Computational Methods for Control of Queueing Models in Bounded Domains

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(ABSTRACT)

The study of stochastic queueing networks is quite important due to the many applications including transportation, telecommunication, and manufacturing industries. Since there is often no explicit solution to these types of control problems, numerical methods are needed. Following the method of Boué-Dupuis, we use a Dynamic Programming approach of optimization on a controlled Markov Chain that simulates the behavior of a fluid limit of the original process. The search for an optimal control in this case involves a Skorokhod problem to describe the dynamics on the boundary of closed, convex domain. Using relaxed stochastic controls we show that the approximating numerical solution converges to the actual solution as the size of the mesh in the discretized state space goes to zero, and illustrate with an example.

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Dedication

A Estela

Acknowledgments

I would like to thank Professor Marty Day for his help, time, and immense patience. I would also like to thank Professor Terry Herdman for his deep respect to my personal process and continuous encouragement. To both of them many thanks for their commitment to my success. I also want to thank my friends, in Blacksburg and elsewhere, who made my path enjoyable and kept me sane and human. They are too many to list here; they know who they are. A special gratitude goes to the people of the United States, for a world of possibilities. Finally, to the people of El Salvador and our history, for I am their product.

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Chapter 1

Introduction

The study of queueing networks has a wide variety of applications. We find this in different manufacturing industries, telecommunications, and transportation. One substantial challenge is being able to control the system of queues in order to satisfy specific purposes, for example, to provide the fastest service at the smallest cost possible.

Our work provides a numerical approximation for a deterministic fluid control problem. The dynamics of this system is determined by exogenous arrival rates and service rates; it is complicated by constraints in the minimum and maximum capacity of each queue. Given an initial state, the amount of units waiting for service at each queue, the control mechanism will attempt to empty the system while minimizing a prescribed cost functional which depends on the configuration of the system at a given point in time.

In the following paragraphs we first provide a broad overview of the problem, and we explain how our work fits in the context of the existing literature, and finally we provide an overview of the structure of this document.

Consider a network where service is provided at several stations in the form of different tasks. Customers line up in queues demanding services of one task at a time. Once a customer receives service of one task, it moves to the next queue to wait for service of another task from another server or the same server (reentrant queues) or leaves the system. Customers enter the system at arrival rates following a Poisson process. In general a customer could enter the system at any queue. At each queue, a customer waits in line and once at the front of the queue the customer receives service at an exponentially distributed amount of time; tasks may be performed at different service rates. The rate at which customers arrive at each queue depends on the exogenous arrival rate for that queue and the effective service rate of the preceding queue. One constraint on the queues is that each of them has a maximum capacity. Another obvious constraint is that at each queue there cannot be a negative number of customers. It is possible to have other linear constraints to determine the set of all possible combinations of the number of customers at each queue, the state space

G, which is a convex, closed and bounded subset of $\mathbb{R}^n$ including 0. Here n is the number of queues. The configuration of the system determines how the system reacts to an attempt from any queue to violate these constraints. For example, when the system attempts to exceed the capacity of one the queues, one scenario would be to drop the customer out of the system. Another possibility is to push the customer back into the previous line. One option to enforce these constraints is via the design of controls. Instead, we choose to incorporate a reflection mapping directly in the dynamics of the system. The distribution of service at the service station is determined by a control (measurable) function that takes values from a convex and compact control space U. This describes a stochastic queueing control problem.

Control problems of stochastic queueing networks are often approximated by fluid, deterministic models because of the close connection between the stability of the stochastic network and the corresponding fluid model (see Bäuerle [8]). Also we note that often the latter are easier to solve and produce the desired stability [13]. These fluid limit processes have been studied and applied very amply (see [1, 2, 5, 13, 14, 17, 21, 31]). Under some conditions such as smoothness of the value function, the relationship between the solution of the Dynamic Programming Principle and the solution of the Hamilton-Jacobi Bellman equation is established; in many cases, however, the value function is not "smooth enough" and either the solution is only local or existence/uniqueness cannot be guaranteed. Crandall and Lions [16] introduced the notion of a viscosity solution for the Hamilton Jacobi equation, allowing for global solutions where the classical method of characteristics failed.

An inherent characteristic of the fluid models arising from queuing networks is the presence of reflecting boundaries. This is commonly addressed by incorporating reflection mappings of the appropriate class of functions. The extent of the study of this problem comprises a variety of domains and types of functions. Harrison and Reiman [26] utilize this approach, the Skorokhod problem, to heavy traffic limits in queueing networks with constant reflections on a face. However they consider only continuous functions on the nonnegative orthant, whereas Dupuis and Ishii [22] admit functions that are right continuous with limits on the left and with domains on more general convex sets.

Uniqueness theorems for many types of Hamilton-Jacobi equations are standard in the theory of viscosity solutions (see [7]). However the particular form of the equation for our problem falls outside the scope of those results. Recently however there has been some progress on extending uniqueness results to problems of our type (see [20]). Since we will not be using those results there is no need to describe them further here. We only mention them as a possible alternative approach to our problem. For that reason, and as a general tool, numerical solutions are attempted. One approximation scheme is a finite difference approximation to the HJB equation. Crandall and Lions [15] developed this method for the continuous function problem on an open set. Souganidis in [34] generalized this method to other approximation schemes.

An alternative to solving the optimal control problem of the fluid deterministic model via solving the HJB equation is to approximate its solution by solving a discrete controlled

Markov Chain-based problem. The Markov Chain approximation technique was introduced around 1977 by Kushner and later used by Kushner and Dupuis [28]. In [12], Boué and Dupuis present a thorough treatment of such an approximation technique for deterministic control problems, under certain assumptions, in particular, continuity of the value function. They provided a proof of convergence of the solution of the approximating problem to the actual solution of the deterministic control problem via relaxed controls. In their work they deal with unbounded controls and although their state space is bounded, they do not consider reflecting dynamics. In our study we restrict the control space to a convex and compact set, which is more realistic for the queuing network applications, but we consider the complexity of reflecting dynamics. As part of our work, we include the proof of continuity of the value function. We continue to use the stochastic relaxed control representation.

In Chapter 2, we begin by describing the basic components of our deterministic/fluid problem and of its limiting approximation version, as well as the technical hypotheses related to the latter. We lay out the specifics of the dynamics governing the state as the solution of a differential equation with initial condition, the characteristics of the state space described as the intersection of a set of linear inequalities, and the basic properties of the control space. We also include a description of the Skorokhod problem, the mechanism to enforce the constraints on the state space (the reflection dynamics), hypotheses on the bounds and Lipschitz conditions of the velocity function and the cost function, and the definitions of the cost and value functions as well.

A basic assumption commonly made is that the value function is continuous. We prove this assertion in Chapter 3 by relying on results such as convergence of the cost functional and compactness of the set of relaxed controls from Chapter 5. We also show the link between the value function and the solution to the Dynamic Programming Principle as an alternative approach to showing uniqueness of the solution to the optimal control problem.

Chapter 4 describes the Markov Chain (MC) method we use to solve the control problem in a discretized state space as an approximation to the fluid model of Chapter 2. We specify the conditions that are necessary for consistency with the fluid model and describe how the Markov Chain is constructed by specifying the transition probabilities and time lapses in two similar methods. We define the value function on this discretized model, define a discretization error function, and describe the computational method to solve the discrete space optimal control problem as well as the conditions to guarantee its solution.

The core chapter of this work is Chapter 5. In this chapter we show that the value function in the discrete model converges to the value function of the continuous model under the stated conditions. For this purpose we represent the control functions as a compact set of relaxed controls, show existence and uniqueness of the solution of the corresponding equation, and prove the continuity of the mapping that defines the cost functional from the initial state and the functions of relaxed controls and approximation errors.

Finally, Chapter 6 shows an example to illustrate how the numerical method is implemented, as well as the verification of the required hypotheses.

Chapter 2

Hypotheses

We begin by describing the basic components of our problem and the technical hypotheses regarding them. We draw from the work of Day [17, 18] for the representation of the state space and the description of the dynamics. The state space $G \subset \mathbb{R}^n$ will be a compact, convex polyhedron containing 0. ∂G will denote its boundary and G^o its interior. G is defined by a system of linear inequalities

$$x \cdot n_i \geq c_i, \quad c_i \leq 0 \tag{2.0.1}$$

where $i = 1, 2, \ldots, N$; n_i is assumed to be a unit vector for each i, and $I(x) = \{i : x \cdot n_i = c_i\}$ is the set of active constraints for $x \in G$. The boundary ∂G is the union of "faces" $\partial_i G = \{x \in G : x \cdot n_i = c_i\}$. Associated with each constraint i there is a vector d_i that has to be provided in the problem statement and which is normalized by

$$d_i \cdot n_i = 1. \tag{2.0.2}$$

The role of the restoration vector d_i is to provide a direction to drive the system back into the state space when trying to escape through face i. For each $x \in \partial G$ we define the normalized convex combinations of the d_i for the active constraints, that is

$$d(x) = \left\{ \sum_{i \in I(x)} \alpha_i d_i : \alpha_i \geq 0, \left\| \sum_{i \in I(x)} \alpha_i d_i \right\| = 1 \right\}. \tag{2.0.3}$$

We can expand the definition (2.0.3) to all G by defining $d(x) = 0$ if $x \in G^0$. We need to make some technical hypotheses on the n_i and d_i to insure important properties of the Skorokhod problem (see Section 2.2 below for details).

In the following two sections we describe the dynamics of the system and provide a more detailed explanation of the Skorokhod problem.

2.1 System dynamics

Consider a network consisting of n queues. Customers arrive from outside the system to the queues at (possibly time dependent) rates given by the column vector (all our vectors are column vectors) $\beta = [\beta_1, \ldots, \beta_n]$, not necessarily the same rate at each queue. The servers allocate their effort according to a measurable function $u(\cdot) \in \mathcal{U} = \{u : \mathbb{R}_+ \to U, u \text{ measurable}\}$. The function $u(t) = [u_1, \ldots, u_n]$ takes values in a convex and compact metric space of admissible control values U. Suppose there are m servers. Server i offers service to queues $\{S_i\}$. Notice that $\cup_{i=1}^{m} S_i = \{1, 2, \ldots, n\}$. Each server has to allocate its total capacity among its queues; for example, if one of the servers has queues S_j, then $\sum_{i \in S_j} u_i = 1$. Queue i receives service at a maximum rate s_i which once affected by the control will influence the rate at which customers arrive to the next queue or leave the system. This combined effect is summarized in a matrix M, one column per queue. The matrix M affects the vector $u(t)$ to generate the appropriate contributions to the dynamics $\dot{x}$.

For the sake of illustration, we use an example of one server and two queues, with exogenous arrivals only in one queue as depicted in Figure 2.1. The same system will be used in Chapter 6 as an example of how to apply the numerical method, which is the primary purpose of this work. Many other configurations are possible; we refer the reader to [18, 21, 31] for other examples of networks.

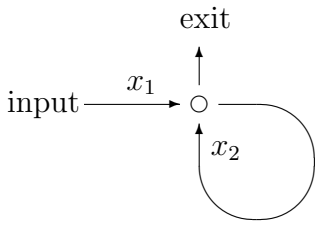

Figure 2.1: One server, two queues

In this system, matrix M takes the form

$$M = \begin{bmatrix} s_1 & 0 \\ -s_1 & s_2 \end{bmatrix}.$$

This class of examples is described more generally by a preliminary description of the state

$x(t)$, that is the length of the queues at time t, given by

$$\dot{x}(t) = \beta - M\, u(t). \tag{2.1.1}$$

We now impose the non-negativity condition together with a bound on the capacity of each queue, say $0 \leq x_i \leq C_i$. The Skorokhod problem provides a way to model these constraints by adding some multiples of the corresponding restoration vector (also called reflecting vector to the system dynamics). For the same example mentioned above, if $x_1 = C_1$, we need to push out of the system any extra request for service coming into queue 1 thus a reflecting vector $\tilde{\mathbf{d}}_\mathbf{1} = [-1\ 0]^T$, whereas a reflecting vector $\mathbf{d_1} = [1\ -1]^T$ would take action if $x_1 = 0$ with an effect that would be equivalent to pulling out service request from queue 2 and reallocating it to queue 1. A more practical interpretation is that of slowing down the activity in the server that affects queue 1. The remaining reflecting vectors are

$$\mathbf{d_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_2 = 0 \text{ and } \tilde{\mathbf{d}}_\mathbf{2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ x_2 = C_2.$$

as shown in Figure 2.2.

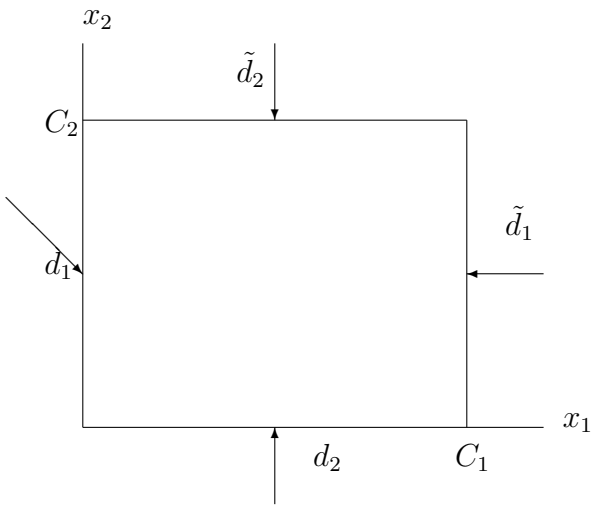

Figure 2.2: Reflecting vectors

More formally, this corresponds to the Skorokhod dynamics

$$\dot{x}(t) = \pi(x(t),\, v(t)) \tag{2.1.2}$$

where $v(t)$ (which we will refer to as "velocity") is the same as the $\dot{x}(t)$ in (2.1.1). Actually, v is a function of x and u ($v = v(x,\, u)$). The function $\pi(x,\, v)$ is the velocity projection map

that for a control u and any $x \in G$, we define as follows.

$$\pi(x,\, v) = \begin{cases} v, & \text{if } x \in G^o \\ v + \sum_I \alpha_i\, d_i, & \text{if } x \in \partial G, \end{cases} \tag{2.1.3}$$

where the α_i satisfy the complementarity conditions

$$\alpha_i \geq 0, \quad n_i \cdot w \geq 0, \quad \alpha_i (n_i \cdot w) = 0, \tag{2.1.4}$$

with $w = \pi(x,\, v) = v + \sum_I \alpha_i\, d_i$, for $x \in \partial G$.

Now we expand the definition of the dynamics of our model as

$$\begin{cases} \dot{x}(t) = \pi(x(t),\, v(x(t), u(t))) \\ x(0) = x_0 \in G \end{cases} \tag{2.1.5}$$

together with (2.1.1), (2.1.3), and (2.1.4).

2.2 The Skorokhod problem

We present here the Skorokhod Problem (SP) whose definition (Definition 2.2.1) we transcribe from [2] (also see [28]) with the notation adapted to match ours. We should see this definition as an expansion of (2.1.5):

$$\begin{cases} \dot{x}(t) = \pi(x(t),\, \dot{y}(t)) \\ x(0) = x_0 \in G. \end{cases} \tag{2.2.1}$$

For a function of bounded variation $\eta : [0,\, \infty) \to \mathbb{R}^n$, we let $|\eta|(t)$ denote the total variation over the interval $[0,\, t]$. Let $D_+([0,\, \infty) : \mathbb{R}^n)$ be the set of functions that are continuous from the right with limits from the left, with domain $[0, \infty)$ and range R^n. Then the Skorokhod problem is defined as follows.

Definition 2.2.1. *Let $y \in D_+([0,\, \infty) : \mathbb{R}^n)$ be given. Then $(x,\, y,\, \eta)$ solves the Skorokhod problem for y (with respect to G and d_i, $i = 1, 2, \ldots, N$) if $x(0) = y(0)$, and if for all $t \in [0,\, \infty)$*

1. $x(t) = y(t) + \eta(t)$,
2. $x(t) \in G$,
3. $|\eta|(t) < \infty$,
4. $|\eta|(t) = \int_{[0,\, t]} 1_{\{x(s) \in \partial G\}} d|\eta|(s)$,

5. *There exists a Borel measurable function* $\gamma : [0, \infty) \to \mathbb{R}^n_+$ *with* $\gamma(t) \in d(x)$ *as in (2.0.3)* ($d|\eta|-$*almost everywhere) as and such that*

$$\eta(t) = \int_{[0,\, t]} \gamma(s)\, d|\eta|(s).$$

Given a function $y : [0, T] \to \mathbb{R}^n$ with $y(0) \in G$, the Skorokhod problem consists of finding a function $x : [0, T] \to G$ such that

$$x(t) = y(t) + \int_{[0,\, t)} \gamma(s)\, d|\eta|(s).$$

We want to refer to the solution map using the notation

$$\begin{aligned} \Gamma : \mathbb{R}^n &\to G \\ \Gamma(y(\cdot)) &= x(\cdot) \end{aligned} \tag{2.2.2}$$

is called the Skorokhod map ([17]).

Our model uses

$$y(t) = y(0) + \int_0^t v(x(s),\, u(s))\, ds$$

as the input for the Skorokhod problem.

With this notation, we can represent (2.1.5) as

$$\begin{cases} x(t) = y(t) + \int_{(0,\, t]} \gamma(s)\, d|\eta|(s) \\ y(t) = x_0 + \int_0^t v(x(s),\, u(s))\, ds \\ y(0) = x_0 \end{cases} \tag{2.2.3}$$

or equivalently

$$\begin{cases} x(\cdot) = \Gamma(y(\cdot)) \\ y(t) = x_0 + \int_0^t v(x(s),\, u(s))\, ds \\ y(0) = x_o. \end{cases} \tag{2.2.4}$$

The integral in the first equation of (2.2.3) takes care of the reflecting boundary property, with γ determining the direction of the reflection and η the intensity.

As Dupuis and Ishii point out in [22], when considering differential games associated with queueing systems (as in [1] and [21]) only absolutely continuous functions are relevant, as it is the case according to the definition of our control problem; in this case the velocity projection map $\dot{x}(t) = \pi(x(t),\, \dot{y}(t))$ with initial conditions is the differential version of the Skorkhod problem.

The following assumption is fundamental for uniqueness of the solution to the Skorokhod Problem (SP).

Assumption 2.1 *There exists a compact, convex set $B \subseteq \mathbb{R}^n$ with $0 \in B^0$, such that for each $i = 1, \ldots, N$ and $z \in \partial B$, and any inward normal v to B at z,*

$$|z \cdot n_i| < 1 \quad \textit{implies} \quad v \cdot d_i = 0.$$

One of the consequences of this assumption is Theorem 2.2 below, as found in [22], which states the nature of the Lipschitz continuity of Γ.

Existence of solution to the Skorokhod problem is related to the following assumption.

