A Spatial Dynamic Approach to Three-Dimensional Gravity-Capillary Water Waves

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(ABSTRACT)

Three-dimensional gravity-capillary steady waves on water of finite-depth, which are uniformly translating in a horizontal propagation direction and periodic in a transverse direction, are considered. The exact Euler equations are formulated as a spatial dynamic system in which the variable used for the propagating direction is the time-like variable. The existence of the solutions of the system is determined by two non-dimensional constants: the Bond number b and λ (the inverse of the square of the Froude number). The property of Sobolev spaces and the spectral analysis show that the spectrum of the linear part consists of isolated eigenvalues of finite algebraic multiplicity and the number of purely imaginary eigenvalues are finite. The distribution of eigenvalues is described by b and λ. Assume that C_1 is the curve in (b, λ)-plane on which the first two eigenvalues for three-dimensional waves collide at the imaginary axis, and that the intersection point of the curve C_1 with the line $\lambda = 1$ is $(b_0, 1)$ where $b_0 > 0$. Two cases $(b_0, 1)$ and $(b, \lambda) \in C_1$ where $0 < b < b_0$, are investigated. A center-manifold reduction technique and a normal form analysis are applied to show that for each case the dynamical system can be reduced to a system of ordinary differential equations with finite dimensions. The dominant system for the case $(b_0, 1)$ is coupled Schrödinger-KdV equations while it is a Schrödinger equation for another case $(b, \lambda) \in C_1$. Then, from the existence of the homoclinic orbit connecting to the two-dimensional periodic solution (called generalized solitary wave) for the dominant system, it is obtained that such generalized solitary wave solution persists for the original system by using the perturbation method and adjusting some appropriate constants.

Dedication

This dissertation is dedicated first and foremost to my wife Yan Yang. Her tolerance and determination to assume the housework and take care of Felix, our wonderful son, while I devoted myself to this research, amazes me. Also, to my parents, my brothers' and my sisters' families, I owe a great debt of gratitude. I look forward to returning my favor and my love! Thank you.

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Chapter 1

Introduction

The problem of water waves has been studied for at least more than a century because of their practical importance and because they offer an ideal setting for a variety of phenomena of the nonlinear wave motions. Here, by water, we mean that the fluid is incompressible, inviscid and the flow is irrotational. The full equation governing the motion of water waves is the Laplace equation subject to the nonlinear boundary conditions. There are three main methods to deal with them: numerical computations, formal approximations without rigorous proofs, and the analytical study of the solutions of the equations which include the effects neglected by approximate models. In my thesis, I only focus on the analytical study of water waves.

Since the free surface and the nonlinear boundary conditions cause many difficulties when water waves are analyzed theoretically, the approximate models become very important and are always studied first. There are two models: a linear model and a weakly nonlinear model (see the book by Whitham [117]). The basic idea of them can be summarized as follows. All functions are expanded in powers of some small parameters (for example, the parameters are chosen as the ratio of typical wavelength and mean depth or the ratio of typical amplitude and mean depth) which are then plugged into the governing equations and the boundary conditions. A sequence of equations of successive orders can be obtained. After a few steps, the dominant equations can be derived and discussed, such as the Korteweg-de Vries (KdV) equations, the Boussinesq equations, the nonlinear Schrödinger equations, Kadomtsev-Petviashvili (KP) equations, the Davey-Stewartson (DS) equations or their variants.

The analytic study of water waves has been started for a long time. Here we only concentrate on the travelling waves (steady waves). It mainly includes the following aspects (see the review by Dias & Iooss [36]):

1. Existence proofs for the travelling waves based on methods of local analysis using the implicit function theorem

Here "local" means that only the solution set in a small neighborhood of the trivial solution is considered. The complex potential function and conformal mappings are used to reduce the problem to the existence of an analytic function satisfying nonlinear boundary conditions and then the implicit function theorem is applied to give the existence of nontrivial solutions under some conditions. Because of the unknown free boundary, which is a part of the solution, an idea to overcome this difficulty is to use the compltex potential function or the streamfunction as the independent variable (see Nekrasov [90] and Levi-Civita [75]). However, this method can not be applied to the three-dimensional case since the complex potential function or the streamfunction can not be used and the free boundary condition can not be transformed into a simple form.

2. Existence proofs for the steady waves based on variational formulations

   The first contribution to the study of water waves using variational formulations was made by Luke [79], who published a formal variational principle that recovers the water wave equations. Then there has been a great deal of activities concerning variational formulations of exact and model water wave equations (for example, Groves & Toland [57]). The variational formulations can be exploited in two ways: (i) it may be possible to use the direct methods of the calculus of variations to find critical points of the variational functional and hence solutions of the problem. This approach is used in the existence theories for doubly periodic waves by Craig & Nicholls [33] and fully localized solitary waves by Groves & Sun [56]; (ii) the system is formulated as an (typically ill-posed) evolutionary equation where the variable used for the propagating direction is chosen as a time-like variable. Then the Hamiltonian structure of center manifolds introduced by Mielke [83] is applied so that the original problem is changed to a system of ordinary differential equations with finite dimensions. Groves & Mielke [55] applied this method to prove the existence of the generalized solitary waves. This variational formulation method is very powerful because it can be applied to not only two-dimensional but also three-dimensional water wave problems.

