 3.4.4. *Maintain the setting of Theorem 3.4.1. Let r and s be integers, and let $\eta = r\lambda_2 + s\mu_2 \in \Lambda_2$. Let b and c be integers such that $c \geq 0$ and c is as small as possible such that $\pm\eta \in b\lambda + c\mu + 2\Lambda_1$. Then c, the coset number of η with respect to λ and μ, is the smallest nonnegative integer congruent to $\pm(pr - qs)$ modulo $2d'$.*

The congruence in Lemma 3.4.4 is modulo $2d'$. All that we need to know about p and q are their values modulo $2d'$. Note that the congruence is linear in r and s. So if p and q are multiplied by a common unit modulo $2d'$, then the coset number c_i (up to the $\pm$ sign) is also multiplied by that common unit. More precisely, suppose that c_1, c_2, c_3, c_4 arise from slope p/q. Let $p'/q' \in \widehat{\mathbb{Q}}$ and suppose that, just as for p/q, the order of the image of $q'\lambda_2 + p'\mu_2$ in Λ_2/Λ_1 is d. Also suppose that there exists an integer u which is a unit modulo $2d'$ such that $p' \equiv up \mod 2d'$ and $q' \equiv uq \mod 2d'$. The point of this discussion is that if c_1', c_2', c_3', c_4' are the coset numbers as in Theorem 3.4.1 for p'/q', then c_1', c_2', c_3', c_4' are congruent to $\pm uc_1', \pm uc_2', \pm uc_3', \pm uc_4'$ module $2d'$, not necessarily in order.

3.4.2 Coset numbers for the main example

Recall that for our the main example $\Lambda_2 = \mathbb{Z}^2$ and $\Lambda_1 = \langle (2,-1),(0,3)\rangle$. We begin with an essential simple closed curve δ in $S^2 \setminus P_f$ with slope p/q. Assume that $\gcd(p,q)=1$. Let d be the degree of the restriction of f to any connected component of $f^{-1}(\delta)$. By Theorem 3.4.1, d is the smallest positive integer such that there exist integers $x,\ y$ for which

$$x(2,-1)+y(0,3)=d(q,p).$$

Solving for x and y, we obtain

$$x=\frac{dq}{2} \qquad \text{and} \qquad y=\frac{d(2p+q)}{6}$$

Suppose that $q \equiv 0 \mod 2$. In this case, x is an integer for every d and $(2p+q)\equiv 0 \mod 2$. Thus, for y to be an integer, the only condition on d is that $d \equiv 0 \mod 3$ if $2p+q \not\equiv 0 \mod 3$. Now suppose that $q \not\equiv 0 \mod 2$. Since we want x to be an integer, $d \equiv 0 \mod 2$ must hold. Thus, for y to be an integer the only condition on d is that $d \equiv 0 \mod 3$ if $2p+q \not\equiv 0 \mod 3$. This discussion tells us that the values of d are $1,2,3,6$.

Now we determine the number of various types of components of $f^{-1}(\delta)$. The elements $(0,0)$, $(2,0)$, $(0,3)$ and $(2,3)$ of Λ_2 map to the four elements of P_2. Calculating $pr-qs$ for these four values yields $0,\ 2p,\ -3q$ and $2p-3q$. In what follows, we find the reduced residues of ± 1 times these values module $2d'$.

Noting that $\Lambda_2 = \langle (1,0),(0,1)\rangle = \langle (1,1),(0,1)\rangle$ and $\Lambda_1 = \langle (2,-1),(0,3)\rangle = \langle 2(1,1),3(0,1)\rangle$, one sees that $\Lambda_2/\Lambda_1 \cong \mathbb{Z}/6\mathbb{Z}$. Since $dd' = |\Lambda_2/\Lambda_1| = 6$ and the values of d are $1,2,3,6$, we have the following four cases.

Case 1: $d=6$.
We first show that q is odd. Proceed by contradiction. Suppose $q=2k$ for some integer k. Then $x=6k$ and so $y=2(p+k)$, whence $6k(2,-1)+2(p+k)(0,3)=6(q,p)$. So the equation $\tilde{x}(2,-1)+\tilde{y}(0,3)=3(q,p)$ has integer solutions for $\tilde{x}=3k$ and $\tilde{y}=p+k$. This contradicts the minimality of $d=6$. Note that in this case $2p+q\not\equiv 0 \mod 3$. Otherwise, $x=3q$ and $y=(2p+q)=3k$ for some integer k, then $3q(2,-1)+3k(0,3)=6(q,p)$. So the equation $\tilde{x}(2,-1)+\tilde{y}(0,3)=2(q,p)$ has integer solutions for $\tilde{x}=q$ and $\tilde{y}=k$. This contradicts (again) the minimality of $d=6$. Clearly $d'=1$ and $2d'=2$. Since q is odd, so are $-3q$ and $2p-3q$. So reducing $0,2p,-3q$ and $2p-3q$ modulo 2 yields $0,0,1,1$.

Case 2: $d=3$.
Since $x=3q/2$ is an integer, then $q\equiv 0 \mod 2$. Thus $p\equiv 1 \mod 2$. Clearly the reduced residues associated to 0 and $2p$ are 0 and 2 respectively. Note that in this case $2d'=4$. Also, note that $2p+q\not\equiv 0 \mod 3$; otherwise we get a contradiction with the minimality of $d=3$. If $q\equiv 0 \mod 4$, then $\pm 3q \equiv 0 \mod 4$. Since $2p\equiv 2 \mod 4$ and $3q\equiv 0 \mod 4$, then $2p-3q\equiv 2 \mod 4$ and $-(2p-3q)\equiv 2 \mod 4$. So in this case, the reduced residues are $0,0,2,2$. If $q\equiv 2 \mod 4$, then $\pm 3q\equiv 2 \mod 4$. Since $2p\equiv 2 \mod 4$ and $3q\equiv 2$

mod 4, then $2p-3q \equiv 0 \mod 4$ and $-(2p-3q) \equiv 0 \mod 4$. Thus, in this case, the reduced residues are also $0, 0, 2, 2$.

Case 3: $d = 2$.
Clearly $x = q$ and $y = (2p+q)/3$. Since $y \in \mathbb{Z}$, $2p + q \equiv 0 \mod 3$ and so $q \not\equiv 0 \mod 3$ (otherwise 3 divides $\gcd(p, q)$). If $q \equiv 0 \mod 2$, then $x = 2k$ and $y = 2\tau$ for some integers k and τ. Then $2k(2, -1) + 2\tau(0, 3) = 2(q, p)$. So the equation $\tilde{x}(2, -1) + \tilde{y}(0, 3) = 1(q, p)$ has integer solutions for $\tilde{x} = k$ and $\tilde{y} = \tau$. This contradicts the minimality of $d = 2$. Therefore $q \not\equiv 0 \mod 2$ and so q is a unit modulo 6.

Suppose that $q \equiv 1 \mod 6$, then $\pm 3q \equiv 3 \mod 6$ so the respective coset number is 3. Clearly $q \equiv 1 \mod 3$ and since $2p + q \equiv 0 \mod 3$, we have $2p \equiv -1 \mod 3$ and so $p \equiv 1 \mod 3$. Then $2p \equiv 2 \mod 6$ and $-2p \equiv 4 \mod 6$; so the respective coset number $c = \min\{2, 4\} = 2$. Since $2p \equiv 2 \mod 6$ and $-3q \equiv 3 \mod 6$, then $2p - 3q \equiv 5 \mod 6$ and $-(2p - 3q) \equiv 1 \mod 6$; so the respective coset number $c = \min\{5, 1\} = 1$. So up to sign our values reduce to $0, 1, 2, 3 \bmod 6$. According to the remark after Lemma 3.4.4, multiplying q by a unit modulo 6 amounts to multiplying these four values by the same unit. The units modulo 6 are represented by ± 1. Since multiplication by -1 does nothing, we obtain $0, 1, 2, 3$. So if $q \equiv \pm 1 \mod 12$ or $q \equiv \pm 7 \mod 12$, the coset numbers are $0, 1, 2, 3$.

Case 4: $d = 1$**.**
Clearly $2d' = 12$. Since $x = q/2$ and $y = (2p + q)/6$ are integers, $q \equiv 0 \mod 2$ and $2p + q \equiv 0 \mod 6$. Then p is odd and $2p + q \equiv 0 \mod 3$. Then $q \not\equiv 0 \mod 3$ (otherwise 3 divides $\gcd(p, q)$). So up to sign $q \equiv 2 \mod 12$ or $q \equiv 4 \mod 12$.

Suppose that $q \equiv 2 \mod 12$. Then $q = 12k + 2$ for some integer k. Since $2p + q \equiv 0 \mod 3$, then $2p + 12k + 2 = 3m$ for some integer m. If $m \equiv 0 \mod 4$, then $2p + 2 \equiv 0 \mod 12$ and so $p + 1 \equiv 0 \mod 6$. Thus $p + 1 \equiv 0 \mod 12$ or $p + 1 \equiv 6 \mod 12$. Therefore $p \equiv -1, 5 \mod 12$. If $m \equiv 1 \mod 4$, then $3m \equiv 3 \mod 12$ so $2p + 2 \equiv 3 \mod 12$ which is impossible. If $m \equiv 2 \mod 4$, then $3m \equiv 6 \mod 12$ so $2p + 2 \equiv 6 \mod 12$; hence $2p \equiv 4 \mod 12$ and so p is even, which is a contradiction. If $m \equiv 3 \mod 4$, then $3m \equiv 9 \mod 12$ so $2p + 2 \equiv 9 \mod 12$, which is impossible. Therefore, in this case, $p \equiv -1, 5 \mod 12$. So up to a sign, our four values are $0, 2, 4, 6 \bmod 12$.

Suppose that $q \equiv 4 \mod 12$. Then $q = 12k + 4$ for some integer k. Since $2p + q \equiv 0 \mod 3$, then $2p + 12k + 4 = 3m$ for some integer m. If $m \equiv 0 \mod 4$, then p is even which is a contradiction. If $m \equiv 1 \mod 4$, then $3m \equiv 3 \mod 12$ so $2p + 4 \equiv 3 \mod 12$ which is impossible. If $m \equiv 2 \mod 4$ then $3m \equiv 6 \mod 12$; thus $2p + 4 \equiv 6 \mod 12$. Then $p \equiv 1 \mod 6$ and so $p \equiv 1, 7 \mod 12$. If $m \equiv 3 \mod 4$, then $3m \equiv 9 \mod 12$ so $2p + 4 \equiv 9 \mod 12$ which is impossible. Therefore, in this case, $p \equiv 1, 7 \mod 12$. So up to a sign, our four values are $0, 0, 2, 2 \bmod 12$. This completes the computation for $d = 1$.

$q \mod 12$	$2p+q \mod 3$	d	c_1, c_2, c_3, c_4	essl	perl	null
$\pm1, \pm3, \pm5$	±1	6	0, 0, 1, 1	1	0	0
$0, \pm2, \pm4, \pm6$	±1	3	0, 0, 2, 2	2	0	0
$\pm1, \pm5$	0	2	0, 1, 2, 3	1	2	0
±2	0	1	0, 2, 4, 6	2	4	0
±4	0	1	0, 0, 2, 2	2	0	4

TABLE 1. Degrees, coset numbers and numbers of components for the main example.

Now we apply Theorem 3.4.1 to the main example. Table 1 displays the results of the previous computations. The first column gives us q modulo 12. The second gives $2p+q$ modulo 3. The third column gives us the degree of the restriction of f to every connected component of the inverse image of an essential simple closed curve in $S^2 \setminus P_f$ with slope p/q. the next column gives us the values of the coset numbers $c_1,\ c_2,\ c_3,\ c_4$ which appear in Theorem 3.4.1. The last three columns give the numbers of essential components, peripheral components and null homotopic components in this inverse image. It is furthermore true that this map f is expanding. By Lemma A.1 of [19], there are no Levy cycles. Now, suppose there exists an f-invariant multicurve $\Gamma = \{\gamma\}$. Then its Thurston multiplier can be either $\lambda_\Gamma = 1/6$ or $\lambda_\Gamma = 2/3$ or $\lambda_\Gamma = 1/2$. Since all of them are less than 1, by Thurston characterization theorem we conclude that the map f is combinatorially equivalent to a rational map.

Unfortunately the analysis given above does not provide any information with respect to the fixed point of the Teichmüller map Σ_f. In the next chapter, using the **slope map of** f and properties of horoballs in $\mathbb{H}$, we prove that f is combinatorially equivalent to a rational map without using the fact that f is an expanding map. Moreover we provide a location of the fixed point of Σ_f in $\mathbb{H}$.

Chapter 4

Slope function and horoballs in Teichmüller Space

Every homotopy class of essential nonperipheral simple closed curves in a 4-punctured sphere is assigned a slope in $\widehat{\mathbb{Q}}$ which characterizes the homotopy class. Taking pullbacks essentially obtains for every nearly Euclidean Thurston map an induced map from $\widehat{\mathbb{Q}}$ to itself. In this situation, a Thurston obstruction is a fixed point of this map. In this chapter we compute the induced map for the main example given in Chapter 3 and we also provide a location of the fixed point of the corresponding Teichmüller map in the Teichmüller space.

4.1 Slope function

Let f be a nearly Euclidean Thurston map. As in Section 3.3, we fix an ordered basis (λ_2, μ_2) of Λ_2 by which we define slopes of multicurves in $S^2 \setminus P_2$. Recall that in this setting $P_2 = P_f$. Let p and q be relative prime integers, so that $\frac{p}{q} \in \widehat{\mathbb{Q}}$. Let δ be a multicurve in $S^2 \setminus P_2$ with slope $\frac{p}{q}$. Every connected component of $f^{-1}(\delta)$ is in $S^2 \setminus P_2$. Moreover all components in $f^{-1}(\delta)$ that are essential and nonperipheral are in the same homotopy class as curves in $S^2 \setminus P_2$. Now we define a **slope function** $\sigma_f : \widehat{\mathbb{Q}} \to \widehat{\mathbb{Q}} \cup \{\emptyset\}$ as follows.

- If no connected component of $f^{-1}(\delta)$ is essential and nonperipheral in $S^2 \setminus P_2$, then we set $\sigma_f(\frac{p}{q}) = \emptyset$.

- If some connected component α of $f^{-1}(\delta)$ is essential and nonperipheral in $S^2 \setminus P_2$, then we set

$$\sigma_f(\tfrac{p}{q}) = \text{slope of } \alpha \text{ in } S^2 \setminus P_2.$$

This defines σ_f, independent of the choices of δ and α. By Proposition 3.1.4, f can be expressed as a composition $f = h \circ g$ of functions, where $g : S^2 \to S^2$ is a Euclidean Thurston map and $h : S^2 \to S^2$ is any orientation-preserving homeomorphism such that $h(P_1) = P_2$. We choose h so that h fixes $P_1 \cap P_2$. Suppose that $P_1 = \{x_1, x_2, x_3, x_4\}$. For every $k \in \{1, 2, 3, 4\}$ let β_k be an arc S^2 which joins x_k to $h(x_k)$. Since h fixes $P_1 \cap P_2$, we may choose these arcs so that they are disjoint. Every connected component of $q_j^{-1}(p_j^{-1}(\beta_k))$ contains exactly one element of Λ_j. If β_k is nontrivial, then the restriction of $p_j \circ q_j$ to such a component is a branched covering map onto β_k with degree 2. We call every such connected component a **spin mirror** for $p_j \circ q_j$. This terminology is explained in Section 5 of [9]. We furthermore assume that every spin mirror for $p_1 \circ q_1$ is a piecewise-linear arc in $\mathbb{R}^2$. The next two theorems provide a way to compute the slope function σ_f. For details of the proofs see Section 5 of [9].

Theorem 4.1.1. *Let f be a nearly Euclidean Thurston map in the setting of Section 3.1. Let $\frac{p}{q} \in \widehat{\mathbb{Q}}$. Let δ be an essential simple closed curve in $S^2 \setminus P_2$ with slope $\frac{p}{q}$ relative to the basis (λ_2, μ_2) of Λ_2. Suppose that α is an essential nonperipheral component of $f^{-1}(\delta)$ in $S^2 \setminus P_2$. Let d be the degree with which f maps α to δ. Let $\lambda = q\lambda_2 + p\mu_2$. Let v be any point in $\mathbb{R}^2$ such that $p_1(q_1(v)) \in \alpha$ and v is not contained in a spin mirror for $p_1 \circ q_1$. It is possible to choose δ so that the line segment joining v and $w = v + 2d\lambda$ is a lift of α to $\mathbb{R}^2$ under $p_1 \circ q_1$ and it intersects the spin mirrors for $p_1 \circ q_1$ transversely in finitely many points. Let S be the line segment joining v and w. Let $\lambda_1, \dots, \lambda_n$ be the midpoints of the spin mirrors which meet S in order. Then $\sigma_f(\frac{p}{q})$ is the slope of the line segment joining v and $w' = (-1)^n + 2\sum_{i=1}^{n}(-1)^{i+1}\lambda_i$ relative to either of the two ordered bases of λ_1, determined by (λ_2, μ_2) and the choice of spin mirrors.*

Theorem 4.1.2. *Let f be a nearly Euclidean Thurston map in the setting of Section 3.1. Let $\frac{p}{q} \in \widehat{\mathbb{Q}}$. As in Theorem 4.1.1, let $\lambda = q\lambda_2 + p\mu_2$ and let μ be an element of Λ_2 such that λ and μ form a basis of Λ_2. Also, let c_1, c_2, c_3, c_4 be the coset numbers for $q_1^{-1}(p_1^{-1}(P_2))$ relative to λ and μ. We assume that $\sigma_f(\frac{p}{q}) \neq \emptyset$, equivalently, $c_2 \neq c_3$ by Theorem 3.4.1. Let L be a line in $\mathbb{R}^2$ which is a Γ_1-translate given in parametric form by either $(x, y) = t\lambda + c_2\mu$ or $(x, y) = t\lambda + c_3\mu$. Let v and w be distinct elements of $L \cap q_1^{-1}(p_1^{-1}(P_2))$ such that no element of $q_1^{-1}(p_1^{-1}(P_1 \cup P_2))$ is strictly between v and w. Let S be the closed line segment which joins v and w. We assume that the interior of S intersects the spin mirrors for $p_1 \circ q_1$ transversely in finitely many points. Let $\lambda_1, \dots, \lambda_n$ be the midpoints of these spin mirrors which meet the interior of S in order. Since v, $w \in q_1^{-1}(p_1^{-1}(P_2))$, both v and w are contained in spin mirrors for $p_1 \circ q_1$. Let λ_0 and λ_{n+1} be the midpoints of these two spin mirrors. Then $\sigma_f(\frac{p}{q})$ is the slope of the line segment joining 0 and $\sum_{i=0}^{n}(-1)^i(\lambda_{i+1} - \lambda_i)$ relative to either of the two ordered bases of Λ_1 determined by (λ_2, μ_2) and the choice of spin mirrors.*

One advantage of Theorem 4.1.2 over Theorem 4.1.1 is that in Theorem 4.1.2 both v and w are in Λ_2, whereas in Theorem 4.1.1 neither is. Another advantage is that in Theorem 4.1.2 the line segment joining v and w is shorter than the one in Theorem 4.1.1, resulting in a shorter computation.

4.2 Slope function for the main example

In this section we show for the main example that Theorem 4.1.2 provides an algorithm for computing the slope function which is easy to implement by computer.

Let p and q be integers so that $\gcd(p,q)=1$. Table 1 shows that $c_2 \neq c_3$, so $\sigma_f(\frac{p}{q})$ is never $\emptyset$. We begin by choosing appropriate lattice points v and w as in Theorem 4.1.2.

Recall that for the main example $\Lambda_1 = \langle (2,-1),(0,3)\rangle$ and $\Lambda_2 = \mathbb{Z}^2$. Since the basis for Λ_2 is the standard basis $\lambda_2 = (1,0)$ and $\mu_2 = (0,1)$, $\lambda = (q,p)$. The set $q_1^{-1}(p_1^{-1}(P_1 \bigcup P_2))$ is a union of cosets of $2\Lambda_1$ in Λ_2 and the following elements are distinct representatives for these cosets.

$$(0,0),(0,3),(2,-1),(2,2),(2,0),(2,-2),(2,1),(2,3).$$

One sees that

$$\{(0,0),(0,3)\} \subset q_1^{-1}(p_1^{-1}(P_1 \cap P_2)),$$
$$\{(2,-1),(2,2)\} \subset q_1^{-1}(p_1^{-1}(P_1 \setminus P_2)),$$
$$\{(2,3),(2,1),(2,0),(2,-2)\} \subset q_1^{-1}(p_1^{-1}(P_2 \setminus P_1)).$$

We first determine all cases in which is possible to choose $v=(0,0)$. Clearly $(0,0) \in q^{-1}(p_1^{-1}(P_2))$, but we also require $(0,0)$ to be in the line L of Theorem 4.1.2. Since L has parametric form $(x,y) = t\lambda + c_2\mu$ and (λ,μ) is an ordered basis of Λ_2, $(0,0)$ is in L if and only if $c_2 = 0$. Table 1 shows that this is in turn equivalent to the condition $q \equiv 0 \mod 4$ if $2p+q \equiv 0 \mod 3$. With $v=(0,0)$ the element w is an integer multiple of $\lambda = (q,p)$; i.e. $w = x(q,p)$ where x is an integer. Without loss of generality, we may assume that x is a positive integer. We want $w = x(q,p)$ to be in $q_1^{-1}(p_1^{-1}(P_2))$ with no elements of $q_1^{-1}(p_1^{-1}(P_1 \cup P_2))$ strictly between $v=(0,0)$ and $w = x(q,p)$. Let (r,s) be one of our eight coset representatives. We are interested in the congruence $(r,s) \equiv x(q,p) \mod 2\Lambda_1$. So we are interested in integers y and z such that

$$(r,s) = x(q,p) + y(4,-2) + z(0,6).$$

The following equations give y and z as rational numbers.

$$y = \frac{1}{4}(r - xq) \quad \text{and} \quad z = \frac{1}{12}(2s + r - x(2p+q))$$

Claim 4.2.1. The numbers y and z are integers if and only if

$$r \equiv xq \mod 4, \qquad s \equiv xp \mod 2, \qquad 2s + r \equiv x(2p+q) \mod 3 \tag{4.2.1}$$

Proof. [$\Rightarrow$] Since y and z are integers, $r - xq \equiv 0 \mod 4$ and $2s + r - x(2p+q) \equiv 0 \mod 12$. Then $2(s - xp) \equiv xq - r \equiv 0 \mod 4$. Therefore, $s - xp \equiv 0 \mod 2$. Clearly $2s + r - x(2p+q) \equiv 0 \mod 3$.

[$\Leftarrow$] Obviously y is an integer. Since $s - xp \equiv 0 \mod 2$, then $2(s - xp) \equiv 0 \mod 4$. By assumption $r - xq \equiv 0 \mod 4$, whence $2s + r - x(2p+q) \equiv 0 \mod 4$. By assumption $2s + r - x(2p+q) \equiv 0 \mod 3$, whence $2s + r - x(2p+q) \equiv 0 \mod 12$ and the claim follows. $\square$

So, assuming that $q \equiv 0 \mod 4$ if $2p + q \equiv 0 \mod 3$, then we may take $v = (0,0)$ and $w = x(q,p)$, where x is the smallest positive integer which satisfies Equation 4.2.1 for some choice of (r,s).

