Essays on Applied Microeconomic Theory

Hojjatallah Ghandi

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Economics

Hans Haller, Chair
Amoz Kats
Djavad Salehi-Isfahani
Nicolaus Tideman

May 1, 2009
Blacksburg, Virginia

Keywords: Price Hysteresis, Constrained Capacity, Oil Market, Redistribution, Social Groups

Essays on Applied Microeconomic Theory

Hojjatallah Ghandi

(ABSTRACT)

The first part of this dissertation investigates the possibility of an output cut by a firm as a result of an increase in demand in industries with constrained capacities. We are specially interested in the crude oil industry, although the paper has implications beyond that market. Two simple closely related models are developed. In both models a firm cuts the output at some point solely because of an increase in demand. We use this fact to explain the sharp decline of the crude oil prices in 1986. There are price and quantity hysteresis in the second model. The price hysteresis has two implications. First, the price path when the demand increases might be different from the price path when the demand decreases. This in turn implies that a temporary shock in the demand for (or supply of) crude oil can cause permanent changes in the price. We claim that the temporary changes in the supply of crude oil in 1973 resulted in the price hysteresis phenomenon described in the second model in such a way that it kept the prices high even after the return of the producers to the market.

The second part investigates the relationship between the taste for public expenditure and the size and distribution of social groups in a society. Societies with ethnic heterogeneity spend less on redistribution and welfare programs and impose lower tax rates relative to homogeneous societies. We construct a theoretical model to explain these facts. There are two social groups in the model: a minority group and a majority group. When members of one group feel empathy for each other but not for members of the other group, then taxes, and redistribution depend upon the size and distribution of those groups. At first, the equilibrium tax rate and redistribution decrease as the size of the minority group increases from zero, then eventually, the relationship between them becomes positive.

Acknowledgments

This work would not have been possible without the help from my two great advisors, Hans Haller and Mark Stegeman. I am deeply grateful to their advice and support. I would like to express my deepest gratitude to the other members of my dissertation committee Amoz Kats, Djavad Salehi-Isfahani, and Nicolaus Tideman for their support. I would also like to thank Catherine Eckel, Carl Kaiser, and Aris Spanos for their help.

I also thank the special people in my life, my love of life Zahra and my lovely son Sina who are always there for me. Foremost appreciation goes to my parents MehrAli and Fatemeh. This work is dedicated to them.

Contents

1 Introduction **1**

1.1 A Brief History of the Crude Oil Industry and Prices 1

1.2 The Determinants of Redistribution 6

2 Price Hysteresis in the Crude Oil Market **11**

2.1 Introduction 11

2.2 Model 15

2.3 Model with Price and Quantity Hysteresis 22

2.4 Implications of the Two Models in the Crude Oil Market 31

2.5 Conclusion 32

3 Social Groups and Redistribution in a Democracy **33**

3.1 Introduction 33

3.2 Model without Ethnicity 37

3.3 Model with Ethnicity 40

3.4 Example 55

3.5 Conclusion 61

4 Public Expenditure in the Presence of Social Groups in a Democracy 63

4.1 Introduction 63

4.2 Analysis 64

4.3 Conclusion 85

5 Bibliography 87

6 Appendix 1: Matlab programs 93

7 Appendix 2: Graph 102

List of Figures

2.1 Demand and capacities 23

2.2 Price hysteresis 27

2.3 Output hysteresis 28

2.4 Output total 28

7.1 t, redistribution size as a function of b, the size of the minority group 102

List of Tables

2.1 Price and Outputs 19

2.2 Price and Outputs in different cases 21

2.3 Price, outputs and leader firm's profit as demand increases 29

2.4 Price, outputs and leader firm's profit as demand decreases 29

Chapter 1

Introduction

1.1 A Brief History of the Crude Oil Industry and Prices

The use of crude oil goes back as far as history can tell. Babylonians and ancient Persians used crude oil to produce asphalt, medicine and light. [1] In China, Japan, and Rome, crude oil was used for heating and lighting. In more recent times and about one millennium ago, Persian alchemist Muhammad Zakariya Razi distilled petroleum to produce kerosene and a few other related chemicals.

The modern history of crude oil starts with the invention of the refining process in the middle of the 19th century and then digging the first oil wells in the US and Romania and building the first refineries around the world. Driven by the demand for kerosene and oil lamps, the oil industry grew very fast during the second half of the 19th century. The US production of crude oil increased by more than 2000 folds during 1860s alone. The Middle East entered the crude oil market in 1908 and after William D'Arcy discovered oil in the city of Masjid Soleiman in southwest of Iran. Many other Middle Eastern countries entered this market

[1] Encyclopedia Britannica, Eleventh Edition, article "Petroleum".

soon after.

The Middle Eastern oil rich countries entered the petroleum market with concession contracts. Per barrel loyalties and in some cases a small percentage of the profit were paid to Kings, Shahs and Amirs of the region for the, most of the time, exclusive rights of exploring, extraction and exporting of crude oil. All of these nations nationalized their oil industry later on and during the 1950s, 60s, and 70s. Meantime and in 1960, the Organization of Petroleum Exporting Countries (OPEC) was formed. The goal of OPEC was to influence prices in this market.

The price of crude oil was very stable and the changes in the price were very smooth during the 1950s, 60s and before the embargo crises. In real terms however, the prices declined slightly. On October 1973, Arab oil exporting countries of the Middle East imposed an embargo on the western countries supporting Israel during the Yom Kippur war. The price of oil jumped from $3 to more than $12 per barrel. This was the very first time that countries of the Middle East influenced prices by cutting their production. The countries that imposed the embargo increased their production to the previous levels a few months later. But the prices remained high and were relatively stable until 1979. Why did the prices remain high even after these countries came back to the market? Two models are developed in the following chapters[2] in order to answer this question. The next disruption of crude oil was caused by the Iranian revolution and the Iran-Iraq war in 1979 and 1980. The crude oil price experienced another huge jump from $12 to almost $35. The prices stayed around $30 for a few years until 1986. In 1986, and as far as the literature can explain for no apparent reason, the crude oil price almost suddenly collapsed. The two models mentioned above are intended to also find a way to explain this collapse. For the rest of the last century, oil prices showed more or less a random pattern. A short lived hike in 1990 and a decrease in 1997 are the two exceptions. From 2000 to 2008 and as a result of higher demands from many countries, oil prices showed a steady increase over time. The record prices were set in April 2008 of $147.

[2] Price Hysteresis in the Crude Oil Market

With the beginning of the financial crises in 2008, the oil prices retreated and they have lost more than 60% of their peak value so far. Oil prices have been studied extensively by economists since the 1970s. Macroeconomists are interested in finding a relationship between oil prices, GDP and inflation. The only negative supply side shocks to the aggregate supply in the US resulted from oil shocks after all. Microeconomists on the other hand, and with little success, have tried to find those reasons that drive the prices up and down.

Three main approaches have been used in the literature to analyze the changes in crude oil prices: the theory of exhaustible resources, the supply and demand method and the backward bending supply curve.

In the first framework, oil is considered to be exhaustible since it is not renewable and it is somewhat limited relative to demand. The exhaustible resources theory considers the oil reserves as assets. A profit maximizing owner of the exhaustible resources has two options. Leave the resources that he has in the ground or sell them and change them to the other forms of assets. For the sake of simplicity, let us assume that the cost of production is negligible, and the investment in the market returns a risk free constant interest rate r. If the annual percentage change in the price is lower than r, the owner will find it profitable to sell more resources and invest in the market in order to maximize his profit. Selling the resources will decrease the price. As the owner continues to sell, the current price goes down and the percentage change in the price will increase. On the other hand, if the annual percentage change in the price is higher than r, the owner will find it profitable to stop selling and the prices will go up. This argument implies that at the equilibrium, the annual percentage change in the price must be equal to r. Hence the price of the exhaustible resources must increase at the rate of r, the interest rate. More importantly, this theory explains why the price doesn't reflect the marginal cost of production. The theory of exhaustible resources has been used many times to construct models in order to predict the price trends, changes and volatilities. The forecasts, however, proved to be erroneous; see (Lynch 2002). In order to make this theory more compatible with reality, new models have been constructed by relaxing some of the assumptions. Some models suggest that the trend can be decreasing and some suggest that the trend has a U shape. Unfortunately, none of these models can successfully predict the changes and trends in the oil market. Critics of this theory, lead by Adelman, deny that the theory of exhaustible resources contains any potential in forecasting the price changes. Adelman denies the exhaustibility of the oil.[3] Hence talking about the exhaustible resources theory, in his point of view, is irrelevant. The second important method of analyzing the crude oil market is based on demand and supply analysis. This method is the most widely used method for modeling the oil market.[4] Plenty of work has been done to estimate the demand for crude oil both in the short and the long run. The focus of many of these works is finding the price and the income elasticity of demand for different countries. The price as we might expect is found to be very inelastic in the short run and more elastic but still inelastic in the long run. [5] Gately and Huntington (2002) claim that since an increase in the price may induce investment in more efficient equipments especially in the long run while the decrease in price does not reverse this process, the price elasticity of demand for crude oil when prices increase is different from the elasticity when prices decrease. Griffin and Schulman (2005) however find that the hypothesis of price symmetry cannot be rejected after controlling for technical changes. A bulk of empirical works studies the income elasticity of demand for crude oil. Based on these studies, income elasticity depends on GDP per capita and hence the country of interest. Also, demand is more income elastic in the long run. The most controversial part of the demand supply analysis is the supply side story. There are two main approaches for modeling the supply side: Economic models and geophysical models. Geophysical models are based on the ground breaking work of American geophysicist Marion King Hubbert. His peak oil theory indicates that the crude oil production will follow a bell-shaped curve. In the pre-peak era, the production increases due to the discovery of new fields and new technologies. In the postpeak era, production decreases due to resource depletion. The time of the peak depends on the total reserves. However as Lynch and Adelman claim, the total reserves are a function of technical advances,

[3] The Real Oil Problem

[4] Bacon, 1991; Dees et al., 2007.

[5] See Fattouh 2007.

cost of extraction and more importantly the crude oil prices. [6] For example, the oil sand in Canada was not considered to be part of the total oil reserves until recently. Economic models make a distinction between OPEC and non- OPEC behavior. Non-OPEC producers are believed to behave competitively. However, there is no consensus over the role of OPEC, especially in setting the prices. Some models suggest that OPEC is a cartel with members that cheat regularly. Adelman calls it a clumsy cartel, and some deny the power of OPEC in setting prices altogether. [7] . Empirical works do not favor any of these or other theories in the literature. Also, the fact that almost all OPEC members produce very close to their capacity questions the OPEC influence in recent price hypes. Although this fact and other evidence question the market power of OPEC, still without any doubt, OPEC members can not be considered as competitive firms since marginal cost of production for OPEC members is way below price. Some models have tried to go beyond classic economics model to explain OPEC behavior. Among them, the idea of backward bending supply introduced by Crémer and Salehi-Isfahani is remarkable. The idea is based on the assumptions that oil is exhaustible, its production is controlled by national governments, and it is a major source of their GNP. For these countries, keeping oil in the ground is one of the few available forms of investment. The other forms are real investment at home and investment abroad. Real investment at home is subject to diminishing returns and investment abroad is subject to political and other risks. When the prices increase, these countries will tend to invest more. One form of investment is just keeping the oil in the ground by cutting production. They conclude that for some prices the supply has a negative slope that they call a backward-bending supply curve and, hence, there are two stable equilibria in the market, one associated with a lower price and one with a higher price.

We develop two models in chapter 2 in order to explain two major phenomena in the crude oil market in the 1970s and 80s. In particular, we answer these two questions: why did the oil prices not go back to the previous levels even after Saudi Arabia and other producers lifted

[6] The Real Oil Problem.

[7] Salehi, Fattouh

the embargo in 1973? And why did the prices decline so sharply in 1986? The two models are based on reasonable assumptions. In both models a firm cuts the output at some point solely because of an increase in demand. This can explain the huge fall of the crude prices in 1986. There are price and quantity hysteresis in the second model. The price hysteresis has two implications. First, ceteris paribus, the price path when the demand increases might be different from the price path when the demand decreases. This in turn implies that a temporary shock in the demand for (or supply of) crude oil can leave permanent changes in the price. We claim that the temporary changes in the supply of crude oil in 1973 resulted in the price hysteresis phenomenon described in the second model in such a way that it kept the prices high even after the return of the producers to the market.

1.2 The Determinants of Redistribution

Redistribution is the transfer of income or wealth from some individuals, usually rich, to others that are usually poor. Hence one expects a relatively rich person to be against redistribution and a relatively poor person to be in favor of it. But the story of redistribution is much more complicated than this.

Consider a society ruled by democracy. Data shows that the distribution of income, and for that matter wealth, is not symmetric. Indeed it is skewed to the right. This means that the income of the median person in the income distribution is less than the average income. But in a democracy, where a majority of votes sets the rules, the median voter is in fact the decisive voter. One might expect the median voter who is a rational utility maximizer and a fully informed individual to vote for a complete redistribution of income. However, this is not in line with our observations in countries like the United States or Germany. In other words, although majorities of people in western democracies have less than average income, they do not support a complete redistribution of income.

There are several theories explaining why a complete redistribution fails to occur in democracies. One of the most remarkable theories is the work of Meltzer and Richards 1981, henceforth MR, in their paper "A rational theory of the size of government". They assume that rational voters, including the median voter, choose between leisure and consumption and that they consider this choice when they are voting for redistribution. Under certain conditions like the normality of leisure, lump sum transfers, and a linear income tax, the decisive voter will vote for less than a complete redistribution. A complete redistribution does not maximize the median voter's utility and indeed nobody works when there is a complete redistribution. [8]

In one sense, MR is a static model. The relative relations of people in the distribution of income do not change. In other words, a person's income rank in the society stays the same. MR assume that the productivity of each individual is exogenous and predetermined. Poor remain poor and rich remain rich; there is no social mobility. Benabou and Ok (2001) introduced a model that allows for social mobility. In their model, the relative rank of individuals in the income ladder can change. The poor in the society have the potential to change to an individual that is relatively rich in the future. This is called POUM (Prospect Of Upward Mobility). In this case, individuals care about current as well as future income. There are three key assumptions in their paper; first, redistribution policies are long-lasting; second, workers with lower than average incomes are not too risk averse; third, there is a large enough number of poor workers with an optimistic view of their future, as they expect to go from poorer than the average to richer than average. With these three assumptions, it is possible to have poorer than average individuals that reject the redistribution of income because of their belief in the prospect for upward mobility. Relative to the basic MR model, a median voter in the model based on POUM supports a lesser amount of redistribution.

POUM can be used to explain two things:[9] first, individuals and especially the median voter, vote on redistribution based on current as well as future income; second, POUM can explain why a relatively poor and rational utility maximizing median voter is in favor of

[8]Our model in the following chapters is an expansion of this model with more than one social group.
[9]Alesina, Preferences for Redistribution

limited redistribution. Along with the POUM hypothesis, there are other explanations for the support of less redistribution; such as optimism about upward mobility. Over-optimism about upward mobility can be a result of social indoctrinations designed to prevent the adoption of excessive redistribution policies [10] or a self-induced idea to convince yourself or others to work hard. [11]

POUM explains why a relatively poor and rational utility maximizing median voter might be in favor of limited redistribution. Is it possible for a relatively rich rational utility maximizing individual, with an income higher than the average, to vote for redistribution? The answer can be yes and for the following reasons[12]. First, there are externalities in education. An increase in the average level of education can lead to better aggregate productivity, new inventions and other positive externalities. If more equality leads to better education, then even a self-interested rich person might favor more redistribution.[13] Second, inequality creates some negative externalities such as crime. If the cost of increasing security is higher than extra redistribution that results in the same desired level of security, then the rich might favor more redistribution. Third, higher redistribution may lead to less incentive for work. This in turn creates negative externalities. For this reason, the rich may favor a lower level of redistribution. A trade-off between the third reason and the other two will lead the utility maximizing rich to an optimum level of favorite redistribution.

Corneo and Grüner (2002) give one interesting explanation of limits to redistribution in their paper "Social Limits to Redistribution"; They believe that when economic inequality has some informational value for social decisions, people tend to place a limit on redistribution. In their point of view, there are sources of utility that cannot be allocated by the market; individuals assign values to conversation, dinners and parties, playing, and admiration by others. In all of these activities, the most important issue is not the existence of conversations or parties or playing, but with whom the individual is conversing, socializing, and

[10] See Alesina Glaeser 2004

[11] See Benabou and Tirole (2006).

[12] See Alesina, Preferences for Redistribution

[13] See Perotti (1999).

playing. In a situation such as this, the relative position in the income distribution may create some information. In other words, economic inequality allows some separation across "different types." Greater redistribution causes the position in the income distribution to be less informative. However, the informational value of this separation is positive for some and negative for others. If the value of this information is high for individuals close to the median voter, as Corneo and Grüner think it is, then the median voter will vote for less redistribution.

The determinants of redistribution that are discussed so far are all in the context of positive economics. There are other determinants, in the normative economic context, that are worth mentioning: ideology is one of them. For example, a libertarian, who is relatively poor, may support a lower level of redistribution to satisfy his ideology. Sticking to his belief about redistribution might make him happier than a bit more consumption. A relatively rich liberal, may support a higher level of redistribution, even if it decreases his consumption to support his ideology. Fairness is another important issue; an individual's perspective of causes of pre-tax distribution of income may lead to different preferences for redistribution. Here, income that is earned through effort and hard work is considered to be fair, and income that is earned by pure luck is considered to be unfair. If people think that the distributive outcomes of the market economy are fair, then they will favor less redistribution. [14] There are several other theories that explain why a median voter is not in favor of a complete redistribution. One explanation is the costs of taxation; people will vote for less redistribution when these costs are increasing. Lobbying activities of high-income groups constitute another important factor. Moreover, people with higher incomes show more participation in political processes, including voting. Higher rates of participation in the political process by those with higher incomes result in the shift of decisive voter status to individuals with higher incomes.

Another important determinant of redistribution size is interpersonal preferences. [15] One will increase his support for redistribution and welfare programs if the local recipients are

[14] See Alesina and Angeletos (2005).

[15] "Group Loyalty and the Taste for Redistribution" by Luttmer.

from his own ethnic group. There are differences between the preferences for redistribution and the size of welfare programs in US and Europe. In the US, the taste for redistribution differs among different states. Many studies have tried to explain these differences. [16]

Alesina et al. compare the US and the EU welfare expenditure in their paper "Inequality and Happiness: Are Europeans and Americans Different?" and find that the taste for redistribution in the US is different from that in the EU. Although wealthy Europeans, like wealthy Americans, do not oppose inequality, the poor in Europe are unhappier about inequality compared to the poor in America. In the US, the poor believe in the probability of upward mobility and hence they are less concerned about inequality. When the poor are more concerned about inequality, they become favorable toward government intervention in redistributing the wealth and income. Different perceptions about fairness can lead to the same result.

We develop two models in chapter 3 and 4 in order to construct a theoretical relationship between the change in size and distribution of social groups and welfare expenditures in a democratic community. By investigating the relationship between the size of a minority group and the tax rate, we find out that the outcomes do not necessarily change in favor of a group as it grows in size. We find a new method of identifying the median voter when the preferred tax rate and its distribution are endogenous. Under certain conditions, this distribution is a well-behaved function and hence the equilibrium tax rate exists and is unique. We show a method of calculating this equilibrium and at the end we prove that, in a democracy where the decisive voter is the median voter, the equilibrium tax rate change as the size of one group starts to increase. At first, this equilibrium rate will decrease because the taste of the median voter, who is a member of the other group, will change. Then it increases as the smaller group grows because the situation of the median voter changes.

[16] Look at Alesina 2004 , Luttmer.

Chapter 2

Price Hysteresis in the Crude Oil Market

2.1 Introduction

For many decades, economists have tried, with little success, to understand the reasons behind the behavior of the crude oil prices.[1] In most of their history, crude oil prices have followed almost a random path. There are few exceptions. Among the most important exceptions are: the sharp increases of the crude oil prices in 1973, 1980, the 2000-2007 period, and the sharp declines of 1986. We more or less know the reasons behind the sharp increases in the price of crude oil in 1973, 1980 and most recently the 2000-2007 period. The first two are associated with the sharp decline in supply as a result of Arab oil embargo following the Israeli-Arab war of 1973 and the Iranian revolution, respectively. The latest hike in prices is due to the changes in demand, mostly from fast growing economies like China and India. On the other hand, there are no clear explanations for some other important changes in crude oil prices. For example, why didn't the oil prices go back to the previous levels even after

[1] Krugman, NewYork times, Op-Ed article

Saudi Arabia and other producers lifted the embargo? And why did the prices decline so sharply in 1986? This paper is mainly intended to answer these two questions. However, the implications of the models that will be used go beyond crude oil market.

Three main approaches have been used in the literature to analyze the changes in crude oil prices: the theory of exhaustible resources, the supply and demand method and the backward bending supply curve.

In the first framework, oil is considered to be exhaustible since it is not renewable and it is somewhat limited relative to demand. The exhaustible resources theory considers the oil reserves as assets. A profit maximizing owner of the exhaustible resources has two options. Leave the resources that he has in the ground or sell them and change them to the other forms of assets. For the sake of simplicity, let us assume that the cost of production is negligible, and the investment in the market returns a risk free constant interest rate r. If the annual percentage change in the price is lower than r, the owner will find it profitable to sell more resources and invest in the market in order to maximize his profit. Selling the resources will decrease the price. As the owner continues to sell, the current price goes down and the percentage change in the price will increase. On the other hand, if the annual percentage change in the price is higher than r, the owner will find it profitable to stop selling and the prices will go up. This argument implies that at the equilibrium, the annual percentage change in the price must be equal to r. Hence the price of the exhaustible resources must increase at the rate of r, the interest rate. More importantly, this theory explains why the price doesn't reflect the marginal cost of production.

The theory of exhaustible resources has been used many times to construct models in order to predict the price trends, changes and volatilities. The forecasts, however, proved to be erroneous; See (Lynch 2002). In order to make this theory more compatible with reality, new models have been constructed by releasing some of the assumptions. Some models suggest that the trend can be decreasing and some suggest that the trend has a U shape. Unfortunately, none of these models can successfully predict the changes and trends in the oil market. Critics of this theory, lead by Adelman, deny that the theory of exhaustible resources contains any potential in forecasting the price changes. Adelman denies the exhaustibility of the oil. Hence talking about the exhaustible resources theory, in his point of view, is irrelevant.

The second important method of analyzing the crude oil market is based on demand and supply analysis. This method is the most widely used method for modeling the oil market. (Bacon, 1991; Dees et al., 2007). Plenty of work has been done to estimate the demand for crude oil both in the short and the long run. The focus of many of these works is finding the price and the income elasticity of demand for different countries. The price as we might expect is found to be very inelastic in the short run and more elastic but still inelastic in the long run. See (Fattouh 2007). Gately and Huntington(2002) claim that since an increase in the price may induce investment in more efficient equipments especially in long run while the decrease in price does not reverse this process, the price elasticity of demand for crude oil when the prices increase is different from the elasticity when the prices decrease. Griffin and Schulman (2005) however find that the hypothesis of price symmetry cannot be rejected after controlling for technical changes.

A bulk of empirical works studies the income elasticity of demand for crude oil. Based on these studies, income elasticity depends on GDP per capita and hence the country of interest. Also, demand is more income elastic in the long run.

The most controversial part of the demand supply analysis is the supply side story. There are two main approaches for modeling the supply side: Economic-based models and geophysical models. Geophysical models are based on the ground breaking work of American geophysicist Marion King Hubbert. His peak oil theory indicates that the crude oil production will follow a bell-shaped curve. In the pre-peak era, the production increases due to the discovery of the new fields and new technologies. In the post-peak era, production decreases due to resource depletion. The time of the peak depends on the total reserves. However as Lynch and Adelman claim, the total reserves is a function of technical advances, cost of extraction and more importantly the crude oil prices (Adelman). For example, the oil sand in Canada was not considered to be a part of the total oil reserves until recently.

Economic-based models make a distinction between OPEC and non-OPEC behavior. Non-OPEC producers are believed to behave competitively. However, there is no consensus over the role of OPEC in especially setting the prices. Some models suggest that OPEC is a cartel with members that cheat regularly, Adelman calls it a clumsy cartel, and some deny the power of OPEC in setting prices altogether (Salehi, Fattouh). Empirical works do not favor any of these or other theories in the literature. Also, the fact that almost all OPEC members produce very close to their capacity questions the OPEC influence in recent price hypes. Although this fact and other evidences question the market power of OPEC, still without any doubt, since marginal cost of production for OPEC members is way below price, OPEC members can not be considered as competitive firms.