3. Existence of solutions of steady water wave problems by directly transforming the governing equations into a spatial dynamical system without taking advantage of the properties of the Hamiltonian system

   There are two main methods for performing such a change of coordinates, which are much more general since the Hamiltonian structure is not needed. One is given by Levi-Citiva [75]. The new unknown is $\alpha+i\beta$ as an analytic function of $w=\phi+i\psi$ where ϕ and ψ are the velocity potential and the stream function, and $w'(x+iy)=e^{(\beta-i\alpha)}$. The function α represents the angle of the streamline with the horizontal, while β represents the logarithm of the velocity modulus. Then ϕ and ψ are regarded as the independent variables. The main advantage of this method is that it leads to a weakly nonlinear problem, instead of a true quasi-linear partial differential equation problem. Dias & Iooss [36] gave a review on the results by applying this method. For example, Iooss and Kirrmann [67] investigated capillary-gravity waves with infinite depth and Iooss [63] studied the gravity and capillary-gravity periodic travelling waves for two superposed fluid layers, one being of infinite depth. Another method is developed by Kirchgässner [72]. He used the intermediate choice instead of using the stream function or the velocity potential. The propagation direction is chosen as the time-like variable and the center manifold reduction is used to change the original water wave problem to a system of ordinary differential equations with finite dimensions. He gave the bifurcation curve (see Figure 1.1) described by the Bond number and the Froude number and discussed the bifurcations under the small external pressure. Iooss and Kirchgässner [66] adopted this method and showed the existence of solitary waves with oscillations at infinity (called generalized solitary waves) for the two-dimensional water waves when the Froude number is close to 1 and the Bond number is less than 1/3. Dias & Iooss [36] also gave the spatial dynamic approach for three-dimensional water waves, which will be used here.

4. Global results about periodic waves and solitary waves

   The first proof of the existence of large amplitude, periodic water waves with a wave-like bottom is due to Krasovakii [73] and is based on an adaptation of the monotone minorant theorem to a particular version of Nekrasov's equation. Keady and Norbury [70] extended the Krasovskii's set of periodic waves using a global bifurcation theorem. Then, Amick & Toland [5] gave a global theory on bifurcation of large amplitude periodic water waves. They proved that if the mean depth h is fixed, then as $\lambda \to \infty$ the connected sets of periodic waves of wavelength λ on a flow of mean depth h converge, in a certain sense, to a connected set of solitary waves whose asymptotic height is h. In the same year, using the complex potential and the conformal map, they [6] proved the existence of solitary water waves of all amplitudes, from zero up to and including that of the solitary wave of greatest height, each of which has Froude number greater than one, and decays exponentially at infinity. Toland [109], Amick, Fraenkel & Toland [7], and McLeod [81] investigated the largest amplitude solution. Buffoni, Dancer & Toland [25] gave a significant new contribution on this problem and showed a very complicated structure of secondary bifurcation near the solution of largest amplitude using a variational method. Sun [106] derived an integral formulation for two-dimensional periodic travelling gravity waves in two fluids without boundaries and then gave a mathematical proof of the existence of periodic waves of small and large amplitude using the global bifurcation theory. Constantin & Strauss [30] recently considered a more general version of the hydrodynamic problem in which vorticity effects are included. They essentially presented the rotational counterpart of Keady and Norbury's result [70] for a general class of vorticity functions.

5. Stability and instability results

   From the existence we mentioned above, there naturally comes a mathematical problem: stability and instability of the solutions. Especially, we are interested in stability and instability of two kinds of solutions: solitary waves with surface tension and without surface tension; Stoke waves which are periodic waves without surface tension. Phenomenologically, solitary waves appear to be stable whereas Stoke waves are stable only for large enough wavelengths.

(i) Stoke waves. At some critical finite wavelength, the periodic waves of sufficiently small amplitude destabilize, an instability mechanism first discovered in Benjamin[12], Benjamin & Feir [13] and Whitham [116] and known as the Benjamin-Feir instability. However, they did not provide a mathematical proof of the instability. Bridges & Mielke [20] gave a rigorous proof by using a Hamiltonian center-manifold analysis reducing the linear stability problem to an ordinary differential eigenvalue problem. Constantin & Strauss [31] recently studied the stability properties with vorticity and Francius & Kharif [45] considered three-dimensional instabilities in shallow water.

(ii) Solitary waves. Groves, Haragus & Sun [53] showed that the travelling line solitary wave corresponding to the three-dimensional gravity-capillary water wave for large surface tension is linearly unstable (called transverse instability) under spatially inhomogeneous perturbations (three-dimensional perturbation) which are periodic in the direction transverse to the propagation and have long wavelength. It roughly means that the linearization about this solitary wave has spatially bounded solutions, periodic in the transverse direction and localized in the propagation, which grow exponentially in time. Then Pego & Sun [94] gave the same result with three-dimensional perturbations which oscillate along the wave crest with wavenumber in a finite band. Haragus and Scheel [60] also gave the finite-wavelength spectral stability of this solitary wave solution with a two-dimensional perturbation of finite wave-number. Roughly speaking, the real part of the spectrum of the linear operator corresponding to the linearization about the solitary wave is not positive. However, they did not obtain full spectral stability as the wavelength tends to infinity. This result had been enhanced by Mielke [84] in which a nonlinear energetic stability result is obtained under the assumption that the solution exists on a large closed interval. Then Buffoni [24] presented the conditional energetic stability of gravity solitary waves in the presence of weak surface tension.

6. Existence and uniqueness (or non-existence) of solutions to the Cauchy problem for water waves

The fundamental question for water waves is to ask in which function space of data the initial value problems are well posed, and how long one can guarantee a time of existence before the solution breaks, or at least diverges to infinity in the specified topology. There are two kinds of results.

(i) Without surface tension. The earliest mathematical results on the well-posedness of the two-dimensional irrotational water wave problem in an infinitely deep basin with small data in some Sobolev space were given by Nalimov [91]. Then Wu [118, 119] generalized it for general data in two or three dimensions. For the general problem, Christodoulou and Lindblad [28] first obtained the energy estimates based on the geometry of the moving domain with the Rayleigh-Taylor sign condition for rotational flows and Lindblad [76] proved existence of its solutions while Ebin [43] showed that the problem is ill-posed in the absence of this condition.

(ii) With surface tension. Yosihara [120], Iguchi [62] and Ambrose [2] solved the well-posedness irrotational problem in two dimensions under varying assumptions on the initial data. Schweizer [101] proved existence of the general three-dimensional problem and Ambrose & Masmoudi [3] showed that solutions of the two-dimensional irrotational problem converges to solutions of the zero surface problem by writing the equation in terms of the arc length of the fluid boundary as the surface tension goes to zero. Recently, Shatah & Zeng [102] derived estimates to the free boundary problem for the Euler equation with surface tension, or without surface tension provided that Rayleigh-Taylor sign condition holds, and proved that as the surface tension tends to 0, solutions converge to the Euler flow with zero surface tension and Rayleigh-Taylor sign condition. Ambrose & Masmoudi [4] investigated the well-posedness of the initial value problem for a 3D vortex sheet in the presence of surface tension. Coutand & Shkoller [32] considered two and three-dimensional fluids which are not required to be irrotational, and are supposed to be immersed in a vacuum with or without surface tension.