First assume that $q \equiv 0 \mod 4$. Equation 4.2.1 implies that $r \equiv 0 \mod 4$. Thus either $(r,s) = (0,0) \in \Lambda_1$ or $(r,s) = (0,3) \in \Lambda_1$. Then either

$$(0,0) = x(q,p) + 2y(2,-1) + 2z(0,3) \quad \text{or} \quad (0,0) = x(q,p) + 2y(2,-1) + (2z-1)(0,3)$$

In any case, x is the order of the image of $\lambda = (q,p) \in \Lambda_2/\Lambda_1$. Theorem 3.1 and Table 1 imply that $x = 1$ if $2p + q \equiv 0 \mod 3$ and $x = 3$ if $2p + q \equiv \pm 1 \mod 3$. This gives the first two lines of Table 2.

Next assume that $q \cong 2 \mod 4$. Since $\gcd(p,q) = 1$, $p \equiv 1 \mod 2$. If $x = 1$, then Equation 4.2.1 yields

$$r \equiv q \mod 4, \qquad s \equiv p \mod 2, \qquad 2s + r \equiv (2p+q) \mod 3$$

and so

$$r \equiv 2 \mod 4, \qquad s \equiv 1 \mod 2, \qquad 2s + r \equiv (2p+q) \mod 3.$$

These congruences have a solution with $(r,s) \in q_1^{-1}(p_1^{-1}(P_2))$ if and only if either $(r,s) \in \{(2,1),(2,3)\}$ and $2p + q \cong \pm 1 \mod 3$. This gives line 6 of Table 2. If $x = 2$, then Equation 4.2.1 yields

$$r \equiv 2q \mod 4, \qquad s \equiv 2p \mod 2, \qquad 2s + r \equiv 2(2p+q) \mod 3.$$

Then $r \equiv 0 \mod 4$ and $s \equiv 0 \mod 2$. Hence $(0,0) = (r,s) = 2(q,p) + y(4,-2) + z(0,6)$. Thus $(q,p) = -y(2,-1) - z(0,3) \in \Lambda_1$. Hence $2p + q = -6z \equiv 0 \mod 3$. Since $c_2 = 0$ is equivalent to the condition $q \equiv 0 \mod 4$ if $2p + q \equiv 0 \mod 3$, then $q \equiv 0 \mod 4$ must hold. But this contradicts the assumption $q \equiv 2 \mod 4$. Thus there is no acceptable value of w with $x = 2$.

Now suppose that $q \equiv \pm 1 \mod 4$. Then $r \equiv xq \equiv \pm x \mod 4$. If x is odd then r has to be odd, which is impossible. So x is even. Note that in this case p also has to be even. If $x = 2$, then Equation 4.2.1 yields

$$r \equiv 2 \mod 4, \qquad s \equiv 0 \mod 2, \qquad 2s + r \equiv 2(2p + q) \mod 3.$$

There is a solution with $(r, s) \in q_1^{-1}(p_1^{-1}(P_2))$ if and only if $(r, s) \in \{(2, 0), (2, -2)\}$ and $2p + q \equiv \pm 1 \mod 3$. Since $q \equiv \pm 1 \mod 4$ implies $2p + q \not\equiv 0 \mod 3$ we do not need to check any other case for x.

We have handled all cases in which it is possible to choose $v = (0, 0)$. It remains to discuss the cases for which $q \not\equiv 0 \mod 4$ and $2p + q \equiv 0 \mod 3$. Table 1 shows that these are precisely the cases in which $c_1 < c_2 < c_3 < c_4$. Moreover, Table 1 shows that $c_1 < c_2 < c_3 < c_4$ is either $0 < 1 < 2 < 3$ or $0 < 2 < 4 < 6$. Since the parametric form of the line L of Theorem 4.1.2 is either $(x, y) = t\lambda + c_2(1, 0)$ or $(x, y) = t\lambda + c_3(1, 0)$, we may choose the line L such that its parametric form is

$$(x, y) = t\lambda + (2, 0).$$

Recall that $\lambda = (q, p)$. So we are interested in the congruence $(r - 2, s) \equiv x(q, p) \mod 2\Lambda_1$. More precisely, we are interested in integers y and z such that

$$(r - 2, s) = x(q, p) + y(4, -2) + z(0, 6).$$

The following equations give y and z as rational numbers.

$$y = \frac{1}{4}(r - 2 - xq) \quad \text{and} \quad z = \frac{1}{12}(2s + r - 2 - x(2p + q))$$

Claim 4.2.2. The numbers y and z are integers if and only if

$$r - 2 \equiv xq \mod 4, \qquad s \equiv xp \mod 2, \qquad 2s + r - 2 \equiv x(2p + q) \mod 3 \tag{4.2.2}$$

Proof. Proceed as in Claim 4.2.1. □

Assume that $q \not\equiv 0 \mod 4$ and $2p + q \equiv 0 \mod 3$ and $x \equiv 1 \mod 2$. It is easy to see that $r \not\equiv 2 \mod 4$, otherwise $0 \equiv xq \mod 4$. So $(r, s) = (0, 0)$ or $(r, s) = (0, 3)$. Since $2p+q \equiv 0 \mod 3$, the last congruence in Equation 4.2.2 implies that $2s+r-2 \equiv 0 \mod 3$. So if $(r, s) = (0, 0)$ we have $-2 \equiv 0 \mod 3$, which is a contradiction; if $(r, s) = (0, 3)$ we get $4 \equiv 0 \mod 3$, which is also a contradiction. Thus $x \equiv 0 \mod 2$. If $x = 2$ the congruences in Equation 4.2.2 are solved by choosing $q \equiv 2 \mod 4$, $2p+q \equiv 0 \mod 3$ and $(r, s) = (2, 0)$. Now if $q \equiv \pm 1 \mod 4$ and $2p + q \equiv 0 \mod 3$ and $x = 2$ it is easy to see that $r \equiv 0 \mod 4$ and $s \equiv 0 \mod 2$, whence $(r, s) = (0, 0)$ but this contradicts the last congruence given in Equation 4.2.2. Since x has to be even, our next possibility is $x = 4$. Assuming that $q \equiv \pm 1 \mod 4$, $2p + q \equiv 0 \mod 3$ and $x = 4$, it is not difficult to see that $(r, s) = (2, 0)$ gives a solution for Equation 4.2.2. This completes the computations of Table 2.

$q \mod 4$	$2p+q \mod 3$	v	w
0	0	$(0,0)$	(q,p)
0	± 1	$(0,0)$	$(3q,3p)$
± 1	0	$(2,0)$	$(2+4q,4p)$
± 1	± 1	$(0,0)$	$(2q,2p)$
2	0	$(2,0)$	$(2+2q,2p)$
2	± 1	$(0,0)$	(q,p)

Table 2. Values of v and w for the main example.

Now we find the lattice points $\lambda_1, \ldots, \lambda_n$ which appear in Theorem 4.1.2. Note that a point $(x,y) \in \mathbb{R}^2$ is the center of a spin mirror for $p_1 \circ q_1$ is and only if x is an integer congruent to 2 modulo 4 and there exists an integer Q_x such that $2x + y = 6Q_x$. A $(x,y) \in \mathbb{R}^2$ is in a spin mirror for $p_1 \circ q_1$ if and only if x is an integer congruent to 2 modulo 4 and there exists an integer Q_x and a real number R_x with $|R_x| \leq 2$ such that $x + 2y = 10Q_x + R_x$. Figure 4.1 shows some spin mirrors for $p_1 \circ q_1$.

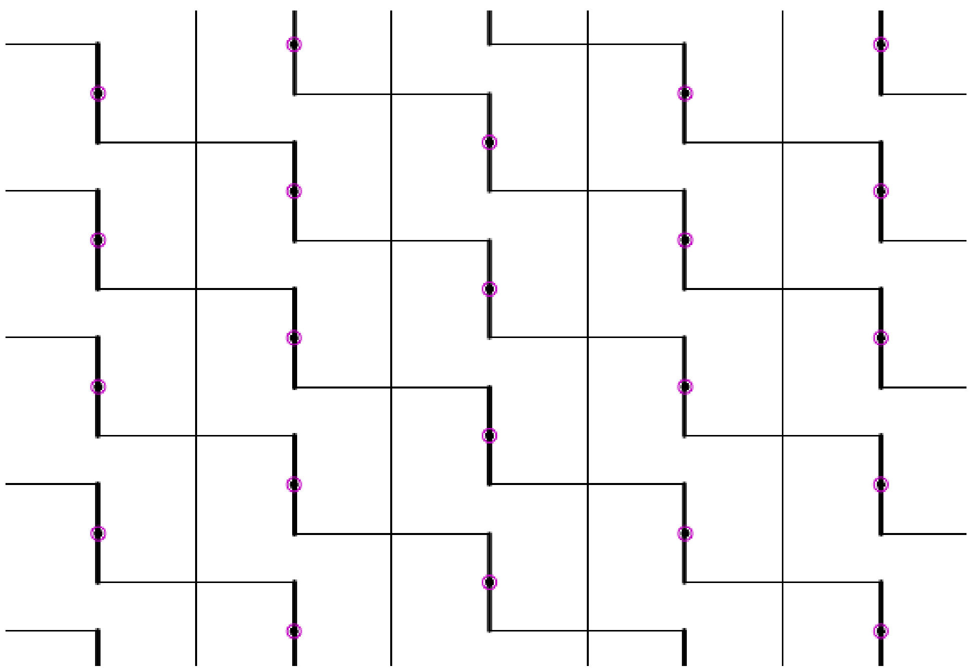


Figure 4.1: Spin mirrors for $p_1 \circ q_1$

Suppose that $v = (0,0)$. Let $w = (w_1, w_2)$. The line segment joining v and w is in the line given by $y = \frac{p}{q}x$. For every integer x, define Q_x and R_x so that

$$\Big(1 + \frac{2p}{q}\Big)x = 6Q_x + R_x \quad \text{with } Q_x \in \mathbb{Z}, R_x \in \mathbb{Q} \ \text{ and } \ -3 < R_x \leq 3.$$

Let $0 < x_1 < x_2 < x_3 < \cdots < x_n < w_1$ be those integers congruent to 2 mod 4 such that $|R_{x_i}| < 2$. Set $x_0 = 0$ and $x_{n+1} = w_1$. If λ_i is in the center of the spin mirror as in Theorem 4.1.2, then

$$\lambda_i = \left(x_i, \frac{6Q_{x_i} - x_i}{2}\right) = \frac{1}{2}x_i(2,-1) + Q_{x_i}(0,3) \quad \text{for} \quad i \in \{0,\ldots,n+1\}.$$

Set

$$N = \sum_{i=0}^{n} (-1)^i (Q_{x_{i+1}} - Q_{x_i})$$

and

$$D = \frac{1}{2}\sum_{i=0}^{n} (-1)^i (x_{i+1} - x_i).$$

By Theorem 4.1.2 we obtain

$$\sigma_f\left(\frac{p}{q}\right) = \frac{N}{D}.$$

Now suppose that $v = (0,0)$. Let $w = (w_1, w_2)$. The line segment joining v and w is in the line given by $y = \frac{p}{q}x - \frac{2p}{q}$. For every integer x, define Q_x and R_x so that

$$\left(1 + \frac{2p}{q}\right)x = 6Q_x + R_x \quad \text{with } Q_x \in \mathbb{Z}, R_x \in \mathbb{Q} \quad \text{and} \quad -3 < R_x \leq 3.$$

Let $2 < x_1 < x_2 < x_3 < \cdots < x_n < w_1$ be those integers congruent to 2 module 4 such that $|R_{x_i}| < 2$. Set $x_0 = 2$ and $x_{n+1} = w_1$. Set

$$N = \sum_{i=0}^{n} (-1)^i (Q_{x_{i+1}} - Q_{x_i})$$

and

$$D = \frac{1}{2}\sum_{i=0}^{n} (-1)^i (x_{i+1} - x_i).$$

Then

$$\sigma_f\left(\frac{p}{q}\right) = \frac{N}{D}.$$

This algorithm can be used by hand in simple cases. As an illustration, we compute $\sigma_f(\frac{1}{4})$. Table 2 shows that $v = (0,0)$ and $w = (4,1)$. Since $(1 + \frac{2\cdot 1}{4}) \cdot 2 = 6 \cdot 0 + 3$, we have that $n = 0$, $x_0 = 0$, $x_1 = 4$, $Q_{x_0} = 0$ and $Q_{x_1} = 1$. So $N = Q_{x_1} - Q_{x_0} = 1 - 0 = 1$ and $D = \dfrac{1}{2}(x_1 - x_0) = 2$. Therefore,

$$\sigma_f\left(\frac{1}{4}\right) = \frac{N}{D} = \frac{1}{2}.$$

4.3 Horoballs in Teichmüller Space

In this section we relate horoballs in $\mathbb{H}$ to moduli of curve families. For more details see Section 6 of [9].

Let f be a NET map. As in Section 3.3, we fix an ordered basis (λ_2, μ_2) of Λ_2 by which we define slopes of multicurves in $S^2 \setminus P_2$. Recall that in this setting $P_2 = P_f$. Let $\mathbb{H}$ be the upper half plane, i.e. $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. We may identify $\mathbb{H}$ with the Teichmüller space of $S^2 \setminus P_2$ as follows. For each $\tau \in \mathbb{H}$, there exists a unique $\mathbb{R}$-linear isomorphism $\varphi : \mathbb{C} \to \mathbb{R}^2$ so that $\varphi(1) = 2\lambda_2$ and $\varphi(\tau) = 2\mu_2$. The map $p_2 \circ q_2 \circ \varphi : \mathbb{C} \to S^2 \setminus P_2$ induces a complex structure on $S^2 \setminus P_2$. As τ varies over $\mathbb{H}$, the resulting isotopy classes of complex structures on $S^2 \setminus P_2$ are distinct, and every complex structure on $S^2 \setminus P_2$ is isotopic to one of them. In this way we regard $\mathbb{H}$ as the Teichmüller space of $S^2 \setminus P_2$.

Remark 4.3.1. For a sphere with four marked points, the Teichmüller and hyperbolic metrics coincide. In this thesis we regard $\mathbb{H}$ as a metric space with respect to the hyperbolic metric.

Horocycles in $\mathbb{H}$ at ∞ are simply horizontal lines, which are given by equations of the form $\mathrm{Im}(z) = m$ for positive real numbers m. So horoballs in $\mathbb{H}$ at ∞ are simply sets of the form

$$\{z \in \mathbb{H} : \mathrm{Im(z)} > m\}.$$

On the other hand, let p and q be relatively prime integers with $q \neq 0$. Horocycles in $\mathbb{H}$ at p/q are Euclidean circles given by equations of the form

$$\frac{\mathrm{Im}(z)}{|qz - p|^2} = m,$$

for positive real numbers m. One sees that the relation between the Euclidean diameter of the horocycle and the number m is

$$D = \frac{1}{q^2 m}.$$

So if p and q are relatively prime integers, then horoballs in $\mathbb{H}$ at $p/q \in \widehat{\mathbb{Q}}$ are the subsets of the form

$$\{z \in \mathbb{H} : \frac{\mathrm{Im}(z)}{|qz - p|^2} > m\}.$$

for positive real numbers m.

Now we define the modulus of a curve family. Let $\tau \in \mathbb{H}$, let $\Lambda_\tau = \langle 1, \tau \rangle$, and let $T_\tau = \mathbb{C}/\Lambda_\tau$. We denote by $\Gamma_{\frac{p}{q},\tau}$ the set of simple closed curves in T_τ with slope $\frac{p}{q}$ with respect to the ordered basis $(1, \tau)$ of Λ_τ. By abuse of notation we write dz for the 1-form on T_τ induced by the standard 1-form on $\mathbb{C}$. For a nonnegative Borel measurable function ρ on T_τ, define

$$L_\rho(\Gamma_{\frac{p}{q},\tau}) = \inf\left\{ \int_\gamma \rho|dz| : \gamma \in \Gamma_{\frac{p}{q},\tau} \right\}$$

and

$$A_\rho = \iint\limits_{T_\tau} \rho^2 |dz|^2.$$

The modulus of the curve family $\Gamma_{\frac{p}{q},\tau}$ on T_τ is

$$\mathrm{mod}_\tau\left(\frac{p}{q}\right) = \inf_\rho \frac{A_\rho}{L^2_\rho(\Gamma_{\frac{p}{q},\tau})}.$$

For every $\frac{p}{q} \in \widehat{\mathbb{Q}}$ and every real $m > 0$, we set $B_m(p/q) = \{\tau \in \mathbb{H} : \mathrm{mod}_\tau(\frac{p}{q}) > m\}$. The following lemma relates the modulus $\mathrm{mod}_\tau(\frac{p}{q})$ and the positive real number $\dfrac{\mathrm{Im}(\tau)}{|q\tau + p|^2}$. For more details see Lemma 5.3 of [9].

Lemma 4.3.1. *For every $\frac{p}{q} \in \widehat{\mathbb{Q}}$ and $\tau \in \mathbb{H}$ we have that*

$$\mathrm{mod}_\tau\left(\frac{p}{q}\right) = \frac{\mathrm{Im}(\tau)}{|q\tau + p|^2}.$$

Corollary 4.3.2. *If $p/q \in \widehat{\mathbb{Q}}$ and if m is a positive real number, then*

$$B_m(p/q) = \{\tau \in \mathbb{H} : \frac{\mathrm{Im}(\tau)}{|p\tau + q|^2} > m\},$$

a horoball in $\mathbb{H}$ at $-q/p$.

Let $\iota : T_\tau \to T_\tau$ be the involution defined by $\iota(z) = -z$. The quotient space $T_\tau/(z \sim \iota(z))$ is topologically a 2-sphere. Denote this quotient by S_τ. Let $p_\tau : T_\tau \to S_\tau$ be the corresponding degree 2 branched covering map and let P_τ be the set of four branch points of p_τ in S_τ. Let $\frac{p}{q} \in \widehat{\mathbb{Q}}$. Denote by $\tilde{\Gamma}_{\frac{p}{q},\tau}$ the set of essential simple closed curves in $S_\tau \setminus P_\tau$ with slope $\frac{p}{q}$. By definition, the elements of $\tilde{\Gamma}_{\frac{p}{q},\tau}$ lift under p_τ to $\Gamma_{\frac{p}{q},\tau}$. We define the modulus of this family just as we defined $\mathrm{mod}_\tau\left(\dfrac{p}{q}\right)$.

Proposition 4.3.3. *In the previous setting,*

$$\inf_\rho \frac{\tilde{A}_\rho}{L^2_\rho\left(\tilde{\Gamma}_{\frac{p}{q},\tau}\right)} = \frac{1}{2}\mathrm{mod}_\tau\left(\frac{p}{q}\right)$$

Proof. When pulling back from S_τ to T_τ, lengths of curves do not change but area doubles. Then,

$$\frac{2\tilde{A}_\rho}{L^2_\rho\left(\tilde{\Gamma}_{\frac{p}{q},\tau}\right)} = \frac{A_\rho}{L^2_\rho\left(\Gamma_{\frac{p}{q},\tau}\right)}.$$

So this new modulus is $\dfrac{1}{2}\mathrm{mod}_\tau\left(\dfrac{p}{q}\right)$. $\square$

Let f be a NET map. Suppose $\frac{p}{q} \in \widehat{\mathbb{Q}}$ and let γ be an essential simple closed curve in $S^2 \setminus P_2$ with slope $\frac{p}{q}$. We define $\delta_f : \widehat{\mathbb{Q}} \to \mathbb{Q}$ as

$$\delta_f\Big(\frac{p}{q}\Big) = \frac{c}{d},$$

where d is the degree with which f maps every connected component of $f^{-1}(\gamma)$ to γ, and c is the number of these connected components which are essential and nonperipheral. Note that a multicurve Γ whose only element is γ is f-stable if and only if either $\sigma_f(p/q) = p/q$ or $\sigma_f(p/q) = \emptyset$. Also, note that if Γ is f-stable, then A^Γ is a 1×1 matrix with entry $\delta_f(p/q)$. Hence Γ is a Thurston obstruction if and only if $p/q \in \mathrm{Fix}(\sigma_f)$ and $\delta_f(p/q) \geq 1$.

Now, let $\Sigma_f : \mathbb{H} \to \mathbb{H}$ be the Thurston map on Teichmüller space induced by f. Let $\tau \in \mathbb{H}$, let $\tau' = \Sigma_f(\tau)$, let $\frac{p'}{q'} = \sigma_f(\frac{p}{q})$ and let $\delta = \delta_f(\frac{p}{q})$. Then an argument based on the subadditivity of moduli proves that

$$\mathrm{mod}_{\tau'}\Big(\frac{p'}{q'}\Big) \geq \delta \mathrm{mod}_{\tau}\Big(\frac{p}{q}\Big).$$

So, if

$$\mathrm{mod}_{\tau}\Big(\frac{p}{q}\Big) > m,$$

then

$$\mathrm{mod}_{\tau'}\Big(\frac{p'}{q'}\Big) \geq \delta m.$$

Hence

$$\Sigma_f\Big(B_m\Big(\frac{p}{q}\Big)\Big) \subseteq B_{\delta m}\Big(\frac{p}{q}\Big).$$

for every positive real number m. The following theorem shows how knowledge of the pullback of a given curve under f translates into an interval of slopes in which the slope of a Thurston obstruction cannot lie. For details of the proof see Section 6 of [9].

Theorem 4.3.4. *In the previous setting we have the following.*

1. *If $\frac{p}{q} \neq \frac{p'}{q'}$, then for every sufficiently large $m > 0$, the closed horoballs $B = \overline{B_m(\frac{p}{q})}$ and $B' = \overline{B_{\delta m}(\frac{p'}{q'})}$ are disjoint. When they are disjoint the set*

$$H = \{\tau \in \mathbb{H} : d(\tau, B) < d(\tau, B')\}$$

   is an open hyperbolic half space which is independent of m.

2. *If $\frac{r}{s}$ is a fixed point of σ_f and $-\frac{s}{r} \in \partial_\infty H$, then $\delta_f(\frac{r}{s}) < 1$, i.e., there is no Thurston obstruction with slope $\frac{r}{s}$.*

3. *If τ_0 is a fixed point of Σ_f, then $\tau_0 \notin H$.*

We provide a brief explanation of the construction of the half space H. If neither $\frac{p}{q}$ nor $\frac{p'}{q'}$ is zero, then the Euclidean diameters of B and B' are $\frac{1}{mp^2}$ and $\frac{1}{\delta m p'^2}$ respectively. So, if $\frac{p}{q} \neq \frac{p'}{q'}$ and $m > 0$ is sufficiently large, then B and B' are disjoint. Assume that m is this large. Let l be the geodesic joining $-\frac{q}{p}$ and $-\frac{q'}{p'}$, let $l_m \subset l$ be the closure of the geodesic segment lying outside $B \cup B'$ and let $l_m^\perp$ be its perpendicular bisector. Note that the hyperbolic distance between the horocycles $\partial B_m(\frac{p'}{q'})$ and $\partial B_{\delta m}(\frac{p'}{q'})$ is $|\ln(\delta)|$, which is independent of m; hence the perpendicular bisector $l_m^\perp$ is also independent of m and so H is an open hyperbolic half space which is independent of m. From now on we denote by $l^\perp$ the perpendicular bisector $l_m^\perp$. If neither $\frac{p}{q}$ nor $\frac{p'}{q'}$ is 0 and the radii of B and B' are unequal, the boundary of the half space H in $\mathbb{H}$ is the geodesic curve $l^\perp$. See Figure 4.2 below.