Some models have tried to go beyond classic economics model to explain the OPEC behavior. Among them, the idea of back-ward bending supply introduced by Crémer and Salehi-Isfahani is remarkable. The idea is based on the assumptions that oil is exhaustible, its production is controlled by national governments, and it is a major source of their GNP. For these countries, keeping oil in the ground is one of the few available forms of investments. The other forms are real investment at home and investment abroad. Real investment at home is subject to diminishing returns and investment abroad is subject to political and other risks. When the prices increase, these countries will tend to invest more. One form of investment is just keeping the oil in the ground by cutting production. They conclude that for some prices the supply has a negative slope that they call a backward-bending supply curve and, hence, there are two stable equilibria in the market, one associated with a lower price and one with a higher price.

2.2 Model

The model that we will use is very simple. But it will serve to illustrate what we are considering. Let every firm in the industry be selling identical goods, which can be crude oil or gas. There are many small firms and one big firm, henceforth the market leader.[2] The industry minus the leader firm is a competitive market. The term competitive firms will be used to refer to these small firms for the rest of this paper. There is no new entry to the market. Furthermore, assume that marginal cost of production is c for competitive firms and zero for the leader firm. The competitive firms will produce their product if the price is higher than the marginal cost of production, and since they don't have any market power i.e. they cannot change the price, they will produce at the full capacity if the price of the good is higher than the marginal cost. The leader has an unlimited capacity. The rest of the industry has a capacity equal to s, meaning that the marginal cost is c when $q < s$ and infinity when $q > s$, where q is the total output of the competitive firms. Obviously, the leader has some form of market power. The leader will set the price such that it maximizes the profit. There is no transaction cost and many buyers in the market have uniform access to the firms. We assume full information. In order to make the model simple, let demand be a linear function of price with the following inverse demand function

$$p = a - Q \tag{2.1}$$

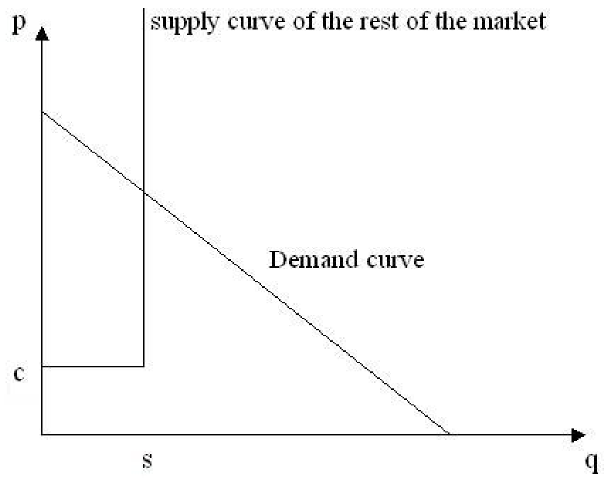
where p is the market price and Q is the total production by all firms including the leader. The demand curve and the supply of the industry minus that of the leader firm are illustrated in Figure 1.[3]

In order to find the price and total output we consider three cases. In the first case, suppose that the ex ante price[4] is less than the marginal cost of production of the competitive firms

[2] The term "market leader" is chosen only to make a distinction between the firm that has unlimited capacity and other firms.

[3] This industry resembles the world crude oil industry with Saudi Arabia (OPEC) being the leader firm.

[4] We will find a way to determine price later in this paper.



that is c. In this case, only the leader firm will produce. The competitive firms will not produce any output since the profit would be negative if they produced and zero if they didn't. The leader firm in this case will face the demand function (2.1) and can act like a monopoly meaning that it sets the price at $a/2$ if[5]

$$a/2 < c. \tag{2.2}$$

Hence the profit, quantity and price are

$$\pi_m = a^2/4, \qquad Q = q_L = a/2, \qquad p = a/2. \tag{2.3}$$

where q_L is the leader's level of output and π_m is the leader firm's profit when it acts like a monopoly.

When $a/2$ is higher than c, then two cases are possible: the leader doesn't let the competitive firms enter the market by charging a price less than their marginal cost of production, or it pays off for the leader to let the others enter the market. In the former case, the leader firm sets the price just under c, let say $c - \epsilon$, to prevent other firms to enter the market. In this

[5] A price higher than c will be discussed later

case, simple calculation shows that price, quantity and profit are [6]

$$p = c, \qquad Q = q_L = a - c, \qquad \pi_U = pq_L = c(a - c) \tag{2.4}$$

where π_U is the leader firm's profit when it undercuts the other firm's prices.

In the last case, it benefits the leader firm to let the other firms enter the market. All the other firms will produce at the full capacity meaning that the demand for the leader firm is equal to the market demand at any price minus the part of demand that is satisfied by the other firms, i.e. $q_L + s = a - p$ where q_L is the demand for the leader's product. In other words, the leader firm residual faces a demand that is

$$q_L = a - s - p \qquad \text{if} \quad a/2 > c \quad \text{and} \qquad p > c \tag{2.5}$$

The leader firm's total revenue and profit is $\pi_L = (a - s - q_L)q_L$, the first order condition implies

$$q_L = p = (a - s)/2, \tag{2.6}$$

and the second order condition holds. Furthermore,

$$Q = q_L + s = (a + s)/2. \tag{2.7}$$

Notice that in this case the price must be higher than the price in the second case in which the leader firm undercuts the other firms' prices and eventually sets the price at $c-\epsilon$. Hence, $(a - s)/2 \geq c$ implying

$$a \geq 2c + s. \tag{2.8}$$

The leader firm's total revenue or profit is

$$\pi_L = q_L p = (a - s)^2/4. \tag{2.9}$$

[6]To be precise, $p = c - \epsilon$, $Q = q_L = a - c + \epsilon$, $\pi_U = pq_L = (c - \epsilon)(a - c + \epsilon)$. However, I will not use ϵ as far as it is possible to avoid unnecessary complications.

The next step is finding the price and the quantity that maximize the profit of the leader firm. As discussed before, if $a/2 < c$, then we are in the first case and the p, Q and q_L are equal to $a/2$ and that's the equilibrium in the market. If $a/2 > c$ but condition (2.8) doesn't hold, i.e. $a < 2c + s$, then the leader firm will set the price right below c and $p = c - \epsilon$, $Q = q_L = a - c + \epsilon$ are the equilibrium price and quantity.[7] If condition (2.8) holds then whether the leader firm will let the others enter the market by setting a price higher than their marginal cost of production c or doesn't let them enter the market by setting a price lower than their marginal cost of production c depends on the profit that the firm will make in each cases. The profits in the two cases must be compared to find the condition for which the firm will change its strategy from undercutting (that is the price right below c) to letting the others enter the market.

Proposition 1 *The leader firm will change its strategy from undercutting to letting the others to enter the market when* $a = 2c + s + 2\sqrt{cs}$*.*

Proof. Consider

$$d\pi = \pi_L - \pi_U = (a - s)^2/4 - c(a - c) \tag{2.10}$$

that is the difference between the profit in the two cases. Then $d\pi(a) = \pi_L(a) - \pi_U(a) = (a-s)^2/4 - c(a-c) = a^2/4 - (c+s/2)a + (s^2/4 + c^2)$. The expression is purposefully ordered in terms of a. When $d\pi(a) = 0$, the firm is indifferent between the two strategies. By solving $d\pi(a) = 0$, we have

$$a = 2c + s \pm 2\sqrt{cs}. \tag{2.11}$$

Since c and s are positive numbers, the equation always has two answers. The smaller answer, $a = 2c + s - 2\sqrt{cs}$, contradicts (2.8) so it cannot be an answer. Therefore, $a = 2c + s + 2\sqrt{cs}$ is the only acceptable answer to our problem. Notice that $d\pi(a)$ is a convex function of a, hence $d\pi(a) > 0$ for all $a > 2c + s + 2\sqrt{cs}$ implying that $\pi_L > \pi_U$ for this range of a. This means that it will pay the leader firm to change its strategy from undercutting to letting the others enter the market when $a = 2c + s + 2\sqrt{cs}$. ■

[7] In fact $p = c - \epsilon$ and $Q = q_L = a - c + \epsilon$, however we can neglect ϵ without losing any generality.

Table 2.1: Price and Outputs

	case 1	case 2	case 3
a	$a < 2c$	$2c < a < 2c + s + 2\sqrt{cs}$	$2c + s + 2\sqrt{cs} < a$
Market structure	Monopoly	Bertrand	Price Leadership
p	$a/2$	c	$(a - s)/2$
q_L	$a/2$	$a - c$	$(a - s)/2$
Q	$a/2$	$a - c$	$(a + s)/2$
profit	$a^2/4$	$c(a - c)$	$(a - s)^2/4$

Based on this proposition and previous discussions, the equilibrium price, quantities and profit can be calculated. For all a such that $2c < a < 2c + s + 2\sqrt{cs}$, the leader firm will set the price just below the marginal cost of the production c. The leader firm maximizes its profit by undercutting the price of the other firms. The total production of the industry is equal to $a - c$. The game is similar to a Bertrand game. For all $a > 2c + s + 2\sqrt{cs}$ the firm will set the price higher than c, in fact the equilibrium price and quantities as shown above are $p = q_L = (a - s)/2$ and $Q = (a + s)/2$.

Table 2.1 summarizes all of the three cases that we have discussed.

I intend to show that it is possible in some markets to have situations under which when demand increases, at least one firm has incentives to reduce production in order to make more profit. Furthermore, as a result of the production cuts by this firm, the total production in the industry may decrease. In other words, it is possible to see a reduction in the quantity supplied when the demand increases. The following proposition investigates this possibility in our model.

Proposition 2 *If demand increases so that the leader firm changes the strategy from Bertrand, i.e. case 2, to Price Leadership, i.e. case 3, then the price will jump up and the leader firm's production as well as the total production in the market will decrease.*

Proof. As mentioned above and summarized in table (2.1), when $2c + s + 2\sqrt{cs} < a$, the leader firm plays its Bertrand strategy, and sets the price just below the marginal cost of the other firms. When $a = 2c + s + 2\sqrt{cs}$, the firm is indifferent between its Bertrand strategy and setting a price like a leader: in both cases the firm makes the same profit. As it is shown in Table (2.2), the firm chooses the Bertrand strategy $p = c$, if it chooses the price leadership $p = (a-s)/2$, substituting $a = 2c+s+2\sqrt{cs}$ in p and by simplifying yields $p = c+\sqrt{cs}$ which is greater than the price in Bertrand case. When the firm switches the strategy, its output will decrease from $q_L = a - c$ to $q_L = (a - s)/2$. Substituting $a = 2c + s + 2\sqrt{cs}$ in q_L and simplifying $q_L = a-c = c+s+2\sqrt{cs}$ for the Bertrand strategy and $q_L = (a-s)/2 = c+\sqrt{cs}$ for the price leadership strategy, shows that the leader firm's output will decrease by $s+\sqrt{cs}$. The other firms will supply the market with output equal to s, implying that the total output will decrease by $\sqrt{cs}$. Table (**??**) details these and other changes. ■

Some economists argue, and rightly so, that OPEC sets the quantity and not the price in order to maximize its profit. The following proposition indicates that the results of the previous model remain the same if the leader firm changes its strategy from price setting to quantity setting.

Proposition 3 *In the previous model, the price and quantity setting strategies by the leader*

Table 2.2: Price and Outputs in different cases

	Case 1 to 2		Case 2 to 3	
a	$a = 2c$		$a = 2c + s + 2\sqrt{cs}$	
	Case 1	Case 2	Case 2	Case 3
Market structure	Monopoly	Bertrand	Bertrand	Price Leadership
p	$a/2 = c$	c	c	$(a-s)/2 = c + \sqrt{cs}$
q_L	$a/2 = c$	$a - c = c$	$a - c = c + s + 2\sqrt{cs}$	$(a-s)/2 = c + \sqrt{cs}$
Q	$a/2 = c$	$a - c = c$	$a - c = c + s + 2\sqrt{cs}$	$(a+s)/2 = c + s + \sqrt{cs}$
Profit	$a^2/4 = c^2$	$c(a-c) = c^2$	$c(a-c) = c(c + s + 2\sqrt{cs})$	$(a-s)^2/4 = c(c + s + 2\sqrt{cs})$

firm lead to the same result.

Proof. Consider case 1 and case 3 in the previous tables. The leader firm is acting like a monopoly and a price leader, respectively. In both cases, the quantity of production and the price are related by the demand or residual demand curves. Hence the relationship between the price and quantity is a linear relationship. The leader firm's profit is a quadratic and concave function in price and quantity. The maximum of this function does not change with the change in the endogenous variable. Hence price and quantity setting leads to the same results. In case 2, there is no linear relationship between price and quantity anymore. Suppose that the leader firm sets the quantity instead of price. Simple argument and calculation show that $Q = q_L = a-c+\epsilon$ will lead to a maximum profit. $Q = q_L = a-c+\epsilon$ corresponds to $p = c - \epsilon$, that is exactly the same price that the leader firm would charge if it set price instead of quantity. ■

2.3 Model with Price and Quantity Hysteresis

In the previous section it is assumed that all firms have full information about the demand curve. However, in the oil and many other markets, firms have limited information about the market demand. So how does a firm maximize its profit? In this section, let assume that at any moment the firm increases (or decreases) the price slightly. If the profit that it makes increases (or decreases) then the firm will increase the price, otherwise the firm will stick to the previous price.[8] In some markets, it doesn't matter whether firms know or do not know the demand curve. An example can demonstrate the mechanism. Consider a monopoly market structure with a market demand equal to $Q = a - p$. If the monopoly knew what the demand is, it would set the price at $p = a/2$ immediately. If the monopoly does not know the demand and maximizes its profit by changing the price slightly each time, then if the price is less than $a/2$, the monopoly will make more profit each time it increases the price slightly until the price reaches $a/2$ and if the price is more than $a/2$, the monopoly will earn more profit each time it decreases the price slightly until the price reaches $a/2$.

Returning to the model from the previous section, suppose that the leader firm does not know what the demand is.[9] In every moment, the leader firm increases the price by ϵ and compares the profits. If $\pi_L(p + \epsilon)$ is larger than $\pi_L(p)$ then it sets the price at $p + \epsilon$, otherwise it will decrease the price by ϵ, compare the profits and set the price at the point that maximizes the profit. Furthermore, let the leader firm have a maximum capacity s_L, meaning that its marginal cost is zero up to this maximum capacity and then it is infinity. The assumptions are the same as before for the rest of the firms, demand and buyers. Figure 2.1 details these assumptions. We make two more assumptions to avoid unnecessary complications. The two

[8] Let's skip the discussion about the initial price.

[9] The other firms' information about the demand does not change the equilibrium in our model.

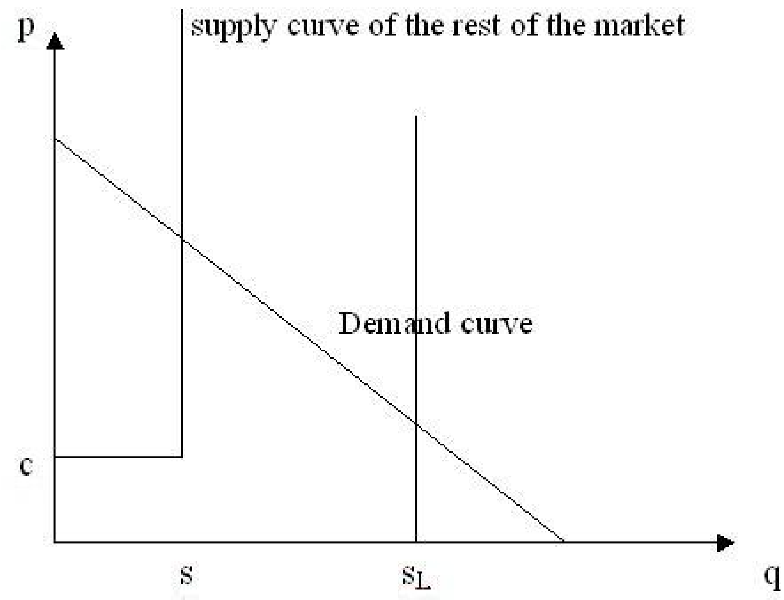


Figure 2.1: Demand and capacities

assumptions are[10]

$$s > c \qquad \text{and} \qquad s_L > 2c. \tag{2.12}$$

Similar to the previous section, several cases are possible and as we will see, the equilibrium price and quantity when demand goes up might be different from the equilibrium when demand goes down. Starting from $a = 0$, let the demand increase. At first a is less than $2c$. In this case, if the price is less than $a/2$, the firm will earn more profit every time it increases the price by ϵ. If the price is higher than $a/2$, the firm will earn more profit every time it decreases the price by ϵ. Therefore, $p = a/2$ is the equilibrium. $p = a/2$ and $a < 2c$ implies $p < c$, where c is the marginal cost of the other firms, and hence none of them will operate. As before, $\pi_m = a^2/4$ and $Q = q_L = a/2$. When $a = 2c$, then if the firm sets the price at c,

[10]Assumption $s_L > 2c$ assures us that the leader firm will not reach its full capacity before letting the others in. The case for which the leader firm reaches its full capacity before letting the others in is not that interesting.

the other firms will enter the market. The profit that the firm makes is a random number between zero and c^2. This is why the leader firm will find it profitable to set the price just below the marginal cost of the other firms in the market, that is equal to $c - \epsilon$. This will prevent the other firms from entering the market and at the same time it will make a profit equal to c^2 minus let's say ϵ.[11] For $a > 2c$, whenever the firm increases the price by an ϵ, the other firms will enter the market. As table 2.4 shows, several cases are possible for the different rang of a. For $2c < a < s + s_L + c$, the leader firm will charge a price $c - \epsilon$ and it will not find it profitable to increase the price by ϵ because every time it increases the price, the other firms will enter the market and hence the demand for the leader firm's output will decrease by s while the price increases by ϵ causing $\pi_L = pq_L$ to decrease.[12] Charging a price bellow c by the leader firm doesn't mean that the rest of the firms will not enter the market when $2c < a < s+s_L+c$. In fact in some cases the other firms will enter the market, in which case there is not a unique price anymore.[13] The reason is as follows. For $2c < a < c + s_L$, the price is $c-\epsilon$, the only firm in the market is the leader firm and its output is $a-c$. When $a = c + s_L$, the leader firm produces at the full capacity s_L. For $c + s_L < a < s + s_L + c$, the leader firm will set its price at $c-\epsilon$ and produce at the full capacity. However its production cannot meat the market demand. Some, but not all other firms, will enter the market and their output will be equal to $a-s_L-c$. [14] The leader firm will not increase the price to c, since if it does that then more of the other firms will enter the market and hence the leader firm's profit will decrease.[15] At $a = s + s_L + c$, all firms in the market will produce at full capacity.

[11]The profit is $(c + \epsilon)(c - \epsilon)$ to be exact.

[12]The change in demand can be less than s in some cases. This will be discussed in the following paragraphs.

[13]The leader firm will charges a price $c - \epsilon$ and the small firms charge a price equal to c.

[14]The price is c, hence the total demand is $a-c$, the leader firm's output is s_L that is less than the market demand at $p = c-\epsilon$, some of the other firms will enter the market, they will charge a price equal to c that is an ϵ higher than the price that the leader firm is charging and they will produce such that the total output becomes equal to the quantity demanded that is $a - c$. Hence the other firm's output is equal to $a - s_L - c$.

[15]The output of the other firms will be a random number between $a - s_L - c$ and s and the output of the leader firm will decrease accordingly. Its output is a random number between $a - s - c$ and s_L. The profit that the leader firm makes is a random number between $c(a - s - c)$ and cs_L with an expected value less than cs_L that is the profit that the leader firm makes if it set the price at $c - \epsilon$.

The leader firm faces a demand equal to $q = a - s - p = s_L + c - p$. The midpoint of this demand curve is $p_{mid} = q_{mid} = (s_L + c)/2$. From assumption (2.12), we know that $s_L > 2c$, implying that $c < (s_L + c)/2$. This means that $p = c$ is in the inelastic part of the demand curve, hence the leader firm will make a little bit more profit every time it increases the price by ϵ and it will do so until the price and its output are $p_{mid} = q_{mid} = (s_L + c)/2$. From assumption (2.12) we know that $s_L > 2c$, implying that $(s_L + c)/2 < s_L$. This means that the firm will decrease its output as soon as demand increases to $q = s + s_L + c - p$. The total output will shrink as a result. From this point and up to the point where $a \leq s + 2s_L$, the market structure is similar to a price leadership market structure. Since the market demand is $q = a - p$ and the rest of the market produce at the full capacity that is s, the leader firm faces a demand $q = a - s - p$. Hence it will set the price and quantity at $p = q = (a - s)/2$. When $a = s + 2s_L$, and since $p = q_L = (a - s)/2 = s_L$, the leader firm output will be again equal to s_L. The interesting point here is that, two different prices are possible for the same level of production by firms.[16] Indeed there is no one by one relationship between price and quantity supplied for $(s_L + c)/2 + s \leq Q$ while a is increasing. For $a > s + 2s_L$, all firms are already producing at the full capacity. Hence the output will not change. The price however will increase at a faster pace and $p = a - s - s_L$.[17]

Now let us investigate the equilibrium price and outputs when the demand is decreasing. These equilibra are different in some cases from when the demand is increasing. Notice that the result of the following discussion is summarized in table 2.4. Starting with $a \geq s + 2s_L$, and like before all firms are producing at the full capacity and $p = a - s - s_L$. As a is decreasing and while $s + 2c < a < s + 2s_L$ the market structure will be like a price leadership. It is very important to notice that the range of a for which the market structure is like a price leadership market structure is different when demand is constantly increasing from when the demand is constantly decreasing.[18] The leader firm faces a demand $q = a - s - p$ and it will

[16]Notice that the market demands are not the same in these two cases.

[17]Since $p = a - Q$ and $Q = s + s_L$.

[18]We don't find it necessary to discuss the situations in which a is both decreasing and increasing in the ranges that are discussed here.

set the price at $p = q_L = (a - s)/2$ until a approaches $a = s + 2c$. At this point if the firm charges the price $p = (a - s)/2 = c$, its profit is c^2.[19] However, when the firm decreases the price by ϵ, i.e. when $p = c - \epsilon$, the other firms will exit the market resulting in an increase in the demand for the products of the leader firm by s. Thus its profit will be $c(s + c)$ minus let say an ϵ [20] that is obviously much higher than when it charges $p = (a - s)/2$. Hence the leader firm will decrease the price by an ϵ.[21] For $2c < a < s + 2c$ like before the firm will set the price at $p = c - \epsilon$, and for $a < 2c$ the firm again set the price as if it is monopoly. Figure 2.2 shows the changes in the price vs. the changes in demand. As it is shown in the figure, there is a price hysteresis in this market. The price hysteresis occurs when $s + 2c < a < s + c + s_L$.

The changes in the output of the leader firm are depicted in Figure 2.3. The hysteresis effect

[19] $p = q_L = c$

[20] To be more precise, its profit is $(c - \epsilon)(s + c + \epsilon) = c(s + c) - s\epsilon - \epsilon^2$.

[21] To be more precise, and from a theoretical point of view, the firm will cut the price to $p = c - \epsilon$ when $a = s + 2c + 2\epsilon$. The reason is as follow: when $a = s + 2c + 2\epsilon$ the firm can charge the price $p = (a - s)/2 + \epsilon = c + \epsilon$. Remember that all other firms are in the market at this price. In the next moment, the firm change the price by an ϵ and compares the profits that it makes at each price level. If the firm sets the price at $p = c + 2\epsilon$, it obviously will lose some profit since $p = c + 2\epsilon$ is located in the elastic part of the demand the firm is facing. However if the firm decreases the price by an ϵ, i.e. when the price that it is charging is $p = c$, then the quantity demanded for its product is a random number between c and $c + s$ (more precisely $c + 2\epsilon$ and $c + s + 2\epsilon$). The reason is obvious, the buyers do not see any difference between the firms, the total quantity demanded is $c + s$ and they randomly choose the firms in their transactions. Hence the profit that the leader firm makes is a random number between c^2 and $c(c + s)$ (more precisely $(c + 2\epsilon)^2$ and $c(c + s + 2\epsilon)$) with an expected value that is higher than the profit when it set the price at $p = c + \epsilon$ that is $(c + \epsilon)(c + 2\epsilon)$. Hence it set the price at $p = c$. In the next moment, the firm will again change the price by an ϵ and again compares the profits that it makes at each price level. An increase in the price decreases the profit for the same reasons that we discussed here. However if the firm sets the price at $p = c - \epsilon$, all other firms exit the market. The quantity demanded for the products of the leader firm will increase to $s + c + 3\epsilon$ and the profit that it will make is $(c - \epsilon)(s + c + 3\epsilon)$ that is obviously higher than the expected value of the profit that it makes when it charges $p = c - \epsilon$. Hence the firm will set the price at $p = c - \epsilon$.

