Chapter 2

Shell Equations

2.1 Introduction

In this chapter, we derive the governing equations of motion describing the static and dynamic behaviors of a thin elastic shell under pressure. We follow Sanders' shell theory (Sanders, 1959) because of its consistency and good accuracy. First, the fundamental theorems of surfaces, which relate the parameters of a shell, are presented. Thereafter, we present Love's first approximation and other simplifications to be used in the course of the derivation. Love's first approximation is then applied to simplify the three-dimensional stress-strain relations, and the three-dimensional stress quantities are converted into two-dimensional ones. It has been realized in the past that since the initial stresses due to the applied pressure on a shell may be large, it is important to use the nonlinear strain-displacement equations (Soedel, 1986). The nonlinear strain-displacement relation couples the pressure with the stiffness terms in the equations of motion. First, we derive nonlinear strain-displacement relations from the three-dimensional elasticity theory considering nonlinearities in the in-plane strains only. These relations are later used in obtaining the governing equations of motion of a shell under pressure using Hamilton's principle. To this

end, we find variations of strain energy, kinetic energy, and the work done by different forces acting on a shell element. These variations are combined to find the shell equations and boundary conditions. The equations are separated into static and dynamic parts. These equations were derived earlier by Budiansky (1968) using tensors. In this study, we will use these equations for subsequent analyses. As special cases, we also derive the equations presented by other researchers (Sanders, 1963; Soedel, 1986; Plaut et al., 2000). The equations are presented for a general shell and can be specialized for other geometries (e.g. beams, plates, circular cylinders) with or without pressure as per the need. Since the thickness of an inflatable structure is very small, bending moments in the governing equations can be neglected, as they are very small compared to the in-plane stresses. However, the methodologies presented herein will not ignore the bending moments so as to keep the method applicable to a broader class of shell, i.e., with a relatively thicker wall.

2.2 Gauss-Codazzi Conditions

A shell is defined by a reference surface, thickness of the reference surface, and its edges. Figure 2.1 shows a shell with the associated coordinate system (α_1, α_2, and ζ).

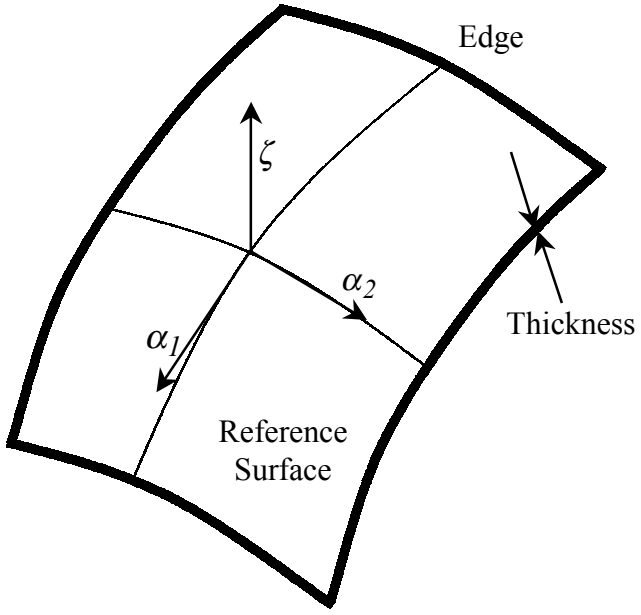

Fig. 2.1: Shell reference surface with the coordinate system.

The reference surface, which defines the shape of a shell, is described by the two Lamé parameters, A_1 and A_2, and the two principal radii of curvatures, R_1 and R_2. In order to define a valid surface, these quantities must satisfy the following three differential equations, known as Gauss-Codazzi conditions (Kraus, 1967):

$$\frac{1}{R_2}\frac{\partial A_1}{\partial \alpha_2} = \frac{\partial}{\partial \alpha_2}\left(\frac{A_1}{R_1}\right), \qquad \frac{1}{R_1}\frac{\partial A_2}{\partial \alpha_1} = \frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{R_2}\right), \tag{2.1}$$

$$\frac{\partial}{\partial \alpha_1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial \alpha_1}\right) + \frac{\partial}{\partial \alpha_2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial \alpha_2}\right) = -\frac{A_1 A_2}{R_1 R_2}. \tag{2.2}$$

Equations (2.1) and (2.2) are known as Codazzi conditions and Gauss condition, respectively. A simple manipulation of Eqs. (2.1) gives another form of Codazzi conditions (Soedel, 1986), which will be applied later in simplifying strain-displacement relationships:

$$\frac{\partial[A_1(1+\zeta/R_1)]}{\partial \alpha_2} = \left(1+\frac{\zeta}{R_2}\right)\frac{\partial A_1}{\partial \alpha_2}, \qquad \frac{\partial[A_2(1+\zeta/R_2)]}{\partial \alpha_1} = \left(1+\frac{\zeta}{R_1}\right)\frac{\partial A_2}{\partial \alpha_1}. \tag{2.3}$$

For different shell structures the radii of curvature and the Lamé parameters will be in general different and will be functions of α_1 and α_2. However, they must always satisfy the Gauss-Codazzi conditions.

2.3 Assumptions

In order to deduce the theory of the thin elastic shell from the three dimensional elasticity, a few simplifying assumptions, known as Love's first approximation, are used. These assumptions are stated as follows (Kraus, 1967):

1. The thickness of the shell is negligible compared to its radii of curvature.
2. The deflection of the shell is small.
3. The transverse normal stress is negligible.

4. The normal to the reference surface of the shell remains normal to the deformed surface.
5. The normal to the reference surface undergoes negligible change in length during deformation.

We assume that the pressure remains constant as the shell vibrates. The prestresses are assumed to be of membrane-type, i.e., uniform throughout the thickness. However, the present theory can be easily extended to include the bending-type of prestresses. In addition, we assume that the prestresses do not change with time. These assumptions are reasonable as the shell thickness is assumed small and the shell is assumed to be undergoing small vibration. In the following sections, the above assumptions will be accompanied by some additional definitions used by Sanders in order to remove some of the inconsistencies in the other shell theories such as Reissner's version of Love's theory (Reissner, 1941). The inconsistencies are related to 1) satisfying the "sixth equilibrium equation", and 2) zero strains corresponding to small rigid body motions.

2.4 Constitutive Laws

Let σ_{ij}^{r} be the initial stresses (caused by the pressure) and σ_{ij} be the vibratory stresses, and let ε_{ij} denote the strains, where $i, j = 1, 2, 3$. If the two suffixes are the same, these are called direct stresses/strains and if the suffixes are different, they are called shear stress/strains. From the symmetry of the 3-D elastic stresses and strains, we obtain

$$\sigma_{ij} = \sigma_{ji}, \qquad \varepsilon_{ij} = \varepsilon_{ji}. \tag{2.4}$$

The engineering strains (γ_{12}, γ_{13}, and γ_{23}) are defined as twice of corresponding tensor shear strains, i.e.,

$$\gamma_{23} = 2\varepsilon_{23}, \qquad \gamma_{13} = 2\varepsilon_{13}, \qquad \gamma_{12} = 2\varepsilon_{12}. \tag{2.5}$$

Assuming that the material obeys Hooke's Law and is isotropic, the stress-strain relations for a three-dimensional element are

$$\varepsilon_{11} = \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})],$$

$$\varepsilon_{22} = \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})],$$

$$\varepsilon_{33} = \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})], \tag{2.6}$$

$$\gamma_{23} = \frac{\sigma_{23}}{G}, \qquad \gamma_{13} = \frac{\sigma_{13}}{G}, \qquad \gamma_{12} = \frac{\sigma_{12}}{G}.$$

where E, G, and ν are the elastic modulus, shear modulus, and Poisson's ratio of the shell material, respectively. Following Love's first approximation, we substitute $\sigma_{33} = \gamma_{23} = \gamma_{13} = 0$ in Eqs. (2.6) to obtain

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu\sigma_{22}), \qquad \varepsilon_{22} = \frac{1}{E}(\sigma_{22} - \nu\sigma_{11}), \qquad \gamma_{12} = \frac{\sigma_{12}}{G}. \tag{2.7}$$

Since the stress and strain cannot be put to zero at the same time, we get nonzero ε_{33}, σ_{23}, and σ_{13}. While σ_{23} and σ_{13} are obtained from the equations of motions, the normal strain ε_{33} can be obtained from the following equation:

$$\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}). \tag{2.8}$$

The above equation can be used in calculating the constriction of the shell thickness during vibration (Soedel, 1986). Solving Eqs. (2.7) for stresses yields

$$\sigma_{11} = \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22}), \qquad \sigma_{22} = \frac{E}{1-\nu^2}(\varepsilon_{22} + \nu\varepsilon_{11}), \qquad \sigma_{12} = G\gamma_{12}. \tag{2.9}$$

The above-mentioned three-dimensional stresses are next converted to two-dimensional ones.

2.5 Stress Resultants and Stress Couples

The thickness of the shell is small, and hence the three-dimensional stresses can be integrated over thickness to obtain the two-dimensional stress resultants and stress couples:

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ Q_{13} \end{Bmatrix} = \int_{\zeta} \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} \left(1 + \frac{\zeta}{R_2} \right) d\zeta \,, \qquad \begin{Bmatrix} N_{22} \\ N_{21} \\ Q_{23} \end{Bmatrix} = \int_{\zeta} \begin{Bmatrix} \sigma_{22} \\ \sigma_{21} \\ \sigma_{23} \end{Bmatrix} \left(1 + \frac{\zeta}{R_1} \right) d\zeta \,,$$

$$\begin{Bmatrix} M_{11} \\ M_{12} \end{Bmatrix} = \int_{\zeta} \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \end{Bmatrix} \zeta \left(1 + \frac{\zeta}{R_2} \right) d\zeta \,, \qquad \begin{Bmatrix} M_{22} \\ M_{21} \end{Bmatrix} = \int_{\zeta} \begin{Bmatrix} \sigma_{22} \\ \sigma_{21} \end{Bmatrix} \zeta \left(1 + \frac{\zeta}{R_1} \right) d\zeta \,, \qquad (2.10)$$

where N_{11}, N_{22}, N_{12}, and N_{21} are the vibratory in-plane stress resultants, Q_{13} and Q_{23} are the transverse shear stress resultants, and M_{11}, M_{22}, M_{12}, and M_{21} are the bending and twisting moment resultants. These quantities are shown in Fig. 2.2 and Fig. 2.3. Since the order of moment arm lengths corresponding to transverse shear stresses (σ_{13}, σ_{23}) is the dimension of the differential shell element, the couples M_{13} and M_{23} will be of an order of magnitude less than the other couples. Hence M_{13} and M_{23} are neglected. Definitions similar to vibratory in-plane stress resultants can be also written for initial in-plane stress resultants (N^r_{11}, N^r_{22}, N^r_{12}, N^r_{21}):

$$\begin{Bmatrix} N^r_{11} \\ N^r_{12} \end{Bmatrix} = \int_{\zeta} \begin{Bmatrix} \sigma^r_{11} \\ \sigma^r_{12} \end{Bmatrix} \left(1 + \frac{\zeta}{R_2} \right) d\zeta \,, \qquad \begin{Bmatrix} N^r_{22} \\ N^r_{21} \end{Bmatrix} = \int_{\zeta} \begin{Bmatrix} \sigma^r_{22} \\ \sigma^r_{21} \end{Bmatrix} \left(1 + \frac{\zeta}{R_1} \right) d\zeta \,. \qquad (2.11)$$

The corresponding bending and twisting moment resultants for initial stresses are not needed as the prestresses are assumed to be of membrane-type. The initial in-plane stress resultants

are shown in Fig. 2.4. The symbol p denotes pressure, which is assumed positive when acting along the normal as shown.

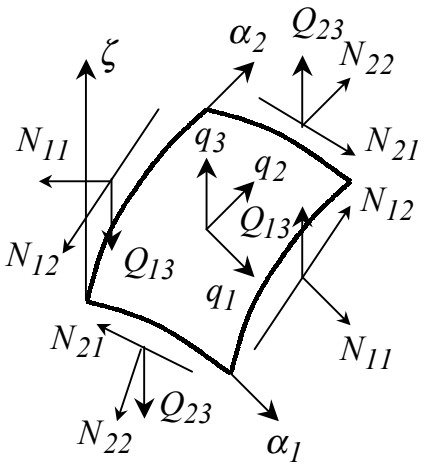

Fig. 2.2: In-plane stress resultants and external loadings.

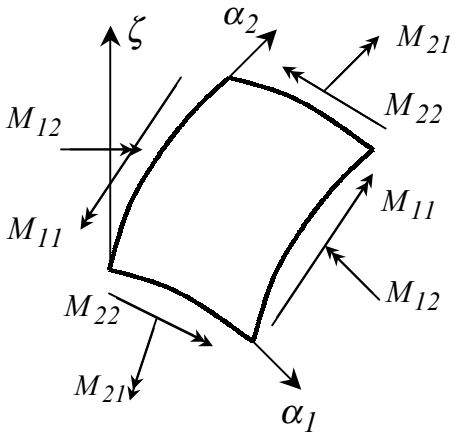

Fig. 2.3: Bending and twisting moment resultants.

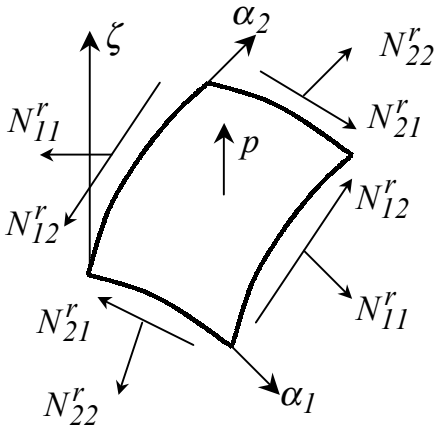

Fig. 2.4: Initial in-plane stress resultants and the pressure loading.

2.6 Nonlinear Strain-Displacement Relations

The strain-displacement relations for any three-dimensional elastic body in an orthogonal coordinate system are (Saada, 1974):

$$\begin{aligned}
\varepsilon_{11} &= e_{11} + \tfrac{1}{2}(e_{11}^2 + e_{21}^2 + e_{31}^2), \\
\varepsilon_{22} &= e_{22} + \tfrac{1}{2}(e_{22}^2 + e_{12}^2 + e_{32}^2), \\
\varepsilon_{33} &= e_{33} + \tfrac{1}{2}(e_{33}^2 + e_{13}^2 + e_{23}^2), \\
\gamma_{12} &= \gamma_{21} = e_{12} + e_{21} + e_{11}e_{12} + e_{22}e_{21} + e_{31}e_{32}, \\
\gamma_{13} &= \gamma_{31} = e_{13} + e_{31} + e_{11}e_{13} + e_{33}e_{31} + e_{21}e_{23}, \\
\gamma_{23} &= \gamma_{32} = e_{23} + e_{32} + e_{22}e_{23} + e_{33}e_{32} + e_{12}e_{13},
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
e_{11} &= \frac{1}{H_1}\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{H_1 H_2}\frac{\partial H_1}{\partial \alpha_2} + \frac{W}{H_1 H_3}\frac{\partial H_1}{\partial \zeta}, \\
e_{22} &= \frac{1}{H_2}\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{H_1 H_2}\frac{\partial H_2}{\partial \alpha_1} + \frac{W}{H_2 H_3}\frac{\partial H_2}{\partial \zeta}, \\
e_{33} &= \frac{1}{H_3}\frac{\partial W}{\partial \zeta} + \frac{U_2}{H_2 H_3}\frac{\partial H_3}{\partial \alpha_2} + \frac{U_1}{H_1 H_3}\frac{\partial H_3}{\partial \alpha_1},
\end{aligned} \tag{2.13}$$

$$e_{21} = \frac{1}{H_1}\frac{\partial U_2}{\partial \alpha_1} - \frac{U_1}{H_1 H_2}\frac{\partial H_1}{\partial \alpha_2}, \qquad e_{12} = \frac{1}{H_2}\frac{\partial U_1}{\partial \alpha_2} - \frac{U_2}{H_1 H_2}\frac{\partial H_2}{\partial \alpha_1},$$

$$e_{13} = \frac{1}{H_3}\frac{\partial U_1}{\partial \zeta} - \frac{W}{H_1 H_3}\frac{\partial H_3}{\partial \alpha_1}, \qquad e_{31} = \frac{1}{H_1}\frac{\partial W}{\partial \alpha_1} - \frac{U_1}{H_1 H_3}\frac{\partial H_1}{\partial \zeta},$$

$$e_{32} = \frac{1}{H_2}\frac{\partial W}{\partial \alpha_2} - \frac{U_2}{H_2 H_3}\frac{\partial H_2}{\partial \zeta}, \qquad e_{23} = \frac{1}{H_3}\frac{\partial U_2}{\partial \zeta} - \frac{W}{H_2 H_3}\frac{\partial H_3}{\partial \alpha_2},$$

where U_1, U_2, and W are the displacements, and H_1, H_2, and H_3 are the Lamé coefficients of the elastic body along the coordinate lines α_1, α_2, and ζ, respectively. For a thin shell, the Lamé coefficients H_1, H_2, and H_3 are given by

$$H_1 = A_1(1+\zeta/R_1), \qquad H_2 = A_2(1+\zeta/R_2), \qquad H_3 = 1. \tag{2.14}$$