Assumption 2.2.2. *Let $F = I(x)$, $x \in \partial G$. We define N_F to be the $n \times |F|$ matrix whose columns are n_i (note that $|F| \leq N$ from above), and Γ_F is the $n \times |F|$ matrix whose columns are d_i, then we also assume that*

$$N_F^T \Gamma_F \tag{2.2.5}$$

is coercive. This means that for any $a_i \in \mathbb{R}$, $i \in F$, not all zeros,

$$\left(\sum_F a_i n_i \right)^T \left(\sum_F a_i d_i \right) > 0. \tag{2.2.6}$$

Theorem 2.2. *Assume the conditions of Assumption 2.1 and (2.2.5); let (ϕ_1, ψ_1, η_1) and (ϕ_2, ψ_2, η_2) be any two solutions of the corresponding SP, where $(\psi_1, \psi_2) \in D([0, T]; \mathbb{R}^n)^2$. Then there exists $C_\Gamma < \infty$ (which is independent of T) such that*

$$\sup_{t \in [0,T]} |\eta_1(t) - \eta_2(t)| \leq C_\Gamma \sup_{t \in [0,T]} |\psi_1(t) - \psi_2(t)|,$$

$$\sup_{t \in [0,T]} |\phi_1(t) - \phi_2(t)| \leq C_\Gamma \sup_{t \in [0,T]} |\psi_1(t) - \psi_2(t)|.$$

In particular, C_Γ is bounded above by the diameter of any set B satisfying Assumption 2.1, plus 1.

In the same paper, Dupuis and Ishii assume the existence of a projection from $\mathbb{R}^n$ into G to prove the existence of solution to the Skorokhod problem on a polyhedral cone. See Theorems (3.2) and (3.3) of [22]. Day provides a unified treatment of existence in [17] via the set of complementary conditions (2.1.4), also under Assumption 2.1, for closed convex polyhedra. We refer the reader to [18] for an example on the verification of Assumption 2.1. For the rest of our work, we make this assumption.

2.3 Hypotheses on a Controlled Fluid System

Let

$$\tau_\epsilon = \inf \{t \geq 0 : \|x_{x_0, u}(t)\| < \epsilon\} \tag{2.3.1}$$

where $x_{x_0,\,u}$ is the process with that begins at x_0 and uses a control u. Notice that as $\epsilon \downarrow 0$, τ_ϵ increases.

Our definition of τ_0 is that

$$\tau_0(x(\cdot)) = \lim_{\epsilon \downarrow 0} \tau_\epsilon \quad (= +\infty \text{ allowed}). \tag{2.3.2}$$

This is the first time the system approaches the origin in the limit. It is important for us to consider this definition given that our process $x(\cdot)$ is defined on the set $D_+([0,\,\infty) : \mathbb{R}^n)$. Due to the discontinuities, in general $\tau_0(x(\cdot)) \leq \inf\{t \geq 0 : x(t) = 0\}$; equality would hold, however, for continuous functions $x(\cdot)$. We need to point out that for any control if $y(0) = 0$ then $\tau_0 = 0$.

Consider the cost functional

$$J(x_o, u(\cdot)) = \int_0^{\tau_0} L(y(t),\, u(t))\, dt \tag{2.3.3}$$

with value function

$$V(x) = \inf_{u(\cdot) \in \mathcal{U}} J(x, u(\cdot)), \quad \forall\, x \in G. \tag{2.3.4}$$

From the definition of the cost functional and of τ_0 it is clear that

$$J(0, u(\cdot)) = 0.$$

The formulation of our control problem assumes the following hypotheses of the control set U, the cost function L and the velocity v.

$$\begin{aligned}
&\text{(i)} && U \text{ is compact} \\
&\text{(ii)} && |v(x,u)| \leq C_1, \quad \forall\, (x,u) \in G \times U; v \in C(\mathbb{R}^n \times U : \mathbb{R}^n) \\
&\text{(iii)} && |v(x,u) - v(y,u)| \leq C_v |x - y| \quad \forall\, x,\, y \in G, u \in U \\
&\text{(iv)} && \inf_{x \in G,\, u \in U;\, |x| > \delta} L(x,\, u) > 0, \quad \text{each } \delta > 0;\; L \in C(\mathbb{R}^n \times U : \mathbb{R}^+) \\
&\text{(v)} && L(x,\, u) \leq C_2, \quad \forall\, (x,u) \in G \times U \\
&\text{(vi)} && |L(x,u) - L(y,u)| \leq C_L\, m(|x - y|) \quad \forall\, x,\, y \in G,\, \forall\, u \in U,\; m(t) \to 0 \text{ as } t \downarrow 0
\end{aligned} \tag{2.3.5}$$

for some constants C_v, C_1, C_2 and C_L. Usually m is assumed to be nondecreasing.

Chapter 3

Continuity of the Value Function and Dynamic Programming

A common and basic assumption in the literature is that the value function $V(x)$ of the problem in consideration is continuous. In Section 3.2 we show that this is the case under the assumptions of Chapter 2 and some additional assumptions.

This result is particularly important when using a Dynamic Programming approach to optimal control problems. The value function (2.3.4) satisfies a functional equation called the Dynamic Programming Principle (DPP) and its infinitesimal version is the Hamilton-Jacobi-Bellman (HJB) equation, which is a first order nonlinear partial differential equation. The basic idea is that a function that solves the HJB equation (often in the viscosity sense) is a solution to the DPP and therefore the optimal solution to the control problem. Moreover, if the limit of a sequence of discrete versions of the HJB equation is a continuous function, then this function is the optimal solution of the control problem. We refer the reader to [7] for more in depth coverage of this approach. A brief discussion of the dynamic programming method will be the focus of Section 3.1.

3.1 Dynamic Programming and the Hamilton-Jacobi Bellman Equation

The purpose of this section is to briefly present the Dynamic Programming method and the Dynamic Programming Principle (DPP) derived from the former and its connection with the Hamilton-Jacobi Bellman (HJB) equation. The importance of the DPP in our work is two-fold: On the one hand we use the DPP in the proof of upper semicontinuity of the value function $V(x)$ as part of the proof of Theorem (3.2.1); on the other hand, it will be used in the numerical method to be developed in Section 4.3.

Dynamic Programming is an optimization method developed by Richard Bellman. For an introduction on this topic see Fleming and Rishel [25]. The focus on the value function as the central quantity comes from the Dynamic Programming method.

Bardi and Capuzzo-Dolcetta [7] explain the derivation of the Dynamic programming Principle in the following manner. Assume that for each state x there is an optimal control u^* so that

$$V(x(0)) = J(x,\, u^*) = \int_0^{\tau_o} L(y(s),\, u^*(s))\, ds.$$

Then, for any $t > 0$,

$$V(x(0)) = \int_0^t L(y(s),\, u^*(s))\, ds + \int_t^{\tau_o} L(y(s),\, u^*(s))\, ds.$$

Under some conditions and using an argument based on the semigroup property we have

$$V(x(0)) = \int_0^t L(y(s),\, u^*(s))\, ds + V(y_{u^*}(t))$$

where $V(y_{u^*}(t))$ is the value function at the state y reached at time t following an optimal control u^*.

If a function V solves the DPP, under certain conditions it solves the Hamilton-Jacobi Bellman equation in the classical or viscosity sense. This connection as well as many applications have received ample attention [1, 12, 19, 30, 6, 16, 15].

For the specific conditions of our problem, that is, under assumptions of Section 2.3 it can be shown that for any $0 \leq t$

$$V(x_0) = \inf_{u(\cdot)\in\mathcal{U}} \left\{ \int_0^{t\wedge\tau_o} L(y(s),\, u(s))\, ds + V(y(t\wedge\tau_o)) \right\}. \tag{3.1.1}$$

This equation is called the Dynamic Programming Principle (DPP).

Also notice that by 2.3.5(ii) and (iii) for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\inf_{x\in G, |x|\geq\epsilon} V(x) \geq \delta. \tag{3.1.2}$$

Following the work of Kushner and Dupuis [28] and Boué and Dupuis [12], in Chapter 4 we use the DPP in the numerical solution to the Markov Chain-based problem that approximates the solution to the control problem from Chapter 2. The value function and the dynamic programming principle for the Markov chain problem are (4.0.13) and (4.0.14) respectively.

3.2 Continuity of the value function

We continue to consider the problem introduced in Chapter 2 and all the hypotheses, in particular (2.3.5). We also assume the results of the following lemmas from Chapter 5:

Lemma 5.2.1 (compactness of the controls,) Lemma 5.2.3 (density of the control space,) and Lemma 5.2.4 (existence and uniqueness of solution of the Skorokhod problem and continuity of the mapping from an initial value and a control function into the process $x_{x_{0,\,u(\cdot)}}(\cdot)$ from (2.1.5)). The main result of this chapter is the following theorem.

Theorem 3.2.1. *For a given $x_0 \in G$ assume*

(a) *$V(x) < \infty$ for all $x \in G$ and*

(b) *$V(x)$ is continuous at 0.*

Then $V(\cdot)$ is continuous.

The proof of the theorem will use the following lemmas.

Lemma 3.2.2. *Under the hypotheses of Theorem 3.2.1 and $\|x_0\| > \epsilon$ in G, suppose $x(t) = x_{x_o,\,u(\cdot)}$ and $J(x_0,\, u(\cdot)) < \infty$. Then either $\tau_0 < \infty$ or 0 is a limit point of $x(t)$ as $t \to \infty$.*

Proof. Suppose the conclusion is not true, that is, $\tau_0 = \infty$ and there exists $\epsilon > 0$ with $\|x(t)\| \geq \epsilon$ all $t < \infty$. We will show that (a) would be violated. Recall (3.1.2). Since $\delta \leq V(x_0) \leq \lim_{T\to\infty} \int_0^T L(x(s),\, u(s))\, ds$ there exists $T_1 \geq 1$ with $\int_0^{T_1} L(x(s),\, u(s))\, ds > \delta/2$ by (2.3.5 (iii)). We have $\|x(T_1)\| \geq \epsilon$, $\delta < V(x(T_1)) \leq \lim_{T\to\infty} \int_{T_1}^T L(x(s),\, u(s))\, ds$ thus there exists $T_2 > T \vee T_1$, $\int_{T_1}^{T_2} L(x(s),\, u(s))\, ds > \delta/2$. Continuing in this way there exists $T_n \uparrow \infty$ with $\int_{T_{n-1}}^{T_n} L(x(s),\, u(s))\, ds > \delta/2$. Therefore

$$\int_0^{T_n} L(x(s),\, u(s))\, ds > n\,\delta/2 \Rightarrow J(x_0,\, u(\cdot)) = \infty$$

in contradiction with the hypothesis that $J(x_0,\, u(\cdot)) < \infty$. □

Lemma 3.2.3. *Under the hypotheses of Theorem 3.2.1 and for any $x_0 \in G$, $u(\cdot) : [0,\, \infty) \to U$, $t < \tau_0$ the cost functional J satisfies*

$$J(x_0,\, u(\cdot)) \geq \int_0^t L(x(s),\, u(s))\, ds$$

Proof. From the definition of the cost functional and the value function we have

$$\begin{aligned} J(x_0,\, u(\cdot)) &= \int_0^t L(x(s),\, u(s))\, ds + \lim_{T\to\infty} \int_t^{T\wedge\tau_0} L(x(s),\, u(s))\, ds \\ &\geq \int_0^t L(x(s),\, u(s))\, ds + 0 \qquad\qquad L > 0. \end{aligned}$$

□

We now present a proof for Theorem 3.2.1.

Proof. We establish upper and lower semicontinuity separately. First we prove upper semicontinuity. Consider $x_0 \in G$ and $\epsilon > 0$. There exists $\delta > 0$ so that $\sup_{|x|<\delta} V(x) < \epsilon$. We select a control $u(\cdot)$ so that

$$J(x_0,\, u(\cdot)) < V(x_0) + \epsilon. \tag{3.2.1}$$

By Lemma 3.2.2 there exists $T < \infty$ so that $|x(T)| < \delta$. Consider any sequence $y_n \to x_0$ in G. Then $x_n(\cdot) \to x(\cdot)$ uniformly on $[0,\, T]$, where $x(t) = x_{x_0,\, u(\cdot)}(t)$ and $x_n(t) = x_{y_n,\, u(\cdot)}(t)$ by Lemma 5.2.4 of Chapter 5. For n sufficiently large $|y_n(T)| < \delta$ and $V(y_n(T)) < \epsilon$. We can pass to the limit $n \to \infty$ in

$$\begin{aligned} V(y_n) &\le \int_0^T L(y_n(t),\, u(t))\, dt + V(y_n(T)) \qquad \text{by (3.1.1)} \\ &\le \int_0^T L(y_n(t),\, u(t))\, dt + \epsilon \end{aligned}$$

to conclude

$$\begin{aligned} \overline{\lim}\, V(y_n) &\le \int_0^T L(x(t),\, u(t))\, dt + \epsilon \\ &\le J(x_0,\, u(\cdot)) + \epsilon \\ &\le V(x_0) + 2\,\epsilon \end{aligned}$$

where the second inequality follows from Lemma 3.2.3 and the third inequality is a result of (3.2.1) . Since $\epsilon > 0$ and $y_n \to x_0$ were arbitrary, we conclude that

$$\overline{\lim_{y \to x_0}}\, V(y) \le V(x_0)$$

which is the desired upper semicontinuity.

Next we need to show that V is lower semicontinuous. Consider $x_0 \in G$, $\epsilon > 0$ and let $y_n \to x_0$ with

$$V(y_n) \to \underline{\lim_{y \to x_0}}\, V(y).$$

Let $u_n(\cdot)$ be an ϵ-optimal control for $y_n : J(y_n, u_n(\cdot)) < \epsilon + V(y_n)$. Let $x_n(t) = x_{y_n,\, u_n(\cdot)}(t)$. By Lemma 3.2.3, for every $T < \infty$ we have

$$\int_0^T L(x_n(t),\, u_n(t))\, dt \le J(y_n,\, u_n(\cdot)) < \epsilon + V(y_n).$$

By Lemmas 5.2.1 to 5.2.4 of Chapter 5 there is a $u(\cdot)$: $[0,\, \infty) \to U$ and a subsequence so that for every T, $x_{n'}(\cdot) \to x(\cdot) = x_{x_0,\, u(\cdot)}(\cdot)$ uniformly on $[0,\, T]$ and

$$\int_0^T L(x_{n'}(t),\, u_{n'}(t))\, dt \to \int_0^T L(x(t),\, u(t))\, dt.$$

Therefore, for every T

$$\int_0^T L(x(t),\, u(t))\, dt \leq \epsilon + \underline{\lim}_{y \to x} V(y),$$

which implies

$$V(x) \leq J(x_0,\, u(\cdot)) \leq \epsilon + \underline{\lim}_{y \to x} V(y).$$

So $V(x) \leq \underline{\lim}_{y \to x} V(y)$ providing the desired lower semicontinuity. □

Chapter 4

Markov Chain Method

The motivation to develop numerical methods to approximate the solution of the control problems that we are studying is that usually it is not possible to find an explicit solution for the value function V. Besides, with the increasing technological capability, it is also good to develop these numerical methods as a general tool to solve these control problems. In this chapter we will discuss one such method, the Markov Chain method, applied to our problem. The main idea behind the Markov Chain (MC) method is to approximate the original continuous controlled process $x(t)$ by a controlled finite-state Markov chain $\{X_k^h\}$ where k is the step index and h is the grid spacing parameter. This MC is defined so that certain local properties of the approximating chain are similar to or consistent with those of the original control process, and so that the cost associated with the controlled MC will be close to the cost for the original control problem.

We begin by discretizing the state space into $G^h = G \cap h\,\mathbb{Z}^n$, $h > 0$. We are going to define controlled probabilities that describe how the chain moves from grid point to grid point. These transitions are jumps $X_k \to X_{k+1}$. X_{k+1} will always be limited to one of the immediate neighbors of X_k. (The meaning of "neighbor" depends on the specific way in which the probabilities are defined.) In addition to G^h we need to include points of $h\,\mathbb{Z}^n$ not in G but which are neighbors to the points in G^h. These points will form the set $\widetilde{G}^h$. We will represent these points by $\widetilde{Y}^h$ to distinguish them from those in G and also for a cleaner bookkeeping of the stages. For instance, if from $X_k^h \in G^h$ the state moves to a point in $\widetilde{G}^h$, we call this point $\widetilde{Y}_{k+1}^h$. Finally, for the points in $\widetilde{G}^h$ we want to add their projection onto the boundary (see (4.0.9) below); these points form the set $\partial\bar{G}^h$. The projection of $\widetilde{Y}_{k+1}^h$ into $\partial\bar{G}^h$ is denote by X_{k+1}^h. The complete grid consists of

$$\overline{G^h} = G^h \cup \widetilde{G}^h \cup \partial\bar{G}^h. \tag{4.0.1}$$

In Figure 4.1 we illustrate these definitions with an example. The state space G for this example is the region bounded by the axes x_1 and x_2, and the lines $x_1 = C_1$, $x_2 = C_2$, and $33C_2x_1 + 28C_1x_2 - 49C_1C_2 = 0$. In this picture we have set the values of $h = 0.1$, $C_1 = 1.1$,

and $C_2 = 0.7$. The asterisks represent G^h, the empty dots constitute $\widetilde{G}^h$ and the solid dots represent the set $\partial\bar{G}^h$.