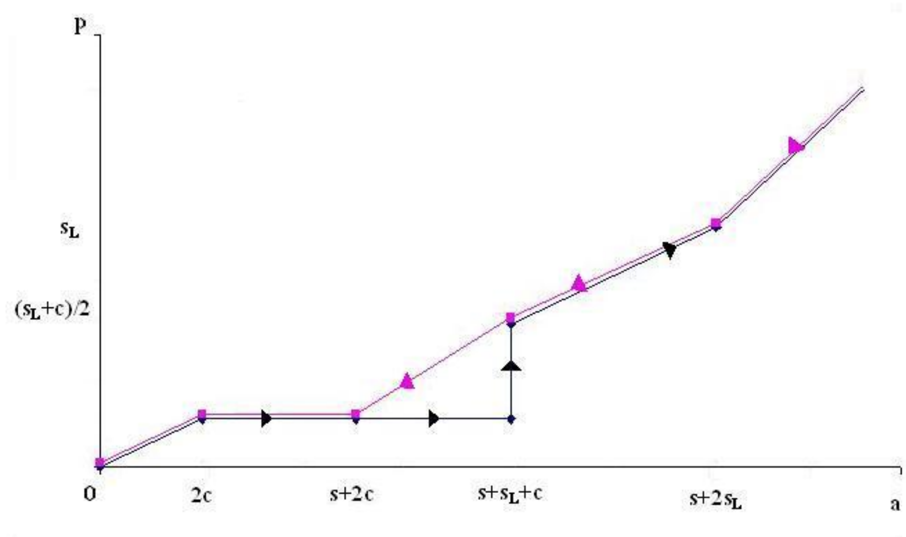


Figure 2.2: Price hysteresis

can be seen in this figure too. More importantly, as demand increases at around $a = s + s_L + c$ the leader firm decreases its output suddenly and sharply. The decrease in the leader firm's output causes the total output in the market to fall. The fall in the total output is depicted in figure 2.4.

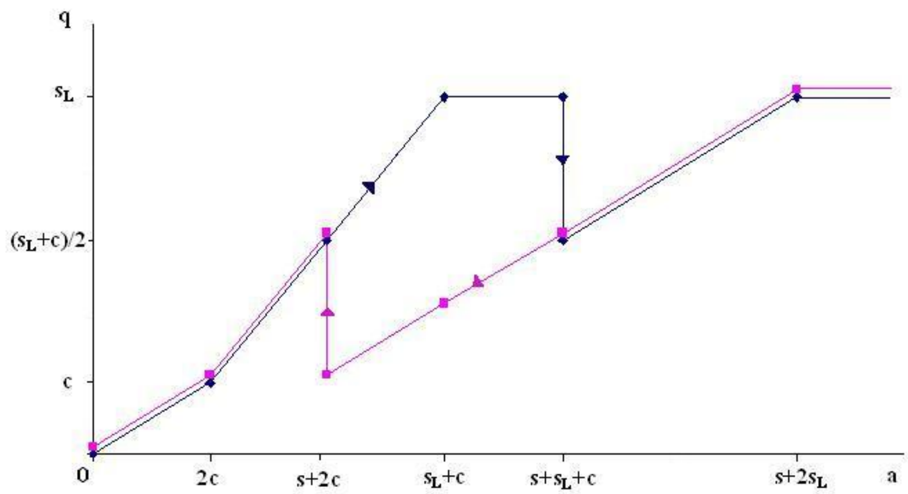


Figure 2.3: Output hysteresis

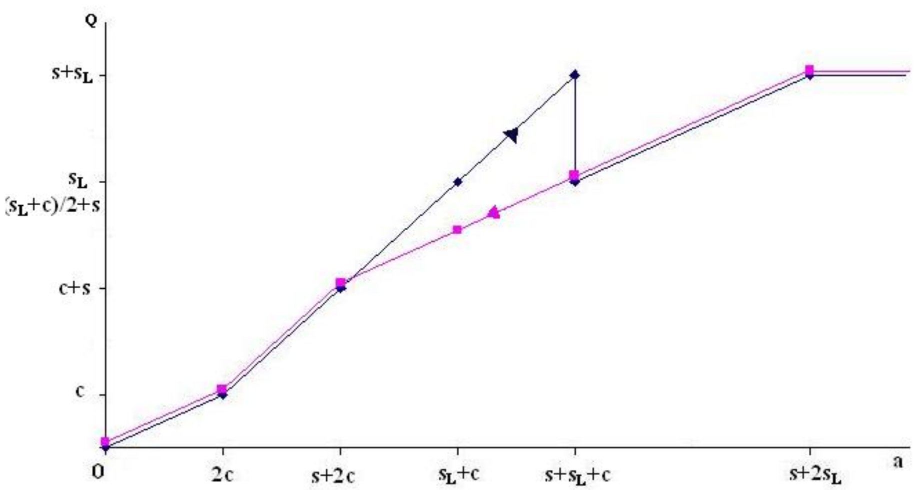


Figure 2.4: Output total

Table 2.3: Price, outputs and leader firm's profit as demand increases

a	$a < 2c$	$a = 2c$	$2c < a < c + s_L$	$a = c + s_L$	$c + s_L < a < s + s_L + c$
p	$a/2$	$a/2 = c$	c	c	c
q_L	$a/2$	$a/2 = c$	$a - c$	s_L	s_L
Q	$a/2$	$a/2 = c$	$a - c$	s_L	$a - c$
π	$a^2/4$	c^2	$c(a - c)$	cs_L	cs_L

Table 2.3 continued.

	$a = s + s_L + c$		$s + s_L + c < a < s + 2s_L$	$a = s + 2s_L$	$s + 2s_L < a$
	c	$(a - s)/2 = (s_L + c)/2$	$(a - s)/2$	s_L	$a - s - s_L$
	s_L	$(a - s)/2 = (s_L + c)/2$	$(a - s)/2$	s_L	s_L
	$s + s_L$	$(s_L + c)/2 + s$	$(a + s)/2$	$s + s_L$	$s + s_L$
	cs_L	$(s_L + c)^2/4$	$(a - s)^2/4$	s_L^2	$s_L(a - s - s_L)$

Table 2.4: Price, outputs and leader firm's profit as demand decreases

a	$s + 2s_L < a$	$a = s + 2s_L$	$s + 2c < a < s + 2s_L$	$a = s + 2c$	
p	$a - s - s_L$	s_L	$(a - s)/2$	$(a - s)/2 = c$	c
q_L	s_L	s_L	$(a - s)/2$	$(a - s)/2 = c$	$s + c$
Q	$s + s_L$	$s + s_L$	$(a + s)/2$	$s + c$	$s + c$
π	$s_L(a - s - s_L)$	s_L^2	$(a - s)^2/4$	c^2	$c(s + c)$

Table 2.4 continued.

$2c < a < s + 2c$	$a = 2c$	$a < 2c$
c	$a/2 = c$	$a/2$
$a - c$	$a/2 = c$	$a/2$
$a - c$	$a/2 = c$	$a/2$
$c(a - c)$	c^2	$a^2/4$

Like the previous section, let us investigate the change in the results in a case that the leader firm sets the quantity instead of the price in order to maximize its profit. In this case and at every moment, the leader firm increases the quantity by ϵ and compares the profits. If $\pi_L(q_L+\epsilon)$ is bigger than $\pi_L(q_L)$ then it sets the quantity at $q_L+\epsilon$; otherwise it will decrease the quantity by ϵ, and compare the profits and set the quantity at the point that maximizes the profit.

Consider tables 2.3 and 2.4. In the entire range of a, with exception of when the leader firm is undercutting the price, the leader firm is acting like a monopoly or a price leadership. In these cases and like before, the quantity of production and the price are related by the demand or residual demand curves. Hence the relationship between the price and quantity is a linear relationship. The leader's firm profit is a quadratic and concave function in price and quantity. The maximum of this function does not change with the change in the endogenous variable. Hence price and quantity setting lead to the same results. There is no linear relationship between price and quantity in cases where the leader firm is setting the price at $p = c - \epsilon$. The range of a in these cases are: $2c \leq a \leq s + s_L + c$ and $s + 2s_L \leq a$ in table 2.3 and $s + 2s_L \leq a$ and $2c \leq a \leq s + 2c$ in table 2.4. There is a small difference between price setting and quantity setting in these ranges. When the firm is setting the prices, it always can ask for a price that is higher or lower by an ϵ for its products. But since the leader firm's capacity has a limit in this section, sometimes the firm can only decrease the quantity of production by ϵ. A simple investigation similar to previous arguments and calculations shows that despite this small change, quantity setting leads to the same results as price setting.

2.4 Implications of the Two Models in the Crude Oil Market

Crude oil market is similar to the models that we have made in the two previous sections. Many economists make a distinction between OPEC and non-OPEC behavior. Non-OPEC producers are believed to behave competitively. The largest producer of oil in OPEC is, by far, Saudi Arabia. The marginal cost of producing crude oil in Saudi Arabia is much lower than the rest of the market specially if one compares it with the marginal cost of production in non-OPEC countries. Also, the production capacity of the rest of the market cannot be changed easily at least in the short term. Our models reflect these properties of the crude oil industry. Both models can explain the sharp decline of the prices in 1986. Remember that crude prices increased in 1979 from $12 to more than $30 as a result of the Iranian revolution. From 1979 to 1985, OPEC and in particular Saudi Arabia were acting similar to a price leader. The prices were high. Consumers were searching for alternatives to crude oil, and Non-OPEC producer were increasing their production. At some point in 1986, Saudi Arabia decided that it was not profitable anymore for the kingdom to act like a price leader. Using our first model, this means that they switched from case 3 to case 2. They increased their output, the total output in the market increased and the prices fell sharply.

The second model can be used to explain the behavior of the crude oil prices during 1970s. In 1973, the price of crude oil increased from $3 to $12 after the Arab oil embargo that followed the Israeli-Arab war of 1973. But why did the oil prices in 1973 not go back to the previous levels even after Saudi Arabia and other producers lifted the embargo? The second model suggests that we may have a price hysteresis in the crude oil market. The price hysteresis has two implications. First, ceteris paribus, the price path when the demand increases might be different from the price path when the demand decreases. This in turn implies that a temporary shock in the demand for (or supply of) crude oil can leave permanent changes in the price. In other words, the temporary changes in the supply of crude oil in 1973 resulted in the price hysteresis phenomenon described in the second model in such a way that it kept the prices high even after the return of the producers to the market.

2.5 Conclusion

This paper investigates the possibility of an output cut by a firm as a result of an increase in demand in industries with constraint capacities. We are specially interested in the crude oil industry, although the paper has implications beyond this market. The results of the paper are used to explain two major phenomena in the crude oil market in the 1970s and 80s. In particular, we answer these two questions: why did the oil prices not go back to the previous levels even after Saudi Arabia and other producers lifted the embargo in 1973? And why did the prices decline so sharply in 1986?

Two simple close models are developed in order to answer these questions. In both models a firm cuts the output at some point solely because of an increase in demand. This can explain the huge fall of the crude prices in 1986. There are price and quantity hysteresis in the second model. The price hysteresis has two implications. First, ceteris paribus, the price path when the demand increases might be different from the price path when the demand decreases. This in turn implies that a temporary shock in the demand for (or supply of) crude oil can leave permanent changes in the price. We claim that the temporary changes in the supply of crude oil in 1973 resulted in the price hysteresis phenomenon described in the second model in such a way that it kept the prices high even after the return of the producers to the market. The implications of the results of this paper are not limited to the crude oil market. The results might be used to explain the behavior of firms in any industry with capacity constraints.

Chapter 3

Social Groups and Redistribution in a Democracy

3.1 Introduction

Empirical studies suggest that societies with ethnic heterogeneity spend less on redistribution and welfare programs and impose lower tax rates relative to homogeneous societies. For example, there are differences between the welfare programs in US and Europe. In this paper, we analyse the relationship between the size and distribution of ethnic groups in a population and welfare expenditures, redistribution and tax rate when there is a democratic society. We assume that there are two groups in the population: a minority group and a majority group. By investigating the relationship between the size of the minority group and redistribution, we find out that the outcomes do not necessarily change in favor of a group as it grows in size.

Redistribution occurs in both homogeneous and heterogeneous communities. Before we investigate the relationship between the size of social groups and the size of redistribution, it is necessary to know why redistribution occurs in a homogeneous population. In theory, the size of the redistribution and tax rate reflects the relation between the decisive voter's income and the average income. If a country is a democracy, then the median voter is in fact the decisive voter. Empirical studies show that the income distribution is skewed to the right and the median voter's income is less than the average income. Hence, we would expect a median voter who is a rational utility maximizer and a fully informed individual to vote for a complete redistribution of incomes. However, in the real world, a complete redistribution does not occur.

There are several theories explaining why a complete redistribution fails to occur in democracies. Meltzer and Richard give one explanation in their paper "A rational theory of the size of government. "[1] They assume that rational voters[2] choose between leisure (or labor time) and consumption for a given tax rate and that they consider this choice when they are voting for redistribution. Under certain conditions like the normality of leisure, the decisive voter will vote for less than a complete redistribution. In their model, nobody works when there is a complete redistribution.

The other reason that is mentioned in the literature to explain why the poor is not in favor of high level of redistribution is the hope of the poor for moving up in the income hierarchy. This idea is often referred to as the *"prospect of upward mobility"* or (POUM) hypothesis. Benabou and Ok have developed a model based on this hypothesis in their paper "Social Mobility and the Demand for Redistribution: the POUM Hypothesis."[3]. According to their model, when redistribution policies are more stable than the individuals' income, then there is a range of individuals with income below the average that may become adverse to redistribution policies simply because they expect to have a higher income in the future. As a result, a majority of people may actually end up being against redistribution. The probability of being against redistribution under the POUM hypothesis is related to the

[1]Meltzer, A. and Richard, S., "A Rational Theory of the Size of Government.", *J. Polit. Econ., Oct. 1981.*

[2]Including the median voter

[3]Benabou Rolan. Ok, Efe A., "Social Mobility and the Demand for Redistribution: The POUM Hypothesis.",*Quarterly Journal of Economics, May 2001, v116, i2, p447.*

level of risk aversion. The more risk loving the individuals are, the more they are oppose to government intervention for redistribution. In other words, risk aversion makes people favor redistribution.

Corneo and Grüner give one interesting explanation of limits to redistribution in their paper "Social limits to redistribution."[4] They believe that when economic inequality has some informational value for social decisions, people tend to place a limit to redistribution. In their point of view, there are sources of utility that cannot be allocated by the market. Conversation, dinners and parties, playing, and admiration by others, have some values that individuals assign. In all of these activities, the most important issue is not the existence of conversations or parties or playing, but with whom the individual is conversing, socializing, and playing. In a situation such as this, the relative position in the income distribution may create some information. In other words, economic inequality allows some separation across *"different types."* Greater redistribution causes the position in the income distribution to be less informative. However, the informational value of this separation is positive for some and negative for others. If the value of this information is high for individuals close to the median voter, as Corneo and Grüner think it is, then the median voter will vote for less redistribution.

There are several other theories that explain why the median voter is not in favor of a complete redistribution. One explanation is the costs of taxation. People will vote for less redistribution when these costs are increasing. Lobbying activities of high-income groups constitute another important factor. Moreover, people with higher incomes show more participation in political processes, including voting. Higher rates of participation in the political process by those with higher income result in the shift of decisive voter status to individuals with higher incomes.[5] However, none of the theoretical models that have been used to inves-

[4]Corneo, Giacomo. Grüner, Hans Peter., "Individual preferences for political redistribution.", *Journal of Public Economics No. 83 (2002) 83–107.*

[5]These results are based on the assumption that monetary benefits are the only things that matters for voters.

tigate redistribution can explain the difference between the redistribution in homogeneous and heterogeneous communities. For example the EU and US governments provide different levels of public goods and redistribution. Even in the US itself, the level of the public goods provided by different counties varies. Alesina et al. compare US and EU welfare expenditure in their paper "Inequality and Happiness: Are Europeans and Americans Different?"[6] and find that the taste for redistribution in the US is different from that in the EU. Although wealthy Europeans, like wealthy Americans, do not oppose inequality, the poor in Europe are unhappier about inequality compared to the poor in America. In the US, the poor believe in the probability of upward mobility and hence they are less concerned about inequality. When the poor are more concerned about inequality, they become favorable toward government intervention in redistributing the wealth and income. In another paper, Alesina et al. [7] use the POUM hypothesis to explain this difference in preferences. The paper concludes that the POUM hypothesis can explain the popularity of government intervention against inequality in Europe.

One behavioral work that is more related to the focus of this paper is "Group Loyalty and the Taste for Redistribution" by Luttmer, [8] in which he uses survey data in the US to investigate interpersonal preferences. He concludes that attitude toward welfare spending is affected by both self-interests and interpersonal preferences. One will increase his or her support for welfare if the local recipients of the welfare are from his or her own ethnic group. Luttmer uses this finding to explain why homogeneous states show a higher level of welfare expenditure.

[6] Alesina, Alberto. Tella, Rafael Di. MacCulloch Robert., "Inequality and Happiness: Are Europeans and Americans Different?," *NBER, working paper 8198,*.

[7] Alesina, Alberto. Glaeser, Edward. and Sacerdote, Bruce., "Why Doesn't The US Have A European-Style Welfare State?," *Harvard Institute of Economic Research. Discussion Paper Number 1933,*.

[8] Luttmer, Erzo F. P., "Group Loyalty and the Taste for Redistribution.", *Journal of Political Economy, Vol. 109 (3) pp. 500-528.*

3.2 Model without Ethnicity

Following Meltzer and Richard (1981, henceforth MR),[9] consider a continuous population of persons, having unit measure. Each person is characterized by his exogenous productivity $x \in X \equiv (x_{min}, x_{max}) \subset \Re_{++}$. $F : X \to [0,1]$ describes the exogenous distribution of x in the population. F has a continuous and strictly positive density $f \equiv F'$.

The population plays a game in two stages. In the first stage, a voting game establishes a common tax rate $t \in [0,1]$. In the second stage, each person chooses how many hours to work, $n \in [0,1]$. This decision leaves him $1-n$ hours of leisure. A person x who works for n hours earns (pre-tax) income

$$y = nx \tag{3.1}$$

The government collects fraction t of this income and distributes equal shares to everyone. The disbursement to each person is thus

$$r = t\bar{y}, \tag{3.2}$$

where $\bar{y}$ denotes average income across the population. This paper studies the determinants of r in equilibrium. After accounting for taxation and redistribution, a person with income y consumes

$$c = (1-t)y + r \tag{3.3}$$

This gives him utility

$$u(c,n) \equiv c + \frac{\sigma_0 \left(1 - n^{1+\sigma}\right)}{1+\sigma} \tag{3.4}$$

where $\sigma_0 > 0$ and $\sigma \in (0,1)$ are exogenous constants. This utility function, different from that employed by MR, comes from Diamond (1998). It is strictly concave in leisure ($l = 1-n$) and exhibits a constant elasticity of labor supply $1/\sigma$.

The first stage of the game determines t, and everyone enters stage 2 with complete information about the subgame initiated by the realization of t. In this subgame, each person

[9]Meltzer, A. and Richard, S., "A Rational Theory of the Size of Government.", *J. Polit. Econ., Oct. 1981.*

x chooses $n \in [0,1]$ to maximize his utility (3.4), subject to (3.1)-(3.3). Eliminating c, he thus chooses n to maximize

$$u = (1-t)nx + t\bar{y} + \frac{\sigma_0\left[1 - n^{1+\sigma}\right]}{1+\sigma}. \tag{3.5}$$

Given t, the labor decisions of other persons affect him only through $\bar{y}$, which he treats as exogenous because he is small. Therefore, person x's first-order condition is:

$$\frac{\partial u}{\partial n} = (1-t)x - \sigma_0 n^{\sigma} = 0. \tag{3.6}$$

Because $\frac{\partial^2 u}{\partial n^2} < 0$ for all $n \in (0,1)$, he has a strictly dominant strategy:

$$n(x;t) \equiv \left(x\frac{1-t}{\sigma_0}\right)^{\frac{1}{\sigma}} \tag{3.7}$$

Equation (3.7) describes x's labor supply, given the tax rate t. From (3.1), his income is

$$y(x;t) \equiv x\left(x\frac{1-t}{\sigma_0}\right)^{\frac{1}{\sigma}} \tag{3.8}$$

These two equations imply a familiar result: individuals' income and the time that they work are both ordered by productivity. In other words, there is a positive relationship between n, y and x. From (3.2) and (3.3), redistribution and consumption in this society are:[10]

$$\begin{aligned} r(t) &\equiv t\bar{y}(t) && (3.9)\\ where\ \bar{y}(t) &\equiv \int_{x_{min}}^{x_{max}} y(x;t)f(x)dx \\ c(x;t) &\equiv (1-t)y(x;t) + r(t) && (3.10) \end{aligned}$$

Anticipating these stage 2 outcomes, society chooses a single tax rate t in stage 1. To determine the equilibrium tax rate, consider the question: which tax rate would a given person $x \in X$ prefer (to impose on everyone)? Because (3.7) shows that labor supply is a differentiable function of t, applying the envelope theorem to (3.5) shows that

$$\frac{du}{dt} = t\frac{d\bar{y}(t)}{dt} + \bar{y} - nx < \bar{y} - y, \tag{3.11}$$

[10] Population is equal to one.

the inequality being an immediate implication of (3.1) and (3.8). Therefore, a person who will earn an above-average income y in stage 2 prefers $t = 0$ in stage 1. That is intuitive, because he will pay more in taxes than he gets from redistribution. A person who will earn below-average income does not, however, prefer $t = 1$, because the implication that $y = r = 0$ drives utility to the no-production lower bound $u = \frac{\sigma_0}{1+\sigma}$. It is a well-established empirical fact that average income exceeds the median voter's income, which suggests that, in the present model, realistic specifications of F will cause most persons to prefer $t > 0$.

Substituting (3.7)-(3.10) into (3.5) and simplifying expresses person x's utility as a function of t:

$$\begin{aligned} \widehat{u}(x;t) &\equiv c(x;t) + \frac{\sigma_0\left[1 - n(x;t)^{1+\sigma}\right]}{1+\sigma} \qquad (3.12) \\ &= \frac{\sigma_0}{1+\sigma}\left[1 + \sigma\left(x\frac{1-t}{\sigma_0}\right)^{\frac{1+\sigma}{\sigma}}\right] + t\left(\frac{1-t}{\sigma_0}\right)^{\frac{1}{\sigma}} \int_{x_{\min}}^{x_{\max}} z^{\frac{1+\sigma}{\sigma}} f(z)dz. \end{aligned}$$

Proposition 1 describes some properties of the tax rate t that maximizes $\widehat{u}(x;t)$, given x.

Proposition 4 *The preferred tax rate of a person x is unique and described by a continuous function $T : X \to [0, 1)$. For some $x' \in X$, $T(x) = 0$ for $x \geq x'$. For $x < x'$, $T(x)$ is differentiable and strictly decreasing.*

Proof. $\widehat{u}(x;t)$ can be expressed as $\alpha(x)\,(1-t)^{\frac{1+\sigma}{\sigma}} + \beta t(1-t)^{\frac{1}{\sigma}} + \gamma$, where $\alpha(x)$, $\beta > 0$, and γ do not depend on t. Then $\frac{\partial \widehat{u}(x;t)}{\partial t} = \frac{1}{\sigma}(1-t)^{\frac{1-\sigma}{\sigma}}\left[(\beta - \alpha(x))(1+\sigma)(1-t) - \beta\right]$. Observe that $\frac{\partial \widehat{u}(x;t)}{\partial t} < 0$ for $t \approx 1$. Therefore either $\frac{\partial \widehat{u}(x;t)}{\partial t} < 0$ for all $t > 0$, in which case $t = 0$ is the unique maximizer of $\widehat{u}(x;t)$, or $\frac{\partial \widehat{u}(x;t)}{\partial t} = 0$ at a unique $t \in (0, 1)$, which must be the unique maximizer of $\widehat{u}(x;t)$. That establishes uniqueness. Let $T : X \to [0, 1)$ describe the t that maximizes $\widehat{u}(x;t)$, given x. Because $\widehat{u}(x;t)$ is twice-continuously differentiable for $t < 1$, the implicit function theorem shows that T is differentiable wherever $T(x) > 0$, and T' has the sign of $\frac{\partial^2 \widehat{u}}{\partial t \partial x}$. The sign of $\frac{\partial^2 \widehat{u}}{\partial t \partial x}$ matches the sign of $-\alpha'(x)$, which an easy calculation shows to be negative. Therefore, for all $x \in X$, either $T'(x) < 0$ or $T(x) = 0$. The argument at

(3.11) shows that $T(x) = 0$ for sufficiently large x; let x' denote the smallest such x. The result follows. ■

The initial assumptions on F imply that there exists some well-defined person x who has median productivity in the population, and Proposition 1 implies that this same person has median preferences over t. Without exploring the stage 1 game in detail, assume that it returns the most preferred t of this median person (i.e., voter).