Using Eqs. (2.14), we evaluate e_{11} in Eqs. (2.13) as

$$e_{11} = \frac{1}{A_1(1+\zeta/R_1)}\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_1(1+\zeta/R_1)A_2(1+\zeta/R_2)}\frac{\partial [A_1(1+\zeta/R_1)]}{\partial \alpha_2} + \frac{W}{A_1(1+\zeta/R_1)}\frac{\partial [A_1(1+\zeta/R_1)]}{\partial \zeta}. \tag{2.15}$$

Substituting the Codazzi conditions, Eqs. (2.3), in the above equation and after some simplifications, we get

$$e_{11} = \frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{W}{R_1}\right). \tag{2.16}$$

Similarly, we evaluate other quantities in Eqs. (2.13). The results are summarized in the following equations (Teng and Hong, 1998):

$$e_{11} = \frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{W}{R_1}\right),$$

$$e_{22} = \frac{1}{(1+\zeta/R_2)}\left(\frac{1}{A_2}\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{W}{R_2}\right),$$

$$e_{21} = \frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial U_2}{\partial \alpha_1} - \frac{U_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\right), \qquad e_{12} = \frac{1}{(1+\zeta/R_2)}\left(\frac{1}{A_2}\frac{\partial U_1}{\partial \alpha_2} - \frac{U_2}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\right),$$

$$e_{31} = \frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial W}{\partial \alpha_1} - \frac{U_1}{R_1}\right), \qquad e_{32} = \frac{1}{(1+\zeta/R_2)}\left(\frac{1}{A_2}\frac{\partial W}{\partial \alpha_2} - \frac{U_2}{R_2}\right),$$

$$e_{33} = \frac{\partial W}{\partial \zeta}, \qquad e_{13} = \frac{\partial U_1}{\partial \zeta}, \qquad e_{23} = \frac{\partial U_2}{\partial \zeta}. \tag{2.17}$$

From the second statement of Love's first approximation, the displacement field can be represented linearly, i.e.,

$$U_1(\alpha_1,\alpha_2,\zeta) = u_1(\alpha_1,\alpha_2) + \zeta \left.\frac{\partial U_1}{\partial \zeta}\right|_{\zeta=0},$$
$$U_2(\alpha_1,\alpha_2,\zeta) = u_2(\alpha_1,\alpha_2) + \zeta \left.\frac{\partial U_2}{\partial \zeta}\right|_{\zeta=0}, \tag{2.18}$$
$$W(\alpha_1,\alpha_2,\zeta) = w(\alpha_1,\alpha_2) + \zeta \left.\frac{\partial W}{\partial \zeta}\right|_{\zeta=0},$$

where u_1, u_2, and w are the components of the displacement vector of a point on the reference surface along the α_1, α_2, and ζ directions. Since the linear component, e_{33}, of the normal strain, ε_{33}, is generally an order of magnitude larger than its nonlinear component, the fifth assumption of Love's first approximation is applied to the linear component of the normal strain. This leads to

$$e_{33} = \left.\frac{\partial W}{\partial \zeta}\right|_{\zeta=0} = 0. \tag{2.19}$$

Similarly, from the fourth assumption, we obtain that the linear components of γ_{31} and γ_{23} are zero, implying

$$\frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial W}{\partial \alpha_1} - \frac{U_1}{R_1}\right) + \frac{\partial U_1}{\partial \zeta} = 0,$$
$$\frac{1}{(1+\zeta/R_2)}\left(\frac{1}{A_2}\frac{\partial W}{\partial \alpha_2} - \frac{U_2}{R_2}\right) + \frac{\partial U_2}{\partial \zeta} = 0. \tag{2.20}$$

From the above relations, we can find the slopes of the displacements at the reference surface as

$$\beta_1 = \left.\frac{\partial U_1}{\partial \zeta}\right|_{\zeta=0} = \frac{u_1}{R_1} - \frac{1}{A_1}\frac{\partial w}{\partial \alpha_1},$$

$$\beta_2 = \left.\frac{\partial U_2}{\partial \zeta}\right|_{\zeta=0} = \frac{u_2}{R_2} - \frac{1}{A_2}\frac{\partial w}{\partial \alpha_2}, \tag{2.21}$$

where β_1 and β_2 represent the rotations of the tangents to the middle surface oriented along the coordinate lines α_1 and α_2, respectively. Using Eqs. (2.18), (2.19), and (2.21), we can write

$$\begin{aligned} U_1(\alpha_1,\alpha_2,\zeta) &= u_1(\alpha_1,\alpha_2) + \zeta\ \beta_1(\alpha_1,\alpha_2), \\ U_2(\alpha_1,\alpha_2,\zeta) &= u_2(\alpha_1,\alpha_2) + \zeta\ \beta_2(\alpha_1,\alpha_2), \\ W(\alpha_1,\alpha_2,\zeta) &= w(\alpha_1,\alpha_2). \end{aligned} \tag{2.22}$$

Substituting Eqs. (2.22) into Eqs. (2.17) yields the following relations (Teng and Hong, 1998):

$$\begin{aligned} &e_{11} = \varepsilon_1^o + \zeta\,\kappa_1, && e_{22} = \varepsilon_2^o + \zeta\,\kappa_2, && e_{33} = 0, \\ &e_{21} = \varepsilon_3^o + \zeta\,\kappa_3, && e_{12} = \varepsilon_4^o + \zeta\,\kappa_4, && e_{13} = \beta_1, \\ &e_{31} = \varepsilon_5^o + \zeta\,\kappa_5, && e_{32} = \varepsilon_6^o + \zeta\,\kappa_6, && e_{23} = \beta_2. \end{aligned} \tag{2.23}$$

where

$$\varepsilon_1^o = \frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{w}{R_1}\right), \quad \kappa_1 = \frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1} + \frac{\beta_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\right),$$

$$\varepsilon_2^o = \frac{1}{(1+\zeta/R_2)}\left(\frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{w}{R_2}\right), \quad \kappa_2 = \frac{1}{(1+\zeta/R_2)}\left(\frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2} + \frac{\beta_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\right),$$

$$\varepsilon_3^o = \frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} - \frac{u_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\right), \quad \kappa_3 = \frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial \beta_2}{\partial \alpha_1} - \frac{\beta_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\right),$$

$$\varepsilon_4^o = \frac{1}{(1+\zeta/R_2)}\left(\frac{1}{A_2}\frac{\partial u_1}{\partial \alpha_2} - \frac{u_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\right), \qquad \kappa_4 = \frac{1}{(1+\zeta/R_2)}\left(\frac{1}{A_2}\frac{\partial \beta_1}{\partial \alpha_2} - \frac{\beta_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1}\right),$$

$$\varepsilon_5^o = -\frac{1}{(1+\zeta/R_1)}\beta_1, \qquad \kappa_5 = -\frac{1}{(1+\zeta/R_1)}\frac{\beta_1}{R_1},$$

$$\varepsilon_6^o = -\frac{1}{(1+\zeta/R_2)}\beta_2, \qquad \kappa_6 = -\frac{1}{(1+\zeta/R_2)}\frac{\beta_2}{R_2}. \tag{2.24}$$

Now we substitute Eqs. (2.23) in Eqs. (2.12) and neglect the terms having ζ^2. The remaining part can be separated into two groups based upon dependency on ζ. The group of terms that does not contain ζ represents the changes in lengths of the shell element. The other group represents the changes in the curvatures and the torsion of the reference surface. To demonstrate this, we substitute e_{11}, e_{21}, and e_{31} from Eqs. (2.23) in ε_{11} to get

$$\varepsilon_{11} = \varepsilon_1^o + \zeta\,\kappa_1 + \tfrac{1}{2}[(\varepsilon_1^o + \zeta\,\kappa_1)^2 + (\varepsilon_3^o + \zeta\,\kappa_3)^2 + (\varepsilon_5^o + \zeta\,\kappa_5)^2]. \tag{2.25}$$

After neglecting terms containing ζ^2 and separating the terms in the two groups, we obtain

$$\varepsilon_{11} = \varepsilon_1^o + \frac{1}{2}[(\varepsilon_1^o)^2 + (\varepsilon_3^o)^2 + (\varepsilon_5^o)^2] + \zeta(\kappa_1 + \varepsilon_1^o\,\kappa_1 + \varepsilon_3^o\,\kappa_3 + \varepsilon_5^o\,\kappa_5). \tag{2.26}$$

Similarly we can obtain ε_{22} and ε_{12}. The following equations summarize the results:

$$\varepsilon_{11} = \varepsilon_1^t + \zeta\,\kappa_1^t, \qquad \varepsilon_{22} = \varepsilon_2^t + \zeta\,\kappa_2^t, \qquad \gamma_{12} = \gamma_{12}^t + \zeta\,\kappa_{12}^t. \tag{2.27}$$

where

$$\varepsilon_1^t = \varepsilon_1^o + \frac{1}{2}[(\varepsilon_1^o)^2 + (\varepsilon_3^o)^2 + (\varepsilon_5^o)^2], \qquad \kappa_1^t = \kappa_1 + \varepsilon_1^o\,\kappa_1 + \varepsilon_3^o\,\kappa_3 + \varepsilon_5^o\,\kappa_5,$$

$$\varepsilon_2^t = \varepsilon_2^o + \frac{1}{2}[(\varepsilon_2^o)^2 + (\varepsilon_4^o)^2 + (\varepsilon_6^o)^2], \qquad \kappa_2^t = \kappa_2 + \varepsilon_2^o \kappa_2 + \varepsilon_4^o \kappa_4 + \varepsilon_6^o \kappa_6,$$

$$\gamma_{12}^t = \varepsilon_3^o + \varepsilon_4^o + \varepsilon_1^o \varepsilon_4^o + \varepsilon_2^o \varepsilon_3^o + \varepsilon_5^o \varepsilon_6^o, \tag{2.28}$$

$$\kappa_{12}^t = \kappa_3 + \kappa_4 + \kappa_1 \varepsilon_4^o + \varepsilon_1^o \kappa_4 + \kappa_2 \varepsilon_3^o + \varepsilon_2^o \kappa_3 + \kappa_5 \varepsilon_6^o + \varepsilon_5^o \kappa_6.$$

The superscript t in the above equation denotes the total quantity, i.e., summation of both linear and nonlinear terms. Equations (2.27) and (2.28) represent exact nonlinear strain-displacement relations. In order to simplify, we introduce some more approximations. Since we assumed the initial stresses to be of membrane-type, i.e., no bending-type prestresses, it is sufficient to retain only linear terms in the expressions of changes in curvature and torsion. This implies

$$\kappa_1^t = \kappa_1, \qquad \kappa_2^t = \kappa_2, \qquad \kappa_{12}^t = \kappa_3 + \kappa_4. \tag{2-29}$$

In order to satisfy the "sixth equilibrium equation" and the zero strains due to the small rigid body motions, Sanders (1959) defined a new quantity β_n that represents the rotation about the normal to the reference surface, given by

$$\beta_n = \frac{1}{2 A_1 A_2}\left[\frac{\partial(A_2 u_2)}{\partial \alpha_1} - \frac{\partial(A_1 u_1)}{\partial \alpha_2}\right]. \tag{2.30}$$

The new strain quantities are defined as follows:

$$\varepsilon_3^s = \varepsilon_3^o - \frac{\beta_n}{1+\zeta/R_1}, \qquad \varepsilon_4^s = \varepsilon_4^o + \frac{\beta_n}{1+\zeta/R_2},$$

$$\kappa_3^s = \kappa_3 - \frac{\beta_n/R_1}{1+\zeta/R_1}, \qquad \kappa_4^s = \kappa_4 + \frac{\beta_n/R_2}{1+\zeta/R_2}. \tag{2.31}$$

Expanding ε_3^s, we get

$$\varepsilon_3^s = \frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} - \frac{u_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\right) - \frac{1}{2A_1A_2(1+\zeta/R_1)}\left(A_2\frac{\partial u_2}{\partial \alpha_1} + u_2\frac{\partial A_2}{\partial \alpha_1} - A_1\frac{\partial u_1}{\partial \alpha_2} - u_1\frac{\partial A_1}{\partial \alpha_2}\right)$$

(2.32)

After simplification, we get

$$\varepsilon_3^s = \frac{1}{2(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial u_1}{\partial \alpha_2} - \frac{u_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} - \frac{u_2}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\right). \tag{2.33}$$

Similarly, we expand ε_4^s to find

$$\varepsilon_4^s = \frac{1}{2(1+\zeta/R_2)}\left(\frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial u_1}{\partial \alpha_2} - \frac{u_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} - \frac{u_2}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\right). \tag{2.34}$$

One can show by neglecting ζ/R_1 and ζ/R_2 with respect to *1* that (Sanders, 1959)

$$\varepsilon_4^s = \varepsilon_3^s, \qquad \kappa_3^s - \kappa_4^s = \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\varepsilon_3^s. \tag{2.35}$$

Using the new definition of strains and neglecting ζ/R_1 and ζ/R_2 in comparison with *1*, we get the in-plane strains as

$$\begin{aligned}
\varepsilon_1^t &= \varepsilon_1^o + \frac{1}{2}[(\varepsilon_1^o)^2 + (\varepsilon_3^s)^2 + 2\varepsilon_3^s\beta_n + (\beta_1)^2 + (\beta_n)^2], \\
\varepsilon_2^t &= \varepsilon_2^o + \frac{1}{2}[(\varepsilon_2^o)^2 + (\varepsilon_3^s)^2 - 2\varepsilon_3^s\beta_n + (\beta_2)^2 + (\beta_n)^2], \\
\gamma_{12}^t &= \varepsilon_3^o + \varepsilon_4^o + \varepsilon_3^s(\varepsilon_1^o + \varepsilon_2^o) - \beta_n(\varepsilon_1^o - \varepsilon_2^o) + \beta_1\beta_2,
\end{aligned} \tag{2.36}$$

and the bending and twisting strains can be obtained from Eqs. (2.19). In the next section, we use the above strain definitions in order to derive the governing equations of a shell vibrating under a constant pressure.

2.7 Hamilton's Principle

We derive the equations of motion of the shell using Hamilton's Principle. Let's assume that a shell, subjected to a body force vector $\boldsymbol{F}$ and a surface force vector $\boldsymbol{T}$, is changing its states between times t_o and t_1. Hamilton's Principle states that the actual path taken by a dynamic system is such that (Kraus, 1967)

$$\delta \int_{t_o}^{t_1} (\Pi - K_E) dt = 0 , \tag{2.37}$$

where Π is the potential energy and K_E is the kinetic energy. Let U_E be the strain energy and $\boldsymbol{U}$ be the displacement vector at equilibrium. The potential energy is defined as

$$\Pi = U_E - \int_S \boldsymbol{T} \cdot \boldsymbol{U} \, dS - \int_V \boldsymbol{F} \cdot \boldsymbol{U} \, dV . \tag{2.38}$$

The kinetic energy is given by

$$K_E = \frac{1}{2} \int_V \rho \dot{\boldsymbol{U}} \cdot \dot{\boldsymbol{U}} \, dV . \tag{2.39}$$

Now we can write Eq. (2.37) as

$$\delta \int_{t_o}^{t_1} \left\{ U_E - \int_S \boldsymbol{T} \cdot \boldsymbol{U} \, dS - \int_V \boldsymbol{F} \cdot \boldsymbol{U} \, dV - \frac{1}{2} \int_V \rho \dot{\boldsymbol{U}} \cdot \dot{\boldsymbol{U}} \, dV \right\} dt = 0 . \tag{2.40}$$

In the following sections, we consider the terms in Eq. (2.40) individually.

