

Well-posedness results for a class of complex flow problems
in the high Weissenberg number limit

Xiaojun Wang

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Michael Renardy, Chair
Jeffrey T. Borggaard
Yuriko Renardy
Robert C. Rogers
Shu-Ming Sun

May, 2012
Blacksburg, Virginia

Keywords: stress boundary layer, scaling analysis, multiscale modeling, Lagrangian
coordinates, curvilinear coordinates, mollifier, symmetric hyperbolic system
Copyright 2012, Xiaojun Wang

Well-posedness results for a class of complex flow problems in the high Weissenberg number limit

Xiaojun Wang

(ABSTRACT)

For *simple* fluids, or *Newtonian* fluids, the study of the Navier-Stokes equations in the high Reynolds number limit brings about two fundamental research subjects, the Euler equations and the Prandtl's system. The consideration of infinite Reynolds number reduces the Navier-Stokes equations to the Euler equations, both of which are dealing with the entire flow region. Prandtl's system consists of the governing equations of the boundary layer, a thin layer formed at the wall boundary where viscosity cannot be neglected.

In this dissertation, we investigate the upper convected Maxwell(UCM) model for *complex* fluids, or *non-Newtonian* fluids, in the high Weissenberg number limit. This is analogous to the Newtonian fluids in the high Reynolds number limit. We present two well-posedness results.

The first result is on an initial-boundary value problem for incompressible hypoelastic materials which arise as a high Weissenberg number limit of viscoelastic fluids. We first assume the stress tensor is rank-one and develop energy estimates to show the problem is locally well-posed. Then we show the more general case can be handled in the same spirit. This problem is closely related to the incompressible ideal magneto-hydrodynamics(MHD) system.

The second result addresses the formulation of a time-dependent elastic boundary layer through scaling analysis. We show the well-posedness of this boundary layer by transforming to Lagrangian coordinates. In contrast to the possible ill-posedness of Prandtl's system in Newtonian fluids, we prove that in non-Newtonian fluids the stress boundary layer problem is well-posed.

Dedication

To my parents

To my wife Suxing and my daughter Hannah

To Yinwei Zhan, Yuesheng Xu and Dening Li, who set me on the road.

Acknowledgments

I am deeply indebted to my advisor, Prof. Michael Renardy, for his constant guidance and generous support. It is my best luck to have had him as my advisor. I also wish to thank Prof. Shu-Ming Sun for his beneficial courses and warm encouragements. I am grateful to all my committee members, Prof. Robert Rogers, Prof. Jeff Borggaard and Prof. Yuriko Renardy for their time and effort. I also thank Prof. George A. Hagedorn, Prof. Traian Iliescu and Prof. Peter Haskell for their help in one way or another.

Blacksburg, May, 2012

Xiaojun Wang

Contents

- 1 Introduction** **1**
- 1.1 General 1
- 1.2 Complex fluids 1
 - 1.2.1 Physical description 1
 - 1.2.2 Mathematical description 2
- 1.3 High Weissenberg number limit 5

- 2 High Weissenberg Number Limit** **7**
- 2.1 Governing system 7
 - 2.1.1 Governing equations 7
 - 2.1.2 Comparison with MHD 10
 - 2.1.3 Main results and plan 12
- 2.2 Construction of solution 13
 - 2.2.1 Metric space 13
 - 2.2.2 Iterate scheme 14
 - 2.2.3 Lemmas 16
- 2.3 Proofs of lemmas 20
- 2.4 Proof of the data dependence 25
- 2.5 On Corollary 1 28

- 3 Boundary Layers in complex fluids** **29**
- 3.1 Introduction 29

| | | |
|-------|--|----|
| 3.2 | Formulation of boundary layers | 31 |
| 3.2.1 | Scaling analysis | 31 |
| 3.2.2 | Curved boundary | 33 |
| 3.2.3 | Lagrangian description | 35 |
| 3.3 | Results | 36 |
| 3.3.1 | Lemmas | 36 |
| 3.3.2 | Main theorem | 37 |

Chapter 1

Introduction

1.1 General

Most of the notations here follow Renardy[50], Renardy and Rogers[51]. For the material on Sobolev space, see [51]. For introductory materials on Newtonian fluids, see Chorin, et al. [5]. For known results on Sobolev embedding, see Temam [64]. For introduction on tensor and coordinate transformation, see Bird [1], Truesdell et al. [65] and Sommerfeld [63]. For general results on first order symmetric hyperbolic systems, see Friedrichs [14], Kreiss [27].

1.2 Complex fluids

Fluids with complex microstructure abound in daily life. One encounters complex fluids in biological science, material science as well as in many industrial processes, e.g., in the chemical, food, and oil industries. Examples include polymers, glassy liquids, particulate suspensions, electro-and magneto-responsive suspensions, foams, emulsions and blends, liquid crystals, surfactant solutions and so on. We are interested in the flow of polymeric liquids, such as molten plastics, lubricants, paints, and many biological fluids. These fluids have mechanical properties that are intermediate between ordinary liquids and ordinary solids. They are called *viscoelastic* fluids or *soft-solid* [29].

1.2.1 Physical description

Chemical composition is responsible for the flow behavior of any fluid. The *Newtonian fluids*, or *simple fluids*, such as water and air, usually have small constituent particles. The influence of a fluid's microstructure on its flow appears as viscosity. Its governing equation is mainly about the interplay of inertial force, pressure, viscous force and, if free surface exists, surface

tension. But a *non-Newtonian fluid*, or *complex fluid* has complex microstructure, generally with particles of much larger scale. The microstructure of a complex fluid has substantial influence on its flow motion and the phenomena for such a fluid are much more varied and complex than those of a Newtonian fluid.

The Weissenberg effects, die swell and tubeless-siphon are the most well-known phenomena for non-Newtonian fluids. These phenomena can be viewed as manifestations of the particular viscoelastic mechanisms such as normal stress effects and high elongation viscosities. When one rotates a rod in a vessel containing a Newtonian fluid, the free surface is depressed near the rotating rod due to inertial effects. In contrast, for a non-Newtonian fluid with measurable normal stresses, the free surface will actually rise, and the fluid climbs up the rod [50]. The reason is that in the rotation the normal stress component arises in the circular direction of the flow and a tension in the flow direction leads to a pressure force pushing the fluid inward. This pressure can be sufficiently large to overcome the centrifugal force and make fluids climb on the rod.

If a fluid that contains large molecules is stretched, the molecules and the particles align in the direction of the stretching process. This results in a substantial increase in the extensional viscosity of the fluid. That causes the famous tubeless siphon effect—the fluid can drag itself out of a beaker, see [50]. The special microstructure of non-Newtonian causes all kinds of mechanisms, hence phenomena, that are different from the Newtonian case.

A viscoelastic fluid shows intermediate properties between a solid and a liquid. To what extent it shows solid-like or liquid-like behavior depends not only upon properties such as normal stress and elongational viscosity but also the time scale of the flow. Weissenberg number, or Deborah number¹, the ratio of a relaxation time of the fluids to a time scale of the flow, is an important dimensionless measure of elasticity. A prophetess named Deborah in the Old Testament said “the mountains flowed before the Lord”. Marcus Reiner uses this story to suggest that the time scale is important in rheology [41]. On the other hand, when a viscoelastic fluid experiences deformation in a very short time, it shows the “bouncing” phenomenon, which is typically a solid-like behavior(<http://web.mit.edu/nmf/>).

1.2.2 Mathematical description

For general incompressible fluids, the conservation of momentum gives rise to the equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) = \operatorname{div} \mathbf{T} - \nabla p + \mathbf{f}, \quad (1.1)$$

while conservation of mass requires ²

¹There are subtle differences between the definitions of these two numbers from the point of view of an engineer [8]. Here we consider time-dependent flow and it does not make a difference in our analysis.

²Here we assume constant temperature and omit the balance of energy.

$$\nabla \cdot \mathbf{v} = 0. \quad (1.2)$$

Here $\rho, \mathbf{v}, \mathbf{T}, p, \mathbf{f}$ are the density, velocity, extra stress tensor, isotropic pressure and body force, respectively. The stress tensor \mathbf{T} represents the force which the material develops in response to deformation. To complete the mathematical formulation, we need a constitutive law relating \mathbf{T} to the motion.

Let $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$ denote the rate-of-strain tensor, which is the symmetric part of velocity gradient. A Newtonian fluid has the constitutive law

$$\mathbf{T} = 2\eta \mathbf{D}, \quad (1.3)$$

where viscosity η is a constant which implies the stress is shear-rate-independent. Hence by (1.2) we have the Navier-Stokes equation

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = \eta \Delta \mathbf{v} - \nabla p + \mathbf{f}, \quad (1.4)$$

The Navier-Stokes equation has to be generalized in order to study more complex fluids. People consider the generalized Newtonian fluid, in which case the viscosity depends on the rate-of-strain tensor; namely $\mathbf{T} = \eta(|\mathbf{D}|)(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$. This model is useful in many applications because it incorporates the idea of a shear-rate-dependent viscosity. Engineers have used this type of models for over 50 years to get approximate solutions to flow problems as well as heat- and mass-transfer problems. However, its use is strictly speaking limited to steady-state shearing flow [1] and it fails to accommodate the “memory character”, or “elastic response” of viscoelastic fluids. This means the stress at any time depends upon the history of its motion. A great deal of physical insight about “elastic effects” has been obtained by studying the “ordered flow”, which is an expansion about the Newtonian fluid. But it is of limited use to both engineers and chemists because it cannot describe the full range of time dependent behavior, see [1]. Non-Newtonian models aim at taking this point into consideration.

For a general non-Newtonian fluid, the stress can be much more complicated. The constitutive law can be put into a functional $\mathcal{G}(\mathbf{T}, \mathbf{T}_t, \nabla \mathbf{T}, \mathbf{v}, \nabla \mathbf{v}, \dots) = 0$. In many cases, the relationship between stress and deformation for a complex fluid is nonlinear, is unknown, or is under dispute [29]. Below we list some well-known models which include linear and non-linear, differential and integral ones. For more details, see [1, 42, 50].

Maxwell's Model: $\frac{\partial \mathbf{T}}{\partial t} + \lambda \mathbf{T} = \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$.

Boltzmann's Model: $\mathbf{T}(\mathbf{x}, t) = \int_{-\infty}^t G(t-s)(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) ds$

UCM model: $\lambda \left(\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{T} - (\nabla \mathbf{v}) \mathbf{T} - \mathbf{T} (\nabla \mathbf{v})^T \right) + \mathbf{T} = \eta(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$

Phan-Thien Tanner: $\dots + \kappa(tr \mathbf{T}) \mathbf{T} = \dots$ ³

³When an ellipsis appears in a formula here, it represents the terms already there in the upper convected Maxwell model.

Giesekus: $\dots + \kappa \mathbf{T}^2 = \dots$

Johnson-Segalman: $\dots + (1 - a)(\mathbf{T}\mathbf{D} + \mathbf{D}\mathbf{T}) = \dots$

Dumbbell Model(Molecular theory/Fokker-Planck equation):

$$\mathbf{T}(\mathbf{x}, t) = n \int_{\mathbb{R}^3} R \mathbf{F}(R) \psi(R, \mathbf{x}, t) dR,$$

$$\int_{\mathbb{R}^3} \psi(R, \mathbf{x}, t) dR = 1$$

$$\frac{\partial \psi}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{x}}) \psi = \frac{2kT}{\eta} \Delta_R \psi + \operatorname{div}_R [-\nabla \mathbf{v}(\mathbf{x}, t) \cdot R \psi + \frac{2}{\eta} \mathbf{F}(R) \psi]$$

Multi-Mode: $\mathbf{T} = \sum \mathbf{T}_i$

Oldroyd B model: $\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \operatorname{div} \mathbf{T} + \nu \Delta \mathbf{v} - \nabla p + \mathbf{f}$

The Maxwell's and Boltzmann's model are linear in differential and integral forms, respectively. The UCM model is a nonlinear modification of Maxwell's model. This model can also be motivated by molecular theories [1, 50]. The upper convected time derivative $\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{T} - (\nabla \mathbf{v}) \mathbf{T} - \mathbf{T} (\nabla \mathbf{v})^T$ contains several nonlinear terms which are of importance. In our notation, we adopt $(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j}$. The Oldroyd B model has a stress that is the linear superposition of a UCM and a Newtonian contribution. Other popular differential models differ from the UCM by adding additional nonlinearities.

The linear viscoelastic models start considering the time-dependent stress behavior, see [11]. However they are restricted to flows with very small displacement gradients. They violate the "frame-independence-principle", which states that material properties, including stresses, are independent of the frame of reference [65].

The nonlinear models mentioned above satisfy frame indifference, but they are sufficiently complex that very few flow problems can be solved analytically. They also present great difficulties even in numerical analysis of the high Weissenberg number problems(HWNP). The upper convected Maxwell model is a good starting point for study: On the one hand it has a simpler form than most of the other nonlinear models, and on the other hand it is a model delicate enough to contain the necessary nonlinearities.

For the Navier-Stokes equations, there is a well-developed theory of existence, qualitative dynamics and numerical approximation[28][64], although there are still open problems for the "millionaire-to-be". For viscoelastic fluids, there are many more unresolved issues on modeling, numerical simulation, and mathematical analysis. Virtually everything that is known about basic existence has been established in the past thirty years, see [18, 23, 42, 50, 53, 59] and the references therein.

The work of this thesis is based on the Upper Convected Maxwell(UCM) model. For vis-

coelastic fluids of UCM model, the governing system is

$$\begin{aligned} \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) &= \nabla \cdot \mathbf{T} - \nabla p, \\ \nabla \cdot \mathbf{v} &= 0, \\ \lambda\left(\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} - (\nabla \mathbf{v})\mathbf{T} - \mathbf{T}(\nabla \mathbf{v})^T\right) + \mathbf{T} &= \eta(\nabla \mathbf{v} + (\nabla \mathbf{v})^T). \end{aligned} \quad (1.5)$$

Here \mathbf{v} , \mathbf{T} , p are velocity, stress and isotropic pressure while ρ , λ , η have physical meanings of fluids density, relaxation time and viscosity. Assuming proper scales $L, U, \frac{L}{U}, \frac{\eta U}{L}$ for length, velocity, time and stress, this can be written as the dimensionless system

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} &= \frac{1}{\mathbf{Re}} \nabla \cdot \mathbf{T} - \nabla p, \\ \nabla \cdot \mathbf{v} &= 0, \\ \frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} - (\nabla \mathbf{v})\mathbf{T} - \mathbf{T}(\nabla \mathbf{v})^T + \frac{1}{\mathbf{Wi}}\mathbf{T} &= \frac{1}{\mathbf{Wi}}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T), \end{aligned} \quad (1.6)$$

where $\mathbf{Re} = \rho UL/\eta$ is the Reynolds number and $\mathbf{Wi} = \lambda U/L$ is the Weissenberg number.

1.3 High Weissenberg number limit

We are interested in a limiting system of the UCM model, in which case Weissenberg number \mathbf{Wi} goes to infinity. This is analogous to the Navier-Stokes equations approaching Euler equations in Newtonian flow.

The flow of Newtonian fluids at high Reynolds number presents a host of mathematical difficulties in both numerical and theoretical analysis. Formally, if we take the Reynolds number to infinity, we obtain the Euler equations. That raises questions about the limiting system, which include well-posedness, stability, even brand new issues like Onsager's conjecture[38]. Another difficulty of high Reynolds number flow is the formation of singular layers along boundaries and separating streamlines, where the validity of the Euler equations breaks down.

It turns out that difficulties in these two aspects exist also for the non-Newtonian flows in high Weissenberg number limit. Since we focus particularly on near-wall behavior, we have strong interests in situations dominated by shear flow. Since normal stresses in shear flow are of order \mathbf{Wi} rather than order one, we shall scale the stresses with an additional factor \mathbf{Wi} to obtain

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} &= \frac{\mathbf{Wi}}{\mathbf{Re}} \nabla \cdot \mathbf{T} - \nabla p, \\ \nabla \cdot \mathbf{v} &= 0, \\ \frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} - (\nabla \mathbf{v})\mathbf{T} - \mathbf{T}(\nabla \mathbf{v})^T + \frac{1}{\mathbf{Wi}}\mathbf{T} &= \frac{1}{\mathbf{Wi}^2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T). \end{aligned} \quad (1.7)$$

Here $E = \frac{\mathbf{Wi}}{\mathbf{Re}}$ is called the elasticity constant. This plays a key role in our problem. When \mathbf{Wi} is finite and a wall boundary is present, we need to impose the no-slip boundary condition on the system (1.7). The first rigorous result for the nonlinear models at finite Weissenberg number was given in [43]. In the high Weissenberg number limit of non-Newtonian fluids we get a system that plays a role analogous to the Euler equations in Newtonian fluids[47, 54]. We can only impose the no-penetration condition on it. The study of stress boundary layers formation at high Weissenberg number limit was initialized in [44, 20, 45]. In [46, 48, 49], Boundary layer analysis for steady flow has been put into a neat framework. We work on the non-steady, i.e., time-dependent case, which has a very different origin from the steady one [57]. The general idea is to make proper variable substitutions to put the most significant variables to order one. We then omit the high order small terms to get an approximating equations. Advances in scaling and boundary layer analysis can be expected to shed light on the multiscale modeling [10, 32, 39].

In this dissertation, we concern ourselves with the time-dependent system in the high Weissenberg number limit. We shall investigate

1. Well-posedness of the limiting UCM model[67]–Chapter 2
2. Stress boundary layer formation and its well-posedness[56, 57]–Chapter 3.

Chapter 2

High Weissenberg Number Limit

The goal of this chapter is a well-posedness result for the equations arising in the high Weissenberg number limit of the upper convected Maxwell fluids. This limiting system for viscoelastic fluids is analogous to the Euler equations for Newtonian fluids [67].

2.1 Governing system

In Section 2.1, we introduce the governing equations. In Section 2.2, we discuss the analogy of our problem with ideal MHD, which was first pointed out by Ogilvie and Proctor [35]. In Section 2.3, we state our main result and outline the plan of the chapter.

2.1.1 Governing equations

We recall the nonlinear UCM model of viscoelastic fluids

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = E \operatorname{div} \mathbf{T} - \nabla p + \mathbf{f}, \quad (2.1)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2.2)$$

$$\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{T} - (\nabla \mathbf{v}) \mathbf{T} - \mathbf{T} (\nabla \mathbf{v})^T + \frac{1}{\mathbf{Wi}} \mathbf{T} = \frac{1}{\mathbf{Wi}^2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T). \quad (2.3)$$

where $\mathbf{v}, \mathbf{T}, p, \mathbf{f}$ are the dimensionalized versions of velocity, extra stress tensor, isotropic pressure and body force, respectively. \mathbf{Wi} is the dimensionless Weissenberg number. $E = \mathbf{Wi}/\mathbf{Re}$ is the elasticity constant. In this chapter, we are interested in a limit where \mathbf{Wi} and \mathbf{Re} tend to infinity simultaneously, but E remains fixed. Hence it can be eliminated by a rescaling of \mathbf{T} , and we simply set it equal to 1.

In (2.3), letting $\mathbf{S} \triangleq \mathbf{T} + \mathbf{Wi}^{-2}\mathbf{I}$, we have

$$\frac{\partial \mathbf{S}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{S} - (\nabla \mathbf{v})\mathbf{S} - \mathbf{S}(\nabla \mathbf{v})^T + \mathbf{Wi}^{-1}(\mathbf{S} - \mathbf{Wi}^{-2}\mathbf{I}) = 0. \quad (2.4)$$

Proposition 1. The matrix \mathbf{S} is symmetric, and, for physically relevant solutions at finite Weissenberg number, it must be positive definite.

Proof: The initial value of \mathbf{S} is symmetric by default. Taking transpose of \mathbf{S} in (2.4), we see \mathbf{S} and \mathbf{S}^T satisfy the same equation, hence they must be same whenever the solution exists. To see \mathbf{S} stays positive definite, we use the Jacobi's formula

$$\frac{d}{dt} \det(\mathbf{S}) = \det(\mathbf{S}) \operatorname{tr}(\mathbf{S}^{-1} \frac{d\mathbf{S}}{dt}) = \det(\mathbf{S}) \operatorname{tr}(\mathbf{S}^{-1} [\frac{\partial \mathbf{S}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{S}]). \quad (2.5)$$

From (2.4), noting that $\operatorname{tr}(\mathbf{S}^{-1} \nabla \mathbf{v} \mathbf{S}) = \operatorname{tr}(\nabla \mathbf{v}) = \operatorname{div} \mathbf{v} = 0$, we have

$$\frac{d}{dt} \det \mathbf{S} = \det(\mathbf{S}) \operatorname{tr}(\frac{\mathbf{S}^{-1}}{\mathbf{Wi}^3} - \frac{\mathbf{I}}{\mathbf{Wi}}). \quad (2.6)$$

\mathbf{S} ceases to be positive definite only if some eigenvalue of \mathbf{S} , hence $\det \mathbf{S}$, approaches zero. But that would make some eigenvalue for \mathbf{S}^{-1} become large. Hence the right hand side of (2.6) would become positive and $\det \mathbf{S}$ would increase and get away from zero.