3.3 Model with Ethnicity

When society comprises different groups, and members of one group feel empathy for each other but not for members of other groups, then taxes, redistribution, income, and work depend upon the size and distribution of those groups. This section investigates these relationships for the case of a society which has two groups: a minority group A, and a majority group B.

Assume that the members of group $g \in \{A, B\}$ are distributed, according to their productivity, by the exogenous distribution $F_g : X \to [0, 1)$. Then $F_A(x) + F_B(x) = F(x)$ for all x. Because A represents the minority group: $F_A(x_{\max}) \leq F_B(x_{\max})$. Assume that $f_A(x) \equiv F'_A(x)$ and $f_B(x) \equiv F'_B(x)$ are continuous and strictly positive.

To model empathy, we make three key assumptions. First, the selfish utility function (3.4) is augmented by a term representing the utility of others in the ethnic group. Second, a person feels empathy only for persons of lower productivity in his group. A person would not, for example, sacrifice \$1 to give \$10 to someone in the same ethnic group who has higher income. This assumption is a simple way to capture the idea that empathy extends primarily to those who are less fortunate. In the context of tax policy, it implies that a poor person will not vote to reduce taxes and redistribution, reducing his own consumption, for the sake of helping richer persons in his group. (He might, in contrast, vote for lower taxes in the expectation that the resulting increase in labor supply will enhance total tax revenue enough to leave him a net beneficiary.) The third, purely simplifying assumption, concerns the magnitude of empathy: if the continuous population represents say, N total persons, then the weight that any person in the group places on the utility of any one poorer person in his group is $\frac{1}{N}$ times the weight that he places on his own utility.[11]

A person x in group $g \in \{A, B\}$ thus has the following empathetic utility function, which comprises the selfish utility function (3.4) and the new empathetic component v:

$$U(u_g(\cdot), n, x) \equiv u(x, n) + v_g(u_g(\cdot), x) \tag{3.13}$$

$$\textit{where } v_g(u(\cdot), x) \equiv \int_{x_{\min}}^{x} u(z) f_g(z) dz, \tag{3.14}$$

$u_g : X \rightarrow \Re_+$ describes the utility profile of group g, and $u(x, n)$ represents his own utility.

In stage 2, each person x in group g now faces the problem of choosing $n \in [0, 1]$ to maximize $U(u_g(\cdot), n, x)$, given t and subject to (3.1)-(3.3). He neither observes nor controls $u_g(z)$ for $z \neq x$, but that does not affect his decision, because one small person has no impact on the value of $v(u_g(\cdot), x)$. Therefore, person x simply maximizes $u(x, n)$. In other words, he is solving (in stage 2) exactly the same problem as in the non-empathetic model, and this problem must have the same solution. Therefore, the stage 2 outcomes are exactly the same as before, implying that group g's consumption profile is described by (3.10). The resulting utility (substituting (3.4) into (3.13)) and then substituting in the stage two solutions (3.7), (3.10), and (3.12)) is the empathetic analog to (3.12).

$$\widehat{U}_g(x; t) \equiv \widehat{u}(x; t) + v_g(u(\cdot; t), x). \tag{3.15}$$

Using (3.8), (3.10), and (3.14)

$$\begin{aligned} v_g(u(\cdot; t), x) = {} & \frac{\sigma_0}{1+\sigma} \left[F_g(x) + \sigma \left(\frac{1-t}{\sigma_0} \right)^{\frac{1+\sigma}{\sigma}} \int_{x_{\min}}^{x} z^{\frac{1+\sigma}{\sigma}} f_g(z) dz \right] + \\ & t \left(\frac{1-t}{\sigma_0} \right)^{\frac{1}{\sigma}} F_g(x) \int_{x_{\min}}^{x_{\max}} z^{\frac{1+\sigma}{\sigma}} f(z) dz \end{aligned} \tag{3.16}$$

[11] We expect to relax this assumption in later versions of the paper.

To express person x's empathetic utility as a function of t, combine (3.8), (3.9), (3.12), (3.15), and (3.16):

$$\begin{aligned}\widehat{U}_g(x;t) \;=\; & \frac{\sigma_0}{1+\sigma}\left[1+\sigma\left(x\frac{1-t}{v_0}\right)^{\frac{1+\sigma}{\sigma}}\right] \\ & +\frac{\sigma_0}{1+\sigma}\left[F_g(x)+\sigma\left(\frac{1-t}{\sigma_0}\right)^{\frac{1+\sigma}{\sigma}}\int_{x_{\min}}^{x} z^{\frac{1+\sigma}{\sigma}} f_g(z)dz\right] \\ & +t\left(\frac{1-t}{\sigma_0}\right)^{\frac{1}{\sigma}}\left[1+F_g(x)\right]\int_{x_{\min}}^{x_{\max}} z^{\frac{1+\sigma}{\sigma}} f(z)dz\end{aligned} \tag{3.17}$$

The first term represents his selfish utility before disbursements, the second term represents the utility of poorer persons in his ethnic group before disbursements, and the third term represents selfish and empathetic benefits from disbursements. By a simple rearrangement we have:

$$\begin{aligned}\widehat{U}_g(x;t) \;=\; & \frac{\sigma_0}{1+\sigma}\left[1+F_g(x)\right] \\ & +\frac{\sigma_0^{-\frac{1}{\sigma}}\sigma}{(1+\sigma)}(1-t)^{\frac{1+\sigma}{\sigma}}\left[x^{\frac{1+\sigma}{\sigma}}+\int_{x_{\min}}^{x} z^{\frac{1+\sigma}{\sigma}} f_g(z)dz\right] \\ & +t\left(\frac{1-t}{\sigma_0}\right)^{\frac{1}{\sigma}}\left[1+F_g(x)\right]\int_{x_{\min}}^{x_{\max}} z^{\frac{1+\sigma}{\sigma}} f(z)dz\end{aligned} \tag{3.18}$$

It is useful to collapse some of this expression into the functions $\alpha_g : X \to \Re_{++}$, $\beta_g : X \to \Re_{++}$,and $\gamma_g : X \to \Re_{++}$, which are defined implicitly by (3.19):

$$\widehat{U}_g(x;t) = \alpha_g(x)\left(1-t\right)^{\frac{1+\sigma}{\sigma}} + \beta_g(x)t(1-t)^{\frac{1}{\sigma}} + \gamma_g(x). \tag{3.19}$$

If person x is a member of group g, then his preferred tax rate in stage 1 is whatever t maximizes $\widehat{U}_g(x;t)$.

Proposition 5 *In the model with ethnicity: The preferred tax rate of a person x, in group $g \in G$, is unique and described by a continuous function $T_g : X \to [0,1)$. For some $x'_g \in X$, $T_g(x) = 0$ for $x \geq x'_g$. For $x < x'_g$, $T_g(x)$ is differentiable and strictly decreasing:*

$$T_g(x) = \frac{\alpha_g(x) - \frac{\sigma}{1+\sigma}\beta_g(x)}{\alpha_g(x) - \sigma\beta_g(x)} \tag{3.20}$$

Proof. Fix $g \in G$. Most of the proof follows the proof of Proposition 1, except that $\widehat{U}_g(x;t)$ replaces $\widehat{u}(x;t)$ and β and γ are now functions of x. The formula for $T_g(x)$ comes from solving $\frac{\partial \widehat{U}_g(x;t)}{\partial t} = 0$, using (3.19). The only other substantial change arises in the demonstration that $\frac{\partial^2 \widehat{U}_g(x;t)}{\partial t \partial x} < 0$ wherever $T_g(x) > 0$. Proposition 1 shows that $\frac{\partial \widehat{u}(x;t)}{\partial t \partial x} < 0$. Fix $x' \in X$ such that $T_g(x') > 0$. Let $t' \equiv T_g(x')$, be the maximizer of $\widehat{U}_g(x';t)$. Then from (3.14) and (3.15): $\frac{\partial \widehat{U}_g(x';t)}{\partial t}\Big|_{t=t'} = \frac{\partial \widehat{u}(x';t)}{\partial t}\Big|_{t=t'} + \int_{x_{min}}^{x'} \frac{\partial u(z;t)}{\partial t}\Big|_{t=t'} f_g(z)dz = 0$. Suppose that $\frac{\partial u(x';t)}{\partial t}\Big|_{t=t'} > 0$. Then $\frac{\partial^2 u(x;t)}{\partial t \partial x} < 0$ implies that $\frac{\partial u(x;t)}{\partial t}\Big|_{t=t'} > 0$ for all $x < x'$, implying that $\int_{x_{min}}^{x'} \frac{\partial u(z;t)}{\partial t}\Big|_{t=t'} f_g(z)dz > 0$, implying that $\frac{\partial \widehat{u}(x';t)}{\partial t}\Big|_{t=t'} < 0$, a contradiction. Therefore, it must be $\frac{\partial u(x';t)}{\partial t}\Big|_{t=t'} \leq 0$. The proof of Proposition 1 shows that $\frac{\partial \widehat{u}(x;t)}{\partial t \partial x} < 0$, and (3.14) and (3.15) show that $\frac{\partial^2 \widehat{U}_g(x;t)}{\partial t \partial x} = \frac{\partial^2 \widehat{u}(x;t)}{\partial t \partial x} + \frac{\partial u(x';t)}{\partial t}\Big|_{t=t'} f_g(x)$, so it follows that $\frac{\partial^2 \widehat{U}_g(x;t)}{\partial t \partial x} < 0$, as desired. The proof concludes as for Proposition 1. ■

For any group g and tax rate t, let $L_g(t) \subset X$ denote the set of persons whose preferred tax rate is at least t, and let $M_g(t) \subset X$ denote the set of persons in group g whose preferred tax rate is at most t. A *median* tax rate is any rate t^* such that at least half of the population is in the union of groups $M_A(t^*)$ and $M_B(t^*)$, and at least half of the population is in the union of groups $L_A(t^*)$ and $L_B(t^*)$.

Proposition 6 *There exists a unique median tax rate* $t^* \in [0, 1)$*, and each group includes some person who most prefers* t^**.*

Proof. For any $t > 0$, Proposition 2 implies that the measure of individuals preferring exactly t is zero, implying that $\mu(M_A(t) \cup M_B(t)) + \mu(L_A(t) \cup L_B(t)) = 1$, where μ denotes the measure of the set. Therefore, $t > 0$ is a median tax rate if and only if $\mu(M_A(t) \cup M_B(t)) = \frac{1}{2}$. Let $t_{\max} < 1$ denote the highest tax rate preferred by any person. Then Proposition 2 implies that $\mu(M_A(t) \cup M_B(t))$ is continuous and strictly increasing in t for $t \in [0, t_{\max}]$ and equal to one for $t \geq t_{\max}$. If $\mu(M_A(t) \cup M_B(t)) = \frac{1}{2}$ for some $t^* > 0$, then these facts imply that t^* is a median tax rate and must also be the unique median rate, because $\mu(M_A(t) \cup M_B(t)) > \frac{1}{2}$ for $t > t^*$ and $\mu(M_A(t) \cup M_B(t)) < \frac{1}{2}$ for $t < t^*$. If not, then $\mu(M_A(t) \cup M_B(t)) > \frac{1}{2}$ for all $t > 0$, implying that the only possible median rate is $t = 0$; in this case, $\mu(M_A(0) \cup M_B(0)) \geq \frac{1}{2}$ and by definition $\mu(L_A(0) \cup L_B(0)) \geq \mu(M_A(0) \cup M_B(0))$, implying that $t = 0$ is indeed a median rate. That establishes existence and uniqueness. Let $t^* \in [0, t_{\max})$ denote the unique median rate. Because a person $x = x_{\min}$ has the same utility function (3.17) regardless of his group, $t_{\max}$ is his preferred tax rate regardless of his group, and Proposition 2 implies that someone in each group prefers t, for any $t \in [0, t_{\max})$. ■

The next step is finding the equilibrium tax rate t^* that is the preferred tax rate of the median voters. Notice that the distribution of the preferred tax rate is not given. Furthermore, since the individual x preferred tax rate depends on whether she is a member of group A or group B, there is no unique individual x that prefers t^*, that is the median of the preferred tax distribution.[12] Indeed, based on the proof of proposition 3, each group includes some person who most prefers t^*. Since there are two groups, there is at least one person in each group that prefers t^*. Suppose that individual $x = z_A$, a member of minority group A, and individual $x = z_B$, a member of majority group B, are these two persons. In other words, $T_A(z_A) = t^*$ and $T_B(z_B) = t^*$. The nice properties of the utility function and distribution of productivity can help us to determine these median voters as well as the equilibrium tax rate t^*. The following proposition elaborates a method of finding the equilibrium tax rate.

Proposition 7 *The equilibrium tax rate t^* and unique median voters z_A and z_B can be found by solving this system of equations:*

$$F_A(z_A) + F_B(z_B) = 1/2$$

$$T_A(z_A) = T_B(z_B) = t^* \tag{3.21}$$

Proof. Proposition 2 suggests that $T_g(x)$ is a strictly decreasing function of x, which implies that $T_g(x)$ is equal to t^* for only one x. This implies the uniqueness of z_A and z_B. The same proposition implies that every person $x < z_A$,that is a member of minority group, prefers a tax rate t that is higher than $T_A(z_A)$. An individual $x < z_B$, that is a member of majority

[12]It's important to notice that the median of the distribution of x is not the median voter.

group, prefers a tax rate t that is higher than $T_B(z_B)$. $T_A(z_A)$ must be equal to $T_B(z_B)$, otherwise at least one of z_A or z_B is not the median voter anymore. Hence a population of $F_A(z_A)$ in the minority group and a population of $F_B(z_B)$ in the majority group prefers a tax rate that is at least equal to $T_A(z_A)$ (or $T_B(z_B)$) and the rest of the population prefers a lower tax rate. If z_A and z_B are median voters then it is obvious that $F_A(z_A)+F_B(z_B) = 1/2$, and the proof is complete. ■

Notice that, there are only some special cases in which z_A and z_B are equal. In a case that they are equal, they must be equal to x_{med}. i.e. $z_A = z_B = x_{med}$, where x_{med} is the median of productivity distribution.[13] Other than these special cases and in general, the median of $f(x)$ is not necessarily the median voter. The reason is that the individual x preferred t will change if she changes her group or if there is a change in the distribution of the two groups. Hence, in general, z_A and z_B are not equal. From proposition 6 and proposition 7, it is obvious that if $z_A > x_{med}$ then z_B must be smaller than x_{med} and vice versa.

It is a well-established empirical fact that average income exceeds the median voter's income. Realistic specifications of F_g leads to $y_{med} < \bar{y}$, where $y_{med} \equiv y(x_{med};t)$ is the income of x_{med} who is person x with median productivity in the population. In the present model, we assume that F_gs are realistic, implying that y_{med} is less than $\bar{y}$.

Proposition 8 *The median tax rate t^* is positive.*

Proof. Let us first show that $\frac{dU_g(x_{med};t)}{dt}\Big|_{t=0} > 0$. From (3.13): $\frac{dU_g(x_{med};t)}{dt}\Big|_{t=0} = \frac{du_g(x_{med};t)}{dt}\Big|_{t=0} + \frac{dv_g(x_{med};t)}{dt}\Big|_{t=0}$. To show $\frac{dU_g(x_{med};t)}{dt}\Big|_{t=0} > 0$, It is sufficient to show that $\frac{du_g(x_{med};t)}{dt}\Big|_{t=0} > 0$, and $\frac{dv_g(x_{med};t)}{dt}\Big|_{t=0} > 0$. First consider $\frac{du_g(x_{med};t)}{dt}\Big|_{t=0}$. Applying the envelope theorem to (3.5) shows that

$$\frac{du_g(x_{med};t)}{dt}\Big|_{t=0} = (t\frac{d\bar{y}(t)}{dt} + \bar{y} - nx_{med})\Big|_{t=0} = \bar{y} - y_{med}, \tag{3.22}$$

that is positive. Therefore, x_{med} who will earn a belove-average income y_{med} in stage 2

[13] It's not difficult to find conditions under which the median of $f(x)$ is the median voter. An example is when $F_A(x) = F_B(x)$ for all x.

prefers $t > 0$ in stage 1. That is intuitive, because he will pay less in taxes than he gets from redistribution.

Now consider $\frac{dv_g(x_{med};t)}{dt}\Big|_{t=0}$. From (3.14),

$\frac{dv_g(x_{med};t)}{dt}\Big|_{t=0} = \int_{x_{min}}^{x_{med}} \frac{du(z;t)}{dt}\Big|_{t=0} f_g(z)dz.$

Note that from (3.8) $y(x;t) < \bar{y}$ for all $x < x_{med}$ because $y(x;t)$ is increasing in x and $y_{med} < \bar{y}$. Hence, from proposition 1, we conclude that $\frac{du(x;t)}{dt}\Big|_{t=0} > 0$ for all $x < x_{med}$. This implies that $\frac{dv_g(x_{med};t)}{dt}\Big|_{t=0} = \int_{x_{min}}^{x_{med}} \frac{du(z;t)}{dt}\Big|_{t=0} f_g(z)dz > 0$ implying that $\frac{dU_g(x_{med};t)}{dt}\Big|_{t=0} > 0$.

This means that individual x_{med} prefers a positive tax rate no matter which group he or she belongs to. In other words $T_g(x_{med}) > 0$ for $g \in \{A, B\}$. However this x_{med} is not necessarily the decisive (or median) voter. As we stated before, there are two median voters: z_A and z_B with the preferred tax rate $t^* > 0$. In special cases $z_A = z_B = x_{med}$. In these cases $t^* > 0$ since x_{med} prefers a positive tax rate. Otherwise, as we stated before, at least one of this median voters is smaller than x_{med}, i.e. either $z_A \leq x_{med}$ or $z_B \leq x_{med}$. However, according to 7 $T_A(z_A) = T_B(z_B) = t^*$, that means both of them prefer the same tax rate. Suppose that $z_A \leq x_{med}$; proposition 2 suggests that $T_A(z_A) \geq T_A(x_{med}) > 0$ implying that $t^* > 0$, and if $z_B \leq x_{med}$ then again $T_B(z_B) \geq T_B(x_{med}) > 0$ implying that $t^* > 0$, and the proof is complete. ■

The next part of the paper finds the change in t^*, the equilibrium tax rate, when there is a change in distribution of one of the social groups.

We will limit the changes in the distribution of different groups to an increase in the size of one those social groups. We need a few assumptions in order to capture these changes. First, let's assume that the distribution of group g is $f_g(x, \theta)$, where θ is a parameter that captures the changes in the distribution of group g. Second, let's assume that b_g is the size of group g. Since the whole population is equal to one, b_g would be equal to the fraction of the population that belongs to group g. We can conclude that

$$b_g = \int_{x_{\min}}^{x_{\max}} f_g(\eta, \theta) d\eta \tag{3.23}$$

Or we can simply say that b_g is a function of θ,

$$b_A \equiv L(\theta) \tag{3.24}$$

Third, let assume that

$$\frac{\partial b_A}{\partial \theta} = \frac{\partial L(\theta)}{\partial \theta} \geq 0, \tag{3.25}$$

$$\frac{\partial f_A(x, \theta)}{\partial \theta} \geq 0 \tag{3.26}$$

for all $x \in X$.

The first inequality is just an assumption about θ.[14] The second one however, imposes a restriction on how the group A's population can increase. Combined with the first inequality, it assumes that when there is an increase in the size of group A, every member of group A will stay in the group and some members of the other group will join group A.[15] In other words, if there is an increase in the population of group A, then its density will increase for all $x \in X$.

The following lemma and proposition finds a relationship between the change in the population of one group and the equilibrium tax rate.

Lemma 9 *The preferred tax rate of a member of group g decreases as the size of the other group increases.*

Proof. From 4.12 and 4.13, $\frac{\partial f_A(x)}{\partial b_A} \geq 0$. $f(x) = f_A(x) + f_B(x)$ and $f(x)$ doesn't change with a change in b_A, implying that $\frac{\partial f_B(x)}{\partial b_A} \leq 0$. Consider $\breve{U_B}(x; t; b_A)$ that is defined by 3.16. Let $T_B(x; b_A)$ describe the t that maximizes $\breve{U_B}(x; t; b_A)$, given x. Because $\breve{U_B}(x; t; b_A)$ is twice-continuously differentiable for $t < 1$, the implicit function theorem shows that T_B is differentiable wherever $T_B(x; b_A) > 0$, and $\frac{\partial T_B}{\partial b_A}$ has the minus sign of $\frac{\partial \breve{U_B}(x;t;b_A)}{\partial f_A} * \frac{\partial \breve{U_B}(x;t;b_A)}{\partial t}|_{t=T_B}$.

[14] Since θ is an arbitrary parameter, we can assume that there is a positive relationship between the size of group A and θ.

[15] We do not think that this assumption is a very strong assumption.

Easy calculation using $f(x) = f_A(x) + f_B(x)$ and 3.16 shows that the first part is negative. From the discussion in proposition 2, the second part is negative, implying that $\frac{\partial T_B}{\partial b_A}$ is negative. A similar argument can be used to show $\frac{\partial T_A}{\partial b_B}$ is negative. ■

Let discuss two implications of this lemma. First, similar argument can be used to show that $\frac{\partial T_g(x;b_A)}{\partial b_g}$ is positive. Second, for b_A close to zero, $\frac{\partial T_g(x;b_A)}{\partial b_g} \geq 0$ implies that a person x in group A has a lower preferred tax rate compare to same person x in group B. This argument implies that, for b_A close to zero, if two individuals x_A and x_B, where x_A is a member of group A and x_B is a member of group B prefer the same tax rate, then $x_A \leq x_B$.