2.7.1 Variation of the Strain Energy

The strain energy is defined as the volume integration of the strain energy density function *P*, i.e.,

$$U_E = \int_V P\, dV\,. \tag{2.41}$$

The material is assumed to follow Hooke's Law (Fig. 2.2). Since σ_{ij}^r is constant in time and σ_{ij} is proportional to ε_{ij}, the strain energy density function can be given by (Fig. 2.4, Soedel, 1986)

$$\begin{aligned} P = \tfrac{1}{2}(&\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + \sigma_{23}\varepsilon_{23} + \sigma_{32}\varepsilon_{32} + \sigma_{31}\varepsilon_{31} + \sigma_{13}\varepsilon_{13} \\ &+ \sigma_{12}\varepsilon_{12} + \sigma_{21}\varepsilon_{21}) + (\sigma_{11}^r\varepsilon_{11} + \sigma_{12}^r\varepsilon_{12} + \sigma_{21}^r\varepsilon_{21} + \sigma_{22}^r\varepsilon_{22}). \end{aligned} \tag{2.42}$$

In the above equation, prestresses σ_{33}^r, σ_{23}^r, σ_{32}^r, σ_{31}^r, and σ_{13}^r, related to the transverse direction, are assumed negligible. Now, using symmetry of stress and strain tensors, Eq. (2.42), and the definitions of the engineering strains, Eq. (2.43), we can write the above equation as

$$P = \tfrac{1}{2}(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + \sigma_{23}\gamma_{23} + \sigma_{13}\gamma_{13} + \sigma_{12}\gamma_{12}) + \sigma_{11}^r\varepsilon_{11} + \sigma_{12}^r\gamma_{12} + \sigma_{22}^r\varepsilon_{22}.$$

(2-43)

According to Love's first approximation, ε_{33} and σ_{33} are negligibly small, which leads to dropping the terms corresponding to $\delta\varepsilon_{33}$ in Eq. (2.43). Though from Love's first approximation $\delta\gamma_{23}$ and $\delta\gamma_{13}$ are also zero, they are not dropped from the strain energy expression in order to obtain the nonzero transverse shear stresses (σ_{13}, σ_{23}) from the

governing equations. The variation of the strain energy density function can now be written as

$$\delta P = \frac{\partial P}{\partial \varepsilon_{11}}\delta\varepsilon_{11} + \frac{\partial P}{\partial \varepsilon_{22}}\delta\varepsilon_{22} + \frac{\partial P}{\partial \gamma_{23}}\delta\gamma_{23} + \frac{\partial P}{\partial \gamma_{13}}\delta\gamma_{13} + \frac{\partial P}{\partial \gamma_{12}}\delta\gamma_{12}\,. \tag{2.44}$$

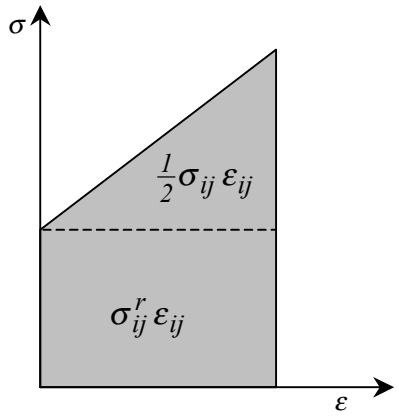

Fig. 2.5: Initial and vibratory stress as a function of strain.

In order to find the derivatives of P with respect to strains, we use Eq. (2.43) in conjunction with the stress-strain relationships (Eqs. (2.9)). From Eq. (2.43), we can find the derivative of P with respect to ε_{11} as

$$\frac{\partial P}{\partial \varepsilon_{11}} = \frac{1}{2}\left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial \varepsilon_{11}}\varepsilon_{11} + \frac{\partial \sigma_{22}}{\partial \varepsilon_{11}}\varepsilon_{22}\right) + \sigma^r_{11}\,. \tag{2.45}$$

Using Eqs. (2.9), we obtain

$$\frac{\partial P}{\partial \varepsilon_{11}} = \frac{1}{2}\left(\sigma_{11} + \frac{E}{1-\nu^2}\varepsilon_{11} + \nu\frac{E}{1-\nu^2}\varepsilon_{22}\right) + \sigma^r_{11} = \sigma_{11} + \sigma^r_{11}\,. \tag{2.46}$$

Similarly, other derivatives in Eq. (2.44) can be calculated. Summarizing, we get

$$\frac{\partial P}{\partial \varepsilon_{11}} = \sigma_{11} + \sigma^r_{11}, \qquad \frac{\partial P}{\partial \varepsilon_{22}} = \sigma_{22} + \sigma^r_{22}\,,$$

$$\frac{\partial P}{\partial \gamma_{12}} = \sigma_{12} + \sigma_{12}^{r}, \qquad \frac{\partial P}{\partial \gamma_{23}} = \sigma_{23}, \qquad \frac{\partial P}{\partial \gamma_{13}} = \sigma_{13}. \tag{2.47}$$

This leads to the following expression for the variation of strain energy:

$$\begin{aligned} \delta U_E = & \int_V [\sigma_{11}\,\delta\varepsilon_{11} + \sigma_{22}\,\delta\varepsilon_{22} + \sigma_{12}\,\delta\gamma_{12} + \sigma_{23}\,\delta\gamma_{23} + \sigma_{13}\,\delta\gamma_{13}]dV \\ & + \int_V [\sigma_{11}^{r}\,\delta\varepsilon_{11} + \sigma_{22}^{r}\,\delta\varepsilon_{22} + \sigma_{12}^{r}\,\delta\gamma_{12}]dV. \end{aligned} \tag{2.48}$$

Equation (2.48) contains variation only in three-dimensional strains. These variations are next written in terms of two-dimensional strains. For example, using Eqs. (2.27), (2.29), and (2.36), we can write ε_{11} as

$$\varepsilon_{11} = \varepsilon_1^o + \frac{1}{2}[(\varepsilon_1^o)^2 + (\varepsilon_3^s)^2 + 2\varepsilon_3^s\,\beta_n + (\beta_1)^2 + (\beta_n)^2] + \zeta\,\kappa_1. \tag{2.49}$$

Now the variation of ε_{11} can be written as

$$\delta\varepsilon_{11} = \delta\varepsilon_1^o + \varepsilon_1^o\,\delta\varepsilon_1^o + \varepsilon_3^s\,\delta\varepsilon_3^s + \beta_n\,\delta\varepsilon_3^s + \varepsilon_3^s\,\delta\beta_n + \beta_1\,\delta\beta_1 + \beta_n\,\delta\beta_n + \zeta\,\delta\kappa_1, \tag{2.50}$$

which is simplified to give

$$\delta\varepsilon_{11} = (1 + \varepsilon_1^o)\,\delta\varepsilon_1^o + (\varepsilon_3^s + \beta_n)\,\delta\varepsilon_3^s + \beta_1\delta\beta_1 + (\varepsilon_3^s + \beta_n)\,\delta\beta_n + \delta\kappa_1. \tag{2.51}$$

As mentioned earlier, since the initial stresses may be large, it is necessary to use the nonlinear strain-displacement equations. The nonlinear strain-displacement equations will be used only in association with the initial stresses (second integral of Eq. (2.48)). This has a twofold advantage: 1) it maintains the proper homogeneity in the order of the equation, and 2) the resulting governing equations remain linear. Substituting $\delta\varepsilon_{11}$ in Eq. (2.48) produces

$$\int_V \{\sigma_{11} + \sigma_{11}^r (1+\varepsilon_1^o)\}\,\delta\varepsilon_1^o dV + \int_V \sigma_{11}^r \{(\varepsilon_3^s + \beta_n)\,\delta\varepsilon_3^s + \beta_1 \delta\beta_1 + (\varepsilon_3^s + \beta_n)\,\delta\beta_n\} dV \,.$$

(2.52)

Substitution of strain-displacement relations in the above equation yields

$$\int_V \{\sigma_{11} + \sigma_{11}^r (1+\varepsilon_1^o)\}\left[\frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial \delta u_1}{\partial \alpha_1} + \frac{\delta u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{\delta w}{R_1}\right) + \frac{\zeta}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial \delta\beta_1}{\partial \alpha_1}\right.\right.$$
$$\left.\left. + \frac{\delta\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\right] dV + \int_V \sigma_{11}^r \{(\varepsilon_3^s + \beta_n)\,\delta\varepsilon_3^s + \beta_1 \delta\beta_1 + (\varepsilon_3^s + \beta_n)\,\delta\beta_n\} dV.$$

(2.53)

The volume integral in the above equation can be written as the triple integral in the following manner:

$$\int_{\alpha_2}\int_{\alpha_1}\int_{\zeta} \{\sigma_{11} + \sigma_{11}^r (1+\varepsilon_1^o)\}\left[\frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial \delta u_1}{\partial \alpha_1} + \frac{\delta u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{\delta w}{R_1}\right) + \frac{\zeta}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial \delta\beta_1}{\partial \alpha_1}\right.\right.$$
$$\left.\left. + \frac{\delta\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\right] A_1 A_2 \left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right) d\zeta\, d\alpha_1\, d\alpha_2 + \int_{\alpha_2}\int_{\alpha_1}\int_{\zeta} \sigma_{11}^r \{(\varepsilon_3^s + \beta_n)\,\delta\varepsilon_3^s + \beta_1 \delta\beta_1 + (\varepsilon_3^s$$
$$+ \beta_n)\,\delta\beta_n\} A_1 A_2 \left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right) d\zeta\, d\alpha_1\, d\alpha_2.$$

(2.54)

After some simplification and neglecting ζ / R_1 and ζ / R_2 compared to *1* in the second integral, we get

$$\int_{\alpha_2}\int_{\alpha_1}\int_{\zeta} \{\sigma_{11} + \sigma_{11}^r (1+\varepsilon_1^o)\}\left(1+\frac{\zeta}{R_2}\right)\left[\left(\frac{1}{A_1}\frac{\partial \delta u_1}{\partial \alpha_1} + \frac{\delta u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{\delta w}{R_1}\right) + \zeta\left(\frac{1}{A_1}\frac{\partial \delta\beta_1}{\partial \alpha_1} + \frac{\delta\beta_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}\right)\right] A_1$$
$$A_2\, d\zeta\, d\alpha_1\, d\alpha_2 + \int_{\alpha_2}\int_{\alpha_1}\int_{\zeta} \sigma_{11}^r \left(1+\frac{\zeta}{R_2}\right)\{(\varepsilon_3^s + \beta_n)\,\delta\varepsilon_3^s + \beta_1 \delta\beta_1 + (\varepsilon_3^s + \beta_n)\,\delta\beta_n\} A_1 A_2\, d\zeta\, d\alpha_1\, d\alpha_2.$$

(2.55)

Now integrating over the thickness and neglecting the moments due to prestresses, we obtain

$$\int_{\alpha_2}\int_{\alpha_1}\left[\{N_{11}+N_{11}^{r}(1+\varepsilon_1^{o})\}\left(A_2\frac{\partial\delta u_1}{\partial\alpha_1}+\delta u_2\frac{\partial A_1}{\partial\alpha_2}+\frac{A_1A_2}{R_1}\delta w\right)+M_{11}\left(A_2\frac{\partial\delta\beta_1}{\partial\alpha_1}+\frac{\partial A_1}{\partial\alpha_2}\delta\beta_2\right)\right]d\alpha_1\,d\alpha_2$$
$$+\int_{\alpha_2}\int_{\alpha_1}N_{11}^{r}\{(\varepsilon_3^{s}+\beta_n)\,\delta\varepsilon_3^{s}+\beta_1\delta\beta_1+(\varepsilon_3^{s}+\beta_n)\,\delta\beta_n\}A_1\,A_2\,d\alpha_1\,d\alpha_2.$$

(2.56)

Similarly, the contribution of $\delta\varepsilon_{22}$ is

$$\int_{\alpha_2}\int_{\alpha_1}\left[\{N_{22}+N_{22}^{r}(1+\varepsilon_2^{o})\}\left(A_1\frac{\partial\delta u_2}{\partial\alpha_2}+\delta u_1\frac{\partial A_2}{\partial\alpha_1}+\frac{A_1A_2}{R_2}\delta w\right)+M_{22}\left(A_1\frac{\partial\delta\beta_2}{\partial\alpha_2}+\frac{\partial A_2}{\partial\alpha_1}\delta\beta_1\right)\right]d\alpha_1\,d\alpha_2$$
$$+\int_{\alpha_2}\int_{\alpha_1}N_{22}^{r}\{(\varepsilon_3^{s}-\beta_n)\,\delta\varepsilon_3^{s}+\beta_2\delta\beta_2-(\varepsilon_3^{s}-\beta_n)\,\delta\beta_n\}A_1\,A_2\,d\alpha_1\,d\alpha_2.$$

(2.57)

Again, using Eqs. (2.27), (2.29), and (2.36), we can write $\delta\gamma_{12}$ as

$$\delta\gamma_{12}=\delta\varepsilon_3^{o}+\delta\varepsilon_4^{o}+\zeta\left(\delta\kappa_3+\delta\kappa_4\right)+(\varepsilon_1^{o}+\varepsilon_2^{o})\delta\varepsilon_3^{s}+(\varepsilon_3^{s}-\beta_n)\delta\varepsilon_1^{o}+(\varepsilon_3^{s}+\beta_n)\delta\varepsilon_2^{o}$$
$$-(\varepsilon_1^{o}-\varepsilon_2^{o})\delta\beta_n+\beta_2\,\delta\beta_1+\beta_1\,\delta\beta_2.$$

(2.58)

As before, the geometric nonlinearity is considered only with the prestress term. Keeping this in mind and substituting $\delta\gamma_{12}$ in the shear stress related terms of Eq. (2.48), we get

$$\int_{V}(\sigma_{12}+\sigma_{12}^{r})\{\delta\varepsilon_3^{o}+\delta\varepsilon_4^{o}+\zeta\left(\delta\kappa_3+\delta\kappa_4\right)\}+\sigma_{12}^{r}\{(\varepsilon_1^{o}+\varepsilon_2^{o})\delta\varepsilon_3^{s}+(\varepsilon_3^{s}-\beta_n)\delta\varepsilon_1^{o}+(\varepsilon_3^{s}+\beta_n)\delta\varepsilon_2^{o}$$
$$-\left(\varepsilon_1^{o}-\varepsilon_2^{o}\right)\delta\beta_n+\beta_2\,\delta\beta_1+\beta_1\,\delta\beta_2\}\,dV.$$

(2.59)

Substituting the strain-displacement relations and representing the volume integral in terms of a triple integral, we get

$$\int_{\alpha_2}\int_{\alpha_1}\int_{\zeta}\Bigg[(\sigma_{12}+\sigma_{12}^{r})\Bigg\{\frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial\delta u_2}{\partial\alpha_1}-\frac{\delta u_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)+\frac{1}{(1+\zeta/R_2)}\left(\frac{1}{A_2}\frac{\partial\delta u_1}{\partial\alpha_2}-\frac{\delta u_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)\Bigg\}$$

$$+(\sigma_{12}+\sigma_{12}^{r})\zeta\left\{\frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial\delta\beta_2}{\partial\alpha_1}-\frac{\delta\beta_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)+\frac{1}{(1+\zeta/R_2)}\left(\frac{1}{A_2}\frac{\partial\delta\beta_1}{\partial\alpha_2}-\frac{\delta\beta_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)\right\}\Bigg]A_1A_2$$

$$\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)d\zeta\,d\alpha_1\,d\alpha_2+\int_{\alpha_2}\int_{\alpha_1}\int_{\zeta}\sigma_{12}^{r}\{(\varepsilon_1^{o}+\varepsilon_2^{o})\delta\varepsilon_3^{s}+(\varepsilon_3^{s}-\beta_n)\delta\varepsilon_1^{o}+(\varepsilon_3^{s}+\beta_n)\delta\varepsilon_2^{o}-$$

$$(\varepsilon_1^{o}-\varepsilon_2^{o})\delta\beta_n+\beta_2\,\delta\beta_1+\beta_1\,\delta\beta_2\}A_1A_2\left(1+\frac{\zeta}{R_1}\right)\left(1+\frac{\zeta}{R_2}\right)d\zeta\,d\alpha_1\,d\alpha_2. \tag{2.60}$$