In the following, we discuss in more details the solitary waves in fluids which are incompressible, inviscid and irrotational. In 1834, Russell observed a solitary wave in a canal between Edinburgh and Glasgow. Then, he made some experiments, and in 1844 he published his observations [98]. There was a controversy if such a wave of a permanent homoclinic form could exist if a dissipation was neglected. Airy [1] argued that the dispersion would destroy this wave. It was accepted that such waves exist when Boussinesq [18] and Rayleigh [96] found approximations to such a wave by using a perturbation analysis. Friedrichs and Hyers [46] showed rigorously the existence of solitary waves for two-dimensional water waves without surface tension.

For the capillary-gravity water waves with finite depth, the existence of the solutions of the system is determined by two important non-dimensional constants, the Bond number b and the Froude number F (let $\lambda = F^{-2}$), which in turn give the number of eigenvalues on the imaginary axis of the complex plane for the corresponding linearized operator around a uniform flow.

The distribution of the eigenvalues for two-dimensional capillary-gravity water waves is given by Figure 1.1. If $\lambda \neq 1$ and $b \neq \frac{1}{3}$, a stationary KdV equation can be formally derived while for λ close to 1 and b close to $\frac{1}{3}$, the derivation of the stationary KdV equation fails and a fifth order ordinary differential equation appears and is considered as a perturbed stationary KdV equation. Amick & Kirchgässner [8] used the complex potential as the independent variable and the implicit function theorem to look for solitary waves near the curve $E_r \backslash \{(\frac{1}{3}, 1)\}$. They found a solitary wave of depression just above this curve and

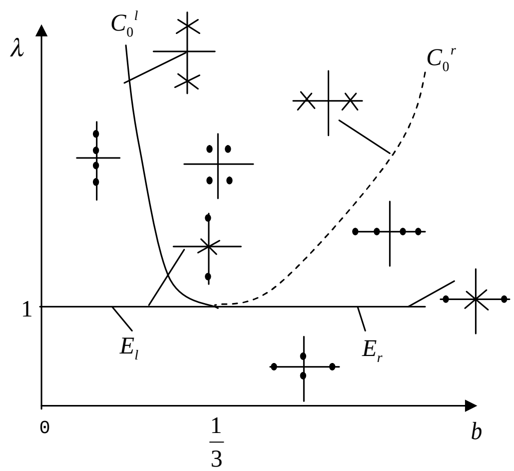

Figure 1.1: The curves C_0^l, E_l, E_r consist of points in (b, λ)-parameter space at which the number of purely imaginary eigenvalues of linearized problem changes; at C_0^r four nonzero, real eigenvalues become complex without passing through zero. Dots and crossed denote respectively simple and double eigenvalues.

proved it to be unique. Buffoni, Groves & Toland [27] studied the parameter region to the left of the curve C_0^r and near the point $(\frac{1}{3}, 1)$. They applied variational formulations and a center manifold technique and changed the problem into a system of ordinary differential equations with dimension 4. Then they used the Hamiltonian structure and proved the existence of infinitely many distinct solitary wave solutions. These solutions are waves of depression which have an arbitrary but finite number of troughs, between which there are distributed smaller local maxima and minima, and which have an exponentially decaying oscillatory tail at infinity. For the region that lies the right side of $C_0^l \backslash \{(\frac{1}{3}, 1)\}$, Iooss & Kirchgässner [65] first discussed it. They showed that in this region, there are solitary waves corresponding to homoclinic solutions of a fourth-order, reversible dynamical system. They applied a normal form analysis and demonstrated that at each order the normal form admits two reversible homoclinic solutions provided certain coefficients have the correct signs. Dias & Iooss [35] calculated the coefficients in question and thereby verified the condition on their signs. Iooss & Pérouème [68] completed the existence proof by showing that the reversible homoclinic solutions of the normal form persist for the full dynamical system. Buffoni & Groves [26] also studied this case by using variational formulations and the center manifolds. They showed that there are infinitely many geometrically distinct modulated solitary waves associated with critical points of the water wave problem. Notice that by moving sufficiently high up along the curve C_0^l one can treat the water of arbitrarily large depth. The limiting case (infinite depth) has been considered by Iooss & Kirmann [67], who obtained the solitary waves decaying algebraically rather than exponentially to zero at infinity. Beale [11], Sun [105], Iooss & Kirchgässner [66], and Lombardi [78] used different methods and gave the existence of generalized solitary waves which have an oscillatory tail of exponentially small amplitude at infinity just below the curve $E_l \backslash \{(\frac{1}{3}, 1)\}$. Beale [11] used the complex velocity and Sun [105] regarded the streamfunction as the independent variable, and then both considered the existence of solutions of the corresponding partial differential equations. Iooss & Kirchgässner [66] applied the spatial dynamic approach and the center manifold technique which changed it into a system of ordinary differential equations. By the normal form analysis, they proved the persistence of the homoclinic orbit decaying the periodic solution for the normal form, which corresponds to the generalized solitary water wave. Lombardi [78] studied the general system in the Hilbert space with the fixed point theorem and applied the result to the water wave problem. Sun & Shen [107] considered the streamfunction as the independent variable which changes the problem into the existence of solutions of partial different equations, and by the fixed point theorem, proved the existence of solitary wave solutions for λ is greater than but close to 1 and the Bond number b is greater than but close to $\frac{1}{3}$ from some direction. Finally, Mielke, Holmes & O'Reilly [89] discussed the possibility of the chaotic behavior when $b < \frac{1}{3}$ and λ is close to 1, but found none (see Mielke [88]). Therefore, the existence of two-dimensional waves near the bifurcation curves $C_0 = C_0^r \bigcup C_0^l$ and $\lambda = 1$ has been proved rigorously. All these results and other results for other two dimensional water wave problems can be found in the review [36] by Dias & Iooss.