Proposition 10 *For a population with a very small minority group, the equilibrium tax rate has a negative relation with the minority population .*

Proof. For a fixed tax rate $0 < t < t_{max}$, the set of persons in group g with a preferred tax rate higher than t is $L_g(t; b_A) = F_g(T_g^{-1}(t; b_A); b_A)$, where F_g is the fraction of population distribution that is a member of group g, b_A is the group A population and $T_g^{-1}(t; b_A)$ is the person x in group g with a preferred tax rate t, it is the inverse of function $T_g(x)$. Thus, the measure of individuals with a preferred tax rate higher than t is $\mu(\bigcup_g L_g(t; b_A)) = F_A(T_A^{-1}(t; b_A); b_A) + F_B(T_B^{-1}(t; b_A); b_A)$, where μ denotes the measure of the set. As b_A rises, the measure μ will change. The change is $\frac{\partial \mu(\bigcup_g L_g(t;b_A))}{\partial b_A} = \frac{\partial F_A(T_A^{-1}(t;b_A);b_A)+F_B(T_B^{-1}(t;b_A);b_A)}{\partial b_A}$. This proposition investigates the change in equilibrium tax rate at around $b_A = 0$. Let us first find $\frac{\partial \mu(\bigcup_g L_g(t;b_A))}{\partial b_A}\Big|_{b_A=0}$, that is $\frac{\partial \mu(\bigcup_g L_g(t;b_A))}{\partial b_A}\Big|_{b_A=0} =$ $[\frac{\partial F_A(T_A^{-1}(t;b_A);b_A)}{\partial b_A} + \frac{\partial F_A(T_A^{-1}(t;b_A);b_A)}{\partial x} * \frac{\partial T_A^{-1}((t;b_A);b_A)}{\partial b_A} +$

$\frac{\partial F_B(T_B^{-1}(t;b_A);b_A)}{\partial b_A} + \frac{\partial F_B(T_B^{-1}(t;b_A);b_A)}{\partial x} * \frac{\partial T_B^{-1}((t;b_A);b_A)}{\partial b_A}]\Big|_{b_A=0}$. From 4.12 and 4.13, it is obvious that $\frac{\partial F_A(T_A^{-1}(t;b_A);b_A)}{\partial b_A}$ is always positive and $\frac{\partial F_B(T_B^{-1}(t;b_A);b_A)}{\partial b_A}$ is always negative. However, since $F(x) = F_A(x) + F_B(x)$ doesn't change with a change in b_A and $T_B^{-1}(t; b_A) > T_A^{-1}(t; b_A)$ for small b_A (look at the lemma and its implications)the absolute value of $\frac{\partial F_B(T_B^{-1}(t;b_A);b_A)}{\partial b_A}$ is greater than the absolute value of $\frac{\partial F_A(T_A^{-1}(t;b_A);b_A)}{\partial b_A}$, implying that $\frac{\partial F_B(T_B^{-1}(t;b_A);b_A)}{\partial b_A} + \frac{\partial F_A(T_A^{-1}(t;b_A);b_A)}{\partial b_A}$ is negative. $\frac{\partial F_A(T_A^{-1}(t;b_A);b_A)}{\partial x}$ is negligible around $b_A = 0$ because $f_A(x) = \frac{\partial F_A(x)}{\partial x}$ is very small, implying that the second term is negligible. Now let us look at the last term that is $\frac{\partial F_B(T_B^{-1}(t;b_A);b_A)}{\partial x} * \frac{\partial T_B^{-1}((t;b_A);b_A)}{\partial b_A}$. From CDF properties, it is obvious that $\frac{\partial F_B(T_B^{-1}(t;b_A);b_A)}{\partial x}$ is positive and from lemma 1, $\frac{\partial T_B^{-1}((t;b_A);b_A)}{\partial b_A}$ is negative, implying that the last term is negative. Thus $\frac{\partial \mu(\bigcup_g L_g(t;b_A))}{\partial b_A} |_{b_A=0} \leq 0$ for all $0 < t < t_{max}$ especially for t_o^*, the equilibrium tax rate before the change in b_A. Notice that from the median voter hypothesis, before the change in b_A, $\mu(\bigcup_g L_g(t_o^*;b_A))$ was equal to 1/2. After the change in b_A, $\mu(\bigcup_g L_g(t_o^*;b_A))$ is less than 1/2, simply because $\frac{\partial \mu(\bigcup_g L_g(t_o^*;b_A))}{\partial b_A} |_{b_A=0} \leq 0$. This implies that, by the definition of median voter, t_o^* is not median voter anymore. In other words, the measure of the set of persons whose preferred tax rate is at least t_o^* is less than 1/2, implying that the equilibrium tax rate, t^*, will decrease. This conclude the proposition. ■

Proposition 11 *The equilibrium tax rate is a function of the population of the minority group A and the change in the equilibrium tax rate can be found by the following formula:*

$$\frac{\partial t^*}{\partial b_A} = -\frac{\frac{\partial [F_A(z_A;\theta)+F_B(z_B;\theta)]}{\partial b_A} - f_A(z_A;\theta)(\frac{\partial z_A}{\partial b_A}) - f_B(z_B;\theta)(\frac{\partial z_B}{\partial b_A})}{f_A(z_A;\theta)/(\frac{\partial T_A(z_A;\theta)}{\partial z_A}) + f_B(z_B;\theta)/(\frac{\partial T_B(z_B;\theta)}{\partial z_B})} \tag{3.27}$$

Proof. Consider the following system of equations:

$K_1 = t^* - T_A(z_A;\theta) = 0$

$K_2 = t^* - T_B(z_B;\theta) = 0$

$K_3 = F_A(z_A;\theta) + F_B(z_B;\theta) - 1/2 = 0$

$K_4 = b_A - L(\theta) = 0$

The first two equations reflect equation (3.20) and the second equation in the system of equations (3.21). Equation K_3 is another form of the first equation in the same system. The last equation in our system of equations is the definition of $L(\theta)$ in (4.11). $\frac{\partial t^*}{\partial b_A}$ is the change in the equilibrium tax rate as a result of a change in the population of group A. By applying the implicit function theorem to the system of equations K_1 to K_4

$$\frac{\partial t^*}{\partial b_A} = -\frac{\det\begin{bmatrix} \frac{\partial K_1}{\partial b_A} & \frac{\partial K_1}{\partial z_A} & \frac{\partial K_1}{\partial z_B} & \frac{\partial K_1}{\partial \theta} \\ \frac{\partial K_2}{\partial b_A} & \frac{\partial K_2}{\partial z_A} & \frac{\partial K_2}{\partial z_B} & \frac{\partial K_2}{\partial \theta} \\ \frac{\partial K_3}{\partial b_A} & \frac{\partial K_3}{\partial z_A} & \frac{\partial K_3}{\partial z_B} & \frac{\partial K_3}{\partial \theta} \\ \frac{\partial K_4}{\partial b_A} & \frac{\partial K_4}{\partial z_A} & \frac{\partial K_4}{\partial z_B} & \frac{\partial K_4}{\partial \theta} \end{bmatrix}}{\det\begin{bmatrix} \frac{\partial K_1}{\partial t^*} & \frac{\partial K_1}{\partial z_A} & \frac{\partial K_1}{\partial z_B} & \frac{\partial K_1}{\partial \theta} \\ \frac{\partial K_2}{\partial t^*} & \frac{\partial K_2}{\partial z_A} & \frac{\partial K_2}{\partial z_B} & \frac{\partial K_2}{\partial \theta} \\ \frac{\partial K_3}{\partial t^*} & \frac{\partial K_3}{\partial z_A} & \frac{\partial K_3}{\partial z_B} & \frac{\partial K_3}{\partial \theta} \\ \frac{\partial K_4}{\partial t^*} & \frac{\partial K_4}{\partial z_A} & \frac{\partial K_4}{\partial z_B} & \frac{\partial K_4}{\partial \theta} \end{bmatrix}}$$

.

By inserting equations K_1 to K_4 from the system of equations we have:

$$\frac{\partial t^*}{\partial b_A} = -\frac{\det\begin{bmatrix} 0 & -\frac{\partial T_A(z_A;\theta)}{\partial z_A} & 0 & -\frac{\partial T_A(z_A;\theta)}{\partial \theta} \\ 0 & 0 & -\frac{\partial T_B(z_B;\theta)}{\partial z_B} & -\frac{\partial T_B(z_B;\theta)}{\partial \theta} \\ 0 & \frac{\partial F_A(z_A;\theta)}{\partial z_A} & \frac{\partial F_B(z_B;\theta)}{\partial z_B} & \frac{\partial F_A(z_A;\theta)+F_B(z_B;\theta)}{\partial \theta} \\ 1 & 0 & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}}{\det\begin{bmatrix} 1 & -\frac{\partial T_A(z_A;\theta)}{\partial z_A} & 0 & -\frac{\partial T_A(z_A;\theta)}{\partial \theta} \\ 1 & 0 & -\frac{\partial T_B(z_B;\theta)}{\partial z_B} & -\frac{\partial T_B(z_B;\theta)}{\partial \theta} \\ 0 & \frac{\partial F_A(z_A;\theta)}{\partial z_A} & \frac{\partial F_B(z_B;\theta)}{\partial z_B} & \frac{\partial F_A(z_A;\theta)+F_B(z_B;\theta)}{\partial \theta} \\ 0 & 0 & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}}$$

By simplifying some of the elements of the matrices in the above equation we have:

$$\frac{\partial t^*}{\partial b_A} = -\frac{\det\begin{bmatrix} 0 & -\frac{\partial T_A(z_A;\theta)}{\partial z_A} & 0 & -\frac{\partial T_A(z_A;\theta)}{\partial \theta} \\ 0 & 0 & -\frac{\partial T_B(z_B;\theta)}{\partial z_B} & -\frac{\partial T_B(z_B;\theta)}{\partial \theta} \\ 0 & f_A(z_A;\theta) & f_B(z_B;\theta) & \frac{\partial F_A(z_A;\theta)+F_B(z_B;\theta)}{\partial \theta} \\ 1 & 0 & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}}{\det\begin{bmatrix} 1 & -\frac{\partial T_A(z_A;\theta)}{\partial z_A} & 0 & -\frac{\partial T_A(z_A;\theta)}{\partial \theta} \\ 1 & 0 & -\frac{\partial T_B(z_B;\theta)}{\partial z_B} & -\frac{\partial T_B(z_B;\theta)}{\partial \theta} \\ 0 & f_A(z_A;\theta) & f_B(z_B;\theta) & \frac{\partial F_A(z_A;\theta)+F_B(z_B;\theta)}{\partial \theta} \\ 0 & 0 & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}}$$

By calculating the determinants and simplifying we have:

$$\frac{\partial t^*}{\partial b_A} = -\frac{-((\frac{\partial T_A(z_A;\theta)}{\partial z_A})(\frac{\partial T_B(z_B;\theta)}{\partial z_B})(\frac{\partial F_A(z_A;\theta)+F_B(z_B;\theta)}{\partial \theta}))}{-\frac{\partial L(\theta)}{\partial \theta}(f_A(z_A;\theta)(\frac{\partial T_B(z_B;\theta)}{\partial z_B}) + f_B(z_B;\theta)(\frac{\partial T_A(z_A;\theta)}{\partial z_A}))} +$$

$$\frac{+(f_A(z_A;\theta))(\frac{\partial T_A(z_A;\theta)}{\partial \theta})(\frac{\partial T_B(z_B;\theta)}{\partial z_B}) + (f_B(z_B;\theta))(\frac{\partial T_B(z_B;\theta)}{\partial \theta})(\frac{\partial T_A(z_A;\theta)}{\partial z_A})}{-\frac{\partial L(\theta)}{\partial \theta}(f_A(z_A;\theta)(\frac{\partial T_B(z_B;\theta)}{\partial z_B}) + f_B(z_B;\theta)(\frac{\partial T_A(z_A;\theta)}{\partial z_A}))}$$

By dividing both numerator and denominator by $(\frac{\partial T_A(z_A;\theta)}{\partial z_A})(\frac{\partial T_B(z_B;\theta)}{\partial z_B})$ we get the following equation:

$$\frac{\partial t^*}{\partial b_A} = -\frac{-\frac{\partial F_A(z_A;\theta)+F_B(z_B;\theta)}{\partial \theta}}{-\frac{\partial L(\theta)}{\partial \theta}(f_A(z_A;\theta)/(\frac{\partial T_A(z_A;\theta)}{\partial z_A}) + f_B(z_B;\theta)/(\frac{\partial T_B(z_B;\theta)}{\partial z_B}))} +$$

$$\frac{+(f_A(z_A;\theta))(\frac{\partial T_A(z_A;\theta)}{\partial \theta})/(\frac{\partial T_A(z_A;\theta)}{\partial z_A}) + (f_B(z_B;\theta))(\frac{\partial T_B(z_B;\theta)}{\partial \theta})/(\frac{\partial T_B(z_B;\theta)}{\partial z_B})}{-\frac{\partial L(\theta)}{\partial \theta}(f_A(z_A;\theta)(\frac{\partial T_B(z_B;\theta)}{\partial z_B}) + f_B(z_B;\theta)(\frac{\partial T_A(z_A;\theta)}{\partial z_A}))}$$

And more simplifying will result in the following equation:

$$\frac{\partial t^*}{\partial b_A} = -\frac{\frac{\partial F_A(z_A;\theta)+F_B(z_B;\theta)}{\partial \theta} - f_A(z_A;\theta)(\frac{\partial z_A}{\partial \theta}) - f_B(z_B;\theta)(\frac{\partial z_B}{\partial \theta})}{\frac{\partial L(\theta)}{\partial \theta}(f_A(z_A;\theta)/(\frac{\partial T_A(z_A;\theta)}{\partial z_A}) + f_B(z_B;\theta)/(\frac{\partial T_B(z_B;\theta)}{\partial z_B}))}$$

By dividing both numerator and denominator by $\frac{\partial L(\theta)}{\partial \theta}$ and inserting $b_A = L(\theta)$ into the equation and simplifying it we get the following result:

$$\frac{\partial t^*}{\partial b_A} = -\frac{\frac{\partial [F_A(z_A;\theta)+F_B(z_B;\theta)]}{\partial b_A} - f_A(z_A;\theta)(\frac{\partial z_A}{\partial b_A}) - f_B(z_B;\theta)(\frac{\partial z_B}{\partial b_A})}{f_A(z_A;\theta)/(\frac{\partial T_A(z_A;\theta)}{\partial z_A}) + f_B(z_B;\theta)/(\frac{\partial T_B(z_B;\theta)}{\partial z_B})}$$

The proof of the proposition is complete. ■

Consider the first equation in the system of equations 3.21. Notice that F_A, z_A, F_B and z_B are functions of b_A. Hence the first equation in the system of equations 3.21 can be rephrased as $F_A(z_A(t;b_A);b_A) + F_B(z_B(t;b_A);b_A) = 1/2$. The following equation is the total derivative of this equation:

$$\frac{\partial F_A(z_A(t;b_A);b_A)}{\partial b_A} + \frac{\partial F_A(z_A(t;b_A);b_A)}{\partial x} * \frac{\partial z_A(t;b_A)}{\partial b_A} + \frac{\partial F_B(z_B(t;b_A);b_A)}{\partial b_A} + \frac{\partial F_B(z_B(t;b_A);b_A)}{\partial x} * \frac{\partial z_B(t;b_A)}{\partial b_A} = 0 \tag{3.28}$$

Proposition 12 *If $z_A < z_B$, then $\partial z_A/\partial b_A > 0$.*

Proof. There are four possible movement of median voters as a result of a change in the population of group A.

$(I)\partial z_A/\partial b_A > 0$ and $\partial z_B/\partial b_A > 0$.

$(II)\partial z_A/\partial b_A > 0$ and $\partial z_B/\partial b_A < 0$.

$(III)\partial z_A/\partial b_A < 0$ and $\partial z_B/\partial b_A > 0$.

$(IV)\partial z_A/\partial b_A < 0$ and $\partial z_B/\partial b_A < 0$.

(IV) is inconsistent with (3.28).The reason is as follow: From 4.12 and 4.13, it is obvious that $\frac{\partial F_A(z_A(t;b_A);b_A)}{\partial b_A}$ is always positive and $\frac{\partial F_B(z_A(t;b_A);b_A)}{\partial b_A}$ is always negative. However, since $F(x) = F_A(x) + F_B(x)$ doesn't change with the change in b_A and $z_A < z_B$ (this is the assumption of this proposition)the absolute value of $\frac{\partial F_B(z_B(t;b_A);b_A)}{\partial b_A}$ is greater than the absolute value of $\frac{\partial F_A(z_A(t;b_A);b_A)}{\partial b_A}$, implying that $\frac{\partial F_B(z_B(t;b_A);b_A)}{\partial b_A} + \frac{\partial F_A(z_A(t;b_A);b_A)}{\partial b_A}$ is negative. This sum is the sum of two components of equation (3.28). On the other hand, from CDF properties, it is obvious that $\frac{\partial F_B}{\partial x}$ and $\frac{\partial F_A}{\partial x}$ are positive and (IV) indicates that $\partial z_A/\partial b_A < 0$ and $\partial z_B/\partial b_A < 0$, implying that the other two components of equation (3.28) i.e. $\frac{\partial F_A(z_A(t;b_A);b_A)}{\partial x} * \frac{\partial z_A(t;b_A)}{\partial b_A}$ and $\frac{\partial F_B(z_B(t;b_A);b_A)}{\partial x} * \frac{\partial z_B(t;b_A)}{\partial b_A}$ are negative, that is impossible since from equation (3.28) the sum is equal to zero.

(III) is inconsistent with other results of the paper. Consider the median voter in group A that is z_A. Based on (III), $\partial z_A/\partial b_A < 0$ implying that the median voter moves to the left. First of all and from Lemma 1, and since the population of this group has increased, every member of the group will prefer a higher tax rate including the old median voter. Second, the new median voter is on the left of the old median voter. Proposition 2 implies that this new median voter's preferred tax rate is higher than the old median voter in the group implying that the new median voter in group A definitely prefers a higher tax rate. Now consider the median voter in group B that is z_B. Based on (III), $\partial z_A/\partial b_A > 0$ implying that the median voter moves to the right. First, from Lemma 1 and since the population of this group has increased, every member of the group will prefer a lower tax rate including the old median voter. Second, the new median voter is on the right of the old median voter. Proposition 2 implies that this new median voter's preferred tax rate is lower than the old median voter in the group implying that the new median voter in group B definitely prefers a lower tax rate. This means that the median voter in one group prefers a higher tax rate and the median voter in the other group prefers a lower tax rate that is inconsistent with the second equation in the system of equations in proposition 4. This means that only I and II are consistent. This concludes the proposition. ■

Proposition 13 *The median tax rate* $t^*(b_A) = T_A(z_A(b_A); b_A) = T_B(z_B(b_A); b_A)$ *is a quasi-convex function of* b_A *and has only one minimum.*

Proof. From proposition (3.27), it is possible to find all of the minimums and maximums of the t^*. $\frac{\partial t^*}{\partial b_A} = 0$ implies

$$\frac{\partial[F_A(z_A;\theta) + F_B(z_B;\theta)]}{\partial b_A} = f_A(z_A;\theta)(\frac{\partial z_A}{\partial b_A}) + f_B(z_B;\theta)(\frac{\partial z_B}{\partial b_A}). \tag{3.29}$$

All of the terms in this equation are the same terms in equation (3.28). Let $\tau_1 = \frac{\partial[F_A(z_A;\theta)+F_B(z_B;\theta)]}{\partial b_A}$, $\tau_2 = f_A(z_A;\theta)(\frac{\partial z_A}{\partial b_A})$, and $\tau_3 = f_B(z_B;\theta)(\frac{\partial z_B}{\partial b_A})$. From (3.28), $\tau_1 + \tau_2 + \tau_3 = 0$ and from (3.29) $\tau_1 = \tau_2 + \tau_3$ at the minimum or maximum. This two equations immediately imply that $\tau_1 = 0$ and $\tau_2 = -\tau_3$ at min or max. $\tau_1 = 0$ means that $\frac{\partial F_A(z_A;\theta)}{\partial b_A} = -\frac{\partial F_B(z_B;\theta)}{\partial b_A}$ that is possible only if $z_A = z_B = x_{med}$ where x_{med} is the median of the population. From lemma 1 we know that $z_A < z_B$ for b_A around 0. From proposition 9, when $z_A < z_B$, then $\partial z_A/\partial b_A > 0$ i.e. the median voter in the group A moves to the right as long as $z_A < z_B$. Eventually $z_A = z_B$ and that is where there is the first minimum or maximum. But since from proposition 7 we know that for b_A around 0, then $\frac{\partial t^*}{\partial b_A} < 0$, and since there is no min or max until $z_A = z_B$, it must be true that $\frac{\partial t^*}{\partial b_A} < 0$ as long as $z_A < z_B$ and until $z_A = z_B$. Notice that t^* is a continuous function of b_A. Thus the first extreme point is a min not a max.

A completely similar argument can be used to show that $z_A > z_B$ for b_A around 1. And when $z_A > z_B$, then $\partial z_A/\partial b_A < 0$ i.e. the median voter in the group B moves to the left as long as $z_A > z_B$. This with the similar argument that was mentioned in above implies that $z_A = z_B$ can occur only once implying that the there is only one extreme point that must be a min and this concludes the proposition. ■

As we have mentioned in the introduction, the negative relationship between the tax rate and the size of a social group is known empirically, but to the best of our knowledge it is not theoretically elaborated. Our model shows, and the last proposition in particular proves, that under a series of reasonable assumptions, one can explain why this negative relationship exists. However, this negative relationship exists only when the size of a group is small. The symmetric nature of the problem implies that this negative relationship must eventually become a positive relationship when the small group grows large enough. The next example shows both the negative and positive effects. The example assumes a certain specific distribution of the whole population and each group, and concludes that the relationship is negative at first and then positive.

3.4 Example

Suppose that x has a uniform distribution with $x_{min} = 0$, $x_{max} = 2$, $\sigma_0 = 2$ and $\sigma = 1$. In a case that there is no empathy we will have this proposition:

Proposition 14 *The preferred size of t, for individual x is $t = \frac{-4+3x^2}{-8+3x^2}$ and since it's decreasing in x, the individual $x = 1$ is the median voter and hence the equilibrium tax rate is $t = \frac{1}{5}$.*

Proof. From 3.4 the utility function is:

$$u(c, l) = c + 2\{1 - n^2/2\} \tag{3.30}$$

Individuals maximize this utility subject to

$$c = (1 - t)nx + r$$

By inserting c into the utility function to maximize

$$u(c, l) = (1 - t)nx + r + 2\{1 - \frac{n^2}{2}\}, \tag{3.31}$$

In stage 2 individual x choose n to maximize 3.31 for the given t and r. By getting the derivative of u with respect to n, and putting $\partial u/\partial n = 0$, we have:

$$n = \frac{1}{2}(1 - t)\, x \tag{3.32}$$

Thus the income is

$$y = nx = \frac{1}{2}(1 - t)\, x^2 \tag{3.33}$$

Notice that income is ordered by x. Then calculate the average income and disbursement:

$$\bar{y} = \int_0^2 \frac{1}{2}(1 - t)\, x^2 dF(x) = 1/2 * \int_0^2 \frac{1}{2}(1 - t)\, x^2 dx = \frac{2}{3}(1 - t) \tag{3.34}$$

$$r = t\bar{y} = \frac{2}{3}t\left(1-t\right) \tag{3.35}$$

Hence, the amount of the tax that individual x pays is $ty = \frac{1}{2}t\left(1-t\right)x^2$. A non altruistic individual x would prefer $t = 0$ if $r \leq ty$or $\bar{y} \leq y$ otherwise his preferred size of tax rate would be bigger than zero. As it is mentioned before, a familiar empirical result is that the median voter's income is less than the average income. Hence if the decisive voter is the median voter, the amount of redistribution will be bigger than zero, meaning that the decisive voter would prefer $t > 0$. This example is consistent with those empirical results.

In stage 1, there is a decisive voter who determines the tax rate t. This decisive voter chooses a tax rate that maximizes his utility. By inserting n and r from (3.32) and (3.35) into utility (3.31) we have: $u(x,t) = t\int_0^2 \frac{1}{2}\left(1-t\right)x^2 dF(x) + \frac{1}{2}(1-t)\left(1-t\right)x^2 + 2\{1 - \frac{\left(\frac{1}{2}(1-t)x\right)^2}{2}\}$ $= \frac{t}{2}\int_0^2 \frac{1}{2}\left(1-t\right)x^2 dx + \frac{1}{2}\left(1-t\right)^2 x^2 + 2(1 - \frac{\left(\frac{1}{2}(1-t)x\right)^2}{2})$. Thus $u(x,t) = \frac{2}{3}t - \frac{2}{3}t^2 + \frac{1}{4}x^2 - \frac{1}{2}x^2 t + \frac{1}{4}x^2 t^2 + 2$. By getting the derivative of this decisive voter with respect to t will give us the equilibrium tax rate. Hence the individual x's preferred tax rate is: $t = \frac{-4+3x^2}{-8+3x^2}$ Simple calculation shows that t is a decreasing function of x. Hence, if the decision making process is based on majority rule, $x = 1$ is the decisive voter. This implies that the equilibrium size of redistribution is: $t = \frac{-4+3}{-8+3} = \frac{1}{5}$. The other variables of the example are $n = \frac{1}{2}\left(1-t\right)x = \frac{1}{2}\left(1-\frac{1}{5}\right)x = \frac{2}{5}x$ with $n_{min} = 0$, $n_{max} = 4/5$, $\bar{y} = \frac{2}{3}\left(1-\frac{1}{5}\right) = \frac{8}{15}$and $r = \frac{2}{3}\frac{1}{5}\left(1-\frac{1}{5}\right) = \frac{8}{75}$and the proof is complete. ■

Now suppose that we have the same population with the same assumptions but this time individuals care about other individuals well being as well as their own.

The new utility, U, has two components: u, the part that maximizes individual's own utility and a new part which shows empathy. To make the example easier, we assume that individual x cares about his utility and other individuals' consumption. Furthermore assume that individual x cares only about those individuals with lower productivity. Hence the new utility U is:

$$U = u + \int_0^x cdF(x) \tag{3.36}$$

where $u = \left(r + (1-t)nx + 2(1 - \frac{n^2}{2})\right)$ like before. So:

$$U = \left(r + (1-t)nx + 2(1 - \frac{n^2}{2})\right) + \int_0^x \left(r + (1-t)nx\right) dF(x) \tag{3.37}$$

Proposition 15 *The preferred size of t, for individual x is $t = \frac{-4-2x+3x^2+x^3}{-8-4x+3x^2+x^3}$, and since it's decreasing in x, the individual $x = 1$ is the median voter and hence the equilibrium tax rate is $t = \frac{1}{4}$.*

Proof.

From (3.37) $U = \left(r + (1-t)nx + 2(1 - \frac{n^2}{2})\right) + 1/2 \int_0^x cdx$. By simplifying $U = r + nx - nxt + 2 - n^2 + 1/2 \int_0^x cdx$.