Using the definition of the stress resultants and neglecting ζ/R_1 and ζ/R_2 compared to *1* in the second integral, we get

$$\int_{\alpha_2}\int_{\alpha_1}\Bigg[(N_{12}+N_{12}^{r})\left(\frac{1}{A_1}\frac{\partial\delta u_2}{\partial\alpha_1}-\frac{\delta u_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)+(N_{21}+N_{21}^{r})\left(\frac{1}{A_2}\frac{\partial\delta u_1}{\partial\alpha_2}-\frac{\delta u_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)+M_{12}\left(\frac{1}{A_1}\right.$$

$$\left.\frac{\partial\delta\beta_2}{\partial\alpha_1}-\frac{\delta\beta_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)+M_{21}\left(\frac{1}{A_2}\frac{\partial\delta\beta_1}{\partial\alpha_2}-\frac{\delta\beta_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)A_1A_2\,d\alpha_1\,d\alpha_2+\int_{\alpha_2}\int_{\alpha_1}N_{12}^{r}\{(\varepsilon_1^{o}+\varepsilon_2^{o})\delta\varepsilon_3^{s}+$$

$$(\varepsilon_3^{s}-\beta_n)\delta\varepsilon_1^{o}+(\varepsilon_3^{s}+\beta_n)\delta\varepsilon_2^{o}-(\varepsilon_1^{o}-\varepsilon_2^{o})\delta\beta_n+\beta_2\,\delta\beta_1+\beta_1\,\delta\beta_2\}A_1A_2\,d\alpha_1\,d\alpha_2. \tag{2.61}$$

In Love's theory, the eight stress resultants and couples N_{11}, N_{22}, N_{12}, N_{21}, M_{11}, M_{22}, M_{12}, and M_{21} were reduced to six by letting $N_{12}=N_{21}$ and $M_{12}=M_{21}$. However, after imposing these equalities, the stress resultants N_{12} and N_{21} and stress couples M_{12} and M_{21} do not satisfy the equilibrium condition given by the following equation, known as the sixth equilibrium equation:

$$N_{21} + N_{21}^r - N_{12} - N_{12}^r + \frac{M_{21}}{R_2} - \frac{M_{12}}{R_1} = 0\,. \tag{2.62}$$

One should note that in the above equation, the prestress terms have been also considered. If one is interested in satisfying this equation and reducing the number of stress resultants and couples as well, the resultants and couples will have to be defined in a new way. Here, we follow the approach of Sanders (1959) and modify Eq. (2.61) using the sixth equilibrium equation and the rotation of reference surface. To this end, the left side of the Eq. (2.62) is multiplied by β_n and added to the integrand of Eq. (2.61), i.e.,

$$\int_{\alpha_2}\int_{\alpha_1}\Bigg[(N_{12}+N_{12}^r)\left(\frac{1}{A_1}\frac{\partial\delta u_2}{\partial\alpha_1}-\frac{\delta u_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)+(N_{21}+N_{21}^r)\left(\frac{1}{A_2}\frac{\partial\delta u_1}{\partial\alpha_2}-\frac{\delta u_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)+M_{12}\left(\frac{1}{A_1}\frac{\partial\delta\beta_2}{\partial\alpha_1}\right.$$
$$\left.-\frac{\delta\beta_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)+M_{21}\left(\frac{1}{A_2}\frac{\partial\delta\beta_1}{\partial\alpha_2}-\frac{\delta\beta_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)+\left(N_{21}+N_{21}^r-N_{12}-N_{12}^r+\frac{M_{21}}{R_2}-\frac{M_{12}}{R_1}\right)\delta\beta_n\Bigg]A_1A_2$$
$$d\alpha_1\,d\alpha_2+\int_{\alpha_2}\int_{\alpha_1}N_{12}^r\{(\varepsilon_1^o+\varepsilon_2^o)\delta\varepsilon_3^s+(\varepsilon_3^s-\beta_n)\delta\varepsilon_1^o+(\varepsilon_3^s+\beta_n)\delta\varepsilon_2^o-(\varepsilon_1^o-\varepsilon_2^o)\,\delta\beta_n+\beta_2\,\delta\beta_1+\beta_1\,\delta\beta_2\}$$
$$A_1A_2\,d\alpha_1\,d\alpha_2.$$

(2.63)

After some simplification, we obtain

$$\int_{\alpha_2}\int_{\alpha_1}\Bigg[(N_{12}+N_{12}^r)\left(\frac{1}{A_1}\frac{\partial\delta u_2}{\partial\alpha_1}-\frac{\delta u_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}-\delta\beta_n\right)+(N_{21}+N_{21}^r)\left(\frac{1}{A_2}\frac{\partial\delta u_1}{\partial\alpha_2}-\frac{\delta u_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\delta\beta_n\right)$$
$$+M_{12}\left(\frac{1}{A_1}\frac{\partial\delta\beta_2}{\partial\alpha_1}-\frac{\delta\beta_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}-\frac{\delta\beta_n}{R_1}\right)+M_{21}\left(\frac{1}{A_2}\frac{\partial\delta\beta_1}{\partial\alpha_2}-\frac{\delta\beta_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+\frac{\delta\beta_n}{R_2}\right)\Bigg]A_1A_2\,d\alpha_1\,d\alpha_2+$$
$$\int_{\alpha_2}\int_{\alpha_1}N_{12}^r\{(\varepsilon_1^o+\varepsilon_2^o)\delta\varepsilon_3^s+(\varepsilon_3^s-\beta_n)\delta\varepsilon_1^o+(\varepsilon_3^s+\beta_n)\delta\varepsilon_2^o-(\varepsilon_1^o-\varepsilon_2^o)\,\delta\beta_n+\beta_2\,\delta\beta_1+\beta_1\,\delta\beta_2\}A_1A_2$$
$$d\alpha_1\,d\alpha_2.$$

(2.64)

Using the definition of modified shear strain in Eq. (2.31) and ignoring ζ/R_1 and ζ/R_2 with respect to *1*, we can write

$$\int_{\alpha_2}\int_{\alpha_1}\{(N_{12}+N_{12}^r)\,\delta\varepsilon_3^s+(N_{21}+N_{21}^r)\delta\varepsilon_4^s+M_{12}\,\delta\kappa_3^s+M_{21}\,\delta\kappa_4^s\}A_1A_2\,d\alpha_1\,d\alpha_2+$$
$$\int_{\alpha_2}\int_{\alpha_1}N_{12}^r\{(\varepsilon_1^o+\varepsilon_2^o)\delta\varepsilon_3^s+(\varepsilon_3^s-\beta_n)\delta\varepsilon_1^o+(\varepsilon_3^s+\beta_n)\delta\varepsilon_2^o-(\varepsilon_1^o-\varepsilon_2^o)\,\delta\beta_n+\beta_2\,\delta\beta_1+\beta_1\,\delta\beta_2\}A_1A_2\,d\alpha_1\,d\alpha_2. \tag{2.65}$$

Now from the relations given in Eqs. (2.35), we obtain

$$\int_{\alpha_2}\int_{\alpha_1}\left[\frac{(N_{12}+N_{12}^r+N_{21}+N_{21}^r)}{2}2\,\delta\varepsilon_3^s+\frac{M_{12}}{2}\left\{\delta\kappa_3^s+\delta\kappa_4^s+\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\delta\varepsilon_3^s\right\}+\right.$$
$$\left.\frac{M_{21}}{2}\left\{\delta\kappa_4^s+\delta\kappa_3^s-\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\delta\varepsilon_3^s\right\}\right]A_1A_2\,d\alpha_1\,d\alpha_2+\int_{\alpha_2}\int_{\alpha_1}N_{12}^r\{(\varepsilon_1^o+\varepsilon_2^o)\delta\varepsilon_3^s+ \tag{2.66}$$
$$(\varepsilon_3^s-\beta_n)\delta\varepsilon_1^o+(\varepsilon_3^s+\beta_n)\delta\varepsilon_2^o-(\varepsilon_1^o-\varepsilon_2^o)\,\delta\beta_n+\beta_2\,\delta\beta_1+\beta_1\,\delta\beta_2\}A_1A_2\,d\alpha_1\,d\alpha_2.$$

The quantity $\left(\frac{M_{12}-M_{21}}{2}\right)\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\delta\varepsilon_3^s$ can be ignored, as it would be small. This gives

$$\int_{\alpha_2}\int_{\alpha_1}\left\{\frac{(N_{12}+N_{12}^r+N_{21}+N_{21}^r)}{2}2\,\delta\varepsilon_3^s+\left(\frac{M_{12}+M_{21}}{2}\right)(\delta\kappa_3^s+\delta\kappa_4^s)A_1A_2\,d\alpha_1\,d\alpha_2\right.$$
$$+\int_{\alpha_2}\int_{\alpha_1}N_{12}^r\{(\varepsilon_1^o+\varepsilon_2^o)\delta\varepsilon_3^s+(\varepsilon_3^s-\beta_n)\delta\varepsilon_1^o+(\varepsilon_3^s+\beta_n)\delta\varepsilon_2^o-(\varepsilon_1^o-\varepsilon_2^o)\,\delta\beta_n+\beta_2\,\delta\beta_1+ \tag{2.67}$$
$$\left.\beta_1\,\delta\beta_2\}A_1A_2\,d\alpha_1\,d\alpha_2.\right.$$

Following Sanders (1959), we define the following stress resultants:

$$\widetilde{N}_{12}=\frac{1}{2}(N_{12}+N_{21}),\qquad \widetilde{N}_{12}^r=\frac{1}{2}\left(N_{12}^r+N_{21}^r\right),$$
$$\widetilde{M}_{12}=\frac{1}{2}(M_{12}+M_{21}),\qquad \widetilde{\kappa}_{12}=\frac{1}{2}\left(\kappa_3^s+\kappa_4^s\right). \tag{2.68}$$

In the light of these new definitions, we can rewrite Eq. (2.67) as

$$\int_{\alpha_2}\int_{\alpha_1}\{(\widetilde{N}_{12}+\widetilde{N}^r_{12})2\,\delta\varepsilon^s_3+\widetilde{M}_{12}2\delta\widetilde{\kappa}_{12}\}A_1A_2\,d\alpha_1\,d\alpha_2+\int_{\alpha_2}\int_{\alpha_1}N^r_{12}\{(\varepsilon^o_1+\varepsilon^o_2)\delta\varepsilon^s_3+$$
$$(\varepsilon^s_3-\beta_n)\delta\varepsilon^o_1+(\varepsilon^s_3+\beta_n)\delta\varepsilon^o_2-(\varepsilon^o_1-\varepsilon^o_2)\,\delta\beta_n+\beta_2\,\delta\beta_1+\beta_1\,\delta\beta_2\}A_1A_2\,d\alpha_1\,d\alpha_2. \tag{2.69}$$

As mentioned earlier, the contributions of $\delta\gamma_{13}$ and $\delta\gamma_{23}$ are sought even if from Love's assumption they should be zero. Since these quantities are quite small, we consider only the linear parts of γ_{13} and γ_{23}. From equations (2.12), (2.17), and (2.22), we get

$$\gamma_{13}=\frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial W}{\partial\alpha_1}-\frac{U_1}{R_1}\right)+\frac{\partial U_1}{\partial\zeta}=\frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}-\frac{u_1+\zeta\,\beta_1}{R_1}\right)+\beta_1. \tag{2.70}$$

Calculating the variation of the potential energy associated with $\delta\gamma_{13}$, we get

$$\int_{\alpha_2}\int_{\alpha_1}\int_{\zeta}\sigma_{13}\left[\frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial\delta w}{\partial\alpha_1}-\frac{\delta u_1+\zeta\,\delta\beta_1}{R_1}\right)+\delta\beta_1\right]A_1\left(1+\frac{\zeta}{R_1}\right)A_2\left(1+\frac{\zeta}{R_2}\right)d\zeta\,d\alpha_1\,d\alpha_2. \tag{2.71}$$

The above equation can be rewritten as

$$\int_{\alpha_2}\int_{\alpha_1}\int_{\zeta}\sigma_{13}\left[\frac{1}{(1+\zeta/R_1)}\left(\frac{1}{A_1}\frac{\partial\delta w}{\partial\alpha_1}-\frac{\delta u_1}{R_1}\right)+\frac{1}{(1+\zeta/R_1)}\delta\beta_1\right]A_1\left(1+\frac{\zeta}{R_1}\right)A_2\left(1+\frac{\zeta}{R_2}\right)d\zeta\,d\alpha_1\,d\alpha_2. \tag{2.72}$$

Neglecting ζ/R_1 in comparison with *1* and utilizing the definitions of the stress resultants, we obtain

$$\int_{\alpha_2}\int_{\alpha_1}Q_{13}\left(A_2\frac{\partial\delta w}{\partial\alpha_1}-A_1A_2\frac{\delta u_1}{R_1}+A_1A_2\,\delta\beta_1\right)d\alpha_1\,d\alpha_2. \tag{2.73}$$

Similarly, the contribution of $\delta\gamma_{23}$ is

$$\int_{\alpha_2}\int_{\alpha_1} Q_{23}\left(A_1\frac{\partial\delta w}{\partial\alpha_2}-A_1A_2\frac{\delta u_2}{R_2}+A_1A_2\,\delta\beta_2\right)d\alpha_1\,d\alpha_2\,. \tag{2.74}$$

Combining and rearranging the contributions of all the terms in Eqs. (2.48) and writing β_n and ε_3^s in terms of displacements using Eqs. (2.30) and (2.33), we get

$$\begin{aligned}\delta U_E = \int_{\alpha_2}\int_{\alpha_1}\Bigg[&\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}\left(A_2\frac{\partial\delta u_1}{\partial\alpha_1}+\frac{\partial A_1}{\partial\alpha_2}\delta u_2+\frac{A_1A_2}{R_1}\delta w\right)+\{N_{22}+N_{22}^r\\
&(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\left(A_1\frac{\partial\delta u_2}{\partial\alpha_2}+\frac{\partial A_2}{\partial\alpha_1}\delta u_1+\frac{A_1A_2}{R_2}\delta w\right)+\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)\\
&+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}\left(A_2\frac{\partial\delta u_2}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\delta u_1+A_1\frac{\partial\delta u_1}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\delta u_2\right)+M_{11}\left(A_2\frac{\partial\delta\beta_1}{\partial\alpha_1}\right.\\
&\left.+\frac{\partial A_1}{\partial\alpha_2}\delta\beta_2\right)+M_{22}\left(A_1\frac{\partial\delta\beta_2}{\partial\alpha_2}+\frac{\partial A_2}{\partial\alpha_1}\delta\beta_1\right)+\tilde{M}_{12}\left(A_2\frac{\partial\delta\beta_2}{\partial\alpha_1}-\frac{\partial A_1}{\partial\alpha_2}\delta\beta_1+A_1\frac{\partial\delta\beta_1}{\partial\alpha_2}-\frac{\partial A_2}{\partial\alpha_1}\delta\beta_2+\right.\\
&\left.\frac{1}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\left(\frac{\partial(A_2\delta u_2)}{\partial\alpha_1}-\frac{\partial(A_1\delta u_1)}{\partial\alpha_2}\right)\right)+Q_{13}\left(A_2\frac{\partial\delta w}{\partial\alpha_1}-A_1A_2\frac{\delta u_1}{R_1}+A_1A_2\,\delta\beta_1\right)+Q_{23}\left(A_1\frac{\partial\delta w}{\partial\alpha_2}-\right.\\
&\left.A_1A_2\frac{\delta u_2}{R_2}+A_1A_2\,\delta\beta_2\right)+(N_{11}^r\beta_1+\tilde{N}_{12}^r\beta_2)\,A_1A_2\,\delta\beta_1+(N_{22}^r\beta_2+N_{12}^r\,\beta_1)\,A_1A_2\,\delta\beta_2+\frac{1}{2}\{(N_{11}^r\\
&(\varepsilon_3^s+\beta_n)-N_{22}^r(\varepsilon_3^s-\beta_n)-\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}\left(\frac{\partial(A_2\delta u_2)}{\partial\alpha_1}-\frac{\partial(A_1\delta u_1)}{\partial\alpha_2}\right)\Bigg]d\alpha_1\,d\alpha_2.\end{aligned} \tag{2.75}$$

In order to remove the differentiation of the variation of displacements, we perform integration-by-parts. For example, we use the following integration:

$$\int_{\alpha_2}\int_{\alpha_1}(N_{11}+N_{11}^r)A_2\frac{\partial\delta u_1}{\partial\alpha_1}d\alpha_1\,d\alpha_2=\int_{\alpha_2}(N_{11}+N_{11}^r)A_2\,\delta u_1\,d\alpha_2-\int_{\alpha_2}\int_{\alpha_1}\frac{\partial[(N_{11}+N_{11}^r)\,A_2]}{\partial\alpha_1}\delta u_1\,d\alpha_1\,d\alpha_2. \tag{2.76}$$