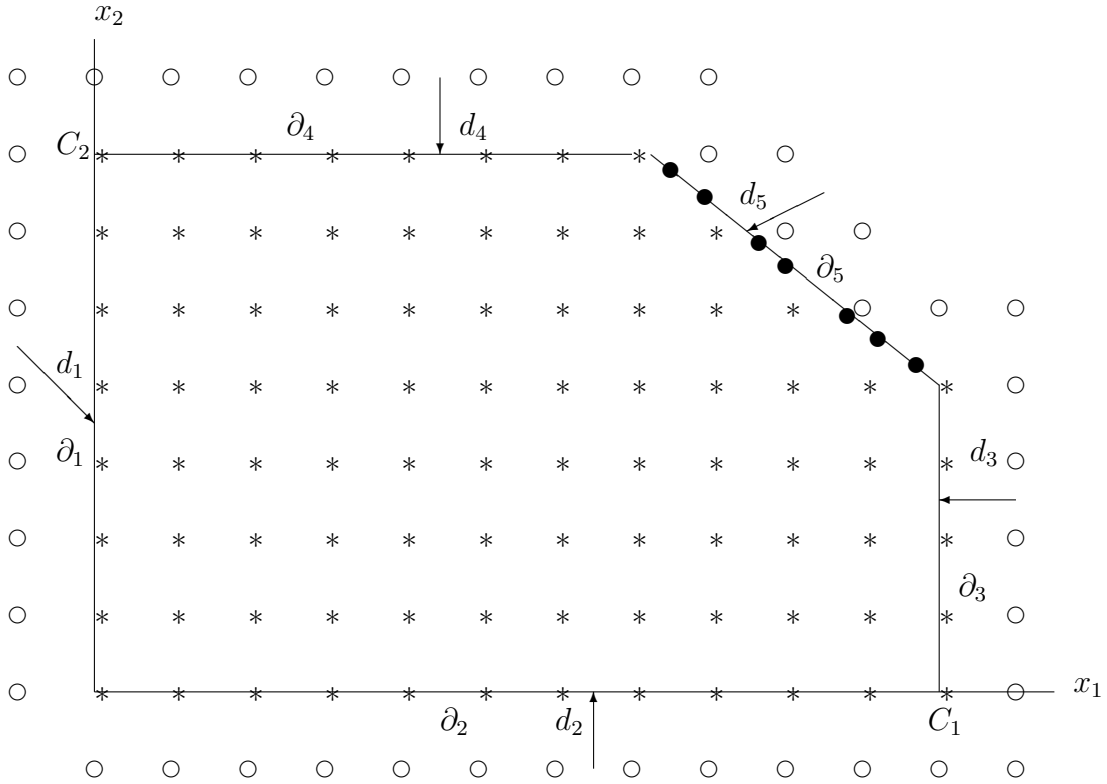

Figure 4.1: Different Types of Grid Points

Let $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\{\mathcal{F}_k\}$ (an increasing family $\{\mathcal{F}_k: k \in \mathbb{Z}^+\}$ of sub-$\sigma-$algebras of $\mathcal{F}$) be the underlying probability space on which the Markov Chain X_k^h $(k = 0, 1, 2, \dots)$ is defined (see Section 4.2). For a given grid (depending on the notion of immediate neighbors), the Markov Chain is accompanied by a time increment process t_k^h, so that X_k^h is viewed as an approximation $X_k^h = x(t_k^h)$ i.e. both the spatial and temporal increments are controlled. Thus for each $x \in \overline{G^h}$ we need to specify both $p(x, y|u)$ and the time increment $\Delta t(x, u)$ and proceed to construct a controlled chain as follows. We start at $(t_0^h, X_0^h) = (0, x_0)$. Suppose we have a control policy for the Markov Chain. By this we mean a sequence $\{u_k^h\}$ of control values $u_k^h \in U$. These controls are any measurable function of the history of $\{X_i^h : i = k, k-1, ..., 0\}$. That is, u_k^h is an $\mathcal{F}_k-$measurable random variable. The chain stays in the state X_k^h in $G^h \cup \partial\bar{G}^h$ for a time $\Delta\, t^h(X_k^h, u_k^h)$ before it "jumps" to the next state X_{k+1}^h with probability $p(X_k^h, X_{k+1}^h|u_k^h)$. In general, the values of $\Delta t^h(X_k^h, u_k^h)$ may depend on the values of X_k^h and/or u_k^h. So, if we set $t_0^h = 0$, then t_{k+1}^h will be defined by $t_{k+1}^h = t_k^h + \Delta\, t^h(X_k^h, u_k^h)$ for $k = 0, 1, 2, \dots$ These jumps initiated and completed at grid points in $G^h \cup \partial\bar{G}^h$ without intermediate steps will be called "standard" jumps. There will be

another kind of jump, "reflection jump," that is subject to different conditions than (4.0.2) and (4.0.3). The specific definitions of p^h and $\Delta\, t^h$ for the standard jumps are case dependent and we will explain some posible choices in Section 4.1, but in general it is required that they comply with the following "local consistency" conditions. Let $x = X_k^h$ and $u = u_k^h$, then

$$\begin{cases} \text{(i)} \quad \mathrm{E}\left[X_{k+1}^h - X_k^h \,|\, \mathcal{F}_k\right] & = \sum_{y \in h\mathbb{Z}^n} (y - x)\, p^h(x,\, y|u) \\ & = v(x,u)\Delta t^h(x,u) + o\left(\Delta t^h\,(x,u)\right) \\ \text{(ii)}\ \mathrm{cov}\left[X_{k+1}^h - X_k^h \,|\, \mathcal{F}_k\right] & = o\left(\Delta t^h\,(x,u)\right) \end{cases} \tag{4.0.2}$$

and

$$\lim_{h\to 0} \sup_{\{x,\,u\}} \Delta t^h(x,\, u) \to 0. \tag{4.0.3}$$

For these standard jumps we will also require that each $\Delta t^h(X_k^h,\, u_k^h)$ is bounded below, that is, for each i and every h there is an $\epsilon > 0$ such that

$$\epsilon \leq \Delta t^h(X_k^h,\, u_k^h). \tag{4.0.4}$$

Conditions (4.0.2) - (4.0.4) will have to be satisfied for each specific implementation.

For one of these standard jumps, let

$$\Delta e_k = X_{k+1}^h - (X_k^h + v(X_k^h,\, u_k^h)\, \Delta t^h(X_k^h,\, u_k^h)). \tag{4.0.5}$$

The interpretation of $\Delta\, e_k$ is that it represents the local (random) error to account for the discrepancy between X_{k+1}^h and the point in G where the control u_k^h will take the system by following the direction of $v(X_k^h,\, u_k^h)$ for a time $\Delta t^h\,(X_k^h,\, u_k^h)$. When compared the expectation of (4.0.5) with (4.0.2 (i)) we can conclude that

$$\mathrm{E}[\Delta e_k] = o\left(\Delta t^h\,(X_k^h, u_k^h)\right).$$

In fact, for the chains that we will construct, the probabilities and time lapses satisfy these stronger properties:

$$\begin{aligned} \text{(i)} \quad & \mathrm{E}\left[X_{k+1}^h - X_k^h \,|\mathcal{F}_k\right] = v(x,u)\Delta t^h(x,u) \\ \text{(ii)}\ & \mathrm{cov}\left[X_{k+1}^h - X_k^h \,|\, \mathcal{F}_k\right] = O(h^2) \end{aligned} \tag{4.0.6}$$

and

$$\lim_{h\to 0} \sup_{\{x,\,u\}} \Delta t^h(x,\, u) \to 0. \tag{4.0.7}$$

Part (i) of (4.0.6) can be expressed as

$$\mathrm{E}\left[\Delta e_k \middle| \mathcal{F}_k\right] = 0. \tag{4.0.8}$$

Now we describe the second kind of jump: reflection jumps.

For a jump that passes through a point in $\widetilde{G}^h$, the action of the Skorokhod problem on the grid is represented by an "instantaneous" jump (i.e. $\Delta t^h\,(X_k^h,\, u_k^h) = 0$) to a point in the boundary of G following the direction prescribed by the corresponding reflection vector (2.0.3). These jumps will be referred to as "reflection" jumps. Thus, to mimic the effect of the Skorokhod map, to each $\widetilde{Y}^h \in \widetilde{G}^h$ we assign a point X^h in $\partial\bar{G}^h$ and a "time" $\Delta\eta$ such that $p^h(\widetilde{Y}^h,\, X^h) = 1$ (uncontrolled) and the projection of $\widetilde{Y}^h$ is

$$X^h = \widetilde{Y}^h + \gamma(X^h)\,\Delta\eta. \tag{4.0.9}$$

(This is the discrete projection Π of [17].)

We generalize (4.0.9) to any jump originating anywhere in the grid with the distinction that $\gamma(X^h) = 0$ and $\Delta\,\eta = 0$ if $X^h \in G^h$, that is, for a standard jump. Notice that our definitions of the chain probabilities will be such that a point X^h in $\partial\bar{G}^h \setminus G^h$ can be accessed only from a point $\widetilde{Y}^h$ in $\widetilde{G}^h$. Also notice that every reflection jump consists of two parts, one of which has $\Delta\, t^h(X_k^h,\, u_k^h) > 0$ keeping the time to go from X_k^h to X_{k+1}^h bounded below by (4.0.4).

We will also adopt the following convention. We denote the process $\{Y_k^h\}$ as the one that follows $\{X_k^h\}$ except for the effect of all the previous reflection mapping. Thus, recursively we set

$$\begin{cases} Y_0^h &= X_0^h, \\ Y_k^h &= Y_{k-1}^h + v(X_{k-1}^h,\, u_{k-1}^h)\,\Delta\, t^h(X_{k-1}^h,\, u_{k-1}^h) + \Delta\, e_{k-1} \\ X_k^h &= X_{k-1}^h + v(X_{k-1}^h,\, u_{k-1}^h)\,\Delta\, t^h(X_{k-1}^h,\, u_{k-1}^h) + \Delta\, e_{k-1} + \gamma(X_k^h)\,\Delta\eta_k. \end{cases} \tag{4.0.10}$$

Clearly, we have

$$\Delta e_k = Y_{k+1}^h - (Y_k^h + v(X_k^h,\, u_k^h)\,\Delta t^h(X_k^h,\, u_k^h)). \tag{4.0.11}$$

We now pose an optimal control problem for this chain, intended to approximate the original control problem (2.2.3) of Chapter 2. Given the function L as used in (2.3.3) satisfying conditions (2.3.5 (iii), (iv), and (v)), a grid spacing h, and a control policy $\{u_i^h\}$ we define the cost functional and the value function as

$$W^h(x_0,\, \{u_k^h\}) = \mathrm{E}_{x_0}\left[\sum_{j=0}^{N^h-1} L(X_j^h,\, u_j^h)\,\Delta\, t^h(X_j^h,\, u_j^h)\right] \tag{4.0.12}$$

$$V^h(x_0) = \inf_{u^h} W^h(x_0,\, \{u_k^h\}) \tag{4.0.13}$$

for the discrete process $\{X_k^h\}$ where N^h is the first j for which $X_j^h = 0$ and E_{x_0} represents the expected value given a specific initial state $x_0 = X_0^h$.

For each $x \in \overline{G^h}$, the Dynamic Programming Equation (DPE) characterizing V^h, as used in [12] and derived in [28], is

$$V^h(x) = \inf_{u\in U}\left\{\sum_{y\in\overline{G^h}} V^h(y)\,p(x,y|u) + L(x,u)\,\Delta t^h(x,u)\right\}, \qquad V^h(0) = 0. \tag{4.0.14}$$

This equation for V^h can be solved by an iterative method such as Jacobi or Gauss-Seidel iterations (discussion of those calculations as well as conditions for existence and uniqueness will be in Section 4.3). The idea is that V^h should approximate V if h is small. We establish this approximation in Chapter 5, with examples provided in Chapter 6. In the remainder of this chapter we

1. describe in more detail some possible definitions of $p(x,\, y|u)$,
2. explain how to interpret the result of such a chain constructed in terms of a filtered probability space $(\Omega,\, \mathcal{F},\, P)$ with filtration $\{\mathcal{F}_k\}$,
3. discuss existence and uniqueness for (4.0.14) and
4. explain how the calculations for V^h are carried out, and why the iteration procedures used for that converge.

4.1 Description of the Markov Chain

In this section we will describe in more detail how to obtain different definitions of the probabilities $p(x,\, y|u)$ and the time increments $\Delta t^h(x,\, u)$ for the construction of Markov chains. The fundamental requirement for the controlled chain $\{X_k^h, k < \infty\}$ is that it be locally consistent with the process $\dot{x}(t) = v(x(t),\, u(t))$, for $x \in G^h$ and v satisfying conditions (2.3.5 (i) and (ii)). Recall that the locally consistency condition requires compliance with (4.0.6), (4.0.7), and (4.0.4).

A number of different approaches can be taken to explicitly define the transition probabilities and time increments (see [28] for some examples). It is important that these definitions be made in such a way that the minimum over the controls in the MC calculation of (4.0.14) or equivalent formulation of the problem, can be evaluated efficiently. Such formulations may depend on the structure of $v(x,\, u)$ and $L(x,\, u)$.

In the following we present two different approaches to define the transition probabilities and time increments. Both follow the same strategy, that is, to proceed along $y = x + t\, v(x,\, u)$ until the "cell" boundary is reached (except in the special cases when $v(x,\, u)$ is too small or null); the time "t" to reach the boundary is $\Delta t(x,\, u)$, then write this value y as a convex combination of the grid points on that "face" of the cell. We use the shape of the cell to distinguish between the two methods which basically differ in the metric used. The first method we describe uses the 1-norm (for a vector $x \in \mathbb{R}^n$, $\|x\|_1 = \sum_{i=1}^n |x(i)|$.) This generates a diamond-like cell. Hence we call this the diamond method. In the second method we use the infinity-norm (if $x \in \mathbb{R}^n$ then $\|x\|_\infty = \max\{|x(i)|,\, i = 1,\, 2,\, \ldots,\, n\}$,) producing regular boxes as cells. We call this the box method. Both of these are effective for the example of Chapter 6, although the diamond method is somewhat simpler to implement.

In the discussion below, let $h > 0$ be the grid spacing in every direction. Let e_j be the coordinate vectors in $\mathbb{R}^n$, i.e., the vector with 1 in the j^{th} entry and zeros everywhere else.

4.1.1 The diamond method

For this method we only allow jumps in the direction of the coordinate axes, that is from $x = X_k^h \in G^h$ we can only jump to $y = x \pm h\, e_j$ for only one $j = 1, 2, \ldots n$. Given the control u, the velocity $v(x, u)$ is determined and we define the transition probabilites

$$p_0(x, y|u) = \begin{cases} \frac{|v_j(x,u)|}{\|v(x,u)\|_1}, & \text{if } y = x + \operatorname{sgn}(v_j(x, u))\, h\, e_j \\ 0, & \text{otherwise} \end{cases} \tag{4.1.1}$$

where the subscript j on v refers to the j^{th} coordinate of $v(x, u)$. This is if x is a "standard gridpoint". If $x \in \partial\, \overline{G}^h \setminus G^h$ the formula needs to be modified to accommodate for irregular distance from x, say $y = x + \sum_{j=1}^n \operatorname{sgn}(v_j(x, u))\, h_j\, e_j$. We temporarily make the additional hypothesis that $\inf_{\{x,\, u\}} \|v(x, u)\|_1 \geq K$, for some positive constant K, and set the time increment

$$\Delta_0\, t^h(x, u) = \frac{h}{\|v(x, u)\|_1}. \tag{4.1.2}$$

After discussing this case we will remove the assumption of $\inf_{\{x,\, u\}} \|v(x, u)\|_1 \geq K$.

Notice that indeed (4.0.6 (i)) holds for (4.1.1) since

$$\begin{aligned} \sum_{y \in h\, \mathbb{Z}^n} (y - x) p_0(x, y|u) &= \sum_{j=1}^n h \operatorname{sign}(v_j(x, u))\, e_j\, \frac{|v_j(x, u)|}{\|v(x, u)\|_1} \\ &= \sum_{j=1}^n |v_j(x, u)| \operatorname{sign}(v_j(x, u))\, e_j\, \frac{h}{\|v(x, u)\|_1} \\ &= v(x, u) \Delta\, t^h(x, u). \end{aligned}$$

To show (4.0.6 (ii)) holds we introduce the following notation:
For $j = 1, 2, \ldots, n$ let

$$y_m^h = \begin{cases} x + h\, e_j, & m = j \\ x - h\, e_j, & m = n + j, \end{cases}$$

$$\Delta X = X_{k+1}^h - X_k^h, \qquad \Delta X_m = y_m^h - x \quad (m = 1, 2, \ldots, 2\, n),$$

and let $p(\Delta\, X_m) = p_0(x, y_m^h|u)$, hence

$$p(\Delta\, X_m) = \begin{cases} \frac{|v_j(x,u)|}{\|v(x,u)\|_1}, & \text{if } \Delta\, X_m = \operatorname{sign}(v_j(x, u))\, h\, e_j \\ 0, & \text{otherwise.} \end{cases}$$

Then (4.0.6 (ii)) is equivalent to

$$
\begin{aligned}
&\mathrm{E}\left[(\Delta X - \mathrm{E}[\Delta X])\ (\Delta X - \mathrm{E}[\Delta X])^T \,|X_k^h = x,\, u_k^h = u\right] \\
=&\sum_{m=1}^{2n} (\Delta X_m - \mathrm{E}[\Delta X])\ (\Delta X_m - \mathrm{E}[\Delta X])^T\ p(\Delta X_m) \\
=&\sum_{j=1}^{n} \left(h\, e_j + h\, \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)\ \left(h\, e_j + h\, \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)^T\ p(\Delta X_j) \\
&+ \sum_{j=1}^{n} \left(h\, e_j - h\, \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)\ \left(h\, e_j - h\, \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)^T\ p(\Delta X_j) \\
=&\sum_{j=1}^{n} h^2\ \left(e_j + \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)\ \left(e_j + \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)^T\ p(\Delta X_j) \\
&+ \sum_{j=1}^{n} h^2\ \left(e_j - \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)\ \left(e_j - \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)^T\ p(\Delta X_j) \\
=&h^2 \sum_{j=1}^{n} \left(e_j + \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)\ \left(e_j + \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)^T\ p(\Delta X_j) \\
&+ h^2 \sum_{j=1}^{n} \left(e_j - \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)\ \left(e_j - \frac{v(x,\, u)}{\|v(x,\, u)\|_1}\right)^T\ p(\Delta X_j) \\
=&O(h^2).
\end{aligned}
$$

Since we have assumed that $\inf_{\{x,\, u\}} \|v(x,\, u)\|_1 \geq K > 0$, then $\frac{1}{\|v(x,\, u)\|_1} \leq K^{-1}$. Therefore, for any x and u we have that $h\, \frac{1}{\|v(x,\, u)\|_1} \leq h\, K^{-1}$ which is the same as saying that $\Delta_0\, t^h(x,\, u) \leq h\, K^{-1}$. So, it is easy to see that indeed

$$\lim_{h \to 0} \sup_{\{x,\, u\}} \Delta_0\, t^h(x,\, u) \to 0. \tag{4.1.3}$$

The problem with $v(x,\, u)$ being too small ($\|(v(x,\, u)\|_1 << h$) is that $\Delta_0\, t^h(x,\, u)$ would be extremely large, contrary to $\sup_{\{x,\, u\}} \Delta_0\, t^h(x,\, u) \to 0$. This would be a problem for the convergence analysis of Chapter 5. In order to prevent this, we set the time increment to be the minimum between the original value as in (4.1.2) and a prescribed constant, call it $\Delta\, T$. Thus

$$\widetilde{\Delta\, t} = \min\left\{\Delta_0\, t^h(x,\, u),\, \Delta\, T\right\}. \tag{4.1.4}$$

An obvious choice for this alternative $\Delta\, T$ is a positive constant multiple of h, say $K'\, h$. If the minimum is $\Delta_0\, t^h(x,\, u)$, the probabilities are taken to be $p_0(x\, y|u)$ as defined in (4.1.1), otherwise, we can define the new probabilities based on the original ones as follows. Given the original probabilities $p_0(x,\, y|u)$ associated with $\Delta_0\, t^h(x,\, u)$ and if $\widetilde{\Delta\, t} = \min\{\Delta_0\, t^h(x,\, u),\, \Delta\, T\}$

then

$$\widetilde{p}(x,\, y|u) = \begin{cases} \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h(x,\, u)} p_0(x,\, y|u) & \text{for } y \neq x \\ 1 - \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h(x,\, u)} & \text{for } y = x \end{cases} \tag{4.1.5}$$

(assuming $p_0(x,\, x|u) = 0$) satisfy (4.0.6 (i)) by construction.

If $v(x,\, u) = 0$ we arbitrarily set $\Delta\, t^h(x,\, u) = h$ and assign equal probabilities to each grid point that determines the cell, that is

$$\widetilde{p}(x,\, y|u) = \begin{cases} \frac{1}{2n}, & \text{if } y = x \pm h\, e_j, \quad j = 1,\, 2,\, \ldots,\, n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.6}$$

It can easily be verified that contidions (4.0.6) hold.