□

Under high Weissenberg Number asymptotics, formally letting $\mathbf{Wi}^{-1} = 0$, (2.4) becomes

$$\frac{\partial \mathbf{S}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{S} - (\nabla \mathbf{v})\mathbf{S} - \mathbf{S}(\nabla \mathbf{v})^T = 0. \quad (2.7)$$

However, we can only assume that \mathbf{S} is positive semidefinite. In this case, there exists a tensor \mathbf{C} such that

$$\mathbf{S} = \mathbf{C}\mathbf{C}^T. \quad (2.8)$$

With

$$\tilde{\mathbf{C}} = \frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{C} - (\nabla \mathbf{v})\mathbf{C}, \quad (2.9)$$

(2.7) and (2.8) imply

$$\tilde{\mathbf{C}}\mathbf{C}^T + \mathbf{C}\tilde{\mathbf{C}}^T = \mathbf{0}. \quad (2.10)$$

For (2.7) to be true, it is sufficient to require $\tilde{\mathbf{C}} = \mathbf{0}$, i.e.,

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{C} - (\nabla \mathbf{v})\mathbf{C} = \mathbf{0}. \quad (2.11)$$

or in vector form:

$$\frac{\partial \mathbf{C}_i}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{C}_i - (\mathbf{C}_i \cdot \nabla)\mathbf{v} = \mathbf{0}, i = 1, 2, 3. \quad (2.12)$$

Here \mathbf{C}_i are the column vectors of \mathbf{C} .

Using the Einstein summation convention, we have $\mathbf{S} = \mathbf{C}\mathbf{C}^T = \mathbf{C}_i\mathbf{C}_i^T$. Hence

$$\nabla \cdot \mathbf{T} = \nabla \cdot \mathbf{S} = \nabla \cdot (\mathbf{C}_i\mathbf{C}_i^T) = (\mathbf{C}_i \cdot \nabla)\mathbf{C}_i + (\nabla \cdot \mathbf{C}_i)\mathbf{C}_i. \quad (2.13)$$

Substituting into (2.3), we have the equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{C}_i \cdot \nabla)\mathbf{C}_i = (\nabla \cdot \mathbf{C}_i)\mathbf{C}_i - \nabla p + \mathbf{f}. \quad (2.14)$$

The objective of this chapter is the analysis of the system (2.2), (2.12), (2.14). As an intermediate step, we formulate a simpler problem, and we eventually show that once we solve the simpler problem (Theorem 1), the general case is actually an easy generalization (Corollary 1).

We shall now assume that \mathbf{S} is a rank-one tensor, namely

$$\mathbf{S} = \mathbf{q}\mathbf{q}^T, \mathbf{q}(x, t) \in \mathbb{R}^n, \mathbf{q} \neq \mathbf{0}. \quad (2.15)$$

We can now satisfy (2.7) by requiring that

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{q} - (\mathbf{q} \cdot \nabla)\mathbf{v} = 0. \quad (2.16)$$

Substituting $\nabla \cdot \mathbf{T} = \nabla \cdot (\mathbf{q}\mathbf{q}^T)$ into (2.3), we have the equation of motion

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{q} \cdot \nabla)\mathbf{q} = (\nabla \cdot \mathbf{q})\mathbf{q} - \nabla p + \mathbf{f}. \quad (2.17)$$

Now we seek solutions of (2.2), (2.16) and (2.17) for $t > 0$ and $x \in \Omega$, where Ω is a bounded domain in \mathbb{R}^3 . We impose initial conditions

$$\mathbf{v}(x, 0) = \mathbf{v}_0(x), \mathbf{q}(x, 0) = \mathbf{q}_0(x). \quad (2.18)$$

and the boundary conditions

$$\mathbf{v} \cdot \mathbf{n} = 0, \mathbf{q} \cdot \mathbf{n} = 0, x \in \partial\Omega. \quad (2.19)$$

The boundary conditions mean that no fluid crosses the boundary and that the stress tensor is aligned parallel to the boundary.

Most of this chapter will be concerned with the analysis of the initial-boundary value problem consisting of (2.2) and (2.16)-(2.19).

Remarks:

1. In principle, the term $\mathbf{Wi}^{-1}(\mathbf{S} - \mathbf{Wi}^{-2}\mathbf{I})$, which we neglected in the equations, is a term involving no derivatives, which does not affect the well-posedness of the problem. The difference between the case of finite Weissenberg number and infinite Weissenberg number is in the assumptions on \mathbf{S} . For finite Weissenberg number \mathbf{S} is strictly positive definite. The existence result in [43] depends on this to guarantee a strict ellipticity condition. In the high Weissenberg number limit, polymer molecules typically line up in a preferred direction, and the result is a stress dominated by one component. In the limit $\mathbf{Wi} \rightarrow \infty$, the appropriate assumption for most flows of physical interest is a stress of rank one.
2. The loss of strict ellipticity affects the boundary conditions which can be imposed. For finite Weissenberg number, we can impose Dirichlet conditions for the velocity, but in our case, we can impose only the normal component. The condition $\mathbf{q} \cdot \mathbf{n} = 0$ is not really “imposed.” Rather, it is preserved by the equations as long as it holds for the initial data, same as proof of lemma 4 below. It is reasonable to require this condition, since the flow at a wall is viscometric, and the dominant stress component for high Weissenberg number is the first normal stress.

2.1.2 Comparison with MHD

It is easy to see the formal analogy between MHD and our problem (2.2) and (2.16)-(2.19). We collect the equations to have

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{q} \cdot \nabla)\mathbf{q} = (\nabla \cdot \mathbf{q})\mathbf{q} - \nabla p + \mathbf{f}, \\ \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{q} - (\mathbf{q} \cdot \nabla)\mathbf{v} = 0, \\ \nabla \cdot \mathbf{v} = 0, \\ \mathbf{v}(x, 0) = \mathbf{v}_0(x); \mathbf{q}(x, 0) = \mathbf{q}_0(x), \\ \mathbf{v} \cdot \mathbf{n} = 0; \mathbf{q} \cdot \mathbf{n} = 0, x \in \partial\Omega. \end{array} \right. \quad (2.20)$$

We emphasize the fact that the normal components of \mathbf{v} and \mathbf{q} vanish on the boundary and this makes the boundary characteristic. It is an important part of our argument to show that this property persists in our iterative scheme, see the proof of Lemma 4 in Section 3.

We recall the incompressible magnetohydrodynamics (MHD) system

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{q} \cdot \nabla)\mathbf{q} = \epsilon \Delta \mathbf{v} - \nabla(p + \frac{1}{2}|\mathbf{q}|^2) + \mathbf{f}, \\ \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{q} - (\mathbf{q} \cdot \nabla)\mathbf{v} = \eta \Delta \mathbf{q}, \\ \nabla \cdot \mathbf{v} = 0, \nabla \cdot \mathbf{q} = 0, \\ \mathbf{v}(x, 0) = \mathbf{v}_0(x); \mathbf{q}(x, 0) = \mathbf{q}_0(x), \end{array} \right. \quad (2.21)$$

where $\mathbf{v}, \mathbf{q}, \mathbf{f}, \epsilon, \eta$ represent the fluid velocity field, the magnetic field, the body force, the constant kinematic viscosity and the constant magnetic diffusivity, respectively, with $\epsilon \geq 0, \eta \geq 0$. If the system is set in \mathbb{R}^n , then (2.21) is a Cauchy problem. If we consider the

problem in some domain in \mathbb{R}^n , we have an IBVP and we need proper boundary conditions for it. For example, in the case that $\epsilon = 0, \eta = 0$, we may impose the same boundary condition as in (2.20)

$$\mathbf{v} \cdot \mathbf{n} = 0; \mathbf{q} \cdot \mathbf{n} = 0, x \in \partial\Omega. \quad (2.22)$$

In other cases, (2.22) is generally not enough for problem (2.21) to be well-posed.

Magnetohydrodynamics finds a wide range of industrial applications, from liquid metals to cosmic plasmas. For an in-depth derivation of the MHD model and a detailed application to plasmas, see [16]. We organize our review into three cases:

Case I. $\epsilon > 0, \eta > 0$. The kinematic viscosity and the magnetic diffusivity are taken into account. In this case, for the Cauchy problem (2.21), J. Cannone et al. study the uniqueness and regularity of Leray-Hopf's weak solutions in [3], while C. He and Z. Xin [19] give two classes of sufficient conditions which guarantee the weak solutions are regular.

Case II. $\epsilon = \eta = 0$. The MHD equations (2.21) are called ideal MHD in the sense that the effect of fluid viscosity and magnetic diffusivity is neglected. It describes the motion of a perfectly conducting fluid interacting with a non-resistive magnetic field. In this case, similar to getting the regularity for the Euler equation by controlling the vorticity, Q. Jiu and C. He [22] and J. Wu [69] study sufficient conditions which guarantee the regularity of solutions of 3D ideal MHD. Zhang and Liu [70] study the blow-up criterion of smooth solutions to the Cauchy problem (2.21). P. Secchi studies the initial boundary value problem when Ω is a bounded domain in [60]. He also shows, for $\Omega = \mathbb{R}_+^3$, the existence of a regular solution in the anisotropic Sobolev space $H_{**}^m(\Omega)$, $m \geq 5$ he introduced in [62]. If a solution exists on some bounded domain Ω , we can linearize (2.21)-(2.22) about this solution to get a perturbed equation whose regularity is a consequence of M. Ohno and T. Shirota [36].

Case III. One of ϵ, η is zero, but not both. We say the equations have partial viscous terms; in a sense they are like a combination of Euler and Navier-Stokes equations. In this case the smoothing effect of the viscous part plays a critical role. J. Fan and T. Ozawa [12] prove some regularity conditions for (2.21) as a Cauchy problem. For the IBVP with exterior domain or half plane case, E. Case et al. [4] prove the existence of classical solutions global in time. For smooth initial data and without boundary, it has been proved that the solution of the Navier-Stokes equations converges to that of the Euler equations as the viscosity goes to zero. It is natural to ask what happens when Case I becomes Case II. In this regard, J. Wu [68] investigates the Cauchy problem while G. Chen et al. [6] study the IBVP with a nonhomogeneous vorticity boundary condition.

It is interesting to see the obvious structural analogy between our setting (2.20) for the UCM fluid in the limit of infinite Weissenberg number and the Case II for ideal MHD. One difference is that there is no physical reason to require $\nabla \cdot \mathbf{q} = 0$ in our case. On the other hand, even in our problem $\nabla \cdot \mathbf{q}$ will equal zero if it is zero initially. If $\nabla \cdot \mathbf{q}$ is not zero, we shall be able to derive estimates for $\nabla \cdot \mathbf{q}$, which will play a crucial role in our proof.

2.1.3 Main results and plan

We make the following assumptions of smoothness and compatibility:

(S1) The domain $\Omega \subset \mathbb{R}^3$ is a bounded open set lying locally on one side of its boundary $\partial\Omega$ which is of class C^{m+2} , $m \geq 3$.

(S2) $\mathbf{v}_0 \in H^m$, $\mathbf{q}_0 \in H^m$, $\nabla \cdot \mathbf{q}_0 \in H^m$.

(S3) For some $T > 0$, we have $\mathbf{f} \in \bigcap_{k=0}^m W^{k,1}([0, T]; H^{m-k}(\Omega))$.

(C) $\mathbf{v}_0 \cdot \mathbf{n} = 0$, $\mathbf{q}_0 \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\nabla \cdot \mathbf{v}_0 = 0$.

We prove the following results:

Theorem 1 Assume that (S1)-(S3) and (C) hold. Then there is a $T' \in (0, T]$ such that the problem (2.20) has a unique solution with the regularity

$$(\mathbf{v}, \mathbf{q})^T \in \bigcap_{k=0}^m C^k([0, T']; H^{m-k}(\Omega)). \quad (2.23)$$

Moreover, the solutions depend on the data continuously in $C([0, T'], H^m)$. More specifically, suppose we have a sequence of data $\mathbf{v}_0^n, \mathbf{q}_0^n, \mathbf{f}^n$ which satisfy the requirements for $\mathbf{v}_0, \mathbf{q}_0, \mathbf{f}$. Correspondingly we have a sequence of solutions $(\mathbf{v}^n, \mathbf{q}^n)$ in $C([0, T^n], H^m)$. Despite the different intervals of time existence, we prove that if the following is true:

1. $\lim_{n \rightarrow \infty} (\mathbf{v}_0^n, \mathbf{q}_0^n) \rightarrow (\mathbf{v}_0, \mathbf{q}_0)$ in H^m
2. $\lim_{n \rightarrow \infty} \nabla \cdot \mathbf{q}_0^n \rightarrow \nabla \cdot \mathbf{q}_0$ in H^m ,
3. $\lim_{n \rightarrow \infty} \mathbf{f}^n \rightarrow \mathbf{f}$ in $L^1([0, T], H^m)$, then on some common interval $[0, T^*]$ the solutions $(\mathbf{v}^n, \mathbf{q}^n)$ exist and $(\mathbf{v}^n, \mathbf{q}^n) \rightarrow (\mathbf{v}, \mathbf{q})$ in $C([0, T^*], H^m)$.

As far as we know, the closest result to ours is in [60] where P. Secchi shows the well-posedness of solutions for MHD Case II using techniques based on Kato's perturbation theory, see also Veiga[66]. In several aspects, our proof is different. In our problem there is no divergence free assumption on \mathbf{q} . That forces an extra estimation of β which approximates $\nabla \cdot \mathbf{q}$ in our scheme, see Section 2.1. For the same reason in the data dependence part we need the assumption $\lim_{n \rightarrow \infty} \nabla \cdot \mathbf{q}_0^n \rightarrow \nabla \cdot \mathbf{q}_0$ in H^m to guarantee the continuous dependence in $C([0, T^*], H^m)$ of the solution on the data. In our proof of existence and uniqueness, we take advantage of results in Majda[33]. For continuous dependence, while Secchi's approach can be applied to our problem, we take a different approach based on an idea of Bona and Smith [2], see also [25] for a similar application of the same method. After we tackle the case of a rank one stress tensor, our analysis can be smoothly extended to the more general case given by (2.2)(2.12)(2.14). More specifically, we make the following assumptions:

(S4) $\mathbf{v}_0 \in H^m$, $\mathbf{C}_0 \in H^m$, $\nabla \cdot \mathbf{C}_0 \in H^m$.

(C1) $\mathbf{v}_0 \cdot \mathbf{n} = 0$, $\mathbf{C}_{i0} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\nabla \cdot \mathbf{v}_0 = 0$.

Here \mathbf{C}_{i0} is the i th column of \mathbf{C}_0 . And we have

Corollary 1. Assume that (S1),(S3),(S4) and (C1) hold. Then there is a $T' \in (0, T]$ such that the problem (2.14),(2.2) and (2.12), together with corresponding initial values, has a unique solution with the regularity

$$(\mathbf{v}, \mathbf{C}) \in \bigcap_{k=0}^m C^k([0, T']; H^{m-k}(\Omega)) \quad (2.24)$$

Moreover the solutions depend continuously on the initial data.

In Section 2.2, we introduce the iteration scheme and the metric space we use to construct the solution. To show the existence part of **Theorem 1**, it suffices to show the lemmas which are stated in Section 2.2. Then in Section 2.3-2.4 we complete the proofs of those lemmas as well as the continuous dependence of the solution on the data. In Section 2.5 we describe the proof for **Corollary 1**.

2.2 Construction of solution

To prove the existence and uniqueness of the solution, we generate a sequence of functions approximating the solution. As is commonly done for PDEs of hyperbolic type, we first show boundedness of the iterates in a higher order norm and then use knowledge of this bound to show convergence in a lower order norm. We shall use some of the terminology introduced in [43].

2.2.1 Metric space

Let $Z(M, T')$ be the set of all functions $\mathbf{l} = (\mathbf{v}, \mathbf{q}) : \Omega \times [0, T'] \rightarrow \mathbb{R}^6$ with the following properties:

$$\mathbf{l} \in W^{0,\infty}([0, T']; H^m(\Omega)) \cap W^{1,\infty}([0, T'], H^{m-1}(\Omega)); \quad (2.25)$$

$$\|\mathbf{l}\|_{0,m} + \|\mathbf{l}\|_{1,m-1} \leq M, \quad (2.26)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (2.27)$$

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0, \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad (2.28)$$

$$\mathbf{v}(x, 0) = \mathbf{v}_0(x), \mathbf{q}(x, 0) = \mathbf{q}_0(x). \quad (2.29)$$

Here $\|\cdot\|_{k,l}$ denotes the norm in $W^{k,\infty}([0, T']; H^l(\Omega))$. On $Z(M, T')$ we define the metric

$$d(\mathbf{l}, \tilde{\mathbf{l}}) = \|\mathbf{l} - \tilde{\mathbf{l}}\|_{0,1}. \quad (2.30)$$

As in [43], when M is sufficiently large we can show $Z(M, T')$ is not empty and is complete with metric d .

In (2.17), we take the divergence and the normal component on the boundary to obtain a Neumann problem for p :

$$\Delta p = \nabla \cdot (\mathbf{f} + (\mathbf{q} \cdot \nabla)\mathbf{q} + (\nabla \cdot \mathbf{q})\mathbf{q} - (\mathbf{v} \cdot \nabla)\mathbf{v}),$$

with boundary condition

$$\frac{\partial p}{\partial n} = \{\mathbf{f} + (\mathbf{q} \cdot \nabla)\mathbf{q} + (\nabla \cdot \mathbf{q})\mathbf{q} - (\mathbf{v} \cdot \nabla)\mathbf{v}\} \cdot \mathbf{n}, x \in \partial\Omega.$$

On the other hand, if we view p as known, (2.20) is a hyperbolic problem for \mathbf{v} and \mathbf{q} . This suggests that we construct our solution by iterating between these two problems. To ensure sufficient regularity at each step, it will be necessary to introduce an additional scalar function $\beta(x, t)$ which approximates $\nabla \cdot \mathbf{q}$ in the iteration.

We shall adopt the terminology of a ‘‘controllable’’ function in [43].

Definition 1. A continuous function $\psi(M, T', \alpha, \beta, \dots) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \dots \rightarrow \mathbb{R}$ is called controllable if there are continuous functions $\tau(M, \alpha, \beta, \dots)$ and $\omega(\alpha, \beta, \dots)$ such that $\tau > 0$ and $\psi(M, T', \alpha, \beta, \dots) \leq \omega(\alpha, \beta, \dots)$ as long as $T' \leq \tau(M, \alpha, \beta, \dots)$.

The essence of Definition 1 is evident. For fixed α, β , we can always bound ψ by adjusting T' even if M changes. This definition is interwoven with the metric space. When we adjust M, T' for ψ , we also adjust the space $Z(M, T')$.

2.2.2 Iterate scheme

Starting with $(\mathbf{v}^n, \mathbf{q}^n) \in Z(M, T')$, we try to construct a mapping $\Sigma : (\mathbf{v}^n, \mathbf{q}^n) \mapsto (\mathbf{v}^{n+1}, \mathbf{q}^{n+1})$ on $Z(M, T')$.

Step 1. For given \mathbf{v}^n determine $\beta^n(x, t)$ from

$$\frac{\partial}{\partial t} \beta^n(x, t) + (\mathbf{v}^n \cdot \nabla) \beta^n(x, t) = 0, \quad (2.31)$$

$$\beta^n(x, 0) = \nabla \cdot \mathbf{q}_0 \quad (2.32)$$

Step 2. For known \mathbf{q}^n and β^n , determine φ^n from

$$\Delta\varphi^n = \beta^n - \nabla \cdot \mathbf{q}^n \quad (2.33)$$

$$\frac{\partial\varphi^n}{\partial n} = 0 \quad (2.34)$$

Notice this problem is well-defined because we can easily derive $\int_{\Omega} \beta^n dx = 0$ from (2.31)-(2.32) and $\mathbf{q}^n \cdot \mathbf{n} = 0$.