Repeating the above operation on each of the terms containing the displacement derivatives, we get

$$\int_{\alpha_2}\int_{\alpha_1}\Bigg[-\frac{\partial[\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}A_2]}{\partial\alpha_1}\delta u_1+\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}\Bigg(\frac{\partial A_1}{\partial\alpha_2}$$

$$\delta u_2+\frac{A_1A_2}{R_1}\delta w\Bigg)-\frac{\partial[\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}A_1]}{\partial\alpha_2}\delta u_2+\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+$$

$$\beta_n)\}\Bigg(\frac{\partial A_2}{\partial\alpha_1}\delta u_1+\frac{A_1A_2}{R_2}\delta w\Bigg)-\frac{1}{2}\frac{\partial}{\partial\alpha_1}[\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+$$

$$\varepsilon_2^o)\}A_2]\delta u_2-\frac{1}{2}\frac{\partial}{\partial\alpha_2}[\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}A_1]\delta u_1$$

$$-\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}\Bigg(\frac{\partial A_1}{\partial\alpha_2}\delta u_1+\frac{\partial A_2}{\partial\alpha_1}\delta u_2\Bigg)-$$

$$\frac{\partial(M_{11}A_2)}{\partial\alpha_1}\delta\beta_1+M_{11}\frac{\partial A_1}{\partial\alpha_2}\delta\beta_2-\frac{\partial(M_{22}A_1)}{\partial\alpha_2}\delta\beta_2+M_{22}\frac{\partial A_2}{\partial\alpha_1}\delta\beta_1-\frac{\partial(\tilde{M}_{12}A_2)}{\partial\alpha_1}\delta\beta_2-\frac{\partial(\tilde{M}_{12}A_1)}{\partial\alpha_2}\delta\beta_1$$

$$-\tilde{M}_{12}\Bigg(\frac{\partial A_1}{\partial\alpha_2}\delta\beta_1+\frac{\partial A_2}{\partial\alpha_1}\delta\beta_2\Bigg)+\frac{1}{2}\Bigg\{-\frac{\partial}{\partial\alpha_1}\Bigg[\tilde{M}_{12}\Bigg(\frac{1}{R_2}-\frac{1}{R_1}\Bigg)\Bigg]A_2\delta u_2+\frac{\partial}{\partial\alpha_2}\Bigg[\tilde{M}_{12}\Bigg(\frac{1}{R_2}-\frac{1}{R_1}\Bigg)\Bigg]A_1\delta u_1\Bigg\}$$

$$-\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}\delta w-Q_{13}A_1A_2\frac{\delta u_1}{R_1}+Q_{13}A_1A_2\,\delta\beta_1-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}\delta w-Q_{23}A_1A_2\frac{\delta u_2}{R_2}+Q_{23}A_1A_2\,\delta\beta_2+$$

$$(N_{11}^r\beta_1+N_{12}^r\beta_2)A_1A_2\,\delta\beta_1+(N_{22}^r\beta_2+N_{12}^r\beta_1)A_1A_2\,\delta\beta_2-\frac{1}{2}\frac{\partial}{\partial\alpha_1}\{N_{11}^r(\varepsilon_3^s+\beta_n)-N_{22}^r(\varepsilon_3^s-\beta_n)$$

$$-\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}A_2\delta u_2+\frac{1}{2}\frac{\partial}{\partial\alpha_2}\{N_{11}^r(\varepsilon_3^s+\beta_n)-N_{22}^r(\varepsilon_3^s-\beta_n)-\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}A_1\delta u_1\Bigg]d\alpha_1\,d\alpha_2$$

$$+\int_{\alpha_1}[\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}A_1\,\delta u_2+\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)$$

$$+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}A_1\,\delta u_1+M_{22}A_1\,\delta\beta_2+\tilde{M}_{12}A_1\,\delta\beta_1+Q_{23}A_1\delta w-\frac{1}{2}\{(N_{11}^r(\varepsilon_3^s+\beta_n)-N_{22}^r(\varepsilon_3^s-\beta_n)-$$

$$\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}\,A_1\delta u_1-\frac{\tilde{M}_{12}}{2}\Bigg(\frac{1}{R_2}-\frac{1}{R_1}\Bigg)A_1\delta u_1]d\alpha_1+\int_{\alpha_2}[\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}A_2\,\delta u_1+$$

$$\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}A_2\,\delta u_2+M_{11}A_2\,\delta\beta_1+\tilde{M}_{12}A_2\delta\beta_2+$$

$$Q_{13}A_2\delta w+\frac{1}{2}\{(N_{11}^r(\varepsilon_3^s+\beta_n)-N_{22}^r(\varepsilon_3^s-\beta_n)-\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}A_2\delta u_2+\frac{\tilde{M}_{12}}{2}\Bigg(\frac{1}{R_2}-\frac{1}{R_1}\Bigg)A_2\delta u_2\,]d\alpha_2. \tag{2.77}$$

Rearranging the terms in Eq. (2.77), we get

$$\int_{\alpha_2}\int_{\alpha_1}\Bigg[\Bigg\{-\frac{\partial[\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}A_2]}{\partial\alpha_1}+\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\frac{\partial A_2}{\partial\alpha_1}-$$

$$\frac{1}{2}\frac{\partial}{\partial\alpha_2}[\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}A_1]-\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+$$

$$N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}\frac{\partial A_1}{\partial\alpha_2}-Q_{13}\frac{A_1A_2}{R_1}+\frac{1}{2}\frac{\partial}{\partial\alpha_2}\{N_{11}^r(\varepsilon_3^s+\beta_n)-$$

$$N_{22}^r(\varepsilon_3^s-\beta_n)-\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}A_1-\frac{A_1}{2}\frac{\partial}{\partial\alpha_2}\left[\tilde{M}_{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]\Bigg\}\delta u_1$$

$$+\Bigg\{-\frac{\partial[\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}A_1]}{\partial\alpha_2}+\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}\frac{\partial A_1}{\partial\alpha_2}$$

$$-\frac{1}{2}\frac{\partial}{\partial\alpha_1}[\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}A_2]-\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+$$

$$N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}\frac{\partial A_2}{\partial\alpha_1}-Q_{23}\frac{A_1A_2}{R_2}-\frac{1}{2}\frac{\partial}{\partial\alpha_1}\{N_{11}^r(\varepsilon_3^s+\beta_n)-$$

$$N_{22}^r(\varepsilon_3^s-\beta_n)-\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}A_2-\frac{A_2}{2}\frac{\partial}{\partial\alpha_1}\left[\tilde{M}_{12}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\right]\Bigg\}\delta u_2+$$

$$\Bigg\{\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}\frac{A_1A_2}{R_1}+\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\frac{A_1A_2}{R_2}-$$

$$\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}\Bigg\}\delta w$$

$$+\Bigg\{-\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+M_{22}\frac{\partial A_2}{\partial\alpha_1}-\tilde{M}_{12}\frac{\partial A_1}{\partial\alpha_2}-\frac{\partial(\tilde{M}_{12}A_1)}{\partial\alpha_2}+Q_{13}A_1A_2+A_1A_2(N_{11}^r\beta_1+\tilde{N}_{12}^r\beta_2)\Bigg\}\delta\beta_1+$$

$$\Bigg\{-\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+M_{11}\frac{\partial A_1}{\partial\alpha_2}-\tilde{M}_{12}\frac{\partial A_2}{\partial\alpha_1}-\frac{\partial(\tilde{M}_{12}A_2)}{\partial\alpha_1}+Q_{23}A_1A_2+A_1A_2(N_{22}^r\beta_2+\tilde{N}_{12}^r\beta_1)\Bigg\}\delta\beta_2\Bigg]d\alpha_1d\alpha_2$$

$$+\int_{\alpha_1}[\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\delta u_2+\{(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r\varepsilon_1^o$$

$$-\frac{\tilde{M}_{12}}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\}\delta u_1+M_{22}\delta\beta_2+\tilde{M}_{12}\delta\beta_1+Q_{23}\delta w]A_1d\alpha_1+\int_{\alpha_2}[\{N_{11}+N_{11}^r(1+\varepsilon_1^o)$$

$$+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}\delta u_1+\{(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+2\tilde{N}_{12}^r\varepsilon_2^o+\frac{\tilde{M}_{12}}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\}\delta u_2$$

$$+M_{11}\delta\beta_1+\tilde{M}_{12}\delta\beta_2+Q_{13}\delta w]A_2\,d\alpha_2. \tag{2.78}$$

This completes the derivation of the variation of the strain energy.

2.7.2 Variation of the Kinetic Energy

In Eq. (2.39), if we substitute the displacements U_1, U_2, and W, we get

$$K_E = \tfrac{1}{2}\int_V \rho\,(\dot{U}_1 \boldsymbol{t}_1 + \dot{U}_2 \boldsymbol{t}_2 + \dot{W}\boldsymbol{t}_3)\cdot(\dot{U}_1 \boldsymbol{t}_1 + \dot{U}_2 \boldsymbol{t}_2 + \dot{W}\boldsymbol{t}_3)\,dV\,. \tag{2.79}$$

where $\boldsymbol{t}_1$, $\boldsymbol{t}_2$, and $\boldsymbol{t}_3$ are the unit vectors along the coordinate lines α_1, α_2, and ζ, respectively. Now using Eq. (2.22), and representing the volume integral in terms of a triple integral, we get

$$K_E = \tfrac{1}{2}\int_{\alpha_2}\int_{\alpha_1}\int_{\zeta} \rho[(\dot{u}_1 + \zeta\,\dot{\beta}_1)^2 + (\dot{u}_2 + \zeta\,\dot{\beta}_2)^2 + (\dot{w})^2]\,A_1\,A_2\,d\zeta\,d\alpha_1\,d\alpha_2\,. \tag{2.80}$$

where we neglected ζ / R_1 and ζ / R_2 in comparison to unity. After integrating across the thickness, i.e., $\zeta = -h/2$ to $\zeta = h/2$, we can write the above equation as

$$K_E = \frac{\rho h}{2}\int_{\alpha_2}\int_{\alpha_1}[(\dot{u}_1^2 + \dot{u}_2^2 + \dot{w}^2) + \frac{h^2}{12}(\dot{\beta}_1^2 + \dot{\beta}_2^2)]\,A_1\,A_2\,d\alpha_1\,d\alpha_2\,. \tag{2.81}$$

Now we evaluate the variation of the kinetic energy term of Eq. (2.40):

$$\int_{t_o}^{t_1}\delta K_E\,dt = \frac{\rho h}{2}\delta\int_{t_o}^{t_1}\int_{\alpha_2}\int_{\alpha_1}[(\dot{u}_1^2 + \dot{u}_2^2 + \dot{w}^2) + \frac{h^2}{12}(\dot{\beta}_1^2 + \dot{\beta}_2^2)]\,A_1\,A_2\,d\alpha_1\,d\alpha_2\,dt\,. \tag{2.82}$$

The above equation can be rewritten as

$$\int_{t_o}^{t_1} \delta K_E \, dt = \rho h \int_{t_o}^{t_1} \int_{\alpha_2} \int_{\alpha_1} [(\dot{u}_1 \delta u_1 + \dot{u}_2 \delta u_2 + \dot{w} \delta w) + \frac{h^2}{12} (\dot{\beta}_1 \delta \beta_1 + \dot{\beta}_2 \delta \beta_2)] A_1 A_2 \, d\alpha_1 \, d\alpha_2 \, dt \, . \tag{2.83}$$

To eliminate the time derivative in the variations, we perform integration-by-parts to obtain

$$\begin{aligned} \int_{t_o}^{t_1} \delta K_E \, dt = {} & \rho h \int_{\alpha_2} \int_{\alpha_1} [(\dot{u}_1 \delta u_1 + \dot{u}_2 \delta u_2 + \dot{w} \delta w) + \frac{h^2}{12} (\dot{\beta}_1 \delta \beta_1 + \dot{\beta}_2 \delta \beta_2)] A_1 A_2 \, d\alpha_1 \, d\alpha_2 \Big|_{t_o}^{t_1} \\ & - \rho h \int_{t_o}^{t_1} \int_{\alpha_2} \int_{\alpha_1} [(\ddot{u}_1 \delta u_1 + \ddot{u}_2 \delta u_2 + \ddot{w} \delta w) + \frac{h^2}{12} (\ddot{\beta}_1 \delta \beta_1 + \ddot{\beta}_2 \delta \beta_2)] A_1 A_2 \, d\alpha_1 \, d\alpha_2 \, dt. \end{aligned} \tag{2.84}$$

Recognizing that the virtual displacement vanishes at times t_o and t_1 and neglecting the rotatory inertia terms, $\rho h^3 \ddot{\beta}_1 / 12$ and $\rho h^3 \ddot{\beta}_2 / 12$, which are usually very small, we obtain

$$\int_{t_o}^{t_1} \delta K_E \, dt = -\rho h \int_{t_o}^{t_1} \int_{\alpha_2} \int_{\alpha_1} [(\ddot{u}_1 \delta u_1 + \ddot{u}_2 \delta u_2 + \ddot{w} \delta w)] A_1 A_2 \, d\alpha_1 \, d\alpha_2 \, dt. \tag{2.85}$$

The above equation gives the variation of kinetic energy term in Eq. (2.40).

2.7.3 Variation of the Work Done by Pressure

As the shell vibrates, the enclosed volume changes. We assume that the pressure force acts normal to the deformed surface during the vibration. This is called the follower action of the pressure force (Brush and Almroth, 1975). Variation of the work done by the pressure p can be written as (Budiansky, 1968)

$$\delta W_p = \int_{\alpha_2} \int_{\alpha_1} p[\beta_1 \delta u_1 + \beta_2 \delta u_2 + (\varepsilon_1^o + \varepsilon_2^o) \delta w] A_1 A_2 \, d\alpha_1 \, d\alpha_2. \tag{2.86}$$

2.7.4 Variations of the Work Done by External and Edge Forces

Let's assume that q_1, q_2, and q_3 are the static equivalents of the external forces along the coordinate lines α_1, α_2, and ζ, respectively. These forces are applied on the mid-surface and are equivalent to the body forces and the surface forces. Denoting the variation of the total work due to these forces by δW_L, we write

$$\delta W_L = \int_{\alpha_2}\int_{\alpha_1} [q_1\,\delta U_1(\alpha_1,\alpha_2,0) + q_2\,\delta U_2(\alpha_1,\alpha_2,0) + q_3\,\delta W(\alpha_1,\alpha_2,0)] A_1 A_2 d\alpha_1\, d\alpha_2. \tag{2.87}$$

Representing this in terms of the displacements of the reference surface, we get

$$\delta W_L = \int_{\alpha_2}\int_{\alpha_1} [q_1\,\delta u_1 + q_2\,\delta u_2 + q_3\,\delta w]\, A_1 A_2 d\alpha_1\, d\alpha_2. \tag{2.88}$$

On the edge with constant α_1, let $\bar{\sigma}_{11}$, $\bar{\sigma}_{12}$, and $\bar{\sigma}_{13}$ be the vibratory stresses in the α_1, α_2, and ζ directions, respectively. The variation δW_{e_1} of the total work done due to these edge stresses is

$$\delta W_{e_1} = \int_{\alpha_2}\int_{\zeta} [\bar{\sigma}_1\,\delta U_1 + \bar{\sigma}_{12}\,\delta U_2 + \bar{\sigma}_{13}\,\delta W] A_2 (1+\zeta/R_2)\, d\zeta\, d\alpha_2. \tag{2.89}$$

Using Eq. (2.22), we get

$$\delta W_{e_1} = \int_{\alpha_2}\int_{\zeta} [\bar{\sigma}_1 (\delta u_1 + \zeta\,\delta\beta_1) + \bar{\sigma}_{12} (\delta u_2 + \zeta\,\delta\beta_2) + \bar{\sigma}_{13}\,\delta w] A_2 (1+\zeta/R_2) d\zeta\, d\alpha_2. \tag{2.90}$$

Integrating over the thickness, we get

$$\delta W_{e_1} = \int_{\alpha_2} [\overline{N}_{11}\,\delta u_1 + \overline{M}_{11}\,\delta\beta_1 + \overline{N}_{12}\,\delta u_2 + \overline{M}_{12}\,\delta\beta_2 + \overline{Q}_{13}\;\delta w] A_2\, d\alpha_2. \qquad (2.91)$$