In the original process we had that

$$x + v(x,\, u)\, \Delta_0\, t^h(x,\, u) = \sum_{y^h \in h\, \mathbb{Z}^n} y^h\, p_0(x,\, y^h|u).$$

The new time increment $\widetilde{\Delta}\, t$ would take us from x to a point $\widetilde{y} = x + \widetilde{\Delta}\, t\, v(x,\, u)$ in the interior of the cell, requiring $n+1$ points $y_j^h \quad (j = 0,\, 1,\, 2,\, \ldots,\, n)$ to express $\widetilde{y}$ as a convex combination of them. Notice that

$$\begin{aligned} x + \widetilde{\Delta}\, t\, v(x,\, u) &= x + \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h\,(x,\, u)} \Delta_0\, t^h\,(x,\, u)\, v(x,\, u) \\ &= \left(1 - \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h\,(x,\, u)}\right) x + \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h\,(x,\, u)} (x + \Delta_0\, t^h(x,\, u)\, v(x,\, u)) \\ &= \left(1 - \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h\,(x,\, u)}\right) x + \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h\,(x,\, u)}\, y \\ &= \left(1 - \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h(x,\, u)}\right) x + \sum_{y \in h\, \mathbb{Z}^h} \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h\,(x,\, u)} p_0(x,\, y|u)\, y \\ &= \widetilde{p}_0\, x + \sum_{i=1}^{n} \widetilde{p}_i\, y_i^h. \end{aligned}$$

The proof that (4.0.6 (ii)) holds is the same as above when $\widetilde{\Delta}\, t = \Delta\, t^h(x,\, u) = h/\|v(x,\, u)\|_1$. Suppose that $\widetilde{\Delta}\, t = \Delta\, T = K'\, h$. As before, we adopt the following notation. For $j = 0,\, 1,\, 2,\, \ldots,\, n$ let

$$y_m^h = \begin{cases} x, & m = 0, \\ x + h\, e_j, & m = j, \text{ and} \\ x - h\, e_j, & m = n + j; \end{cases}$$

$$\Delta X = X^h_{k+1} - X^h_k, \quad \Delta X_m = y^h_m - x \;=\; \pm\, h\, e_j \quad (m = 1,\, 2,\, \ldots,\, 2\,n), \text{ and } \Delta\, X_0 = 0,$$

and let

$$\widetilde{p}(\Delta\, X_m) = \widetilde{p}(x,\, y|u) = \begin{cases} \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h(x,\, u)}\, p_0(x,\, y|u), & \text{for } m \neq 0, \\ 1 - \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h(x,\, u)}, & \text{otherwise.} \end{cases} \tag{4.1.7}$$

For the case where $\widetilde{\Delta} t = K' h < \Delta_0 t^h$ we have

$$\begin{aligned}
&\mathrm{E}\left[(\Delta\, X - \mathrm{E}[\Delta\, X])\; (\Delta\, X - \mathrm{E}[\Delta\, X])^T\, |X^h_k = x,\, u^h_k = u\right] \\
&= \sum_{m=0}^{2\,n} (\Delta\, X_m - \mathrm{E}[\Delta\, X])\; (\Delta\, X_m - \mathrm{E}[\Delta\, X])^T\; \widetilde{p}(\Delta\, X_m) + (-\widetilde{\Delta}\, t\, v(x,\, u))\, (-\widetilde{\Delta}\, t\, v(x,\, u)^T)\, \widetilde{p}(x,\, x|u) \\
&= h^2 \sum_{j=1}^{n} (e_j + K'\, v(x,\, u))\; (e_j + K'\, v(x,\, u))^T\; \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h(x,\, u)} p_0(\Delta\, X_j) \\
&\quad + h^2 \sum_{j=1}^{n} (e_j - K'\, v(x,\, u))\; (e_j - K'\, v(x,\, u))^T\; \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h(x,\, u)} p_0(\Delta\, X_j) \\
&\quad + (-\widetilde{\Delta}\, t\, v(x,\, u))\, (-\widetilde{\Delta}\, t\, v(x,\, u))^T \left(1 - \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h(x,\, u)}\right) \\
&= O(h^2) + (K'\, h)^2\, v(x,\, u)\, v(x,\, u)^T \left(1 - \frac{\widetilde{\Delta}\, t}{\Delta_0\, t^h(x,\, u)}\right) \\
&= O(h^2).
\end{aligned}$$

We conclude that $\lim_{h \to 0} \widetilde{\Delta}\, t = 0$.

An alternative definition of the time lapse and probabilities for this case is to use an instantaneous jump ($\Delta\, t^h(x,\, u) = 0$) and $p(x,\, y|u) = 1$ for $y = x$ and 0 otherwise, but then we wouldn't have that $L(x,\, u)\, \Delta\, t^h(x,\, u) \neq 0$ which we need for Assumption 1.2 in Section 4.3.

As we will see in Example 6.1, we can use this method along with the specific definitions of $v(x,\, u)$ and $L(x,\, u)$ to generate an expression on the right side of (4.0.14) that is "piecewise linear fractional" in terms of the components of the control u, allowing us to evaluate the infimum by comparing only at a finite number of values of the control. This is what we mean by "efficient evaluation" in Section 4.1.

4.1.2 The box method

A variant of the above approach is what we call the box method, described in this subsection.

We allow jumps to any grid point at a distance h from the current point using the metric induced by the infinity norm. Hence if the current state is x, we can make a random jump to any $y = x + h \sum_{j=1}^{n} \pm b_j\, e_j$ with $b_j = 0$ or 1.

For a given control $u_i^h = u$, we obtain the velocity $v(x, u)$ (assume $v(x, u) \neq 0$) and define the time increment $\Delta t^h(x, u)$ such that $y = x + \Delta\, t^h(x, u)\, v(x, u)$ is on a face of the "cell" (box) and then write y as the probabilistic average of the closest n grid points from that face. This gives a piecewise linear fractional expression of the components of the control u. If $|v|$ does not have a positive lower bound, we make the same modifications as (4.1.4) and (4.1.5) above. The consistency conditions can be verified in a very similar way as we did for the Diamond Method.

4.2 Filtered Probability Space

In Section 4.1 we created a controlled finite state Markov Chain $\{X_k^h\}$ by defining transition probabilities and time increments that satisfy local consistency conditions. In this section we explain how to interpret this MC in terms of a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with filtration $\{\mathcal{F}_k\}$.

The initial state of the network and the dynamics on the network (arrivals to each queue and service provided) is the experiment generating a probability space Ω on which the random variables X_n (state of the system) is defined. Associated with this probability space are the $\sigma-$algebra $\mathcal{F}$ and the probability measure $\mathbf{P}$ conforming the probability triplet $(\Omega, \mathcal{F}, \mathbf{P})$.

More formally, let Ω be the set of all sequences of neighboring grid points. Then for $\omega \in \Omega$

$$\begin{aligned} \omega &= (x_0, x_1, x_2, \dots), \\ X_k^h(\omega) &= x_k \text{ the } k^{th} \text{ entry of } \omega, \\ \mathcal{F}_k &= \sigma(X_i : i \leq k). \end{aligned}$$

Given a sequence $\{u_k\}$ of random variables (u_k being $\mathcal{F}_k-$measurable) with $u_k \in U$, we make the following claim.

Claim 4.2.1. *There exists a probability measure* $\mathbf{P}$ ($\mathbf{P}^{(u_\omega)}$) *so that*

$$\mathbf{P}(X_k^h = i|\mathcal{F}_{k-1}) = p(X_{k-1}^h, i|u_{k-1}^h).$$

The justification to this claim is that this is an application of the Kolmogorov consistency theorem.

There will be a different probability space for each h and for each choice of stochastic controls $\{u_k^h\}$; however, what is important is the very existence of $(\Omega, \mathcal{F}, \mathbf{P})$ and of X_{k+1}^h for a given $\{u_k^h\}$ sequence. Given this, we will be able to construct the appropriate probability measures on the sequence of $\{X_k^h\}$, as we will see in Chapter 6.

4.3 Calculation for the approximating value function

The purpose of creating the MC in Section 4.1 is to be able to approximate the solution of (2.3.4) by the solution of the control problem (4.0.12) and (4.0.13) for the MC, as restated here. Given the function L as used in (2.3.3) satisfying conditions (2.3.5 (iii) - (v)) we define the cost functional and the value function as

$$W^h(x_0,\, \{u_i^h\}) = \mathrm{E}_{x_0}\left[\sum_{j=0}^{N^h-1} L(X_j^h,\, u_j^h)\,\Delta\, t^h(X_j^h,\, u_j^h)\right] \tag{4.0.12}$$

$$V^h(x_0) = \inf_{u^h} W^h(x_0,\, \{u_i^h\}) \tag{4.0.13}$$

for the discrete process $\{X_k^h\}$ where N^h is the first j for which $X_j^h = 0$ and E_{x_0} represents the expected value given a specific initial state $x_0 \in G^h \cup \partial\, \widetilde{G}^h$.

We use Dynamic Programming (DP) as the optimization technique, based on the priciple of optimality. Roughly speaking, the principle of optimality states that if we assume that at step k we follow an optimal decision policy from step $k{+}1$ on with optimal value $V(X_{k+1}^h)$, the optimal decision to be made at step k is the one that produces the minimum cost, including the cost of going from X_k^h to X_{k+1}^h. This is what the Dynamic Programming Equation (DPE) states: For $X_k^h = x$,

$$V^h(x) = \inf_{u\in U}\left[\sum_{y\in\overline{G^h}} V^h(y)\, p(x,\, y|u) + L(x,\, u)\,\Delta\, t^h(x,\, u)\right], \qquad V^h(0) = 0. \tag{4.0.14}$$

We can write the expression for the cost functional and the value function using vector notation, with vectors of dimension $\mathrm{N} = \left|\overline{G^h}\right|$, the amount of grid points numbered from 1 to N.

Let $P(u)$ be the $N \times N$ transition probability matrix of entries $p(x,\, y|u)$ for each $(x,\, y)$ in the grid, with $x = 0$ being an absorbing state, that is, $p(0,\, 0|u) = 1$ for any control. The component located on the i^{th} row and j^{th} column of $P(u)$ is the probability of going from the i^{th} node to the j^{th} node given that control u is being used. $C(u)$ denotes the cost vector for each x in the grid, that is $C(x,\, u) = L(x,\, u)\,\Delta\, t^h(x,\, u)$, with $C(0) = 0$.

We can write (4.0.14) as

$$V = \inf_{u(x)\in U}[P(u)\, V + C(u)] \tag{4.3.1}$$

with the infimum taken component by component, and its corresponding non-optimal version

$$J_{k+1}(u_k) = P(u_k)\, J_k + C(u_k). \tag{4.3.2}$$

Several computational techniques to carry out this minimization are explained in Chapter 6 of [28]. We will limit our discussion to approximations in value space, that is (4.3.3) below.

In particular we employ the Jacobi method [28], so-called because of its similarities with the Jacobi method to solve a matrix equation.

The Jacobi method consists of standard fixed point iterations. To solve (4.3.1), the iterative nature of the Jacobi method (and other classical methods) requires an initial "guess" cost vector V_0. Then we define the sequence

$$V_{k+1} = \inf_{u(x)\in U}[P(u)\,V_k + C(u)] \tag{4.3.3}$$

and iterate this formula until an acceptable error level is reached (or up to a maximum number of iterations) updating all the components of V_{k+1} once at each iteration. We expect the iterates V_k to converge to the true value function $V = V^h$ of (4.0.13). Another common method is the approximation in policy space. See [28] for more details on this method.

The following questions are addressed below: (1) Does (4.0.14) have a unique fixed point (i.e. solution)? (2) Under which conditions does the iteration (4.3.3) converge to a solution of (4.0.14)? The issue of how well the solution V^h of (4.0.13), thus achieved, approximates the solution of the original problem (2.3.4) is addressed in Chapter 5.

D. P. Bertsekas in [10] helps us answer questions (1) and (2). The only hypothesis imposed in his formulation that can be problematic for our setting is that the set of controls available at each state is required to be finite (p 79 of [10]). To satisfy this we will take advantage of the linear fractional structure of the value function thanks to the specific configuration of our cost function and definition of the transition probabilities and time lapses. We will discuss this in Section 6.1. The important feature of the value function as a linear fractional function of u, as we can see in the example, is that the minimum of the cost functional takes place when all the effort of the control is placed on one queue per server, thus we will have a finite number of controls to check.

Before we look at the conditions required to guarrantee uniqueness and existence of a solution to the optimization problem on the MC, we refer the reader to the definition of *proper policy* (Definition A.1.3 of Appendix A).

The assumptions Bertsekas makes to derive the desire result are:

Assumption 4.3.1. *There exists at least one proper policy.*

Assumption 4.3.2. *For every improper policy μ, the corresponding cost $J_\mu(i)$ is ∞ for at least one state i.*

Note that under our hypotheses about the cost function L (2.3.5 (iii)) and properties of the time lapse $\Delta\, t^h$ for every improper policy u the corresponding cost $J(u)$ in (4.3.2) is infinity for at least one state. That is because every component of the cost $C(u)$ is strictly positive except the one corresponding to the terminal state. So we only need to verify Assumption 4.3.1, and this needs to be done "by hand."

For the vector formulation of our problem and as a consequence of Bertsekas' Proposition 1.2 ([10], p. 83) we conclude the following.

Proposition 4.3.3. *Under Assumptions 4.3.1 and 4.3.2 and hypothesis (2.3.5 (iv))*

1. *Equation (4.3.1) has a unique solution, and*
2. $\lim_{k\to\infty} V_k = V$ *for every initial cost vector* V_0 *in the recursive equation (4.3.3).*

Therefore, the implementation of our method will require a check that we satisfy Assumption 4.3.1 in order to achieve convergence of the numerical method. We will do this for an example in Section 6.2.

Chapter 5

Convergence

The main goal of this chapter is to show that the approximating value function $V^h(x)$ defined by (4.0.13) converges to the actual value function $V(x)$ given in (2.3.4).

We begin in Section 5.1 by defining a continuous time interpolation of the controlled MC. The continuous time interpolation of the control will be considered as an element of a set of stochastic relaxed controls. With the appropriate weak topology the set of such controls is compact (Section 5.2). The continuous time interpolation process $X^h(\cdot)$ depends continuously on the control and an "error process" (defined in (5.1.4)) that will vanish as the grid spacing goes to zero. This allows us to take the limit as the grid spacing (h) goes to zero of the value V^h of the controlled chain to show its convergence to the value function of the original control problem.

5.1 Continuous Time Interpolation of a Controlled Markov Chain

Based on the discrete MC ($\{X_j^h, u_j^h\}$) we define the piecewise continuous time interpolations $X^h(\cdot)$, $Y^h(\cdot)$, and $u^h(\cdot)$ as follows. For a fixed grid spacing $h > 0$ let

$$t_k^h = \sum_{j=1}^{k} \Delta t^h(X_{j-1}^h, u_{j-1}^h), \qquad t_0^h = 0,$$

$$X^h(t) = X_k^h \text{ if } t \in \left[t_k^h, t_{k+1}^h\right), \tag{5.1.1}$$

$$Y^h(t) = Y_k^h \text{ if } t \in \left[t_k^h, t_{k+1}^h\right), \tag{5.1.2}$$

with X_k^h and Y_k^h as in (4.0.10). Also let

$$u^h(t) = u_k^h \text{ if } t \in \left[t_k^h, t_{k+1}^h\right) \tag{5.1.3}$$

and define the accumulated error at any time t by

$$e^h(t) = Y^h(t) - \left[X_0^h + \int_0^t v(X^h(s),\, u^h(s))\, ds \right]. \tag{5.1.4}$$

Thus, we can rewrite (4.0.11) as

$$\Delta\, e_n = Y_{n+1}^h - \left(Y_n^h + \int_{t_n^h}^{t_{n+1}^h} v(X^h(t),\, u^h(t))\, dt \right)$$

from which it is easy to see that

$$e^h(t_n^h) = \sum_{i=0}^{n-1} \Delta\, e_i. \tag{5.1.5}$$

Note however that $e^h(t)$ is not constant on the intervals $[t_k^h,\, t_{k+1}^h)$. A modification with that property will appear as e^H in (5.1.7) below.

It follows from (4.0.8) that $e^h(t_k^h)$ is an $\mathcal{F}_k$ martingale on the discrete time scale $k = 0,\, 1,\, 2,\, \ldots$ (see Definition A.4.1 in Appendix A). For simplicity we set $e_k = e^h(t_k^h)$.

Recall from Chapter 4 ((4.0.5) and (4.0.9)) that

$$\begin{cases} \widetilde{Y}_{k+1}^h = X_k^h + v(X_k^h,\, u_k^h)\, \Delta\, t^h(X_k^h,\, u_k^h) + \Delta\, e_k \\ X_{k+1}^h = \widetilde{Y}_{k+1}^h + \gamma(X_{k+1}^h)\, \Delta\, \eta_k \end{cases}$$

and

$$Y_{k+1}^h = \widetilde{Y}_{k+1}^h.$$

This construction implies that $X^h(\cdot) = \Gamma(Y^h(\cdot))$ with no approximation error because of the instantaneous nature of the reflection mapping when applied to the piecewise constant process $Y^h(\cdot)$.

With this notation we can express the continuous time interpolation process as

$$\begin{cases} X^h(\cdot) & = \Gamma(Y^h(\cdot)) \\ Y^h(t) & = X_0^h + \int_0^t v(X^h(s),\, u^h(s))\, ds + e^h(t) \\ X_0^h & = x_0^h. \end{cases} \tag{5.1.6}$$

Let $D = D([0,\, \infty))$ be the set of functions $y : [0, \infty) \to \mathbb{R}^n$ which are right continuous with left limits (CADLAG). We give D the metric of uniform convergence on each $[0,\, T]$. With this metric, D is a complete metric space. Thus, $X^h(\cdot)$ and $e^h(\cdot)$ are typical elements of D.

Given a value $0 < t < \infty$, let η_t be the random variable that maps an element ω in the probability space Ω (Section 4.2) to the integer k such that $t \in [t_k^h,\, t_{k+1}^h)$, $\eta_t(\omega) = k$. We see

that η_t is a stopping time with respect to the filtration $\mathcal{F}_k$. Note that for any $t \in [0, T]$, η_t is a bounded stopping time by virtue of (4.0.4). (The argument for this is given in the proof of the next lemma.) Hence e_{η_t} is integrable by *Doob's Optional-Stopping Theorem* ([32], p. 157).