Individuals choose n to maximize U given r and t. The first order condition is

$\frac{d}{dn}\left(r + nx - nxt + 2 - n^2 + 1/2 \int_0^x cdx\right) = x - xt - 2n = 0$, hence $n = \frac{1}{2}\left(x\left(1-t\right)\right)$ like before. Now suppose that individual x is a decisive voter that determines the amount of t. He chooses t that maximizes his utility

$U = \left(r + (1-t)nx + 2(1 - \frac{n^2}{2})\right) + 1/2 \int_0^x \left(r + (1-t)nx\right) dx$, where $r = t\bar{y} = t \int_0^2 \frac{1}{2}\left(1-t\right) x x dF(x) = 1/2 * t \int_0^2 \frac{1}{2}\left(1-t\right) x^2 dx$. Hence, $r = \left(\frac{2}{3}t\left(1-t\right)\right)$. Hence,

$U = \left(\left(\frac{2}{3}t\left(1-t\right)\right) + \frac{1}{2}(1-t)\left(\left(1-t\right)x\right)x + 2(1 - \frac{\left(\frac{1}{2}(1-t)x\right)^2}{2})\right) +$
$1/2 \int_0^x \left(\left(\frac{2}{3}t\left(1-t\right)\right) + \frac{1}{2}(1-t)\left(\left(1-t\right)x\right)x\right) dx$
$= \frac{2}{3}t - \frac{2}{3}t^2 + \frac{1}{4}x^2 - \frac{1}{2}x^2 t + \frac{1}{4}x^2 t^2 + 2 + \frac{1}{3}tx - \frac{1}{3}t^2 x + \frac{1}{12}x^3 - \frac{1}{6}x^3 t + \frac{1}{12}x^3 t^2$

The first order condition $\frac{dU}{dt} = 0$ leads to

$t = \frac{-4+3x^2-2x+x^3}{-8+3x^2-4x+x^3}$.

Since t is decreasing in x for $0 \le x \le 2$, individual $x = 1$ is indeed the median voter and hence the equilibrium tax rate is $t = \frac{1}{4}$. Note that, compare to the situation in which there is no empathy, the equilibrium size of t has increased. Other variables are $n = \left(\frac{1}{2}\left(1-\frac{1}{4}\right)x\right) = \frac{3}{8}x$ with $n_{min} = 0$, $n_{max} = \frac{6}{8}$ and $\bar{y} = \frac{1}{2}$ and $r = \frac{2}{3}\frac{1}{4}\left(1-\frac{1}{4}\right) = \frac{1}{8}$, and the proof is complete. ■

When there are different groups in the society and every member of one group has empathy only toward members of his own group but not for members of other groups, then the tax rate, income, and work are functions of the size and distribution of those groups. This next part investigates the tax rate as a function of the size of the minority group in a society that comprises two groups: a minority and a majority group. As in the previous sections, there are a minority group A and a majority group B in a society. Assume that the productivity distribution of minority and majority group are known. In this example, we assume that the average productivity of minority group is less than the average productivity of majority group.

Suppose b is the proportion of minority group population in the society. Furthermore, assume that the population productivity has a uniform distribution with $x_{min} = 0$, $x_{max} = 2$. Suppose that a fraction $b(1 - x/2)$ of individuals with productivity x are members of group A that is the minority group and thus a fraction $1/2 - b(1 - x/2)$ of these individuals are members of group B. Hence the minority population with $x < 1$ is three times bigger than the minority population with $x > 1$.

As a result, utility for a member of Group A is:

$$U = u + \int_0^x b(1 - x/2)cdx \tag{3.38}$$

and utility for a member of Group B is:

$$U = u + \int_0^x (1/2 - b(1 - x/2))cdx \tag{3.39}$$

Proposition 16 *The preferred size of t, for individual x that is a member of group A is $t_A = \frac{16-12x^2+3bx^4-8bx^3-4bx^2+16bx}{32-12x^2+3bx^4-8bx^3-8bx^2+32bx}$ and the preferred size of t, for individual x that is a member of group B is $t_B = \frac{-16+16bx+12x^2+3bx^4+4x^3-8bx^3-4bx^2-8x}{12x^2+3bx^4+4x^3-8bx^3-8bx^2-16x-32+32bx}$. Both t_A and t_B are decreasing in x. t_A is increasing in b and t_B is decreasing in b.*

Proof. For a member of group A with productivity x, $U = u + \int_0^x b*(1-x/2)cdx$ by inserting $u = \left(r + (1-t)nx + 2(1-\frac{n^2}{2})\right)$, $r = \frac{2}{3}t\,(1-t)$and $n = \frac{1}{2}x\,(1-t)$ into the utility function

$U = \left(\left(\frac{2}{3}t\,(1-t)\right) + (1-t)\left(\frac{1}{2}x\,(1-t)\right)x + 2(1 - \frac{\left(\frac{1}{2}x(1-t)\right)^2}{2})\right) +$

$b\int_0^x(1-x/2)*\left(\left(\frac{2}{3}t\,(1-t)\right) + (1-t)\left(\frac{1}{2}x\,(1-t)\right)x\right)dx$ and simplifying we have:

$U = \frac{2}{3}t - \frac{2}{3}t^2 + \frac{1}{4}x^2 - \frac{1}{2}x^2t + \frac{1}{4}x^2t^2 + 2 - \frac{1}{16}bx^4 + \frac{1}{8}bx^4t - \frac{1}{16}bx^4t^2 + \frac{1}{6}x^3b - \frac{1}{3}x^3bt$

$+\frac{1}{6}x^3bt^2 - \frac{1}{6}btx^2 + \frac{1}{6}bt^2x^2 + \frac{2}{3}txb - \frac{2}{3}t^2xb$

By applying the first order condition $t_A = \frac{16-12x^2+3bx^4-8bx^3-4bx^2+16bx}{32-12x^2+3bx^4-8bx^3-8bx^2+32bx}$. t_A is the preferred tax rate for individual x that is a member of minority group. Simple calculations show that $\frac{dt}{db} \geq 0$ and $\frac{dt}{dx} \leq 0$ for all $0 \leq x \leq 2$ and $0 \leq b \leq 1/2$.

These two inequality shows that t ,preferred tax rate for a member of group A is increasing as the size of his group increases and decreasing as his productivity increases.

Now consider the utility for a member of Group B that is $U = u + \int_0^x(1/2 - b(1-x/2))cdx$. By inserting $u = \left(r + (1-t)nx + 2(1-\frac{n^2}{2})\right)$, $r = \frac{2}{3}t\,(1-t)$and $n = \frac{1}{2}x\,(1-t)$ in the utility function and simplifying $U = \frac{2}{3}t - \frac{2}{3}t^2 + \frac{1}{4}x^2 - \frac{1}{2}x^2t + \frac{1}{4}x^2t^2 + 2 + \frac{1}{16}bx^4 - \frac{1}{8}bx^4t + \frac{1}{16}bx^4t^2 +$ $\frac{1}{12}x^3 - \frac{1}{6}x^3t + \frac{1}{12}x^3t^2 - \frac{1}{6}bx^3 + \frac{1}{3}bx^3t - \frac{1}{6}bx^3t^2 + \frac{1}{6}btx^2 - \frac{1}{6}bt^2x^2 + \frac{1}{3}tx - \frac{1}{3}t^2x - \frac{2}{3}btx + \frac{2}{3}bt^2x$

By applying the first order condition $t_B = \frac{-16+16bx+12x^2+3bx^4+4x^3-8bx^3-4bx^2-8x}{12x^2+3bx^4+4x^3-8bx^3-8bx^2-16x-32+32bx}$. t_B is the preferred tax rate of individual x that is a member of majority group. Simple calculations show that $\frac{dt}{db} \leq 0$ and $\frac{dt}{dx} \leq 0$. These two inequalities indicate that t -preferred tax rate- for a member of group B is decreasing as the size of minority group is increasing or as his productivity is increasing. ■

Now we want to find the median voter. Suppose that an individual $x = y$, a member of minority group and an individual $x = z$, a member of majority group are median voters.

Proposition 17 *The equilibrium tax rate can be found by solving this systems of equations:*

$$\int_0^y b(1-x/2)dx + \int_0^z \left(\frac{1}{2} - b(1-x/2)\right) dx = 1/2$$

$$t_A = t_B$$

by solving this system of equations we can see that the equilibrium size of redistribution will decrease initially and then increase as the size of the minority increases.

Proof. To see why, note that every individual with $x < y$ that is a member of minority group and individual with $x < z$ that is a member of majority group will support higher t. In other words a population of $\int_0^y b(1-x/2)dx + \int_0^z \left(\frac{1}{2} - b(1-x/2)\right) dx$ will support higher taxes and a population of $\int_y^2 b(1-x/2)dx + \int_z^2 \left(\frac{1}{2} - b(1-x/2)\right) dx$ will support less taxes. Since the population is equal to one we know that the following equation needs to be true, $\int_0^y b(1-x/2)dx + \int_0^z \left(\frac{1}{2} - b(1-x/2)\right) dx = 1/2$ or $-\frac{1}{4}by^2 + by + \frac{1}{4}bz^2 + \frac{1}{2}z - bz = \frac{1}{2}$. On the other hand an individual y that is a member of minority group, has a preferred tax rate $t_A = \frac{16-12y^2+3by^4-8by^3-4by^2+16by}{32-12y^2+3by^4-8by^3-8by^2+32by}$ and an individual z that is a member of minority group, has a preferred size of redistribution $t_B = \frac{-16+16bz+12z^2+3bz^4+4z^3-8bz^3-4bz^2-8z}{12z^2+3bz^4+4z^3-8bz^3-8bz^2-16z-32+32bz}$. For the median voters $t_A = t_B$. By simplifying these equations we get:

$\frac{-16+16bz+12z^2+3bz^4+4z^3-8bz^3-4bz^2-8z}{12z^2+3bz^4+4z^3-8bz^3-8bz^2-16z-32+32bz} = \frac{16-12y^2+3by^4-8by^3-4by^2+16by}{32-12y^2+3by^4-8by^3-8by^2+32by}$

$-\frac{1}{4}by^2 + by + \frac{1}{4}bz^2 + \frac{1}{2}z - bz = \frac{1}{2}$

By solving this system of equations we can find the equilibrium size of t as a function of b. This system can be solved using a simulation method.

Appendix 1 shows the matlab program for solving the system of equations in the example, and figure 1 shows the relationship between the equilibrium size of t and the minority population. ■

3.5 Conclusion

We investigated the relationship between the change in size and distribution of social groups and welfare expenditures in a democratic community. There are two groups in the population: a minority group and a majority group. We assume that the size of the minority group is an indicator of the changes in the population combination, and the tax rate is an indicator of expenditure on welfare programs and redistribution. By investigating the relationship between the size of a minority group and the tax rate, we find out that the outcomes do not necessarily change in favor of a group as it grows in size. We assume that the productivity distributions of the society and all groups in it are given and exogenous. However this is t, the tax rate, that individuals are voting for. In a democracy, this t is in fact the median of the distribution of the preferred tax rate. In our model, the distribution of the preferred tax rate is an endogenous function and is determined by the type of utility function and the distributions of productivity as well as some other functions, such as the production technology. It is the median of distribution of the preferred tax rate that determines the equilibrium of t, not the median of productivity. We find a new method of identifying the median voter when the preferred tax rate and its distribution are endogenous. Under certain conditions, this distribution is a well-behaved function and hence the equilibrium tax rate exists and is unique. We show a method of calculating this equilibrium and at the end we prove that, in a democracy where the decisive voter is the median voter, the equilibrium tax rate change as the size of one group starts to increase. At first, this equilibrium rate will decrease because the taste of the median voter, who is a member of the other group, will change. Then it increases as the smaller group grows because the situation of the median voter changes. This negative relationship between the tax rate and the size of a social group is known empirically, but to the best of our knowledge it is not theoretically elaborated. Our model shows, and the last proposition in particular proves, that under a series of reasonable assumptions, one can explain why this negative relationship exists. However, this negative relationship exists only when the size of a group is small. The symmetric nature of the problem implies that this negative relationship must eventually become a positive relationship when the small group grows large enough.

In the last section, we elaborate a numerical example to show both of these negative and positive effects. The example assumes certain specific distributions for the population and each group and concludes that the relationship between the size of the minority group and the equilibrium tax rate is negative at first and then it becomes positive. In this specific example, the minimum of the equilibrium size redistribution occurs when the smaller group population is about 37% of the total population.

Chapter 4

Public Expenditure in the Presence of Social Groups in a Democracy

4.1 Introduction

This paper is intended to construct a theoretical relationship between the taste for public expenditure and the size and distribution of social groups in a society. There are a number of empirical papers which show that there is a relationship between the level of public good and welfare expenditures and the population combination.

One empirical work that is related to the focus of this paper is "Group Loyalty and the Taste for Redistribution" by Luttmer, in which he uses survey data in the US to investigate interpersonal preferences. He concludes that attitude toward welfare spending is affected by both self-interests and interpersonal preferences. One will increase his or her support for welfare if the local recipients of the welfare are from his or her own ethnic group. Luttmer uses this finding to explain why homogeneous states show a higher level of welfare expenditure.

There are a number of papers written by Alesina and others which find that the taste for redistribution in the United States is different from that in the European Union. Some of these papers find an empirical relationship between the size of minority groups and the level of welfare expenditure.

Although the negative relationship between the size of welfare expenditure and the size of a minority group is known empirically, but to the best of our knowledge it is not theoretically elaborated. Our claim is that we need to expand our understanding of the median voter theory to be able to construct a model that can explain these relationships.

Like most analytical works in public choice, this paper is based upon a model of majority decision making. The difference is that as a result of having different social groups in the model, the uniqueness of the median voter is not guaranteed anymore; in fact, there are n different median voters. All of them, though, have the same taste for public expenditure. The next chapter shows how we can find these median voters and how their taste for redistribution might change as a result of a change in the size of social groups.

4.2 Analysis

Consider a society with a continuum of individuals. Individuals are different in one characteristic that we call x. x belongs to $X \subset \Re$

Let's assume that X is a convex and closed set. An individual with characteristic x is called individual x. [1] For now we assume that X has a minimum, x_{min}, and a maximum, x_{max}. So $x \in X = \{x \mid x_{min} \leq x \leq x_{max}\}$. x has a distribution $f(x)$ and $f(x) > 0$ for all $x \in X - \{x_{min}, x_{max}\}$.

There are n different social groups in this society. Let's call them A_1,..., A_n. A is the set of all groups and it's partitioned to A_1 to A_n.[2] Let z_i be an individual x who is a member of group A_i where $x = z_i$. The distribution of z_i is $f_i(z_i)$. Without losing generality, we assume

[1] x can be any characteristic of the individual x like income, wealth or productivity.

[2] This means that we consider a nonempty intersection of any two groups as a separate social group.

that

$$f_i(z_i) > 0 \text{ for all } i \text{ and all } x_{min} < z_i < x_{max}.$$

Notice that $\sum_{i=1}^{n} f_i(z_i) = f(x)$, since A is partitioned to n groups.

We assume that this community is a democracy. People engage in direct voting to make a public choice. Let's assume that people are voting for t, which is the size of an arbitrary policy.[3,4] A policy will be adopted if a simple majority of people vote for it. In this case, a median voter's preferred size of t is the outcome of the voting system.

All individuals are rational utility maximizers. Their utility is affected by x, t, the group that they belong to, the distribution of all groups,[5] and a consumption bundle called c. c depends on x, t, l and y, where l and y are two endogenous vectors.

An individual $z_i = x$ will maximize

$$U = U(c(x, t, l, y), g(f_i(z_i), f_{-i}(z_{-i}))) \tag{4.1}$$

where

$y = (y_1, y_2, ..., y_{m"})$

$l = (l_1, l_2, ..., l_{m'})$

$c : \Re^{m'+m"+2} \mapsto \Re^m$

$g : \Re^n \mapsto \Re^n$,

[3] It means that the vote is not a "yes or no" vote for a very specific detailed policy, rather it's about how much of a certain thing the society needs.

[4] Examples of t are the size of the government, the size of redistribution or welfare, public good, education, or even military expenditure.

[5] Individuals care about the utility of the other members of their group as well as their own.

$U : \Re^{n+m} \mapsto \Re$ and

$f_{-i}(z_{-i}) = (f_1(z_1), ..., f_{i-1}(z_{i-1}), f_{i+1}(z_{i+1}), ..., f_n(z_n))$.

We assume that U is a C^2 function in t and x.

There is a game under which this economy will determine t. This game is a sequential game. In the second stage of this game, individuals choose l to maximize their utility for a given t, x, and y. [6], [7]

y will become a known vector as soon as all individuals determine the l that maximizes their utility. [8] Note that since x and i might be different for different individuals, l might be different for different individuals. However y is a global vector meaning that it's the same for all individuals.

Every individual has a preferred size of t. Let's call it t_p^x. t_p^x is a level of t that maximizes the utility of individual x. [9] In other words if he could choose t he would choose t_p^x. By looking at the utility function, it's pretty obvious that t_p^x depends on x, y, the group that he belongs to, as well as the distribution of all of different social groups in the society. Hence $t_p^{z_i}$ would be the preferred size of t for an individual $x = z_i$ who is a member of group A_i.

In the first stage of the game, the decisive voter determines t, knowing the reaction of other individuals in stage two. This means that the decisive voter knows the level of y for every t that he would set. Furthermore, the decisive voter is subject to the same t that other individuals are. Hence he inserts $y(t)$ into his utility function (4.1) and chooses t that maximizes her utility that is:

[6] Without losing any generality, we can assume that there is a unique l that maximizes this utility.

[7] x is like the total endowment and imposes a restriction, so individual x will maximize his utility subject to x.

[8] For example, consider the consumer reaction to a new quantity tax on gas. l is like every individual level of consumption of gas and other goods while y could be the total level of gas consumption or the total money that is paid in the form of taxes.

[9] Let's assume that U is a single picked function of t.

$$U = U(c(x, t, l(x, t, g), y(t)), g(f_i(z_i), f_{-i}(z_{-i}))) \tag{4.2}$$

By getting the derivative of this utility with respect to t, we can find the equilibrium size of policy t, then we can find l for all individuals and y for the whole economy.

The problems that we have not addressed so far are: a method that describes how we can find t, the equilibrium size of the policy in question, existence and uniqueness of it and how we can calculate the change in t when there is a change in the distribution of one of the social groups.

Let's start with the existence and uniqueness of t. As we have mentioned, every individual has a preferred size of t that we call t_p^x. Consider individual z_i, which is an individual x that is a member of group A_i, then $t_p^{z_i}$ is his preferred size of t, i.e.

$$t_p^{z_i} = argmaxU(c(z_i, t, l(z_i, t, g), y(t)), g_i(f_i(z_i), f_{-i}(z_{-i}))) \tag{4.3}$$

Let's assume that $t_p^{z_i}$ exists and it is unique for all z_i. Define:

$$k_i(x) = t_p^{z_i} \tag{4.4}$$

$k_i(x)$ is a function that maps x space to t space[10]. We assume that $k_i(x)$ is a monotonic function, in other word $t_p^{z_i}$ is ordered by x (or z_i), ceteris paribus. [11] Define $h_i(t)$ to be the distribution of $t_p^{z_i}$ in the t space. Then we have the following proposition.

[10] $k_i(x) = t_p^{z_i}$ simply means that $argmaxU$ depends on x and the group that x belongs to.

[11] For example suppose that x is individuals' income and t is the education expenditure, then $t_p^{z_i}$ is ordered by x means that education is a normal (or an inferior) good for all individual $x \in X$.

Proposition 18 *The equilibrium size of policy t exists.*

Proof.

Let's assume that B_i is the range of $k_i(x)$. Remember that X is a convex and closed set and there is an individual x who belongs to group A_i for all $x \in X$. The implicit function theorem implies that $k_i(x)$ is a continuous and hence B_i is a convex and closed set. Furthermore $h_i(t) > 0$ for all $t \in B_i$ and it's continuous. This means that $h_i(t)$ is an integrable function. Let's t_m be the median voter in the t space, t_m can be calculated by solving this equation:

$$\int_{-\infty}^{t_m} \sum_{i=1}^{n} h_i(\alpha) d\alpha = 1/2 \tag{4.5}$$

Now consider the function

$$\int_{-\infty}^{t} \sum_{i=1}^{n} h_i(\alpha) d\alpha \tag{4.6}$$

Since $h_i(t) \geq 0$ for all i, (4.6) is a monotonic and continuous function. Furthermore it is between 0 and 1. From the intermediate value theorem we know that there exist a t for which (4.6) is equal to 1/2, this means that t_m exists. ■

Define

$$B = \bigcup_{i=1}^{n} B_i$$

and

$$C = \{B_1, B_2, B_3, ..., B_n\}$$

and let C^s be a subset of the power set of C that is not equal to the null set or C itself i.e.

$$C^s \subset P(C) - \{\emptyset, C\}$$

where $P(C)$ is the power set of C.

Proposition 19 *Assume that for every C^s there exists a $B_i \subset C - C^s$ such that:*

$$B_i \cap C^s \neq \emptyset \tag{4.7}$$

then t_m, the median voter in the t space, is unique.[12]

Proof. If the assumption in the proposition holds, then the set B is a convex and closed set[13]. Then the function (4.6) will be a strictly increasing function in t and since it is a continuous function with a minimum equal to zero and a maximum equal to one, the t_m that solves equation (4.5) would have a unique answer meaning that the median voter is unique. ■

So far we have discussed the situation of the median voter in the t space. The problem that arises is that usually the t space is not observable. However x and social groups are visible.[14] As we have assumed before, we may know that $t_p^{z_i}$ is ordered by x,[15] i.e. ceteris paribus the preferred size of the policy t is ordered by the characteristic x. This however does not imply that t_p^x is ordered by x.[16] Hence the median of x is not the median voter. So who is the median voter in the x space? This part of the paper is intended to answer this question.

[12]This is just a sufficient condition, it's not a necessary one.

[13]Remember that B_i is a convex and closed set for all i.

[14]For example, suppose that the society wants to decide about the size of education expenditure, from a researcher's point of view, their income and the group that they belong to is observable but their preferred size of education expenditure is not.

[15]Consider the previous example, this means that if we only consider the members of group i, then their preferred size of education expenditure is order by income.

[16]Again consider the previous example, this means that the preferred size of education expenditure is not order by income of all individual x.

Let's assume that

$$z_i^* = k_i^{-1}(t_m) \tag{4.8}$$

where as before, $k_i(x)$ is a function that maps x space to t space and it gives us the preferred size of t for a member of group A_i. Equation (4.8) indicates that the preferred size of t for z_i^*, a member of group A_i, is equal to the median of the distribution of the preferred size of t in the t space.

Now we are ready to find the median voters in the x space. Suppose that an individual $x = z_i^*$, a member of group A_i is a median voters.

Proposition 20 *The equilibrium size of policy t and median voters z_1^* to z_n^* can be found by solving this system of equations:*

$$\sum_{i=1}^{n} \int_{-\infty}^{z_i} f_i(\beta) d\beta = 1/2$$

and

$$t_p^{z_i} = t_p^{z_j} \tag{4.9}$$

for all $i, j \in \{1, 2, ..., n\}$.

Proof. Let $Z^* = (z_1^*, z_2^*, ..., z_i^*, ..., z_n^*)$ be the answer of the system of equations (4.9). Every individual $x < z_i^*$, that is a member of group A_i, will support a size t that is lower (higher) than t_m ,the preferred size of t for z_i^*. This means that $k_i(x)$ will map every $x < z_i^*$ to a t in the t space that is less (more) than t_m.[17] Every individual with $x > z_i^*$ that is a member of group A_i will support a size t that is higher (lower) than t_m. This means that $k_i(x)$ will map every $x > z_i^*$ to a t in the t space that is more (less) than t_m. The above statements are valid for all i. As a result a population of

$$\sum_{i=1}^{n} \int_{-\infty}^{z_i^*} f_i(z_i) dz_i$$

[17] Based on the assumption that $k_i(x)$ is a monotonic function, $t_p^{z_i}$ is ordered by x (or z_i).

will support lower (higher) t and a population of

$$\sum_{i=1}^{n}\int_{z_i^*}^{\infty} f_i(z_i)dz_i$$

will support higher (lower) t.

From (4.5) we know that

$$\int_{-\infty}^{t_m}\sum_{i=1}^{n} h_i(\alpha)d\alpha = 1/2,$$

so we immediately can conclude that

$$\sum_{i=1}^{n}\int_{-\infty}^{z_i^*} f_i(z_i)dz_i = 1/2.$$

For the second equation, we know by definition that $z_i^* = k_i^{-1}(t_m)$ or for all i. In other words, $t_m = k_i(z_i^*)$, this means that z_i^* is a median voter and since t_m is unique, we can conclude that the preferred size t is the same for all z_i^* no matter which group they belong to[18]. Hence

$$t_p^{z_i^*} = t_p^{z_j^*}$$

for all $i, j \in \{1, 2, ..., n\}$, and the proof is complete. ■

Notice that the system of equations (4.9) includes n equations in which z_is are the independent variables and $z_i^* = z_i$ are the answers to the system of equations.