Similarly, if we assume that $\overline{\sigma}_{22}$, $\overline{\sigma}_{21}$, and $\overline{\sigma}_{23}$ are the stresses in the α_2, α_1, and ζ directions, respectively, on the edge with constant α_2, we get the variation of the total work done as

$$\delta W_{e_2} = \int_{\alpha_1} [\overline{N}_{22}\,\delta u_2 + \overline{M}_{22}\,\delta\beta_2 + \overline{N}_{21}\,\delta u_1 + \overline{M}_{21}\,\delta\beta_1 + \overline{Q}_{23}\;\delta w] A_1\, d\alpha_1. \qquad (2\text{-}92)$$

2.7.5 Combining All the Energy Variations

Now that the variations of the potential energy, kinetic energy, work done by the pressure, and work done by the external and edge forces have been found, we can put them in Eq. (2.40) to obtain

$$\int_{t_o}^{t_1}\int_{\alpha_2}\int_{\alpha_1}\Bigg[\Bigg\{-\frac{\partial[\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}A_2]}{\partial\alpha_1}+\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\frac{\partial A_2}{\partial\alpha_1}-$$

$$\frac{1}{2}\frac{\partial}{\partial\alpha_2}[\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}A_1]-\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+$$

$$N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}\frac{\partial A_1}{\partial\alpha_2}-Q_{13}\frac{A_1A_2}{R_1}+\frac{1}{2}\frac{\partial}{\partial\alpha_2}\{N_{11}^r(\varepsilon_3^s+\beta_n)-N_{22}^r(\varepsilon_3^s$$

$$-\beta_n)-\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}A_1-\frac{A_1}{2}\frac{\partial}{\partial\alpha_2}\left[\tilde{M}_{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]-p\beta_1A_1A_2+A_1A_2(\rho h\ddot{u}_1-q_1)\Bigg\}\delta u_1$$

$$+\Bigg\{-\frac{\partial[\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}A_1]}{\partial\alpha_2}+\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}\frac{\partial A_1}{\partial\alpha_2}$$

$$-\frac{1}{2}\frac{\partial}{\partial\alpha_1}[\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}A_2]-\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+$$

$$N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}\frac{\partial A_2}{\partial\alpha_1}-Q_{23}\frac{A_1A_2}{R_2}-\frac{1}{2}\frac{\partial}{\partial\alpha_1}\{N_{11}^r(\varepsilon_3^s+\beta_n)-N_{22}^r(\varepsilon_3^s$$

$$-\beta_n)-\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}A_2-\frac{A_2}{2}\frac{\partial}{\partial\alpha_1}\left[\tilde{M}_{12}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\right]-p\beta_2A_1A_2+A_1A_2(\rho h\ddot{u}_2-q_2)\Bigg\}\delta u_2+$$

$$\Bigg\{\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}\frac{A_1A_2}{R_1}+\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\frac{A_1A_2}{R_2}-$$

$$\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}-p(\varepsilon_1^o+\varepsilon_2^o)A_1A_2+A_1A_2(\rho h\ddot{w}-q_3)\Bigg\}\delta w+\Bigg\{-\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+M_{22}\frac{\partial A_2}{\partial\alpha_1}-$$

$$\tilde{M}_{12}\frac{\partial A_1}{\partial\alpha_2}-\frac{\partial(\tilde{M}_{12}A_1)}{\partial\alpha_2}+Q_{13}A_1A_2+A_1A_2(N_{11}^r\beta_1+\tilde{N}_{12}^r\beta_2)\Bigg\}\delta\beta_1+\Bigg\{-\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+M_{11}\frac{\partial A_1}{\partial\alpha_2}-$$

$$\tilde{M}_{12}\frac{\partial A_2}{\partial\alpha_1}-\frac{\partial(\tilde{M}_{12}A_2)}{\partial\alpha_1}+Q_{23}A_1A_2+A_1A_2(N_{22}^r\beta_2+\tilde{N}_{12}^r\beta_1)\Bigg\}\delta\beta_2\Bigg]d\alpha_1d\alpha_2dt$$

$$+\int_{t_o}^{t_1}\int_{\alpha_1}[\{N_{22}+N_{22}^r(1+\varepsilon_2^o)-\overline{N}_{22}+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\delta u_2+\{(\tilde{N}_{12}+\tilde{N}_{12}^r-\overline{N}_{21})+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r\varepsilon_1^o-$$

$$\frac{\tilde{M}_{12}}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\Bigg\}\delta u_1+(M_{22}-\overline{M}_{22})\delta\beta_2+(\tilde{M}_{12}-\overline{M}_{21})\delta\beta_1+(Q_{23}-\overline{Q}_{23})\delta w]A_1d\alpha_1dt$$

$$+\int_{t_o}^{t_1}\int_{\alpha_2}[\{N_{11}+N_{11}^r(1+\varepsilon_2^o)-\overline{N}_{11}+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}\delta u_1+\{(\tilde{N}_{12}+\tilde{N}_{12}^r-\overline{N}_{12})+N_{11}^r(\varepsilon_3^s+\beta_n)+\tilde{N}_{12}^r\varepsilon_2^o+$$

$$\frac{\tilde{M}_{12}}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\Bigg\}\delta u_2+(M_{11}-\overline{M}_{11})\delta\beta_1+(\tilde{M}_{12}-\overline{M}_{12})\delta\beta_2+(Q_{13}-\overline{Q}_{13})\delta w]A_2\,d\alpha_2dt.$$

(2.93)

Since the variations of the displacements are arbitrary, the above equation requires that the coefficients of the displacement variations vanish individually. The above equation contains three main integrals. The first one, which is the integration over the area of the shell, will give the equations of motions, while the last two will constitute the boundary conditions.

2.7.6 Shell Equations

Setting the coefficients of δu_1, δu_2, δw, $\delta\beta_1$, and $\delta\beta_2$ equal to zero in the first integration of Eq. (2.93), we obtain the following five equations of motion:

$$-\frac{\partial[\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}A_2]}{\partial\alpha_1}+\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\frac{\partial A_2}{\partial\alpha_1}-\frac{1}{2}$$
$$\frac{\partial}{\partial\alpha_2}[\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}A_1]-\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+$$
$$N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}\frac{\partial A_1}{\partial\alpha_2}-Q_{13}\frac{A_1A_2}{R_1}+\frac{1}{2}\frac{\partial}{\partial\alpha_2}\{N_{11}^r(\varepsilon_3^s+\beta_n)-N_{22}^r$$
$$(\varepsilon_3^s-\beta_n)-\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}A_1-\frac{A_1}{2}\frac{\partial}{\partial\alpha_2}\left[\tilde{M}_{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]-p\,\beta_1\,A_1A_2+A_1\,A_2\,(\rho\,h\ddot{u}_1-q_1)=0$$

$$-\frac{\partial[\{N_{22}+N_{22}^r(1+\varepsilon_2^o)+\tilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}A_1]}{\partial\alpha_2}+\{N_{11}+N_{11}^r(1+\varepsilon_1^o)+\tilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}\frac{\partial A_1}{\partial\alpha_2}-\frac{1}{2}$$
$$\frac{\partial}{\partial\alpha_1}[\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}A_2]-\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^r)+$$
$$N_{11}^r(\varepsilon_3^s+\beta_n)+N_{22}^r(\varepsilon_3^s-\beta_n)+\tilde{N}_{12}^r(\varepsilon_1^o+\varepsilon_2^o)\}\frac{\partial A_2}{\partial\alpha_1}-Q_{23}\frac{A_1A_2}{R_2}-\frac{1}{2}\frac{\partial}{\partial\alpha_1}\{N_{11}^r(\varepsilon_3^s+\beta_n)-N_{22}^r$$
$$(\varepsilon_3^s-\beta_n)-\tilde{N}_{12}^r(\varepsilon_1^o-\varepsilon_2^o)\}A_2-\frac{A_2}{2}\frac{\partial}{\partial\alpha_1}\left[\tilde{M}_{12}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\right]-p\,\beta_2\,A_1A_2+A_1\,A_2\,(\rho\,h\ddot{u}_2-q_2)=0,$$

$$\{N_{11}+N_{11}^{r}(1+\varepsilon_1^{o})+\tilde{N}_{12}^{r}(\varepsilon_3^{s}-\beta_n)\}\frac{A_1A_2}{R_1}+\{N_{22}+N_{22}^{r}(1+\varepsilon_2^{o})+\tilde{N}_{12}^{r}(\varepsilon_3^{s}+\beta_n)\}\frac{A_1A_2}{R_2}-$$
$$\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}-p(\varepsilon_1^{o}+\varepsilon_2^{o})A_1A_2+A_1A_2(\rho h\ddot{w}-q_3)=0,$$

$$-\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+M_{22}\frac{\partial A_2}{\partial\alpha_1}-\tilde{M}_{12}\frac{\partial A_1}{\partial\alpha_2}-\frac{\partial(\tilde{M}_{12}A_1)}{\partial\alpha_2}+Q_{13}A_1A_2+A_1A_2(N_{11}^{r}\beta_1+\tilde{N}_{12}^{r}\beta_2)=0,$$

$$-\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+M_{11}\frac{\partial A_1}{\partial\alpha_2}-\tilde{M}_{12}\frac{\partial A_2}{\partial\alpha_1}-\frac{\partial(\tilde{M}_{12}A_2)}{\partial\alpha_1}+Q_{23}A_1A_2+A_1A_2(N_{22}^{r}\beta_2+\tilde{N}_{12}^{r}\beta_1)=0. \tag{2.94}$$

One can also reduce the five equations of motion to three by eliminating Q_{13} and Q_{23}. This gives

$$\frac{\partial[\{N_{11}+N_{11}^{r}(1+\varepsilon_1^{o})+\tilde{N}_{12}^{r}(\varepsilon_3^{s}-\beta_n)\}A_2]}{\partial\alpha_1}+\frac{1}{2}\frac{\partial}{\partial\alpha_2}[\{2(\tilde{N}_{12}+\tilde{N}_{12}^{r})+N_{11}^{r}(\varepsilon_3^{s}+\beta_n)+N_{22}^{r}$$
$$(\varepsilon_3^{s}-\beta_n)+\tilde{N}_{12}^{r}(\varepsilon_1^{o}+\varepsilon_2^{o})\}A_1]+\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^{r})+N_{11}^{r}(\varepsilon_3^{s}+\beta_n)+N_{22}^{r}(\varepsilon_3^{s}-\beta_n)+\tilde{N}_{12}^{r}(\varepsilon_1^{o}$$
$$+\varepsilon_2^{o})\}\frac{\partial A_1}{\partial\alpha_2}-\{N_{22}+N_{22}^{r}(1+\varepsilon_2^{o})+\tilde{N}_{12}^{r}(\varepsilon_3^{s}+\beta_n)\}\frac{\partial A_2}{\partial\alpha_1}+\frac{1}{R_1}\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{1}{R_1}\frac{\partial(\tilde{M}_{12}A_1)}{\partial\alpha_2}+$$
$$\frac{\tilde{M}_{12}}{R_1}\frac{\partial A_1}{\partial\alpha_2}-\frac{M_{22}}{R_1}\frac{\partial A_2}{\partial\alpha_1}+\frac{A_1}{2}\frac{\partial}{\partial\alpha_2}\left[\tilde{M}_{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]-\frac{A_1A_2}{R_1}(N_{11}^{r}\beta_1+\tilde{N}_{12}^{r}\beta_2)-\frac{1}{2}\frac{\partial}{\partial\alpha_2}\{N_{11}^{r}$$
$$(\varepsilon_3^{s}+\beta_n)-N_{22}^{r}(\varepsilon_3^{s}-\beta_n)-\tilde{N}_{12}^{r}(\varepsilon_1^{o}-\varepsilon_2^{o})\}A_1+p\beta_1A_1A_2-A_1A_2(\rho h\ddot{u}_1-q_1)=0,$$

$$\frac{\partial[\{N_{22}+N_{22}^{r}(1+\varepsilon_2^{o})+\tilde{N}_{12}^{r}(\varepsilon_3^{s}+\beta_n)\}A_1]}{\partial\alpha_2}+\frac{1}{2}\frac{\partial}{\partial\alpha_1}[\{2(\tilde{N}_{12}+\tilde{N}_{12}^{r})+N_{11}^{r}(\varepsilon_3^{s}+\beta_n)+N_{22}^{r}$$
$$(\varepsilon_3^{s}-\beta_n)+\tilde{N}_{12}^{r}(\varepsilon_1^{o}+\varepsilon_2^{o})\}A_2]+\frac{1}{2}\{2(\tilde{N}_{12}+\tilde{N}_{12}^{r})+N_{11}^{r}(\varepsilon_3^{s}+\beta_n)+N_{22}^{r}(\varepsilon_3^{s}-\beta_n)+\tilde{N}_{12}^{r}(\varepsilon_1^{o}$$
$$+\varepsilon_2^{o})\}\frac{\partial A_2}{\partial\alpha_1}-\{N_{11}+N_{11}^{r}(1+\varepsilon_1^{o})+\tilde{N}_{12}^{r}(\varepsilon_3^{s}-\beta_n)\}\frac{\partial A_1}{\partial\alpha_2}+\frac{1}{R_2}\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+\frac{1}{R_2}\frac{\partial(\tilde{M}_{12}A_2)}{\partial\alpha_1}+$$
$$\frac{\tilde{M}_{12}}{R_2}\frac{\partial A_2}{\partial\alpha_1}-\frac{M_{11}}{R_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{A_2}{2}\frac{\partial}{\partial\alpha_1}\left[\tilde{M}_{12}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\right]-\frac{A_1A_2}{R_2}(N_{22}^{r}\beta_2+\tilde{N}_{12}^{r}\beta_1)+\frac{1}{2}\frac{\partial}{\partial\alpha_1}\{N_{11}^{r}$$
$$(\varepsilon_3^{s}+\beta_n)-N_{22}^{r}(\varepsilon_3^{s}-\beta_n)-\tilde{N}_{12}^{r}(\varepsilon_1^{o}-\varepsilon_2^{o})\}A_2+p\beta_2A_1A_2-A_1A_2(\rho h\ddot{u}_2-q_2)=0,$$

$$\frac{\partial}{\partial\alpha_1}\left[\frac{1}{A_1}\left(\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(\tilde{M}_{12}A_1)}{\partial\alpha_2}+\tilde{M}_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1}\right)\right]+\frac{\partial}{\partial\alpha_2}\left[\frac{1}{A_2}\left(\frac{\partial(M_{22}A_1)}{\partial\alpha_2}\right.\right.$$
$$\left.\left.+\frac{\partial(\tilde{M}_{12}A_2)}{\partial\alpha_1}+\tilde{M}_{12}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}\right)\right]-\{N_{11}+N_{11}^{r}(1+\varepsilon_1^{o})+\tilde{N}_{12}^{r}(\varepsilon_3^{s}-\beta_n)\}\frac{A_1A_2}{R_1}-$$
$$\{N_{22}+N_{22}^{r}(1+\varepsilon_2^{o})+\tilde{N}_{12}^{r}(\varepsilon_3^{s}+\beta_n)\}\frac{A_1A_2}{R_2}-\frac{\partial}{\partial\alpha_1}[A_2(N_{11}^{r}\beta_1+\tilde{N}_{12}^{r}\beta_2)]-\frac{\partial}{\partial\alpha_2}[A_1$$
$$(N_{22}^{r}\beta_2+\tilde{N}_{12}^{r}\beta_1)]+p(\varepsilon_1^{o}+\varepsilon_2^{o})A_1A_2-A_1A_2(\rho h\ddot{w}-q_3)=0.$$

(2.95)

In the following sub-sections, we separate these equations into static and dynamic parts.

2.7.6.1 Static Equations

Vibration of a shell under prestress can be analyzed analogously to a spring-mass system under the effect of gravity. In a linear analysis, we can separate the dynamic and static analyses. First, we set the vibration of the shell to be zero. This yields the following static equations:

$$\frac{\partial(N_{11}^{r}A_2)}{\partial\alpha_1}+\frac{\partial(\tilde{N}_{12}^{r}A_1)}{\partial\alpha_2}+\tilde{N}_{12}^{r}\frac{\partial A_1}{\partial\alpha_2}-N_{22}^{r}\frac{\partial A_2}{\partial\alpha_1}=0,$$

$$\frac{\partial (N_{22}^r A_1)}{\partial \alpha_2} + \frac{\partial (\tilde{N}_{12}^r A_2)}{\partial \alpha_1} + \tilde{N}_{12}^r \frac{\partial A_2}{\partial \alpha_1} - N_{11}^r \frac{\partial A_1}{\partial \alpha_2} = 0, \tag{2.96}$$

$$\frac{N_{11}^r}{R_1} + \frac{N_{22}^r}{R_2} = p.$$

The solution of these equations gives initial deflections and initial stresses.