Lemma 5.1.1. *The process $e^h(\cdot)$ converges in distribution to the zero process with respect to the uniform metric on $t \in [0, T]$.*

Proof. To show $e^h(t)$ converges to the zero process, define

$$e^H(t) = Y^h(t) - x_0^h - \int_0^{t_k^h} v(X(s),\, u(s))\, ds, \text{ for } t \in [t_k^h,\, t_{k+1}^h). \tag{5.1.7}$$

The connection between $e^h(\cdot)$ and $e^H(\cdot)$ is

$$e^H(t) = e^h(t) + \int_{t_k^h}^{t} v(X(s),\, u(s))\, ds.$$

This makes e^H constant on $[t_k^h,\, t_{k+i}^h)$ with $e^h(t_k^h) = e^H(t_k^h)$. We can also view $e^H(t)$ as the martingale e_n stopped at η_t: $e^H(t) = e_{\eta_t}$. This would not hold in general with the extra term $\int_{t_k^h}^{t} v(X(s),\, u(s))\, ds$.

Notice that for any $t \in [0, T]$, η_t is a bounded stopping time. In particular η_T is bounded. Indeed, notice that

$$\eta_T = \inf\left\{ n : \sum_{i=0}^{n} \Delta t^h(X_{t_i}^h,\, u_{t_i}^h) \geq T \right\}$$

that is $T \in [t_{\eta_T}^h,\, t_{\eta_T+1}^h)$. By (4.1.4) there is an N such that

$$N > \frac{T}{\widetilde{\Delta}\, t}$$

so that

$$T < N\, \widetilde{\Delta}\, t \leq t_N^h.$$

This implies that $\eta_T < N$. It follows that

$$\sup_{[0,\,T]} \|e^H(t)\| = \sup_{0 \leq n < \eta_T} \|e_n\| \leq \sup_{0 \leq n \leq N} \|e_n\|. \tag{5.1.8}$$

For the proof consider any $\epsilon > 0$. We have the following.

$$\begin{aligned}
P_{x_0^h}\left\{\sup_{t\in[0,T]}\|e^H(t)\|\geq\epsilon\right\} &\leq P_{x_0^h}\left\{\sup_{0\leq n<N}\|e_n\|\geq\epsilon\right\}\\
&\leq \frac{4}{\epsilon^2}\mathrm{E}_{x_0^h}\left[\|e_N\|^2\right]\\
&= \frac{4}{\epsilon^2}\mathrm{E}_{x_0^h}\left[\left\|\sum_{i=0}^{N-1}\Delta e_i\right\|^2\right]\\
&= \frac{4}{\epsilon^2}\mathrm{E}_{x_0^h}\left[\left\|\sum_{i=0}^{N-1}Y_{i+1}^h - Y_i^h - v(X_i^h,\,u_i^h)\,\Delta\,t^h(X_i^h,u_i^h)\right\|^2\right]\\
&= \frac{4}{\epsilon^2}\mathrm{E}_{x_0^h}\left[\sum_{i=0}^{N-1}\left\|Y_{i+1}^h - Y_i^h - v(X_i^h,\,u_i^h)\,\Delta\,t^h(X_i^h,u_i^h)\right\|^2\right]\\
&= \frac{4}{\epsilon^2}\sum_{i=0}^{N-1}\mathrm{E}\left[\left\|Y_{i+1}^h - Y_i^h - v(X_i^h,\,u_i^h)\,\Delta\,t^h(X_i^h,u_i^h)|\mathcal{F}_i\right\|^2\right]\\
&= \frac{4}{\epsilon^2}\sum_{i=0}^{N-1}C\,h^2 \qquad \text{for some } C>0\\
&= \frac{4}{\epsilon^2}NC\,h^2.
\end{aligned}$$

We use (5.1.8) to justify the first line of the proof. The next line is Doob's inequality. The next equations are due to equation (5.1.5), the definition of $\Delta\,e_i$, $\mathcal{L}^2$ martingale property that we can apply since Δe_i are uniformly integrable, and the consistency condition (4.0.6) (ii), in that order. Then, by letting $C' = \frac{4NC}{\epsilon^2}$ we have

$$P_{x_0^h}\left\{\sup_{t\in[0,T]}\|e^H(t)\|\geq\epsilon\right\}\leq h^2\,C'.$$

Hence $\sup_{t\in[0,T]}\|e^H(t)\|$ converges in probability to 0 as $h\downarrow 0$ and by Theorem A.2.6, $e^H(\cdot)$, as a $D-$value process, converges in distribution to the zero process.

Also, we note that

$$\begin{aligned}
\left\|\int_{t_{k^h}^h}^{t} v(X(s),\,u(s))\,ds\right\| &\leq \left\|v(t^h(X_{k^h}^h,\,u_{k^h}^h)\right\|\Delta t^h(X_{k^h}^h,\,u_{k^h}^h)\\
&\leq C_1\,\Delta t^h(X_{k^h}^h,\,u_{k^h}^h).
\end{aligned}$$

And by (4.1.3) it follows that $e^h(t)$ converges to $e^H(t)$ as h goes to zero. □

For $X(\cdot) \in D$ define $\tau_\epsilon = \inf\{t \geq 0 : \|X(t)\| < \epsilon\}$. Then τ_ϵ increases as $\epsilon \downarrow 0$. Our definition for τ_0 is that

$$\tau_0(X) = \lim_{\epsilon \downarrow 0} \tau_\epsilon \quad (= +\infty \text{ allowed.}) \tag{5.1.9}$$

In general $\tau_0(X) \leq \inf\{t \geq 0 : X(t) = 0\}$, but they are equal for $X(\cdot) \in C$. This is the same definition of τ_0 as in (2.3.2) applied to the process $X^h(\cdot)$. We write the cost functional and value function for the Markov Chain problem as

$$W^h(x_0^h,\, u^h) = \mathrm{E}_{x_0^h}\left[\int_0^{\tau_0(X^h(t))} L(X^h(t),\, u^h(t))\, dt\right], \tag{5.1.10}$$

$$V^h(x_0^h) = \inf_{\{u^h\}} W^h(x_0^h,\, u^h). \tag{5.1.11}$$

Before we move on to show the convergence of relaxed controls, we want to direct the reader's attention to Appendix A where we state some definitions and results that we will be using in the discussion below.

5.2 Relaxed controls. Compactness and Continuity Properties

Consider the probability space $(\Omega,\, \mathcal{F},\, P)$ on which the random variables in the introduction to this chapter are defined. Also consider a sequence $\{h_n\}$, each $h_n \in (0,\, 1)$ with $n = 1,\, 2,\, 3,\, \ldots$ such that $h_n \downarrow 0$ as $n \to \infty$. For an initial state x_0 assume that $x_0^n = X^h(0) \to x_0$ as $h \to 0$.

Recall that a control sequence $\{u_i^h\}$ consists of $\mathcal{F}_k-$measurable random variables u_k^h. The definition of $u^h(t)$ and $X^h(\cdot)$ (5.1.3, 5.1.6), imply that $u^h(t)$ is itself a stochastic process defined on the probability space Ω.

Define the space $\mathcal{R}$ of relaxed controls to be the set of (Borel) measures ν on $[0,\, \infty) \times U$ with the property that

$$\nu([0,\, T] \times U) = T \text{ for every } T \in [0,\, \infty).$$

We will often write

$$\int_0^T \int_U f(t,\, u)\, d\nu$$

in place of

$$\int_{[0,\, T]\times U} f(t,\, u)\, d\nu(t,\, u).$$

We give $\mathcal{R}$ the topology of weak convergence on $[0,\, T] \times U$ for each T. That is $\nu_n \Rightarrow \nu$ when

$$\int_0^T \int_U f(t,\, u)\, d\nu_n \to \int_0^T \int_U f(t,\, u)\, d\nu$$

for each T and each continuous bounded function $f : [0,\, \infty) \times U \to \mathbb{R}$.

With this characterization we state Lemmas 5.2.1 and 5.2.3 below. See Appendix B for their proofs.

Lemma 5.2.1. *$\mathcal{R}$ is compact.*

Since $\mathcal{R}$ is totally bounded, by Proposition A.2.3 of Appendix A we have

Corollary 5.2.2. *$\mathcal{R}$ is separable.*

If there is a measurable function $u : [0,\, \infty) \to U$ so that

$$\nu(A) = \int_0^\infty \chi_A\,(t,\, u(t))\, dt \tag{5.2.1}$$

for all $A \subseteq [0,\, \infty) \times U$ we call ν "the relaxed representation" of the standard control $u(\cdot)$, or simply a "standard control."

Lemma 5.2.3. *The standard controls are dense in $\mathcal{R}$.*

The following lemmas provide us with the necessary structure for the convergence properties to be discussed below.

Lemma 5.2.4. *Given $x \in G$, $\nu \in \mathcal{R}$, and $e \in D$ there exist unique $X(\cdot)$, $Y(\cdot) \in D$ solving*

$$\begin{cases} X(\cdot) &= \Gamma(Y(\cdot)), \\ Y(t) &= x + \int_0^t \int_U v(X(s),\, u)\, d\nu + e(t). \end{cases} \tag{5.2.2}$$

Denote $X(\cdot) = X_{x,\,\nu,\,e}(\cdot)$. The map $(x,\, \nu,\, e) \to X_{x,\,\nu,\,e}(\cdot)$ is continuous with respect to the topology on D identified above. Moreover, on each $[0,\, T]$ this is Lipschitz continuous in e, uniformly w.r.t. x and ν.

Proof. We first show existence. Let $c = \min\{T,\, \frac{1}{2C_\Gamma C_v}\}$ and $X_1,\, X_2 \in D([0,\, c])$. Define $\Phi : X(t) \to \Gamma(x + \int_0^t \int_U v(X(s),\, u)\, d\nu + e(t))$. Clearly $\Phi : D \to D$. Moreover, Φ is a contraction on $[0,\, c]$ because

$$\begin{aligned} \sup_{[0,\,c]} \|\Phi(X_2) - \Phi(X_1)\| &= \sup_{[0,\,c]} \left\| \Gamma\left(x + \int_0^t \int_U v(X_2,\, u)\, d\nu + e(t)\right) - \Gamma\left(x + \int_0^t \int_U v(X_1,\, u)\, d\nu + e(t)\right) \right\| \\ &\leq C_\Gamma \sup_{[0,\,c]} \left\| \int_0^t \int_U v(X_2,\, u)\, d\nu - \int_0^t \int_U v(X_1,\, u)\, d\nu \right\| \\ &= C_\Gamma \sup_{[0,\,c]} \left\| \int_0^t \int_U \left[v(X_2,\, u) - v(X_1,\, u)\right]\, d\nu \right\| \end{aligned}$$

$$\leq C_\Gamma\, C_v\, c \sup_{[0,\,c]} \|X_2(t) - X_1(t)\|$$
$$\leq \frac{1}{2} \sup_{[0,\,c]} \|X_2(t) - X_1(t)\|.$$

Then, by the contraction mapping theorem, there is a unique fixed point $X \in D([0,\, c])$ solution to (5.2.2). Notice that the first inequality is due to the Lipschitz continuity of Γ(with Lipschitz constant C_Γ) whereas the second to the last inequality uses (2.3.5 (i)).

If $c = T$ we are done. Otherwise, we repeat this argument on $[c,\, 2c]$ using $X(c)$ as the initial conditions at $t = c$. Repeating this on each $[nc,\, (n+1)c]$ we get existence on any $[0,\, T]$.

Next we show uniqueness and Lispchitz continuity in e. Suppose $(X_i,\, Y_i)$ are solutions both with the same x, ν, and e_i, $i = 1, 2$ respectively. Let $\delta^X(t) = \sup_{[0,\,t]} |X_2(s) - X_1(s)|$, $\delta^Y(t) = \sup_{[0,\,t]} |Y_2(s) - Y_1(s)|$, $\delta^e(t) = \sup_{[0,\,t]} |e_2(s) - e_1(s)|$. Since $X_i = \Gamma(Y_i)$, Lipschitz continuity of Γ implies

$$\delta^X(t) \leq C_\Gamma\, \delta^Y(t).$$

Since $Y_i = x + \int_0^t \int_U v(X_i(s),\, u)\, d\nu + e_i(t)$, we have

$$Y_2(t) - Y_1(t) = [e_2(t) - e_1(t)] + \int_0^t \int_U [v(X_2(s),\, u) - v(X_1(s),\, u)]\, d\nu \tag{5.2.3}$$

thus from (2.3.5 (ii))

$$\delta^Y(t) \leq \delta^e(t) + \int_0^t C_v\, \delta^X(s)\, ds$$
$$\leq \delta^e(t) + \int_0^t C_v C_\Gamma\, \delta^Y(s)\, ds.$$

Gronwall's inequality (Appendix A, Theorem A.3) yields

$$\delta^Y(t) \leq \delta^e(t) + \int_0^t C_v\, C_\Gamma\, e^{C_v\, C_\Gamma (t-s)}\, \delta^e(s)\, ds$$
$$\leq \delta^e(t) e^{C_v C_\Gamma t}.$$

In particular $\delta^Y(T) \leq \delta^e(T)\, e^{C_v\, C_\Gamma\, T}$.

If $e_1 = e_2$, $\delta^e(T) = 0$, so $\delta^Y(T) = 0$, and $\delta^X(T) = 0$. This proves uniqueness. In general, this shows X, Y are both Lipschitz continuous on $[0,\, T]$ with respect to e on $[0,\, T]$. Notice that the Lipschitz constant is independent of ν. To extend this to Lipschitz continuity in $(x,\, e)$ for ν fixed we just need to consider x_1 and x_2 and add $[x_2 - x_1]$ on the right hand side of (5.2.3).

To prove joint continuity, suppose $x_n \to x$, $\nu_n \to \nu$, and $e_n \to e$ (uniformly) in their respecive senses and let $(X_n,\, Y_n)$, $(X,\, Y)$ be the corresponding solutions. We want to show $X_n \to X$,

$Y_n \to Y$ uniformly on each $[0,\, T]$. Let

$$\delta_n^X(t) = \sup_{[0,\,t]} |X(s) - X_n(s)|, \quad \delta_n^Y(t) = \sup_{[0,\,t]} |Y(s) - Y_n(s)|, \quad \delta_n^e(t) = \sup_{[0,\,t]} |e(s) - e_n(s)|$$

and

$$\delta_n^*(t) = |x - x_n| + \left| \int_0^t \int_U v(X(s),\, u)\, d\nu - \int_0^t \int_U v(X(s),\, u)\, d\nu_n \right|.$$

Now $f(t,\, u) = v(X(t),\, u)$ is continuous at s if $X(t)$ is continuous at s. Since $X \in D$, X has at most a countable set J of discontinuities, and so the discontinuities of f are contained in $J \times U$. Since $\nu(\{s\} \times U) = 0$, it follows that $\nu(J \times U) = 0$. Since f is bounded ($(X,\, u) \in G \times U$ which is compact), $\nu_n \Rightarrow \nu$ on $[0,\, t] \times U$ implies $\delta_n^*(t) \to 0$. The boundedness of v (and $\nu([0,\, T] \times U) = T$ all $\nu \in \mathcal{R}$) implies that the δ_n^* are equicontinuous. In fact they are Lipschitz with constant $2C_1$. Therefore $\delta_n^* \to 0$ uniformly on each $[0,\, T]$.

Now

$$\begin{aligned} Y(t) - Y_n(t) = x - x_n &+ \int_0^t \int_U [v(X(s),\, u) - v(X_n(s),\, u)]\, d\nu_n \\ &+ \left[\int_0^t \int_U v(X(s),\, u)\, d\nu - \int_0^t \int_U v(X(s),\, u)\, d\nu_n \right] + e(t) - e_n(t) \end{aligned}$$

and again we have

$$\delta_n^X(t) \le C_\Gamma\, \delta_n^Y(t).$$

Therefore

$$\begin{aligned} \delta_n^Y(t) &\le \delta_n^*(t) + \int_0^t C_v \delta_n^X(s)\, ds + \delta_n^e(t) \\ &\le \delta_n^*(t) + \int_0^t C_v\, C_\Gamma\, \delta_n^Y(s)\, ds + \delta_n^e(t), \end{aligned}$$

which by Gronwall inequality implies

$$\delta_n^Y(T) \le \delta_n^*(T) + \int_0^T C_v\, C_\Gamma\, e^{C_v\, K(T-s)} \delta_n^*(s)\, ds + \delta_n^e(T).$$

Since $\delta_n^* \to 0$ uniformly on $[0,\, T]$, it follows that both $\delta_n^Y(T) \to 0$ and $\delta_n^X(T) \to 0$. □

Lemma 5.2.5. *For each T, the map*

$$(x,\, \nu,\, e) \to \int_0^T \int_U L(X_{x,\,\nu,\,e}(t),\, u)\, d\nu$$

is continuous, and

$$(x,\, \nu,\, e) \to \int_0^{T \wedge \tau_0(X_{x,\,\nu,\,e})} \int_U L(X_{x,\,\nu,\,e}(t),\, u)\, d\nu$$

is lower semicontinuous.

Proof. The proof of the first part is as before: Suppose $(x_n,\, \nu_n,\, e_n) \to (x,\, \nu,\, e)$ and let X_n, X be the respective solutions. Then we know $X_n \to X$ in D (uniformly on compact sets).

$$\begin{aligned}&\int_0^T \int_U L(X_n(s),\, u)\, d\nu_n - \int_0^T \int_U L(X(s),\, u)\, d\nu \\ &= \left[\int_0^T \int_U L(X(s),\, u)\, d\nu_n - \int_0^T \int_U L(X(s),\, u)\, d\nu\right] + \int_0^T \int_U [L(X_n(s),\, u) - L(X(s),\, u)]\, d\nu_n.\end{aligned}$$

The second term on the right hand side is bounded by $\int_0^T C_L\, m(X_n(s) - X(s))\, ds$ which goes to 0 by the uniform convergence of $X_n \to X$. The first term also goes to 0 for the same reason as $\delta_n^* \to 0$ in the proof of Lemma 5.2.4: $f(t,\, u) = L(X(t),\, u)$ is bounded and continuous except on a set $J \times U$, where J is the (countable) set of discontinuities of X. Since $\nu(J \times U) = 0$, it follows that

$$\int_0^T \int_U f\, d\nu_n \to \int_0^T \int_U f\, d\nu.$$

This means that the first term above goes to 0.

Now, suppose that $x_n \to x$, $\nu_n \to \nu$, and $e_n \to e$ (in their respective norms).
Let

$$\begin{aligned} X_n &= X_{x_n, \nu_n, e_n}, & \tau_n &= \tau_0(X_n); \\ X &= X_{x, \nu, e}, & \tau_0 &= \tau_0(X). \end{aligned}$$

Let $\tau_* = \underline{\lim}\, \tau_n$. By passing to a subsequence we can assume $\tau_n \to \tau_*$. We only need to consider $\tau_* \le T$. We know $X_n(\cdot) \to X(\cdot)$ uniformly on $[0,\, T]$. Given any ϵ, for n sufficiently large $\sup_{[0,\,T]} |X_n - X| < \frac{\epsilon}{2}$, $\tau_n < \tau_* + \epsilon$ and there is $t < \tau_n$ with $|X_n(t)| < \frac{\epsilon}{2}$. So $|X(t)| < \epsilon$ for some $t < \tau_* + \epsilon$. Thus we have $\tau_\epsilon(X) < \tau_* + \epsilon$ and it follows that $\tau_0 \le \tau_*$. Consider any $S < \tau_0$, then

$$\begin{aligned} \underline{\lim} \int_0^{T \wedge \tau_n} L(X_n,\, u)\, d\nu_n &\ge \underline{\lim} \int_0^{S \wedge \tau_n} L(X_n,\, u)\, d\nu_n && \text{Since } L > 0 \\ &= \lim \int_0^S L(X_n,\, u)\, d\nu_n && \text{Since } \tau_n \to \tau_* > S \\ &= \int_0^S L(X,\, u)\, d\nu. \end{aligned}$$

Letting $S \uparrow \tau_0 \wedge T$ we conclude that

$$\underline{\lim} \int_0^{T \wedge \tau_n} L(X_n,\, u)\, d\nu_n \ge \int_0^{T \wedge \tau_0} L(X,\, u)\, d\nu.$$

□

5.3 Convergence of the Value Function

This section is the core of this chapter. We will show that the value function on the grid $(V^h(\cdot))$ convergences to the value function of the continuous process $(V(\cdot))$ as the grid becomes finer.