Rigorous treatments for three-dimensional travelling water waves have been developed only recently (see the survey by Groves [51]). Craig & Nicholls [33] considered the three-dimensional travelling gravity water waves periodic in two horizontal directions and showed the existence of the doubly periodic water waves while Groves & Sun [56] discussed fully localized solitary waves of three-dimensional gravity-capillary water waves. In 2002, Groves, Haragus & Sun [54] showed the dimension-breaking bifurcation of two-dimensional solitary waves for large surface tension by using variational formulations and the Lyapunov centre theorem. Later, Groves & Haragus [52] considered the water wave problem by choosing an arbitrary horizontal spatial direction as the time-like variable, and Groves [50] investigated the water waves periodic only in the propagate direction with variational formulations and the center manifold reduction. Haragus & Kirchgässner [59] adopted a different method without variational formulations−−the spatial dynamic system to study both cases: periodic in the transverse direction and periodic in the propagation direction. In particular, Groves & Mielke [55] considered the water waves periodic in the transverse direction y with period P. They used variational formulations and obtained the spatial dynamic system with the Hamiltonian structure. By the spectral analysis, the distribution of the eigenvalues is given in Figure 1.2 by (b, λ)-plane where the eigenvalues σ of the linearized operator satisfy

$$(\lambda - b\tau_k^2)\tau_k \sin\tau_k - \sigma^2 \cos\tau_k = 0 \qquad (\lambda = F^{-2}) \tag{1.1}$$

with $\tau_k^2 = \sigma^2 - \frac{4\pi^2 k^2}{P^2}$ for some nonnegative integer k. Figure 1.2 gives the values of (b, λ) at which the number of purely imaginary eigenvalues changes, where C_k is the curve corresponding to the wave number k in the y-direction with $y \in [0, P]$ (i.e., the wave is periodic in y with a period $P_1 = P/k$). The solid dots correspond to two-dimensional eigenvalues

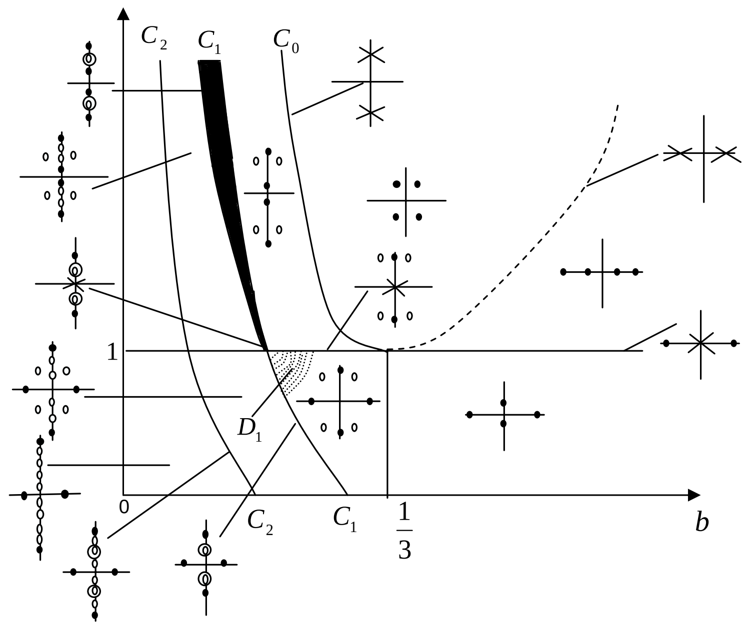

Figure 1.2: The solid and hollow dots represent two-dimensional and three-dimensional eigenvalues, respectively. Crosses are double two-dimensional eigenvalues, double circles are quadruple three-dimensional eigenvalues. The curves $C_0, C_1, C_2, \cdots$ consist of points in (b, λ)-parameter space at which four purely imaginary eigenvalues become complex without passing through zero. The dashed curve consists of points at which four two-dimensional real eigenvalues become complex without passing through zero.

(whose eigenfunctions are independent of y), which are solutions of (1.1) with $k = 0$, while the hollow dots denote three-dimensional eigenvalues (whose eigenfunctions are dependent on y), which are solutions of (1.1) with $k \neq 0$. Since three-dimensional steady waves are invariant under the reflection $y \to -y$, we may restrict to the waves which are even in the y-direction and have Fourier cosine series expansions. Then, the three-dimensional eigenvalues are geometrically simple, while the purely imaginary eigenvalues are also algebraically simple if (b, λ) stays away from the curves $C_k, k = 0, 1, 2, \ldots$ and $\{\lambda = 1\}$. The existence of the curves $C_k, k = 0, 1, 2, \ldots$ and the line $\lambda = 1$ is associated with bifurcation phenomena. If we let the intersection point of C_k with the line $\lambda = 1$ be b_k, then Figure 1.2 shows that for $(b, \lambda) \in C_k$, the number of eigenvalues is different for $b > b_k$ (denoted by C_k^-) or $0 < b < b_k$ (denoted by C_k^+). Here we note that the curve C_k depends upon the period P. If we let $C_k = C_k(P)$, from (1.1) two curves $C_k(P)$ and $C_1(P_0)$ are same where $P_0 = P/k$. Therefore, to discuss three-dimensional solutions with a period P in y bifurcating from the zero solution for (b, λ) near C_k, we may just study the solutions near the bifurcating curve C_1 with a different period $P_0 = P/k$. Groves & Mielke [55] proved that when (b, λ) is near

C_1^- from the right side there exists a three-dimensional generalized solitary wave which is periodic in the transverse direction and decays to a two-dimensional periodic wave at infinity in the propagation direction.

We are here interested in another half C_1^+ of C_1 and the critical point $(b_0, 1)$ which is the intersection point of the curve C_1 and the straight line $\lambda = 1$ (see Figure 1.2). We directly use the spatial coordinates to change the governing system to a spatial dynamic system where the variable used for the propagating direction is the time-like variable instead of applying variational formulations and the properties of a Hamiltonian system by Groves & Mielke [55]. The spectral analysis yields that the eigenvalues of the linear part are isolated and the number of the pure imaginary eigenvalues is finite, which is same as the one given by Groves & Mielke [55]. Then the center manifold reduction shows that the dynamical system can be reduced to a system of ordinary differential equations with finite dimensions.