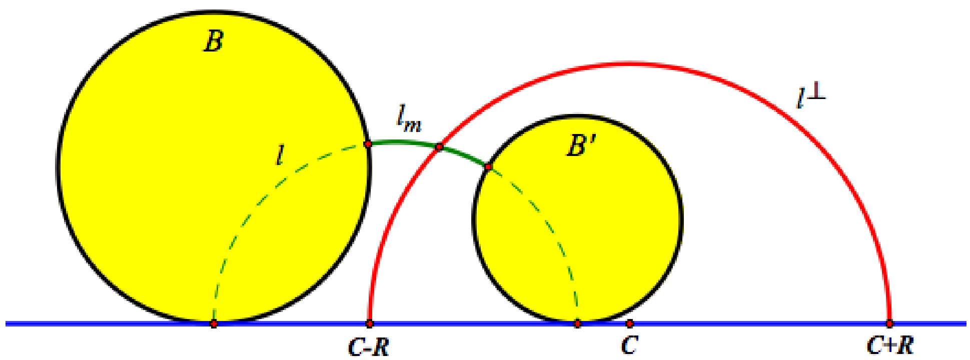


Figure 4.2: The generic situation in Theorem 4.3.4

By using this generic situation, in [9], J.W.Canon et al. deduce the following formulas

$$C = \frac{-pq + \delta p'q'}{p^2 - \delta^2 p'^2} \quad \text{and} \quad R = \left|\frac{(pq' - p'q)\sqrt{\delta}}{p^2 - \delta^2 p'^2}\right|.$$

Note that if $\delta > \frac{p^2}{p'^2}$, then B has larger Euclidean radius than B' and H is the region in $\mathbb{H}$ outside the Euclidean circle with center C and radius R. In this case we say that H is **unbounded**. If $\delta < \frac{p^2}{p'^2}$, then H is the region in $\mathbb{H}$ within this circle. In this case we say that H is **bounded**. Finally, if $\delta = \frac{p^2}{p'^2}$, then B and B' have the same Euclidean radius and $l^\perp$ is a Euclidean half-line. One endpoint of $l^\perp$ is ∞ and the other point is $-\frac{1}{2}(\frac{q}{p} + \frac{q'}{p'})$. So if $\delta = \frac{p^2}{p'^2}$, then

$$H = \{\tau \in \mathbb{H} : \mathrm{Re}(\tau) < -\frac{1}{2}(\frac{q}{p} + \frac{q'}{p})\} \quad \text{if} \quad \frac{p}{q} < \frac{p'}{q'}$$

and

$$H = \{\tau \in \mathbb{H} : \mathrm{Re}(\tau) > -\frac{1}{2}(\frac{q}{p} + \frac{q'}{p})\} \quad \text{if} \quad \frac{p}{q} > \frac{p'}{q'}.$$

4.4 Location of the fixed point for the main example

Using the results presented in the previous sections we provide the location of the fixed point for the main example. From now on f is the subdivision map given in the main example and σ_f is the slope map corresponding to f with respect to the ordered basis $\{(1,0),(0,1)\}$ of Λ_2.

Claim 4.4.1. For any $k \in \mathbb{Z}$, $\sigma_f(3k) = 2k$, $\sigma_f(3k+1) = 2k+1$ and $\sigma_f(3k+2) = 2k+2$.

Proof. Since $q = 1$, $q \equiv 1 \mod 4$. Suppose $p = 3k$, then $2p+q \equiv 6k+1 \equiv 1 \mod 3$. By Table 2, $v = (0,0)$ and $w = (2, 6k)$ and so $x_0 = 0$ and $x_1 = w_1 = 2$. Then $Q_{x_0} = 0$ and $Q_{x_1} = 2k$. Thus $N = (2k - 0)$ and $D = 0.5(2-0)$. Hence $\sigma_f(3k) = 2k$. If $p = 3k+1$, then $2p+q \equiv 6k+3 \equiv 0 \mod 3$. By Table 2, $v = (2,0)$ and $w = (6, 12k+4)$ and so $x_0 = 2$ and $x_1 = w_1 = 6$. Then $Q_{x_0} = 0$ and $Q_{x_1} = 4k+2$. Thus, $N = (4k+2-0)$ and $D = 0.5(6-2)$. Hence $\sigma_f(3k+1) = 2k+1$. Finally, if $p = 3k+2$, then $2p+q \equiv 6k+5 \equiv -1 \mod 3$. In this case, $v = (0,0)$ and $w = (2, 6k+4)$ and so $x_0 = 0$ and $x_1 = w_1 = 2$. Then, $Q_{x_0} = 0$ and $Q_{x_1} = 2k+1$. Hence, $N = (2k+2-0)$ and $D = 0.5(2-0)$. Therefore, $\sigma_f(3k+2) = 2k+2$. $\square$

Remark 4.4.1. Note that $\sigma_f(n) = n - [n/3]$. Hence, $\mathrm{Fix}(\sigma_f) \bigcap \mathbb{Z} = \{0,1,2\}$. Furthermore, using Tables 1 and 2 for this example, one sees that if $n \not\equiv 1 \mod 3$, then $\delta_f(n/1) = 1/6$ while if $n \equiv 1 \mod 3$, then $\delta_f(n/1) = 1/2$.

Claim 4.4.2. For any $k \in \mathbb{Z}^+$, $k \geq 2$ we have

$$\sigma_f\Big(\frac{2^k+1}{2^k}\Big) = \frac{2^{k-1}+1}{2^{k-1}}$$

Proof. Set $p = 2^k + 1$ and $q = 2^k$. It is clear that $q \equiv 0 \mod 4$ and $2p+q \equiv \pm 1 \mod 3$. By Table 2, $v = (0,0)$ and $w = (3q, 3p)$. In this case the line segment joining v and w is in the line given by $y = \frac{p}{q}x$. For every integer x, we defined Q_x and R_x so that

$$\Big(1 + \frac{2p}{q}\Big)x = 6Q_x + R_x \text{ with } Q_x \in \mathbb{Z}, R_x \in \mathbb{Q} \text{ and } -3 < R_x \leq 3.$$

Recall that

$$\sigma_f\Big(\frac{p}{q}\Big) = \frac{N}{D}$$

where

$$N = \sum_{i=0}^{n} (-1)^i (Q_{x_{i+1}} - Q_{x_i}),$$

$$D = \frac{1}{2}\sum_{i=0}^{n} (-1)^i (x_{i+1} - x_i),$$

and $x_0 = 0 < x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} = w_1$ are those integers congruent to 2 mod 4 such that $|R_{x_i}| < 2$.

Suppose $x \equiv 2 \mod 4$. We first show that if $2^k < x < 2 \cdot 2^k$ then $|R_x| > 2$. Split the proof into two cases.

If $2^k < x < (3/2) \cdot 2^k$, then $2 < \frac{2x}{2^k} < 3$. Note that $\left(1 + \dfrac{2p}{q}\right)x = 3x + \dfrac{2x}{2^k}$. Since x is even, $3x$ is a multiple of 6. This forces $R_x = \frac{2x}{2^k}$. So $2 < |R_x| < 3$.

If $(3/2) \cdot 2^k < x < 2 \cdot 2^k$, then $3 < \frac{2x}{2^k} < 4$. Note that $\left(1 + \dfrac{2p}{q}\right)x = 3x + \dfrac{2x}{2^k} = 3x + 6 + (\dfrac{2x}{2^k} - 6)$. Since x is even, $3x + 6$ is a multiple of 6. This forces $R_x = \frac{2x}{2^k} - 6$. So $2 < |R_x| < 3$.

We now show that if $0 \leq x < 2^k$ or $2 \cdot 2^k < x < 3 \cdot 2^k$, then $|R_x| < 2$. As above, analyze two cases.

If $0 \leq x < 2^k$, then $0 \leq \frac{2x}{2^k} < 2$. Note that $\left(1 + \dfrac{2p}{q}\right)x = 3x + \dfrac{2x}{2^k}$. Since x is even, $3x$ is a multiple of 6. This forces $Q_x = \frac{x}{2}$ and $R_x = \frac{2x}{2^k}$. So $0 < |R_x| < 2$.

If $2 \cdot 2^k < x < 3 \cdot 2^k$, then $4 < \frac{2x}{2^k} < 6$. Note that $\left(1 + \dfrac{2p}{q}\right)x = 3x + \dfrac{2x}{2^k} = 3x + 6 + (\dfrac{2x}{2^k} - 6)$ This forces $Q_x = \frac{x}{2} + 1$ and $R_x = \frac{2x}{2^k} - 6$. So $0 < |R_x| < 2$.

Since $x_{n+1} = w_1 = 3q = 3(2^k)$, one sees that $Q_{x_{n+1}} = \frac{x_{n+1}}{2} + 1$. Now, it follows that

$$N = \sum_{i=0}^{n} (-1)^i (Q_{x_{i+1}} - Q_{x_i}) = [2^k + 2 - (2^{k-1} - 1)] - [2^k + 4 - (2^k - 2)] = 2^{k-1} + 1$$

and

$$2D = \sum_{i=0}^{n} (-1)^i (x_{i+1} - x_i) = [2 \cdot 2^k + 2 - (2^k - 2)] - [4] = 2^k.$$

Therefore,

$$\sigma_f\left(\frac{2^k + 1}{2^k}\right) = \frac{N}{D} = \frac{2^{k-1} + 1}{2^{k-1}}$$

$\square$

Claim 4.4.3. For any $k \in \mathbb{Z}^+$, $k \geq 2$ we have

$$\sigma_f\left(\frac{2^k - 1}{2^k}\right) = \frac{2^{k-1} - 1}{2^{k-1}}$$

Proof. Proceed as above. $\square$

Remark 4.4.2. In the previous setting, if $p = 2^k \pm 1$ and $q = 2^k$ then $q \equiv 0 \mod 4$ and $2p + q \equiv \pm 1 \mod 3$. Thus, the Thurston multiplier is $\delta_f(p/q) = 2/3$. Note that in this case the Thurston multiplier is independent of k.

Claim 4.4.4. **Open covering from the Half Spaces: $H_{3k}, |k| \geq 3$.**
For each $k \in \mathbb{Z} \setminus \{0\}$, let $\frac{p_k}{q_k} = \frac{3k}{1}$, $\frac{p'_k}{q'_k} = \sigma_f(\frac{p_k}{q_k}) = \frac{2k}{1}$ and $\delta = \delta_f(\frac{p_k}{q_k}) = 1/6$. Let H_{3k} be the corresponding open hyperbolic half space as in Theorem 4.3.4. Let $l^{\perp}_{(3k)}$ be the geodesic curve that determines H_{3k}; i.e. $l^{\perp}_{(3k)} = \partial H_{3k} \cap \mathbb{H}$. Let $a_k < b_k$ be the endpoints of $l^{\perp}_{(3k)}$. Then the following statements hold.

1. For each $k \in \mathbb{Z} \setminus \{0\}$, H_{3k} is bounded.
2. If $k \geq 3$, then $a_k < a_{k+1} < b_k < b_{k+1} < 0$.
3. If $k \leq -3$, then $0 < a_{k-1} < a_k < b_{k-1} < b_k$.
4. $\lim\limits_{|k| \to \infty} a_k = 0$.

Proof.

1. It follows from the inequality
$$\frac{1}{6} = \delta < \frac{p_k^2}{p_k'^2} = \left(\frac{3k}{2k}\right)^2 = \frac{3}{2}.$$

2. Using the formulas for the center and the radius of $l^{\perp}_{(3k)}$, we have
$$a_k = -\frac{1}{k}\Big(\frac{1+\sqrt{\delta}}{3+2\sqrt{\delta}}\Big) \quad \text{and} \quad b_k = -\frac{1}{k}\Big(\frac{1-\sqrt{\delta}}{3-2\sqrt{\delta}}\Big),$$
for any $k \geq 1$. Hence $a_k < a_{k+1}$ and $b_k < b_{k+1} < 0$. It remains to show that
$$-\frac{1}{k+1}\Big(\frac{1+\sqrt{\delta}}{3+2\sqrt{\delta}}\Big) < -\frac{1}{k}\Big(\frac{1-\sqrt{\delta}}{3-2\sqrt{\delta}}\Big)$$
which is equivalent to the inequality
$$\frac{1}{k}\Big(\frac{1-\sqrt{\delta}}{3-2\sqrt{\delta}}\Big) < \frac{1}{k+1}\Big(\frac{1+\sqrt{\delta}}{3+2\sqrt{\delta}}\Big)$$
and equivalent to
$$(1-\sqrt{\delta})(3+2\sqrt{\delta}) < k\Big[(1+\sqrt{\delta})(3-2\sqrt{\delta}) - (1-\sqrt{\delta})(3+2\sqrt{\delta})\Big].$$
So we require $(1-\sqrt{\delta})(3+2\sqrt{\delta}) < k \cdot 2\sqrt{\delta}$. This last inequality holds for $k \geq 3$.

3. Using the formulas for the center and the radius of $l^{\perp}_{(3k)}$, we have

$$a_k = -\frac{1}{k}\Big(\frac{1-\sqrt{\delta}}{3-2\sqrt{\delta}}\Big) \quad \text{and} \quad b_k = -\frac{1}{k}\Big(\frac{1+\sqrt{\delta}}{3+2\sqrt{\delta}}\Big),$$

for any $k \leq -1$. Then proceed as in the proof of statement 2.

4. It follows from the form of a_k.

□

Claim 4.4.5. **Open covering from the Half Spaces: $H^+_{p_k/q_k}$, $p_k = 2^k + 1$ and $q_k = 2^k$.** For each $k \in \mathbb{N} \setminus \{0\}$, let $\frac{p_k}{q_k} = \frac{2^k+1}{2^k}$, $\frac{p'_k}{q'_k} = \sigma_f(\frac{p_k}{q_k}) = \frac{2^{k-1}+1}{2^{k-1}}$ and $\delta = \delta_f(\frac{p_k}{q_k}) = 2/3$. Let $H^+_{p_k/q_k}$ be the corresponding open hyperbolic half space as in Theorem 4.3.4. Let $l^{\perp}_{(+,k)}$ be the geodesic curve that determines $H^+_{p_k/q_k}$; i.e. $l^{\perp}_{(+,k)} = \partial H^+_{p_k/q_k} \cap \mathbb{H}$. Let $a_k < b_k$ be the endpoints of $l^{\perp}_{(+,k)}$. Then the following statements hold.

1. For each $k \geq 2$, $H^+_{p_k/q_k}$ is bounded.
2. If $k \geq 2$, then $-1 < a_{k+1} < a_k < b_{k+1} < b_k$.
3. $\lim\limits_{k\to\infty} a_k = -1$.

Proof.

1. It follows from the inequality

$$\frac{2}{3} = \delta < \frac{p_k^2}{p_k'^2} = \Big(\frac{2^k+1}{2^{k-1}+1}\Big)^2.$$

2. Using the formulas for the center and the radius of $l^{\perp}_{(+,k)}$, we have

$$-1 < a_k = \frac{2^k - 2^{k-1}\sqrt{\delta}}{-(2^k+1)+(2^{k-1}+1)\sqrt{\delta}} < b_k = \frac{2^k + 2^{k-1}\sqrt{\delta}}{-(2^k+1)-(2^{k-1}+1)\sqrt{\delta}}$$

for any $k \geq 2$. It is furthermore true that

$$a_k = -\frac{1}{1+\frac{1}{2^k}\Big(\frac{1-\sqrt{\delta}}{1-\sqrt{\delta}/2}\Big)} \quad \text{and} \quad b_k = -\frac{1}{1+\frac{1}{2^k}\Big(\frac{1+\sqrt{\delta}}{1+\sqrt{\delta}/2}\Big)}.$$

Hence $a_{k+1} < a_k$ and $b_{k+1} < b_k < 0$. It remains to show that $a_k < b_{k+1}$ for all $k \geq 2$. This is equivalent to show that

$$-\frac{1}{1+\frac{1}{2^k}\left(\frac{1-\sqrt{\delta}}{1-\sqrt{\delta}/2}\right)} < -\frac{1}{1+\frac{1}{2^{k+1}}\left(\frac{1+\sqrt{\delta}}{1+\sqrt{\delta}/2}\right)}$$

This inequality is equivalent to $2 < 3\sqrt{\delta}$ which holds because $\delta = 2/3$.

3. It follows from the form of a_k.

$\square$

Claim 4.4.6. **Open covering from the Half Spaces: $H^-_{p_k/q_k}$, $p_k = 2^k - 1$ and $q_k = 2^k$.** For each $k \in \mathbb{N} \setminus \{0,1\}$, let $\frac{p_k}{q_k} = \frac{2^k-1}{2^k}$, $\frac{p'_k}{q'_k} = \sigma_f(\frac{p_k}{q_k}) = \frac{2^{k-1}-1}{2^{k-1}}$ and $\delta = \delta_f(\frac{p_k}{q_k}) = 2/3$. Let $H^-_{p_k/q_k}$ be the corresponding open hyperbolic half space as in Theorem 4.3.4. Let $l^{\perp}_{(-,k)}$ be the geodesic curve that determines $H^-_{p_k/q_k}$; i.e. $l^{\perp}_{(-,k)} = \partial H^-_{p_k/q_k} \cap \mathbb{H}$. Let $a_k < b_k$ be the endpoints of $l^{\perp}_{(-,k)}$. Then the following statements hold.

1. For each $k \geq 2$, $H^-_{p_k/q_k}$ is bounded.

2. If $k \geq 2$, then $-1 < a_{k+1} < a_k < b_{k+1} < b_k$.

3. $\lim\limits_{k\to\infty} a_k = -1$.

Proof.

1. It follows from the inequality

$$\frac{2}{3} = \delta < \frac{p_k^2}{p_k'^2} = \left(\frac{2^k-1}{2^{k-1}-1}\right)^2.$$

2. In this case, we have

$$a_k = \frac{2^k + 2^{k-1}\sqrt{\delta}}{-(2^k-1)-(2^{k-1}-1)\sqrt{\delta}} \quad \text{and} \quad b_k = \frac{2^k - 2^{k-1}\sqrt{\delta}}{-(2^k-1)+(2^{k-1}-1)\sqrt{\delta}}.$$

Now, proceed as in the previous claim.

3. It follows from the form of a_k.

$\square$

Due to the previous claims, we have the following facts:

1. Since $8/75$ is the center of $l^{\perp}_{(-9/1)}$, the open cover $\mathcal{U}_1 = \{\partial_\infty H_{3k} : k \leq -3\}$ covers the set $(0, 8/75]$.

2. Since $8/75$ is the center of $l^{\perp}_{(9/1)}$, the open cover $\mathcal{U}_2 = \{\partial_\infty H_{3k} : k \geq 3\}$ covers the set $[-8/75, 0)$.

3. Since $-16/19$ is the center of $l^{\perp}_{(5/4)}$, the open cover $\mathcal{U}_3 = \{\partial_\infty H^{+}_{p_k/q_k} : p_k = 2^k + 1, q_k = 2^k, k \geq 2\}$ covers the set $(-1, -16/19]$.

4. Since $-32/25$ is the center of $l^{\perp}_{(3/4)}$, the open cover $\mathcal{U}_4 = \{\partial_\infty H^{-}_{p_k/q_k} : p_k = 2^k - 1, q_k = 2^k, k \geq 2\}$ covers the set $[-32/25, -1)$.

It is furthermore true that

5. $\partial_\infty H_{1/4}$ is the complement of the interval $(-2\sqrt{2}, 2\sqrt{2})$.

6. The set $[-2\sqrt{2}, -32/25] \bigcup [-16/19, -8/75] \bigcup [8/75, 2\sqrt{2}]$ can be covered by finitely many open boundaries $\partial_\infty H$'s that come from bounded half spaces. More precisely, in Figure 4.3 –from left to right– we have:

   $[-2\sqrt{2}, -32/25]$ is covered by $1/2, 5/8$ and $3/4$ (blue circles).

   $[-16/19, -8/75]$ is covered by $5/4, 11/8, 3/2, 7/4, 5/2, 4$ and 7 (red circles).

   $[8/75, 2\sqrt{2}]$ is covered by $-13/2, -9/2, -2$ and $-1/2$ (green circles).

Main conclusion. The open set $\mathbb{R} \setminus \{-1, 0\}$ is covered by countably many open intervals that come from the boundary of open half-spaces. Statement 2 of Theorem 4.3.4 implies that $\delta_f(\frac{r}{s}) < 1$ for all $\frac{r}{s} \in \mathrm{Fix}(\sigma_f) \setminus \{0, 1\}$. On the other hand, Table 1 shows that

$$\delta_f\left(\frac{0}{1}\right) = \frac{1}{6} \quad \text{and} \quad \delta_f\left(\frac{1}{1}\right) = \frac{1}{2}.$$

Therefore, we conclude that f has no Thurston obstructions and so it is equivalent to a rational map.

$\frac{p}{q}$	$\frac{p'}{q'}$	δ	C	R	H bounded?
$\frac{1}{2}$	$\frac{1}{1}$	$\frac{2}{3}$	-4	$3\sqrt{\frac{2}{3}}$	yes
$\frac{5}{8}$	$\frac{1}{0}$	2	$-\frac{40}{23}$	$\frac{8}{23}\sqrt{\frac{2}{3}}$	yes
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{2}{3}$	$-\frac{32}{25}$	$\frac{6}{25}\sqrt{\frac{2}{3}}$	yes
$\frac{5}{4}$	$\frac{3}{2}$	$\frac{2}{3}$	$-\frac{48}{57}$	$\frac{6}{57}\sqrt{\frac{2}{3}}$	yes
$\frac{11}{8}$	$\frac{1}{0}$	2	$-\frac{88}{119}$	$\frac{8}{119}\sqrt{2}$	yes
$\frac{3}{2}$	$\frac{1}{1}$	$\frac{2}{3}$	$-\frac{16}{25}$	$\frac{3}{25}\sqrt{\frac{2}{3}}$	yes
$\frac{7}{4}$	$\frac{3}{2}$	2	$-\frac{16}{31}$	$\frac{2}{31}\sqrt{2}$	yes
$\frac{5}{2}$	$\frac{2}{1}$	2	$-\frac{6}{17}$	$\frac{1}{17}\sqrt{2}$	yes
$\frac{4}{1}$	$\frac{3}{1}$	$\frac{1}{2}$	$-\frac{5}{23}$	$\frac{2}{23}\sqrt{\frac{1}{2}}$	yes
$\frac{7}{1}$	$\frac{5}{1}$	$\frac{1}{2}$	$-\frac{9}{73}$	$\frac{4}{73}\sqrt{\frac{1}{2}}$	yes
$-\frac{13}{2}$	$-\frac{4}{1}$	2	$\frac{18}{137}$	$\frac{5}{137}\sqrt{2}$	yes
$-\frac{9}{2}$	$-\frac{3}{1}$	$\frac{2}{3}$	$\frac{48}{225}$	$\frac{9}{225}\sqrt{\frac{2}{3}}$	yes
$-\frac{2}{1}$	$-\frac{1}{1}$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{2}{7}\sqrt{\frac{1}{2}}$	yes
$-\frac{1}{2}$	$\frac{0}{1}$	2	2	$\sqrt{2}$	yes
$\frac{1}{4}$	$\frac{1}{2}$	2	0	$2\sqrt{2}$	no

Table 3. Half-space data for the main example.