In the previous section we introduced the concept of standard controls (5.2.1). We want to emphasize the fact that the controls $u^h(t)$ of (5.1.3) are random variables depending on $\omega \in (\Omega,\, \mathcal{F},\, P)$ so that given a control function $u^h_\omega(t)$, we define its stochastic relaxed control representation $\nu^h_\omega \in \mathcal{R}$ on $[0,\, \infty) \times U$ by

$$\nu^h_\omega(A) = \int_0^\infty \chi_A(t,\, u^h_\omega(t))\, dt \tag{5.3.1}$$

for all $A \subseteq [0,\, \infty) \times U$ measurable. In other words, for a given choice of (stochastic) control sequence u^h_n and initial $x^h \in G^h$, the Markov Chain followed by continuous time interpolation produces $\mathcal{R}$ and D valued random variables ν^h_ω and e^h_ω defined on the probability space $(\Omega,\, \mathcal{F},\, P)$. We make explicit the dependence of the process $X^h(t)$ on these parameters; we denote it by $X_{x^h,\, \nu^h_\omega,\, e^h_\omega}(t)$ and shorten its notation by setting

$$X^h_\omega(t) = X_{x^h,\, \nu^h_\omega,\, e^h_\omega}(t).$$

Let us use Θ to denote probability measures on D, and Λ to denote probability measures on $\mathcal{R}$. Let Θ^h and Λ^h denote the distributions of e^h_ω and ν^h_ω respectively. These are the marginals of the pair $(\nu^h_\omega,\, e^h_\omega)$.

In terms of these relaxed controls we can express (5.1.6) as

$$\begin{cases} X^h_\omega(\cdot) = \Gamma\left(Y^h_\omega(\cdot)\right), \\ Y^h_\omega(t) = x^h_0 + \int_0^t \int_U v(X^h_\omega(s),\, u)\, d\nu^h_\omega(s,\, u)) + e^h_\omega(t), \\ X^h_0 = x^h_0 \in G^h \end{cases} \tag{5.3.2}$$

with value function

$$V^h(x_0) = \inf_{\{u^h_\omega(\cdot)\} \in \mathcal{U}} \mathrm{E}_{x^h_0} \left[\int_0^{\tau^h_0(X^h_\omega(t))} \int_U L(X^h_\omega(t),\, u^h_\omega(t))\, d\nu^h_\omega \right]. \tag{5.3.3}$$

In order to prove convergence of the value function on the grid to the value function on the whole domain G we need the following hypotheses.

Chain Hypotheses. Let $\overline{G^h} = G^h \cup \partial\, \overline{G}^h \cup \widetilde{G}^h$ as in (4.0.1).
a) There exists $K, \delta > 0$ so that

$$\sup_{x^h \in \overline{G^h}} |V^h(x^h)| \le K \text{ all } 0 < h < \delta.$$

b) Given $\epsilon > 0$ there exists $\delta > 0$ so that

$$|V^h(x)| < \epsilon \text{ for all } x \in \overline{G^h} \text{ with } |x| < \delta \text{ and } h < \delta.$$

Theses hypotheses guarantee continuity at the origin and an upper bound on the value function on the grid $(V^h(\cdot))$ for any grid spacing. We have not developed a convenient sufficient condition for this. Consequently these hypotheses will need to be verified on case-by-case basis.

Theorem 5.3.1. *Assume the Chain Hypotheses and the hypotheses of Theorem 3.2.1 hold. Then for any $x \in G$, $x_h \to x$, and $h > 0$, $V^h(x^h)$ converges to $V(x)$ as h goes to 0.*

Proof. The proof of $V(x) = \lim_{h \downarrow 0} V^h(x)$ is in two parts. First we will show that $V(x) \leq \underline{\lim}_{h \downarrow 0} V^h(x_h)$. Let $X^n_\omega = X_{x_n, \nu^n_\omega, e^n_\omega}$ for some $\mathcal{R}$ and D random variables ν^n_ω, e^n_ω (with distributions Λ^n) and Θ^n, and $e^n \to 0$ in probability (i.e. $\Theta^n \Rightarrow \delta_0$).

Since $\mathcal{R}$ is compact $\{\Lambda^h\}$ is tight so there is a subsequence h_n and a probability measure Λ on $\mathcal{R}$ so that $\Lambda^{h_n} \Rightarrow \Lambda$. Hence, for every bounded and continuous function f on $\mathcal{R}$

$$\int_{\mathcal{R}} f(\nu)\, d\Lambda^{h_n} \to \int_{\mathcal{R}} f(\nu)\, d\Lambda.$$

We also know that as $h \downarrow 0$, $e^h_\omega \to 0$ (in D) in distribution (Lemma 5.1.1), that is to say that $\Theta^h \implies \delta_0$, the probability measure concentrated in the identically 0 function in D.

Pick any sequence $h_n \downarrow 0$. For simplicity use only n instead of h_n; for example we write $V^n(x_n)$ for $V^{h_n}(x_{h_n})$. Choose a controlled Markov Chain X^n with $X^n(0) = x_n$ and

$$W^n(x_n) \leq V^n(x_n) + \frac{1}{n}.$$

Recall that by Theorem 5.2.5, for each T the map

$$\Phi : (x,\, \nu,\, e) \to \int_0^{T \wedge \tau_0} \int_U L(X_{x,\, \omega,\, e},\, u)\, d\nu$$

is lower semicontinuous.

Let Q^n be the joint distribution of $(x^n,\, \nu^n_\omega,\, e^n_\omega)$. By Theorem A.2.14 of Appendix A, $Q^n \Rightarrow \delta_x \times \Lambda \times \delta_0$. It follows that

$$\mathrm{E}\left[\int_0^{T \wedge \tau_0} \int_U L(X_{x^n, \nu^n_\omega\, e^n_\omega},\, u)\, d\nu^n_\omega\right] = \mathrm{E}^{Q^n}\left[\Phi(x,\, \nu,\, e)\right]$$

and by Lemma A.2.18 of Appendix A,

$$\mathrm{E}^{Q}[\Phi] \leq \underline{\lim}\, \mathrm{E}^{Q^n}\left[\Phi(x,\, \nu,\, e)\right].$$

The preceding implies that for each $T = k$ there is ν_k so that

$$\int_0^{k\wedge\tau_0(X_{x,\,\nu_k})} \int_U L(X_{x,\,\nu_k}(t),\, u)\, d\nu_k < \underline{\lim}\, V^h(x_h). \tag{5.3.4}$$

$\mathcal{R}$ being compact allows us to select a convergent subsequence $\nu_{k'} \to \nu$. For each T, using the lower semicontinuity again we have

$$\begin{aligned}
\int_0^{T\wedge\tau_0(X)} \int_U L(X_{x,\nu}(t),\, u)\, d\nu &\le \underline{\lim}_{k'} \int_0^{T\wedge\tau_0(X_{\nu_{k'}})} \int_U L(X_{\nu_{k'}},\, u)\, d\nu_{k'} \\
&\le \underline{\lim}_{k'} \int_0^{k'\wedge\tau_0(X_{\nu_{k'}})} \int_U L(X_{\nu_{k'}},\, u)\, d\nu_{k'} \qquad \text{since } L > 0 \\
&\le \underline{\lim}\, V^n(x_n).
\end{aligned}$$

The last inequality holds because of (5.3.4). Since $T < \infty$ is arbitrary,

$$\int_0^{\tau_0(X_{x,\,\nu})} L(X_{x,\,\nu},\, u)\, d\nu \le \underline{\lim}\, V^n(x_n)$$

hence

$$\inf_{\nu\in\mathcal{R}} \int_0^{\tau_0(X_{x^n,\,\nu})} L(X_{x,\nu}(t),\, u)\, d\nu \le \underline{\lim}\, V^h(x_h)$$

and then using the density of the standard controls (and continuity of V at 0) we conclude that

$$V(x) \le \underline{\lim}\, V^h(x_h).$$

We now show that $\overline{\lim}_{h\downarrow 0} V^h(x_h) \le V(x)$. We can assume that $V(x) < \infty$ (otherwise the result is trivial). Consider any $\epsilon > 0$. Let K, δ be as in the Chain Hypotheses. By the definition of $V(\cdot)$, there is some control u^ϵ such that $J(x,\, u^\epsilon(\cdot)) < \epsilon + V(x)$. Let $X^\epsilon = X_{x,\,u^\epsilon}$.

Since X^ϵ is continuous there exists $0 < T < \tau_0(X^\epsilon)$ at which $|X^\epsilon(T)| < \frac{\delta}{3}$. If there were no such T, then $|X^\epsilon(t)| \ge \frac{\delta}{3}$ for all t, so $L(X^\epsilon(t),\, u^\epsilon(t)) \ge C$ for some $C > 0$ (2.3.5 (iii)) and $\tau_0 = +\infty$ imply $J(x,\, u^\epsilon(\cdot)) = +\infty$, contrary to $J(x,\, u^\epsilon(\cdot)) < V(x) + \epsilon < \infty$. Thus we have that

$$\int_0^T L(X^\epsilon(s),\, u^\epsilon(s))\, ds < J(x,\, u^\epsilon(\cdot)) < V(x) + \epsilon. \tag{5.3.5}$$

By Lemma 5.2.3, $u^\epsilon(\cdot)$ can be approximated in the topology of $\mathcal{R}$ by a continuous standard control, say $u^*(\cdot)$, such that with $X^* = X_{x,\,u^*}$

$$\sup_{[0,\,T]} |X^\epsilon(t) - X^*(t)| < \frac{\epsilon}{3}$$

and

$$\int_0^T L(X^*(t),\, u^*(t))\, dt < \int_0^T L(X^\epsilon(t), u^\epsilon(t))\, dt + \epsilon. \tag{5.3.6}$$

The uniform approximations of $X(\cdot)$ on $[0,\,T]$ and of $\int_0^T L(X(t),\,u(t))\,dt$ are consequence of Lemma 5.2.4 and Lemma 5.2.5.

Now, let us use u^* to define a control sequence $\{u_n^h\}$ for the MC: $u_n^h = u^*(t_n^h)$. Given $x^h \to x$, let ν_ω^h, e_ω^h be the (random) elements of $\mathcal{R}$ and D associated with the resulting controlled chain $X^h(t)$. In the following we will show that as h goes to 0, ν_ω^h converges in distribution to ν^*, the element of $\mathcal{R}$ associated with u^*. This will allow us to conclude that ν_ω^h converges in probability to ν^* since convergence in distribution (to a deterministic element) implies convergence in probability (page 27 of [11]).

To establish $\nu_\omega^h \Rightarrow \nu^*$, recall that we defined $u_n^h = u^*(t_n^h)$. The continuous time interpolation (which is what defines ν_ω^h) is $u^h(t) = u_n^h(t) = u^*(t_n^h)$ for that n such that $t_n^h \le t < t_{n+1}^h$. Although n and t_n^h depend on ω in the underlying probability space, $u^h(t) \to u^*(t)$ uniformly with respect to ω because of the consistency condition (4.0.3). Let $m(h) \doteq \sup_{\{x,\,u\}} \Delta\, t^h(x,\,u)$ then in the above, $t - m(h) \le t_n^h \le t$ regardless of n or ω. Now $u^*(\cdot)$ is uniformly left continuous on $[0,\,T]$ that is, given $\epsilon' > 0$ there is a $\delta' > 0$ so that $|u^*(s) - u^*(t)| < \epsilon'$ all $0 \le t - \delta' < s < t \le T$. So if $m(h) < \delta'$ we have $|u_\omega^h(t) - u^*(t)| < \epsilon'$ for all ω. Consider any continuous function $f : [0,\,T] \times U \to \mathbb{R}$. Since $[0,\,T] \times U$ is compact, f is uniformly continuous and therefore

$$\mathrm{E}\left[\int_0^T \int_U f(t,\,u)\,d\nu_\omega^h\right] - \int_0^T \int_U f(t,\,u)\,d\nu^* = \mathrm{E}\left[\int_0^T |f(t,\,u_\omega^h(t)) - f(t,\,u^*(t))|\,dt\right] \to 0.$$

This implies $\nu_\omega^h \Rightarrow \nu^*$ as desired. Also, as $h \to 0$, $e_\omega^h \to 0$ (Lemma 5.1.1). Then the convergence of $e_\omega^h \to 0$ and $\nu_\omega^h \to \nu$ imply the following two convergences in probability, by Lemma 5.2.4:

$$\sup_{[0,\,T]} |X^h(\cdot) - X^*(\cdot)| \to 0 \quad \text{and}$$

$$\int_0^T \int_U L(X^h(s),\,u)\,d\nu_\omega^h \to \int_0^T L(X^*(s),\,u^*(s))\,ds. \tag{5.3.7}$$

Consider a control sequence obtained by modifying u_n^h as follows.
If at time $t = T$, X^h has not reached $\{|X| < \delta\}$, then for $t_n^h > T$ use a control so that

$$\mathrm{E}\left[\int_T^{\tau_0} L(X^h,\,u^h)\,dt\right] \le K.$$

The existence of such a control is guaranteed by Chain Hypothesis (a).
If $|X_n^h| < \delta$ for some $t_n^h \le T$, then from t_n^h forward use a control for which we get

$$\mathrm{E}\left[\int_{t_n^h}^{\tau_0} L(X^h,\,u^h)\,dt\right] < \epsilon.$$

Such a control exists due to the Chain Hypothesis (b). We now have

$$V^h(x^h) \le \mathrm{E}\left[\int_0^T \int_U L(X^h(s),\,u)\,d\nu_\omega^h\right] + \epsilon + K\,P(|X^h(\cdot)| > \delta \text{ on } [0,\,T]).$$

Note that as $h \downarrow 0$ as $T \to \infty$, $P(|X^h(\cdot)| > \delta) \to 0$, for sufficiently large T, thus we can bound the last term by ϵ. It follows that (see (5.3.7), (5.3.6), and (5.3.5))

$$\begin{aligned}\overline{\lim}\, V^h(x^h) &\le \int_0^T L(X^*(t),\, u^*(t))\, dt + \epsilon \\ &\le \int_0^T L(X^\epsilon(t),\, u^\epsilon(t))\, dt + 2\epsilon \\ &< V(x) + 3\epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary this completes the proof. □

Note that the Chain Hypotheses together with the upperbound $V \le \underline{\lim}\, V^h$ implies:
(a) $V(x) \le K$ all $x \in G$
(b) $V(x) < \epsilon$ all $x \in G$ with $|x| < \delta$.
But (b) was one of the hypotheses of Theorem 3.2.1 that we needed to assume for the proof, so we do not really have an independent argument for continuity of V here.

Chapter 6

An Example

In the previous chapters we have developed a theory and explained the mechanism to find the solution of the problem described in Chapter 2. We will present in this chapter one example to illustrate how to define the Markov Chain probabilities and time lapses, verify the conditions, and implement the numerical method.

6.1 Description of a problem and its solution

Consider the system consisting of one server with a reentrant queue, as shown in Figure 2.1. We motivate this example by thinking about a printer that only uses two colors, say blue and red, but can only print one color on a page at a time: first blue and then red. Each stack (queue) has a maximum capacity, $C_1 = 1.2$ and $C_2 = 0.8$ respectively. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

be the state vector in the state space $G = [0,\, 1.2] \times [0,\, 0.8]$, where

x_1 = number of pages waiting for blue ink
and
x_2 = number of pages waiting for red ink.

For this example we use

$$v(x,\, u) = \beta - Mu \quad \text{with} \quad U = [0,\, 1] \times [0,\, 1],$$

$$\beta = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} \text{ and } M = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Notice the values of β, the disturbance or load. This says that external arrivals happen only in queue 1 at a constant rate, and we have no external arrivals into queue 2. Since there is no departure from the system after queue 1 is serviced, all its output is transfer to queue 2. Using these values (2.1.1) takes the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{6.1.1}$$

with $u_1(t) + u_2(t) = 1$. However (6.1.1) describes the dynamics only when neither of the queues is empty or has reached the maximum capacity, that is, for $x \in G^o$, the interior of G that we define to be

$$G = (0,\, 1.2) \times (0,\, 0.8).$$

When one of the queues becomes empty or is at maximum capacity we need to correct for what the control would attempt to do if left unchecked. For instance if $x_1 = 1.2$, we need to push out of the system any extra request for service coming into queue 1 thus a reflecting vector $\widetilde{d}_1 = [-1\ 0]^T$. The remaining reflecting vectors are

$$d_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ x_1 = 0;\ d_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, x_2 = 0;\ \widetilde{d}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ x_2 = 0.8$$

as shown in Figure 2.2. In particular notice that the effect of $\widetilde{d}_2$ is to drop out of the system any page that attempts to exceed the maximum capacity of queue 2. These are the reflecting vectors necessary to explain the Skorokhod formulation for this specific problem.

Given the cost function

$$L(x,\, u) = \frac{1}{2}\|x(t)\|^2 + 1.1$$

(independent of u) our goal is to design service allocation policies for $x(t)$ to satisfy all the demand while minimizing the cost functional

$$J(x_0,\, u) = \int_0^{\tau_0} \left[\frac{1}{2} \parallel x(t) \parallel^2 + 1.1\right] dt.$$

The dynamic programming approach consists of studying the value function

$$V(x) = \min_{u \in U} J(x, u).$$

With this example in mind we are now going to implement the calculations described in Chapter 4.

To create the Markov Chain we take the following steps.