This proposition indicates that the median of $f(x)$ is not necessarily the median voter[19]. The reason is that the individual x taste for t will change if she changes her group or if there is a change in the distribution of the groups under investigation.

The next part of the paper is intended to find the change in t_m, the equilibrium size of policy t, when there is a change in distribution of one of the social groups.

[18]Otherwise at least one of them is not the median voter anymore.

[19]Although we can imagine condition under which the median of f(x) is the median voter. For example, if $f_i(x)/f(x) = constant$ for all i then the median of $f(x)$ is the median voter.

We will limit the changes in the distribution of different groups to an increase in the size of one those social groups. We need a few assumptions in order to capture these changes. First, let's assume that the distribution of group A_i is $f_i(x, \theta)$, where θ is a parameter that captures the changes in the distribution of group i. Second, let's assume that b_i is the size of group A_i. Since the whole population is equal to one, b_i would be equal to the fraction of the population that belongs to group A_i. We can conclude that

$$b_i = \int_{-\infty}^{+\infty} f_i(\eta, \theta) d\eta \tag{4.10}$$

Or we can simply say that b_i is a function of θ,

$$b_i = L(\theta) \tag{4.11}$$

Third, let assume that

$$\frac{\partial b_i}{\partial \theta} = \frac{\partial L(\theta)}{\partial \theta} \geq 0 \tag{4.12}$$

,

$$\frac{\partial f_i(x, \theta)}{\partial \theta} \geq 0 \tag{4.13}$$

for all $x \in X$, and

$$\frac{\partial f_j(x, \theta)}{\partial \theta} \leq 0 \tag{4.14}$$

for all $x \in X$ and all $j \neq i$.

The first inequality is just an assumption about θ.[20] The second one however, imposes a restriction on how the group A_i's population can increase. Combined with the first inequality, it assumes that when there is an increase in the size of group A_i, every member of group A_i will stay in the group and some members of other groups will join group A_i.[21] In other words, if there is an increase in the population of group A_i, then its distribution will increase for all x that belongs to the group.

[20] Since θ is an arbitrary parameter, we can assume that there is a positive relationship between the size of the group and θ.

[21] We do not think that this assumption is a very strong assumption.

Now we are ready to find a relationship between the change in the population of one group and the equilibrium size of the policy t, that is t_m. The following proposition shows a relationship between the two.

Proposition 21 *When the size of one social group changes, the equilibrium size of the policy t or simply t_m will change and we can calculate this change by using the following formula:*

$$\frac{\partial t_m}{\partial b_i} = -\frac{\left(\frac{\partial \sum_{j=1}^{n} \int_{-\infty}^{z_j^*} f_j(\beta,\theta) d\beta}{\partial b_i}\right) - \sum_{j=1}^{n} f_j(z_j^*) \left(\frac{\partial z_j^*}{\partial b_i}\right)}{\sum_{j=1}^{n} f_j(z_j^*) / \left(\frac{\partial k_j(z_j^*)}{\partial z_j^*}\right)} \tag{4.15}$$

Proof. We will start the proof with $n = 2$ and then we will expand it to $n > 2$.

Consider the following system of equations:

$K_1 = t_m - k_i(z_i^*) = 0$

$K_2 = t_m - k_j(z_j^*) = 0$

$K_3 = \int_{-\infty}^{z_i^*} f_i(\beta, \theta) d\beta + \int_{-\infty}^{z_j^*} f_j(\beta, \theta) d\beta - 1/2 = 0$

$K_4 = b_i - L(\theta) = 0$

The first two equations are reflecting equation (4.4) and the second equation in the system of equations (4.9). Equation K_3 is another form of the first equation in the same system. The last equation in our system of equations is the definition of $L(\theta)$ in (4.11).

The implicit function theorem helps us to find out how the equilibrium size of a policy t will change when there is a change in the population of group A_i. Change in b_i reflects the change in the population of group A_i, and change in t_m reflects the change in the equilibrium size of the policy. Hence $\frac{\partial t_m}{\partial b_i}$ shows the change in the equilibrium size of policy t as a result of a change in population combination. We investigate this change by applying the implicit function theorem to the system of equations $K_1 = 0$ to $K_4 = 0$. Consider equations $K_1 = 0$ to $K_4 = 0$, then we know that:

$$\frac{\partial t_m}{\partial b_i} = -\frac{\det\begin{bmatrix} \frac{\partial K_1}{\partial b_i} & \frac{\partial K_1}{\partial z_i^*} & \frac{\partial K_1}{\partial z_j^*} & \frac{\partial K_1}{\partial \theta} \\ \frac{\partial K_2}{\partial b_i} & \frac{\partial K_2}{\partial z_i^*} & \frac{\partial K_2}{\partial z_j^*} & \frac{\partial K_2}{\partial \theta} \\ \frac{\partial K_3}{\partial b_i} & \frac{\partial K_3}{\partial z_i^*} & \frac{\partial K_3}{\partial z_j^*} & \frac{\partial K_3}{\partial \theta} \\ \frac{\partial K_4}{\partial b_i} & \frac{\partial K_4}{\partial z_i^*} & \frac{\partial K_4}{\partial z_j^*} & \frac{\partial K_4}{\partial \theta} \end{bmatrix}}{\det\begin{bmatrix} \frac{\partial K_1}{\partial t_m} & \frac{\partial K_1}{\partial z_i^*} & \frac{\partial K_1}{\partial z_j^*} & \frac{\partial K_1}{\partial \theta} \\ \frac{\partial K_2}{\partial t_m} & \frac{\partial K_2}{\partial z_i^*} & \frac{\partial K_2}{\partial z_j^*} & \frac{\partial K_2}{\partial \theta} \\ \frac{\partial K_3}{\partial t_m} & \frac{\partial K_3}{\partial z_i^*} & \frac{\partial K_3}{\partial z_j^*} & \frac{\partial K_3}{\partial \theta} \\ \frac{\partial K_4}{\partial t_m} & \frac{\partial K_4}{\partial z_i^*} & \frac{\partial K_4}{\partial z_j^*} & \frac{\partial K_4}{\partial \theta} \end{bmatrix}}$$

By inserting equations K_1 to K_4 from the system of equations we have:

$$\frac{\partial t_m}{\partial b_i} = -\frac{\det\begin{bmatrix} 0 & -\frac{\partial k_i(z_i^*)}{\partial z_i^*} & 0 & -\frac{\partial k_i(z_i^*)}{\partial \theta} \\ 0 & 0 & -\frac{\partial k_j(z_j^*)}{\partial z_j^*} & -\frac{\partial k_j(z_j^*)}{\partial \theta} \\ 0 & \frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial z_i^*} & \frac{\partial \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial z_j^*} & \frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta + \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial \theta} \\ 1 & 0 & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}}{\det\begin{bmatrix} 1 & -\frac{\partial k_i(z_i^*)}{\partial z_i^*} & 0 & -\frac{\partial k_i(z_i^*)}{\partial \theta} \\ 1 & 0 & -\frac{\partial k_j(z_j^*)}{\partial z_j^*} & -\frac{\partial k_j(z_j^*)}{\partial \theta} \\ 0 & \frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial z_i^*} & \frac{\partial \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial z_j^*} & \frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta + \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial \theta} \\ 0 & 0 & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}}$$

By simplifying some of the elements of the matrices in the above equation we have:

$$\frac{\partial t_m}{\partial b_i} = -\frac{\det\begin{bmatrix} 0 & -\frac{\partial k_i(z_i^*)}{\partial z_i^*} & 0 & -\frac{\partial k_i(z_i^*)}{\partial \theta} \\ 0 & 0 & -\frac{\partial k_j(z_j^*)}{\partial z_j^*} & -\frac{\partial k_j(z_j^*)}{\partial \theta} \\ 0 & f_i(z_i^*,\theta) & f_j(z_j^*,\theta) & \frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta + \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial \theta} \\ 1 & 0 & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}}{\det\begin{bmatrix} 1 & -\frac{\partial k_i(z_i^*)}{\partial z_i^*} & 0 & -\frac{\partial k_i(z_i^*)}{\partial \theta} \\ 1 & 0 & -\frac{\partial k_j(z_j^*)}{\partial z_j^*} & -\frac{\partial k_j(z_j^*)}{\partial \theta} \\ 0 & f_i(z_i^*,\theta) & f_j(z_j^*,\theta) & \frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta + \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial \theta} \\ 0 & 0 & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}}$$

By calculating the determinants and simplifying we have:

$$\frac{\partial t_m}{\partial b_i} = -\frac{-((\frac{\partial k_i(z_i^*)}{\partial z_i^*})(\frac{\partial k_j(z_j^*)}{\partial z_j^*})(\frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta + \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial \theta}))}{-\frac{\partial L(\theta)}{\partial \theta}(f_i(z_i^*,\theta)(\frac{\partial k_j(z_j^*)}{\partial z_j^*}) + f_j(z_j^*,\theta)(\frac{\partial k_i(z_i^*)}{\partial z_i^*}))} +$$

$$\frac{-(f_i(z_i^*,\theta))(\frac{\partial k_i(z_i^*)}{\partial \theta})(\frac{\partial k_j(z_j^*)}{\partial z_j^*}) - (f_j(z_j^*,\theta))(\frac{\partial k_j(z_j^*)}{\partial \theta})(\frac{\partial k_i(z_i^*)}{\partial z_i^*})}{-\frac{\partial L(\theta)}{\partial \theta}(f_i(z_i^*,\theta)(\frac{\partial k_j(z_j^*)}{\partial z_j^*}) + f_j(z_j^*,\theta)(\frac{\partial k_i(z_i^*)}{\partial z_i^*}))}$$

By dividing both numerator and denominator by $(\frac{\partial k_i(z_i^*)}{\partial z_i^*})(\frac{\partial k_j(z_j^*)}{\partial z_j^*})$ we get the following equation:

$$\frac{\partial t_m}{\partial b_i} = -\frac{-\frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta + \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial \theta}}{-\frac{\partial L(\theta)}{\partial \theta}(f_i(z_i^*,\theta)/(\frac{\partial k_i(z_i^*)}{\partial z_i^*}) + f_j(z_j^*,\theta)/(\frac{\partial k_j(z_j^*)}{\partial z_j^*}))} +$$

$$\frac{-(f_i(z_i^*,\theta))(\frac{\partial k_i(z_i^*)}{\partial \theta})/(\frac{\partial k_i(z_i^*)}{\partial z_i^*}) - (f_j(z_j^*,\theta))(\frac{\partial k_j(z_j^*)}{\partial \theta})/(\frac{\partial k_j(z_j^*)}{\partial z_j^*})}{-\frac{\partial L(\theta)}{\partial \theta}(f_i(z_i^*,\theta)/(\frac{\partial k_i(z_i^*)}{\partial z_i^*}) + f_j(z_j^*,\theta)/(\frac{\partial k_j(z_j^*)}{\partial z_j^*}))}$$

And more simplifying will result in the following equation:

$$\frac{\partial t_m}{\partial b_i} = -\frac{\frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta + \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial \theta} - f_i(z_i^*,\theta)(\frac{\partial z_i^*}{\partial \theta}) - f_j(z_j^*,\theta)(\frac{\partial z_j^*}{\partial \theta})}{\frac{\partial L(\theta)}{\partial \theta}(f_i(z_i^*,\theta)/(\frac{\partial k_i(z_i^*)}{\partial z_i^*}) + f_j(z_j^*,\theta)/(\frac{\partial k_j(z_j^*)}{\partial z_j^*}))}$$

By dividing both numerator and denominator by $\frac{\partial L(\theta)}{\partial \theta}$ we get the following equation:

$$\frac{\partial t_m}{\partial b_i} = -\frac{(\frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta + \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial \theta} - f_i(z_i^*,\theta)(\frac{\partial z_i^*}{\partial \theta}) - f_j(z_j^*,\theta)(\frac{\partial z_j^*}{\partial \theta}))/\frac{\partial L(\theta)}{\partial \theta}}{(f_i(z_i^*,\theta)/(\frac{\partial k_i(z_i^*)}{\partial z_i^*}) + f_j(z_j^*,\theta)/(\frac{\partial k_j(z_j^*)}{\partial z_j^*}))}$$

By inserting $b_i = L(\theta)$ into the equation and simplifying it we get the final result:

$$\frac{\partial t_m}{\partial b_i} = -\frac{\frac{\partial \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta + \int_{-\infty}^{z_j^*} f_j(\beta,\theta)d\beta}{\partial b_i} - f_i(z_i^*,\theta)(\frac{\partial z_i^*}{\partial b_i}) - f_j(z_j^*,\theta)(\frac{\partial z_j^*}{\partial b_i})}{f_i(z_i^*,\theta)/(\frac{\partial k_i(z_i^*)}{\partial z_i^*}) + f_j(z_j^*,\theta)/(\frac{\partial k_j(z_j^*)}{\partial z_j^*})}$$

and the proof is complete for $n = 2$.

Now we are ready to prove the theorem for a general case where $n > 2$. When there are n social groups then we have to consider the following system of equations with $n+2$ equations:

$K_1 = t_m - k_1(z_1^*) = 0$

$K_2 = t_m - k_2(z_2^*) = 0$

...

$K_n = t_m - k_n(z_n^*) = 0$

$K_{n+1} = \sum_{i=1}^{n} \int_{-\infty}^{z_i} f_i(\beta,\theta)d\beta - 1/2 = 0$

$K_{n+2} = b_i - L(\theta) = 0$

Again the first n equations are reflecting equation (4.4) and the second equation in the system of equations (4.9). Equation K_{n+1} is another form of the first equation in the same system. The last equation in our system of equations is the definition of $L(\theta)$ in (4.11).

And like before, we want to find out how the equilibrium size of a policy t will change when there is a change in the population of group A_i. Change in b_i reflects the change in the population of group A_i; and change in t_m reflects the change in the equilibrium size of the policy. Hence, we should calculate $\frac{\partial t_m}{\partial b_i}$. We investigate this change by applying the implicit function theorem to the system of equations $K_1 = 0$ to $K_{n+2} = 0$.

$$\frac{\partial t_m}{\partial b_i} = -\frac{\det \begin{bmatrix} \frac{\partial K_1}{\partial b_i} & \frac{\partial K_1}{\partial z_1^*} & \cdots & \frac{\partial K_1}{\partial z_n^*} & \frac{\partial K_1}{\partial \theta} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial K_n}{\partial b_i} & \frac{\partial K_n}{\partial z_1^*} & \cdots & \frac{\partial K_n}{\partial z_n^*} & \frac{\partial K_n}{\partial \theta} \\ \frac{\partial K_{n+1}}{\partial b_i} & \frac{\partial K_{n+1}}{\partial z_1^*} & \cdots & \frac{\partial K_{n+1}}{\partial z_n^*} & \frac{\partial K_{n+1}}{\partial \theta} \\ \frac{\partial K_{n+2}}{\partial b_i} & \frac{\partial K_{n+2}}{\partial z_1^*} & \cdots & \frac{\partial K_{n+2}}{\partial z_n^*} & \frac{\partial K_{n+2}}{\partial \theta} \end{bmatrix}}{\det \begin{bmatrix} \frac{\partial K_1}{\partial t_m} & \frac{\partial K_1}{\partial z_1^*} & \cdots & \frac{\partial K_1}{\partial z_n^*} & \frac{\partial K_1}{\partial \theta} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial K_n}{\partial t_m} & \frac{\partial K_n}{\partial z_1^*} & \cdots & \frac{\partial K_n}{\partial z_n^*} & \frac{\partial K_n}{\partial \theta} \\ \frac{\partial K_{n+1}}{\partial t_m} & \frac{\partial K_{n+1}}{\partial z_1^*} & \cdots & \frac{\partial K_{n+1}}{\partial z_n^*} & \frac{\partial K_{n+1}}{\partial \theta} \\ \frac{\partial K_{n+2}}{\partial t_m} & \frac{\partial K_{n+2}}{\partial z_1^*} & \cdots & \frac{\partial K_{n+2}}{\partial z_n^*} & \frac{\partial K_{n+2}}{\partial \theta} \end{bmatrix}}$$

By inserting equations K_1 to K_{n+2} from the system of equations we have:

$\frac{\partial t_m}{\partial b_i} =$

$$
-\frac{\det\begin{bmatrix}
0 & -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 & -\frac{\partial k_1(z_1^*)}{\partial \theta} \\
0 & 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 & -\frac{\partial k_2(z_2^*)}{\partial \theta} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} & -\frac{\partial k_n(z_n^*)}{\partial \theta} \\
0 & \frac{\partial \int_{-\infty}^{z_1^*} f_1(\beta,\theta)d\beta}{\partial z_1^*} & \frac{\partial \int_{-\infty}^{z_2^*} f_2(\beta,\theta)d\beta}{\partial z_2^*} & \cdots & \frac{\partial \int_{-\infty}^{z_n^*} f_n(\beta,\theta)d\beta}{\partial z_n^*} & \frac{\partial \sum_{i=1}^{n} \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial \theta} \\
1 & 0 & 0 & \cdots & 0 & -\frac{\partial L(\theta)}{\partial \theta}
\end{bmatrix}}{\det\begin{bmatrix}
1 & -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 & -\frac{\partial k_1(z_1^*)}{\partial \theta} \\
1 & 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 & -\frac{\partial k_2(z_2^*)}{\partial \theta} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} & -\frac{\partial k_n(z_n^*)}{\partial \theta} \\
0 & \frac{\partial \int_{-\infty}^{z_1^*} f_1(\beta,\theta)d\beta}{\partial z_1^*} & \frac{\partial \int_{-\infty}^{z_2^*} f_2(\beta,\theta)d\beta}{\partial z_2^*} & \cdots & \frac{\partial \int_{-\infty}^{z_n^*} f_n(\beta,\theta)d\beta}{\partial z_n^*} & \frac{\partial \sum_{i=1}^{n} \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial \theta} \\
0 & 0 & 0 & \cdots & 0 & -\frac{\partial L(\theta)}{\partial \theta}
\end{bmatrix}}
$$

By simplifying some of the elements of the matrices in the above equation we have:

$$\frac{\partial t_m}{\partial b_i} = -\frac{\det\begin{bmatrix} 0 & -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 & -\frac{\partial k_1(z_1^*)}{\partial \theta} \\ 0 & 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 & -\frac{\partial k_2(z_2^*)}{\partial \theta} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} & -\frac{\partial k_n(z_n^*)}{\partial \theta} \\ 0 & f_1(z_1^*,\theta) & f_2(z_2^*,\theta) & \cdots & f_n(z_n^*,\theta) & \frac{\partial \sum_{i=1}^n \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial \theta} \\ 1 & 0 & 0 & \cdots & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}}{\det\begin{bmatrix} 1 & -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 & -\frac{\partial k_1(z_1^*)}{\partial \theta} \\ 1 & 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 & -\frac{\partial k_2(z_2^*)}{\partial \theta} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} & -\frac{\partial k_n(z_n^*)}{\partial \theta} \\ 0 & f_1(z_1^*,\theta) & f_2(z_2^*,\theta) & \cdots & f_n(z_n^*,\theta) & \frac{\partial \sum_{i=1}^n \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial \theta} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}} \tag{4.16}$$

We need to calculate the two big determinants that we have in the above formula. Let's start with the determinant of the matrix in the numerator of the equation. Let's call it $det(N)$.

$$det(N) = \det\begin{bmatrix} 0 & -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 & -\frac{\partial k_1(z_1^*)}{\partial \theta} \\ 0 & 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 & -\frac{\partial k_2(z_2^*)}{\partial \theta} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} & -\frac{\partial k_n(z_n^*)}{\partial \theta} \\ 0 & f_1(z_1^*,\theta) & f_2(z_2^*,\theta) & \cdots & f_n(z_n^*,\theta) & \frac{\partial \sum_{i=1}^n \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial \theta} \\ 1 & 0 & 0 & \cdots & 0 & -\frac{\partial L(\theta)}{\partial \theta} \end{bmatrix}$$

Consider the first column of the above matrix; all elements of this column are zero but the last element $a_{n+2,1}$, that is equal to one. Hence:

$$det(N) = (-1)^{n+3} * a_{n+2,1} * M_{n+2,1} \tag{4.17}$$

where $M_{n+2,1}$ is the $(n+2,1)th$ minor of the matrix and:

$$M_{n+2,1} = \det \begin{bmatrix} -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 & -\frac{\partial k_1(z_1^*)}{\partial \theta} \\ 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 & -\frac{\partial k_2(z_2^*)}{\partial \theta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} & -\frac{\partial k_n(z_n^*)}{\partial \theta} \\ f_1(z_1^*,\theta) & f_2(z_2^*,\theta) & \cdots & f_n(z_n^*,\theta) & \frac{\partial \sum_{i=1}^n \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial \theta} \end{bmatrix}$$

Let's do some elementary row operations to change the matrix to an upper triangle matrix and then find the determinant. By multiplying all rows i for $i = 1$ to n by $f_i(z_i^*,\theta)/\frac{\partial k_i(z_i^*)}{\partial z_i^*}$ and adding them to the last row we have[22]:

$$M_{n+2,1} =$$

$$\det \begin{bmatrix} -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 & -\frac{\partial k_1(z_1^*)}{\partial \theta} \\ 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 & -\frac{\partial k_2(z_2^*)}{\partial \theta} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} & -\frac{\partial k_n(z_n^*)}{\partial \theta} \\ 0 & 0 & \cdots & 0 & \frac{\partial \sum_{i=1}^n \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial \theta} - \sum_{i=1}^n f_i(z_i^*,\theta)\frac{\partial k_i(z_i^*)}{\partial \theta}/\frac{\partial k_i(z_i^*)}{\partial z_i^*} \end{bmatrix}$$

Now we can calculate $M_{n+2,1}$ just by multiplying the diagonal elements of the matrix, hence we have:

$$M_{n+2,1} =$$

$$(-1)^n(\prod_{i=1}^n \frac{\partial k_i(z_i^*)}{\partial z_i^*}) * (\frac{\partial \sum_{i=1}^n \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial \theta} - \sum_{i=1}^n f_i(z_i^*,\theta)\frac{\partial k_i(z_i^*)}{\partial \theta}/\frac{\partial k_i(z_i^*)}{\partial z_i^*}) \tag{4.18}$$

[22] Adding a multiple of one row to another row does not change the determinant.

Now we can calculate the determinant of the numerator by using (4.18) and (4.17), hence we have:

$det(N) = (-1)^{n+3} * a_{n+2,1} * M_{n+2,1} =$

$(-1)^{n+3} * 1 * (-1)^{n}(\prod_{i=1}^{n} \frac{\partial k_i(z_i^*)}{\partial z_i^*}) * (\frac{\partial \sum_{i=1}^{n} \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial\theta} - \sum_{i=1}^{n} f_i(z_i^*,\theta)\frac{\partial k_i(z_i^*)}{\partial\theta}/\frac{\partial k_i(z_i^*)}{\partial z_i^*}) =$

$(-1)^{2n+3}(\prod_{i=1}^{n} \frac{\partial k_i(z_i^*)}{\partial z_i^*}) * (\frac{\partial \sum_{i=1}^{n} \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial\theta} - \sum_{i=1}^{n} f_i(z_i^*,\theta)\frac{\partial z_i^*}{\partial\theta})$

Hence:

$$det(N) = (-1)^{2n+3}(\prod_{i=1}^{n} \frac{\partial k_i(z_i^*)}{\partial z_i^*}) * (\frac{\partial \sum_{i=1}^{n} \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial\theta} - \sum_{i=1}^{n} f_i(z_i^*,\theta)\frac{\partial z_i^*}{\partial\theta}) \tag{4.19}$$

We need to calculate the other big determinants that we have in the formula 4.16, the determinant in the denominator. Let's call it $det(D)$.

$$det(D) = \det \begin{bmatrix} 1 & -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 & -\frac{\partial k_1(z_1^*)}{\partial\theta} \\ 1 & 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 & -\frac{\partial k_2(z_2^*)}{\partial\theta} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} & -\frac{\partial k_n(z_n^*)}{\partial\theta} \\ 0 & f_1(z_1^*,\theta) & f_2(z_2^*,\theta) & \cdots & f_n(z_n^*,\theta) & \frac{\partial \sum_{i=1}^{n} \int_{-\infty}^{z_i^*} f_i(\beta,\theta)d\beta}{\partial\theta} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{\partial L(\theta)}{\partial\theta} \end{bmatrix}$$

Consider the last row of the above matrix, all elements of this row are zero but the last element $a_{n+2,n+2}$ that is equal to $-\frac{\partial L(\theta)}{\partial\theta}$. Hence:

$$det(D) = (-1)^{2n+4} * a_{n+2,n+2} * M_{n+2,n+2} \tag{4.20}$$

where $M_{n+2,n+2}$ is the $(n+2, n+2)th$ minor of the above matrix and:

$$M_{n+2,n+2} = \det \begin{bmatrix} 1 & -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 \\ 1 & 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} \\ 0 & f_1(z_1^*,\theta) & f_2(z_2^*,\theta) & \cdots & f_n(z_n^*,\theta) \end{bmatrix}$$

Let's do some elementary column and row operations. We want to change the matrix to an upper triangle matrix and then find the determinant. We will swap column one with two, then two[23] with three, and so on until we swap column $n-1$ with column n. In this way we have swapped the columns n times.[24] In other words, we have shifted every column to the left, except for the last column that has been replaced the first one. Note that every time we swap two column, we multiply its determinant by (-1). Hence:

$$M_{n+2,n+2} = (-1)^n \det \begin{bmatrix} -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 & 1 \\ 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} & 1 \\ f_1(z_1^*,\theta) & f_2(z_2^*,\theta) & \cdots & f_n(z_n^*,\theta) & 0 \end{bmatrix}$$

Let's do some elementary row operations to complete the change of the matrix to an upper triangle matrix and then find the determinant. By multiplying all rows i for $i = 1$ to n by $f_i(z_i^*,\theta)/\frac{\partial k_i(z_i^*)}{\partial z_i^*}$ and adding them to the last row, that is row $n+1$ we have[25]:

[23]That was column one before the first swapping

[24]There are $n+1$ columns in the matrix.