2.7.6.2 Dynamic Equations

Subtracting the static equations from the total equations results in the equations of motion:

$$\frac{\partial (N_{11} A_2)}{\partial \alpha_1} + \frac{\partial (\tilde{N}_{12} A_1)}{\partial \alpha_2} + \tilde{N}_{12} \frac{\partial A_1}{\partial \alpha_2} - N_{22} \frac{\partial A_2}{\partial \alpha_1} + \frac{1}{R_1} \frac{\partial (M_{11} A_2)}{\partial \alpha_1} + \frac{1}{R_1} \frac{\partial (\tilde{M}_{12} A_1)}{\partial \alpha_2} + \frac{\tilde{M}_{12}}{R_1} \frac{\partial A_1}{\partial \alpha_2}$$
$$- \frac{M_{22}}{R_1} \frac{\partial A_2}{\partial \alpha_1} + \frac{A_1}{2} \frac{\partial}{\partial \alpha_2} \left[\tilde{M}_{12} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) \right] + \frac{\partial [\{N_{11}^r \varepsilon_1^o + \tilde{N}_{12}^r (\varepsilon_3^s - \beta_n)\} A_2]}{\partial \alpha_1} + \frac{1}{2} \frac{\partial}{\partial \alpha_2} [\{N_{11}^r (\varepsilon_3^s$$
$$+ \beta_n) + N_{22}^r (\varepsilon_3^s - \beta_n) + \tilde{N}_{12}^r (\varepsilon_1^o + \varepsilon_2^o)\} A_1] + \frac{1}{2} \{N_{11}^r (\varepsilon_3^s + \beta_n) + N_{22}^r (\varepsilon_3^s - \beta_n) + \tilde{N}_{12}^r (\varepsilon_1^o + \varepsilon_2^o$$
$$)\} \frac{\partial A_1}{\partial \alpha_2} - \{N_{22}^r \varepsilon_2^o + \tilde{N}_{12}^r (\varepsilon_3^s + \beta_n)\} \frac{\partial A_2}{\partial \alpha_1} - \frac{A_1 A_2}{R_1} (N_{11}^r \beta_1 + \tilde{N}_{12}^r \beta_2) - \frac{1}{2} \frac{\partial}{\partial \alpha_2} \{N_{11}^r (\varepsilon_3^s + \beta_n) -$$
$$N_{22}^r (\varepsilon_3^s - \beta_n) - \tilde{N}_{12}^r (\varepsilon_1^o - \varepsilon_2^o)\} A_1 + p \beta_1 A_1 A_2 - A_1 A_2 (\rho h \ddot{u}_1 - q_1) = 0,$$

$$\frac{\partial (N_{22} A_1)}{\partial \alpha_2} + \frac{1}{2} \frac{\partial (\tilde{N}_{12} A_2)}{\partial \alpha_1} + \tilde{N}_{12} \frac{\partial A_2}{\partial \alpha_1} - N_{11} \frac{\partial A_1}{\partial \alpha_2} + \frac{1}{R_2} \frac{\partial (M_{22} A_1)}{\partial \alpha_2} + \frac{1}{R_2} \frac{\partial (\tilde{M}_{12} A_2)}{\partial \alpha_1} + \frac{\tilde{M}_{12}}{R_2} \frac{\partial A_2}{\partial \alpha_1}$$
$$- \frac{M_{11}}{R_2} \frac{\partial A_1}{\partial \alpha_2} + \frac{A_2}{2} \frac{\partial}{\partial \alpha_1} \left[\tilde{M}_{12} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) \right] + \frac{\partial [\{N_{22}^r \varepsilon_2^o + \tilde{N}_{12}^r (\varepsilon_3^s + \beta_n)\} A_1]}{\partial \alpha_2} + \frac{1}{2} \frac{\partial}{\partial \alpha_1} [\{N_{11}^r (\varepsilon_3^s$$
$$+ \beta_n) + N_{22}^r (\varepsilon_3^s - \beta_n) + \tilde{N}_{12}^r (\varepsilon_1^o + \varepsilon_2^o)\} A_2] + \frac{1}{2} \{N_{11}^r (\varepsilon_3^s + \beta_n) + N_{22}^r (\varepsilon_3^s - \beta_n) + \tilde{N}_{12}^r (\varepsilon_1^o + \varepsilon_2^o$$
$$)\} \frac{\partial A_2}{\partial \alpha_1} - \{N_{11}^r \varepsilon_1^o + \tilde{N}_{12}^r (\varepsilon_3^s - \beta_n)\} \frac{\partial A_1}{\partial \alpha_2} - \frac{A_1 A_2}{R_2} (N_{22}^r \beta_2 + \tilde{N}_{12}^r \beta_1) + \frac{1}{2} \frac{\partial}{\partial \alpha_1} \{N_{11}^r (\varepsilon_3^s + \beta_n) -$$
$$N_{22}^r (\varepsilon_3^s - \beta_n) - \tilde{N}_{12}^r (\varepsilon_1^o - \varepsilon_2^o)\} A_2 + p \beta_2 A_1 A_2 - A_1 A_2 (\rho h \ddot{u}_2 - q_2) = 0,$$

$$\frac{\partial}{\partial\alpha_1}\left[\frac{1}{A_1}\left(\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(\widetilde{M}_{12}A_1)}{\partial\alpha_2}+\widetilde{M}_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1}\right)\right]+\frac{\partial}{\partial\alpha_2}\left[\frac{1}{A_2}\left(\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+\frac{\partial(\widetilde{M}_{12}A_2)}{\partial\alpha_1}\right.\right.$$
$$\left.\left.+\widetilde{M}_{12}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}\right)\right]-\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)A_1A_2-\{N_{11}^r\varepsilon_1^o+\widetilde{N}_{12}^r(\varepsilon_3^s-\beta_n)\}\frac{A_1A_2}{R_1}-\{N_{22}^r\varepsilon_2^o+\widetilde{N}_{12}^r$$
$$(\varepsilon_3^s+\beta_n)\}\frac{A_1A_2}{R_2}-\frac{\partial}{\partial\alpha_1}[A_2(N_{11}^r\beta_1+\widetilde{N}_{12}^r\beta_2)]-\frac{\partial}{\partial\alpha_2}[A_1(N_{22}^r\beta_2+\widetilde{N}_{12}^r\beta_1)]+p(\varepsilon_1^o+\varepsilon_2^o)A_1A_2-$$
$$A_1A_2(\rho h\ddot{w}-q_3)=0. \tag{2.97}$$

Another form of the above equations of motion, given in Budiansky (1968), can be obtained by using the static equations, Eqs. (2.96), and Gauss-Codazzi conditions, Eqs. (2.1) and (2.2):

$$\frac{\partial(N_{11}A_2)}{\partial\alpha_1}+\frac{\partial(\widetilde{N}_{12}A_1)}{\partial\alpha_2}+\widetilde{N}_{12}\frac{\partial A_1}{\partial\alpha_2}-N_{22}\frac{\partial A_2}{\partial\alpha_1}+\frac{1}{R_1}\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{1}{R_1}\frac{\partial(\widetilde{M}_{12}A_1)}{\partial\alpha_2}+\frac{\widetilde{M}_{12}}{R_1}\frac{\partial A_1}{\partial\alpha_2}$$
$$-\frac{M_{22}}{R_1}\frac{\partial A_2}{\partial\alpha_1}+\frac{A_1}{2}\frac{\partial}{\partial\alpha_2}\left[\widetilde{M}_{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]+A_1A_2N_{11}^r\left(\frac{1}{A_1}\frac{\partial\varepsilon_1}{\partial\alpha_1}+\frac{\gamma_{12}}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}-\frac{\beta_1}{R_1}\right)+2A_1A_2N_{12}^r$$
$$\left(\frac{1}{A_2}\frac{\partial\varepsilon_1}{\partial\alpha_2}-\frac{\gamma_{12}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)+A_1A_2N_{22}^r\left(\frac{1}{2A_2}\frac{\partial\gamma_{12}}{\partial\alpha_2}+\frac{(\varepsilon_1^o-\varepsilon_2^o)}{A_\phi A_s}\frac{\partial A_2}{\partial\alpha_1}-\frac{1}{A_2}\frac{\partial\beta_n}{\partial\alpha_2}\right)+p\beta_1A_1A_2-$$
$$A_1A_2(\rho h\ddot{u}_1-q_1)=0,$$

$$\frac{\partial(N_{22}A_1)}{\partial\alpha_2}+\frac{1}{2}\frac{\partial(\widetilde{N}_{12}A_2)}{\partial\alpha_1}+\widetilde{N}_{12}\frac{\partial A_2}{\partial\alpha_1}-N_{11}\frac{\partial A_1}{\partial\alpha_2}+\frac{1}{R_2}\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+\frac{1}{R_2}\frac{\partial(\widetilde{M}_{12}A_2)}{\partial\alpha_1}+\frac{\widetilde{M}_{12}}{R_2}\frac{\partial A_2}{\partial\alpha_1}$$
$$-\frac{M_{11}}{R_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{A_2}{2}\frac{\partial}{\partial\alpha_1}\left[\widetilde{M}_{12}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\right]+A_1A_2N_{22}^r\left(\frac{1}{A_2}\frac{\partial\varepsilon_2}{\partial\alpha_2}+\frac{\gamma_{12}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}-\frac{\beta_2}{R_2}\right)+2A_1A_2N_{12}^r$$
$$\left(\frac{1}{A_1}\frac{\partial\varepsilon_2}{\partial\alpha_1}-\frac{\gamma_{12}}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)+A_1A_2N_{11}^r\left(\frac{1}{2A_1}\frac{\partial\gamma_{12}}{\partial\alpha_1}+\frac{(\varepsilon_2^o-\varepsilon_1^o)}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{1}{A_1}\frac{\partial\beta_n}{\partial\alpha_1}\right)+p\beta_2A_1A_2-$$
$$A_1A_2(\rho h\ddot{u}_2-q_2)=0$$

$$\frac{\partial}{\partial\alpha_1}\left[\frac{1}{A_1}\left(\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(\widetilde{M}_{12}A_1)}{\partial\alpha_2}+\widetilde{M}_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1}\right)\right]+\frac{\partial}{\partial\alpha_2}\left[\frac{1}{A_2}\left(\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+\frac{\partial(\widetilde{M}_{12}A_2)}{\partial\alpha_1}\right.\right.$$
$$\left.\left.+\widetilde{M}_{12}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}\right)\right]-\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)A_1A_2-A_1A_2N_{11}^r\left(\frac{\varepsilon_1^o}{R_1}+\frac{1}{A_1}\frac{\partial\beta_1}{\partial\alpha_1}+\frac{\beta_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\right)-A_1A_2N_{12}^r$$
$$\left(\frac{1}{A_2}\frac{\partial\beta_1}{\partial\alpha_2}+\frac{1}{A_1}\frac{\partial\beta_2}{\partial\alpha_1}-\frac{\beta_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}-\frac{\beta_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}-\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\beta_n+\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{\gamma_{12}}{2}\right)-A_1A_2N_{22}^r\left(\frac{\varepsilon_2^o}{R_2}+\right.$$
$$\left.\frac{1}{A_2}\frac{\partial\beta_2}{\partial\alpha_2}+\frac{\beta_1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\right)+A_1A_2p(\varepsilon_1+\varepsilon_2)-A_1A_2(\rho h\ddot{w}-q_3)=0. \tag{2.98}$$

Note that the transverse shear stress resultants Q_{13} and Q_{23} can be obtained using Eqs. (2.94). These equations can be solved in conjunction with the boundary conditions derived below to obtain vibratory stresses and the deflections from the initial equilibrium state.

2.7.7 Boundary Conditions

Setting the coefficients of the displacement variations to zero in the line integrals (last two integrals in Eq. (2.93)) in the expression of the variation of total energy, we obtain the boundary conditions. However, this gives ten boundary conditions, while the order of the equations of motion is eight. In order to reduce the number of boundary conditions, we combine the three shearing stress resultants into two. To this end, $\delta\beta_1$ is expressed in terms of displacements in the first line integral. Rearrangements of terms gives

$$\int_{t_o}^{t_1}\int_{\alpha_1}[\{N_{22}+N_{22}^r(1+\varepsilon_2^o)-\overline{N}_{22}+\widetilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\delta u_2+\{(\widetilde{N}_{12}+\widetilde{N}_{12}^r-\overline{N}_{21})+N_{22}^r(\varepsilon_3^s-\beta_n)+\widetilde{N}_{12}^r\varepsilon_1^o-$$
$$\frac{\widetilde{M}_{12}}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)+\frac{\widetilde{M}_{12}}{R_1}-\frac{\overline{M}_{21}}{R_1}\}\delta u_1+(M_{22}-\overline{M}_{22})\delta\beta_2+(Q_{23}-\overline{Q}_{23})\delta w]A_1d\alpha_1dt$$
$$-\int_{t_o}^{t_1}\int_{\alpha_1}(\widetilde{M}_{12}-\overline{M}_{21})\frac{\partial\delta w}{\partial\alpha_1}\,d\alpha_1\,dt. \tag{2.99}$$

Applying integration-by-parts in the second integration, the above equation can be written as

$$\int_{t_o}^{t_1}\int_{\alpha_1}[\{N_{22}+N_{22}^r(1+\varepsilon_2^o)-\overline{N}_{22}+\widetilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\delta u_2+\{(\widetilde{N}_{12}+\widetilde{N}_{12}^r-\overline{N}_{21})+N_{22}^r(\varepsilon_3^s-\beta_n)+\widetilde{N}_{12}^r\varepsilon_1^o-$$
$$\frac{\widetilde{M}_{12}}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)+\frac{\widetilde{M}_{12}}{R_1}-\frac{\overline{M}_{21}}{R_1}\}\delta u_1+(M_{22}-\overline{M}_{22})\delta\beta_2+(Q_{23}-\overline{Q}_{23})\delta w]A_1 d\alpha_1 dt$$
$$-\int_{t_o}^{t_1}(\widetilde{M}_{12}-\overline{M}_{21})\delta w\,dt+\int_{t_o}^{t_1}\int_{\alpha_1}\frac{\partial(\widetilde{M}_{12}-\overline{M}_{21})}{\partial\alpha_1}\delta w\,d\alpha_1\,dt. \tag{2.100}$$

Recognizing the fact that the arbitrary displacement δw vanishes at the end points t_o and t_1, and rearranging, we get

$$\int_{t_o}^{t_1}\int_{\alpha_1}[\{N_{22}+N_{22}^r(1+\varepsilon_2^o)-\overline{N}_{22}+\widetilde{N}_{12}^r(\varepsilon_3^s+\beta_n)\}\delta u_2+\{(\widetilde{N}_{12}+\widetilde{N}_{12}^r-\overline{N}_{21})+N_{22}^r(\varepsilon_3^s-\beta_n)+$$
$$\widetilde{N}_{12}^r\varepsilon_1^o-\frac{\widetilde{M}_{12}}{2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)+\frac{\widetilde{M}_{12}}{R_1}-\frac{\overline{M}_{21}}{R_1}\}\delta u_1+(M_{22}-\overline{M}_{22})\delta\beta_2+$$
$$\left(Q_{23}+\frac{1}{A_1}\frac{\partial\widetilde{M}_{12}}{\partial\alpha_1}-\overline{Q}_{23}-\frac{1}{A_1}\frac{\partial\overline{M}_{21}}{\partial\alpha_1}\right)\delta w]A_1 d\alpha_1 dt. \tag{2.101}$$

Now we can write the four boundary conditions at the edge with constant α_2 as

$$N_{22}+N_{22}^r(1+\varepsilon_2^o)+\widetilde{N}_{12}^r(\varepsilon_3^s+\beta_n)=\overline{N}_{22} \quad \text{or} \quad u_2=\overline{u}_2,$$
$$\widetilde{N}_{12}+\widetilde{N}_{12}^r+\left(\frac{3}{2R_1}-\frac{1}{2R_2}\right)\widetilde{M}_{12}+\widetilde{N}_{12}^r\varepsilon_1^o+N_{22}^r(\varepsilon_3^s-\beta_n)=\overline{N}_{21}+\frac{\overline{M}_{21}}{R_1} \quad \text{or} \quad u_1=\overline{u}_1,$$
$$Q_{23}+\frac{1}{A_1}\frac{\partial\widetilde{M}_{12}}{\partial\alpha_1}=\overline{Q}_{23}+\frac{1}{A_1}\frac{\partial\overline{M}_{21}}{\partial\alpha_1} \quad \text{or} \quad w=\overline{w},$$
$$M_{22}=\overline{M}_{22} \quad \text{or} \quad \beta_2=\overline{\beta}_2.$$