(i) For (b_1, λ_1) on the curve C_1^+, the purely imaginary part of the spectrum consists of two-dimensional eigenvalues 0 which can be deleted, $\pm is_{00}, \pm is_{01}$, and three-dimensional eigenvalues $\pm is_1$. Here, $\pm is_{00}$, $\pm is_{01}$ are simple and $\pm is_1$ are double. When (b, λ) is non-tangentially close to (b_1, λ_1) from the right side, the normal form analysis gives that the dominant system has a subsystem of Schrödinger equations corresponding to three-dimensional eigenvalues $\pm is_1$. Following the idea by Groves & Mielke [55], we first find a homoclinic orbit approaching the origin for the subsystem. Then using Lyapunov-Schmidt method, we prove that the whole system has a two-dimensional periodic solution with a small amplitude, which determines the form of the generalized solitary wave solution at infinity. Finally, adjusting the Bernoulli constant, using the perturbation theorem and the fixed point theorem show that such homoclinic orbit for the subsystem persists when higher order terms and other equations are included, i.e., there exists a three-dimensional generalized solitary water wave which is periodic in the transverse direction y and approaches a two-dimensional periodic water wave at infinity in the propagation direction x. Let $\lambda = \lambda_1 + \mu$ where $\mu > 0$ is small. The free surface η has the following form (see Figure 1.3)

$$\sqrt{\mu}\text{sech}(\sqrt{q_1\mu}x)\cos(s_1 x)\cos(2\pi y/P) + O(\mu e^{-\sqrt{q_1\mu}|x|}) + \varsigma(x)S_1^{(p)}(x) + O(\mu^{5/2}e^{-r\sqrt{\mu}|x|})$$

where q_1 and r are nonzero positive constants, $\varsigma(x)$ is a smooth even cut-off function with $\varsigma(x) = 0$ for $|x| \leq 1$ and $\varsigma(x) = 1$ for $|x| \geq 2$, and $S_1^{(p)}(x)$ is of order $O(\mu^5)$. Here, $\sqrt{\mu}\text{sech}(\sqrt{q_1\mu}x)\cos(s_1 x)\cos(2\pi y/P)$ and $S_1^{(p)}(x)$ correspond to the homoclinic orbit of the normal form and the periodic solution respectively.

(ii) For the case $(b_0, 1)$, the purely imaginary part of the spectrum consists of a zero eigenvalue with Jordan chain of length 4 which can be reduced to 2, two-dimensional eigenvalues $\pm is_{20}$ which are simple and three-dimensional eigenvalues $\pm is_{10}$ which are double. The basic idea to prove the existence of a generalized solitary water wave solution is same as one for the case C_1^+ when (b, λ) is close to $(b_0, 1)$ from the region D_1^- (see Figure 1.2). The main difference is that the dominant system here has a subsystem of coupled Schrödinger-KdV equations corresponding to the eigenvalue 0 and the three-dimensional eigenvalues $\pm is_{10}$

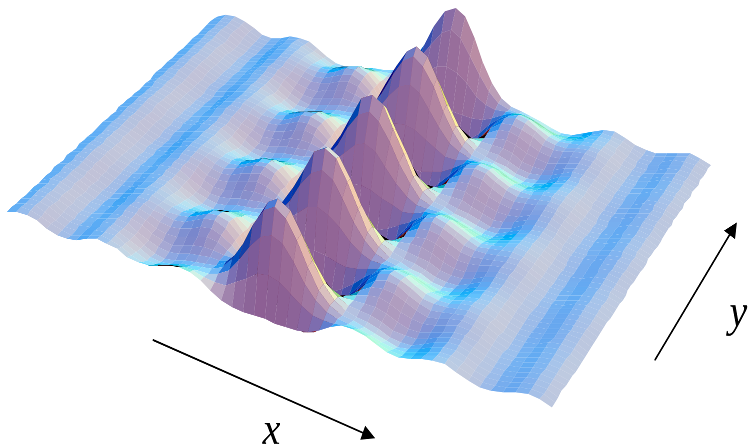

Figure 1.3: A three-dimensional generalized solitary water wave, periodic in the y-direction, approaches a two-dimensional periodic water wave in the x-direction (also see Figure 2 in [55]). The corresponding dominant system is the Schrödinger equation.

while the dominant one in the case (i) and Groves & Mielke's paper [55] has a Schrödinger equation. In order to get a homoclinic orbit for this subsystem, we activate the period $P+\alpha$ in the transverse direction and consider α as an extra small parameter besides the Bernoulli constant. Moreover, the generalized solitary water wave solution bifurcates from a nontrivial solution while one in the case (i) and Groves & Mielke's paper [55] bifurcates from a trivial solution 0. Let $b = b_0 + \mu, \lambda = 1 - k_2\mu$ and $\alpha = k_3\mu$ where $\mu > 0$ is small and $k_2 > 0, k_3$ are constants. The free surface η has the following form (see Figure 1.4)

$$\mu d + \mu \mathrm{sech}^2(\sqrt{c\mu}x) + \mu \mathrm{sech}(\sqrt{c\mu}x)\tanh(\sqrt{c\mu}x)\cos(s_{10}x)\cos(2\pi y/P) \\ + O(\mu^{3/2}(1 + e^{-\sqrt{c\mu}|x|})) + \varsigma(\sqrt{\mu}x)S_2^{(p)}(\sqrt{\mu}x) + O(\mu^{23/16}e^{-\gamma\sqrt{\mu}|x|})$$

where c and γ are nonzero positive constants and $S_2^{(p)}(x)$ is of order $O(\mu^{11/4})$. Here μd, $\mu \mathrm{sech}^2(\sqrt{c\mu}x)$ $+\mu \mathrm{sech}(\sqrt{c\mu}x)\tanh(\sqrt{c\mu}x)\cos(s_{10}x)\cos(2\pi y/P)$ and $S_2^{(p)}(\sqrt{\mu}x)$ correspond to the equilibrium and the homoclinic orbit of the normal form and the periodic solution respectively.