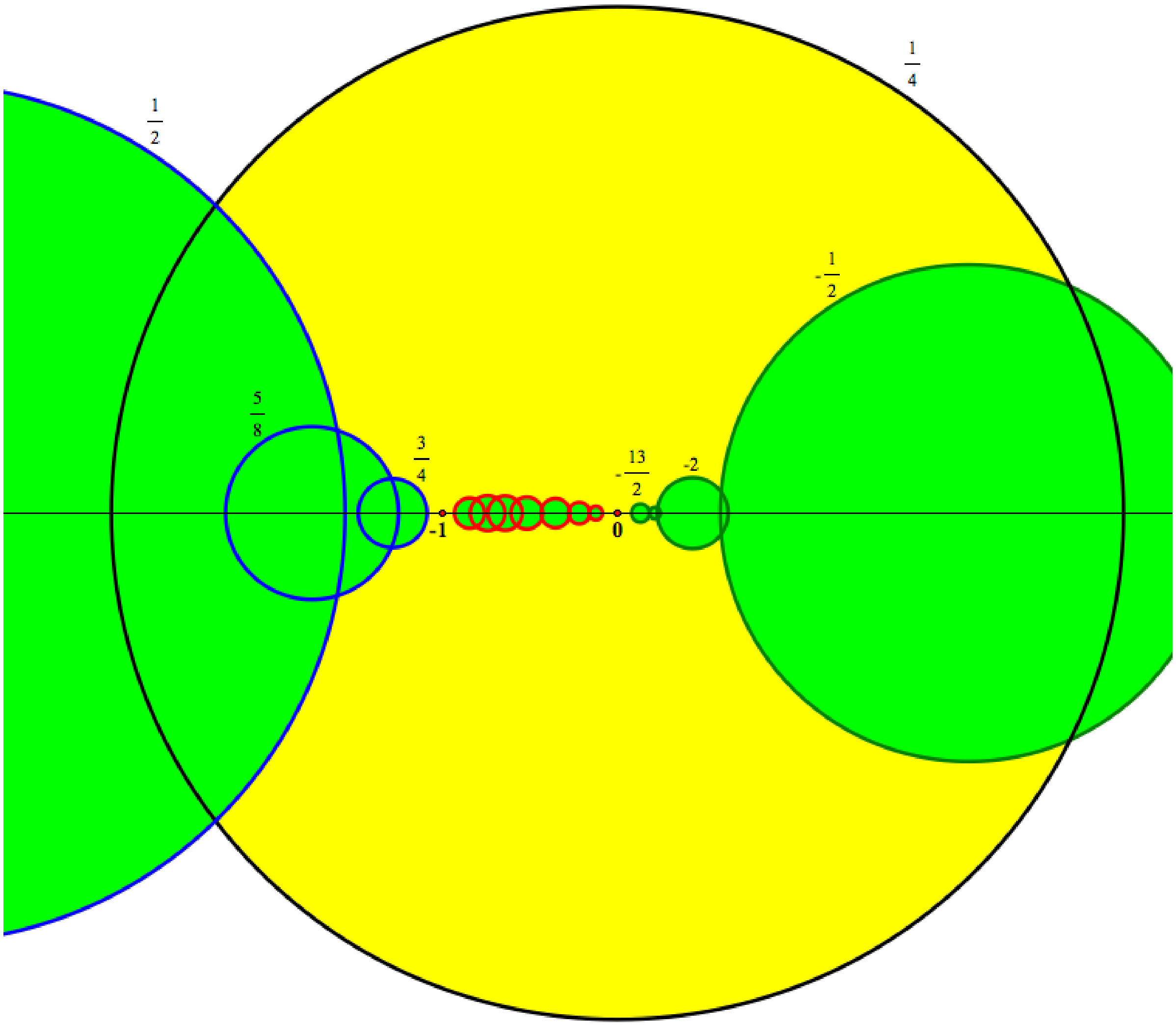


Figure 4.3: Some half-spaces for the main example.

According to Figure 4.3 the fixed point of the Teichmüller map Σ_f lies within the yellow region contained in $\mathbb{H}$, but in the complement of the union of the remaining half-spaces.

Chapter 5

Constant Pullback Map

In this chapter we investigate when a nearly Euclidean Thurston map has the property that the associated pullback map on Teichmüller space is constant. We also provide an example of a nearly Euclidean Thurston map whose Teichmüller map is constant but it does not satisfy McMullen's constant conditions.

5.1 Nonseparating sets

Let us recall the definition of **coset numbers** and a theorem presented in Chapter 3.

Let A be a finite abelian group. Let H be a subset of A which is the disjoint union of four pairs $\{\pm h_1\}$, $\{\pm h_2\}$, $\{\pm h_3\}$, $\{\pm h_4\}$. Let B be a subgroup such that A/B is cyclic, and let a be an element of A so that $a + B$ generates A/B. Let n be the order of A/B. For each $k \in \{1, 2, 3, 4\}$ there exists a unique $c \in \{0, \cdots, n/2\}$ such that

$$(ca + B) \cap \{\pm h_k\} \neq \emptyset$$

Let c_1, c_2, c_3, c_4 be these four integers ordered so that $0 \leq c_1 \leq c_2 \leq c_3 \leq c_4$. These four numbers are called *coset numbers for H relative to B and a or relative to the generator $a + B$ and A/B.* Based on this concept and the setting of Section 3.1, in [9] J. Cannon et al. proved the following result.

Theorem 5.1.1. *Let f be a nearly Euclidean Thurston map. Let δ be an essential simple closed curve in $S^2 \setminus P_f$ with slope p/q, where p and q are relatively prime integers not both 0. Let $\lambda = q\lambda_2 + p\mu_2 \in \Lambda_2$. Let d be the order of the image of $\lambda \in \Lambda_2/\Lambda_1$. Let d' be the positive integer such that $dd' = |\Lambda_2/\Lambda_1| = \deg(f)$. Since p and q are relatively prime, there exists $\mu \in \Lambda_2$ such that λ and μ form a basis of Λ_2. Let c_1, c_2, c_3, c_4 be the coset numbers for the elements of $q_1^{-1}(p_1^{-1}(P_f))$ relative to λ and μ. Then the number of essential components in $f^{-1}(\delta)$ is $c_3 - c_2$.*

In Theorem 5.1 of [1], X. Buff et al. characterized when the Thurston pullback map is constant. Combining statements 1 and 4 of Theorem 5.1 of [1] implies that the Teichmüller map of f is constant if and only if for every essential simple closed curve δ in $S^2 \setminus P_f$ every connected component of $f^{-1}(\delta)$ is either peripheral or trivial in $S^2 \setminus P_f$. This fact and Theorem 5.1.1 lead us to consider the following. Let λ and μ be elements of Λ_2 which form a basis of Λ_2. Let c_1, c_2, c_3, c_4 be the coset numbers for $q_1^{-1}(p_1^{-1}(P_f))$ relative to λ and μ. Theorem 5.1.1 now shows that the Teichmüller map of f is constant if and only if $c_2 = c_3$ for every choice of λ and μ. This result is the motivation of the definition of **nonseparating set**. Let A be a finite Abelian group. A subset H of A is called **nonseparating** if and only if it satisfies the following conditions:

- H is a disjoint union of the form $H = H_1 \coprod H_2 \coprod H_3 \coprod H_4$, where each H_i has the form $H_i = \{\pm h_i\}$.
- Let B be a cyclic subgroup of A such that A/B is cyclic. Let c_1, c_2, c_3, c_4 be the coset numbers for H relative to B and some generator of A/B. The main condition is that $c_2 = c_3$ for every such choice of B and generator of A/B.

Lemma 5.1.2. *Let $\phi : \mathbb{Z} \oplus \mathbb{Z} \to A$ be a surjective group homomorphism from $\mathbb{Z} \oplus \mathbb{Z}$ to a finite Abelian group A. Let $a \in A$ and let B be a cyclic subgroup of A. Then the quotient group A/B is cyclic and the image of $a \in A/B$ generates A/B if and only if there exists a basis of $\mathbb{Z} \oplus \mathbb{Z}$ consisting of elements α and α' with $\phi(\alpha) = a$ and $\phi(\alpha')$ is a generator of B.*

For the details of the proof, see [9].

Theorem 5.1.3. *Let f be a nearly Euclidean Thurston map in the setting of Section 3.1. Then the Teichmüller map of f is constant if and only if $p_1^{-1}(P_f)$ is a nonseparating subset of $\Lambda_2/2\Lambda_1$.*

Proof. Let B be a cyclic subgroup of $A = \Lambda_2/2\Lambda_1$ such that A/B is cyclic. Let a be an element of A such that $a + B$ generates A/B. The canonical projection $\phi : \Lambda_2 \to A$ is a surjective homomorphism. By Lemma 5.1.2 there exists a basis of Λ_2 consisting of elements λ' and μ' such that $\mu' + 2\Lambda_1 = a$ and $\lambda' + 2\Lambda_1$ generates B. Conversely, if λ, μ form a basis of Λ_2, then $B = \langle \lambda + 2\Lambda_1 \rangle$ is a cyclic subgroup of A so that A/B is cyclic and $\mu + 2\Lambda_1$ is an element in A whose image in A/B generates A/B. □

This theorem suggests an algorithm. In order to get nearly Euclidean Thurston maps whose Teichmüller maps are constant we may consider the following steps:

Step 1. Construct a finite Abelian group A generated by two elements with $A/2A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ such that A has a nonseparating subset H.

Step 2. Construct lattices $\Lambda_1 \subseteq \Lambda_2 \subset \mathbb{R}^2$ such that $\Lambda_2/2\Lambda_1 \cong A$.

Step 3. Construct an isomorphism from Λ_2 to Λ_1, which in effect produces a nearly Euclidean Thurston map g corresponding to Λ_1 and Λ_2.

Step 4. Identify S^2 with R^2/Γ_j, $j = 1, 2$ and construct an orientation-preserving homeomorphism $h : S^2 \to S^2$ such that $h(P_2) = p_1(H)$. Here P_2 means P_g.

Step 5. Set $f := h \circ g$. By Lemma 3.1.6, if f has four postcritical points then it is a nearly Euclidean Thurston map. Moreover, $P_f = h(P_g)$ and so $p_1^{-1}(P_f) = H$. By Theorem 5.1.3 the Teichmüller map of f is constant.

Step 6. Because $|\Lambda_2/\Lambda_1| = \deg(f)$, $|\Lambda_2/2\Lambda_1|/|\Lambda_1/2\Lambda_1| = \deg(f)$. Thus $|A| = 4\deg(f)$.

5.1.1 Lemmas

The next two lemmas provide a simple way to produce nonseparating subsets from known ones. For details of the proof, see Section 10 of [9].

Lemma 5.1.4. *Let A be a finite Abelian group, and let $H = \{\pm h_1, \pm h_2, \pm h_3, \pm h_4\}$ be a nonseparating subset of A. Let h be an element of order 2 in A and let $H' = H + h = \{\pm(h_1 + h), \pm(h_2 + h), \pm(h_3 + h), \pm(h_4 + h)\}$. Then H' is a nonseparating subset of A.*

Lemma 5.1.5. *If A is a finite Abelian group and if A' is a subgroup of A, then every subset of A' which is nonseparating for A' is nonseparating for A.*

The next lemma is the converse of Lemma 5.1.5. For details of the proof , see Appendix A.

Lemma 5.1.6. *Let A be a finite Abelian group generated by two elements and let A' be a subgroup of A. If H is a subset of A' which is nonseparating for A, then H is nonseparating for A'.*

5.1.2 Examples of nonseparating sets

Example 1. Let $A = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. The subset $H = A \setminus \{(0, 0)\}$ is a nonseparating set of A. In fact, let B be a cyclic subgroup of A so that A/B is cyclic. Then $B \cong A/B \cong \mathbb{Z}/3\mathbb{Z}$. Given a generator $a + B$ of A/B we have only three cosets: B, $a + B$ and $2a + B$. It is obvious that B contains exactly one pair of mutually inverse elements of order 3. So $c_1 = 0$ and $c_2 = c_3 = c_4 = 1$. By Lemma 5.1.5 , any Abelian group generated by two elements that contains a subgroup isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ contains a nonseparating subset.

Example 2. Let $A = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The subset $H = \{(0, 0), \pm(1, 0), \pm(2, 0), \pm(1, 1)\}$ is a nonseparating subset of A. For details of the proof see Example 9.3 of [9]. By Lemma 5.1.4, $H_0 = \{(0, 0), \pm(1, 0), \pm(2, 0), \pm(1, 2)\}$ is a nonseparating subset of $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

Example 3. In this example we show that $H_1 = \{\pm(1, 0), \pm(0, 1), \pm(1, 2), \pm(2, 1)\}$ is a nonseparating subset of $A = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Let B be a cyclic subgroup of A such that A/B is cyclic. Then $B \cong A/B \cong \mathbb{Z}/4\mathbb{Z}$. In this case we have only six possibilities for B: $\langle(1,0)\rangle$, $\langle(0,1)\rangle$, $\langle(1,2)\rangle$, $\langle(2,1)\rangle$, $\langle(3,1)\rangle$ and $\langle(1,1)\rangle$.

- If $B = \langle(1,0)\rangle = \{(1,0),(2,0),(3,0),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(0,1), \pm(1,1), \pm(1,3), \pm(2,1)\}$. In any case, $c_1 = 0, c_2 = c_3 = 1$.
- If $B = \langle(0,1)\rangle = \{(0,1),(0,2),(0,3),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(1,0), \pm(1,1), \pm(1,2), \pm(1,3)\}$. In any case, $c_1 = 0, c_2 = c_3 = 1$.
- If $B = \langle(1,2)\rangle = \{(1,2),(2,0),(3,2),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(0,1), \pm(1,1), \pm(1,3), \pm(2,1)\}$. In any case, $c_1 = 0, c_2 = c_3 = 1$.
- If $B = \langle(2,1)\rangle = \{(2,1),(0,2),(2,3),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(1,0), \pm(1,1), \pm(1,2), \pm(1,3)\}$. In any case, $c_1 = 0, c_2 = c_3 = 1$.
- If $B = \langle(1,1)\rangle = \{(1,1),(2,2),(3,3),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(1,0), \pm(0,1), \pm(1,2), \pm(2,1)\}$. In any case, $c_1 = c_2 = c_3 = c_4 = 1$.
- If $B = \langle(1,3)\rangle = \{(1,3),(2,2),(3,1),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(1,0), \pm(0,1), \pm(1,2), \pm(2,1)\}$. In any case, $c_1 = c_2 = c_3 = c_4 = 1$.

Example 4. In this example we show that $H_2 = \{\pm(1,0), \pm(0,1), \pm(1,1), \pm(1,3)\}$ is also a nonseparating subset of $A = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Let B be a cyclic group of A such that A/B is cyclic. Then $B \cong A/B \cong \mathbb{Z}/4\mathbb{Z}$. We have only six possibilities for B: $\langle(1,0)\rangle$, $\langle(0,1)\rangle$, $\langle(1,2)\rangle$, $\langle(2,1)\rangle$, $\langle(3,1)\rangle$ and $\langle(1,1)\rangle$.

- If $B = \langle(1,0)\rangle = \{(1,0),(2,0),(3,0),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(0,1), \pm(1,1), \pm(1,3), \pm(2,1)\}$. In any case, $c_1 = 0, c_2 = c_3 = c_4 = 1$.
- If $B = \langle(0,1)\rangle = \{(0,1),(0,2),(0,3),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(1,0), \pm(1,1), \pm(1,2), \pm(1,3)\}$. In any case, $c_1 = 0, c_2 = c_3 = c_4 = 1$.
- If $B = \langle(1,2)\rangle = \{(1,2),(2,0),(3,2),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(0,1), \pm(1,1), \pm(1,3), \pm(2,1)\}$. In any case, $c_1 = c_2 = c_3 = 1$.
- If $B = \langle(2,1)\rangle = \{(2,1),(0,2),(2,3),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(1,0), \pm(1,1), \pm(1,2), \pm(1,3)\}$. In any case, $c_1 = c_2 = c_3 = 1$.
- If $B = \langle(1,1)\rangle = \{(1,1),(2,2),(3,3),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(1,0), \pm(0,1), \pm(1,2), \pm(2,1)\}$. In any case, $c_1 = 0, c_2 = c_3 = 1$.
- If $B = \langle(1,3)\rangle = \{(1,3),(2,2),(3,1),(0,0)\}$, then $a + B$ generates A/B only if $a \in \{\pm(1,0), \pm(0,1), \pm(1,2), \pm(2,1)\}$. In any case, $c_1 = 0, c_2 = c_3 = 1$.

Remark 5.1.1. In a personal communication Professor Walter Parry informed me that up to translation by an element of order 2, H_0, H_1 and H_2 are the only distinct nonseparating subsets of $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

Example 5. This example is due to Walter Parry. Let k be an integer with $k \geq 3$. Let $A = \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and let $H = \{\pm(1,0), \pm(2^{k-2},0), \pm(2^{k-2},1), \pm(2^{k-1}-1,0)\}$. We show that H is a nonseparating subset of A. Let B be a cyclic subgroup of A such that A/B is cyclic. Then either $|B| = 2^k$ or $|B| = 2$. First suppose that $|B| = 2^k$. Then $A/B \cong \mathbb{Z}/2\mathbb{Z}$. Given a generator $a+B$ of A/B we have only two cosets: B and $a+B$. In this case to show that H does not separate c_2 from c_3 it suffices to prove that B does not contain exactly two elements of H. If $(1,0) \in B$, then $(2^{k-2},0)$ and $(2^{k-2}-1,0) \in B$. So if $(1,0) \in B$, then B does not contain exactly two elements of H. Because $(2^{k-2}-1,0)$ generates $\mathbb{Z}/2^k\mathbb{Z}$, the same is true if $(2^{k-2}-1,0) \in B$. So if B contains exactly two elements of H, then these elements are $(2^{k-2},0)$ and $(2^{k-2},1)$. But then $(0,1) \in B$. This is impossible.

Now suppose that $|B| = 2$. Then either $B = \langle(0,1)\rangle$ or $B = \left\langle 2^{k-1},1\right\rangle$. Let $a \in A$ such that $a+B$ generates A/B. The first component of a is odd, otherwise $a+B$ does not generates A/B. Then the first component of a is either $4r+1$ or $4r-1$ for some integer r. So $2^{k-2}a + B = \pm(2^{k-2},0) + B$. Then the coset number of $\pm(2^{k-2},0)+B$ is 2^{k-2}. Similarly, because either $B = \langle(0,1)\rangle$ or $B = \left\langle(2^{k-1},1)\right\rangle$, the coset number of $\pm(2^{k-2},1)+B$ is 2^{k-2}. Let m be the coset number of $(1,0)+B$. Then m is the integer in $\{0,\cdots,2^{k-1}\}$ such that $m(a+B) = \pm(1,0)+B$. Because $(1,0)+B$ generates A/B, it follows that m is odd. So

$$(2^{k-1}-m)(a+B) = 2^{k-1}a - ma + B = (2^{k-1},0) \pm (1,0) + B = \pm(2^{k-1}-1,0) + B.$$

Since m is an odd integer in $\{0,\cdots,2^{k-1}\}$ then either $1 \leq m < 2^{k-2}$ or $2^{k-2} < m < 2^{k-1}$.

If $1 \leq m < 2^{k-2}$ then

$$c_1 = m < c_2 = c_3 = 2^{k-2} < c_4 = 2^{k-1} - m.$$

If $2^{k-2} < m < 2^{k-1}$ then

$$c_1 = 2^{k-1} - m < c_2 = c_3 = 2^{k-2} < c_4 = m.$$

This proves that H is a nonseparating subset of A. On the other hand, the subgroup $\left\langle(2^{k-2},0)\right\rangle \oplus \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Lemma 5.1.4 and *Example 2*, the subset $H' = \{(0,0), \pm(2^{k-2},0), \pm(2^{k-2},1), \pm(2^{k-1},0)\}$ is also a nonseparating subset of A. But there is no element h of order 2 in A such that $H = H' + h$. To see this, proceed by contradiction. If there exists an element h of order 2 in A such that $H = H' + h$, then $h = 0 + h$ must be an element of H. Since $k \geq 3$, H contains no element of order 2. This yields a contradiction.

5.2 Nonexistence results

The following nonexistence results can be found in Section 10 of [9].

Theorem 5.2.1. *There does not exist a nearly Euclidean Thurston map with degree* 2 *whose Teichmüller map is constant.*

Theorem 5.2.2. *Let A be a finite Abelian group such that $A/2A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $2A$ is a cyclic group with odd order. Then A does not contain a nonseparating subset.*

Theorem 5.2.3. *There does not exist a nearly Euclidean Thurston map with degree an odd square-free integer and constant Teichmüller map.*

Our first nonexistence result is the following.

Theorem 5.2.4. *Let A be a finite Abelian group generated by two elements such that $A/2A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. If $|A| = 4p^2$ with p prime and $p \geq 5$, then A does not contain a nonseparating subset.*

Proof. By the Fundamental Theorem of Finite Abelian Groups and the assumptions on the group A, we only have two possibilities, either $A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2p^2\mathbb{Z}$ or $A \cong \mathbb{Z}/2p\mathbb{Z} \oplus \mathbb{Z}/2p\mathbb{Z}$. If $A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2p^2\mathbb{Z}$, then $2A \cong \mathbb{Z}/p^2\mathbb{Z}$ which is a cyclic group with odd order. Then, by Theorem 5.2.2, A does not contain a nonseparating set. The rest of the proof is by contradiction. Without loss of generality assume that $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Now, suppose that A contains a nonseparating subset $H = H_1 \coprod H_2 \coprod H_3 \coprod H_4$. Here each H_i has the form $H_i = \{\pm h_i\}$. Let $\langle H \rangle$ be the subgroup generated by the set H. Then we have two cases, either $0 \oplus 0 \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \subset \langle H \rangle$ or $(0 \oplus 0 \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) \cap (A \setminus \langle H \rangle) \neq \emptyset$

Case 1. $0 \oplus 0 \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \subset \langle H \rangle$.
Count the subgroups of order p in $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. There are in total $p+1$ subgroups of order p. Let $\psi : A \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\phi : A \to \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ be the canonical projections defined by $\psi(a, b, c, d) = (a, b)$ and $\phi(a, b, c, d) = (c, d)$ respectively. Also, consider the set of differences $D := \{\phi(h_i) \pm \phi(h_j) : i, j \in \{1, 2, 3, 4\} \text{ and } i < j\}$. We begin by proving the theorem under the assumption that there exists a subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ with order p such that $G \cap D \subseteq \{0\}$.

Suppose there exists a subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ with order p such that $G \cap D \subseteq \{0\}$. It is clear that $(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z})/G$ is cyclic. By Lemma A.6, we may, and do, assume that $\langle \phi(h_1), \phi(h_2) \rangle = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and h_2, h_3 and h_4 all differ by an element of order 2. Choose B so that it contains h_2. Choose a arbitrarily such that $a + B$ generates A/B. Then $h_2 + B$, $h_3 + B$, $h_4 + B$ lie in the subgroup of order 2 in A/B. They are not all equal because B does not contain three elements of order 2. So either $c_1 = c_2 = 0$ and $c_4 = p$ or $c_1 = 0$ and $c_3 = c_4 = p$. Since $h_1 + B$ does not have order 2 in A/B, we finally have a contradiction.

Now the argument separates into cases. In every case we obtain either a contradiction or a subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ with order p such that $G \cap D \subseteq \{0\}$. By the above, this suffices to prove the theorem. Let ψ and ϕ as above and for each $i \in \{1,2,3,4\}$ set $G_i := \langle \phi(h_i) \rangle$.

Subcase 1 Two of the elements $\phi(h_1), \phi(h_2), \phi(h_3), \phi(h_4)$ are 0. We verify that then D contains at most five elements. Since $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ contains at least six subgroups with order p, there exists a subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ with order p such that $G \cap D \subseteq \{0\}$. This handles the subcase in which two of the elements $\phi(h_1), \phi(h_2), \phi(h_3), \phi(h_4)$ are 0.

Subcase 2 One of the elements $\phi(h_1), \phi(h_2), \phi(h_3), \phi(h_4)$ is 0. Without loss of generality $\phi(h_1) = 0$. In this case we may translate H by an element of order 2 if necessary so that $h_1 = 0$. Now we choose B so that it contains h_2. Then $c_1 = c_2 = 0$. So $c_3 = 0$. Without loss of generality $h_3 \in B$. Next choose B' so that it contains h_4. Then $c'_1 = c'_2 = 0$. So $c'_3 = 0$. Without loss of generality $h_3 \in B'$. If $\phi(h_3) = 0$, then subcase 2 reduces to subcase 1. If $\phi(h_3) \neq 0$, then $\phi(h_1), \phi(h_2), \phi(h_3), \phi(h_4) \in \langle h_3 \rangle$. This contradicts the assumption that $0 \oplus 0 \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \subset \langle H \rangle$. This handles the subcase in which one of the elements $\phi(h_1), \phi(h_2), \phi(h_3), \phi(h_4)$ is 0.