Step 3. Let $\hat{\mathbf{q}}^n = \mathbf{q}^n + \nabla\varphi^n$. Then

$$\nabla \cdot \hat{\mathbf{q}}^n = \beta^n, \hat{\mathbf{q}}^n \cdot \mathbf{n} = 0. \quad (2.35)$$

Step 4. Given $\mathbf{v}^n, \mathbf{q}^n, \hat{\mathbf{q}}^n$ and β^n we determine p^n by

$$\Delta p^n = \nabla \cdot (\mathbf{f} + (\mathbf{q}^n \cdot \nabla)\hat{\mathbf{q}}^n + \beta^n \mathbf{q}^n - (\mathbf{v}^n \cdot \nabla)\mathbf{v}^n), \quad (2.36)$$

$$\frac{\partial p^n}{\partial n} = (\mathbf{f} + (\mathbf{q}^n \cdot \nabla)\hat{\mathbf{q}}^n + \beta^n \mathbf{q}^n - (\mathbf{v}^n \cdot \nabla)\mathbf{v}^n) \cdot \mathbf{n}. \quad (2.37)$$

Step 5. Then $\hat{\mathbf{v}}^{n+1}, \hat{\mathbf{q}}^{n+1}$ are determined from

$$\begin{aligned} & \frac{\partial \hat{\mathbf{v}}^{n+1}}{\partial t} + (\mathbf{v}^n \cdot \nabla)\hat{\mathbf{v}}^{n+1} - (\mathbf{q}^n \cdot \nabla)\hat{\mathbf{q}}^{n+1} = \beta^n \mathbf{q}^n - \nabla p^n + \mathbf{f} \\ & + \left[((\mathbf{q}^n \cdot \nabla)\mathbf{n}_\epsilon) \cdot (\hat{\mathbf{q}}^{n+1} - \hat{\mathbf{q}}^n) - ((\mathbf{v}^n \cdot \nabla)\mathbf{n}_\epsilon) \cdot (\hat{\mathbf{v}}^{n+1} - \mathbf{v}^n) \right] \mathbf{n}_\epsilon, \end{aligned} \quad (2.38)$$

$$\begin{aligned} & \frac{\partial \hat{\mathbf{q}}^{n+1}}{\partial t} + (\mathbf{v}^n \cdot \nabla)\hat{\mathbf{q}}^{n+1} - (\mathbf{q}^n \cdot \nabla)\hat{\mathbf{v}}^{n+1} = \\ & \left[((\mathbf{q}^n \cdot \nabla)\mathbf{n}_\epsilon) \cdot (\hat{\mathbf{v}}^{n+1} - \mathbf{v}^n) - ((\mathbf{v}^n \cdot \nabla)\mathbf{n}_\epsilon) \cdot (\hat{\mathbf{q}}^{n+1} - \mathbf{q}^n) \right] \mathbf{n}_\epsilon, \end{aligned} \quad (2.39)$$

with

$$(\hat{\mathbf{v}}^{n+1}, \hat{\mathbf{q}}^{n+1})(x, 0) = (\mathbf{v}_0, \mathbf{q}_0)(x). \quad (2.40)$$

Here \mathbf{n}_ϵ is a smooth vector function on \mathbb{R}^n which has support in an ϵ neighborhood of the boundary and which is equal to \mathbf{n} on the boundary. We will show that the solution satisfies $\hat{\mathbf{v}}^{n+1} \cdot \mathbf{n} = 0, \hat{\mathbf{q}}^{n+1} \cdot \mathbf{n} = 0$

Step 6. Now for known $\hat{\mathbf{v}}^{n+1}$ solve for ψ^n by

$$\Delta\psi^{n+1} = -\nabla \cdot \hat{\mathbf{v}}^{n+1}, \quad (2.41)$$

$$\frac{\partial\psi^{n+1}}{\partial n} = 0. \quad (2.42)$$

Step 7. Finally, let

$$\mathbf{v}^{n+1} = \hat{\mathbf{v}}^{n+1} + \nabla\psi^{n+1}, \mathbf{q}^{n+1} = \hat{\mathbf{q}}^{n+1}. \quad (2.43)$$

Thus, we have

$$\nabla \cdot \mathbf{v}^{n+1} = 0, \mathbf{v}^{n+1} \cdot \mathbf{n} = 0, \mathbf{q}^{n+1} \cdot \mathbf{n} = 0. \quad (2.44)$$

Our eventual task is to show that the mapping $\Sigma : (\mathbf{v}^n, \mathbf{q}^n) \mapsto (\mathbf{v}^{n+1}, \mathbf{q}^{n+1})$ is a contraction on $Z(M, T')$. Then we show that the fixed point of the contraction is indeed the solution of (2.20).

Remarks: The reason why the term involving \mathbf{n}_ϵ is added in the hyperbolic system is to preserve the boundary condition. We shall explain this in detail in the proofs below.

2.2.3 Lemmas

It is easy to verify the problem in each step is well-defined. We have to show the iterate scheme is well-defined, namely, for some fixed M, T' , the mapping Σ maps $Z(M, T')$ into $Z(M, T')$. We use C, C_M, C_Ω as generic constants throughout this chapter. And $\phi, \phi_1, \phi_2, \dots, \phi_7$ are all controllable functions that are independent of the iteration steps.

Lemma 1. Let $\mathbf{v}^n \in Z(M, T')$ be given. Then (2.31)-(2.32) has a unique solution β^n which satisfies

$$\sup_{0 \leq t \leq T'} \|\beta^n(\cdot, t)\|_{H^m} \leq e^{C_M T'} \|\nabla \cdot \mathbf{q}_0\|_{H^m}, \quad (2.45)$$

where C_M is a constant depending upon M . If there is another equation

$$\frac{\partial}{\partial t} \tilde{\beta}^n(x, t) + (\tilde{\mathbf{v}}^n \cdot \nabla) \tilde{\beta}^n(x, t) = 0,$$

$$\tilde{\beta}^n(x, 0) = \nabla \cdot \mathbf{q}_0,$$

where $\tilde{\mathbf{v}}^n \in Z(M, T')$, then we have

$$\|\beta^n - \tilde{\beta}^n\|_{0,0} \leq \phi_1(M, T') \|\mathbf{v}^n - \tilde{\mathbf{v}}^n\|_{0,0}, \quad (2.46)$$

and

$$\|\nabla(\beta^n - \tilde{\beta}^n)\|_{0,0} \leq \phi_2(M, T') \|\mathbf{v}^n - \tilde{\mathbf{v}}^n\|_{0,1}. \quad (2.47)$$

Here the controllable functions $\phi_1(M, T'), \phi_2(M, T') \rightarrow 0$ as $T' \rightarrow 0$.

Lemma 2. Problem (2.33)-(2.34) has a unique solution which satisfies

$$\|\nabla\varphi^n\|_{H^m} \leq C_\Omega (\|\beta^n\|_{H^{m-1}} + \|\mathbf{q}^n\|_{H^m}). \quad (2.48)$$

Hence we have from Step 3

$$\|\hat{\mathbf{q}}^n\|_m \leq \|\mathbf{q}\|_m + \|\nabla\varphi^n\|_m \leq C_{M,\Omega}.$$

Furthermore, if there is another $\tilde{\varphi}^n$ which satisfies

$$\Delta\tilde{\varphi}^n = \tilde{\beta}^n - \nabla \cdot \tilde{\mathbf{q}}^n, \quad \frac{\partial\tilde{\varphi}^n}{\partial n} = 0,$$

then we have

$$\|\nabla(\varphi^n - \tilde{\varphi}^n)\|_{H^1} \leq C_\Omega(\|\beta^n - \tilde{\beta}^n\|_{H^0} + \|\mathbf{q}^n - \tilde{\mathbf{q}}^n\|_{H^1}), \quad (2.49)$$

and for $\hat{\tilde{\mathbf{q}}}^n = \tilde{\mathbf{q}}^n + \nabla\tilde{\varphi}$ we have

$$\|\hat{\tilde{\mathbf{q}}}^n - \tilde{\mathbf{q}}^n\|_{H^1} \leq C_\Omega(\|\beta^n - \tilde{\beta}^n\|_{H^0} + \|\mathbf{q}^n - \tilde{\mathbf{q}}^n\|_{H^1}). \quad (2.50)$$

Lemma 3. Let $(\mathbf{v}^n, \mathbf{q}^n) \in Z(M, T')$ be given and assume that T' is sufficiently small. Let $\beta^n, \hat{\mathbf{q}}^n$ be as in Lemma 1 and Step 3. Then (2.36)-(2.37) has a solution p^n which satisfies

$$\|\nabla p^n\|_{W^{0,1}([0,T'],H^m)} \leq C(M^2T' + \|\mathbf{f}\|_{W^{0,1}([0,T'],H^m)}). \quad (2.51)$$

Furthermore, if there is a second equation for \tilde{p}^n ,

$$\Delta\tilde{p}^n = \nabla \cdot (\mathbf{f} + (\tilde{\mathbf{q}}^n \cdot \nabla)\hat{\tilde{\mathbf{q}}}^n + \tilde{\beta}^n\tilde{\mathbf{q}}^n - (\tilde{\mathbf{v}}^n \cdot \nabla)\tilde{\mathbf{v}}^n)$$

with boundary condition

$$\frac{\partial\tilde{p}^n}{\partial n} = \{\mathbf{f} + (\tilde{\mathbf{q}}^n \cdot \nabla)\hat{\tilde{\mathbf{q}}}^n + \tilde{\beta}^n\tilde{\mathbf{q}}^n - (\tilde{\mathbf{v}}^n \cdot \nabla)\tilde{\mathbf{v}}^n\} \cdot \mathbf{n}, \quad x \in \partial\Omega,$$

then we have for each t , *a.e.*,

$$\|\nabla p^n - \nabla\tilde{p}^n\|_{0,1} \leq \phi_3(M, T')\|\mathbf{k}^n - \tilde{\mathbf{k}}^n\|_{0,1}. \quad (2.52)$$

Here $\phi_3(M, T')$ is controllable and $\mathbf{k}^n = (\mathbf{v}^n, \mathbf{q}^n)$.

To simplify the notations we let $\mathbf{v}, \mathbf{q}, \hat{\mathbf{q}}, \hat{\mathbf{w}}, \hat{\mathbf{s}}, \mathbf{w}, \mathbf{s}, p$ represent $\mathbf{v}^n, \mathbf{q}^n, \hat{\mathbf{q}}^n, \hat{\mathbf{v}}^{n+1}, \hat{\mathbf{q}}^{n+1}, \mathbf{v}^{n+1}, \mathbf{q}^{n+1}, p^n$ respectively. Now for equations (2.38)-(2.40) we have

$$\frac{\partial\hat{\mathbf{w}}}{\partial t} + (\mathbf{v} \cdot \nabla)\hat{\mathbf{w}} - (\mathbf{q} \cdot \nabla)\hat{\mathbf{s}} = \mathbf{h}_1(\hat{\mathbf{w}}, \hat{\mathbf{s}}, \mathbf{v}, \mathbf{q}, \hat{\mathbf{q}}, \beta, \nabla p, \mathbf{n}_\epsilon, \mathbf{f}), \quad (2.53)$$

$$\frac{\partial\hat{\mathbf{s}}}{\partial t} + (\mathbf{v} \cdot \nabla)\hat{\mathbf{s}} - (\mathbf{q} \cdot \nabla)\hat{\mathbf{w}} = \mathbf{h}_2(\hat{\mathbf{w}}, \hat{\mathbf{s}}, \mathbf{v}, \mathbf{q}, \hat{\mathbf{q}}, \beta, \nabla p, \mathbf{n}_\epsilon, \mathbf{f}), \quad (2.54)$$

with

$$(\hat{\mathbf{w}}(x, 0), \hat{\mathbf{s}}(x, 0)) = (\mathbf{v}_0, \mathbf{q}_0)(x), \quad (2.55)$$

where

$$\begin{aligned}\mathbf{h}_1 &= \beta \mathbf{q} - \nabla p + \mathbf{f} + \left[((\mathbf{q} \cdot \nabla) \mathbf{n}_\epsilon) \cdot (\hat{\mathbf{s}} - \hat{\mathbf{q}}) - ((\mathbf{v} \cdot \nabla) \mathbf{n}_\epsilon) \cdot (\hat{\mathbf{w}} - \mathbf{v}) \right] \mathbf{n}_\epsilon, \\ \mathbf{h}_2 &= \left[((\mathbf{q} \cdot \nabla) \mathbf{n}_\epsilon) \cdot (\hat{\mathbf{w}} - \mathbf{v}) - ((\mathbf{v} \cdot \nabla) \mathbf{n}_\epsilon) \cdot (\hat{\mathbf{s}} - \mathbf{q}) \right] \mathbf{n}_\epsilon.\end{aligned}\quad (2.56)$$

Using $\hat{\mathbf{l}} = (\hat{\mathbf{w}}, \hat{\mathbf{s}})$, $\mathbf{k} = (\mathbf{v}, \mathbf{q})$, we put (2.53)-(2.55) into a more compact form

$$\begin{aligned}\frac{\partial \hat{\mathbf{l}}}{\partial t} + \sum_{j=1}^3 A_j(\mathbf{k}) \frac{\partial \hat{\mathbf{l}}}{\partial x_j} &= \mathbf{g}(\hat{\mathbf{l}}, \mathbf{k}, \hat{\mathbf{q}}, \beta, \nabla p, \mathbf{n}_\epsilon, \mathbf{f}), \\ \hat{\mathbf{l}}(x, 0) &= (\mathbf{v}_0, \mathbf{q}_0)^T,\end{aligned}\quad (2.57)$$

with

$$\mathbf{g}(\hat{\mathbf{l}}, \mathbf{k}, \hat{\mathbf{q}}, \beta, \nabla p, \mathbf{n}_\epsilon, \mathbf{f}) = \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix}.\quad (2.58)$$

Lemma 4: Let $(\mathbf{v}, \mathbf{q}) \in Z(M, T')$, p satisfy the bounds in Lemma 3. Assume that T' is sufficiently small. Then (2.57) has a unique solution

$$\hat{\mathbf{l}} \in W^{0,\infty}([0, T']; H^m(\Omega))\quad (2.59)$$

which satisfies a bound of the form

$$\|\hat{\mathbf{l}}\|_{0,m} \leq e^{CT'} (\|\mathbf{l}_0\|_m + h(T')).\quad (2.60)$$

Here $h(T')$ is a function which tends to zero as $T' \rightarrow 0$. Moreover, if there is a second equation in Step 5

$$\begin{aligned}\frac{\partial \tilde{\hat{\mathbf{l}}}}{\partial t} + \sum_{j=1}^3 A_j(\tilde{\mathbf{k}}) \frac{\partial \tilde{\hat{\mathbf{l}}}}{\partial x_j} &= \tilde{\mathbf{g}}(\tilde{\hat{\mathbf{l}}}, \tilde{\mathbf{k}}, \tilde{\hat{\mathbf{q}}}, \tilde{\beta}, \nabla \tilde{p}^n, \mathbf{n}_\epsilon, \mathbf{f}), \\ \tilde{\hat{\mathbf{l}}}(x, 0) &= (\mathbf{v}_0, \mathbf{q}_0)^T\end{aligned}\quad (2.61)$$

then we have an estimate of the form

$$\|\hat{\mathbf{l}} - \tilde{\hat{\mathbf{l}}}\|_{0,1} \leq \phi_4(M, T') (\|\mathbf{k} - \tilde{\mathbf{k}}\|_{0,1} + \|\mathbf{g} - \tilde{\mathbf{g}}\|_{0,1})\quad (2.62)$$

with

$$\|\mathbf{g} - \tilde{\mathbf{g}}\|_{0,1} \leq \phi_5(M, T') (\|\mathbf{k} - \tilde{\mathbf{k}}\|_{0,1} + \|\hat{\mathbf{l}} - \tilde{\hat{\mathbf{l}}}\|_{0,1}).\quad (2.63)$$

This implies

$$\|\hat{\mathbf{l}} - \tilde{\hat{\mathbf{l}}}\|_{0,1} \leq \phi_6(M, T') \|\mathbf{k} - \tilde{\mathbf{k}}\|_{0,1}.\quad (2.64)$$

Here $\phi_4(M, T')$, $\phi_5(M, T')$, $\phi_6(M, T')$ are controllable and $\phi_4, \phi_6 \rightarrow 0$ as $T' \rightarrow 0$.

In Steps 6 and 7, we know $\mathbf{l} = (\mathbf{v}^{n+1}, \mathbf{q}^{n+1}) = \hat{\mathbf{l}} + (\nabla \psi^{n+1}, 0)$.

Lemma 5: From (2.41)-(2.43) we have

$$\|\mathbf{l}\|_{0,m} \leq \phi_7(M, T')(\|\mathbf{l}_0\|_m + h(T')). \quad (2.65)$$

Moreover, in Step 6, if there is another equation

$$\Delta \tilde{\psi}^{n+1} = -\nabla \cdot \tilde{\mathbf{v}}^{n+1}, \quad \frac{\partial \tilde{\psi}^{n+1}}{\partial n} = 0,$$

then we have

$$\|\nabla(\tilde{\psi}^{n+1} - \psi^{n+1})\|_{0,1} \leq C_M \|\tilde{\mathbf{v}}^{n+1} - \hat{\mathbf{v}}^{n+1}\|_{0,1} \leq C_M \|\tilde{\hat{\mathbf{l}}} - \hat{\mathbf{l}}\|_{0,1}. \quad (2.66)$$

Furthermore, if $\tilde{\mathbf{l}} = (\tilde{\mathbf{v}}^{n+1}, \tilde{\mathbf{q}}^{n+1}) = \tilde{\hat{\mathbf{l}}} + (\nabla \tilde{\psi}^{n+1}, 0)$ then It is obvious that

$$\|\tilde{\mathbf{l}} - \mathbf{l}\|_{0,1} \leq \|\tilde{\hat{\mathbf{l}}} - \hat{\mathbf{l}}\|_{0,1} + \|\nabla(\tilde{\psi}^{n+1} - \psi^{n+1})\|_{0,1}. \quad (2.67)$$

Here ϕ_7 is controllable.

From (2.65) we see that $\|\mathbf{l}\|_{0,m} \leq M$ has a controllable bound. On the other hand, if $\hat{\mathbf{l}} \in W^{0,\infty}([0, T'], H^m(\Omega))$, then it is easy to see from (2.57) and (2.41)-(2.44) that $\mathbf{l} \in W^{1,\infty}([0, T'], H^{m-1}(\Omega))$. In other words, Σ maps $Z(M, T')$ into itself.

And by (2.67), combined with (2.66) and (2.64), it is clear that Σ is a contraction if T' is chosen small enough. So the Cauchy sequence finds a limit $\mathbf{l}_s = (\mathbf{v}, \mathbf{q})^T$ in $Z(M, T')$.

But we can not conclude that \mathbf{l}_s solves the equation (2.20) unless we can show $\beta = \nabla \cdot \mathbf{q}$. Certainly we can not claim $\beta^n = \nabla \cdot \mathbf{q}^n$ from (2.31)-(2.32). However let β denote the solution of (2.31)-(2.32) when $\mathbf{v}^n = \mathbf{v}$ and consider (2.39) in the limit. We have $\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{q} - (\mathbf{q} \cdot \nabla) \mathbf{v} = 0$ from (2.39). Taking the divergence we get

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{q}) + (\mathbf{v} \cdot \nabla)(\nabla \cdot \mathbf{q}) = 0, \quad (2.68)$$

$$(\nabla \cdot \mathbf{q})(x, 0) = \nabla \cdot \mathbf{q}_0. \quad (2.69)$$

which means β and $\nabla \cdot \mathbf{q}$ satisfy the same equation with the same initial value, hence they are equal.

To complete the proof of existence and uniqueness, we only need to show that the solution \mathbf{l}_s , which has been proved to belong to $W^{0,\infty}([0, T'], H^m)$, actually belongs to $C([0, T']; H^m(\Omega))$. For then $\mathbf{l}_s \in \bigcap_{k=0}^m C^k([0, T']; H^{m-k}(\Omega))$ is a direct consequence of the hyperbolic equation (2.20). It is easy to check the solution is weakly continuous in t . Then showing the continuity is a standard procedure as in [43]. All those proofs, together with the proof of data dependence, are presented in the next section.