This thesis is organized as follows. Chapter 2 gives the basic hydrodynamics (see Section 2.1) and then introduces the governing equations for the water waves and the spatial coordinates which change the governing equations to a spatial dynamic system (see Section 2.2). Some definitions are also given. In Chapter 3, the properties of the linear operators and the reduced ordinary differential equations are obtained. Section 3.1 gives the resolvent

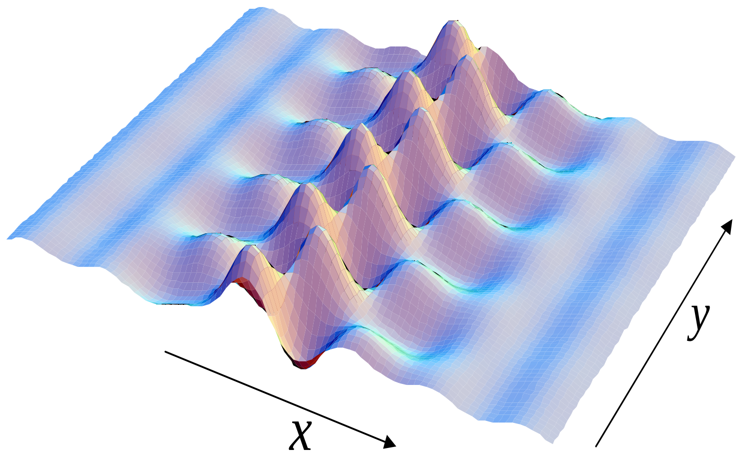

Figure 1.4: A three-dimensional generalized solitary water wave, periodic in the y-direction, approaches a two-dimensional periodic water wave in the x-direction. The corresponding dominant system is the coupled Schrödinger-KdV equations.

estimates of the linear operators and their spectra. A center manifold reduction in Section 3.2 is applied, which changes the dynamic system to a system of ordinary differential equations with finite dimensions. Section 3.3 shows the distribution of eigenvalues of the linear operators. Chapter 4 discusses the case (b, λ) close to the curve C_1^+ and yields the existence of three-dimensional generalized solitary water waves. In Section 4.1, the normal form of the system of ordinary differential equations is given. Section 4.2 presents the two-dimensional periodic solution by using the Lyapunov-Schmidt method and Section 4.3 proves the existence of generalized solitary water waves. The case (b, λ) close to $(b_0, 1)$ from D_1^- is investigated in Chapter 5. Section 5.1 gives the normal form. The scaling in Section 5.2 is applied and the equilibrium is found in Section 5.3. The homoclinic orbit for the normal form in Section 5.4 is obtained by activating the parameter α. Again the Lyapunov-Schmidt method yields the two-dimensional periodic solution in Section 5.5. Section 5.6 yields the existence of three-dimensional generalized solitary water waves following the idea in Section 4.3. Chapter 6 gives the proofs of some lemmas and calculations of some coefficients in Chapter 4 and Chapter 5. Future work is presented in Chapter 7.

Chapter 2

Formulation as a Spatial Dynamic System

2.1 Basic Hydrodynamics

The mathematical theory of water wave motion with a free surface and subjected to gravitational and other forces, together with applications to a wide variety of concrete physical problems, has been studied for more than a century. Here, we do not take into account the effects of viscosity and compressibility. As a consequence of the neglect of internal friction, or in other words of neglect of shear stresses, it is well known that the stress system is a state of uniform compression at each point. The intensity of the compressive stress is called the pressure p. Suppose that the positive z-axis is upward, the x, y-plane is horizontal (usually it will be taken as the undisturbed water surface) and the velocity is $\underline{v} = (u, v, w)$. The equations of motion in terms of the Euler variables can be obtained as follows (See Stoker [103])

$$\begin{aligned} u_t + uu_x + vu_y + wu_z &= -\tfrac{1}{\tilde{\rho}} p_x, \\ v_t + uv_x + vv_y + wv_z &= -\tfrac{1}{\tilde{\rho}} p_y, \\ w_t + uw_x + vw_y + ww_z &= -\tfrac{1}{\tilde{\rho}} p_z - g \end{aligned} \tag{2.1.1}$$

where g is the acceleration of gravity and $\tilde{\rho}$ is the density when we specify the external or body force to consist only of the force of gravity.

Equations (2.1.1) form a set of three nonlinear partial differential equations for the five quantities $u, v, w, \tilde{\rho}$ and p. Since the fluid is assumed to be incompressible, the density $\tilde{\rho}$ can be taken as a known constant. At the same time, the assumption of incompressibility leads to a relatively simple differential equation (also called the equation of continuity) expressing the law of conservation of mass which is

$$\text{div } \underline{v} = u_x + v_y + w_z = 0. \tag{2.1.2}$$

This equation constitutes the needed fourth equation of the determination of the velocity components and the pressure. Equations (2.1.1) and (2.1.2) are sufficient, once appropriate initial and boundary conditions are imposed, to determine the velocity components u, v, w, and the pressure p uniquely.

We also assume that the flow is irrotational, which yields the following equation

$$\text{curl } \underline{v} = (w_y - v_z, u_z - w_x, v_x - u_y) = 0. \tag{2.1.3}$$

Let ϕ be the velocity potential which is a solution of the Laplace equation

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{2.1.4}$$

and satisfies

$$\underline{v} = \text{grad } \phi = (\phi_x, \phi_y, \phi_z).$$

By making use of (2.1.3) and the fact that $\tilde{\rho}$ =constant, it is readily verified that the equations of motion (2.1.1) can be written in the following vector form:

$$\text{grad } \phi_t + \frac{1}{2}\text{grad } (u^2 + v^2 + w^2) = -\text{grad } \frac{p}{\tilde{\rho}} - \text{grad } (gz).$$

Integration of this relation leads to the important equation, called Bernoulli's law:

$$\phi_t + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\tilde{\rho}} + gz = C(t) \tag{2.1.5}$$

where $C(t)$ may depend on t, but not on the spatial variables. Here we may take $C(t) \equiv$ constant C_1 without any essential loss of generality. In fact, if we set $\phi = \phi^* + \int^t (C(\xi) - C_1) d\xi$, then ϕ^* is a harmonic function with grad ϕ=grad ϕ^* and the right hand side of (2.1.5) is equal to C_1. In the following, we take $C(t) \equiv$constant.