1. Construct the grid as described in Chapter 4, using $h = 1.33 \times 10^{-3}$.

2. Following the Diamond method (Section 4.1.1), the transition probabilities and time increments are

$$p\,(x,\,y|u) = \begin{cases} \frac{(\beta - Mu)_i}{\|\beta - Mu\|_1}, & \text{if } y = x + h\,\text{sign}((\beta - Mu)_i)\,e_i; \\ 0, & \text{otherwise} \end{cases} \tag{6.1.2}$$

where e_i are the canonical basis vectors, and time lapse

$$\Delta t^h(x,\,u) = \frac{h}{\parallel \beta - Mu \parallel_1}. \tag{6.1.3}$$

Notice that since $u_1 + u_2 = 1$ and given our definition of β and M we can assure that $\|\beta - Mu\|_1 \neq 0$. We take advantage of the geometry of G and the definition of the reflection vectors, and choose h so that all the boundary grid points ($\partial\bar{G}^h$) are standard grid points (G^h). By doing this, there is no need for defining the probabilities $p(x,\,y|u)$ with $x \in \partial\bar{G}^h \setminus G^h$. The transition probabilities associated with a projection part of the reflection jump will be
if $x_1 > C_1$ then $p(x, (x_1 - h, x_2)) = 1$,
if $x_2 > C_2$ then $p(x, (x_1, x_2 - h)) = 1$,
if $x_2 < 0$ then $p(x, (x_1, x_2 + h)) = 1$,
if $x_1 < 0$ and $x_2 > h$ then $p(x, (x_1 + h, x_2 - h)) = 1$, and
if $x_1 < 0$ and $0 < x_2 \leq h$ then $p(x, (x_1 + h, x_2)) = 1$.

To compute the minimization over u in the discrete dynamic programming equation (4.0.14) we first change its representation to make the implementation more efficient. Let

$$y = x + \sum_{i=1,\,2} \text{sgn}\,(\beta - Mu)_i\,e_i,$$

and use (6.1.2) and (6.1.3). It follows that

$$V^h(x) = \min_u \left[\frac{\sum_{i=1,2} |(\beta - Mu)_i|\,V^h(y) + L(x)\,h}{\parallel \beta - Mu \parallel_1} \right].$$

Since

$$|(\beta - Mu)_i| = \text{sgn}\,(\beta - Mu)_i\,[(\beta - Mu)_i]$$

and define the vectors

$$\mathcal{V}(\beta,\,u) = \text{sgn}\,(\beta - Mu) \;\text{ and }\; V_{\mathcal{V}} = \sum_{i=1,\,2} V^h(x + h\mathcal{V}_i\,e_i)\,\mathcal{V}_i\,e_i.$$

These are respectively the sign of the velocity in the interior of G and the value function at the possible points y where we could move to, multiplied by the sign of the velocity vector in order to obtain the corresponding sign in the compacted notation below. Then

$$V^h(x) = \min_{u \in U} \left\{ \frac{(\beta - Mu)\,V_{\mathcal{V}} + h\,L(x)}{(\beta - Mu) \cdot \mathcal{V}(\beta,\,u)} \right\}. \tag{6.1.4}$$

Notice that (6.1.4) is a piecewise linear fractional function of $u \in U$. As such, it is continuous and monotone between changes of $\mathcal{V}(\beta,\, u)$ with assymptotic behavior when the denominator $(\beta - Mu) \cdot \mathcal{V} = 0$. In our example this never happens. Thus, we will find the minimum value at $\mathbf{u} = [1\ 0]^T$ or $\mathbf{u} = [0\ 1]^T$.

At each $x \in G^h$ we only need to compare a finite number of values and decide which of the finite set of control values achieves the minimum value.

For the the parameter values $h = \frac{1}{750}, C_1 = 1.2,\ C_2 = 0.8$, error $= 1 \times 10^{-6}$, maximum number of iterations or rounds (10,000), initial value $V_0 = 10$, the computer code initializes the different matrices of controls, values of the value function and values of the grid points. At each iteration the computer calculates the cost matrix L. Using some intermediate steps to calculate the different values in (6.1.4), the computer code compares the values from the current value function matrix using different controls; it fixes the controls that provide the minimum and computes the new values for the matrix V. This process is repeated until the maximum number of iterations or stability of V is reached.

Figure 6.1 shows the values of V^h resulting from this calculation. The base of the graph is the state space, with x_2 on the left axis. The vertical axis is the values of V^h. As expected, the maximum cost to empty the system is achieved when the system is at full capacity, as we can see by the highest value at point (1.2, 0.8).

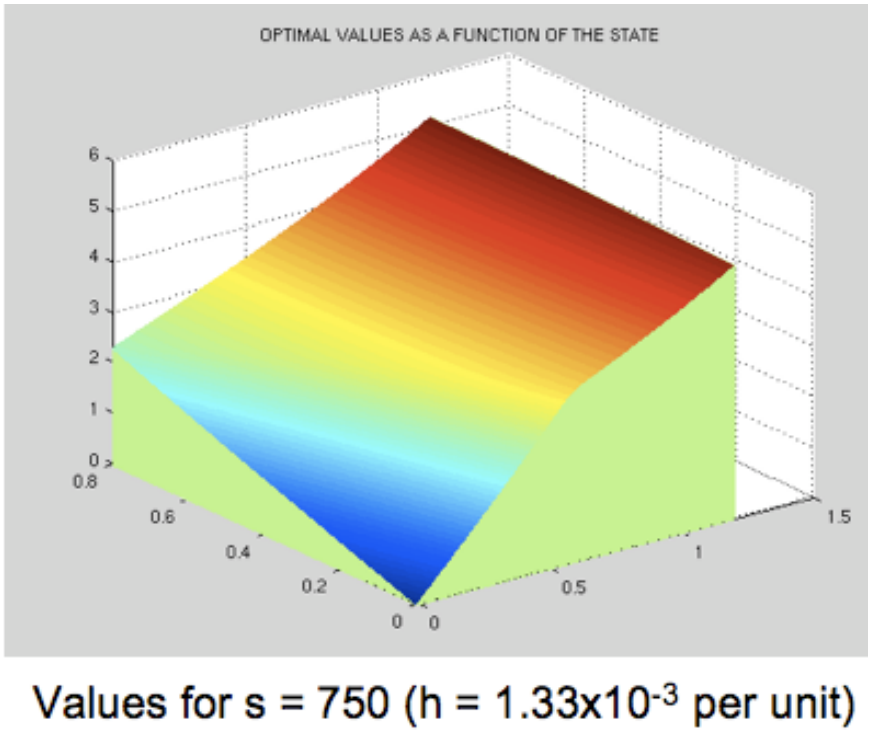

Figure 6.1: Optimal Values

More importantly, Figure 6.2 shows the areas of the state space on which specific controls would produce the optimal solution. Notice the areas red and black; these are a consequence of the discretization. The controls prescribed by the numerical optimization method drives the system to the origin in a zig-zag (down and left) motion corresponding to the controls $\mathbf{u} = [.3\ .7]^T$ and $\mathbf{u} = [.5\ .5]^T$ respectively.

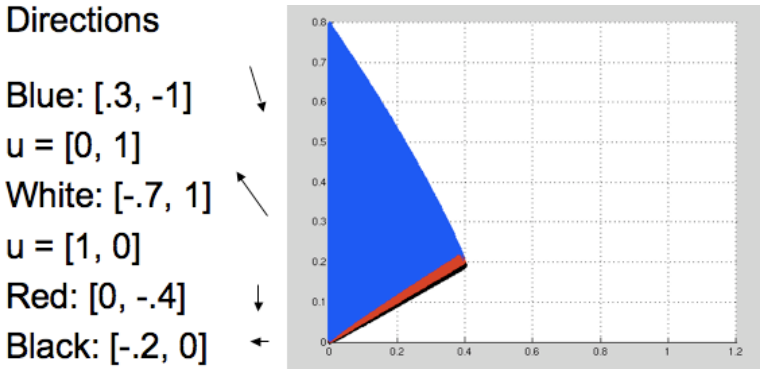

Figure 6.2: Directions and optimal controls

6.2 Verification of hypotheses

We now proceed to verify the conditions that provide a continuous value function (Hypotheses in Chapter 2) as well as the other conditions for convergence of V^h to V (*Chain Hypotheses* and other requirements from Chapter 4 as necessary).

1. $V(x)$ is continuous at 0.
   Assume $x^0 = (x_1, x_2)$ is close to zero, say $0 < \|x\| \leq \delta \leq 1$. We choose a policy that takes x^0 to $x^1 = (0, x_2)$ using control [.5, .5] for a time T_1 and cost C_1 and then to 0 using control [.3, .7] for a time T_2 and cost C_2 for a total cost $J = C_1 + C_2$.
   Using control [.5, .5], we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} .3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{bmatrix} -.2 \\ 0 \end{bmatrix}$$

   Hence, to go from x^0 to x^1 it takes $T_1 = |x_1|/.2 = 5\,|x_1|$ and costs

$$\begin{aligned} \int_0^{5|x_1|} \frac{1}{2}\|x(t)\|^2 + 1.1\,dt &= \int_0^{5|x_1|} \frac{1}{2}\left[(x_1(t))^2 + (x_2(t))^2\right] + 1.1\,dt \\ &\leq \int_0^{5|x_1|} \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + 1.1\,dt \\ &= \frac{1}{2}x_1^2 t + \frac{1}{2}x_2^2 t + 1.1t\Big|_0^{5|x_1|}. \end{aligned}$$

Hence we have the estimate

$$C_1 \leq \frac{5}{2}|x_1|^3 + \frac{5}{2}|x_1|\,x_2^2 + \frac{11}{2}|x_1|.$$

Observe that the inequality of the second line is due to the fact that the control we chose will keep the second component constant, while decreasing the first component to zero.

Next, from $(0,\, x_2)$ we use control $[.3,\, .7]$. To reach the origin it takes

$$T_2 = \frac{5}{2}|x_2| \;\text{ with }\; C_2 \leq \frac{5}{4}|x_2| + \frac{11}{4}|x_2|.$$

Thus, the cost to go from x^0 to $(0,\, 0)$ using these controls (noting that $\|x\| \leq \delta \leq 1$) is

$$\begin{aligned} J &\leq \frac{5}{2}|x_1|^3 + \frac{5}{2}|x_1|\,x_2^2 + \frac{11}{2}|x_1| + \frac{5}{4}|x_2| + \frac{11}{4}|x_2| \\ &\leq \frac{5}{2}\delta^3 + \frac{5}{2}\delta^3 + \frac{11}{2}\delta + \frac{5}{4}\delta^3 + \frac{11}{4}\delta \\ &= \frac{25}{4}\delta^3 + \frac{33}{4}\delta \\ &\leq \frac{58}{4}\delta. \end{aligned}$$

By choosing $\delta \leq \frac{2}{29}\epsilon$, for any $\epsilon > 0$ we attain $V(x) \leq \epsilon$ for any $\delta > 0$ and $\|x\| \leq \delta$.

2. For all $\epsilon > 0,\ \inf_{x\in G, |x|\geq\epsilon} V(x) > 0$.
For any $x \in G$ we have that the cost function is strictly positive, i.e. $L(x) \geq 1.1$. Thus we have

$$\inf_{x\in G, |x|\geq\epsilon} V(x) \geq \int_0^{\tau_0} 1.1\,dt = 1.1\,t\Big|_0^{\tau_0} = 1.1\,\tau_0.$$

It suffices to show that $\tau_0 \neq 0$ or (since $|x| \geq \epsilon > 0$) that $\|v\| < \infty$.
Notice that for any $x \in G^0$, $v(x) = \dot{x} = \beta - M\,u$. That implies

$$\begin{aligned} \left\| \begin{bmatrix} .3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ 1-u_1 \end{bmatrix} \right\| &\leq \left\| \begin{bmatrix} .3 \\ 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} u_1 \\ 1-2\,u_1 \end{bmatrix} \right\| \\ &\leq .3 + |u_1| + |1-2\,u_1| \\ &\leq 2.3 \end{aligned}$$

since $u_1 \in [0,\, 1]$, $|1-2\,u_1|$ reaches its maxima at $u_1 = 0$ and $u_1 = 1$. We conclude that $\|v\| < \infty$. In the worse case scenario, $x \in \partial G$ in which case $\pi(x,\, u) = \beta - Mu + \sum_{i\in I} \alpha_i d_i$ and thus

$$\begin{aligned} \left\| \begin{bmatrix} .3 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ 1-u_1 \end{bmatrix} + \sum_{i\in I} \alpha_i d_i \right\| &\leq \left\| \begin{bmatrix} .3 \\ 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} u_1 \\ 1-2\,u_1 \end{bmatrix} \right\| + \left\| \sum_{i\in I} \alpha_i d_i \right\| \\ &\leq .3 + |u_1| + |1-2\,u_1| + 1 \;\text{ by (2.0.3)} \\ &\leq 3.3. \end{aligned}$$

3. There exists at least one proper policy for the Markov Chain-based control problem. Choose $0 < h < 1$. Then there are

$$N = \left(\frac{1.2}{h} + 2\right)\left(\frac{0.8}{h} + 2\right)$$

grid points. For any $x \in G^h$ (if $x \in \widetilde{G}^h$ it only takes one extra jump to be back in G^h) use control $u^1 = [.5\ .5]^T$ so that $v^1 = [-.2\ 0]^T$ until we reach the axis $x_1 = 0$. That takes at most

$$\frac{x_1}{h} + 1 < \frac{1.2}{h} + 2$$

jumps. Then use control $u^2 = [.3\ .7]^T$ that produces $v^2 = [0\ -.4]^T$ to the origin for a total of

$$\frac{x_2}{h} + 1 < \frac{0.8}{h} + 2$$

jumps. Then the total number of jumps would be at most

$$\frac{1.2}{h} + \frac{0.8}{h} + 4 < N.$$

4. There is a K, $\delta > 0$ so that

$$\sup_{x^h \in \overline{G^h}} \left|V^h(x^h)\right| \leq K, \quad \text{all } 0 < h < \delta.$$

Recall that

$$V^h(x^h) = \inf_u W^h(x,\, u)$$

where

$$W^h(x,\, u) \ = \left[\sum_{y \in G^h} p(x,\, y|u)V(y) + L(x^h)\Delta\, t^h(x,\, u)\right].$$

We choose the same policy as in the previous property. We need to make the following observations that are derived from using the controls as above and our definitions of L and J.

(a) Consider any two different grid points x^i and x^{i+1} (the superscripts are not values of h but time indices: x^i occurs before x^{i+1}.) Then $\|x^i\| \geq \|x^{i+1}\|$ except when $x_2^i = -h$ but this case is not significant for our conclusions since it adds nothing to the cost.

(b) Since $\|x^k\| \geq \|x^{k+1}\|$ then $L(x^k) \geq L(x^{k+1})$ for all x^k and x^{k+1}.

(c) On each segment (horizontal and vertical), the velocity remains constant, so does the total time, regardless of the number of subintervals.

(d) On each segment the more subintervals we have (the smaller h is), the smaller the cost.

Based on these observations, we know that we get the largest possible cost when we start at the point x^0 farthest away from the origin, that is $x^0 = (1.2,\ .8)$. We also know that the cost associated with any regular grid spacing is less than the cost associated with a displacement equal to the largest coordinate of the initial point. For $h^1 = 1.2$ we have

$$C^1 = \left[\frac{1}{2}\|((1.2,\ .8)\|^2 + 1.1\right]\ \frac{1.2}{.2} = 16.05$$

This takes us to $x^1 = (0,\ .8)$. Take $h^2 = 0.8$ that produces

$$C^2 = \left[\frac{1}{2}\|((0,\ .8)\|^2 + 1.1\right]\ \frac{.8}{.4} = 3.$$

Hence $J(x^0,\ u) = 19.05$ is the cost corresponding to an arbitrarily chosen policy and the largest possible initial value. We take $K = 19.05$ and $\delta = 0.1$ for example.

5. Given $\epsilon > 0$ there exists a $\delta > 0$ so that

$$\left|V^h(x)\right| < \epsilon \ \text{ for all } \ x \in \overline{G^h} \ \text{ with } \ |x| < \delta \ \text{ and } \ h < \delta.$$

Since we want x close to zero, we may assume that $\delta \leq 1$ and that $x = (x_1,\ x_2) \neq (0,\ 0)$. Take the same policy u^ϵ as before: to go horizontally to $x^1 = (0,\ x_2)$ with control $u^1 = [.5\ .5]^T$ and then down to the origin with $u^2 = [.3\ .7]^T$. For $\|x\| < \delta \leq 1$ we have

$$J(x,\ u^\epsilon) = C^1 + C^2$$

with

$$\begin{aligned} C^1 &= L(x^1)\,\Delta\, t^h(x^1,\ u^1) \\ &\leq \left(\frac{1}{2}\left[(x_1^1)^2 + (x_2^1)^2\right] + 1.1\right)\ \frac{\delta}{.4} \\ &\leq \left(\delta^2 + 1.1\right)\ \frac{5\,\delta}{2} \end{aligned}$$

$$\begin{aligned} C^2 &= L(x^2)\,\Delta\, t^h(x^2,\ u^2) \\ &\leq \left(\frac{1}{2}(x_2^1)^2 + 1.1\right)\ \frac{\delta}{.2} \\ &\leq \left(\frac{1}{2}\delta^2 + 1.1\right)\ 5\,\delta \end{aligned}$$

$$\begin{aligned} J(x,\ u^\epsilon) &\leq \frac{5}{2}\,\delta^3 + \frac{5.5}{2}\,\delta + \frac{5}{2}\,\delta^3 + 5.5\,\delta \\ &= (5\,\delta^2 + 8.25)\,\delta \\ &\leq \frac{53}{4}\,\delta. \end{aligned}$$

Choose $\delta = \frac{4}{53}\epsilon$.

With this we show that we meet all the conditions required for the limiting fluid deterministic problem to have a unique solution V and for our numerical method to also have a unique solution V^h and to converge to V.

Chapter 7

Conclusion

In order to solve optimal control problems for stochastic queueing networks, we developed a numerical method based on the Markov Chain approximation technique to solve a discretized version of the approximating (deterministic) fluid model. We based our work on the methods by Kushner and Dupuis [28] following the approach of Boué and Dupuis [12] but expanding our solution to problems that incorporate reflecting boundaries (Skorokhod dynamics).

The numerical solution we offer is valid for positive cost functions. The state space is a closed, convex polyhedron that includes the origin. Some conditions are imposed on the dynamics in the interior of the space (the velocity function), such as being bounded and Lispchitz continuous. The cost function carries similar restrictions (see (2.3.5)).

Chapters 2 and 3 deal with the continuous (fluid) control problem. We establish that the value function is continuous (Chapter 3). It was the proof of the convergence of the value function (Chapter 5) that took the heavier mathematical machinery. In order to to this, we rely on the representation of the limiting fluid problem in terms of a representation based on relaxed controls $\mathcal{R}$. We show that the set of relaxed control is compact and separable.

In Chapters 4 and 5 we present the Markov Chain-based approximation to the deterministic problem and provide the support to claim that the numerical method approximates the solution.

Finally, Chapter 6 shows the implementation of our method with a specific example. An outline of the algorithm is presented in that chapter. We also include in that chapter the verification of the conditions for continuity of the value function and convergence of the numerical method.

Our main contribution to the study of optimal control problems that arise from (stochastic) queueing networks is to develop a numerical method for a control of the (deterministic) fluid limit of queueing models in a (polyhedral) bounded domain together with the analysis of convergence, including the proof of continuity of the value function.

As an extension of our work, we can pose the following immediate challenges:

1. An error analysis of our method,
2. a generalized solution to a broader type of cost functions,
3. include a more general type of state space, such as a mixture of smooth (not only straight faces), and
4. expand the method to consider differential game problems.