2.3 Proofs of lemmas

Proof of Lemma 1. We have

$$\frac{\partial}{\partial t}\beta(x, t) + (\mathbf{v} \cdot \nabla)\beta(x, t) = 0. \quad (2.70)$$

Since $\mathbf{v} \cdot \mathbf{n} = 0$, $\nabla \cdot \mathbf{v} = 0$, we have $\frac{\partial}{\partial t} \int_{\Omega} \beta^2(x, t) dx = 0$ which implies $\|\beta(\cdot, t)\|_{H^0} = \|\nabla \cdot \mathbf{q}_0\|_{H^0}$ for $t \in (0, T']$, *a.e.* Following this line and applying the Hölder inequality together with embedding theorem, we can show (2.45)

$$\sup_{0 \leq t \leq T'} \|\beta\|_{H^m} \leq e^{C_M T'} \|\nabla \cdot \mathbf{q}_0\|_{H^m}.$$

Moreover since $m \geq 3$, we see $\|\nabla \beta\|_{\infty} = \sup_{[0, T'] \times \Omega} |\nabla \beta| \leq \phi(M, T')$ is controllable.

Suppose there is another $\tilde{\beta}(x, t)$ satisfying

$$\frac{\partial}{\partial t}\tilde{\beta}(x, t) + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\beta}(x, t) = 0, \tilde{\beta}(x, 0) = \nabla \cdot \mathbf{q}_0.$$

We then have

$$\frac{\partial}{\partial t}(\beta - \tilde{\beta}) + ((\mathbf{v} - \tilde{\mathbf{v}}) \cdot \nabla)\beta + (\tilde{\mathbf{v}} \cdot \nabla)(\beta - \tilde{\beta}) = 0. \quad (2.71)$$

Thus

$$\frac{1}{2} \frac{\partial}{\partial t} \|(\beta - \tilde{\beta})\|_{H^0}^2 \leq \|\mathbf{v} - \tilde{\mathbf{v}}\|_{H^0} \|\nabla \beta\|_{\infty} \|\beta - \tilde{\beta}\|_{H^0}, \quad (2.72)$$

or

$$\frac{\partial}{\partial t} \|\beta - \tilde{\beta}\|_{H^0} \leq C_M \|\mathbf{v} - \tilde{\mathbf{v}}\|_{H^0}. \quad (2.73)$$

Hence on $[0, T']$

$$\|\beta - \tilde{\beta}\|_{0,0} \leq C_M \int_0^{T'} \|\mathbf{v}(s) - \tilde{\mathbf{v}}(s)\|_{H^0} ds \leq \phi_1(M, T') \|\mathbf{v} - \tilde{\mathbf{v}}\|_{0,0}. \quad (2.74)$$

Now differentiating (2.71) w.r.t x_i , we have

$$\frac{\partial}{\partial t} \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} + \frac{\partial(\mathbf{v} - \tilde{\mathbf{v}})}{\partial x_i} \cdot \nabla \beta + (\mathbf{v} - \tilde{\mathbf{v}}) \cdot \nabla \frac{\partial \beta}{\partial x_i} + \frac{\partial \tilde{\mathbf{v}}}{\partial x_i} \cdot \nabla(\beta - \tilde{\beta}) + \tilde{\mathbf{v}} \cdot \nabla \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} = 0. \quad (2.75)$$

Multiplying by $\frac{\partial(\beta - \tilde{\beta})}{\partial x_i}$, and integrating, we find

$$\begin{aligned} & \frac{\partial}{\partial t} \frac{1}{2} \int \left[\frac{\partial(\beta - \tilde{\beta})}{\partial x_i} \right]^2 dx + \int \frac{\partial(\mathbf{v} - \tilde{\mathbf{v}})}{\partial x_i} \cdot \nabla \beta \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} dx + \int (\mathbf{v} - \tilde{\mathbf{v}}) \cdot \nabla \frac{\partial \beta}{\partial x_i} \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} dx \\ & + \int \frac{\partial \tilde{\mathbf{v}}}{\partial x_i} \cdot \nabla(\beta - \tilde{\beta}) \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} dx + \int (\tilde{\mathbf{v}} \cdot \nabla \frac{\partial(\beta - \tilde{\beta})}{\partial x_i}) \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} dx = 0. \end{aligned} \quad (2.76)$$

The last term vanishes. The second term gives

$$\left| \int \frac{\partial(\mathbf{v} - \tilde{\mathbf{v}})}{\partial x_i} \cdot \nabla \beta \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} dx \right| \leq \left\| \frac{\partial(\mathbf{v} - \tilde{\mathbf{v}})}{\partial x_i} \right\|_{L^2} \|\nabla \beta\|_{\infty} \left\| \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} \right\|_{L^2}.$$

The third term gives

$$\begin{aligned} \left| \int (\mathbf{v} - \tilde{\mathbf{v}}) \cdot \nabla \frac{\partial \beta}{\partial x_i} \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} dx \right| &\leq \|(\mathbf{v} - \tilde{\mathbf{v}})\|_{L^6} \|\nabla \nabla \beta\|_{L^3} \left\| \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} \right\|_{L^2} \\ &\leq C \|(\mathbf{v} - \tilde{\mathbf{v}})\|_{H^1} \|\nabla \nabla \beta\|_{H^1} \left\| \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} \right\|_{L^2}. \end{aligned}$$

In this inequality, we use the fact that H^1 is continuously embedded in L^6 and $\|\nabla \nabla \beta\|_{L^3} \leq C \|\nabla \nabla \beta\|_{L^6}$ for we consider the problem on a bounded domain with C^{m+2} boundary.

The fourth term gives

$$\left| \int \frac{\partial \tilde{\mathbf{v}}}{\partial x_i} \cdot \nabla(\beta - \tilde{\beta}) \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} dx \right| \leq \left\| \frac{\partial \tilde{\mathbf{v}}}{\partial x_i} \right\|_{\infty} \|\nabla(\beta - \tilde{\beta})\|_{L^2} \left\| \frac{\partial(\beta - \tilde{\beta})}{\partial x_i} \right\|_{L^2}$$

Combining all these above, we have $\frac{\partial}{\partial t} \|\nabla(\beta - \tilde{\beta})\|_{L^2} \leq C_M (\|\mathbf{v} - \tilde{\mathbf{v}}\|_{H^1} + \|\nabla(\beta - \tilde{\beta})\|_{L^2})$, hence on $[0, T']$ we have, with $\phi_2 = C_M T' e^{C_M T'}$,

$$\|\nabla(\beta - \tilde{\beta})\|_{0,0} \leq \phi_2(M, T') \|\mathbf{v} - \tilde{\mathbf{v}}\|_{0,1}. \quad (2.77)$$

□

Proof of Lemma 2. The proof is trivial and omitted. □

Proof of Lemma 3. Denote by $\bar{\mathbf{f}} = (\mathbf{f} + (\mathbf{q} \cdot \nabla) \hat{\mathbf{q}} + \beta \mathbf{q} - (\mathbf{v} \cdot \nabla) \mathbf{v})$. From (2.36) and (2.37) we have

$$\Delta p = \nabla \cdot \bar{\mathbf{f}}, x \in \Omega; \quad \frac{\partial p}{\partial n} = \bar{\mathbf{f}} \cdot \mathbf{n}, x \in \partial\Omega. \quad (2.78)$$

To make the solution unique, we require $\int_{\Omega} p(x, t) dx = 0$. Then for fixed $t \in [0, T']$ a.e. and for any positive integer m , we have

$$\|\nabla p\|_{H^m(\Omega)} \leq C_{\Omega} (\|\nabla \cdot \bar{\mathbf{f}}\|_{H^{m-1}(\Omega)} + \|\bar{\mathbf{f}} \cdot \mathbf{n}\|_{H^{m-\frac{1}{2}}(\partial\Omega)}). \quad (2.79)$$

We now note the following:

1. From Lemma 1 and Assumption (S2): $\nabla \cdot \mathbf{q}_0 \in H^m$ we conclude $\beta^n \in H^m$.

2. If $\mathbf{f}, \mathbf{g} \in H^s$, $s > \frac{N}{2}$, then $\mathbf{f} \cdot \mathbf{g} \in H^s$.

3. To show

$$\|\nabla \cdot \bar{\mathbf{f}}\|_{H^{m-1}(\Omega)} = \|\nabla \cdot (\mathbf{f} + (\mathbf{q} \cdot \nabla)\hat{\mathbf{q}} + \beta\mathbf{q} - (\mathbf{v} \cdot \nabla)\mathbf{v})\|_{H^{m-1}(\Omega)}, \quad (2.80)$$

we use the following facts

$$\begin{aligned} \mathbf{f} \in H^m &\Rightarrow \nabla \cdot \mathbf{f} \in H^{m-1}, \\ \|\nabla \cdot ((\mathbf{q} \cdot \nabla)\hat{\mathbf{q}})\|_{H^{m-1}} &\leq \left\| \frac{\partial \mathbf{q}_i}{\partial x_j} \frac{\partial \hat{\mathbf{q}}_j}{\partial x_i} \right\|_{H^{m-1}} + \|\mathbf{q} \cdot (\nabla \beta)\|_{H^{m-1}} \\ &\leq \|\mathbf{q}\|_{H^m} \|\hat{\mathbf{q}}\|_{H^m} + \|\mathbf{q}\|_{H^{m-1}} \|\nabla \beta\|_{H^{m-1}}, \\ \nabla \cdot (\beta\mathbf{q}) &= \nabla \beta \cdot \mathbf{q} + \beta \nabla \cdot \mathbf{q} \in H^{m-1}, \\ \|\nabla \cdot ((\mathbf{v} \cdot \nabla)\mathbf{v})\|_{H^{m-1}} &= \left\| \frac{\partial \mathbf{v}_i}{\partial x_j} \frac{\partial \mathbf{v}_j}{\partial x_i} \right\|_{H^{m-1}}. \end{aligned} \quad (2.81)$$

4. To show

$$\|\bar{\mathbf{f}} \cdot \mathbf{n}\|_{H^{m-\frac{1}{2}}(\partial\Omega)} = \|(\mathbf{f} + (\mathbf{q} \cdot \nabla)\hat{\mathbf{q}} + \beta\mathbf{q} - (\mathbf{v} \cdot \nabla)\mathbf{v}) \cdot \mathbf{n}\|_{H^{m-\frac{1}{2}}(\partial\Omega)}, \quad (2.82)$$

we note that

$$\begin{aligned} \mathbf{f} \in H^m(\Omega) &\Rightarrow \mathbf{f} \in H^{m-\frac{1}{2}}(\partial\Omega), \\ \beta\mathbf{q} \cdot \mathbf{n} &= 0, \end{aligned} \quad (2.83)$$

Moreover, because $\hat{\mathbf{q}} \cdot \mathbf{n} = 0$ on the boundary, we have

$$\|(\mathbf{q} \cdot \nabla)\hat{\mathbf{q}} \cdot \mathbf{n}\|_{H^{m-\frac{1}{2}}(\partial\Omega)} = \|\mathbf{q}_i \hat{\mathbf{q}}_j \gamma_{ij}\|_{H^{m-\frac{1}{2}}(\partial\Omega)} \leq C_\gamma \|\mathbf{q}\|_m \|\hat{\mathbf{q}}\|_m \quad (2.84)$$

for some $\gamma \in C^m(\partial\Omega)$. Same for \mathbf{v} .

The analysis above implies

$$\|\nabla p\|_{H^m} \leq C_{\gamma, M, \Omega} (\|\mathbf{f}\|_{H^m} + \|\mathbf{k}\|_{H^m}^2 + \|\mathbf{k}\|_{H^m} + 1), \quad (2.85)$$

hence

$$\|\nabla p\|_{W^{0,1}([0, T']; H^m(\Omega))} \leq C(M^2 T' + \|f\|_{W^{0,1}([0, T']; H^m(\Omega))}). \quad (2.86)$$

Let us now consider a second equation for \tilde{p} . We have

$$\|\nabla p - \nabla \tilde{p}\|_{H^1} \leq C(\|\nabla \cdot (\bar{\mathbf{f}} - \tilde{\mathbf{f}})\|_{H^0(\Omega)} + \|(\bar{\mathbf{f}} - \tilde{\mathbf{f}}) \cdot \mathbf{n}\|_{H^{\frac{1}{2}}(\partial\Omega)}). \quad (2.87)$$

Note that

$$\begin{aligned}\nabla \cdot \bar{\mathbf{f}} &= \nabla \cdot (\mathbf{f} + (\mathbf{q} \cdot \nabla) \hat{\mathbf{q}} + \beta \mathbf{q} + (\mathbf{v} \cdot \nabla) \mathbf{v}) \\ &= \nabla \cdot \mathbf{f} + \frac{\partial q_i}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} + \beta (\nabla \cdot \mathbf{q}) + 2(\nabla \beta) \cdot \mathbf{q} + \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i}\end{aligned}\quad (2.88)$$

and

$$\nabla \cdot \tilde{\mathbf{f}} = \nabla \cdot \mathbf{f} + \frac{\partial \tilde{q}_i}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} + \tilde{\beta} (\nabla \cdot \tilde{\mathbf{q}}) + 2(\nabla \tilde{\beta}) \cdot \tilde{\mathbf{q}} + \frac{\partial \tilde{v}_i}{\partial x_j} \frac{\partial \tilde{v}_j}{\partial x_i}\quad (2.89)$$

So we have

$$\begin{aligned}& \|\nabla \cdot \bar{\mathbf{f}} - \nabla \cdot \tilde{\mathbf{f}}\|_{H^0} \\ & \leq C \left\| \frac{\partial q_i}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} + \beta (\nabla \cdot \mathbf{q}) + 2(\nabla \beta) \cdot \mathbf{q} + \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \right. \\ & \quad \left. - \frac{\partial \tilde{q}_i}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} - \tilde{\beta} (\nabla \cdot \tilde{\mathbf{q}}) - 2(\nabla \tilde{\beta}) \cdot \tilde{\mathbf{q}} - \frac{\partial \tilde{v}_i}{\partial x_j} \frac{\partial \tilde{v}_j}{\partial x_i} \right\|_{H^0} \\ & \leq C \left\| \frac{\partial q_i}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} - \frac{\partial \tilde{q}_i}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} \right\|_{H^0} + C \left\| \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} - \frac{\partial \tilde{v}_i}{\partial x_j} \frac{\partial \tilde{v}_j}{\partial x_i} \right\|_{H^0} \\ & \quad + C \|\beta (\nabla \cdot \mathbf{q}) - \tilde{\beta} (\nabla \cdot \tilde{\mathbf{q}})\|_{H^0} + 2C \|(\nabla \beta) \cdot \mathbf{q} - (\nabla \tilde{\beta}) \cdot \tilde{\mathbf{q}}\|_{H^0}.\end{aligned}\quad (2.90)$$

Now we have four terms to check: ***First two terms**

$$\left\| \frac{\partial q_i}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} - \frac{\partial \tilde{q}_i}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} \right\|_{H^0} \text{ and } \left\| \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} - \frac{\partial \tilde{v}_i}{\partial x_j} \frac{\partial \tilde{v}_j}{\partial x_i} \right\|_{H^0}.$$

We have

$$\begin{aligned}& \left\| \frac{\partial q_i}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} - \frac{\partial \tilde{q}_i}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} \right\|_{H^0} = \left\| \frac{\partial(q_i - \tilde{q}_i)}{\partial x_j} \frac{\partial \hat{q}_j}{\partial x_i} + \frac{\partial \tilde{q}_i}{\partial x_j} \frac{\partial(\hat{q}_j - \tilde{q}_j)}{\partial x_i} \right\|_{H^0} \\ & \leq \left\| \frac{\partial \hat{q}_j}{\partial x_i} \right\|_{\infty} \left\| \frac{\partial(q_i - \tilde{q}_i)}{\partial x_j} \right\|_{H^0} + \left\| \frac{\partial \tilde{q}_i}{\partial x_j} \right\|_{\infty} \left\| \frac{\partial(\hat{q}_j - \tilde{q}_j)}{\partial x_i} \right\|_{H^0} \\ & \leq C (\|\mathbf{q} - \tilde{\mathbf{q}}\|_{H^1} + \|\beta - \tilde{\beta}\|_{H^0}) \\ & \leq \tilde{C} \|\mathbf{k} - \tilde{\mathbf{k}}\|_{H^1}.\end{aligned}\quad (2.91)$$

And similarly

$$\left\| \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} - \frac{\partial \tilde{v}_i}{\partial x_j} \frac{\partial \tilde{v}_j}{\partial x_i} \right\|_{H^0} \leq C \|\mathbf{v} - \tilde{\mathbf{v}}\|_{H^1}.\quad (2.92)$$

***Third term**

$$\begin{aligned}\|\beta (\nabla \cdot \mathbf{q}) - \tilde{\beta} (\nabla \cdot \tilde{\mathbf{q}})\|_{H^0} &\leq \|(\beta - \tilde{\beta}) (\nabla \cdot \mathbf{q})\|_{H^0} + \|\tilde{\beta} \nabla \cdot (\mathbf{q} - \tilde{\mathbf{q}})\|_{H^0} \\ &\leq C_M (\|\beta - \tilde{\beta}\|_{H^0} + \|\mathbf{q} - \tilde{\mathbf{q}}\|_{H^1}) \leq \tilde{C}_M \|\mathbf{k} - \tilde{\mathbf{k}}\|_{H^1}\end{aligned}$$

***Last term**

$$\begin{aligned}\|\nabla \beta \cdot \mathbf{q} - \nabla \tilde{\beta} \cdot \tilde{\mathbf{q}}\|_{H^0} &\leq \|\nabla(\beta - \tilde{\beta}) \cdot \mathbf{q}\|_{H^0} + \|\nabla \tilde{\beta} \cdot (\mathbf{q} - \tilde{\mathbf{q}})\|_{H^0} \\ &\leq C_M \|\mathbf{k} - \tilde{\mathbf{k}}\|_{H^1} \text{ since } \beta, \tilde{\beta} \in H^m\end{aligned}$$

The boundary terms are analyzed in a similar fashion; we omit the details. In summary, we have obtained $\|\nabla p - \nabla \tilde{p}\|_{H^1} \leq C_M \|\mathbf{k} - \tilde{\mathbf{k}}\|_{H^1}$. In view of (2.74), (2.77), the estimations above imply $\|\nabla p - \nabla \tilde{p}\|_{0,1} \leq \phi_3(M, T') \|\mathbf{k} - \tilde{\mathbf{k}}\|_{0,1}$ and the proof of Lemma 3 is complete. \square

Proof of Lemma 4. In the iteration there is a hyperbolic equation system (2.38)-(2.40) where an extra term \mathbf{n}_ϵ is involved. Here we explain what role the trick plays. Note that our problem is posed on a bounded domain with boundary condition. However, we need

to consider the problem on all of space so that we can take spatial derivatives without worrying about the boundary. Then we have to make sure our solution inside the domain is independent of the extension, i.e., at each step we expect $\mathbf{v}^{n+1} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{q}^{n+1} \cdot \mathbf{n}|_{\partial\Omega} = 0$ provided $\mathbf{v}^n \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{q}^n \cdot \mathbf{n}|_{\partial\Omega} = 0$. As it stands, this is not satisfied in the original equation.

But for (2.38)-(2.40), we have on $\partial\Omega$:

$$\frac{\partial(\hat{\mathbf{v}}^{n+1} \cdot \mathbf{n})}{\partial t} + (\mathbf{v}^n \cdot \nabla)(\hat{\mathbf{v}}^{n+1} \cdot \mathbf{n}) - (\mathbf{q}^n \cdot \nabla)(\hat{\mathbf{q}}^{n+1} \cdot \mathbf{n}) = 0 \quad (2.93)$$

$$\frac{\partial(\hat{\mathbf{q}}^{n+1} \cdot \mathbf{n})}{\partial t} + (\mathbf{v}^n \cdot \nabla)(\hat{\mathbf{q}}^{n+1} \cdot \mathbf{n}) - (\mathbf{q}^n \cdot \nabla)(\hat{\mathbf{v}}^{n+1} \cdot \mathbf{n}) = 0 \quad (2.94)$$

Obviously, equations (2.93) and (2.94) form a symmetric hyperbolic system for $(\hat{\mathbf{v}}^{n+1} \cdot \mathbf{n}), (\hat{\mathbf{q}}^{n+1} \cdot \mathbf{n})$. Because of the zero initial data, we have $\hat{\mathbf{v}}^{n+1} \cdot \mathbf{n} = \hat{\mathbf{q}}^{n+1} \cdot \mathbf{n} = 0$ for every n and all t . Thus from the construction of $\mathbf{v}^{n+1}, \mathbf{q}^{n+1}$ in Step 7, it is trivial that $\mathbf{v}^{n+1} \cdot \mathbf{n} = 0, \mathbf{q}^{n+1} \cdot \mathbf{n} = 0$.