The potential equation (2.1.4) together with Bernoulli's law (2.1.5) can be used to take the place of the equations of motion (2.1.1) and the continuity equation (2.1.2) as a means of determining the velocity components u, v, w, and the pressure p: in fact, u, v and w are determined from the solution ϕ of (2.1.4), after which the pressure p can be obtained from (2.1.5).

Now we describe the boundary conditions. Assume that the fluid under consideration to have a boundary surface S which is a *free surface*, i.e. a surface on which the pressure p is prescribed but the form of the surface is not prescribed a prior. We shall in general assume that such a free surface is given by the equation

$$z = \tilde{\eta}(t, x, y), \tag{2.1.6}$$

which yields

$$\phi_x \tilde{\eta}_x - \phi_z + \phi_y \tilde{\eta}_y + \tilde{\eta}_t = 0 \quad \text{on } S. \tag{2.1.7}$$

In addition, the Bernoulli's law (2.1.5) gives the condition:

$$g\tilde{\eta} + \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + \frac{p}{\tilde{\rho}} = \text{constant} \;\; \text{on } S. \tag{2.1.8}$$

Thus the potential function ϕ must satisfy the two nonlinear boundary conditions (2.1.7) and (2.1.8) on the free surface.

2.2 Steady Gravity-Capillary Water Waves

The classical gravity-capillary water waves concern wave motions on the free surface of an inviscid and incompressible fluid of constant density $\tilde{\rho}$ subject to the forces of gravity and surface tension. The fluid is bounded above by a free surface $z = h + \eta(t, x, y)$ and below by a horizontal rigid bottom $z = 0$ where $\eta > -h$ and h represents a reference depth of the fluid at infinity. By (2.1.4), (2.1.7) and (2.1.8), the exact equations governing the motion of the fluid are the following Euler equations (see Dias & Iooss [36]):

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{for } 0 < z < h + \eta, \tag{2.2.1}$$

with the boundary conditions

$$0 = \phi_z \qquad\qquad \text{on } z = 0, \tag{2.2.2}$$

$$\eta_t = \phi_z - \eta_x\phi_x - \eta_y\phi_y \qquad \text{on } z = h + \eta, \tag{2.2.3}$$

$$\begin{aligned} \phi_t = & -\frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) - g\eta + \frac{\kappa}{\tilde{\rho}}\Big[\frac{\eta_x}{(1 + {\eta_x}^2 + \eta_y^2)^{1/2}}\Big]_x \\ & + \frac{\kappa}{\tilde{\rho}}\Big[\frac{\eta_y}{(1 + \eta_x^2 + \eta_y^2)^{1/2}}\Big]_y + \tilde{\varrho}_0 \qquad \text{on } z = h + \eta(x, y) \end{aligned} \tag{2.2.4}$$

where κ is the coefficient of surface tension–the pressure jump is proportional to the curvature and $\tilde{\varrho}_0$ is the Bernoulli constant on the free surface. Note that (2.2.1)-(2.2.4) have a trivial solution $(\eta, \phi) = (0, 0)$ if $\tilde{\varrho}_0 = 0$. The above formulation describes three-dimensional gravity-capillary waves on water of finite depth, but several variations upon this theme are possible. Solutions which do not depend upon the spatial coordinate y are called two-dimensional water waves, solutions with $\kappa = 0$ are called gravity waves and the limiting case $h \to \infty$ is the infinite-depth problem.

Now we introduce some notations.

Definition 2.2.1 1. *Steady waves (travelling waves) are water waves of a special form* $\eta(t, x, y) = \eta(x + c_1t, y + c_2t)$ *and* $\phi(t, x, y, z) = \phi(x + c_1t, y + c_2t, z)$*; in other words they are uniformly translating in the horizontal direction* x *with a velocity* (c_1, c_2)*.*

2. *A solitary wave describes any solution which has a pulse-like profile in its propagation direction x* (see Figure 2.2.1).

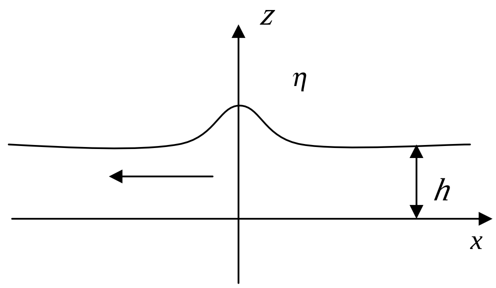

Figure 2.2.1: A solitary wave

3. *Steady water waves periodic in each of two distinct horizontal directions x and y are called doubly periodic steady waves* (see Figure 2.2.2).

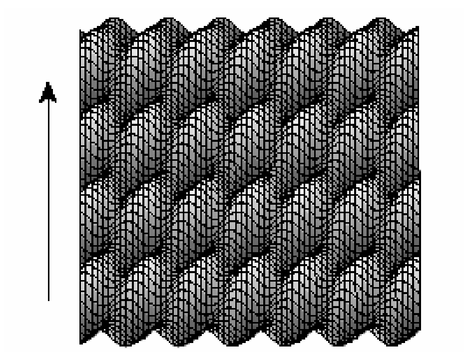

Figure 2.2.2: A doubly periodic travelling wave exhibiting a typical hexagonal pattern.

4. *A line solitary wave is spatially homogeneous in the transverse direction y, while a periodically modulated solitary wave is periodic in the transverse direction y. A fully localized solitary wave on the other hand decays to zero in all spatial directions* (see Figure 2.2.3).