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Appendix A

A.1 Definitions Regarding Controlled Markov Chains

Definition A.1.1. *Consider the discrete-time dynamic system*

$$x_{k+1} = f(x_k,\, u_k),\ k = 0,\, 1,\, \ldots,$$

where for all k, the state x_k is an element of the space G and the control u_k is an element of the space U and it is constrained to take values in a given nonempty subset $U(x_k)$ of U, which depends on the current state x_k. And admissible policy is a sequence of control functions $\pi = \{\mu_0,\, \mu_1,\, \ldots\}$ with $\mu_k : G \to U$, $\mu_k(x_k) \in U(x_k)$, for all $x_k \in G$, $k = 0,\, 1,\, \ldots$. ([9], p. 3)

Definition A.1.2. *A stationary policy is an admissible policy of the form $\pi = \{u,\, u,\, \ldots\}$, that is, a sequence of control functions that take the same action for the same state at every stage. ([10], p. 3)*

Definition A.1.3. *A stationary policy u is said to be proper if, when using this policy, there is a positive probability that the destination (termination or end state) will be reached after at most N-1 steps, with N being the total number of states, regardless of the initial state. A stationary policy that is not proper is said to be improper. ([10], p. 80)*

A.2 Weak Convergence of Probability Measures

Definition A.2.1. *Denote a metric space by S and let $\mathcal{S}$ be the Borel σ-field generated by the open sets of S. If probability measures P_n and P satisfy*

$$\int_S f\, dP_n \to \int_S f\, dP$$

for every bounded, continuous real function f on S, we say that P_n converges weakly to P, and express it by $P_n \Rightarrow P$. ([11], p. 7)

As an extension of Definition A.2.1, we say that finite measures μ_n defined on the metric space S converge weakly to a measure μ (also in S) ($\mu_n \Rightarrow \mu$) if for every bounded, continuous real function f on S, $\int_S f\, d\mu_n \to \int_S f\, d\mu$. In particular, considering $f \equiv 1$, this requires $\mu_n(S) \to \mu(S)$.

Theorem A.2.2. *Let $\{\theta_n\}$ be a sequence in $\mathcal{P}(S)$ as in (B.1.1). Then θ_n converges weakly to θ ($\theta_n \Rightarrow \theta$) in $\mathcal{P}(S)$ if and only if $\mathcal{L}(\theta_n, \theta) \to 0$. Furthermore, with respect to the Lévy-Prohorov metric $\mathcal{P}(S)$ is complete and separable. ([23], p. 374)*

Proposition A.2.3. *A metric space S is compact if and only if it is both complete and totally bounded. ([33], p. 156)*

Definition A.2.4. *We say a sequence of random elements $\{X_n\}$ with values in a metric space S converge in distribution to the random element X if $P_n \Rightarrow P$, where P_n and P are the distributions of X_n and X respectively. ([11], p. 25)*
That is, $\{X_n\} \to X$ in distribution if and only if for any f continuous and bounded on S,

$$\lim_{n\to\infty} E[f(X_n)] = E[f(X)].$$

([23], p. 375)

Note: The notion of convergence in distribution does not require that the random variables X_n and X be defined on a common probability space.

Definition A.2.5. *Let $\{X_n,\, n \in \mathbb{N}\}$ be a sequence of random variables with values in a metric space S that are defined on a sequence of probability spaces $\{(\Omega_n, \mathcal{F}_n,\ P_n),\, n \in N\}$. We say that the sequence $\{X_n\}$ converges in probability to 0 if for every $\epsilon > 0$,*

$$\lim_{n\to\infty} P_n(d(X_n, 0) \geq \epsilon) = 0.$$

([23], p. 375)

Theorem A.2.6. *The sequence of random variables $\{X_n\}$ converges in distribution to 0 if and only if it converges in probability to 0. ([23], p. 376)*

Definition A.2.7. *A probability measure P on $(S, \mathcal{S})$ is regular if for every $\mathcal{S}-$set A and every ϵ there exist a closed set F and an open set G such that $F \subset A \subset G$ and $P(G-F) < \epsilon$. ([11], p. 7)*

Definition A.2.8. *A probability measure P on $(S, \mathcal{S})$ is tight if for each $\epsilon > 0$ there exists a compact set K such that $P(K) \geq 1 - \epsilon$. ([11], p. 8)*

Theorem A.2.9. *A necessary and sufficient condition for $P_n \Rightarrow P$ is that each subsequence $\{P_{n_i}\}$ contain a further subsequence $\{P_{n_{i(m)}}\}$ converging weakly ($m \to \infty$) to P. ([11], p. 20)*

Definition A.2.10. *Let $\mathcal{M}$ be a family of probability measures on $(S, \mathcal{S})$. We call $\mathcal{M}$ 'relatively compact' if every sequence of elements of $\mathcal{M}$ contains a weakly convergent subsequence.([11], p. 57)*

Definition A.2.11. *The family $\mathcal{M}$ is tight if for every ϵ there exists a compact set K such that $P(K) > 1 - \epsilon$ for every P in $\mathcal{M}$.([11], p. 59)*

Theorem A.2.12 (Prohorov's Theorem – Direct)**.** *If $\mathcal{M}$ is tight, then it is relatively compact. ([11], p. 59)*

Theorem A.2.13 (Prohorov's Theorem – Converse)**.** *Suppose that S is separable and complete. If $\mathcal{M}$ is relatively compact, then it is tight.([11], p. 60)*

Theorem A.2.14. *Probability measures on $S' \times S''$ are tight if and only if the two sets of marginal distributions are tight on S' and S''. ([11], p. 65)*

Theorem A.2.15. *Let S be a metric space, and let $P_n, n < \infty$, and P be probability measures on $\mathcal{P}(S)$ satisfying $P_n \Rightarrow P_n$. Let f be a real valued measurable function on S and define D_f to be the measurable set of points at which f is not continuous. Let X_n and X be random variables which induce the measures P_n and P on S, respectively. Then $f(X_n) \Rightarrow f(X)$ whenever $P\{X \in D_f\} = 0$. ([28], p. 249)*

Definition A.2.16. *In a metric space (S, ρ), an extended real-valued function $\psi : S \to \mathbb{R}$ is called lower semicontinuous at a point y if $\psi(y) \neq \infty$ and $\psi(y) \leq \underline{\lim}_{x \to y} \psi(x)$ for every x. ([33], p. 51)*

Lemma A.2.17. *$\psi : S \to \mathbb{R}$ is lower semicontinuous if and only if there exists $f_n \in C(S)$ such that $f_n \uparrow \psi$ pointwise. ([33], p. 50)*

Lemma A.2.18. *If $\mu_n \Rightarrow \mu$ (Weak convergence of probability measures on S) and $\psi : S \to \mathbb{R}$ is bounded lower semicontinuous, then*

$$\int \psi \, d\mu \leq \underline{\lim} \int \psi \, d\mu_n.$$

([11], p. 24)

A.3 Gronwall's inequality

Theorem A.3.1. *Let μ be a Borel measure on $[0, \infty)$, let $\epsilon \geq 0$, and let f be a Borel measurable function that is bounded on bounded intervals and satisfies*

$$0 \leq f(t) \leq \epsilon + \int_{[0,\, t)} f(s) \, \mu(ds), \quad t \geq 0.$$

Then

$$f(t) \leq \epsilon\, e^{\mu[0,\,t)}, \quad t \geq 0.$$

In particular, if $M > 0$ *and*

$$0 \leq f(t) \leq \epsilon + M \int_0^t f(s)\, ds, \quad t \geq 0,$$

then

$$f(t) \leq \epsilon\, e^{M\,t}, \quad t \geq 0.$$

([24], p. 498)

A.4 Some Martingale Facts

Definition A.4.1 (Martingale)**.** *A process* X *is called a martingale if*

1. X *is adapted (for every* $n \in \mathbb{Z}^+, X_n$ *is* $\mathcal{F}_n-$*measurable);*
2. $E(|X_n|) < \infty,\ \forall n;$
3. $E[X_n|\mathcal{F}_{n-1}] = X_{n-1},$ *a.s.* $(n \geq 1).$

([32], p. 144)

Appendix B

B.1 Proofs of some lemmas

Lemma 5.2.1 $\mathcal{R}$ is compact.

Proof. For each $T < \infty$ let $\bar{\nu}^T(A) = \frac{1}{T}\nu(A)$ for all $A \subseteq U \times [0,\, T]$. Then $\mathcal{R}_T = \{\bar{\nu}^T\}$ is the set of probability measures on $[0,\, T] \times U$ on which we define the Lévy-Prohorov metric on $\mathcal{R}_T$

$$A^\epsilon = \{s' : d(s,\, s') < \epsilon \text{ for some } s \in A\}$$

and for $\bar{\nu}_1^T$ and $\bar{\nu}_2^T$ in $\mathcal{R}_T$ let

$$\mathcal{L}^T(\bar{\nu}_1^T,\, \bar{\nu}_2^T) = \inf\{\epsilon > 0 : \bar{\nu}_1^T(A) \le \bar{\nu}_2^T(A^\epsilon) + \epsilon \text{ for all closed } A \subseteq [0,\, T] \times U\}. \tag{B.1.1}$$

For ν_1 and ν_2 in $\mathcal{R}$ define the metric

$$\rho(\nu_1,\, \nu_2) = \sum_{j=1}^{\infty} 2^{-j}\, \frac{\mathcal{L}^j(\bar{\nu}_1^j,\, \bar{\nu}_2^j)}{\mathcal{L}^j(\bar{\nu}_1^j,\, \bar{\nu}_2^j) + 1}. \tag{B.1.2}$$

This makes $\mathcal{R}$ a metric space.

Next we explain why the ρ topology is the same as the weak topology described above.

Under this metric a sequence $\nu_n(\cdot)$ in $\mathcal{R}$ converges weakly to $\nu(\cdot) \in \mathcal{R}$ if

$$\int_0^j \int_U f(t,\, u(t))\, d\nu_n \to \int_0^j \int_U f(t,\, u(t))\, d\nu$$

for any bounded continuous function $f : [0,\, \infty) \times U \to \mathbb{R}$. From the definition of ρ we only have this convergence for each $j \in \mathbb{N}$. However,

$$\int_0^T \int_U f(t,\, u)\, d\nu_n \to \int_0^T \int_U f(t,\, u)\, d\nu \tag{B.1.3}$$

can be expressed as

$$\int_0^j \int_U 1_{[0,\,T]}\, f(t,\, u)\, d\nu_n \to \int_0^j \int_U 1_{[0,\,T]}\, f(t,\, u)\, d\nu$$

for $T < j$. The function $1_{[0,\,T]}\, f(t,\, u)$ is continuous except for $(t,\, u) \in \{T\} \times U$, but since $\nu(\{T\} \times U) = 0$ for any $\nu \in \mathcal{R}$, by Theorem A.2.15 of Appendix A we still get the convergence of (B.1.3).

Thus, under the metric ρ (B.1.2), a sequence ν_n in $\mathcal{R}$ converges to $\nu \in \mathcal{R}$ if and only if the sequence of probability measures converges ($\bar{\nu}_n^T \Rightarrow \bar{\nu}^T$) for all $T < \infty$. Since for each $T < \infty$ the set $[0,\, T] \times U$ is compact, tightness of $\mathcal{R}_T$ is inmediate. Then $\mathcal{R}_T$ is compact and every sequence $\{\bar{\nu}_n^T\}$ has a convergent subsequence $\{\bar{\nu}_{n_k}^T\}$. □

Lemma 5.2.3 *The continuous standard controls are dense in $\mathcal{R}$.*

Proof. The proof is in several stages:
1. Since U is totally bounded, for each $\epsilon > 0$ there exists a finite set $F = \{u_1,\, u_2,\, \ldots,\, u_n\}$ so that $\{B_\epsilon(u_i)\}$ cover U. Form the measurable partition

$$B_1 = B_\epsilon(u_1), \quad B_i = B_\epsilon(u_i) \setminus \cup_{j=1}^{i-1} B_\epsilon(u_j).$$

We can "concentrate" U on $\{u_1,\, u_2,\, \ldots,\, u_n\}$ with the map $\psi_F : U \to U$ defined by $\psi_F(u) = \sum u_i\, 1_{B_i}(u)$ and define $\Phi_F : [0,\, \infty) \times U \to [0,\, \infty) \times U$ by $\Phi_F(t,\, u) = (t,\, \psi_F(u))$. Let $\nu_F = \nu\, \Phi_F^{-1}$. Then for f bounded and continuous function

$$\int_0^T \int_U f(t,\, u)\, d\nu_F = \int_0^T \int_U f(t,\, \psi_F(u))\, d\nu.$$

Take a sequence of F_n associated with $\epsilon = 1/n$ and consider ν_{F_n}. Since $|u - \psi_{F_n}(u)| < \frac{1}{n}$ and f is continuous $|f(t,\, \psi_F(u)) - f(t,\, u)| \to 0$, uniformly on each $[0,\, T] \times U$. Therefore

$$\int_0^T \int_U f(t,\, u)\, d\nu_{F_n} \to \int_0^T \int_U f(t,\, u)\, d\nu$$

Then the collection of ν with "finite support" are dense in $\mathcal{R}$. By this we mean that there is a finite set F so $\nu([0,\, T] \times F) = \nu([0,\, T] \times U)$.

2. Consider $\nu = \nu_F$ as above using $F = \{u_1,\, u_2,\, \ldots,\, u_n\}$. Given $h > 0$ we produce a "piecewise constant" approximation ν_h as follows. For each $I_k = [k\, h,\, (k+1)\, h]$ let $p_i = \frac{1}{h}\nu(I_k \times \{u_i\})$. Then $0 \le p_i$ and $\sum_{i=1}^n p_i = 1$. Define ν_h on $I_k \times U$ by

$$\nu_h(I_k \times U \cap A) = \int_{k\,h}^{(k+1)\,h} \sum_{i=1}^n p_i\, 1_A(t,\, u_i)\, dt.$$

I.e. $\nu_h(du \times dt)$ is $(\sum_{i=1}^n p_i \delta_{u_i}(du))\, dt$ on I_k. Notice that if $f(t,\, u) = f(u)$ does not depend on t then

$$\int_{kh}^{(k+1)h} \int_U f(u)\, d\nu = \int_{kh}^{(k+1)h} \int_U f(u)\, d\nu_h = h \sum_i^n p_i f(u_i).$$

So for $f_h(t,\, u) = f(h\, [\![t/h]\!], u) = f(kh,\, u)$ if $kh \le t < (k+1)h$ we have

$$\int_0^{Nh} \int_U f_h(t,\, u)\, d\nu = \int_0^{Nh} \int_U f_h(t,\, u)\, d\nu_h$$

for any integer N. We also know that as $h \to 0$, $f_h \to f$ uniformly on any $[0,\, T] \times U$. So for $N = [\![T/h]\!]$ we have

$$\int_0^T \int_U f(t,\, u)\, d\nu \le \int_0^{Nh} \int_U f_h(t,\, u)\, d\nu + \|f\|_\infty \cdot h + \|f_h - f\|_{\infty,[0,\,T]} \cdot T$$

from which

$$\left| \int_0^T \int_U f(t,\, u)\, d\nu - \int_0^{Nh} \int_U f_h(t,\, u)\, d\nu \right| \le \|f\|_\infty \cdot h + \|f_h - f\|_{\infty,[0,\,T]} \cdot T$$

and likewise with ν replaced by ν_h. It follows that

$$\int_0^T \int_U f\, d\nu_h \to \int_0^T \int_U f\, d\nu$$

as $h \downarrow 0$. Thus the $\nu = \nu_h$ which are "piecewise constant with finite U support" are dense in $\mathcal{R}$.

3. Given a piecewise constant finitely $U-$supported ν as above, we want to approximate it by a piecewise constant standard control. To do this on each interval $a \le t \le b$ on which

$$\nu(du \times dt) = \left[\sum_{i=1}^n p_i \delta_{u_i}(du) \right] dt$$

we divide $[a,\, b]$ up into n pieces $[\alpha,\, \beta]$, $\beta = \alpha + \frac{b-a}{n}$. On each such piece into intervals of lengths $p_i(\beta - \alpha)$ and define $u_n(t) = u_i$ on each. Then if $f(t,\, u) = f(u)$ in $[\alpha,\, \beta]$ we have

$$\int_\alpha^\beta \int_U f\, d\nu = \int_0^T f(t,\, u_n(t))\, dt.$$

So by an argument like that above we get

$$\int_0^T \int_U f\, d\nu = \lim_{n \to \infty} \int_0^T f(t,\, u_n(t))\, dt.$$

This piecewise constant standard controls are dense in $\mathcal{R}$.

4. Finally, given a piecewise constant standard control $u(t) = u_i$ on $t_{i-1} \le t < t_i \quad (t_i \to \infty)$ we can approximate it by smoothing out the transitions by linear connections between the points

$$P_{i-1}(t_{i-1},\, u_{i-1}) \;\text{ and }\; P_i\left(t_i + \frac{(t_i - t_{i-1}) \wedge 1}{n},\, u_i\right).$$

Since $u_n(t) = u(t)$ on $[0,\, T]$ except for a set of measure $\frac{1}{n}(T+1)$ it follows that

$$\int_0^T f(t,\, u_n(t))\, dt \to \int_0^T f(t,\, u(t))\, dt$$

for all bounded continuous u.

At each stage the convergence of

$$\int_0^T \int_U f\, d\nu_n \to \int_0^T \int_U f\, d\nu$$

for all T and all bounded continuous f means $\nu_n \to \nu$ w.r.t. the metric for $\mathcal{R} : \rho(\nu_n,\, \nu) \to 0$. So by chosing $\rho(\nu_n,\, \nu) < \frac{\epsilon}{3}$ at each stage we obtain $u \in C([0,\, \infty), \times U)$ so that $\rho(\nu_n,\, \nu) < \epsilon$.

□

Vita

José María Menéndez Gómez was born on February 6th, 1968 in Santa Ana, El Salvador. He was the youngest child of the late Maximiliano Menéndez and Yolanda Gómez de Menéndez. In 1985 he obtained his high school diploma from Liceo San Luis. After three years in the seminary with the Marist Brothers, José María went to the Universidad Centroamericana "José Simeón Cañas" where he received his teaching degree in high school mathematics and physics in 1994. In 1996 he graduated from Louisiana State University where he worked on his bachelors of science in mathematics and bachelors of arts in Spanish with a Fulbright-CAMPUS scholarship. At Virginia Tech, José María completed his masters of science degree in mathematics in 2000 and started his doctoral studies in 2002 under the direction of Dr. Martin V. Day.

José María's first experience teaching young children was when he was 8 years old, as a guest in a rural school in El Salvador. This experience marked his life and started a path in the world of education. He taught middle and high school at Escuela San Alfonso and Colegio Nuestra Señora de Fátima, in El Salvador. At the collegiate level, he was an instructor at the Universidad Centroamericana "José Simeón Cañas" and Universidad Dr. José Matías Delgado, in El Salvador as well. In the United States, he taught mathematics at Virginia Tech and currently at The University of Arizona, where he is a CEMELA fellow.