Thus, we do not worry about the boundary condition in the iteration. Moreover, after the extension to the whole space \mathbb{R}^3 , at each step we have a linear symmetric hyperbolic system whose existence and uniqueness can easily be shown. We now prove the requisite estimates for the solution.

We define $\hat{\mathbf{l}}_\alpha = D^\alpha \hat{\mathbf{l}}$ for $|\alpha| \leq m$ and differentiate (2.57) with respect to x, α -times. We compute the equations for $|\alpha| \leq m$,

$$\frac{\partial \hat{\mathbf{l}}_\alpha}{\partial t} + \sum A_j(\mathbf{k}) \frac{\partial \hat{\mathbf{l}}_\alpha}{\partial x_j} = F_\alpha + D^\alpha \mathbf{g}(\hat{\mathbf{l}}, \mathbf{k}, \nabla \cdot \hat{\mathbf{q}}, \nabla p^n, \mathbf{n}_\epsilon, \mathbf{f}), \quad (2.95)$$

$$\mathbf{l}_\alpha(x, 0) = D^\alpha \mathbf{l}_0(x), \quad (2.96)$$

with F_α defined by commutator terms as

$$F_\alpha = \sum_{j=1}^3 [A_j(\mathbf{k}) \frac{\partial \hat{\mathbf{l}}_\alpha}{\partial x_j} - D^\alpha (A_j(\mathbf{k}) \frac{\partial \hat{\mathbf{l}}}{\partial x_j})]. \quad (2.97)$$

Under the assumption on \mathbf{v}, \mathbf{q} , we have (see page 42, Majda [33])

$$\sum_{|\alpha| \leq m} (\|F_\alpha\|_{H^m} + \|D^\alpha \mathbf{g}\|_{H^m}) \leq C_M (\|\hat{\mathbf{l}}\|_m + \|f\|_{H^m}) \quad (2.98)$$

By applying the energy principle to (2.95), using (2.98) and summing over α with $|\alpha| \leq m$ we have

$$\max_{0 \leq s \leq T'} \|\hat{\mathbf{l}}\|_m \leq e^{CT'} (\|\mathbf{l}_0\|_m + h(T')), \quad (2.99)$$

where $h(T') \rightarrow 0$ as $T' \rightarrow 0$.

So (2.60) is proved. The proof of (2.62)-(2.64) is a routine procedure as in [33]. We simply take the difference of equations (2.57) and (2.61) and apply the energy principle and Taylor theorem together with the estimation (2.52). \square

Proof of Lemma 5 From (2.41)-(2.42), we have $\|\nabla\psi\|_{0,m} \leq C_\Omega \|\hat{\mathbf{v}}\|_{0,m} \leq C_\Omega \|\hat{\mathbf{l}}\|_{0,m}$, hence by (2.43)

$$\|\mathbf{l}\|_{0,m} \leq \|\hat{\mathbf{l}}\|_{0,m} + \|\nabla\psi\|_{0,m} \leq \phi_7(M, T') (\|\mathbf{l}_0\|_m + T').$$

And (2.66)-(2.67) are trivial. \square

2.4 Proof of the data dependence

We shall now prove the second part of Theorem 1, namely the continuous dependence of the solution on the initial data. More specifically, consider the equation (2.20) on the whole space, and put it in form of

$$\begin{aligned} \frac{\partial \mathbf{l}}{\partial t} + \sum_{j=1}^3 A_j(\mathbf{l}) \frac{\partial \mathbf{l}}{\partial x_j} &= \mathbf{g}(\mathbf{l}, \nabla \cdot \mathbf{q}, \nabla p, \mathbf{f}), \\ \mathbf{l}(x, 0) &= (\mathbf{v}_0, \mathbf{q}_0)^T, \end{aligned} \quad (2.100)$$

where $\mathbf{f} \in L^1([0, T], H^m)$, $\mathbf{l}_0 = (\mathbf{v}_0, \mathbf{q}_0) \in H^m$, $m > 1 + \frac{N}{2}$. By the first part of Theorem 1, there exists a solution $\mathbf{l} = (\mathbf{v}, \mathbf{q}) \in C([0, T'], H^m)$ for some $0 < T' < T$. Now suppose we have a sequence of data $\mathbf{l}_0^n, \mathbf{f}^n$ which satisfy the requirements for \mathbf{l}_0, \mathbf{f} . Correspondingly, we have a sequence of solutions \mathbf{l}^n in $C([0, T^n], H^m)$ which satisfy

$$\begin{aligned} \frac{\partial \mathbf{l}^n}{\partial t} + \sum_{j=1}^3 A_j(\mathbf{l}^n) \frac{\partial \mathbf{l}^n}{\partial x_j} &= \mathbf{g}(\mathbf{l}^n, \nabla \cdot \mathbf{q}^n, \nabla p^n, \mathbf{f}^n), \\ \mathbf{l}^n(x, 0) &= (\mathbf{v}_0^n, \mathbf{q}_0^n)^T. \end{aligned} \quad (2.101)$$

We shall prove that, despite the different intervals of time existence for different n , there exists a common interval $[0, T^*]$, $T^* \leq \min\{T, T'\}$ on which \mathbf{l}^n exist and $\mathbf{l}^n \rightarrow \mathbf{l}$ in $C([0, T^*], H^m)$ provided

1. $\lim_{n \rightarrow \infty} (\mathbf{v}_0^n, \mathbf{q}_0^n) \rightarrow (\mathbf{v}_0, \mathbf{q}_0)$ in H^m
2. $\lim_{n \rightarrow \infty} \nabla \cdot \mathbf{q}_0^n \rightarrow \nabla \cdot \mathbf{q}_0$ in H^m ,
3. $\lim_{n \rightarrow \infty} \mathbf{f}^n \rightarrow \mathbf{f}$ in $L^1([0, T], H^m)$,

First of all, we note that in the proof of existence, the time interval only depends upon $M, \|\mathbf{l}_0\|_m$ and $\|\mathbf{f}\|_{L^1([0, T], H^m)}$. Since $\lim_{n \rightarrow \infty} \mathbf{l}_0^n \rightarrow \mathbf{l}_0$ in H^m , and $\lim_{n \rightarrow \infty} \mathbf{f}^n \rightarrow \mathbf{f}$ in $L^1([0, T], H^m)$ we

can find a common time interval $[0, T^*]$ on which all the solutions $(\mathbf{v}^n, \mathbf{q}^n) \in C([0, T^*], H^m)$ exist. Now we show the continuous dependence of the solution on the data in $C([0, T^*], H^m)$.

The natural step now would be to take difference of (2.100) and (2.101), and differentiate the difference α times with $|\alpha| \leq m$ and then apply the energy estimate. But it turns out to be insufficient because if we do so we would need to deal with terms like $[A_j(\mathbf{1}) - A_j(\mathbf{1}^\varepsilon)] \frac{\partial^{m+1} \mathbf{1}^n}{\partial x_j^{m+1}}$ over which we have no control. To get around this, we follow a method introduced by Bona and Smith [2]; our application of the method is similar to the application to the Euler equations in [25]. We introduce extra equations

$$\begin{aligned} \frac{\partial \mathbf{1}^\varepsilon}{\partial t} + \sum_{j=1}^3 A_j(\mathbf{1}^\varepsilon) \frac{\partial \mathbf{1}^\varepsilon}{\partial x_j} &= \mathbf{g}^\varepsilon(\mathbf{1}^\varepsilon, \nabla \cdot \mathbf{q}^\varepsilon, \nabla p^\varepsilon, \mathbf{f} * \phi_\varepsilon), \\ \mathbf{1}^\varepsilon(x, 0) &= \mathbf{1}_0^\varepsilon \triangleq \mathbf{1}_0 * \phi_\varepsilon, \end{aligned} \quad (2.102)$$

where $\phi_\varepsilon(x) = \frac{1}{\varepsilon^N} \phi(\frac{x}{\varepsilon})$ is a suitable mollifier.

Taking the difference of (2.100) and (2.102) we have

$$\begin{aligned} \frac{\partial(\mathbf{1}^\varepsilon - \mathbf{1})}{\partial t} + \sum_{j=1}^3 A_j(\mathbf{1}) \frac{\partial(\mathbf{1}^\varepsilon - \mathbf{1})}{\partial x_j} &= \sum_{j=1}^3 [A_j(\mathbf{1}) - A_j(\mathbf{1}^\varepsilon)] \frac{\partial \mathbf{1}^\varepsilon}{\partial x_j} + \mathbf{g}^\varepsilon - \mathbf{g}, \\ (\mathbf{1}^\varepsilon - \mathbf{1})(x, 0) &= (\mathbf{1}_0^\varepsilon - \mathbf{1}_0). \end{aligned} \quad (2.103)$$

Differentiation of (2.103) α -times with $|\alpha| \leq m$ yields

$$\begin{aligned} \frac{\partial}{\partial t} D^\alpha(\mathbf{1}^\varepsilon - \mathbf{1}) + \sum_{j=1}^3 A_j(\mathbf{1}) \frac{\partial}{\partial x_j} D^\alpha(\mathbf{1}^\varepsilon - \mathbf{1}) &= \sum_{j=1}^3 \left([D^\alpha, A_j(\mathbf{1})] \frac{\partial}{\partial x_j} (\mathbf{1}^\varepsilon - \mathbf{1}) \right) \\ &+ \sum_{j=1}^3 D^\alpha \left([A_j(\mathbf{1}) - A_j(\mathbf{1}^\varepsilon)] \frac{\partial \mathbf{1}^\varepsilon}{\partial x_j} \right) + D^\alpha(\mathbf{g}^\varepsilon - \mathbf{g}), \\ D^\alpha(\mathbf{1}^\varepsilon - \mathbf{1})(x, 0) &= D^\alpha(\mathbf{1}_0^\varepsilon - \mathbf{1}_0). \end{aligned} \quad (2.104)$$

Here we use the commutator notation $([D^\alpha, f]g) = D^\alpha(fg) - fD^\alpha g$.

We multiply by $D^\alpha(\mathbf{1}^\varepsilon - \mathbf{1})$, integrate with respect to the spatial variable and add up the resulting equations for all $|\alpha| \leq m$. This leads to

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right)_+ \|\mathbf{1}^\varepsilon - \mathbf{1}\|_m &\leq C_1 \|\mathbf{1}^\varepsilon - \mathbf{1}\|_m + C_2 \|\mathbf{1}^\varepsilon - \mathbf{1}\|_\infty \|\mathbf{1}^\varepsilon\|_{m+1} + \|\mathbf{g}^\varepsilon - \mathbf{g}\|_m \\ &\leq C_1 \|\mathbf{1}^\varepsilon - \mathbf{1}\|_m + C_2 \|\mathbf{1}^\varepsilon - \mathbf{1}\|_{m-1} \|\mathbf{1}^\varepsilon\|_{m+1} \\ &\quad + \|\mathbf{g}^\varepsilon - \mathbf{g}\|_m. \end{aligned} \quad (2.105)$$

Similarly for (2.101) we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right)_+ \|\mathbf{1}^{n,\varepsilon} - \mathbf{1}^n\|_m &\leq C_3 \|\mathbf{1}^{n,\varepsilon} - \mathbf{1}^n\|_m \\ &\quad + C_4 \|\mathbf{1}^{n,\varepsilon} - \mathbf{1}^n\|_{m-1} \|\mathbf{1}^{n,\varepsilon}\|_{m+1} + \|\mathbf{g}^{n,\varepsilon} - \mathbf{g}^n\|_m. \end{aligned} \quad (2.106)$$

In [2], it is shown how to construct ϕ_ε in such a manner that, as $\varepsilon \rightarrow 0$, we have

$$\|\phi_\varepsilon * \mathbf{k}\|_{m+1} \leq \frac{C}{\varepsilon} \|\mathbf{k}\|_m, \quad (2.107)$$

and moreover,

$$\begin{aligned} \|\phi_\varepsilon * \mathbf{k} - \mathbf{k}\|_m &\rightarrow 0, \\ \frac{1}{\varepsilon} \|\phi_\varepsilon * \mathbf{k} - \mathbf{k}\|_{m-1} &\rightarrow 0 \end{aligned} \quad (2.108)$$

uniformly for \mathbf{k} in compact subsets of H^m .

It follows from this that

$$\begin{aligned} \|\mathbf{I}_0^\varepsilon\|_{m+1} &= O\left(\frac{1}{\varepsilon}\right), \\ \|\mathbf{I}_0^{n,\varepsilon}\|_{m+1} &= O\left(\frac{1}{\varepsilon}\right), \\ \|\mathbf{I}_0^\varepsilon - \mathbf{I}_0\|_m &= o(1), \\ \|\mathbf{I}_0^{n,\varepsilon} - \mathbf{I}_0^n\|_m &= o(1), \\ \|\mathbf{f}^\varepsilon - \mathbf{f}\|_{L^1([0,T],H^m)} &= o(1), \\ \|\mathbf{f}^\varepsilon - \mathbf{f}\|_{L^1([0,T],H^{m-1})} &= o(\varepsilon). \end{aligned}$$

Repeating the estimates of the existence proof, with m replaced by $m + 1$, we find that

$$\begin{aligned} \|\mathbf{I}^\varepsilon\|_{m+1} &= O\left(\frac{1}{\varepsilon}\right), \\ \|\mathbf{I}^{n,\varepsilon}\|_{m+1} &= O\left(\frac{1}{\varepsilon}\right). \end{aligned} \quad (2.109)$$

Also, using estimates analogous to those used in the existence proof, we find that

$$\begin{aligned} \|\mathbf{I}^\varepsilon - \mathbf{I}\|_{m-1} &= o(\varepsilon), \\ \|\mathbf{I}^{n,\varepsilon} - \mathbf{I}^n\|_{m-1} &= o(\varepsilon), \\ \|\mathbf{g}^\varepsilon - \mathbf{g}\|_m &\leq C(\|\mathbf{I}^\varepsilon - \mathbf{I}\|_m + \|\mathbf{f}^\varepsilon - \mathbf{f}\|_m + \|\nabla \cdot \mathbf{q}^\varepsilon - \nabla \cdot \mathbf{q}\|_m), \\ \|\mathbf{g}^{n,\varepsilon} - \mathbf{g}^n\|_m &\leq C(\|\mathbf{I}^{n,\varepsilon} - \mathbf{I}^n\|_m + \|\mathbf{f}^\varepsilon - \mathbf{f}\|_m \\ &\quad + \|\nabla \cdot \mathbf{q}^{n,\varepsilon} - \nabla \cdot \mathbf{q}^n\|_m). \end{aligned} \quad (2.110)$$

By combining these estimates with (2.105) and (2.106), we deduce that

$$\|\mathbf{I}^\varepsilon - \mathbf{I}\|_m = o(1), \quad \|\mathbf{I}^{n,\varepsilon} - \mathbf{I}^n\|_m = o(1), \quad (2.111)$$

as $\varepsilon \rightarrow 0$, uniformly in n . We can now complete the proof by applying the triangle inequality

$$\begin{aligned} \|\mathbf{I}^n - \mathbf{I}\|_m &\leq \|\mathbf{I}^n - \mathbf{I}^{n,\varepsilon}\|_m + \|\mathbf{I}^{n,\varepsilon} - \mathbf{I}^\varepsilon\|_m + \|\mathbf{I}^\varepsilon - \mathbf{I}\|_m \\ &\leq o(1) + C\|\mathbf{I}_0^n - \mathbf{I}_0\|_m + o(1), \varepsilon \rightarrow 0. \end{aligned} \quad (2.112)$$

□

2.5 On Corollary 1

For the sake of clarity, we collect the equations to have

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} - (\mathbf{C}_i \cdot \nabla) \mathbf{C}_i = (\nabla \cdot \mathbf{C}_i) \mathbf{C}_i - \nabla p + \mathbf{f}, \\ \frac{\partial \mathbf{C}_i}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{C}_i - (\mathbf{C}_i \cdot \nabla) \mathbf{v} = 0, i = 1, 2, 3, \\ \nabla \cdot \mathbf{v} = 0, \\ \mathbf{v}(x, 0) = \mathbf{v}_0(x); \mathbf{C}(x, 0) = \mathbf{C}_0(x), \\ \mathbf{v} \cdot \mathbf{n} = 0; \mathbf{C}_i \cdot \mathbf{n} = 0, x \in \partial\Omega. \end{cases} \quad (2.113)$$

In (2.14), taking the divergence on both sides we have the following equation for p :

$$\Delta p = \nabla \cdot (\mathbf{f} + (\mathbf{C}_i \cdot \nabla) \mathbf{C}_i + (\nabla \cdot \mathbf{C}_i) \mathbf{C}_i - (\mathbf{v} \cdot \nabla) \mathbf{v}). \quad (2.114)$$

We impose the boundary condition

$$\frac{\partial p}{\partial n} = \{(\mathbf{f} + (\mathbf{C}_i \cdot \nabla) \mathbf{C}_i + (\nabla \cdot \mathbf{C}_i) \mathbf{C}_i - (\mathbf{v} \cdot \nabla) \mathbf{v})\} \cdot \mathbf{n}, x \in \partial\Omega, \quad (2.115)$$

so that this Neumann problem is compatible.

Now a simple observation tells us that (2.113) is still a symmetric hyperbolic system if we view p and the other right hand sides as known quantities. The second equation in (2.113) enables the same estimate as that on β in (2.70). The only difference between (2.20) and (2.113) is that the number of unknowns becomes 12 instead of 6. And (2.114)-(2.115) is still a compatible Neumann problem. So the original proof under a slight modifications on metric space and iteration scheme will work for the new problem. \square

Concluding remarks: It is of interest not just to study the limiting equations for infinite Weissenberg number, but also the manner in which this limit is approached. The term proportional to \mathbf{Wi}^{-1} in (2.4) does not present a problem. The main issue is in the dependence of the initial data on \mathbf{Wi} . As pointed out above, in the infinite Weissenberg number limit, it is appropriate to consider cases where \mathbf{S} is degenerate. On the other hand, for any finite Weissenberg number S must be strictly positive definite. For instance, in a parallel shear flow with shear rate 1, we would have

$$\mathbf{S} = \begin{pmatrix} 2 + \mathbf{Wi}^{-2} & \mathbf{Wi}^{-1} & 0 \\ \mathbf{Wi}^{-1} & \mathbf{Wi}^{-2} & 0 \\ 0 & 0 & \mathbf{Wi}^{-2} \end{pmatrix}. \quad (2.116)$$

Hence in a meaningful physical situation, the initial condition would have to be allowed to depend on \mathbf{Wi} . Above we have derived a result on continuous dependence on the initial data. However, even there we have assumed that $\mathbf{S} \cdot \mathbf{n}$ vanishes on the boundary. If this condition were relaxed, the boundary conditions required for a well posed problem would change, making the analysis much more complicated.

Chapter 3

Boundary Layers in complex fluids

In this chapter we derive a system for the boundary layer of UCM flow in the high Weissenberg limit and prove the well-posedness of this system [56, 57]. A transformation to Lagrangian coordinates is crucial in the argument. A detailed scaling analysis is given. We also show the same boundary layer system can be derived for a curved boundary.

3.1 Introduction

Classical fluid mechanics is based on the Navier-Stokes equations supplemented by a no-slip boundary condition on walls. In the limit of zero viscosity, the Euler equations are obtained. However, the Euler equations do not allow for a no-slip boundary condition, and only the normal component of the velocity can be prescribed to be zero on a wall. It was Prandtl's fundamental insight more than a century ago [40] that for many high Reynolds number flows the Euler equations provide an adequate description except in a thin layer close to the boundary, which is called a boundary layer. By taking advantage of the thinness of this layer, the Navier-Stokes equations can formally be reduced to the system which is now known as the Prandtl equations. For two-dimensional flow, and a boundary placed at $y = 0$,

these equations take the form

$$\begin{aligned}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \nu u_{yy} - \frac{\partial p}{\partial x}, \\
\frac{\partial p}{\partial y} &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
u(x, y, 0) &= u_0(x, y), \\
u(x, 0, t) &= 0, \\
v(x, 0, t) &= 0, \\
u(x, \infty, t) &= u^\infty(x, 0, t).
\end{aligned} \tag{3.1}$$

Here u, v are velocities in x, y directions, p is pressure, ν is viscosity, and $u^\infty(x, y, t)$ represents the given flow in the core region.