Subcase 3 $G_1 = G_2 = G_3 \neq \{0\}$. If $p = 5$, then the elements $\pm\phi(h_1), \pm\phi(h_2)$ and $\pm\phi(h_3)$ are not distinct. So there are at most four elements of the form $\phi(h_4) \pm \phi(h_i)$ with $i \in \{1,2,3\}$. It follows that there exists a subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ with order p such that $G \cap D \subseteq \{0\}$. If $p \geq 7$, then because there are at most six elements of the form $\phi(h_4) \pm \phi(h_i)$ with $i \in \{1,2,3\}$, there again exists a subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ with order p such that $G \cap D \subseteq \{0\}$.

Subcase 4 $G_1 = G_2 \neq \{0\}$. By translating H by an element of order 2 if necessary, we may assume that h_1 has order p. Then h_1 and h_2 are both contained in a subgroup B of A with order $2p$. So $c_1 = c_2 = 0$. So $c_3 = 0$. So either $G_3 = G_1$ or $G_4 = G_1$. Thus subcase 4 reduces to subcase 3.

Subcase 5 $G_i \cap G_j = \{0\}$ for $i \neq j$. By subcase 2 we may assume that none of the elements $\phi(h_1), \phi(h_2), \phi(h_3), \phi(h_4)$ is zero. Suppose for every $i \in \{1,2,3,4\}$ that there are three choices of indices j and k with $j < k$ and a sign such that the elements $\phi(h_j) \pm \phi(h_k)$ is in G_i. Then every element of D is in $G_1 \cup G_2 \cup G_3 \cup G_4$. So there exists a subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ with order p such that $G \cap D \subseteq \{0\}$.

So we may assume that there are not three choices of indices j and k with $j < k$ and a sign such that the element $\phi(h_j) \pm \phi(h_k)$ is in G_1. By translating H by an element of order 2 if necessary, we may assume that h_1 has order p.

In this paragraph we assume that h_1, h_2, h_3, h_4 all have order p and obtain a contradiction. For this, let B be a subgroup of A with order $2p$ which contains h_1. Since $G_1 \cap G_2 = \{0\}$ and h_2 has order p, we may choose $a \in A$ such that $a + B$ generates A/B and $h_2 + B = 2a + B$. We have that $c_1 = 0$. Furthermore, $c_2 \neq 0$ because $G_1 \cap G_j = \{0\}$ for $j \in \{2,3,4\}$. In addition, $c_2 \neq 1$ because h_i has order p for every i. So $c_2 = 2$ because $h_2 + B = 2a + B$. So $c_3 = 2$. Without loss of generality $h_2 + B = h_3 + B$. Hence G_1 contains $\phi(h_2) - \phi(h_3)$. Now we repeat this argument with h_4 instead of h_2. We conclude that G_1 contains one of the elements $\phi(h_4) \pm \phi(h_2)$ or $\phi(h_4) \pm \phi(h_3)$. It follows that there are two and hence three choices of j and k with $j < k$ and signs such that G_1 contains $\phi(h_j) \pm \phi(h_k)$. This contradiction shows that h_1, h_2, h_3 and h_4 do not all have order p.

So we may assume that h_2 has order $2p$. Equivalently, $\psi(h_2) \neq 0$. There are two subgroups B and B' of A with order $2p$ which contain h_1 but not $\psi(h_2)$. Then $a = h_2$ is a an element of A such that $a + B$ generates A/B. Now $c_1 = 0$ and $c_2 = 1$. So $c_3 = 1$. Without loss of generality $h_2 + B = h_3 + B$. So $h_2 - h_3 \in B$. Similarly, $h_2 \pm h_k \in B'$ for some $k \in \{3, 4\}$. If $h_2 + h_3 \in B'$, then $\phi(h_2) \pm \phi(h_3) \in G_1$, whence $\phi(h_2), \phi(h_3) \in G_1$ and this is impossible because $G_i \cap G_j = \{0\}$ for $i \neq j$. If $h_2 \pm h_4 \in B'$, then as in the previous paragraph it follows that there are three choices of j and k with $j < k$ and signs such that G_1 contains $\phi(h_j) \pm \phi(h_k)$, which is impossible. So $h_2 - h_3 \in B \cap B' = G_1$. It follows that $\psi(h_2) = \psi(h_3) \neq 0$. If $\psi(h_4) \neq 0$, then we repeat this argument with h_4 instead h_2 and find that there are three choices of j and k with $j < k$ and signs such that G_1 contains $\phi(h_j) \pm \phi(h_k)$, which is impossible.

We are left with the case in which $\psi(h_1) = \psi(h_4) = 0$ and $\psi(h_2) = \psi(h_3) \neq 0$. Now we choose B to be the subgroup of A with order $2p$ such that $\psi(B)$ does not contain $\psi(h_2)$ and $\phi(B)$ does not contain any of the four elements $\phi(h_1) \pm \phi(h_4)$ or $\phi(h_2) \pm \phi(h_3)$. Regardless of how a generator of A/B is chosen, $c_2 = c_3$. So B contains an element of the form $h_i \pm h_j$. Considering $\psi(B)$ shows that either $i, j \in \{1, 4\}$ or $i, j \in \{2, 3\}$. Considering $\phi(B)$ now yields a contradiction.

This completes the subcase in which $G_i \cap G_j = \{0\}$ for $i \neq j$ and therefore this completes case 1.

Case 2. $(0 \oplus 0 \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) \cap (A \setminus \langle H \rangle) \neq \emptyset$.
This means that $\langle H \rangle$ does not contain a copy of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. Then, $H \subset \langle H \rangle \subset A'$, where A' is a subgroup of A isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. By Lemma A.5, H is a nonseparating set for A'. This contradicts Theorem 5.2.2 because A' is a finite Abelian group for which $2A' \cong \mathbb{Z}/p\mathbb{Z}$ and $A'/2A' \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

This completes case 2 and therefore this completes the proof of the theorem. □

As immediate consequences we have the following corollaries.

Corollary 5.2.5. *For any prime $p \geq 5$ there does not exist a nearly Euclidean Thurston map with degree p^2 whose Teichmüller map is constant.*

Corollary 5.2.6. *Let p be a prime integer with $p \geq 5$. Then $A = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ does not contain a nonseparating set.*

Our second nonexistence result is the following.

Theorem 5.2.7. *Let A be a finite Abelian group generated by two elements such that $A/2A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. If $|A| = 4p^3$ with p prime and $p \geq 7$, then A does not contain a nonseparating subset.*

Proof. By the Fundamental Theorem of Finite Abelian Groups and the assumptions on the group A, we only have two possibilities, either $A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2p^3\mathbb{Z}$ or $A \cong \mathbb{Z}/2p\mathbb{Z} \oplus \mathbb{Z}/2p^2\mathbb{Z}$. If $A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2p^3\mathbb{Z}$, then $2A \cong \mathbb{Z}/p^3\mathbb{Z}$, which is a cyclic group with odd order. Then, by Theorem 5.2.2, A does not contain a nonseparating set. The rest of the proof is by contradiction. Without loss of generality assume that $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$. Now, suppose that A contains a nonseparating subset $H = H_1 \coprod H_2 \coprod H_3 \coprod H_4$. Here each H_i has the form $H_i = \{\pm h_i\}$. Let $\langle H \rangle$ be the subgroup generated by the set H. Then we have two cases, either $0 \oplus 0 \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z} \subset \langle H \rangle$ or $(0 \oplus 0 \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}) \cap (A \setminus \langle H \rangle) \neq \emptyset$

Case 1. $0 \oplus 0 \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z} \subset \langle H \rangle$.
Let $\phi : A \to \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ be the canonical projections defined by $\phi(a, b, c, d) = (c, d)$. Also, consider the set of differences $D := \{\phi(h_i) \pm \phi(h_j) : i, j \in \{1, 2, 3, 4\} \text{ and } i < j\}$. We begin by proving the theorem under the assumption that there exists a cyclic subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ such that $G \cap D \subseteq \{0\}$ and $(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z})/G$ is cyclic.

Suppose there exists a cyclic subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ such that $G \cap D \subseteq \{0\}$ and $(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z})/G$ is cyclic. By Lemma A.6, we may, and do, assume that $\langle \phi(h_1), \phi(h_2) \rangle = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ and h_2, h_3 and h_4 all differ by an element of order 2.

Set $\tilde{A} := \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ and let i_c be the canonical monomorphism $i_c : A \to \tilde{A}$ defined by $i_c(x, y, z, t) = (x, y, pz, t)$. By Lemma 5.1.5 $i_c(H)$ is a nonseparating subset of $\tilde{A}$. Now, set $\mu := i_c(\phi(h_1))$ and $\nu := i_c(\phi(h_2))$. Since $\langle \phi(h_1), \phi(h_2) \rangle = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$, then $\langle \mu, \nu \rangle = i_c(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}) = \langle p \rangle \oplus \mathbb{Z}/p^2\mathbb{Z}$. Furthermore, $i_c(h_2)$, $i_c(h_3)$ and $i_c(h_4)$ all differ by an element of order 2. Since $\nu \in \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ there exists a cyclic subgroup $\tilde{B}$ of $\tilde{A}$ of order $2p^2$ such that $i_c(h_2) \in \tilde{B}$ and $\tilde{A}/\tilde{B}$ is cyclic. Then $i_c(h_2) + \tilde{B}$, $i_c(h_3) + \tilde{B}$, $i_c(h_4) + \tilde{B}$ lie in the subgroup of order 2 in $\tilde{A}/\tilde{B}$. They are not all equal because $\tilde{B}$ does not contain three elements of order 2. So either $c_1 = c_2 = 0$ and $c_4 = p^2$ or $c_1 = 0$ and $c_3 = c_4 = p^2$. Thus, $i_c(h_1) + \tilde{B}$ must have order 2 in $\tilde{A}/\tilde{B}$ and so $2i_c(h_1) \in \tilde{B}$, i.e. $(0, 0, 2\mu) \in \tilde{B}$. Then $(0, 0, \mu) = (1/2)(p^2 + 1)(0, 0, 2\mu) \in B$. Therefore $\langle \mu, \nu \rangle \subseteq \tilde{B}$. This is a contradiction because $\tilde{B}$ is cyclic and $\langle \mu, \nu \rangle = \langle p \rangle \oplus \mathbb{Z}/p^2\mathbb{Z}$.

Now the argument separates into cases. In every case we obtain a cyclic subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \langle p \rangle$ such that $G \cap D \subseteq \{0\}$ and $(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z})/G$ is cyclic. By the above, this suffices to prove the theorem.

First of all, note that there exists $h_i \in H$ so that $\phi(h_i) \notin \mathbb{Z}/p\mathbb{Z} \oplus \langle p \rangle$. Otherwise H is a subset of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus Z/p\mathbb{Z} \oplus \langle p \rangle$. By Lemma A.5, H would be a nonseparating subset of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus Z/p\mathbb{Z} \oplus \langle p \rangle$, which is a finite Abelian group generated by two elements such that $A/2A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This contradicts Theorem 5.2.4. Without loss of generality, we assume that $\phi(h_1) \notin \mathbb{Z}/p\mathbb{Z} \oplus \langle p \rangle$.

On the other hand, since p is prime, the subgroups $G_0 = \langle(1,0)\rangle$, $G_1 = \langle(1,p)\rangle$, $G_2\langle(2,p)\rangle$, $\ldots$, $G_{p-1} = \langle(p-1,p)\rangle$ are cyclic subgroups of $\mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ such that $\cap_{i=0}^{p} G_i = \{0\}$ and $(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z})/G_i$ is cyclic for $i \in \{0,1,2,\ldots,p\}$.

Subcase 1 $\phi(h_i) \notin \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ for all $i \in \{2,3,4\}$.
For any $i, j \in \{1,2,3,4\}$ with $i < j$, the elements $\phi(h_i) + \phi(h_j)$ and $\phi(h_i) - \phi(h_j)$ cannot be both elements of $\mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$. Otherwise, $2\phi(h_i) \in \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ for some $i \in \{1,2,3,4\}$. So $\phi(h_i) \in \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ for some $i \in \{1,2,3,4\}$, which is a contradiction. Hence there are at most 6 elements of D in $\mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$. Since p is prime and $p \geq 7$, there exists a cyclic subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ such that $G \cap D \subset \{0\}$ and $(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z})/G$ is cyclic.

Subcase 2 $\phi(h_2), \phi(h_3) \notin \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ but $\phi(h_4) \in \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$.
Note that $\phi(h_i) \pm \phi(h_4) \notin \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ for $i \in \{1,2,3\}$. Hence there are at most 6 elements of D in $\mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$. Since $p \geq 7$, there exists a cyclic subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ such that $G \cap D \subset \{0\}$ and $(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z})/G$ is cyclic.

Subcase 3 $\phi(h_2) \notin \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ but $\phi(h_3), \phi(h_4) \in \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$.
Then $\phi(h_i) \pm \phi(h_j) \notin \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ for $i \in \{1,2\}$ and $j \in \{3,4\}$. Hence there are at most 4 elements of D in $\mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$. Since $p \geq 7$, there exists a cyclic subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ such that $G \cap D \subset \{0\}$ and $(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z})/G$ is cyclic.

Subcase 4 $\phi(h_2), \phi(h_3), \phi(h_4) \in \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$.
Since $\pm\phi(h_1) \notin \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$, then $\phi(h_1) \pm \phi(h_i) \notin \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ for $i \in \{2,3,4\}$. However, $D' = \{\phi(h_i) \pm \phi(h_j) : i, j \in \{2,3,4\} \text{ and } i < j\} \subset \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$. Since the cardinality of D' is at most 6 and $p \geq 7$, there exists a cyclic subgroup G of $\mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ such that $G \cap D' \subset \{0\}$ and $(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z})/G$ is cyclic. On the other hand, it is clear that

$$D \setminus D' = \{\phi(h_1) \pm \phi(h_i) : i \in \{2,3,4\}\}$$

does not intersect G; otherwise $\phi(h_1) \in \mathbb{Z}/p\mathbb{Z} \oplus \langle p\rangle$ which is a contradiction. So $G \cap D \subset \{0\}$.

This completes case 1.

Case 2. $(0 \oplus 0 \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}) \cap (A \setminus \langle H\rangle) \neq \emptyset$.
This means that $\langle H\rangle$ does not contain a copy of $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$. Then, $H \subset \langle H\rangle \subset A'$, where A' is a subgroup of A generated by 2 elements whose order is $4p^2$. By Lemma A.5, H is a nonseparating set for A'. This contradicts Theorem 5.2.4.

This completes case 2 and therefore this completes the proof of the theorem. □

Remark 5.2.1. The previous argument does not apply to the case $p = 5$. Note that in **subcases 1,2** and **4** the assumption $p \geq 7$ is crucial in order to get a cyclic subgroup G with the desired properties.

Corollary 5.2.8. *For any prime $p \geq 7$ there does not exist a nearly Euclidean Thurston map with degree p^3 whose Teichmüller map is constant .*

5.3 NET maps with constant pullback map

In [9], J. Cannon et al. prove a general existence theorem. If d is an integer with $d > 2$ such that d is divisible by either 2 or 9, then there exists a nearly Euclidean Thurston map with degree d whose Thurston's pullback map is constant. The proof of the case $d = 2k > 4$ relies on the fact that $\mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/2d\mathbb{Z}$ contains a subgroup isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ which contains a nonseparating set. For the case $d = 9k$, one sees that $\mathbb{Z}/2d'\mathbb{Z} \oplus \mathbb{Z}/6d\mathbb{Z}$, where $d' = 3k$, contains a subgroup isomorphic to $\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ which contains a nonseparating set. In particular, this tells us that it is possible to choose f to be nearly Euclidean with odd degree and constant pullback map.

Lemma 5.3.1. *Let $s : S^2 \to S^2$ be an orientation-preserving branched covering map such that* $\deg(s, x) = 2$ *for every $x \in \Omega_s$. If $|V_s| \leq 3$ then* $\deg(s) = 2$ *or* $\deg(s) = 4$.

Proof. If $\deg(s) = 3$, the preimage under s of every element of V_s contains three points counting multiplicity and no such preimage contains two critical points. Then s maps its four critical points bijectively to V_s and so $|V_s| = 4$, which is a contradiction.

If $\deg(s) \geq 5$, we have the following two cases.

First case: $\deg(s)$ *is even.* Then $\deg(s) = 2k$ for some integer k with $k \geq 3$. By the Riemann-Hurwitz formula and the fact that the local degree of s at each of its critical points is 2, then s has exactly $4k - 2$ distinct critical points. If $V_s = \{a, b\}$ then $|\{x \in \Omega_s : s(x) = a\}| \leq k$ and $|\{x \in \Omega_s : s(x) = b\}| \leq k$. So there are at most $2k$ distinct critical points. This yields a contradiction because $k \geq 3$. If $V_s = \{a, b, c\}$ then $|\{x \in \Omega_s : s(x) = a\}| \leq k$, $|\{x \in \Omega_s : s(x) = b\}| \leq k$ and $|\{x \in \Omega_s : s(x) = c\}| \leq k$. Then there are at most $3k$ distinct critical points. Again, this yields a contradiction because $k \geq 3$.

Second case: $\deg(s)$ *is odd.* Then $\deg(s) = 2k + 1$ for some integer k with $k \geq 2$. By the Riemann-Hurwitz formula and the fact that the local degree of s at each of its critical points is 2, then s has exactly $4k$ distinct critical points. If $V_s = \{a, b\}$ then $|\{x \in \Omega_s : s(x) = a\}| \leq k$ and $|\{x \in \Omega_s : s(x) = b\}| \leq k$. So there are at most $2k$ distinct critical points which is a contradiction. If $V_s = \{a, b, c\}$ then $|\{x \in \Omega_s : s(x) = a\}| \leq k$, $|\{x \in \Omega_s : s(x) = b\}| \leq k$ and $|\{x \in \Omega_s : s(x) = c\}| \leq k$. Then there are at most $3k$ distinct critical points which yields a contradiction. $\square$

Proposition 5.3.2. *There exist Thurston maps with constant Thurston pullback map that do not satisfy McMullen's constant conditions.*

Proof. Let f be a nearly Euclidean map with odd degree and constant pullback map. If f satisfies McMullen's constant conditions (see Theorem 2.3.1), there are two orientation-preserving branched covering maps g and s and a set A such that $f = g \circ s$, $|A| \leq 3$ and $V_s \subset A$. Since f is nearly Euclidean, $\deg(s; x) = 2$ for every $x \in \Omega_s$. By Lemma 5.3.1, either $\deg(s) = 2$ or $\deg(s) = 4$. This is impossible because $\deg(f)$ is an odd number. $\square$

The following are examples of nearly Euclidean Thurston maps whose Teichmüller map are constant.

Example 1. We construct an explicit nearly Euclidean rational map of degree 6 whose Teichmüller map is constant. This construction follows the algorithm given right after Theorem 5.1.3. Let $\Lambda_1 = \langle(2,0),(0,3)\rangle$ and $\Lambda_2 = \langle(1,0),(0,1)\rangle$. Clearly $2\Lambda_1 = \langle(4,0),(0,6)\rangle$ and $\Lambda_2/2\Lambda_1 \cong \mathbb{Z}_4 \oplus \mathbb{Z}_6$. Set $A = \mathbb{Z}_4 \oplus \mathbb{Z}_6$. It is clear that $A/2A \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The group A contains the subgroup $\langle(1,0),(0,3)\rangle$, which is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. So A contains a nonseparating set. More precisely, $H = \{(0,0), \pm(1,0), \pm(2,0), \pm(1,3)\}$ is a nonseparating subset for $\mathbb{Z}_4 \oplus \mathbb{Z}_6$. By Lemma 5.1.4, $H' = H + (0,3) = \{(0,3), \pm(1,3), \pm(2,3), \pm(1,0)\}$ is also a nonseparating set for A. Now, set $\mathbb{T}_i = \mathbb{R}^2/2\Lambda_i$ for $i = 1,2$. Also, set $\Gamma_i = \{x \mapsto \pm x + 2\lambda : \ \lambda \in \Lambda_i\}$. The closed rectangles $F_1 = [0,4] \times [0,3]$ and $F_2 = [0,2] \times [0,1]$ are fundamental domains for Γ_1 and Γ_2 respectively.

$$\begin{array}{ccc} \mathbb{T}_1 & \xrightarrow{\tilde{g}} & \mathbb{T}_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ S^2 & \xrightarrow{g} & S^2 \end{array}$$

The map $\tilde{g} : \mathbb{T}_1 \to \mathbb{T}_2$ is defined by $(x,y) + 2\Lambda_1 \mapsto (x,y) + 2\Lambda_2$. In this particular case there is an isomorphism from Λ_2 to Λ_1 defined by $(x,y) \mapsto (2x, 3y)$. Identify $\mathbb{R}^2/\Gamma_1$ and $\mathbb{R}^2/\Gamma_2$ with S^2. In the previous diagram, g is the map on the spheres induced by $\tilde{g}$. We may consider g as the finite subdivision rule of the 2 by 3 rectangular pillowcase with two tile types.

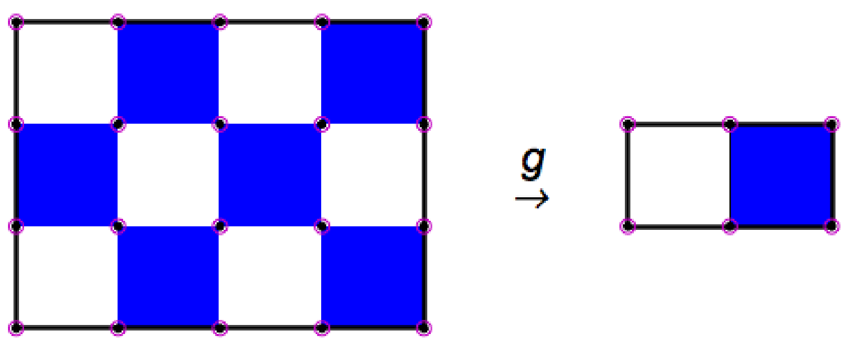


Figure 5.1: Action of the map g.

Now, we can define an orientation-preserving homeomorphism h so that $f = h \circ g$ has constant pullback and $p_1^{-1}(P_f) = H'$. Label the vertices on S^2 as follows, where a, b, c, u and w are real numbers with $0 < a < w < c < 1$ and $0 < b < u$.

$$\begin{array}{ccccc} -1 & \infty & 1 & \infty & -1 \\ -c & iu & c & -iu & -c \\ -w & ib & w & -ib & -w \\ -a & 0 & a & 0 & -a \end{array}$$

Under these conditions, $p_1(H') = \{p_1(0,3), p_1(\pm 1,3), p_1(\pm 2,3), p_1(\pm 1,0)\} = \{-1,\infty,1,0\}$. Note that $g(-a) = -a, g(0) = a, g(ib) = 1, g(-b) = -1$ and so on. Now, let $h : S^2 \to S^2$ be an orientation-preserving homeomorphism such that $h(-a) = 0$, $h(a) = 1$, $h(1) = \infty$, $h(-1) = -1$. We may choose h so that it stabilizes the real axis. The ramification portrait of $f := h \circ g$ is as follows: $f(\pm c) = 0, f(0) = 1, f(\pm iu) = 1, f(\pm w) = -1, f(\pm ib) = f(\infty) = \infty$, all of them mapping with degree 2, while $f(\pm a) = 0$ and $f(\pm 1) = -1$ both mapping with degree 1. So we may assume that

$$f(z) = k\frac{(z^2 - a^2)(z^2 - c^2)^2}{(z^2 + b^2)^2},$$

for some constant k. Set $A = a^2$, $B = b^2$ and $C = c^2$. Then $f(z) = F(z^2)$, where

$$F(z) = k\frac{(z - A)(z - C)^2}{(z + B)^2}$$

Since $f(1) = -1$, k can be written in terms of A, B and C. Let r_1 and r_2 be the roots of $F'(z) = 0$. So far, r_1 and r_2 depend on A, B and C. Since $-1 = f(1) = F(1)$, it will be enough to solve the system $F(1) = -1$, $F(r_1) = 1$ and $F(r_2) = -1$. Using Mathematica, we get

$$f(z) = (-1.436467025...)\frac{(z^2 - 0.00129035...)(z^2 - 0.159374980...)^2}{(z^2 + 0.006861566...)^2}$$

It is clear that the real axis is invariant under f and $P_f = \{-1, 0, 1, \infty\}$, so we may consider f as a finite subdivision map of a finite subdivision rule with two tile types. The first subdivision is shown below.