[25]Adding a multiple of one row to another row does not change the determinant.

$$M_{n+2,n+2} = (-1)^n \det \begin{bmatrix} -\frac{\partial k_1(z_1^*)}{\partial z_1^*} & 0 & \cdots & 0 & 1 \\ 0 & -\frac{\partial k_2(z_2^*)}{\partial z_2^*} & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\frac{\partial k_n(z_n^*)}{\partial z_n^*} & 1 \\ 0 & 0 & \cdots & 0 & \sum_{i=1}^n f_i(z_i^*,\theta)/\frac{\partial k_i(z_i^*)}{\partial z_i^*} \end{bmatrix}$$

Now we can calculate $M_{n+2,n+2}$ just by multiplying the diagonal elements of the matrix, hence we have:

$$M_{n+2,n+2} = (-1)^n * (-1)^n (\prod_{i=1}^n \frac{\partial k_i(z_i^*)}{\partial z_i^*}) * (\sum_{i=1}^n f_i(z_i^*,\theta)/\frac{\partial k_i(z_i^*)}{\partial z_i^*})$$

Hence:

$$M_{n+2,n+2} = (\prod_{i=1}^n \frac{\partial k_i(z_i^*)}{\partial z_i^*}) * (\sum_{i=1}^n f_i(z_i^*,\theta)/\frac{\partial k_i(z_i^*)}{\partial z_i^*}) \tag{4.21}$$

Now we can calculate the determinant of the denominator by using (4.21) and (4.20), hence we have:

$$det(D) = (-1)^{2n+4} * a_{n+2,n+2} * M_{n+2,n+2} =$$

$$(-1)^{2n+4} * (-\frac{\partial L(\theta)}{\partial \theta}) * (\prod_{i=1}^n \frac{\partial k_i(z_i^*)}{\partial z_i^*}) * (\sum_{i=1}^n f_i(z_i^*,\theta)/\frac{\partial k_i(z_i^*)}{\partial z_i^*})$$

So we can conclude that:

$$det(D) = (-\frac{\partial L(\theta)}{\partial \theta}) * (\prod_{i=1}^n \frac{\partial k_i(z_i^*)}{\partial z_i^*}) * (\sum_{i=1}^n f_i(z_i^*,\theta)/\frac{\partial k_i(z_i^*)}{\partial z_i^*}) \tag{4.22}$$

We can simplify (4.16) by using (4.19) and (4.22):

$\frac{\partial t_m}{\partial b_i} = -\frac{det(N)}{det(D)} =$

$$-\frac{(-1)^{2n+3}(\prod_{i=1}^{n}\frac{\partial k_i(z_i^*)}{\partial z_i^*})*(\frac{\partial\sum_{i=1}^{n}\int_{-\infty}^{z_i^*}f_i(\beta,\theta)d\beta}{\partial\theta}-\sum_{i=1}^{n}f_i(z_i^*,\theta)\frac{\partial z_i^*}{\partial\theta})}{(-\frac{\partial L(\theta)}{\partial\theta})*(\prod_{i=1}^{n}\frac{\partial k_i(z_i^*)}{\partial z_i^*})*(\sum_{i=1}^{n}f_i(z_i^*,\theta)/\frac{\partial k_i(z_i^*)}{\partial z_i^*})}$$

By dividing both numerator and denominator by $(\prod_{i=1}^{n}\frac{\partial k_i(z_i^*)}{\partial z_i^*})$ and simplifying we have:

$$\frac{\partial t_m}{\partial b_i} = -\frac{\frac{\partial\sum_{i=1}^{n}\int_{-\infty}^{z_i^*}f_i(\beta,\theta)d\beta}{\partial\theta}-\sum_{i=1}^{n}f_i(z_i^*,\theta)\frac{\partial z_i^*}{\partial\theta}}{(\frac{\partial L(\theta)}{\partial\theta})*(\sum_{i=1}^{n}f_i(z_i^*,\theta)/\frac{\partial k_i(z_i^*)}{\partial z_i^*})}$$

By dividing both numerator and denominator by $\frac{\partial L(\theta)}{\partial\theta}$ we get the following equation:

$$\frac{\partial t_m}{\partial b_i} = -\frac{(\frac{\partial\sum_{i=1}^{n}\int_{-\infty}^{z_i^*}f_i(\beta,\theta)d\beta}{\partial\theta}-\sum_{i=1}^{n}f_i(z_i^*,\theta)\frac{\partial z_i^*}{\partial\theta})/(\frac{\partial L(\theta)}{\partial\theta})}{(\sum_{i=1}^{n}f_i(z_i^*,\theta)/\frac{\partial k_i(z_i^*)}{\partial z_i^*})}$$

And finally by inserting $b_i = L(\theta)$ into the equation and simplifying it we get the result that we are searching for:

$$\frac{\partial t_m}{\partial b_i} = -\frac{\left(\frac{\partial\sum_{j=1}^{n}\int_{-\infty}^{z_j^*}f_j(\beta,\theta)d\beta}{\partial b_i}\right)-\sum_{j=1}^{n}f_j(z_j^*)\left(\frac{\partial z_j^*}{\partial b_i}\right)}{\sum_{j=1}^{n}f_j(z_j^*)/\left(\frac{\partial k_j(z_j^*)}{\partial z_j^*}\right)} \tag{4.23}$$

and the proof of the proposition is complete. ■

Let's have a closer look at equation (4.23). This equation gives us a relationship between the equilibrium size of the policy t and the size of a social group. However, different parts of the equation have different interpretations. Let's look at the numerator of the equation first.

Let's assume that we have two *type*s of individuals. *Type 1* are those who vote for a policy t that is higher (lower) than the equilibrium size of t, i.e. t_m. *Type 2* are those individuals who vote for a policy t that is lower (higher) than t_m. The numerator is about those individuals that change their *type* after a change in the population combination.[26] The first term in the numerator of the equation (4.23), $\frac{\partial \sum_{j=1}^{n} \int_{-\infty}^{z_j^*} f_j(\beta,\theta) d\beta}{\partial b_i}$, is the number of individuals who change their social group and become a member of a new group and at the same time their *type* has changed.[27]

The second term in the numerator of the equation (4.23), $\sum_{j=1}^{n} f_j(z_j^*) \left(\frac{\partial z_j^*}{\partial b_i}\right)$ is the number of individuals whose *type*s have changed, but they have not changed their social groups. Their *type* has changed as a result of a change in the situation of the median voter who is a member of their group and/or as a result of a change in their taste for the size of policy t.[28]

The term in the denominator of the equation (4.23), $\sum_{j=1}^{n} f_j(z_j^*) / \left(\frac{\partial k_j(z_j^*)}{\partial z_j^*}\right)$ describes a relationship between the change in the equilibrium size, t_m in the t space and the change in the situation of the median voter in the x space.

4.3 Conclusion

This paper constructs a theoretical relationship between the size of public expenditure and the size and distribution of social groups in a society. There are n different social groups in the society. Members of one group feel empathy for each other but not for members of other groups. When the society is a democratic society, then the median voter is the

[26] Note that t_m will changes when there is a change in the size of one group. We assume that the *type* of an individual x will change if she was voting for a policy t that is lower (higher) than old t_m before the change in the population combination; and she votes for a policy t that is higher (lower) than the new t_m after the change in the population combination.

[27] Not all of those individual who change their social group change their *type*.

[28] Note that the median voter might not be the same median voter that we had before the change in the combination of the population.

decisive voter and her taste for public expenditure is the one that matters. In the presence of social groups, finding this median voter (voters) becomes a difficult task. The uniqueness of the median voter is not guaranteed anymore; in fact, there are n different median voters. All of them though have the same taste for public expenditure. Proposition 3 suggests a method of finding these median voters. Any change in the size and distribution of social groups will change all individuals' taste for public expenditure. However, only a change in the taste of a fraction of the population really matters. The change in the taste for public expenditure will change the median voters' situation as well as their taste for public expenditure. Hence, public expenditure will change as a result of a change in the size and distribution of social groups. Proposition 4 finds a new method to investigate the relationship between the changes.[29]

[29] Look at Stegeman, Mark. Ghandi, Hojat. "Social Groups and Redistribution in a Democracy" for an example. In this paper, we assume that there are two groups: a minority and a majority group. By investigating the relationship between the size of a group and the size of redistribution, we see that the outcomes of economy do not necessarily change in favor of a group as it becomes bigger.

Chapter 5

Bibliography

Bibliography

[1] Adelman, M.A. (1980), "The Clumsy Cartel", *The Energy Journal*, 1, 43-53.

[2] Adelman, M.A. (1990), "Mineral Depletion, with Special Reference to Petroleum", *Review of Economics and Statistics*, 72, 1-10.

[3] Akerlof, G.A. and R.E. Kranton (2000), "Economics and Identity", *The Quarterly Journal of Economics*, 115, 715–753.

[4] Alesina, A., Baqir R. and W. Easterly (1999), "Public Goods and Ethnic Divisions", *The Quarterly Journal of Economics*, 114, 1243–1284.

[5] Alesina, A., Baqir, R. and C. Hoxby (2004), "Political Jurisdictions in Heterogeneous Communities", *Journal of Political Economy*, 112, 384–396.

[6] Alesina, A. and E.L. Ferrara (2000), "Participation in Heterogeneous Communities", *The Quarterly Journal of Economics*, 115, 847–904.

[7] Alesina, A., E.L. Ferrara (2001), "Preferences for Redistribution in the Land of Opportunities", *NBER*, working paper No. 8267.

[8] Alesina, A., E.L. Ferrara (2004), "Ethnic Diversity and Economic Performance", *Working paper*, working paper No. 10313.

[9] Alesina, A., Glaeser, E. and B. Sacerdote (2001), "Why Doesn't The US Have A European-Style Welfare State?", *Harvard Institute of Economic Research*, Discussion Paper No. 1933.

[10] Alesina, A. and E. Spolaore (1997), "On the Number and Size of the Nations", *The Quarterly Journal of Economics*, 112, 1027–1056.

[11] Alesina, A., Tella, R.D. and R. MacCulloch (2004), "Inequality and Happiness: Are Europeans and Americans Different?", *Journal of Public Economics*, 88, 2009–2042.

[12] Bacon, R. (1991), "Modelling the Price of Oil", *Oxford Review of Economic Policy*, 7, 17-34.

[13] Ball, S., Eckel C., Grossman, P.J. and W. Zame (2001), "Status in Markets", *The Quarterly Journal of Economics*, 116, 161–188.

[14] Becker, G.S. (1971), "The Economics of Discrimination", 2nd ed., *University of Chicago Press*, Chicago, IL.

[15] Benabou R. and E.A. Ok (2001), "Social Mobility and the Demand for Redistribution: The POUM Hypothesis", *Quarterly Journal of Economics*, 116, 447–487.

[16] Bjorvatn, K. and A.W. Cappelen (2003), "Inequality, Segregation, and Redistribution", *Journal of Public Economics*, 87, 1657–1679.

[17] Blount, S. (1995), "When Social Outcomes Aren't Fair: The Effect of Casual Attributions of Preferences", *Organizational Behavior and Human Decision Processes*, 63, 131–144.

[18] Bohn, H. and C. Stuart (2003), "Voting and Nonlinear Taxes in a Stylized Representative Democracy", *CESIFO*, Working Paper No. 1058.

[19] Brock W.A. and J.A. Scheinkman (1985), "Price Setting Supergames with Capacity Constraints", *The Review of Economic Studies*, 52, 371–382.

[20] Caselli, F. and W.J. Coleman (2004), "On the Theory of Ethnic Conflict", *NBER*, working paper No. 5622,

[21] Collier, P. (2001), "Implication of Ethnic Diversity", *Economic Policy*, 16, 127–166.

[22] Corneo, G. and H.P. Grüner (2002), "Individual Preferences for Political Redistribution", *Journal of Public Economics*, 83, 83–107.

[23] Corneo, G. and H.P. Grüner (2000), "Social Limits to Redistribution", *The American Economic Review*, 90, 1491–1507.

[24] Crémer, J. and D. Salehi-Isfahani (1989), "The Rise and Fall of Oil Prices: A Competitive View", *Annales d'conomie et de Statistique*, 15, 427–454.

[25] Crémer, J. and D. Salehi-Isfahani (1991), "Models of the Oil Market", Postrasse (eds.), *Harwood Academic Publishers*, .

[26] Dees S., Karadeloglou, P., Kaufmann, R.K. and M. Sanchez (2007), "Modelling the World Oil Market: Assessment of a Quarterly Econometric Model", *Energy Policy*, 35, 178-191.

[27] Diamond, P.A. (1998), "Optimal Income Taxation: An Example with a U-Shaped Pattern of Optimal Marginal Tax Rates", *American Economic Review*, 88, 83–95.

[28] Fattouh, B. (2007), "The Drivers of Oil Prices: The Usefulness and Limitations of Non-Structural Model, the DemandSupply Framework and Informal Approaches", *Oxford Institute for Energy Studies*

[29] Fattouh, B. (2006), "The Origins and Evolution of the Current International Oil Pricing System: A critical Assessment". In Mabro, R. (ed.), *Oil in the twenty-first century: Issues, challenges, and opportunities*, Oxford University Press, Oxford, UK.

[30] Gately, D. and H. Huntington (2002), "The Asymmetric Effects of Changes in Price and Income on Energy and Oil Demand", *The Energy Journal*, 23, 19-58.

[31] Griffin, J.M. and G.T. Schulman (2005), "Price Asymmetry in Energy Demand Models: A Proxy for Energy-Saving Technical Change", *The Energy Journal*, 26, 1–21.

[32] Glaeser, E.L. (2005), "The Political Economy Of Hatred", *The Quarterly Journal of Economics*, 120, 45–86.

[33] Glaeser, E.L., Laibson, D.I., Scheinkman, J.A. and C.L. Soutter (2000), "Measuring Trust", *Quarterly Journal of Economics*, 115, 811–846.

[34] Gans, J.S. and M. Smart (1996), "Majority Voting with Single-Crossing Preferences", *Journal of Public Economics* 59, 219–237.

[35] Grossman, H.I. (1991), "A General Equilibrium Model of Insurrections", *American Economic Review*, 81, 912–21.

[36] Harms, P. and S. Zink (2003), "Limits to Redistribution in a Democracy: a Survey", *European Journal of Political Economy*, 19, 651–668.

[37] Levitan R. and M. Shubik (1972), "Price Duopoly and Capacity Constraints", *International Economic Review*, 13, 111–122.

[38] Lizzeri, A. and N. Persico (2001), "The Provision of Public Goods Under Alternative Electoral Incentives", *American Economic Review*, 91, 225–239.

[39] Luttmer, E.F.P. (2001), "Group Loyalty and the Taste for Redistribution", *Journal of Political Economy*, 109, 500–528.

[40] Lynch, M.C. (2002), "Forecasting Oil Supply: Theory and Practice", *The Quarterly Review of Economics and Finance*, 42, 373-389.

[41] Meltzer, A. and S. Richard (1981), "A Rational Theory of the Size of Government", *J. Polit. Econ.*, 89, 914–927.

[42] Milanovic, B. (2000), "The Median-Voter Hypothesis, Income Inequality, and Income Redistribution: an Empirical Test with the Required Data", *European Journal of Political Economy*, 16, 367–410.

[43] Mirrlees, J.A. (1971), "An Exploration in the Theoryof Optimum Income Taxation", *Review of Economic Studies*, 38, 135–208.

[44] Mulligan B.C. (2001), "Economic Limits on Rational Democratic Redistribution", *Harris School of Public Policy Studies*, The University of Chicago, Working Paper No. 0107.

[45] Orr, L.L. (1976), "Income Transfers as a Public Good: An Application to AFDC", *American Economic Review*, 66, 359–371.

[46] Persson, T. (2001), "Do Political Institutions Shape Economic Policy?", *NBER*, Working paper No. 8214.

[47] Piketty, T. (1995), "Social Mobility and Redistributive Politics", *Quarterly Journal of Economics*, 116, 551–584.

[48] Roemer, J.E. (1998), "Why the Poor do not Expropriate the Rich: an Old Argument in New Grab", *Journal of Public Economics*, 70, 399–424.

[49] Salehi-Isfahani D. (1995), "Models of the Oil Market Revisited", *Journal of Energy Literature*, 1, 3–21.

[50] Stiglitz, J.E. (1976), "Monopoly and the Rate of Extraction of Exhaustible Resources", *Amer. Econ. Rev.*, 66, 655–661.

[51] Tiebout, C.M. (1956), "A Pure Theory of Local Expenditures", *Journal of Political Economy*, 64, 416–424.

Chapter 6

Appendix 1: Matlab programs

The matlab program for solving the system of equations in the example:

clear all

close all

$B = 0.02 : .001 : .5;$

for $k = 1 : length(B)$

$b = B(k);$

$z = 1.04 : -.0001 : .96;$

$y = (1/(2*b))*(4*b - 2*sqrt(4*b^2 - 4*b^2*z - 2*b + b^2*z.^2 + 2*b*z));$

$j = 0;$ for $i = 1 : length(z)$

$t1(i) = (16 - 12*y(i)^2 + 3*b*y(i)^4 - 8*b*y(i)^3 - 4*b*y(i)^2 + 16*b*y(i))/(32 - 12*y(i)^2 + 3*b*y(i)^4 - 8*b*y(i)^3 - 8*b*y(i)^2 + 32*b*y(i));$

$t2(i) = (-16 + 16*b*z(i) + 12*z(i)^2 + 3*b*z(i)^4 + 4*z(i)^3 - 8*b*z(i)^3 - 4*b*z(i)^2 - 8*z(i))/(12*z(i)^2 + 3*b*z(i)^4 + 4*z(i)^3 - 8*b*z(i)^3 - 8*b*z(i)^2 - 16*z(i) - 32 + 32*b*z(i));$

if $imag(t1(i)) == 0 \& imag(t1(i)) == 0$

$answert1(i) = t1(i);$

$answert2(i) = t2(i);$

end

end

for $l = 1 : length(answert1);$

$xx(l) = answert1(l) - answert2(l);$

end

delta = abs(xx);

[mindelta,ind] = min(delta);

answer(k) = answert2(ind);

end

plot(B,answer)

The matlab program for checking the robustness of the model with ethnicity:

```
clear all
close all
format long
%syms xx %xx is x
sigma0=2;
sigma=1;
xdiv=0.001;
bdiv=0.01;
fs=(sigma/(1+sigma));
invfs=1/fs; %this is the inverse of sigma/(1+sigma)
s0p = sigma0( − 1/sigma); %this is sigma0 with the power of -1 over sigma
ss0=fs*s0p; %sigma0 with the power of -1 over sigma ∗ sigma/(1 + sigma)
xmin=0;
xmax=2;
%zdiv=.001; %zmin=0.95; %zmax=1.05;
B=.02:bdiv:.5; %the minority size
fg=.5;
w=2;
persons=[0:0.001:w];
h=persons;
```

```
h(1,:)=1/2;
ha=h;
hb=h;
prefB=zeros(1,length(B));
for k=1:length(B),
k
b=B(k);
%b=0.01; %This is for a test.....................
%z = 1.04:-.0001:.96;
%z = 0:-.0001:2;
%z=zmin:zdiv:zmax; %find y and z that satisfies the first condition
hb=1/2-b*(1-persons/2); %density function of group b
ha=b*(1-persons/2); %density function of group a
intb=zeros(1,length(persons));
intb(1)=-1/2;
for kk=2:length(persons);
intb(kk)=intb(kk-1)+((hb(kk-1)+hb(kk))/2)*(persons(kk)-persons(kk-1)); %calculate the in-
tegral of majority group end
inta=zeros(1,length(persons));
inta(1)=0;
for kk=2:length(persons);
inta(kk)=inta(kk-1)+((ha(kk-1)+ha(kk))/2)*(persons(kk)-persons(kk-1)); %calculate the in-
```

```
tegral of minority group end
clear matchy; % matchy is the index of potential median voters
clear match;
LL=length(persons);
for kk=1:length(persons);
if intb(kk) ¡= 0
for jj=LL:-1:1;
fool=inta(jj)+intb(kk);
if fool ¿= 0
matchy(1,kk)=jj;
matchy(2,kk)=kk;
fool1=inta(jj)+intb(kk);
LL=jj;
break
break
end end
else
end
end
lmatch=length(matchy);
gg=1;
```

for ff=1:lmatch;

if matchy(1,ff) = 0;

match(1,gg)=matchy(1,ff);

match(2,gg)=matchy(2,ff);

gg=gg+1; end end

alphaa=zeros(1,length(persons));

betaa=zeros(1,length(persons));

alphab=zeros(1,length(persons));

betab=zeros(1,length(persons));

Fg1a=zeros(1,length(persons));

Fg1a(1)=0;

for kk=2:length(persons);

$Fg1a(kk) = Fg1a(kk-1) + ((persons(kk-1)^invfs * ha(kk-1) + persons(kk)^invfs * ha(kk))/2) * (persons(kk) - persons(kk-1))$; %calculate the integral

end

Fg2a=zeros(1);

for kk=2:length(persons);

$Fg2a = Fg2a + ((persons(kk-1)^invfs * h(kk-1) + persons(kk)^invfs * h(kk))/2) * (persons(kk) - persons(kk-1))$; %calculate the integral

end

Fga=inta;

$alphaa = ss0 * persons.^invfs + Fg1a * s0p$;

betaa=s0p*(1+Fga)*(Fg2a);

Fg1=zeros(1,length(persons));

Fg1(1)=0;

for kk=2:length(persons);

$Fg1(kk) = Fg1(kk-1)+((persons(kk-1)^inv fs*hb(kk-1)+persons(kk)^inv fs*hb(kk))/2)*(persons(kk) - persons(kk-1))$; %calculate the integral

end

Fg2=Fg2a;

Fg=intb+1/2;

$alphab = ss0 * persons.^inv fs + Fg1 * s0p$;

betab=s0p*(1+Fg)*(Fg2);

if min(size(match)) == 1;

L0=1;

L1=1;

else

L0=1;

L1=floor(length(match));

end

for i=L0:L1,

kk=match(1,i);

jj=match(2,i);

ptatest=(alphaa(kk)-fs*betaa(kk))/(alphaa(kk)-sigma*betaa(kk));

```
ptbtest=(alphab(jj)-fs*betab(jj))/(alphab(jj)-sigma*betab(jj));

if ptatest ¿= 0

if ptbtest ¿= 0

if ptatest ¡= 1

if ptbtest ¡= 1

ptay(i+1-L0)=(alphaa(kk)-fs*betaa(kk))/(alphaa(kk)-sigma*betaa(kk));

ptby(i+1-L0)=(alphab(jj)-fs*betab(jj))/(alphab(jj)-sigma*betab(jj));

end

end

end

end

end

lptb=length(ptay);

gg=1;

for ff=1:lptb;

if ptay(1,ff)  = 0;

pta(gg)=ptay(ff);

ptb(gg)=ptby(ff);

gg=gg+1;

end

end
```

```
ptvec=abs(pta-ptb);

[non1,index2]=min(ptvec);

prefB(k)=ptb(1,index2);

end

plot(B,prefB)
```

Chapter 7

Appendix 2: Graph

Figure 7.1: t, redistribution size as a function of b, the size of the minority group