One might hope to obtain a simplified procedure for solving high Reynolds number flow problems by solving the Euler equations (or even the simpler special case of potential flow) in the core of the flow domain, and then solving the Prandtl equations near the boundary. This program, however, runs into difficulties related to the question of well-posedness of the Prandtl equations. Oleinik [37] established a well-posedness result under the assumption that the velocity profile in the boundary layer is monotone. Sammartino and Caffisch [58] established an existence result for analytic initial data. We also refer to the review article of Weinan E [9] for further work prior to 2000. Recently, Gérard-Varet and Dormy [15] established that, for general initial data, the Prandtl equations are not well-posed in Sobolev spaces.

Viscoelastic flows exhibit phenomena of instability which share many characteristics of turbulence [17]. In this chapter, we shall focus on the upper convected Maxwell model for the viscoelastic flow. In the limit of high elasticity, a limiting equation can be derived which is similar to the system of ideal magnetohydrodynamics and, like the Euler equations, does not allow the imposition of a no-slip boundary condition. The well-posedness of this system has been established in Chapter 2, also see [67]. The goal of this manuscript is to supplement this analysis with a study of the well-posedness of the accompanying boundary layer equations.

We note that the ill-posedness of the Prandtl equations [15] is linked to shear flow instabilities. Elasticity has a stabilizing effect on high Reynolds number flow instabilities [21, 35, 34, 24, 52]. This stabilizing effect can restore well-posedness of the hydrostatic approximation in situations where this approximation is ill-posed for the Euler equations [55]. We may therefore hope to establish well-posedness in the boundary layer system as well. Indeed, this turns out to be the case. We shall show that a transformation into Lagrangian coordinates transforms the boundary layer system into a semilinear wave equation for which well-posedness can be readily established.

3.2 Formulation of boundary layers

3.2.1 Scaling analysis

We start with the upper convected Maxwell model in dimensionless form:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= \frac{\mathbf{Wi}}{\mathbf{Re}} \nabla \cdot \mathbf{T} - \nabla p, \\ \nabla \cdot \mathbf{v} &= 0, \\ \frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{T} - (\nabla \mathbf{v}) \mathbf{T} - \mathbf{T} (\nabla \mathbf{v})^T + \frac{1}{\mathbf{Wi}} \mathbf{T} &= \frac{1}{\mathbf{Wi}^2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T). \end{aligned} \quad (3.2)$$

In [67], we assume $E = \mathbf{Wi}/\mathbf{Re}$ fixed and formally set $\mathbf{Wi} = \infty$ above, we obtain the limiting system

$$\begin{aligned} \frac{\partial \mathbf{v}^0}{\partial t} + (\mathbf{v}^0 \cdot \nabla) \mathbf{v}^0 &= E \nabla \cdot \mathbf{T}^0 - \nabla p^0, \\ \frac{\partial \mathbf{T}^0}{\partial t} + (\mathbf{v}^0 \cdot \nabla) \mathbf{T}^0 - (\nabla \mathbf{v}^0) \mathbf{T}^0 - \mathbf{T}^0 (\nabla \mathbf{v}^0)^T &= \mathbf{0}, \\ \nabla \cdot \mathbf{v}^0 &= 0. \end{aligned} \quad (3.3)$$

We considered the initial-boundary value problem in a smooth domain Ω , subject to initial conditions for \mathbf{v}^0 and \mathbf{T}^0 , and the boundary condition $\mathbf{v}^0 \cdot \mathbf{n} = 0$. In the proof of the well-posedness of this system, a crucial assumption was that $\mathbf{T}^0 \cdot \mathbf{n} = \mathbf{0}$; it can be shown that the equations preserve this condition if it is satisfied initially. The physical background behind this assumption is that the local flow near a solid wall is always a shear flow, and at high Weissenberg number the extra stress is dominated by the first normal stress, i.e. the stress component tangent to the wall.

For the full equations, however, we have the boundary condition $\mathbf{v} = \mathbf{0}$, not just $\mathbf{v} \cdot \mathbf{n} = 0$. To accommodate this, boundary layers must form near the wall. For discussing these boundary layers, it is convenient to set $\mathbf{S} = \mathbf{T} + \mathbf{I}/\mathbf{Wi}^2$. With this substitution, the constitutive law transforms to

$$\frac{\partial \mathbf{S}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{S} - (\nabla \mathbf{v}) \mathbf{S} - \mathbf{S} (\nabla \mathbf{v})^T + \frac{1}{\mathbf{Wi}} (\mathbf{S} - \frac{1}{\mathbf{Wi}^2} \mathbf{I}) = \mathbf{0},$$

and we have $\text{div } \mathbf{T} = \text{div } \mathbf{S}$ in the momentum equation. The change in boundary conditions is related to the fact that while $\mathbf{S} \cdot \mathbf{n}$ vanishes on the boundary at leading order, for the full equations we have strict positive definiteness of \mathbf{S} .

Here we shall consider the general elasticity constant $E = \mathbf{Wi}/\mathbf{Re}$ with large \mathbf{Wi} . We establish the boundary layer system and show that the boundary layer thickness is $1/\sqrt{\mathbf{WiRe}}$.

Assuming a flat boundary given by $y = 0$, original system (3.2) in component form becomes:

$$\begin{aligned}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{\mathbf{Wi}}{\mathbf{Re}} \left(\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} \right) - \frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= \frac{\mathbf{Wi}}{\mathbf{Re}} \left(\frac{\partial S_{12}}{\partial x} + \frac{\partial S_{22}}{\partial y} \right) - \frac{\partial p}{\partial y}, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial S_{11}}{\partial t} + u \frac{\partial S_{11}}{\partial x} + v \frac{\partial S_{11}}{\partial y} - \frac{\partial u}{\partial x} S_{11} - \frac{\partial u}{\partial y} S_{12} - S_{11} \frac{\partial u}{\partial x} - S_{12} \frac{\partial u}{\partial y} &= -\frac{1}{\mathbf{Wi}} \left(S_{11} - \frac{1}{\mathbf{Wi}^2} \right), \\
\frac{\partial S_{12}}{\partial t} + u \frac{\partial S_{12}}{\partial x} + v \frac{\partial S_{12}}{\partial y} - \frac{\partial u}{\partial x} S_{12} - \frac{\partial u}{\partial y} S_{22} - S_{11} \frac{\partial v}{\partial x} - S_{12} \frac{\partial v}{\partial y} &= -\frac{1}{\mathbf{Wi}} S_{12}, \\
\frac{\partial S_{22}}{\partial t} + u \frac{\partial S_{22}}{\partial x} + v \frac{\partial S_{22}}{\partial y} - \frac{\partial v}{\partial x} S_{12} - \frac{\partial v}{\partial y} S_{22} - S_{12} \frac{\partial v}{\partial x} - S_{22} \frac{\partial v}{\partial y} &= -\frac{1}{\mathbf{Wi}} \left(S_{22} - \frac{1}{\mathbf{Wi}^2} \right). \quad (3.4)
\end{aligned}$$

Consider S_{22} in (3.4). At a flat wall we know $u = v = 0$, $u_x = v_x = 0$ and $v_y = -u_x = 0$. That suggests $S_{22} = O(\mathbf{Wi}^{-2})$. We make the scaling assumption in view of incompressibility

$$\begin{aligned}
t' = t, x' = x, y' = \lambda y, u' = u, v' = \lambda v, p' = p, \\
S'_{11} = \gamma S_{11}, S'_{12} = \kappa S_{12}, S'_{22} = \mathbf{Wi}^2 S_{22} \quad (3.5)
\end{aligned}$$

with λ, γ, κ to be determined. Here $1/\lambda$ represents boundary layer thickness.

Substituting into (3.4) and omit the primes, the moment equations become

$$\begin{aligned}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{\mathbf{Wi}}{\gamma \mathbf{Re}} \left(\frac{\partial S_{11}}{\partial x} + \frac{\lambda \gamma}{\kappa} \frac{\partial S_{12}}{\partial y} \right) - \frac{\partial p}{\partial x}, \\
\frac{1}{\lambda^2} \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) &= \frac{\mathbf{Wi}}{\kappa \lambda \mathbf{Re}} \frac{\partial S_{12}}{\partial x} + \frac{1}{\mathbf{Wi} \mathbf{Re}} \frac{\partial S_{22}}{\partial y} - \frac{\partial p}{\partial y} = 0. \quad (3.6)
\end{aligned}$$

The incompressibility condition stays the same. The constitutive equations in componentwise form are

$$\begin{aligned}
\frac{\partial S_{11}}{\partial t} + u \frac{\partial S_{11}}{\partial x} + v \frac{\partial S_{11}}{\partial y} - \frac{\partial u}{\partial x} S_{11} - \frac{\lambda \gamma}{\kappa} \frac{\partial u}{\partial y} S_{12} - S_{11} \frac{\partial u}{\partial x} - \frac{\lambda \gamma}{\kappa} S_{12} \frac{\partial u}{\partial y} &= -\frac{1}{\mathbf{Wi}} S_{11} + \frac{\gamma}{\mathbf{Wi}^3}, \\
\frac{\partial S_{12}}{\partial t} + u \frac{\partial S_{12}}{\partial x} + v \frac{\partial S_{12}}{\partial y} - \frac{\partial u}{\partial x} S_{12} - \frac{\lambda \kappa}{\mathbf{Wi}^2} \frac{\partial u}{\partial y} S_{22} - \frac{\kappa}{\lambda \gamma} S_{11} \frac{\partial v}{\partial x} - S_{12} \frac{\partial v}{\partial y} &= -\frac{1}{\mathbf{Wi}} S_{12}, \\
\frac{\partial S_{22}}{\partial t} + u \frac{\partial S_{22}}{\partial x} + v \frac{\partial S_{22}}{\partial y} - \frac{\mathbf{Wi}^2}{\kappa \lambda} \frac{\partial v}{\partial x} S_{12} - \frac{\partial v}{\partial y} S_{22} - \frac{\mathbf{Wi}^2}{\kappa \lambda} S_{12} \frac{\partial v}{\partial x} - S_{22} \frac{\partial v}{\partial y} &= -\frac{1}{\mathbf{Wi}} S_{22} + \frac{1}{\mathbf{Wi}}. \quad (3.7)
\end{aligned}$$

To maintain balance between inertial force, stress and pressure to the leading order, (3.6) suggests that $\gamma = \frac{\mathbf{Wi}}{\mathbf{Re}}$ and (3.7) suggests that $\lambda \gamma = \kappa$ as well as $\kappa \lambda = \mathbf{Wi}^2$. It is not hard to solve for the consistent scaling with $\gamma = \frac{\mathbf{Wi}}{\mathbf{Re}}$, $\lambda = \sqrt{\mathbf{Wi} \mathbf{Re}}$, $\kappa = \frac{\mathbf{Wi} \sqrt{\mathbf{Wi}}}{\mathbf{Re}}$.

If we keep only the leading order terms in the rescaled equations, then the momentum equations reduce to

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} - \frac{\partial p}{\partial x}, \\ \frac{\partial p}{\partial y} &= 0.\end{aligned}\tag{3.8}$$

The equation $\partial p/\partial y = 0$ implies that to the leading order the pressure $p(x, t) = p^0(x, 0, t)$ in the boundary layer is the known function given by the outside flow. For the constitutive equation and continuity equations, we have

$$\begin{aligned}\frac{\partial \mathbf{S}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{S} - (\nabla \mathbf{v})\mathbf{S} - \mathbf{S}(\nabla \mathbf{v})^T &= 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0.\end{aligned}\tag{3.9}$$

We note that these equations are different from those formulated for boundary layers in steady flow in [46]. In the steady flow analysis of [46], it was assumed that there is no velocity boundary layer, i.e. u is of order $1/\mathbf{Wi}$ and v is of order $1/\mathbf{Wi}^2$, compared to order 1 and $1/\sqrt{\mathbf{WiRe}}$ above. Moreover, in a steady flow situation, the terms $\partial \mathbf{S}/\partial t$ and $(\mathbf{v} \cdot \nabla)\mathbf{S}$ both vanish on the boundary, and the term \mathbf{S}/\mathbf{Wi} becomes important.

The study of the boundary layer problem therefore reduces to finding spatially periodic solutions (with respect to x) of the equations (3.8) and (3.9) above.

3.2.2 Curved boundary

In an actual flow geometry, the boundary is curved. We shall show now that this does not change the boundary layer equations. For simplicity, we shall stick to the two-dimensional case, where each component of the boundary is a closed curve. We shall use local coordinates q and r , where q denotes arclength along the boundary, and r is distance from the boundary. In three dimensions, we would have to treat the boundary as a manifold, where coordinates can be defined only locally. However, this would not fundamentally alter the analysis.

In three dimensions, let x_i denote Cartesian coordinates, and let p_i denote curvilinear but orthogonal coordinates. We use some transformation rules given in Appendix I of [63]. Let g_k be defined by

$$g_k^2 = \sum_i \left(\frac{\partial x_i}{\partial p_k}\right)^2\tag{3.10}$$

and let $\mathbf{i}_k, \mathbf{e}_k$ be the unit vectors in the Cartesian and p coordinate systems, respectively. Then we have

$$\frac{\partial \mathbf{e}_k}{\partial p_l} = \frac{1}{g_k} \frac{\partial g_l}{\partial p_k} \mathbf{e}_l - \delta_{kl} \sum_h \frac{1}{g_h} \frac{\partial g_k}{\partial p_h} \mathbf{e}_h,\tag{3.11}$$

and

$$\nabla = \sum_k \mathbf{e}_k \frac{1}{g_k} \frac{\partial}{\partial p_k}. \quad (3.12)$$

Hence for a scalar function f we have the gradient

$$\nabla f = f_{x_1} \mathbf{i}_1 + f_{x_2} \mathbf{i}_2 + f_{x_3} \mathbf{i}_3 = \sum_{k=1}^3 \frac{1}{g_k} \frac{\partial f}{\partial p_k} \mathbf{e}_k, \quad (3.13)$$

and the divergence of a vector $\mathbf{f} = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2 + f_3 \mathbf{e}_3$ in the p system is given by

$$\nabla \cdot \mathbf{f} = \frac{1}{g_1 g_2 g_3} \left(\frac{\partial(f_1 g_2 g_3)}{\partial p_1} + \frac{\partial(f_2 g_1 g_3)}{\partial p_2} + \frac{\partial(f_3 g_1 g_2)}{\partial p_3} \right). \quad (3.14)$$

In the two-dimensional case, this reduces to

$$\nabla \cdot \mathbf{f} = \frac{1}{g_1 g_2} \left(\frac{\partial(f_1 g_2)}{\partial p_1} + \frac{\partial(f_2 g_1)}{\partial p_2} \right). \quad (3.15)$$

Convective terms in the equations transform as follows:

$$(\mathbf{a} \cdot \nabla) \mathbf{b} = \sum_{i,k} \mathbf{e}_k \left(\frac{a_i}{g_i} \frac{\partial b_k}{\partial p_i} + \frac{a_k b_i}{g_k g_i} \frac{\partial g_k}{\partial p_i} - \frac{a_i b_i}{g_i g_k} \frac{\partial g_i}{\partial p_k} \right). \quad (3.16)$$

Now we consider a solid boundary of our domain parameterized by the arc length q , $\mathbf{x}^* = \mathbf{x}^*(q) \in C^2$. In the local coordinates (q, r) consider a point $\mathbf{x} = \mathbf{x}^*(q) + r \mathbf{n}$. Then we have

$$\begin{pmatrix} \mathbf{x}_1^*(q) \\ \mathbf{x}_2^*(q) \end{pmatrix} = \begin{pmatrix} \mathbf{n}_2(q) \\ -\mathbf{n}_1(q) \end{pmatrix}, \quad \begin{pmatrix} \mathbf{n}'_1(q) \\ \mathbf{n}'_2(q) \end{pmatrix} = -\rho(q) \begin{pmatrix} \mathbf{n}_2(q) \\ -\mathbf{n}_1(q) \end{pmatrix}. \quad (3.17)$$

Here $\rho(q)$ is the curvature which is assumed to be bounded. When r is small, $\frac{\partial(\mathbf{x}_1, \mathbf{x}_2)}{\partial(q, r)} = 1 - \rho(q)r \neq 0$, hence (q, r) are local coordinates. For this coordinates transformation we have

$$\begin{aligned} g_1 &= \sqrt{\mathbf{x}'_1(q)^2 + \mathbf{x}'_2(q)^2} = 1 - \rho r, \\ g_2 &= 1. \end{aligned} \quad (3.18)$$

The unit vectors in the curvilinear coordinate system are $\mathbf{e}_1 = \boldsymbol{\tau}$, $\mathbf{e}_2 = \mathbf{n}$. With $\mathbf{v} = u \mathbf{e}_1 + v \mathbf{e}_2$, we obtain

$$\nabla \cdot \mathbf{v} = \frac{1}{1 - \rho r} \frac{\partial u}{\partial q} + \frac{\partial v}{\partial r} - \frac{\rho}{1 - \rho r} v \quad (3.19)$$

and

$$\begin{aligned} (\mathbf{v} \cdot \nabla) \mathbf{v} &= \left[\frac{1}{1 - \rho r} u \frac{\partial u}{\partial q} + v \frac{\partial u}{\partial r} - \frac{\rho u v}{1 - \rho r} \right] \mathbf{e}_1 \\ &+ \left[\frac{u}{1 - \rho r} \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial r} + \frac{\rho u^2}{1 - \rho r} \right] \mathbf{e}_2. \end{aligned} \quad (3.20)$$

For the divergence of the stress tensor we find, with $\mathbf{S} = \sum_{ij} S_{ij} \mathbf{e}_i \mathbf{e}_j$,

$$\begin{aligned} \nabla \cdot \mathbf{S} &= \left[\frac{1}{1-\rho r} \frac{\partial S_{11}}{\partial q} + \frac{\partial S_{12}}{\partial r} - \frac{2\rho}{1-\rho r} S_{12} \right] \mathbf{e}_1 \\ &+ \left[\frac{1}{1-\rho r} \frac{\partial S_{12}}{\partial q} + \frac{\partial S_{22}}{\partial r} + \frac{\rho}{1-\rho r} S_{11} - \frac{\rho}{1-\rho r} S_{22} \right] \mathbf{e}_2. \end{aligned} \quad (3.21)$$

Under the scaling

$$\begin{aligned} q' &= q, r' = \mathbf{Wi} r, u' = u, v' = \mathbf{Wi} v, \\ S'_{11} &= S_{11}, S'_{12} = \mathbf{Wi} S_{12}, S'_{22} = \mathbf{Wi}^2 S_{22}, p' = p, \end{aligned} \quad (3.22)$$

the divergence condition becomes

$$\nabla \cdot \mathbf{v}' = \frac{1}{1 - \frac{r'}{\mathbf{Wi}} \rho} \frac{\partial}{\partial q'} u' + \frac{\partial \frac{v'}{\mathbf{Wi}}}{\partial \frac{r'}{\mathbf{Wi}}} + \left(\frac{-\rho}{1 - \frac{r'}{\mathbf{Wi}} \rho} \frac{v'}{\mathbf{Wi}} \right) = 0. \quad (3.23)$$

To the leading order as $\mathbf{Wi} \rightarrow \infty$ we find

$$\nabla \cdot \mathbf{v}' = \frac{\partial u'}{\partial q'} + \frac{\partial v'}{\partial r'} = 0. \quad (3.24)$$

That is, the divergence condition assumes the same form in the (q', r') coordinates as in (3.9) above. In a similar fashion, it can be seen that the momentum and constitutive equations also remain unchanged at leading order.

3.2.3 Lagrangian description

We shall transform (3.8) and (3.9) into Lagrangian coordinates. Let ξ_1 and ξ_2 denote the coordinates of a fluid particle at $t = 0$, and let $x = x(\xi_1, \xi_2, t)$, $y = y(\xi_1, \xi_2, t)$ denote the position of the same particle at a later time. That is, we have

$$\begin{aligned} \frac{\partial x(\xi_1, \xi_2, t)}{\partial t} &= u(x(\xi_1, \xi_2, t), y(\xi_1, \xi_2, t), t), \\ \frac{\partial y(\xi_1, \xi_2, t)}{\partial t} &= v(x(\xi_1, \xi_2, t), y(\xi_1, \xi_2, t), t), \\ x(\xi_1, \xi_2, 0) &= \xi_1, \\ y(\xi_1, \xi_2, 0) &= \xi_2. \end{aligned} \quad (3.25)$$

We introduce the deformation gradient tensor $\mathbf{F} = \begin{pmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} \end{pmatrix}$. It is easy to check that it satisfies $\frac{\partial \mathbf{F}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{F} - (\nabla \mathbf{v}) \mathbf{F} = 0$, $\mathbf{F}(0) = I$ and \mathbf{F} is non-singular all the time.