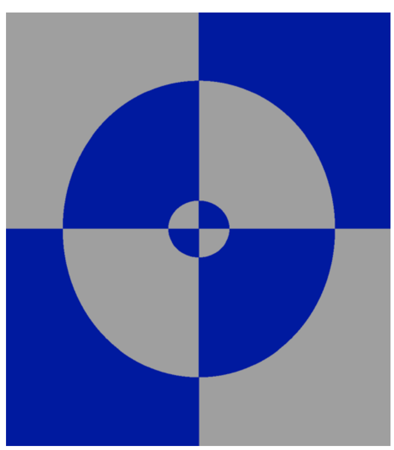


Figure 5.2: First skeleton of the first subdivision of the map f.

Example 2. The previous example can be generalized as follows. Let $\Lambda_1 = \langle (2,0),(0,k)\rangle$ and $\Lambda_2 = \langle (1,0),(0,1)\rangle$. Then $2\Lambda_1 = \langle (4,0),(0,2k)\rangle$ and $\Lambda_2/2\Lambda_1 \cong \mathbb{Z}_4 \oplus \mathbb{Z}_{2k}$. Let $A = \mathbb{Z}_4 \oplus \mathbb{Z}_{2k}$. One sees that $A/2A \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and that A contains the subgroup $\langle (1,0),(0,k)\rangle$

which is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. Then A contains a nonseparating set. More precisely, $H = \{(0,0), \pm(1,0), \pm(2,0), \pm(1,k)\}$ is a nonseparating subset for $\mathbb{Z}_4 \oplus \mathbb{Z}_k$. Let $\mathbb{T}_i = \mathbb{R}^2/2\Lambda_i$ for $i = 1,2$ and let $\Gamma_i = \{x \mapsto \pm x + 2\lambda : \quad \lambda \in \Lambda_i\}$. The closed rectangles $F_1 = [0,4] \times [0,k]$ and $F_2 = [0,2] \times [0,1]$ are fundamental domains for Γ_1 and Γ_2 respectively.

$$\begin{array}{ccc} \mathbb{T}_1 & \xrightarrow{\tilde{g}} & \mathbb{T}_2 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ S^2 & \xrightarrow{g} & S^2 \end{array}$$

The map $\tilde{g} : \mathbb{T}_1 \to \mathbb{T}_2$ is defined by $(x,y) + 2\Lambda_1 \mapsto (x,y) + 2\Lambda_2$. In this particular case there is an isomorphism from Λ_2 to Λ_1 defined by $(x,y) \mapsto (2x, ky)$. Identify $\mathbb{R}^2/\Gamma_1$ and $\mathbb{R}^2/\Gamma_2$ with S^2. In the previous diagram, g is the map on the spheres induced by $\tilde{g}$. Depending on whether k is even or odd, and putting an orientation on S^2, g can be seen as the finite subdivision rule of the 2 by k rectangular pillowcase with two tile types.

Case: k odd, $k \geq 3$. We can factor $g = s_y \circ s_x$, where in the universal cover

$$s_y(x,y) = (x, ky) \quad \text{and} \quad s_x(x,y) = (2x, y)$$

The critical points for s_x are $[(1,0)] \in S^2$ and $[(1,k)] \in S^2$. The critical value set for s_y is

$$V_{s_y} = \{[(0,0)], [(2,0)], [(2,k)], [(0,k)]\} \subset S^2$$

Further, the post-critical set of s_y is precisely V_{s_y} and all post-critical points are fixed by s_y. Now consider an orientation-preserving homeomorphism $h : S^2 \to S^2$ satisfying

$$h : [(0,k)] \mapsto [(1,k)],$$

$$h : [(0,0)] \mapsto [(0,0)],$$
$$h : [(2,0)] \mapsto [(1,0)],$$
$$h : [(2,k)] \mapsto [(2,0)],$$

i.e. $h(P_{s_y}) = p_1(H)$. Define $G := h \circ s_y \circ h^{-1}$. Since G and s_y are topologically conjugate,

$$P_G = h(P_{s_y}) = \{[(1,k)], [(1,0)], [(0,0)], [(2,0)]\}$$

and all post-critical points of G are fixed by G. Let $S : S^2 \to S^2$ be a quadratic map defined by $S = h \circ s_x$, then $S([1,0]) = [(1,0)]$, $S([1,k]) = [(2,0)]$ both mapping with degree 2 and $S[(2,0)] = S[(0,0)] = [(0,0)]$. Thus, the set of critical values of S is $V_S = \{[(1,0)], [(2,0)]\}$. Finally, if we consider the set $A = \{[(0,0)], [(1,0)], [(2,0)]\} \subset S^2$ it is not difficult to check that the set A and maps G and S satisfy Mcmullen's constant conditions. So the map $F = G \circ S$ has constant pullback map, i.e.

$$F = G \circ S = (h \circ s_y \circ h^{-1}) \circ (h \circ s_x) = h \circ g$$

has constant pullback map. On the other hand, we knew this is true because $h(P_g) = h(P_{s_y}) = p_1(H)$ and H is a nonseparating set for the group $\mathbb{Z}_4 \oplus \mathbb{Z}_{2k}$.

Case: k even, $k \geq 2$ We can repeat the same argument as in the odd case; however, the corresponding s_y fixes only two of its post-critical points. The rest of the calculations remain to be the same and considering $A = \{[(0,0)], [(1,0)], [(2,0)]\} \subset S^2$ one sees that $F = G \circ S$ also satisfies the McMullen's constant conditions. Thus, F has constant pullback map.

Example 3. In this example we construct an expanding rational function with degree 4 which is a NET map whose Teichmüller map is constant. Let g be the Lattés map given in Example 9.17 of [9]. The critical points of g are $E_1, E_1', E_2, E_2', E_3, E_3'$ and the postcritical set of g is $\{e_1, e_2, e_3, \infty\}$. Moreover, $g(E_1) = g(E_1') = e_1$, $g(E_2) = g(E_2') = e_2$, $g(E_3) = g(E_3') = e_3$, and $g(e_1) = g(e_2) = g(e_3) = g(\infty) = \infty$. Now let $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be an orientation-preserving homeomorphism such that

$$h(e_1) = e_1,\ h(e_2) = E_1,\ h(e_3) = E_1',\ h(\infty) = \infty.$$

Following the description given in Example 9.17 of [9], one sees that $f = h \circ g$ is a NET map whose Teichmüller map is constant so it is combinatorially equivalent to a rational map, R. One easily verifies that

$$R(E_1) = R(E_1') = e_1,\ \text{mapping with degree 2},$$

$$\begin{aligned} R(E_2) = R(E_2') &= E_1,\ \text{mapping with degree 2},\\ R(E_3) = R(E_3') &= E_1',\ \text{mapping with degree 2},\\ R(e_1) = R(\infty) &= \infty,\ \text{mapping with degree 1}. \end{aligned}$$

So R is a rational map without periodic critical points. Let μ be the Mobiüs transformation that satisfies $\mu(E_1) = 0$, $\mu(E_1') = \infty$, $\mu(e_1) = 1$. The rational map $F = \mu \circ R \circ \mu^{-1}$ has the following branching data:

$$F(0) = F(\infty) = 1,\ \text{mapping with degree 2},$$

$$\begin{aligned} F(\mu(E_2)) = F(\mu(E_2')) &= 0,\ \text{mapping with degree 2},\\ F(\mu(E_3)) = F(\mu(E_3')) &= \infty,\ \text{mapping with degree 2},\\ F(1) = F(\mu(\infty)) &= \mu(\infty),\ \text{mapping with degree 1}. \end{aligned}$$

Setting $\mu(E_2) = \alpha, \mu(E_2') = \beta, \mu(E_3) = \gamma, \mu(E_3') = \lambda$ and using the branching data of F, we have that

$$F(z) = \frac{(z-\alpha)^2(z-\beta)^2}{(z-\gamma)^2(z-\lambda)^2}.$$

The numerator of $F(z)-1$ is given by $(z-\alpha)^2(z-\beta)^2-(z-\gamma)^2(z-\lambda)^2$. Recall that $z=0$ is the unique zero of the numerator of $F(z)-1$ and has multiplicity 2. This implies that

$$(z-\alpha)(z-\beta)+(z-\gamma)(z-\lambda)=2z^2$$

and

$$(z-\alpha)(z-\beta)-(z-\gamma)(z-\lambda) \text{ is a nonzero constant.}$$

Then,

$$\begin{aligned} -(\alpha+\beta)+(\gamma+\lambda)&=0,\\ -(\alpha+\beta)-(\gamma+\lambda)&=0,\\ \alpha\beta+\gamma\lambda&=0. \end{aligned}$$

This forces $\alpha+\beta=0$ and $\gamma+\lambda=0$. So $\alpha=-\beta$, $\gamma=-\lambda$ and $-\beta^2-\lambda^2=0$. Hence, $F(z)$ has the form

$$F(z)=\Big(\frac{z^2-\beta^2}{z^2+\beta^2}\Big)^2$$

with the restriction $F(F(1))=F(1)$. This restriction implies that either

$$\frac{(F(1))^2-\beta^2}{(F(1))^2+\beta^2}=\frac{1-\beta^2}{1+\beta^2} \quad \text{or} \quad \frac{(F(1))^2-\beta^2}{(F(1))^2+\beta^2}=-\frac{1-\beta^2}{1+\beta^2}.$$

Thus, either $(F(1))^2=1$ or $(F(1))^2=\beta^4$.

If $(F(1))^2=1$, since $F(1)\neq 1$, then $F(1)$ has to be -1. Hence, $\Big(\frac{1-\beta^2}{1+\beta^2}\Big)^2=-1$ and so $\beta^2=\pm i$.

If $(F(1))^2=\beta^4$, then $\Big(\frac{1-\beta^2}{1+\beta^2}\Big)^4=\beta^4$. This gives us four polynomial equations:

$$\frac{1-\beta^2}{1+\beta^2}=\beta \quad , \quad \frac{1-\beta^2}{1+\beta^2}=-\beta \quad , \quad \frac{1-\beta^2}{1+\beta^2}=\beta i \quad , \quad \frac{1-\beta^2}{1+\beta^2}=-\beta i$$

Clearly the four solution sets are disjoint. So β can take only finitely many values. We now show that if $\beta^2=i$, then the map

$$F(z)=\Big(\frac{z^2-i}{z^2+i}\Big)^2$$

is an expanding NET map with constant pullback map. To see this, it is enough to consider $g(w)=\Big(\frac{w-i}{w+i}\Big)^2$, $s(z)=z^2$, and $A=\{0,1,\infty\}$. Note that $F=g\circ s$ and g,s and A verify McMullen's constant conditions.

The previous example suggests the following family of rational maps: $F_n(z) = \Big(\dfrac{z^n - i}{z^n + i}\Big)^2$, with $n \geq 2$. Each F_n has no periodic critical points, so each F_n is expanding. Also, note that $F_n = g \circ s_n$ where

$$g(w) = \Big(\frac{w-i}{w+i}\Big)^2 \quad \text{and} \quad s_n(z) = z^n.$$

If n is even and $A = \{0, 1, \infty\}$, then g, s_n and A verify McMullen's constant conditions. Thus, if n is even, F_n has constant pullback map.

Example 4. We construct a nearly Euclidean Thurston map of degree 9 whose Teichmüller map is constant. As before, this construction also follows the algorithm given right after Theorem 5.1.3. Consider the rational Lattès map

$$g(z) = \frac{z(z^4 + 6z^2 - 3)^2}{(3z^4 - 6z^2 - 1)^2}.$$

This is a degree 9 map with postcritical set $P_g = \{0, 1, \infty, -1\}$ and with all postcritical points fixed by g. Using Mathematica, one sees that

$$g(z) = \frac{z(z-\alpha)^2(z+\alpha)^2(z-\beta)^2(z+\beta)^2}{(1-\alpha z)^2(1+\alpha z)^2(1-\beta z)^2(1+\beta z)^2}$$

where $\alpha = \sqrt{-3 + 2\sqrt{3}}$ and $\beta = i\sqrt{3 + 2\sqrt{3}}$. Now, consider the lattice $\Lambda_1 = \langle (3, 0), (0, 3)\rangle$ and define the torus $\mathbb{T}_1 = \mathbb{R}^2/2\Lambda_1$. Then, there exists a branched covering $p_1 : \mathbb{T}_1 \to S^2$ with degree 2 such that the set of branch points of p_1 is $P_1 = \{0, 1, \infty, -1\}$ and

$$p_1((3, 3) + 2\Lambda_1) = 0$$

$$p_1((3, 0) + 2\Lambda_1) = 1$$

$$p_1((0, 0) + 2\Lambda_1) = \infty$$

$$p_1((0, 3) + 2\Lambda_1) = -1$$

Then, the map g lifts to a map $\tilde{g}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{T}_1 & \xrightarrow{\tilde{g}} & \mathbb{T}_1 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_1} \\ S^2 & \xrightarrow{g} & S^2 \end{array}$$

More precisely, $\tilde{g} : \mathbb{T}_1 \to \mathbb{T}_1$ is defined by $(x, y) + 2\Lambda_1 \to (3x, 3y) + 2\Lambda_1$. Now consider the lattice $\Lambda_2 = \langle (1, 0), (0, 1)\rangle$. Since the map $z \to 3z$ is an isomorphism from Λ_2 to Λ_1, we have the following commutative diagram

$$\begin{tikzcd} \mathbb{T}_1 \arrow[r, "\Phi^{-1}\circ\tilde{g}"] \arrow[d, "p_1"'] & \mathbb{T}_2 \arrow[d, "p_2"] \\ S^2 \arrow[r, "g"] & S^2 \end{tikzcd}$$

where $\mathbb{T}_2 = \mathbb{R}^2/2\Lambda_2$, Φ is the isomorphism given by $z+2\Lambda_2 \to 3z+2\Lambda_1$ and $p_2 = p_1\circ\Phi$. Now, identify $\mathbb{R}^2$ with $\mathbb{C}$. Under these assumptions, we may consider the first Riemann sphere (the domain of definition of the map g) as $\mathbb{R}^2/\Gamma_1$ where Γ_1 is the group of transformations acting on $\mathbb{R}^2$: $\Gamma_1 = \{z \to \pm z + 2\lambda : \lambda \in \Lambda_1\}$. Below is a fundamental domain for the action of Γ_1 on $\mathbb{C}$. The dots are elements of Λ_2.The lower left corner is $(0,0)$. Points are labeled (bold) by their images in S^2 under the map $z \to p_1(z+2\Lambda_1)$.

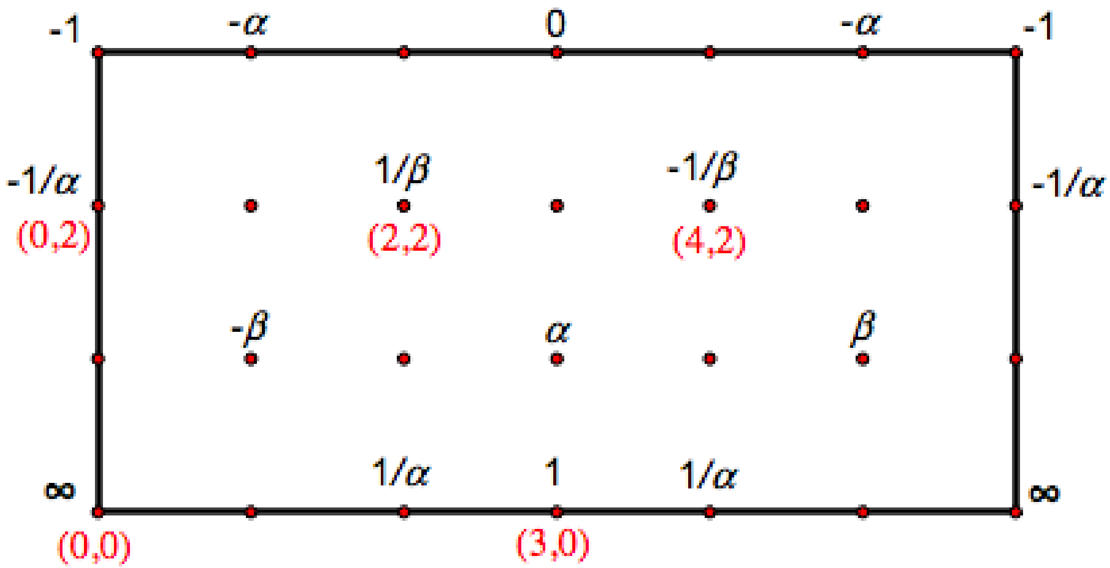


Figure 5.3: Fundamental domain for the action of Γ_1 on $\mathbb{C}$.

On the other hand, we know that $\Lambda_2/2\Lambda_1$ is isomorphic to $\mathbb{Z}_6 \oplus \mathbb{Z}_6$. We know that $H = \{\pm(0,2), \pm(2,2), \pm(2,4), \pm(2,0)\}$ is a nonseparating subset of the group $\mathbb{Z}_6 \oplus \mathbb{Z}_6$. To be more precise,

$$H = \{(0,2),(0,4),(2,2),(4,4),(2,4),(4,2),(2,0),(4,0)\} \subset \mathbb{Z}_6 \oplus \mathbb{Z}_6.$$

We identify $\Lambda_2/2\Lambda_1$ with a subgroup of $\mathbb{T}_1$, and the corresponding identification of H is the subset

$$H = \{\pm(0,2)+2\Lambda_1, \pm(2,2)+2\Lambda_1, \pm(2,4)+2\Lambda_1, \pm(2,0)+2\Lambda_1\}.$$

Let $h : S^2 \to S^2$ be a orientation-preserving homeomorphism such that $h(\infty) = \alpha^{-1}$, $h(1) = \beta^{-1}$, $h(-1) = -\beta^{-1}$, $h(0) = -\alpha^{-1}$. Clearly $h(P_g) \subset g^{-1}(P_g)$. Setting $f := h \circ g$, one sees that f has exactly 4 postcritical points. Moreover, $P_f = \{\pm\alpha^{-1}, \pm\beta^{-1}\}$ and $h(P_g) = p_1(H)$ where H is the nonseparating set (identification) contained in $\mathbb{T}_1$.

$$\begin{array}{ccc} \mathbb{T}_1 & \xrightarrow{\Phi^{-1}\circ\tilde{g}} & \mathbb{T}_2 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle h\circ p_2} \\ S^2 & \xrightarrow{f} & S^2 \end{array}$$

Finally, it is not difficult to see that $p_1^{-1}(P_f) = H$, which is a nonseparating subset of $\Lambda_2/2\Lambda_1$. Therefore, by Theorem 5.1.3, the map $f := h \circ g$ is a nearly Euclidean Thurston map of degree 9 whose Teichmüller map is constant; however this example does not satisfy McMullen's constant conditions.

5.4 Main Theorem

Our main theorem is a nonexistence result. We begin with the following lemma.

Lemma 5.4.1. *Let $N = p_1^{k_1} \cdots p_n^{k_n}$ with $p_i \geq 13$. Let $A = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ such that $a|b$, $N = ab$ and $a > 1$. If $D \subseteq A$ so that $|D| \leq 12$, then there exists a cyclic subgroup G of A such that $G \cap D \subseteq \{0\}$ and A/G is cyclic.*

Proof. Since $a|b$ and $N = ab$, then $a = p_1^{s_1} \cdots p_n^{s_n}$ and $b = p_1^{k_1-s_1} \cdots p_n^{k_n-s_n}$, where $0 \leq 2s_i \leq k_i$. Because $a > 1$, there exists $i_0 \in I := \{1, \cdots, n\}$ for which $s_{i_0} \neq 0$. Set $I_1 := \{i \in I : s_i = 0\}$ and $I_2 := I \setminus I_1$. Note that I_1 may be empty, but I_2 is not empty. So $A \cong C \oplus P$, where $C = \oplus_{i \in I_1} \mathbb{Z}/p_i^{k_i}\mathbb{Z}$ and $P = \oplus_{i \in I_2} \Big(\mathbb{Z}/p_i^{s_i}\mathbb{Z} \oplus \mathbb{Z}/p_i^{k_i - s_i}\mathbb{Z} \Big)$. Without loss of generality assume that $A = C \oplus P$. Now, for each $i \in I_2$ let $\phi_i : A \to \mathbb{Z}/p_i^{s_i}\mathbb{Z} \oplus \mathbb{Z}/p_i^{k_i-s_i}\mathbb{Z}$ be the canonical projection. Since $|\phi_i(D)| \leq 12$ and $p_i \geq 13$, then there exists a cyclic subgroup G_i of $\mathbb{Z}/p_i^{s_i}\mathbb{Z} \oplus \mathbb{Z}/p_i^{k_i-s_i}\mathbb{Z}$ so that $G_i \cap \phi_i(D) \subset \{0\}$ and $\Big(\mathbb{Z}/p_i^{s_i}\mathbb{Z} \oplus \mathbb{Z}/p_i^{k_i-s_i}\mathbb{Z}\Big)/G_i$ is cyclic. Finally, the subgroup $G := \oplus_{i \in I_2} G_i$ is the subgroup desired. $\square$

Theorem 5.4.2. *Let A be a finite Abelian group generated by two elements such that $A/2A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. If $|A| = 4p_1^{k_1}p_2^{k_2} \cdots p_n^{k_n}$ with p_i prime, $p_i \geq 13$ and $|k| = k_1 + k_2 + \cdots + k_n \geq 1$, then A does not contain a nonseparating subset.*

Proof. We proceed by induction on $|k|$. If $|k| = 1$ then $A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2p\mathbb{Z}$. Since $2A$ is cyclic, by Theorem 5.2.2 the conclusion follows. Now, suppose the conclusion holds for any $|k| \in \{1, \cdots, m-1\}$ and assume that $|A| = 4p_1^{k_1}p_2^{k_2} \cdots p_n^{k_n}$, where $k_1 + \cdots + k_n = m$. Considering assumptions on A, there are positive integers a and b so that $A \cong \mathbb{Z}/2a\mathbb{Z} \oplus \mathbb{Z}/2b\mathbb{Z}$, $4ab = |A|$ and a divides b. If $a = 1$, then $2A$ is a cyclic group with odd order. Thus, A does not contain a nonseparating set. From now on, assume that $a > 1$. Without loss of generality we may even assume that $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$. Now, we proceed by contradiction. Suppose that A contains a nonseparating subset $H = H_1 \coprod H_2 \coprod H_3 \coprod H_4$, where each $H_i = \{\pm h_i\}$. Then, either $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z} \subseteq \langle H \rangle$ or $(\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}) \cap (A \setminus \langle H \rangle) \neq \emptyset$.

CASE 1. $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z} \subseteq \langle H \rangle$.
Let ϕ be the canonical projection $\phi : A \to \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ and let $D \subset \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ be the set of differences $D := \{\phi(h_i) \pm \phi(h_j) : i, j \in \{1,2,3,4\} \text{ and } i < j\}$. Note that $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z} = \langle \phi(H) \rangle$. Since the cardinality of D is at most 12 and $p_i \geq 13$, we can apply Lemma 5.4.1 to the group $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$. So there exists a cyclic subgroup G of $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ such that $G \cap D \subseteq \{0\}$ and $(\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z})/G$ is cyclic. By Lemma A.6 we may assume that $\langle \phi(h_1), \phi(h_2) \rangle = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ and h_2, h_3 and h_4 all differ by an element of order 2.