(2.102)

where the over bar quantities are specified at the edges. Considering the static equilibrium state as reference, we can simplify the boundary conditions at the edge with constant α_2 as

$$N_{22} + N_{22}^r \varepsilon_2^o + \tilde{N}_{12}^r (\varepsilon_3^s + \beta_n) = \bar{N}_{22} \quad \text{or} \quad u_2 = \bar{u}_2,$$

$$\tilde{N}_{12} + \left(\frac{3}{2R_1} - \frac{1}{2R_2} \right) \tilde{M}_{12} + \tilde{N}_{12}^r \varepsilon_1^o + N_{22}^r (\varepsilon_3^s - \beta_n) = \bar{N}_{21} + \frac{\bar{M}_{21}}{R_1} \quad \text{or} \quad u_1 = \bar{u}_1,$$

$$Q_{23} + \frac{1}{A_1} \frac{\partial \tilde{M}_{12}}{\partial \alpha_1} = \bar{Q}_{23} + \frac{1}{A_1} \frac{\partial \bar{M}_{21}}{\partial \alpha_1} \quad \text{or} \quad w = \bar{w},$$

$$M_{22} = \bar{M}_{22} \quad \text{or} \quad \beta_2 = \bar{\beta}_2.$$

(103)

Similarly, we can write the four boundary conditions at the edge with constant α_1 as follows:

$$N_{11} + N_{11}^r \varepsilon_1^o + \tilde{N}_{12}^r (\varepsilon_3^s - \beta_n) = \bar{N}_{11} \quad \text{or} \quad u_1 = \bar{u}_1,$$

$$\tilde{N}_{12} + \left(\frac{3}{2R_2} - \frac{1}{2R_1} \right) \tilde{M}_{12} + \tilde{N}_{12}^r \varepsilon_2^o + N_{11}^r (\varepsilon_3^s + \beta_n) = \bar{N}_{12} + \frac{\bar{M}_{12}}{R_2} \quad \text{or} \quad u_2 = \bar{u}_2,$$

$$Q_{13} + \frac{1}{A_2} \frac{\partial \tilde{M}_{12}}{\partial \alpha_2} = \bar{Q}_{13} + \frac{1}{A_2} \frac{\partial \bar{M}_{12}}{\partial \alpha_2} \quad \text{or} \quad w = \bar{w},$$

$$M_{11} = \bar{M}_{11} \quad \text{or} \quad \beta_1 = \bar{\beta}_1.$$

(2.104)

Static equations, Eqs. (2.96), dynamic equations, Eqs. (2.97) or (2.98), along with the boundary conditions, Eqs. (2.103) and (2.104), can describe completely the vibration of a shell under pressure.

2.8 Other Shell Theories as Special Cases

In this section, we present some equations, which were presented by other researchers and can be derived from the above equations as special case.

2.8.1 Shell Equations Presented by Sanders (1963) and Plaut et al. (2000)

In the derivation of the above governing equations, we used Sanders' shell theory with exact geometric nonlinearity in the in-plane strains, Eqs. (2.29) and Eqs. (2.36). From this, we now derive the nonlinear shell theory given by Sander (1963). He assumed small strains and moderately small rotations. Equations (2.36) can be approximated by assuming that the linear in-plane membrane strains (ε_1^o, ε_2^o, ε_3^o, and ε_4^o) are much smaller than the rotations (β_1, β_2, and β_n) to yield the following strain-displacement relations (Teng and Hong,1998):

$$\varepsilon_1^s = \varepsilon_1^o + \frac{1}{2}[(\beta_n)^2 + (\beta_1)^2],$$

$$\varepsilon_2^s = \varepsilon_2^o + \frac{1}{2}[(\beta_n)^2 + (\beta_2)^2],$$

$$\gamma_{12}^s = \varepsilon_3^o + \varepsilon_4^o + \beta_1 \beta_2 . \tag{2.105}$$

In Eqs. (2.105), the superscript *s* denotes the strain corresponding to Sanders' nonlinear shell theory. Also, his derivation was not for a shell under pressure. Hence, we will have to ignore the pressure terms, and prestress terms will be replaced by the vibratory stresses. Keeping these modifications in mind, one can follow the same steps as used in deriving Eqs. (2.97) to derive Sanders' nonlinear shell theory as (Sanders, 1963)

$$\frac{\partial (N_{11}A_2)}{\partial \alpha_1}+\frac{\partial (\widetilde{N}_{12}A_1)}{\partial \alpha_2}+\widetilde{N}_{12}\frac{\partial A_1}{\partial \alpha_2}-N_{22}\frac{\partial A_2}{\partial \alpha_1}+\frac{1}{R_1}\frac{\partial (M_{11}A_2)}{\partial \alpha_1}+\frac{1}{R_1}\frac{\partial (\widetilde{M}_{12}A_1)}{\partial \alpha_2}+\frac{\widetilde{M}_{12}}{R_1}\frac{\partial A_1}{\partial \alpha_2}$$
$$-\frac{M_{22}}{R_1}\frac{\partial A_2}{\partial \alpha_1}+\frac{A_1}{2}\frac{\partial}{\partial \alpha_2}\left[\widetilde{M}_{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]-\frac{A_1A_2}{R_1}(N_{11}\beta_1+\widetilde{N}_{12}\beta_2)-\frac{A_1}{2}\frac{\partial[(N_{11}+N_{22})\beta_n]}{\partial \alpha_2}$$
$$-A_1A_2(\rho h\ddot{u}_1-q_1)=0,$$

$$\frac{\partial (N_{22}A_1)}{\partial \alpha_2}+\frac{\partial (\widetilde{N}_{12}A_2)}{\partial \alpha_1}+\widetilde{N}_{12}\frac{\partial A_2}{\partial \alpha_1}-N_{11}\frac{\partial A_1}{\partial \alpha_2}+\frac{1}{R_2}\frac{\partial (M_{22}A_1)}{\partial \alpha_2}+\frac{1}{R_2}\frac{\partial (\widetilde{M}_{12}A_2)}{\partial \alpha_1}+\frac{\widetilde{M}_{21}}{R_2}\frac{\partial A_2}{\partial \alpha_1}$$
$$-\frac{M_{11}}{R_2}\frac{\partial A_1}{\partial \alpha_2}+\frac{A_2}{2}\frac{\partial}{\partial \alpha_1}\left[\widetilde{M}_{12}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)\right]-\frac{A_1A_2}{R_2}(N_{22}\beta_2+\widetilde{N}_{12}\beta_1)+\frac{A_2}{2}\frac{\partial(N_{11}+N_{22})\beta_n}{\partial \alpha_1}+$$
$$-A_1A_2(\rho h\ddot{u}_2-q_2)=0,$$

$$\frac{\partial}{\partial \alpha_1}\left[\frac{1}{A_1}\left(\frac{\partial (M_{11}A_2)}{\partial \alpha_1}+\frac{\partial (\widetilde{M}_{12}A_1)}{\partial \alpha_2}+\widetilde{M}_{12}\frac{\partial A_1}{\partial \alpha_2}-M_{22}\frac{\partial A_2}{\partial \alpha_1}\right)\right]+\frac{\partial}{\partial \alpha_2}\left[\frac{1}{A_2}\left(\frac{\partial (M_{22}A_1)}{\partial \alpha_2}+\frac{\partial (\widetilde{M}_{12}A_2)}{\partial \alpha_1}\right.\right.$$
$$\left.\left.+\widetilde{M}_{21}\frac{\partial A_2}{\partial \alpha_1}-M_{11}\frac{\partial A_1}{\partial \alpha_2}\right)\right]-\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)A_1A_2-\frac{\partial}{\partial \alpha_1}[A_2(N_{11}\beta_1+\widetilde{N}_{12}\beta_2)]-\frac{\partial}{\partial \alpha_2}[A_1(N_{22}\beta_2+$$
$$\widetilde{N}_{12}\beta_1)]-A_1A_2(\rho h\ddot{w}-q_3)=0.$$

(2.106)

and boundary conditions at the edge with constant α_2 are (Sanders,1963)

$$N_{22}=\overline{N}_{22} \qquad \text{or,} \qquad u_2=\overline{u}_2,$$
$$\widetilde{N}_{12}+\left(\frac{3}{2R_1}-\frac{1}{2R_2}\right)\widetilde{M}_{12}-(N_{11}+N_{22})\frac{\beta_n}{2}=\overline{N}_{21}+\frac{\overline{M}_{21}}{R_1} \qquad \text{or,} \qquad u_1=\overline{u}_1,$$
$$Q_{23}+\frac{1}{A_1}\frac{\partial \widetilde{M}_{12}}{\partial \alpha_1}=\overline{Q}_{23}+\frac{1}{A_1}\frac{\partial \overline{M}_{21}}{\partial \alpha_1} \qquad \text{or,} \qquad w=\overline{w},$$
$$M_{22}=\overline{M}_{22} \qquad \text{or,} \qquad \beta_2=\overline{\beta}_2.$$

(2.107)

Boundary conditions on the edge with constant α_1 can be written by interchanging suffices *1* and *2*. The equations for transverse shear stress resultants Q_{13} and Q_{23} remain the same as given in Eqs. (2.94). If we drop the terms containing β_n in Eqs. (2.106) and change the vibratory stresses in nonlinear terms to the prestresses, we get the equations of motion presented by Plaut et al. (2000). The static equations presented by Plaut et al. (2000) are the same as Eqs. (2.96).

2.8.2 Shell Equations Presented by Soedel (1986)

To derive the nonlinear theory for a shell under prestresses given by Soedel (1986), we neglect β_n compared to β_1 and β_2, and ignore all the squared terms involving u_1 and u_2 compared to w in Eqs. (2.105). This gives

$$\varepsilon_1^d = \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{w}{R_1} + \frac{1}{2A_1^2}\left(\frac{\partial w}{\partial \alpha_1}\right)^2,$$

$$\varepsilon_2^d = \frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{w}{R_2} + \frac{1}{2A_1^2}\left(\frac{\partial w}{\partial \alpha_1}\right)^2,$$

$$\gamma_{12}^d = \frac{1}{A_1}\frac{\partial u_2}{\partial \alpha_1} - \frac{u_1}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{1}{A_2}\frac{\partial u_1}{\partial \alpha_2} - \frac{u_2}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{1}{A_1 A_2}\frac{\partial w}{\partial \alpha_1}\frac{\partial w}{\partial \alpha_2}. \qquad (2.108)$$

The bending and the torsion strains for this case are given by Eqs. (2.29). These strains are the same as those given by Donnell (1934) and hence the superscript *"d"*. In deriving the equations presented by Soedel, we will have to also ignore the changes in definitions due to Sanders. This leads to the following equations (Soedel, 1986):

$$\frac{\partial (N_{11}A_2)}{\partial \alpha_1} + \frac{\partial (N_{21}A_1)}{\partial \alpha_2} + N_{12}\frac{\partial A_1}{\partial \alpha_2} - N_{22}\frac{\partial A_2}{\partial \alpha_1} + \frac{1}{R_1}\frac{\partial (M_{11}A_2)}{\partial \alpha_1} + \frac{1}{R_1}\frac{\partial (M_{21}A_1)}{\partial \alpha_2} + \frac{M_{12}}{R_1}\frac{\partial A_1}{\partial \alpha_2}$$
$$- \frac{M_{22}}{R_1}\frac{\partial A_2}{\partial \alpha_1} - A_1 A_2(\rho h \ddot{u}_1 - q_1) = 0,$$

$$\frac{\partial(N_{22}A_1)}{\partial\alpha_2}+\frac{\partial(N_{12}A_2)}{\partial\alpha_1}+N_{21}\frac{\partial A_2}{\partial\alpha_1}-N_{11}\frac{\partial A_1}{\partial\alpha_2}+\frac{1}{R_2}\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+\frac{1}{R_2}\frac{\partial(M_{12}A_2)}{\partial\alpha_1}+\frac{M_{21}}{R_2}\frac{\partial A_2}{\partial\alpha_1}$$
$$-\frac{M_{11}}{R_2}\frac{\partial A_1}{\partial\alpha_2}-A_1A_2(\rho h\ddot{u}_2-q_2)=0,$$

$$\frac{\partial}{\partial\alpha_1}\left[\frac{1}{A_1}\left(\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(M_{21}A_1)}{\partial\alpha_2}+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_{22}\frac{\partial A_2}{\partial\alpha_1}\right)\right]+\frac{\partial}{\partial\alpha_2}\left[\frac{1}{A_2}\left(\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+\frac{\partial(M_{12}A_2)}{\partial\alpha_1}\right.\right.$$
$$\left.\left.+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_{11}\frac{\partial A_1}{\partial\alpha_2}\right)\right]-\left(\frac{N_{11}}{R_1}+\frac{N_{22}}{R_2}\right)A_1A_2+\frac{\partial}{\partial\alpha_1}[N_{11}^r\frac{A_2}{A_1}\frac{\partial w}{\partial\alpha_1}+N_{21}^r\frac{\partial w}{\partial\alpha_2}]$$
$$+\frac{\partial}{\partial\alpha_2}[N_{22}^r\frac{A_1}{A_2}\frac{\partial w}{\partial\alpha_2}+N_{12}^r\frac{\partial w}{\partial\alpha_1}]-A_1A_2(\rho h\ddot{w}-q_3)=0. \tag{2.109}$$

and boundary conditions at the edge with constant α_2 is (Sanders,1963)

$$N_{22}=\overline{N}_{22} \quad \text{or} \quad u_2=\overline{u}_2,$$
$$N_{21}+\frac{M_{21}}{R_1}=\overline{N}_{21}+\frac{\overline{M}_{21}}{R_1} \quad \text{or} \quad u_1=\overline{u}_1,$$
$$Q_{23}+\frac{1}{A_1}\frac{\partial M_{21}}{\partial\alpha_1}=\overline{Q}_{23}+\frac{1}{A_1}\frac{\partial\overline{M}_{21}}{\partial\alpha_1} \quad \text{or} \quad w=\overline{w},$$
$$M_{22}=\overline{M}_{22} \quad \text{or} \quad \beta_2=\overline{\beta}_2. \tag{2.110}$$

The transverse shear stress resultants, Q_{13} and Q_{23}, are given by the following equations:

$$Q_{13}=\frac{1}{A_1A_2}\left(\frac{\partial(M_{11}A_2)}{\partial\alpha_1}-M_{22}\frac{\partial A_2}{\partial\alpha_1}+M_{12}\frac{\partial A_1}{\partial\alpha_2}+\frac{\partial(M_{21}A_1)}{\partial\alpha_2}\right),$$
$$Q_{23}=\frac{1}{A_1A_2}\left(-\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+M_{11}\frac{\partial A_1}{\partial\alpha_2}-M_{21}\frac{\partial A_2}{\partial\alpha_1}-\frac{\partial(M_{12}A_2)}{\partial\alpha_1}\right) \tag{2.111}$$

The static equations presented by Soedel (1986) for a pure membrane are the same as given in Eqs. (2.96).

2.9 Conclusions

This chapter presented the foundation of this research by deriving the governing equations for vibration of a shell under pressure. The chapter presented almost all the necessary elements needed for deriving the governing equations. The basic theorems of surface, called Gauss-Codazzi conditions, were presented. Then, three-dimensional stress-strain relations and definitions of the two-dimensional stress resultants and stress couples were presented. Thereafter, nonlinear strain-displacement relations and Hamilton's principle were given. In order to obtain the final equations, variations of strain energy, kinetic energy, and work done by external and boundary forces were derived. Using a variational principle, the static and dynamic equations along with boundary conditions were obtained. These equations were derived before by Budiansky (1968) using tensors. However, the derivations of this chapter used line-of-curvature coordinates, which is relatively easier to follow. These equations were then specialized to obtain some other related equations used in the literature for the vibration of a shell. We showed the simplification procedure needed to obtain these approximate equations. Apart from these simplifications, these approximate equations do not contain the follower actions of the pressure force.

In Chapter 3, these equations will be used in deriving the actuator forces and the equations of motion in the presence of piezoelectric patches. In Chapter 4, we will use these equations to solve the free vibration problem of an inflated torus.