Let $\mathbf{C} = \mathbf{F}^{-1}\mathbf{S}\mathbf{F}^{-T}$. It then follows from the constitutive law that $\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{C} = 0$.

The boundary layer system now becomes

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} - P(x, t), \\ \frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{C} &= \mathbf{0}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ u = 0, v = 0, &\text{ at } y = 0, \\ u(x, y, 0) &= u_0(x, y), \\ \mathbf{C}(x, y, 0) &= \mathbf{S}_0(x, y). \end{aligned} \tag{3.26}$$

Here we have introduced $P(x, t)$ for the known pressure gradient $\partial p / \partial x$.

We find that in Lagrangian coordinates, we have $\frac{\partial \mathbf{C}}{\partial t} = 0$, and hence $\mathbf{C}(\xi_1, \xi_2, t) = \mathbf{C}(\xi_1, \xi_2)$. We shall assume throughout that \mathbf{C} is positive semidefinite and there is positive constant C_0 such that $C_{22} \geq C_0 > 0$.

3.3 Results

3.3.1 Lemmas

The following lemma shows how our governing equations transform.

Lemma 1. In Lagrangian coordinates from (3.26) we have for x the second order hyperbolic equation:

$$\begin{aligned} \frac{\partial^2 x}{\partial t^2} &= \nabla \cdot (\mathbf{C}\nabla x) - P(x, t), \\ x(\xi_1, \xi_2, 0) &= \xi_1, \\ \frac{\partial x}{\partial t}(\xi_1, \xi_2, 0) &= u_0(\xi_1, \xi_2), \\ x(\xi_1, 0, t) &= \xi_1. \end{aligned} \tag{3.27}$$

Proof: The horizontal acceleration is given by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 x(\xi_1, \xi_2, t)}{\partial t^2}. \tag{3.28}$$

Next we show that

$$\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{12}}{\partial y} = \frac{\partial}{\partial \xi_i} (C_{ji}(\xi_1, \xi_2) \frac{\partial x}{\partial \xi_j}) = \nabla \cdot (\mathbf{C}\nabla x). \tag{3.29}$$

We write $\operatorname{div}(\mathbf{FCF}^T)$ in components with the Einstein summation convention:

$$\frac{\partial}{\partial x_j}(F_{ik}C_{kl}F_{jl}) = F_{jl}\frac{\partial}{\partial x_j}(C_{kl}F_{ik}) + C_{kl}F_{ik}\frac{\partial F_{jl}}{\partial x_j}. \quad (3.30)$$

For the first term, we note that by the chain rule we have

$$F_{jl}\frac{\partial}{\partial x_j} = \frac{\partial x_j}{\partial \xi_l}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial \xi_l}. \quad (3.31)$$

For the second term we write

$$\frac{\partial F_{jl}}{\partial x_j} = \frac{\partial \xi_k}{\partial x_j}\frac{\partial F_{jl}}{\partial \xi_k} = \frac{\partial \xi_k}{\partial x_j}\frac{\partial^2 x_j}{\partial \xi_k \partial x_l} = \operatorname{tr}(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial \xi_l}). \quad (3.32)$$

Since $\det \mathbf{F} = 1$ by the incompressibility condition, we find

$$\frac{\partial}{\partial \xi_l}(\det \mathbf{F}) = (\det \mathbf{F})\operatorname{tr}(\mathbf{F}^{-1}\frac{\partial \mathbf{F}}{\partial \xi_l}) = 0. \quad (3.33)$$

□

Once we have found x , we can solve for y from the incompressibility condition.

Lemma 2. From the incompressibility condition, we find the following equation for y under the Lagrangian description:

$$\begin{aligned} \frac{\partial x}{\partial \xi_1}\frac{\partial y}{\partial \xi_2} - \frac{\partial x}{\partial \xi_2}\frac{\partial y}{\partial \xi_1} &= 1, \\ y(\xi_1, 0, t) &= 0. \end{aligned} \quad (3.34)$$

This equation is a well-posed transport equation for y with t as a parameter. The details are omitted. □

After we get $x(\xi_1, \xi_2, t)$ and $y(\xi_1, \xi_2, t)$, we can recover $u = x_t, v = y_t$. Hence the deformation tensor $\mathbf{F} = \begin{pmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \xi_2} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \xi_2} \end{pmatrix}$ and extra stress $\mathbf{S} = \mathbf{FCF}^T$.

3.3.2 Main theorem

We now focus on the solution of (3.27). This is simply a semilinear wave equation. We are seeking solutions which are periodic with respect to ξ_1 with a given period, but the behavior for $\xi_2 \rightarrow \infty$ warrants some discussion. We want to match to a given outer solution. Therefore, we assume that the initial condition u_0 has a limit at infinity:

$$\lim_{\xi_2 \rightarrow \infty} u_0(\xi_1, \xi_2) = u^\infty(\xi_1). \quad (3.35)$$

We also assume that C_{11} has a limit,

$$\lim_{\xi_2 \rightarrow \infty} C_{11}(\xi_1, \xi_2) = C_{11}^\infty(\xi_1), \quad (3.36)$$

and that C_{12} and C_{22} and their derivatives are uniformly bounded.

The behavior at infinity is now governed by the following limit problem:

$$\begin{aligned} \frac{\partial^2 x^\infty(\xi_1, t)}{\partial t^2} &= \frac{\partial}{\partial \xi_1} \left(C_{11}^\infty(\xi_1) \frac{\partial x^\infty}{\partial \xi_1} \right) - P(x^\infty, t), \\ x^\infty(\xi_1, 0) &= \xi_1, \\ \frac{\partial x^\infty}{\partial t}(\xi_1, 0) &= u^\infty(\xi_1). \end{aligned} \quad (3.37)$$

Note that this is precisely the x component of the momentum equation for the outer flow at the wall transformed to Lagrangian coordinates.

Combine (3.27) and (3.37), and denote $x - x^\infty$ by X . We have

$$\begin{aligned} X_{tt} &= \nabla \cdot (\mathbf{C} \nabla X) + \Psi(X, \xi_1, \xi_2, t), \\ X(\xi_1, \xi_2, 0) &= 0, \\ \frac{\partial X}{\partial t}(\xi_1, \xi_2, 0) &= \mathbf{f}(\xi_1, \xi_2), \\ X(\xi_1, 0, t) &= \mathbf{g}(\xi_1, t) \end{aligned} \quad (3.38)$$

with

$$\begin{aligned} \Psi &= \nabla \cdot (\mathbf{C} \nabla x^\infty) - \frac{\partial}{\partial \xi_1} \left(C_{11}^\infty(\xi_1) \frac{\partial x^\infty}{\partial \xi_1} \right) - P(X + x^\infty, t) + P(x^\infty, t), \\ \mathbf{f}(\xi_1, \xi_2) &= u_0(\xi_1, \xi_2) - u^\infty(\xi_1), \\ \mathbf{g}(\xi_1, t) &= \xi_1 - x^\infty(\xi_1, t). \end{aligned} \quad (3.39)$$

Now we want to make the boundary and initial conditions homogeneous. Pick a smooth function $\chi = \begin{cases} 1, & 0 \leq \xi_2 \leq 1, \\ 0, & 2 \leq \xi_2, \end{cases}$ defined on $[0, \infty)$ such that $0 \leq \chi \leq 1$.

Let

$$\begin{aligned} Y &= \mathbf{g}(\xi_1, t) \cdot \chi(\xi_2) + t[u_0(\xi_1, \xi_2) - u^\infty(\xi_1) \cdot (1 - \chi(\xi_2))], \\ \bar{x} &= X(\xi_1, \xi_2, t) - Y. \end{aligned} \quad (3.40)$$

We have for \bar{x}

$$\begin{aligned} \bar{x}_{tt} &= \nabla \cdot (\mathbf{C} \nabla \bar{x}) + \Phi(\bar{x}, \xi_1, \xi_2, t), \\ \bar{x}(\xi_1, \xi_2, 0) &= 0, \\ \frac{\partial \bar{x}}{\partial t}(\xi_1, \xi_2, 0) &= 0, \\ \bar{x}(\xi_1, 0, t) &= 0, \end{aligned} \quad (3.41)$$

where

$$\Phi(\bar{x}, \xi_1, \xi_2, t) = \Psi(\bar{x} + Y, \xi_1, \xi_2, t) + \nabla \cdot (\mathbf{C}\nabla Y) - Y_{tt}. \quad (3.42)$$

We have an initial-boundary value problem for a semilinear wave equation of \bar{x} . We shall state our existence result for \bar{x} in L^2 based spaces. Unlike the work of Lasiecka et al. [30], our problem is posed on an unbounded domain and does not satisfy the uniformly hyperbolic condition, namely we do not require $\mathbf{C} > 0$ strictly throughout the domain. To study this IBVP, we shall transform the equation into an equivalent first order system.

Since \mathbf{C} is real positive semi-definite, there is a unique real positive semi-definite \mathbf{A} with $a_{22} \geq c_0 > 0$, denoted by $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$, such that $\mathbf{A}^2 = \mathbf{C}$. Let $U = \bar{x}_t, (V, W)^T = \mathbf{A}\nabla\bar{x}, \vec{V} = (U, V, W)^T$. From (3.41) we have an integro-differential system

$$\vec{V}_t = \mathbf{A}_1 \vec{V}_{\xi_1} + \mathbf{A}_2 \vec{V}_{\xi_2} + \mathbf{B}\vec{V} + \vec{\Phi}, \quad (3.43)$$

with initial and boundary condition

$$\begin{aligned} \vec{V}(\xi_1, \xi_2, 0) &= \vec{0}, \\ U(\xi_1, 0, t) &= \bar{x}_t(\xi_1, 0, t) = 0. \end{aligned} \quad (3.44)$$

$$\begin{aligned} \text{Here } \mathbf{A}_1 &= \begin{pmatrix} 0 & a_{11} & a_{12} \\ a_{11} & 0 & 0 \\ a_{12} & 0 & 0 \end{pmatrix}, \mathbf{A}_2 = \begin{pmatrix} 0 & a_{12} & a_{22} \\ a_{12} & 0 & 0 \\ a_{22} & 0 & 0 \end{pmatrix}, \vec{\Phi} = \begin{pmatrix} \Phi \\ 0 \\ 0 \end{pmatrix}, \\ \mathbf{B} &= \begin{pmatrix} 0 & \frac{\partial a_{11}}{\partial \xi_1} + \frac{\partial a_{12}}{\partial \xi_2} & \frac{\partial a_{12}}{\partial \xi_1} + \frac{\partial a_{22}}{\partial \xi_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Phi = \Phi([\int_0^t U(\xi_1, \xi_2, s) ds], \xi_1, \xi_2, t). \end{aligned}$$

This is a first order linear symmetric hyperbolic system when Φ is given. The dependence of Φ on \bar{x} can then be handled via a standard fixed point argument. In the following, we shall show that on the half plane with given periodic forcing term Φ the system (3.43)-(3.44) has a unique local solution which is periodic.

The theory for hyperbolic systems was developed by Friedrichs [14], Kreiss [26], Lax and Phillips [31], and others. The general idea is to do energy estimate on the equations. Based on proper a priori estimates, one can either define a weak solution then improve its regularity [13], or use different schemes to approximate the real solutions [27]. For general hyperbolic system with characteristic boundary, the full regularity may be lost [7]. People developed anisotropic weighted space of Sobolev type to meet this purpose, see Secchi [61].

Our problem presents a first order symmetric hyperbolic system with characteristic boundary. We show that nevertheless we are still able to get the full regularity in Sobolev spaces because of its special structure.

Theorem 1. Consider the boundary layer system (3.43)-(3.44) on the domain

$$-\infty < \xi_1 < \infty, 0 \leq \xi_2 < \infty, \text{ for } t \geq 0.$$

Let $m \geq 1$ be an integer. We assume that $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}$ and their derivatives up to order m are bounded for $(\xi_1, \xi_2, t) \in (-\infty, \infty) \times [0, \infty) \times [0, T]$. Moreover they are all periodic in ξ_1 with period 1. Let $\Omega = [0, 1] \times [0, \infty)$, $\mathbf{Q} = \Omega \times [0, T]$, and denote by $H_p^m(\mathbf{Q})$, $m \in \mathbb{N}$ the space of all periodic (in ξ_1) functions which have H^m regularity. If $\Phi \in H_p^m(\mathbf{Q})$, then there exists some $T' \in (0, T]$ such that a unique solution of (3.43)-(3.44) exists and satisfies $\vec{V} \in H_p^m(\mathbf{Q})$.

The idea of the proof is to develop a priori estimates for solutions which are assumed smooth, a density argument can be used to extend the result for general Sobolev data in H^m . In the estimates, the inhomogeneous forcing term Φ always contributes terms of the form $(D^\alpha \Phi, D^\beta U) \leq C(\|D^\alpha \Phi\|^2 + \|D^\beta U\|^2)$. Since these terms are easily dealt with, we present the estimates without the forcing term Φ .

Proof: We complete the energy estimate in an elementary way. The L^2 estimate is done by virtue of the special structure of coefficient matrices. The evaluation of higher order Sobolev norms gets into trouble due to the loss of normal derivative at the boundary. For that reason we do the estimate separately. We first work on the L^2 norm. Then we use the integral for the estimate of all derivative except the normal direction. For the normal derivative, we take advantage of the boundary condition and of a constraint on the solution that is preserved by the equations.

1. L^2 estimate

Consider (3.43)-(3.44). Note that boundary conditions are imposed only on U , and not on V and W .

We multiply (3.43) with \vec{V} and integrate it with respect to $\xi \in \Omega$

$$\int_{\Omega} \vec{V}_t \cdot \vec{V} d\xi = \int_{\Omega} \mathbf{A}_1 \vec{V}_{\xi_1} \cdot \vec{V} d\xi + \int_{\Omega} \mathbf{A}_2 \vec{V}_{\xi_2} \cdot \vec{V} d\xi + \int_{\Omega} \mathbf{B} \vec{V} \cdot \vec{V} d\xi. \quad (3.45)$$

Integrating by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\vec{V}|^2 d\xi &= \frac{1}{2} \int_0^\infty \mathbf{A}_1 \vec{V} \cdot \vec{V} \Big|_{\xi_1=0}^1 d\xi_2 - \frac{1}{2} \int_{\Omega} \frac{\partial \mathbf{A}_1}{\partial \xi_1} \vec{V} \cdot \vec{V} d\xi \\ &\quad + \frac{1}{2} \int_0^1 \mathbf{A}_2 \vec{V} \cdot \vec{V} \Big|_{\xi_2=0}^\infty d\xi_1 - \frac{1}{2} \int_{\Omega} \frac{\partial \mathbf{A}_2}{\partial \xi_2} \vec{V} \cdot \vec{V} d\xi \\ &\quad + \frac{1}{2} \int_{\Omega} (\mathbf{B} + \mathbf{B}^*) \vec{V} \cdot \vec{V} d\xi \\ &= \frac{1}{2} \int_{\Omega} [\mathbf{B} + \mathbf{B}^* - \frac{\partial \mathbf{A}_1}{\partial \xi_1} - \frac{\partial \mathbf{A}_2}{\partial \xi_2}] \vec{V} \cdot \vec{V} d\xi. \end{aligned} \quad (3.46)$$

because $\vec{V}(0, \xi_2, t) = \vec{V}(1, \xi_2, t)$ and $\int_0^1 \mathbf{A}_2 \vec{V}(\xi_1, 0, t) \cdot \vec{V}(\xi_1, 0, t) d\xi_1 = 0$.

2. ∂_t estimate

The energy estimate for \vec{V}_t can be done similarly to the L^2 case. Differentiating (3.43) w.r.t t we have

$$(\vec{V}_t)_t = \mathbf{A}_1(\vec{V}_t)_{\xi_1} + \mathbf{A}_2(\vec{V}_t)_{\xi_2} + \mathbf{B}\vec{V}_t. \quad (3.47)$$

Multiply and integrate to have

$$\begin{aligned} \int (\vec{V}_t)_t \cdot \vec{V}_t d\xi &= \int \mathbf{A}_1(\vec{V}_t)_{\xi_1} \cdot \vec{V}_t d\xi \\ &+ \int \mathbf{A}_2(\vec{V}_t)_{\xi_2} \cdot \vec{V}_t d\xi + \int \mathbf{B}\vec{V}_t \cdot \vec{V}_t d\xi. \end{aligned} \quad (3.48)$$

Simplify to have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\vec{V}_t|^2 d\xi &= -\frac{1}{2} \int_{\Omega} \frac{\partial \mathbf{A}_1}{\partial \xi_1} \vec{V}_t \cdot \vec{V}_t d\xi + \frac{1}{2} \int_0^1 \mathbf{A}_2 \vec{V}_t \cdot \vec{V}_t |_{\xi_2=0}^{\infty} d\xi_1 \\ &\quad -\frac{1}{2} \int_{\Omega} \frac{\partial \mathbf{A}_2}{\partial \xi_2} \vec{V}_t \cdot \vec{V}_t d\xi + \frac{1}{2} \int_{\Omega} (\mathbf{B} + \mathbf{B}^*) \vec{V}_t \cdot \vec{V}_t d\xi \\ &= \frac{1}{2} \int_{\Omega} [\mathbf{B} + \mathbf{B}^* - \frac{\partial \mathbf{A}_1}{\partial \xi_1} - \frac{\partial \mathbf{A}_2}{\partial \xi_2}] \vec{V}_t \cdot \vec{V}_t d\xi. \end{aligned} \quad (3.49)$$

Notice that $\mathbf{A}_2 \vec{V}_t \cdot \vec{V}_t = 0$ at the boundary.

3. ∂_{ξ_1} estimate

Differentiate (3.43) w.r.t ξ_1 we get

$$\begin{aligned} (\vec{V}_{\xi_1})_t &= \mathbf{A}_1(\vec{V}_{\xi_1})_{\xi_1} + \frac{\partial \mathbf{A}_1}{\partial \xi_1} \vec{V}_{\xi_1} \\ &+ \mathbf{A}_2(\vec{V}_{\xi_1})_{\xi_2} + \frac{\partial \mathbf{A}_2}{\partial \xi_1} \vec{V}_{\xi_2} + \mathbf{B}\vec{V}_{\xi_1} + \frac{\partial \mathbf{B}}{\partial \xi_1} \vec{V}. \end{aligned} \quad (3.50)$$

Multiply and integrate to get

$$\begin{aligned} \int (\vec{V}_{\xi_1})_t \cdot (\vec{V}_{\xi_1}) d\xi &= \int \mathbf{A}_1(\vec{V}_{\xi_1})_{\xi_1} \cdot \vec{V}_{\xi_1} d\xi + \int \frac{\partial \mathbf{A}_1}{\partial \xi_1} \vec{V}_{\xi_1} \cdot \vec{V}_{\xi_1} d\xi \\ &+ \int \mathbf{A}_2(\vec{V}_{\xi_1})_{\xi_2} \cdot \vec{V}_{\xi_1} d\xi + \int \frac{\partial \mathbf{A}_2}{\partial \xi_1} \vec{V}_{\xi_2} \cdot \vec{V}_{\xi_1} d\xi \\ &+ \int \mathbf{B}\vec{V}_{\xi_1} \cdot \vec{V}_{\xi_1} d\xi + \int \frac{\partial \mathbf{B}}{\partial \xi_1} \vec{V} \cdot \vec{V}_{\xi_1} d\xi. \end{aligned} \quad (3.51)$$

Simplify to get

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\vec{V}_{\xi_1}|^2 d\xi &= -\frac{1}{2} \int_{\Omega} \frac{\partial \mathbf{A}_1}{\partial \xi_1} \vec{V}_{\xi_1} \cdot \vec{V}_{\xi_1} d\xi + \int \frac{\partial \mathbf{A}_1}{\partial \xi_1} \vec{V}_{\xi_1} \cdot \vec{V}_{\xi_1} d\xi \\
&+ \frac{1}{2} \int_0^1 \mathbf{A}_2 \vec{V}_{\xi_1} \cdot \vec{V}_{\xi_1} |_{\xi_2=0}^{\infty} d\xi_1 - \frac{1}{2} \int_{\Omega} \frac{\partial \mathbf{A}_2}{\partial \xi_2} \vec{V}_{\xi_1} \cdot \vec{V}_{\xi_1} d\xi \\
&+ \int \frac{1}{2} [\mathbf{B} + \mathbf{B}^*] \vec{V}_{\xi_1} \cdot \vec{V}_{\xi_1} d\xi \\
&+ \frac{1}{2} \int \frac{\partial (\mathbf{B} + \mathbf{B}^*)}{\partial \xi_1} \vec{V} \cdot \vec{V}_{\xi_1} d\xi + \int \frac{\partial \mathbf{A}_2}{\partial \xi_1} \vec{V}_{\xi_2} \cdot \vec{V}_{\xi_1} d\xi \\
&= \frac{1}{2} \int_{\Omega} \left[\frac{\partial \mathbf{A}_1}{\partial \xi_1} - \frac{\partial \mathbf{A}_2}{\partial \xi_2} + \mathbf{B} + \mathbf{B}^* \right] \vec{V}_{\xi_1} \cdot \vec{V}_{\xi_1} d\xi \\
&+ \frac{1}{2} \int \frac{\partial (\mathbf{B} + \mathbf{B}^*)}{\partial \xi_1} \vec{V} \cdot \vec{V}_{\xi_1} d\xi + \int \frac{\partial \mathbf{A}_2}{\partial \xi_1} \vec{V}_{\xi_2} \cdot \vec{V}_{\xi_1} d\xi. \tag{3.52}
\end{aligned}$$

To estimate \vec{V}_{ξ_1} , we need knowledge of \vec{V}_{ξ_2} because of the term

$$\int \frac{\partial \mathbf{A}_2}{\partial \xi_1} \vec{V}_{\xi_2} \cdot \vec{V}_{\xi_1} d\xi.$$

This turns out to be no problem when we finish the ∂_{ξ_2} estimate below.