If $b = a$, there exists a cyclic subgroup B of A of order $2b$ such that $h_2 \in B$ and A/B is cyclic. This is possible because $\phi(h_2)$ lies in $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ and any element of $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ is a multiple of a basis element of $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$. Since h_2, h_3 and h_4 all differ by an element of order 2, then $h_2 + B$, $h_3 + B$, $h_4 + B$ lie in the subgroup of order 2 in A/B. They are not all equal because B does not contain three elements of order 2. So either $c_1 = c_2 = 0$ and $c_4 = b$ or $c_1 = 0$ and $c_3 = c_4 = b$. Thus, $h_1 + B$ must have order 2 in A/B and so $2h_1 \in B$, i.e. $(0, 0, 2\phi(h_1)) \in B$. Then $(0, 0, \phi(h_1)) = (1/2)(b+1)(0, 0, 2\phi(h_1)) \in B$. Therefore $\langle \phi(h_1), \phi(h_2) \rangle \subseteq B$. This is a contradiction because B is cyclic and $\langle \phi(h_1), \phi(h_2) \rangle = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$.

If $b > a$, set $\tilde{A} := \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ and let i_c be the canonical monomorphism $i_c : A \to \tilde{A}$ defined by $i_c(x, y, z, t) = (x, y, (b/a)z, t)$. By Lemma 5.1.5 $i_c(H)$ is a nonseparating subset of $\tilde{A}$. Now, set $\mu := i_c(\phi(h_1))$ and $\nu := i_c(\phi(h_2))$. Since $\langle \phi(h_1), \phi(h_2) \rangle = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$, then $\langle \mu, \nu \rangle = i_c(\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}) = \langle b/a \rangle \oplus \mathbb{Z}/b\mathbb{Z}$. Furthermore, $i_c(h_2)$, $i_c(h_3)$ and $i_c(h_4)$ all differ by an element of order 2. Since $\nu \in \mathbb{Z}/b\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ there exists a cyclic subgroup $\tilde{B}$ of $\tilde{A}$ of order $2b$ such that $i_c(h_2) \in \tilde{B}$ and $\tilde{A}/\tilde{B}$ is cyclic. Then $i_c(h_2) + \tilde{B}$, $i_c(h_3) + \tilde{B}$, $i_c(h_4) + \tilde{B}$ lie in the subgroup of order 2 in $\tilde{A}/\tilde{B}$. They are not all equal because $\tilde{B}$ does not contain three elements of order 2. So either $c_1 = c_2 = 0$ and $c_4 = b$ or $c_1 = 0$ and $c_3 = c_4 = b$. Thus, $i_c(h_1) + \tilde{B}$ must have order 2 in $\tilde{A}/\tilde{B}$ and so $2i_c(h_1) \in \tilde{B}$, i.e. $(0, 0, 2\mu) \in \tilde{B}$. Then $(0, 0, \mu) = (1/2)(b+1)(0, 0, 2\mu) \in \tilde{B}$. Therefore $\langle \mu, \nu \rangle \subseteq \tilde{B}$. This is a contradiction because $\tilde{B}$ is cyclic and $\langle \mu, \nu \rangle = \langle b/a \rangle \oplus \mathbb{Z}/b\mathbb{Z}$.

CASE 2. $(\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}) \cap (A \setminus \langle H \rangle) \neq \emptyset$. This means that $\langle H \rangle$ does not contain a copy of $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$. Then, $H \subset \langle H \rangle \subseteq A'$ where A' is a proper subgroup of A whose order has the form $4r$. Obviously, r divides $|A|/4$ and $r < |A|/4$. Since A' is a finite Abelian group generated by 2 elements, we can apply the inductive hypothesis to A' and conclude that A' does not contain a nonseparating subset. However, this contradicts Lemma A.5. This completes case 2 and therefore this completes the proof of the theorem. □

Corollary 5.4.3. *Let $n = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ with p_i prime, $p_i \geq 13$. There does not exist a nearly Euclidean Thurston map with degree n whose Teichmüller map is constant.*

Appendix A

Group Theory and nonseparating subsets

Proposition A.1. *Let G be a finite cyclic group of order n and let h be an element of order m in G. Then there exists $g \in G$ such that $\langle g \rangle = G$ and $g^{n/m} = h$.*

Proof. Choose $a \in G$ such that $\langle a \rangle = G$. Then $\langle a^{n/m} \rangle = \langle h \rangle$, so there exists $r \in \mathbb{N}$ such that $a^{nr/m} = h$; of course $\gcd(r, m) = 1$. Also if $d \in \mathbb{N}$, then $a^{nd/m} = h$ if and only if $d \equiv r \mod m$. Since $n = m(n/m)$, analyze the following two cases.

First Case. Every prime divisor of n/m is a divisor of m. Then $\gcd(r, n) = 1$. Otherwise, there exists a prime number p such that $p|r$ and $p|m(n/m)$. Hence $p|r$ and $p|m$, which is a contradiction because $\gcd(r, m) = 1$. Therefore, $\gcd(r, n) = 1$ and so a^r generates G. Also $(a^r)^{n/m} = h$, as required.

Second Case. Not every prime divisor of n/m is a divisor of m. Let q be the product of the primes which divide n/m but do not divide m. Thus, $\gcd(m, q) = 1$ and $\gcd(s, n) = 1$ if and only if $\gcd(s, m) = \gcd(s, q) = 1$. By the Chinese remainder theorem, we may choose $\tau \in \mathbb{N}$ such that $\tau \equiv r \mod m$ and $\tau \equiv 1 \mod q$. Then $\gcd(\tau, n) = 1$, so a^τ generates G. Also $(a^\tau)^{n/m} = h$, as required. □

Proposition A.2. *Every element of $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ is a multiple of a basis element.*

Proof. In fact, let $g \in \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, then $g = (\overline{x}, \overline{y})$. If $\overline{x} = \overline{0}$ or $\overline{y} = \overline{0}$, the proposition follows. Assume $\overline{x} \neq \overline{0}$ and $\overline{y} \neq \overline{0}$, then $g = d(\overline{x/d}, \overline{y/d})$ where $d = \gcd(x, y)$. Since $\gcd(x/d, y/d) = 1$, then $(\overline{x/d}, \overline{y/d})$ is a basis element and the proposition follows. □

Proposition A.3. *Assume that $A = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Let A' be a subgroup of A, and let B' be a cyclic subgroup of A' so that A'/B' is also cyclic. Then there exists a cyclic subgroup B of A such that A/B is cyclic and $A' \cap B = B'$.*

Proof. Since A' is a finite Abelian group generated by two elements, A' is the internal direct sum of its Sylow subgroups. Then $A' = A'_1 + A'_2$ where A'_1 is the subgroup of A' generated by its cyclic Sylow subgroups and A'_2 is the subgroup of A' generated by its noncyclic Sylow subgroups. It is clear that A'_1 is cyclic. Let $|A'_1| = \alpha$ and $|A'_2| = \beta$. Then $\gcd(\alpha, \beta) = 1$ and $\alpha\beta$ divides $n^2 = p_1^{2s_1} \cdots p_r^{2s_r}$. Let $p_{i_1}, \cdots, p_{i_k}$ be the distinct primes that divide α. Now set

$$n_1 = p_{i_1}^{s_{i_1}} \cdots p_{i_k}^{s_{i_k}} \quad \text{and} \quad n_2 = n/n_1 = p_{j_1}^{s_{j_1}} \cdots p_{j_l}^{s_{j_l}}.$$

It is not difficult to see that $\{i_1, \ldots, i_n\} \coprod \{j_1, \ldots j_l\} = \{1, \ldots, r\}$, α divides n_1^2, β divides n_2^2 and $\gcd(n_1, n_2) = 1$. Since $n = n_1 n_2$ and $\gcd(n_1, n_2) = 1$, then the group A is isomorphic to $\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z}$. So there exist subgroups A_1 and A_2 of A such that $A = A_1 + A_2$, where $A_1 \cong \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z}$ and $A_2 \cong \mathbb{Z}/n_2\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z}$. Thus, $A = A_1 + A_2$ with $|A_1| = n_1^2$ and $|A_2| = n_2^2$.

Let B' be a cyclic subgroup of $A' = A'_1 + A'_2$ and let $g \in B'$ be a generator of B'. Then g can be written uniquely as $g = b'_1 + b'_2$ where $b'_i \in A'_i$. Since $o(b_i)$ divides $|A'_i|$ and $\gcd(|A'_1|, |A'_2|) = 1$, we have that $\gcd(o(b'_1), o(b'_2)) = 1$. Then $o(g) = o(b'_1) \cdot o(b'_2)$ and so $B' = \langle b'_1 \rangle + \langle b'_2 \rangle$. This means that B' is the internal direct sum of $B'_1 = \langle b'_1 \rangle$ and $B'_2 = \langle b'_2 \rangle$. It is clear that B'_i is a subgroup of A'_i for $i \in \{1, 2\}$. On the other hand, $A'_i \subset A_i$ for $i \in \{1, 2\}$. In fact, let $x \in A'_1$, then x can be written uniquely as $x = a_1 + a_2$ where $a_i \in A_i$. Then $0 = \alpha x = \alpha a_1 + \alpha a_2$, where $|A'_1| = \alpha$. Thus, $-\alpha a_1 = \alpha a_2 \in A_1 \cap A_2$ and so $\alpha a_2 = 0$. If $a_2 \neq 0$, then $o(a_2)$ divides α which is impossible because $\gcd(\alpha, |A_2|) = 1$. Therefore, $a_2 = 0$ and so $x \in A_1$. For the case $i = 2$, proceed similarly.

By Proposition A.2 , every element of A_1 is a multiple of a basis element. Let $v \in A_1$ be a basis element such that $\langle v \rangle$ contains A'_1. Choose $w \in A_1$ so that $\{v, w\}$ is a basis for A_1. Let $\varphi : A_1 \to \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z}$ be the isomorphism defined on these generators by $\varphi(v) = (1, 0)$ and $\varphi(w) = (0, 1)$. Then A'_1 is isomorphic to $\varphi(A'_1)$ which is a subgroup that is contained in the subgroup $\langle (1, 0) \rangle$ of $\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z}$. Identify b'_1 with $(m, 0) \in \langle (1, 0) \rangle$. We may even assume that m divides n_1. Let $b_1 \in A_1$ so that $\varphi(b_1) = (1, n_1/m) \in \mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z}$. Using these coordinates, we have $\langle \varphi(b_1) \rangle \cap \varphi(A'_1) = \langle (1, n_1/m) \rangle \cap \varphi(A'_1) = \langle (m, 0) \rangle$. Therefore, $\langle b_1 \rangle \cap A'_1 = \varphi^{-1} \langle (m, 0) \rangle = \langle b'_1 \rangle$.

The group A'_2/B'_2 is cyclic. In fact, let $\psi : A'_2 \to A'/B'$ be the canonical projection defined by $x \mapsto x + B'$. Since $A' = A'_1 + A'_2$ and $B' = B'_1 + B'_2$ (both internal direct sums), we conclude that $\text{Ker}(\psi) = B'_2$. Thus A'_2/B'_2 is isomorphic to $\psi(A'_2)$. By assumption A'/B' is cyclic, therefore $\psi(A'_2)$ also is. Since A'_2/B'_2 is cyclic, B'_2 is cyclic and A'_2 is the internal direct sum of the noncyclic p-subgroups of A', then no cyclic subgroup of A'_2 properly contains B'_2. By Proposition A.2, every element of A_2 is a multiple of a basis element; then we can take b_2 to be a basis element of A_2 so that some multiple of b_2 is b'_2. Since $\langle b_2 \rangle \cap A'_2$ is a cyclic subgroup of A'_2 that contains $B'_2 = \langle b'_2 \rangle$, then $\langle b_2 \rangle \cap A'_2 = \langle b'_2 \rangle$.

In the previous setting, let $B := \langle b_1 + b_2 \rangle$. Since $o(b_1)$ and $o(b_2)$ are coprime then $B = \langle b_1 \rangle + \langle b_2 \rangle$. Thus, $A' \cap B = (A'_1 + A'_2) \cap (\langle b_1 \rangle + \langle b_2 \rangle) = \langle b'_1 \rangle + \langle b'_2 \rangle = B'$. Now, note that

$$\frac{A}{B} = \frac{A_1 + A_2}{\langle b_1 \rangle + \langle b_2 \rangle} \cong \frac{A_1}{\langle b_1 \rangle} \oplus \frac{A_2}{\langle b_2 \rangle} \cong \frac{\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z}}{\langle (1, n_1/m) \rangle} \oplus \frac{A_2}{\langle b_2 \rangle}$$

Finally, since $\gcd(n_1, n_2) = 1$ and $(1, n_1/m)$ and b_2 are basis elements of $\mathbb{Z}/n_1\mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z}$ and $A_2 \cong \mathbb{Z}/n_2\mathbb{Z} \oplus \mathbb{Z}/n_2\mathbb{Z}$ respectively, we conclude that A/B is cyclic. □

Proposition A.4. *Let A be a finite Abelian group generated by two elements. Let A' be a subgroup of A, and let B' be a cyclic subgroup of A' so that A'/B' is also cyclic. Then there exists a cyclic subgroup B of A such that A/B is cyclic and $A' \cap B = B'$.*

Proof. Since A is a finite Abelian group generated by two elements, there are integers a and n so that $A \cong \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, where a divides n. Assuming that $A = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, there is a canonical monomorphism $i_c : A \to \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, defined on generators by $i_c(1, 0) = (n/a, 0)$ and $i_c(0, 1) = (0, 1)$. Thus, the group A is isomorphic to the subgroup $\langle n/a \rangle \oplus \mathbb{Z}/n\mathbb{Z}$, where $\langle n/a \rangle$ is the cyclic subgroup of $\mathbb{Z}/n\mathbb{Z}$ generated by n/a. Without loss of generality we may assume that A is a subgroup of $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Let A' be a subgroup of A, and let B' be a cyclic subgroup of A' so that A'/B' is also cyclic. Clearly $B' \subset A' \subset A \subset \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Due to the previous proposition, there exists B a subgroup of $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ such that B and $(\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z})/B$ are cyclic, and $A' \cap B = B'$. Now, set $\tilde{B} = A \cap B$. Obviously $\tilde{B}$ is a cyclic subgroup of A and $A' \cap \tilde{B} = A' \cap (A \cap B) = (A' \cap A) \cap B = A' \cap B = B'$. Finally, using the canonical projection $\phi : A \to (\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z})/B$ defined by $x \mapsto x + B$, one sees that $\mathrm{Ker}(\phi) = A \cap B = \tilde{B}$ and so $A/\tilde{B}$ is isomorphic to $\phi(A)$. Since $(\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z})/B$ is cyclic, $\phi(A)$ also is and therefore $A/\tilde{B}$ is cyclic. □

Lemma A.5. *Let A be a finite Abelian group generated by two elements and let A' be a subgroup of A. If H is a subset of A' which is nonseparating for A, then H is nonseparating for A'.*

Proof. Let B' be a cyclic subgroup of A' such that A'/B' is cyclic. Let a' be an element of A' such that $a' + B'$ generates A'/B'. By Proposition A.4 , there exists B a subgroup of A such that A/B is cyclic and $A' \cap B = B'$. Let n be the order of A/B. Let m be the order of $a' + B \in A/B$. By Proposition A.1, there exists an element a in A such that $a + B$ generates A/B and $(n/m)(a + B) = a' + B$. Let $0 \leq c_1 \leq c_2 \leq c_3 \leq c_4 \leq (1/2)|A'/B'|$ be the coset numbers relative to B' and the generator $a' + B'$ of A'/B'. The previous discussion tells us that $c_i(a' + B') \subset c_i(a' + B) = c_i((n/m)(a + B))$. Thus $c_i(a' + B') \subset c_i(n/m)(a + B)$, where $a + B$ generates A/B, $|A/B| = n$ and the order of $a' + B$ in A/B is m; whence

$$(n/m)c_1 \leq (n/m)c_2 \leq (n/m)c_3 \leq (n/m)c_4.$$

Since $a' \in A'$ and $ma' \in B$, then $ma' \in A' \cap B = B'$. Thus, $m(a' + B') = B'$ and so $|A'/B'|$ divides m. As a consequence, $c_4 \leq m/2$ and so $(n/m)c_4 \leq n/2$. Therefore, the coset numbers of elements of A for a and B are in the same order as the coset numbers of elements of A' for a' and B'. □

In the following lemma a and b are positive odd integers such that $a|b$ and $a > 1$. The group A is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ and $\phi : A \to \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ is the canonical projection.

Lemma A.6. *Suppose that A contains a nonseparating subset $H = \coprod_{i=1}^{4}\{\pm h_i\}$. Let $D = \{\phi(h_i) \pm \phi(h_j) : i, j \in \{1, 2, 3, 4\}$ with $i < j\}$. Assume that there exists a cyclic subgroup G of $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ such that $G \cap D \subseteq \{0\}$ and $(\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z})/G$ is cyclic. If $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z} = \langle \phi(H) \rangle$, then we may assume that $\langle \phi(h_1), \phi(h_2) \rangle = \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ and h_2, h_3 and h_4 all differ by an element of order 2.*

Proof. Define the following three cyclic subgroups of A:

- $E_{(1,0)} = \langle (1, 0) \rangle \oplus G$
- $E_{(0,1)} = \langle (0, 1) \rangle \oplus G$
- $E_{(1,1)} = \langle (1, 1) \rangle \oplus G$

Clearly $A/E_{(1,0)}$ $A/E_{(0,1)}$ and $A/E_{(1,1)}$ are all cyclic. Let $a + E_{(1,0)}$ be a generator of $A/E_{(1,0)}$. Since H is a nonseparating subset of A, without loss of generality we may assume that h_2 and h_3 are elements of $c_2a + E_{(1,0)}$. Then $\phi(h_2) - \phi(h_3) \in G$. So, $\phi(h_2) = \phi(h_3)$ and $h_2 - h_3 = (1, 0, 0, 0)$.

Let $x + E_{(0,1)}$ be a generator of $A/E_{(0,1)}$. We show that $\{\pm h_2, \pm h_3\} \subset c_2x + E_{(0,1)}$ cannot occur. Proceed by contradiction. If $\{h_2, h_3\} \subset c_2x + E_{(0,1)}$ or $\{-h_2, -h_3\} \subset c_2x + E_{(0,1)}$ then $h_2 - h_3 = (0, 1, 0, 0)$ which is impossible. If $\{h_2, -h_3\} \subset c_2x + E_{(0,1)}$ or $\{-h_2, h_3\} \subset c_2x + E_{(0,1)}$ then $h_2 + h_3 = (0, 1, 0, 0)$ and so $2h_2 = (1, 1, 0, 0)$, which yields a contradiction. Similarly, if $y + E_{(1,1)}$ is a generator of $A/E_{(1,1)}$, then $\{\pm h_2, \pm h_3\} \subset c_2y + E_{(1,1)}$ cannot occur.

We now show that $\{\pm h_1, \pm h_4\} \subset c_2x + E_{(0,1)}$ cannot occur. Relabeling, if necessary, it suffices to show that $\{h_1, h_4\} \subset c_2x + E_{(0,1)}$ cannot occur. Proceed by contradiction. Suppose that $\{h_1, h_4\} \subset c_2x + E_{(0,1)}$ then $\phi(h_1) = \phi(h_4)$ and $h_1 - h_4 = (0, 1, 0, 0)$. Let $y + E_{(1,1)}$ be a generator of $A/E_{(1,1)}$. Then one of the following sixteen inclusions must hold.

- $\{h_1, h_3\} \subset c_2y + E_{(1,1)}$
- $\{h_1, -h_3\} \subset c_2y + E_{(1,1)}$
- $\{h_1, h_2\} \subset c_2y + E_{(1,1)}$
- $\{h_1, -h_2\} \subset c_2y + E_{(1,1)}$
- $\{-h_1, h_3\} \subset c_2y + E_{(1,1)}$
- $\{-h_1, -h_3\} \subset c_2y + E_{(1,1)}$
- $\{-h_1, h_2\} \subset c_2y + E_{(1,1)}$

- $\{-h_1, -h_2\} \subset c_2 y + E_{(1,1)}$
- $\{h_4, h_3\} \subset c_2 y + E_{(1,1)}$
- $\{h_4, -h_3\} \subset c_2 y + E_{(1,1)}$
- $\{h_4, h_2\} \subset c_2 y + E_{(1,1)}$
- $\{h_4, -h_2\} \subset c_2 y + E_{(1,1)}$
- $\{-h_4, h_3\} \subset c_2 y + E_{(1,1)}$
- $\{-h_4, -h_3\} \subset c_2 y + E_{(1,1)}$
- $\{-h_4, h_2\} \subset c_2 y + E_{(1,1)}$
- $\{-h_4, -h_2\} \subset c_2 y + E_{(1,1)}$

However, each of them would imply that $\langle \phi(H) \rangle = \langle \phi(h_1) \rangle$. Since $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z} = \langle \phi(H) \rangle$, none of these inclusions occur. So, $\{\pm h_1, \pm h_4\} \subset c_2 x + E_{(0,1)}$ cannot occur. Similarly, if $z + E_{(1,1)}$ is a generator of $A/E_{(1,1)}$, then $\{\pm h_1, \pm h_4\} \subset c_2 z + E_{(1,1)}$ cannot occur either.

Now, without loss of generality either $\{h_3, h_4\} \subset c_2 x + E_{(0,1)}$ or $\{-h_3, h_4\} \subset c_2 x + E_{(0,1)}$. If $\{h_3, h_4\} \subset c_2 x + E_{(0,1)}$, then $h_3 - h_4 = (0, 1, 0, 0)$. Hence $\phi(h_2) = \phi(h_3) = \phi(h_4)$ and the lemma follows. If $\{-h_3, h_4\} \subset c_2 x + E_{(0,1)}$, then $h_3 + h_4 = (0, 1, 0, 0)$. In this case, $\phi(h_2) = \phi(h_3) = -\phi(h_4)$. Relabeling h_4 by $-h_4$, the lemma follows. $\square$

Appendix B

On Thurston maps of degree 2

Let $f : S^2 \to S^2$ be a Thurston map and P_f its postcritical set. Combining statements 1 and 4 of Theorem 5.1 of [1] implies that the pullback map Σ_f is constant if and only if for every essential simple closed curve α in $S^2 \setminus P_f$, every connected component of $f^{-1}(\alpha)$ is either trivial or peripheral in $S^2 \setminus P_f$. We use this result to conclude that there does not exist a Thurston map of degree 2 with at least four postcritical points whose Teichmüller map is constant.

A Thurston map f is a **topological polynomial** if there exists a critical point w, such that $f^{-1}(w) = \{w\}$; we call this point ∞.