4. ∂_{ξ_2} estimate

The crucial estimate is for the term \vec{V}_{ξ_2} . Because of the loss of normal derivative, we can not proceed as before. Instead, we go back to original equation and solve for \vec{V}_{ξ_2} as a function of \vec{V} , \vec{V}_t , \vec{V}_{ξ_1} .

From (3.43) we get

$$\begin{aligned}
U_t &= a_{12} V_{\xi_2} + a_{22} W_{\xi_2} + a_{11} V_{\xi_1} + a_{12} W_{\xi_1} \\
&+ \left[\frac{\partial a_{11}}{\partial \xi_1} + \frac{\partial a_{12}}{\partial \xi_2} \right] V + \left[\frac{\partial a_{12}}{\partial \xi_1} + \frac{\partial a_{22}}{\partial \xi_2} \right] W, \tag{3.53}
\end{aligned}$$

$$V_t = a_{11} U_{\xi_1} + a_{12} U_{\xi_2}, \tag{3.54}$$

$$W_t = a_{12} U_{\xi_1} + a_{22} U_{\xi_2}. \tag{3.55}$$

By (3.55) we have

$$U_{\xi_2} = \frac{1}{a_{22}} (W_t - a_{12} U_{\xi_1}),$$

which leads to

$$\|U_{\xi_2}\|_0 \leq C(\|W_t\|_0 + \|U_{\xi_1}\|_0).$$

To solve for V_{ξ_2} and W_{ξ_2} , we differentiate (3.54) with respect to ξ_1, ξ_2 , respectively, to have

$$V_{\xi_1 t} = a_{11}U_{\xi_1\xi_1} + a_{12}U_{\xi_1\xi_2} + \frac{\partial a_{11}}{\partial \xi_1}U_{\xi_1} + \frac{\partial a_{12}}{\partial \xi_1}U_{\xi_2},$$

$$V_{\xi_2 t} = a_{11}U_{\xi_1\xi_2} + a_{12}U_{\xi_2\xi_2} + \frac{\partial a_{11}}{\partial \xi_2}U_{\xi_1} + \frac{\partial a_{12}}{\partial \xi_2}U_{\xi_2}.$$

Similarly, for (3.55) we have

$$W_{\xi_1 t} = a_{12}U_{\xi_1\xi_1} + a_{22}U_{\xi_1\xi_2} + \frac{\partial a_{12}}{\partial \xi_1}U_{\xi_1} + \frac{\partial a_{22}}{\partial \xi_1}U_{\xi_2},$$

$$W_{\xi_2 t} = a_{12}U_{\xi_1\xi_2} + a_{22}U_{\xi_2\xi_2} + \frac{\partial a_{12}}{\partial \xi_2}U_{\xi_1} + \frac{\partial a_{22}}{\partial \xi_2}U_{\xi_2}.$$

Multiplying with a_{12}, a_{22} and doing simple calculations, we obtain

$$\begin{aligned} & (a_{22}V_{\xi_2} - a_{12}W_{\xi_2} + a_{12}V_{\xi_1} - a_{11}W_{\xi_1})_t \\ &= [a_{12}\frac{\partial a_{11}}{\partial \xi_1} + a_{22}\frac{\partial a_{11}}{\partial \xi_2} - a_{11}\frac{\partial a_{12}}{\partial \xi_1} - a_{12}\frac{\partial a_{12}}{\partial \xi_2}]U_{\xi_1} \\ &+ [a_{12}\frac{\partial a_{12}}{\partial \xi_1} + a_{22}\frac{\partial a_{12}}{\partial \xi_2} - a_{11}\frac{\partial a_{22}}{\partial \xi_1} - a_{12}\frac{\partial a_{22}}{\partial \xi_2}]U_{\xi_2} \\ &= [a_{12}\frac{\partial a_{11}}{\partial \xi_1} + a_{22}\frac{\partial a_{11}}{\partial \xi_2} - a_{11}\frac{\partial a_{12}}{\partial \xi_1} - a_{12}\frac{\partial a_{12}}{\partial \xi_2}]U_{\xi_1} \\ &+ \frac{1}{a_{22}}[a_{12}\frac{\partial a_{12}}{\partial \xi_1} + a_{22}\frac{\partial a_{12}}{\partial \xi_2} - a_{11}\frac{\partial a_{22}}{\partial \xi_1} - a_{12}\frac{\partial a_{22}}{\partial \xi_2}](W_t - a_{12}U_{\xi_1}) \\ &= \mathbf{F}(U_{\xi_1}, W_t, \xi_1, \xi_2). \end{aligned}$$

Since $\vec{V}(\xi_1, \xi_2, 0) = 0$, we have

$$a_{22}V_{\xi_2} - a_{12}W_{\xi_2} = \int_0^t \mathbf{F}(U_{\xi_1}, W_t, \xi_1, \xi_2)ds - a_{12}V_{\xi_1} + a_{11}W_{\xi_1}. \quad (3.56)$$

And (3.53) yields

$$\begin{aligned} a_{12}V_{\xi_2} + a_{22}W_{\xi_2} &= U_t - a_{11}V_{\xi_1} - a_{12}W_{\xi_1} \\ &- \left[\frac{\partial a_{11}}{\partial \xi_1} + \frac{\partial a_{12}}{\partial \xi_2} \right] V + \left[\frac{\partial a_{12}}{\partial \xi_1} + \frac{\partial a_{22}}{\partial \xi_2} \right] W. \end{aligned} \quad (3.57)$$

Together we can now solve for V_{ξ_2} and W_{ξ_2} in terms of $U_t, U_{\xi_1}, V_{\xi_1}, W_{\xi_1}, V, W$, and we know \vec{V}_{ξ_2} in term of other derivatives. We then plug \vec{V}_{ξ_2} back in (3.52) to complete the estimate for \vec{V}_{ξ_1} . Finally, the estimate for \vec{V}_{ξ_2} is done by using the triangle inequality. For the estimate of the integral term $\int_0^t \mathbf{F}(U_{\xi_1}, W_t, \xi_1, \xi_2) ds$, we use Minkowski's inequality. In this way we get the H_1 regularity of the solution. Taking higher order derivatives of (3.43), the H_m energy estimate can be done in same fashion.

Based on the energy estimate, the uniqueness and regularity are easy to see. The existence proof can be obtained along the lines of [27]. Our system corresponds to Case 2 in the discussion given in that reference. Following [27], to show the existence, we would first diagonalize the system so that \mathbf{A}_2 transforms to

$$\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +\lambda \end{pmatrix},$$

where $\lambda = \sqrt{a_{12}^2 + a_{22}^2} \geq c_0 > 0$. Then we consider the perturbed problem with

$$\mathbf{A}_{2\sigma} = \begin{pmatrix} -\lambda + \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & +\lambda + \sigma \end{pmatrix},$$

where $0 < \sigma < \lambda$. This perturbed problem has same number of negative eigenvalue as the original problem, but a noncharacteristic boundary. Standard methods can be used to show that there exists a solution depending on σ , which, for instance, is constructed by finite difference approximating scheme in [27]. The estimates are independent of σ and a limit argument gives the existence of original problem (3.43). We refer to [27], page 297-298 for a detailed discussion.

□

Bibliography

- [1] R. B. Bird, O. Hassager, R.C. Armstrong and C.F. Curtiss, *Dynamics of Polymeric Liquids*, Wiley, New York, 2nd ed. 1987.
- [2] J. L. Bona and R. Smith, The initial-value problem for the Korteweg-de Vries equations, *Philos. Trans. Roy. Soc. London A* **278** (1975), 555-601.
- [3] M. Cannone, C. Miao, N. Prioux and B. Yuan, The Cauchy problem for the magneto-hydrodynamic system, *Banach Center Publ.* **74** (2006), 59-93.
- [4] E. Casella, P. Secchi and P. Trebeschi, Global classical solutions for MHD system, *J. Math. Fluid Mech.* **5** (2003), 70-91.
- [5] A. J. Chorin and J. E. Marsden, *A mathematical introduction to fluid mechanics*. Third edition. Texts in Applied Mathematics, 4. Springer-Verlag, New York, 1993.
- [6] G. Q. Chen, D. Osborn and Z. Qian, The Navier-Stokes equations with the kinematic and vorticity boundary conditions on non-flat boundaries, *Acta Math. Sci.* **29** (2009), 919-948.
- [7] S. X. Chen, On the initial-boundary value problems for quasilinear symmetric hyperbolic system with characteristic boundary, *Chinese Ann. Math.* **3** (1982), 223-232.
- [8] D. M. Dealy, Weissenberg and Deborah numberstheir definition and use, *Rheol. Bull.* **79**, 1418 (2010).
- [9] W. E, Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation, *Acta Math. Sin. (Engl. Ser.)* **16** (2000), no. 2, 207–218.
- [10] W. E, *Principles of multiscale modeling*, Cambridge University Press, Cambridge, 2011.
- [11] M. Fabrizio and A. Morro, *Mathematical problems in linear viscoelasticity*. SIAM Studies in Applied Mathematics, 12. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

- [12] J. Fan and T. Ozawa, Regularity criteria for the magnetohydrodynamic equations with partial viscous terms and the Leray- α -MHD mode, *Kinetic and Related Models* **2** (2009), 293-305.
- [13] K.O. Friedrichs, The identity of weak and strong extensions of differential operators, *Trans. Amer. Math. Soc.* **55** (1944), 132–151.
- [14] K.O. Friedrichs, Symmetric positive linear differential equations, *Comm. Pure Appl. Math.* **11** (1958), 333–418.
- [15] D. Gérard-Varet and E. Dormy, On the ill-posedness of the Prandtl equation, *J. Amer. Math. Soc.* **23** (2010), 591–609.
- [16] H. Goedbloed and S. Poedts, *Principles of Magnetohydrodynamics with Applications to Laboratory and Astrophysical Plasmas*, Cambridge University Press, Cambridge 2004.
- [17] A. Groisman and V. Steinberg, Elastic turbulence in a polymer solution flow, *Nature* **405** (2000), 53–55.
- [18] C. Guillope and J.-C Saut, Mathematical problems arising in differential models for viscoelastic fluids, *Mathematical Topics in Fluid Mechanics*, Longman, Harlow 1992, 64–92.
- [19] C. He and Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, *J. Diff. Eq.* **213** (2005), 235-254.
- [20] E. J. Hinch, The flow of an Oldroyd fluid around a sharp corner, *J. Non-Newtonian Fluid Mech.* **50** (1993), 161-171.
- [21] D.W. Hughes and S.M. Tobias, On the instability of magnetohydrodynamic shear flows, *Proc. Roy. Soc. London A* **457** (2001), 1365–1384.
- [22] Q. Jiu and C. He, Remarks on the regularity to 3-D Ideal magnetohydrodynamic equations, *Acta Math. Sinica* **20** (2004), 695-708.
- [23] D.D. Joseph, *Fluid Dynamics of Viscoelastic Liquids*, Springer, New York, 1990.
- [24] A. Kaffel and M. Renardy, On the stability of plane parallel viscoelastic shear flows in the limit of infinite Weissenberg and Reynolds numbers, *J. Non-Newt. Fluid Mech.* **165** (2010), 1670–1676.
- [25] T. Kato and C.Y. Lai, Nonlinear evolution equations and the Euler flow, *J. Func. Anal.* **56** (1984), 15-28.
- [26] H-O. Kreiss, Initial boundary value problems for hyperbolic systems, *Comm. Pure Appl. Math.* **23** (1970), 277–298.

- [27] H-O. Kreiss and J. Lorenz, Initial-boundary value problems and the Navier-Stokes equations, *Pure and Applied Mathematics* **136**, Academic Press, Inc., Boston, MA, 1989.
- [28] O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon&Breach, New York 1963.
- [29] R. G. Larson, *The Structure and Rheology of Complex Fluids*, Oxford University Press (1999).
- [30] I. Lasiecka, J.-L. Lions and R. Triggiani, Nonhomogeneous boundary value problems for second order hyperbolic operators, *J. Math. Pures Appl. (9)* **65** (1986), no. 2, 149–192.
- [31] P. D. Lax and R.S. Phillips, Local boundary conditions for dissipative symmetric linear differential operators. *Comm. Pure Appl. Math.* **13** (1960), 427–455.
- [32] C. Le Bris and T. Lelivre, Multiscale modelling of complex fluids: a mathematical initiation. *Multiscale modeling and simulation in science*, 49–137, Lect. Notes Comput. Sci. Eng., 66, Springer, Berlin, 2009
- [33] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, New York 1984.
- [34] J. C. Miller, *Shear Flow Instabilities in Viscoelastic Fluids*, PhD thesis, University of Cambridge 2005, <http://cnls.lanl.gov/~jomiller/publications>.
- [35] G. I. Ogilvie and M.R.E. Proctor, On the relation between viscoelastic and magneto-hydrodynamic flows and their instabilities, *J.Fluid Mech.* **476** (2003), 389-409.
- [36] M. Ohno and T. Shirota, On the initial boundary value problem for the linearized MHD equations, *Arch. Rat. Mech. Anal.* **144** (1998), 259-299.
- [37] O.A. Oleinik and V.N. Samokhin, *Mathematical Models in Boundary Layer Theory*, Applied Mathematics and Mathematical Computation, Vol. 15, Chapman & Hall/CRC, Boca Raton, FL, 1999.
- [38] L. Onsager, Statistical hydrodynamics. *Nuovo Cimento* (9) 6, (1949). Supplemento, no. 2(Convegno Internazionale di Meccanica Statistica), 279–287.
- [39] G.A. Pavliotis and A.M. Stuart, *Multiscale methods. Averaging and homogenization*. Texts in Applied Mathematics, 53. Springer, New York, 2008.
- [40] L. Prandtl, *Verhandlungen des III. Internationalen Mathematiker-Kongresses, Heidelberg*, Teubner, Leipzig 1904, 484–491.
- [41] M. Reiner, The Deborah number, *Physics Today* **17** (1) (1964), 62.
- [42] M. Renardy, W.J. Hrusa and J.A. Nohel, *Mathematical Problems in Viscoelasticity*, Pitman Monographs and Surveys in Pure and Applied Mathematics 35, Longman 1987.

- [43] M. Renardy, Local existence of solutions of the Dirichlet initial-boundary value problem for incompressible hypoelastic materials, *SIAM J. Math. Anal.* **21** (1990), 1369-1385.
- [44] M. Renardy, The stresses of an upper convected Maxwell fluid in a Newtonian velocity field near a reentrant corner, *J. Non-Newtonian Fluid Mech.* **50** (1993), pp. 127-134
- [45] M. Renardy, A matched solution for corner flow of the upper convected Maxwell fluid, *J. Non-Newtonian Fluid Mech.* **58**(1995), 8339.
- [46] M. Renardy, High Weissenberg number boundary layers for the UCM fluid, *J. Non-Newtonian Fluid Mech.* **68** (1997), 125–132.
- [47] M. Renardy, The high Weissenberg number limit of the UCM model and the Euler equations, *J. Non-Newtonian Fluid Mech.* **69** (1997), 293-301.
- [48] M. Renardy and T.Hagen, Boundary layer analysis of the Phan-Thien Tanner and Giesekus model in high Weissenberg number flow, *J. Non-Newtonian Fluid Mech.* **73** (1997), pp. 181-189
- [49] M. Renardy, Wall boundary layers for Maxwell liquids, *Arch. Rat. Mech. Anal.* **152** (2000), pp. 93-102
- [50] M. Renardy, *Mathematical Analysis of Viscoelastic Flows*, SIAM CBMS-NSF Regional Conference Series in Applied Mathematics, 2000.
- [51] M. Renardy and R.C. Rogers, *An introduction to partial differential equations*. Second edition. Texts in Applied Mathematics, **13**. Springer-Verlag, New York, 2004.
- [52] M. Renardy, Stability of viscoelastic shear flows in the limit of high Weissenberg and Reynolds numbers, *J. Non-Newtonian Fluid Mech.* **155** (2008), 124–129.
- [53] M. Renardy, Mathematical analysis of viscoelastic fluids, in: C. Dafermos and M. Pokorný (eds.), *Handbook of Differential Equations: Evolutionary Equations 4*, Elsevier 2008, pp. 229-265
- [54] M. Renardy, The high Weissenberg number limit of the upper convected Maxwell fluid, *J. Non-Newtonian Fluid Mech.* **165** (2010), 70-74.
- [55] M. Renardy, Well-posedness of the hydrostatic MHD equations, *J. Math. Fluid Mech.*, to appear.
- [56] M. Renardy and X.J. Wang, Well-posedness of boundary layer equations for time-dependent flow of non-Newtonian fluids, submitted to *Arch. Rat. Mech. Anal.*.
- [57] M. Renardy and X.J. Wang, Boundary layers for the upper convected Maxwell fluid, submitted to *J. Non-Newtonian Fluid Mech.*.

- [58] M. Sammartino and R.E. Caflisch, Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, *Comm. Math. Phys.* **192** (1998), 433-461.
- [59] J.-C Saut, *Lectures on the mathematical theory on viscoelastic fluids*, private communication.
- [60] P. Secchi, On the equations of ideal incompressible magneto-hydrodynamics, *Rend. Sem. Mat. Univ. Padova* **90** (1993), 103-119.
- [61] P. Secchi, The initial-boundary value problem for linear symmetric hyperbolic systems with characteristic boundary of constant multiplicity, *Differential Integral Equations* **9** (1996), no. 4, 671–700.
- [62] P. Secchi, An initial boundary value problem in ideal magneto-hydrodynamics, *Nonlin. Diff. Eq. Appl.* **9** (2002) 441-458.
- [63] A. Sommerfeld, *Mechanik der Deformierbaren Medien*, 6th ed., Akademische Verlagsgesellschaft, Leipzig 1970.
- [64] R. Temam, *Navier-Stokes Equations*, 3rd ed., North Holland, Amsterdam 1984.
- [65] C. Truesdell and W. Noll, *The nonlinear field theories of mechanics*. Second edition. Springer-Verlag, Berlin, 1992.
- [66] H.B. Da Veiga, Kato's perturbation theory and well-posedness for the Euler equations in bounded domain. *Arch. Rat. Mech. Analysis* **104** (1988), 367-382.
- [67] X. Wang and M. Renardy, Well-posedness of the upper convected Maxwell fluid in the limit of infinite Weissenberg number, *Math. Meth. Appl. Sci.* **34** (2011), 125–139.
- [68] J. Wu, Viscous and inviscid magneto-hydrodynamics equations. *J. d'Analyse Math.* **73** (1997), 251-265.
- [69] J. Wu, Bounds and new approaches for the 3D MHD equations, *J. Nonlin. Sci.* **12** (2002), 395-413.
- [70] Z. Zhang and X. Liu, On the blow-up criterion of smooth solutions to the 3D ideal MHD equations, *Acta Math. Appl. Sinica* **20** (2004), 695-700.