Proposition B.1. *Let f be a quadratic topological polynomial. If $|P_f| \geq 4$, then the pullback map Σ_f cannot be constant.*

Proof. If $|P_f| = 4$ then f is a nearly Euclidean Thurston map. By Theorem 5.2.1, the Teichmüller map Σ_f cannot be constant. From now on, assume that $|P_f| \geq 5$. Since $\deg(f) = 2$, f has two critical points; i.e. $\Omega_f = \{a, \infty\}$ where a is some point in $S^2 \setminus \{\infty\}$. We first assume that $z = a$ is not periodic. Enumerate the finite postcritical points as $c_1, c_2, \cdots, c_k$ where $c_i = f^{\circ i}(a)$. Then the ramification portrait for f is:

$$c_0 = a \xrightarrow{2} c_1 \xrightarrow{1} c_2 \xrightarrow{1} c_3 \xrightarrow{1} c_4 \xrightarrow{1} \cdots\cdots \xrightarrow{1} c_k \xrightarrow{1} c_i$$

$$\infty \xrightarrow{2} \infty$$

where $k = |P_f| - 1$ and i is some integer in $\{2, \cdots, k\}$. Note that $k = |P_f| - 1 \geq 4$. Let γ be an arc joining the points ∞ and c_i such that $\gamma \cap (P_f \setminus \{c_i, \infty\}) = \emptyset$. Then $f^{-1}(\gamma)$ is the union of two arcs γ_1 and γ_2 so that $\gamma_1 \cap \gamma_2 = \{\infty\}$, γ_1 joins ∞ and c_k and γ_2 joins ∞ and c_{i-1}. Now let α be the boundary of a small regular neighborhood of the curve γ. By continuity, we may take α to be a simple closed curve so that γ is a core arc for α and $\alpha \cap (P_f \setminus \{c_i, \infty\}) = \emptyset$. Then $\gamma_1 \cup \gamma_2$ is a "core path" for $f^{-1}(\alpha)$. See Figure B.1 below.

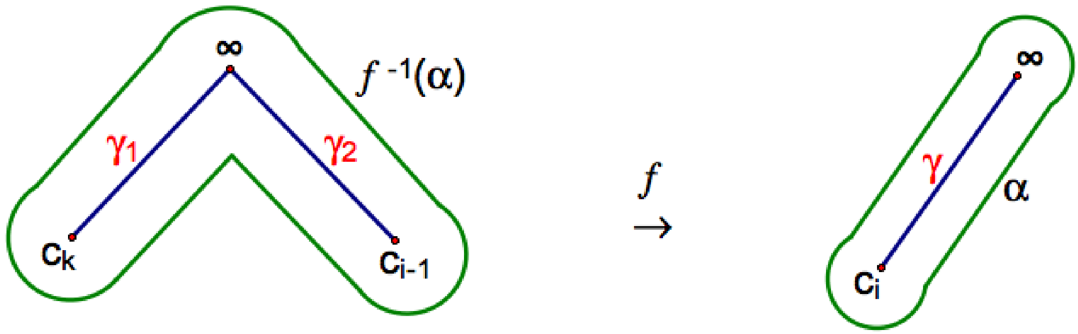


Figure B.1: Small regular neighborhoods of the curve γ and of its preimage

Thus, each connected component of $S^2 \setminus f^{-1}(\alpha)$ contains at least two postcritical points. Therefore, Σ_f cannot be constant.

Now assume that $z = a$ is a periodic critical point. There is a unique point $c \in P_f$ so that $f(c) = a$. It is obvious that f maps c to a with degree 1. Let γ be an arc joining the points ∞ and a. Now let α be the boundary of a small regular neighborhood of the curve γ. By continuity, we may take α to be a simple closed curve so that γ is a core arc for α and $\alpha \cap (P_f \setminus \{a, \infty\}) = \emptyset$. Proceeding as in the non-periodic case, each connected component of $S^2 \setminus f^{-1}(\alpha)$ contains at least two postcritical points. Therefore, Σ_f cannot be constant. $\square$

Theorem B.2. *Let f be a Thurston map of degree 2. If $|P_f| \geq 4$, then Σ_f cannot be constant.*

Proof. Due to the previous proposition, we may assume that f is not a quadratic topological polynomial. So f has two critical points and neither of them is a fixed point. If $|P_f| = 4$, then f is a nearly Euclidean Thurston map and the conclusion holds. From now on assume that $|P_f| \geq 5$. Let $\Omega_f = \{a, b\}$ be the set of critical points of f. Set $\mathcal{O}(a) = \{f^{\circ i}(a) : i \in \mathbb{N}\}$ and $\mathcal{O}(b) = \{f^{\circ i}(b) : i \in \mathbb{N}\}$ and split the analysis into two cases.

Case I. $\mathcal{O}(a) \cap \mathcal{O}(b) = \emptyset$.
1. If $a \in P_f$, let γ be an arc joining the points a and $f(a)$ so that $\gamma \cap (P_f \setminus \{a, f(a)\}) = \emptyset$. If $b \in P_f$, let γ be an arc joining the points b and $f(b)$ so that $\gamma \cap (P_f \setminus \{b, f(b)\}) = \emptyset$. Then, in any case, proceeding as in the quadratic topological polynomial case one sees that Σ_f cannot be constant.

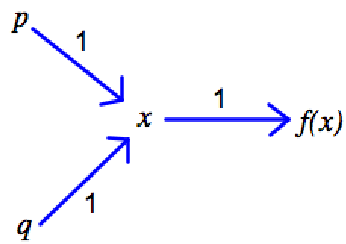
2. If $a \notin P_f$ and $b \notin P_f$, then a and b are both preperiodic critical points and so the mapping scheme of f has two pieces. Then, either $\mathcal{O}(a)$ and $\mathcal{O}(b)$ contain fixed points, i.e.

$$a_0 = a \xrightarrow{2} a_1 \xrightarrow{1} a_2 \xrightarrow{1} a_3 \xrightarrow{1} \cdots\cdots \xrightarrow{1} a_k \xrightarrow{1} a_k$$

and

$$b_0 = b \xrightarrow{2} b_1 \xrightarrow{1} b_2 \xrightarrow{1} b_3 \xrightarrow{1} \cdots\cdots \xrightarrow{1} b_r \xrightarrow{1} b_r$$

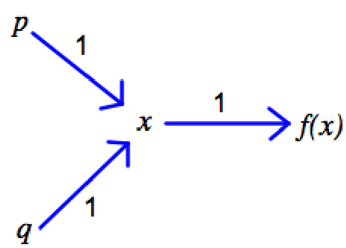
or one of them, say $\mathcal{O}(a)$, contains no fixed point. If $\mathcal{O}(a)$ and $\mathcal{O}(b)$ contain fixed points, namely a_k and b_r, then the Teichmüller map Σ_f cannot be constant. To see this, it is enough to consider the preimage of an appropriate core arc joining a_k and b_r and then proceed as in the quadratic topological polynomial case. If $\mathcal{O}(a)$ has no fixed point, then the portrait of the critical point $z = a$ contains a piece of the form



In this case, let γ be an arc joining the points x and $f(x)$ so that $\gamma \cap (P_f \setminus \{x, f(x)\}) = \emptyset$. Then, proceeding as in the quadratic topological polynomial case, Σ_f cannot be constant.

Case II. $\mathcal{O}(a) \cap \mathcal{O}(b) \neq \emptyset$.
1. Suppose $a \in P_f$ or $b \in P_f$ and proceed as above.

2. If $a \notin P_f$ and $b \notin P_f$. Set $k := \min\{i \in \mathbb{N} : f^{\circ i}(b) \in \mathcal{O}(a)\}$. Since $b \notin P_f$ and f maps b to $f(b)$ with degree 2, $k \geq 2$. Set $x := f^{\circ k}(b)$. Then $x = f(q)$ where $q = f^{\circ k-1}(b)$. By definition of k, x lies on $\mathcal{O}(a) \setminus \{a, f(a)\}$. Thus, $x = f(p)$ for some $p \in \mathcal{O}(a) \cap P_f$. Due to the minimality of k, $p \neq q$. Then the ramification portrait of f contains a piece of the form



As above, let γ be an arc joining the points x and $f(x)$ so that $\gamma \cap (P_f \setminus \{x, f(x)\}) = \emptyset$. Then, proceeding as in the quadratic topological polynomial case, Σ_f cannot be constant. $\square$

Corollary B.3. *Let f be a postcritically finite rational map of degree 2. If $|P_f| \geq 4$, then Σ_f cannot be constant.*

The following is a slight generalization of Proposition B.1.

Proposition B.4. *Let f be a topological polynomial of degree n so that $|P_f| = m \geq 4$. Suppose there exists $c \in P_f \setminus \{\infty\}$ such that $f^{-1}(c)$ contains no critical points. Let $k = |f^{-1}(c) \cap P_f|$. If $m - k \geq 3$ then Σ_f cannot be constant. In particular, if $m - 3 \geq n$ then Σ_f cannot be constant.*

Proof. Since $c \in P_f \setminus \{\infty\}$ and $f^{-1}(c)$ contains no critical points, then $k > 0$. Let γ be an arc joining the points ∞ and c such that $\gamma \cap (P_f \setminus \{c, \infty\}) = \emptyset$. Then $f^{-1}(\gamma)$ is the union of n arcs $\gamma_1, \gamma_2, \cdots, \gamma_n$ so that $\gamma_1 \cap \gamma_2 \cdots \cap \gamma_n = \{\infty\}$, and each γ_i joins ∞ and some preimage of c. Now let α be the boundary of a small regular neighborhood of the curve γ. By continuity, we may take α to be a simple closed curve so that γ is a core arc for α and $\alpha \cap (P_f \setminus \{c, \infty\}) = \emptyset$. Then one connected component of $S^2 \setminus f^{-1}(\alpha)$ contains the set $\{\infty\} \cup (f^{-1}(c) \cap P_f)$; so this connected component of $S^2 \setminus f^{-1}(\alpha)$ contains exactly $k+1$ postcritical points. Thus the other connected component contains $m - (k+1)$ postcritical points. Since $m - (k+1) \geq 2$ we conclude that Σ_f cannot be constant. $\square$

Appendix C

On Expanding Thurston maps

In this appendix we prove that any expanding Thurston map f with $\deg(f) \in \{2, 3\}$ and $|P_f| = 4$ is always combinatorially equivalent to a rational map. We start with the Euclidean case.

Let $\mathbb{Z}^2$ be the standard integral lattice in $\mathbb{R}^2$ and let Γ be the group of isometries of $\mathbb{R}^2$ generated by $x \mapsto x + v$, $v \in \mathbb{Z}^2$, and $x \mapsto -x$ and let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2-by-2 integer matrix with $\det(A) \geq 2$. The quotient space $\mathbb{R}^2/\Gamma$ is homeomorphic to S^2. The map $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $v \to Av$ descends to a branched covering $f_A : S^2 \to S^2$.

Proposition C.1. *Assume that f_A is expanding. If f_A is not combinatorially equivalent to a rational map then* $\deg(f_A) \geq 4$.

Proof. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the standard torus. In the previous setting, the canonical map $\pi : T \to S^2$ is a double cover ramified above P_{f_A}. Also, L_A descends to a covering map of this torus (or f_A lifts to a covering map of this torus). Since L_A is linear, the action on $H_1(T, \mathbb{Z})$ is determined by the matrix A. The eigenvalues of A lie outside the closed unit disk because f_A is expanding. Since f_A is not combinatorially equivalent to a rational map, either both eigenvalues are greater than 1 or both eigenvalues are less than -1. Assume that $\lambda_1 = [(a+d) - \sqrt{(a+d)^2 - 4\det(A)}]/2$ and $\lambda_2 = [(a+d) + \sqrt{(a+d)^2 - 4\det(A)}]/2$ are the real eigenvalues of A. Since $\det(A) = \deg(f_A)$, then $(a+d)^2 \geq 4\deg(f_A)$.

First Case. Both eigenvalues are greater than 1.
Since $\lambda_1 > 1$ then $(a+d) > 2 + \sqrt{(a+d)^2 - 4\deg(f_A)}$, then $a + d > 2$ and $(a+d) - 2 > \sqrt{(a+d)^2 - 4\deg(f_A)}$, whence $\deg(f_A) > (a+d) - 1 \geq 2$. So $\deg(f_A) \geq 3$. Thus $(a+d)^2 \geq 4\deg(f_A) \geq 12$, whence $a + d \geq 4$. Using again the inequality $(a+d) - 2 > \sqrt{(a+d)^2 - 4\deg(f_A)}$, we have $\deg(f_A) > (a+d) - 1 \geq 3$. Therefore, $\deg(f_A) \geq 4$.

Second Case. Both eigenvalues are less than -1.
Since $\lambda_2 < -1$ then $(a+d) < -2 - \sqrt{(a+d)^2 - 4\deg(f_A)}$, then $a+d < -2$ and $(a+d)+2 < -\sqrt{(a+d)^2 - 4\deg(f_A)}$, whence $\deg(f_A) > -(a+d) - 1 \geq 2$. So $\deg(f_A) \geq 3$. Thus $(a+d)^2 \geq 4\deg(f_A) \geq 12$, whence $a+d \leq -4$. Using again the inequality $(a+d)+2 < -\sqrt{(a+d)^2 - 4\deg(f_A)}$, we have $\deg(f_A) > -(a+d) - 1 \geq 3$. Therefore, $\deg(f_A) \geq 4$. $\square$

Remark C.2. *For* $\deg(f_A) = 4$*, there are expanding maps* f_A*—as above—that are not equivalent to rational maps. For an explicit example, see Example 18.5 of [3].*

Remark C.3. *For each prime* $p > 3$*, let* $A_p(x, y) := (ax + by, cx + dy)$ *where* a, b, c, d *are integers such that* $a + d = p$*,* $ad - cb = p$*. Each* A_p *is an expanding Euclidean map with prime degree and is not realizable by a rational map.*

In the next two propositions $|P_f| = 4$. However, we do not need to assume that f is a nearly Euclidean Thurston map.

Proposition C.4. *Let* f *be an expanding Thurston map. If* $\deg(f) \in \{2, 3\}$ *and the orbifold* $\mathcal{O}_f$ *is hyperbolic, then* f *is realizable by a rational map.*

Proof. If $\deg(f) = 2$, proceed by contradiction. Suppose there exists an invariant multicurve $\Gamma = \{\gamma\}$ that is a Thurston obstruction. Let α be any connected component of $f^{-1}(\gamma)$ isotopic to γ in $S^2 \setminus P_f$. Then either $\deg(f : \alpha \to \gamma) = 1$ or $\deg(f : \alpha \to \gamma) = 2$. If $\deg(f : \alpha \to \gamma) = 1$, there is a Levy cycle, which is impossible because f is expanding. Then $\deg(f : \alpha \to \gamma) = 2$. Thus, $\lambda_\Gamma = 1/2$. This is also a contradiction because $\Gamma = \{\gamma\}$ is an obstruction. The case $\deg(f) = 3$ is similar. $\square$

Remark C.5. *An obstructed degree two Thurston map always has a Levy cycle ([13], Cor.3.2). This fails in higher degrees [14]. So, if* f *is expanding with hyperbolic orbifold and degree two, then there are no Levy cycles and so* f *is realizable by a rational map.*

Proposition C.6. *Let* f *be an expanding Thurston map with* $\deg(f) \in \{4, 5\}$ *such that* $\mathcal{O}_f$ *is hyperbolic. If for any* f*-invariant multicurve* Γ *the multiplier* $\lambda_\Gamma \neq 1$*, then* f *is realizable by a rational map.*

Proof. If $\deg(f) = 4$, proceed by contradiction. Suppose there exists an invariant multicurve $\Gamma = \{\gamma\}$ that is a Thurston obstruction. Let α be any connected component of $f^{-1}(\gamma)$ isotopic to γ in $S^2 \setminus P_f$. Since f is expanding, there is no Levy cycle, then $\deg(f : \alpha \to \gamma) \neq 1$. Then either $\deg(f : \alpha \to \gamma) = 2$, $\deg(f : \alpha \to \gamma) = 3$ or $\deg(f : \alpha \to \gamma) = 4$. If $\deg(f : \alpha \to \gamma) = 3$ or $\deg(f : \alpha \to \gamma) = 4$, then the multiplier is either $1/3$ or $1/4$ and so Γ is not an obstruction. Thus $\deg(f : \alpha \to \gamma) = 2$. So the multiplier can be $1/2$ or $2(1/2)$. If the multiplier is $1/2$, Γ is not an obstruction. This implies that the multiplier must be 1; however, by assumption $\lambda_\Gamma \neq 1$. The case $\deg(f) = 5$ is similar. $\square$

On the other hand, there are nearly Euclidean Thurston maps f satisfying all of the following conditions:

1. Expanding map with $\deg(f) = 4$.
2. Hyperbolic Orbifold.
3. Has a Thurston Obstruction.

In this case, there is only one obstruction and its multiplier equals 1. William Floyd pointed out an easy example. The following is a subdivision map with two tile types (squares). The subdivision map is a nearly Euclidean Thurston map (postcritical points are precisely corners of the squares), the mesh goes to zero, it has hyperbolic orbifold, the degree is 4 and the horizontal curve is a Thurston obstruction.

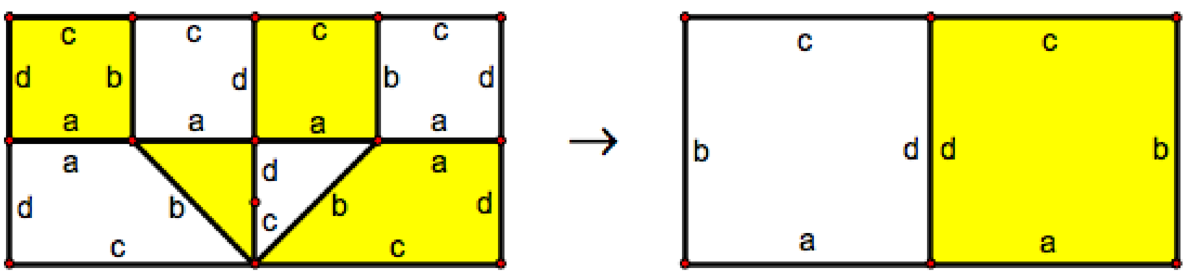


Figure C.1: An obstructed expanding NET map of degree 4, with hyperbolic orbifold.

Appendix D

A remarkable example.

In this appendix we provide an example of a rational Thurston map of degree 4 whose Teichmüller map is constant but it does not satisfy the McMullen's constant conditions.

Consider the rational map $f(z) = \dfrac{z(z^3+2)}{2z^3+1}$. One sees that the critical points of f are

$$x = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad y = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \quad \text{and} \quad z = 1.$$

The local degree of f at every critical point is 3 and the set of fix points of f is $\{x, y, z, 0, \infty\}$. Since 0 is not a critical value, $f^{-1}(0)$ contains exactly four distinct points. More precisely,

$$f^{-1}(0) = \{0, a = -\sqrt[3]{2}x, b = -\sqrt[3]{2}y, c = -\sqrt[3]{2}\}.$$

Let $h : \mathbb{P}^1 \to \mathbb{P}^1$ be the Mobiüs transformation defined by $h(z) = -\sqrt[3]{2}z$. It is clear that $h(x) = a$, $h(y) = b$, $h(z) = c$ and $h(0) = 0$. Now, define $F := h \circ f$. Since h is a homeomorphism, $\deg(F) = 4$. The ramification portrait of F is as follows, $F(x) = a$ mapping with degree 3, $F(y) = b$ mapping with degree 3, $F(z) = c$ mapping with degree 3 and $F(a) = F(b) = F(c) = F(0) = 0$, all mapping with degree 1. Thus, F is an expanding Thurston map and $P_F = \{0, a, b, c\}$. Note that F is not a NET map.

Claim D.1. *The rational map F has constant pullback map.*

Proof. We apply a core arc argument. Due to the symmetry of the mapping scheme and since $|P_F| = 4$, it suffices to analyze the preimage of an arc joining 0 and a. Let γ be an arc joining the points 0 and a such that $\gamma \cap (P_F \setminus \{0, a\}) = \emptyset$. Then $F^{-1}(\gamma)$ is the disjoint union of a triod T and an arc $\tilde{\gamma}$. The ends of the triod T are postcritical points of F while one end of $\tilde{\gamma}$ is the remaining postcritical point of F. So for every essential simple closed curve α in $S^2 \setminus P_F$, every connected component of $F^{-1}(\alpha)$ is either trivial or peripheral in $S^2 \setminus P_F$. $\square$

Since the local degree of F at every critical point is 3, it follows that F does not satisfy McMullen's constant conditions.

List of Symbols.

$\mathbb{R}$ is set of real numbers.
$\mathbb{P}^1 = \overline{\mathbb{C}}$ is the Riemann Sphere.
Ω_f critical set of the Thurston map f.
V_f critical value set of the Thurston map f.
P_f postcritical set of the Thurston map f.
σ_f slope map of f.
$\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$.
$\mathrm{Fix}(\sigma_f)$ is the set of fixed points of the slope map σ_f.
$\mathbb{N} = \{0, 1, 2, \cdots\}$.
$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.
$\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$.
$A = A_1 + A_2$ internal direct sum of subgroups of A.

Bibliography

[1] X. Buff, A. Epstein, S. Koch, and K. Pilgrim, *On Thurston's pullback map.* In *Complex dynamics*, pp 561–583. AK Peters, Wellesley (2009). MR 2508269 (2010g:37071)

[2] X. Buff, G. Cui, L. Tan, *Teichmüller spaces and holomorphic dynamics.* Handbook of Teichmüller theory, Vol. III,ed. Athanase Papadopoulos, EMS, 2011.

[3] M. Bonk and D. Meyer, *Expanding Thurston maps*, Preprint (2010). ArXiv:1009.3647v1.

[4] J. W. Cannon, *The combinatorial Riemann mapping theorem*, Acta Math. **173** (1994), 155–234. MR **95k:**30046

[5] J.W. Cannon and E.L. Swenson, *Recognizing constant curvature discrete groups in dimension 3*, Trans. Amer. Math. Soc. **350** (1998), 809–849.

[6] J.W. Cannon, W. J. Floyd, and W.R. Parry, *Squaring rectangles: the finite Riemann mapping theorem*, The Mathematical Heritage of Wilhelm Magnus — Groups, Geometry & Special Functions, Contemporary Mathematics, vol. 169, Amer. Math. Soc., Providence, 1994, pp. 133-212

[7] J.W. Cannon, W.J. Floyd, R. Kenyon, and W.R. Parry, *Constructing rational maps from finite subdivision rules*, Conform. Geom. Dyn. **7** (2003), 76–102(electronic).

[8] J.W. Cannon, W.J. Floyd, and W.R. Parry, *Finite subdivision rules*, Conform.Geom.Dyn. **5** (2001), 153–196(electronic).

[9] J.W. Cannon, W.J. Floyd, W.R. Parry, and K.M. Pilgrim, *Nearly Euclidean Thurston maps and finite subdivision rules.* ArXiv:1204.3615v1.

[10] A. Douady and J.H. Hubbard, *A proof of Thurston's topological characterization of rational functions*, Acta Math. **171** (1993), 263–297.

[11] O. Forster, *Lectures on Riemann Surfaces.* Springer-Verlag, 1981.

[12] P. Haïssinsky and K. Pilgrim, *Coarse expanding conformal dynamics.* Asterisque, **325**, 2009. MR2662902.

[13] T. Lei, Mating of quadratic polynomials, *Ergodic Theory and Dynamical Systems* **12** (1992) 589–620.

[14] T. Lei and M. Shishikura, *A family of cubic rational maps and matings of cubic polynomials*. Experiment. Math. **9** (2000) 29–53.

[15] S. Levy, *Critically Finite Rational Maps*. Ph.D. thesis, Princeton University, 1985.

[16] J. Milnor, *Pasting together julia sets: A worked out example of mating.* Experimental Mathematics, (13:1):55, 2000.

[17] J. Milnor, *On Lattès Maps.*, In: Dynamics on the Riemann sphere, Eur. Math. Soc., Zürich,(2006), 9–43.

[18] J. Milnor, *Dynamics in One Complex Variable,* Vieweg, 1999, 2nd edition, 2000.

[19] K. Pilgrim, *Thurston obstructions and Ahlfors regular conformal dimension.* J. Math. Pures Appl. (9) 90, **3** (2008), 229–241.

[20] Mitsuhiro Shishikura, *On a theorem of M. Rees for matings of polynomials*. In *The Mandelbrot set, theme and and variations*, pages 289–305. Cambridge Univ. Press, Cambridge, 2000.