5. *A dimension-breaking phenomenon describes the spontaneous emergence of a spatially inhomogeneous solution of a partial differential equation from a solution which is homogeneous in one or more spatial dimensions. For example, $u(x, y, z) = u_1(x, z) + u_2(x, y, z)$ where u_1 is a solution without the transverse direction y.*

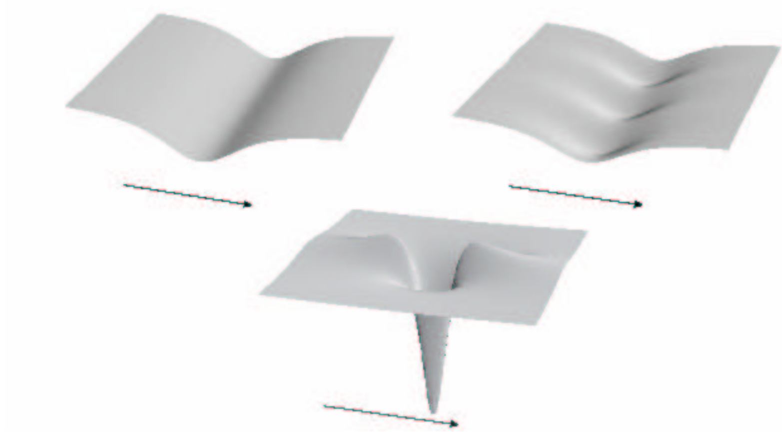

Figure 2.2.3: Clockwise from top to left: line, periodically modulated and fully localized solitary waves.

Remark 2.2.1 *There also exist solitary waves whose pulse-like profile lies in a direction different to that of their propagation direction x;* Figure 2.2.4 *shows two examples of oblique periodically modulated solitary waves of this kind. The figure also illustrates another possible feature of a solitary wave, namely that its pulse-like profile may be made up of multiple individual pulses; waves of this type are called multi-pulse solitary waves.*

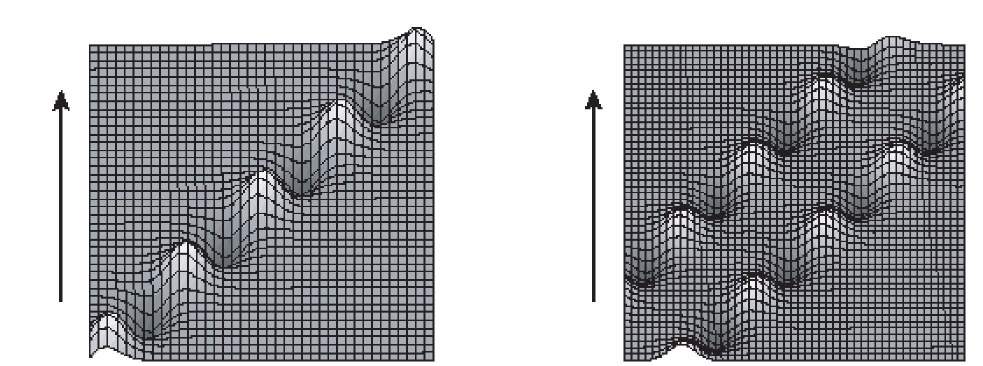

Figure 2.2.4: These solitary waves have a one-pulse (left) and two-pulse (right) profile in one horizontal direction and are periodic in another.

Here, we are interested in surface waves moving with a constant speed $\tilde{c}$ in the x-direction and periodic in the transverse y-direction with a period $P + \alpha$ where P is a constant and α is a small real parameter. Then we just consider the solution of (2.2.1)-(2.2.4) with the special form

$$\eta(t,x,y) = \eta(x+\tilde{c}t, y), \quad \phi(t,x,y,z) = \phi(x+\tilde{c}t, y, z).$$

Substituting this form of η, ϕ into (2.2.1)-(2.2.4) and introducing the dimensionless variables

$$(x', y', z') = \frac{1}{h}(x+\tilde{c}t, \frac{P}{P+\alpha}y, z),$$
$$\eta'(x', y') = \frac{1}{h}\eta(x+\tilde{c}t, y), \quad \phi'(x', y', z') = \frac{1}{\tilde{c}h}\phi(x+\tilde{c}t, y, z),$$

we obtain

$$\begin{array}{ll}
0 = \phi_{xx} + \frac{P^2}{(P+\alpha)^2}\phi_{yy} + \phi_{zz} & 0 < z < 1+\eta(x,y), \\
0 = \phi_z & \text{on } z = 0, \\
0 = \eta_x + \eta_x\phi_x + \frac{P^2}{(P+\alpha)^2}\eta_y\phi_y - \phi_z & \text{on } z = 1+\eta(x,y), \\
0 = \phi_x + \frac{1}{2}(\phi_x^2 + \frac{P^2}{(P+\alpha)^2}\phi_y^2 + \phi_z^2) + F^{-2}\eta & \\
\quad -b\Big[\frac{\eta_x}{(1+\eta_x{}^2+P^2\eta_y^2/(P+\alpha)^2)^{1/2}}\Big]_x & \\
\quad -b\frac{P^2}{(P+\alpha)^2}\Big[\frac{\eta_y}{(1+\eta_x^2+P^2\eta_y^2/(P+\alpha)^2)^{1/2}}\Big]_y + \tilde{\varrho}_1 & \text{on } z = 1+\eta(x,y),
\end{array} \tag{2.2.5}$$

where the primes have been dropped and $\tilde{\varrho}_1 = -\tilde{\varrho}_0/\tilde{c}^2$. Here, $F = \tilde{c}/\sqrt{gh}$ is the Froude number and $b = \frac{\kappa}{\tilde{\rho}h\tilde{c}^2}$ is the Bond number. In general, if we let $\eta \to 0, (\phi_x, \phi_y, \phi_z) \to 0$ as $x \to \pm\infty$, the Bernoulli constant $\tilde{\varrho}_1$ has to be chosen as zero. Otherwise, $\tilde{\varrho}_1$ may depend upon the behavior of the solution at infinity and can be a nonzero constant. Note that (2.2.5) has certain symmetries, which are invariant under translations in the spatial coordinates x and y and in the velocity potential ϕ. There are also two discrete symmetries that play an important role, namely, $y \to -y$ (*reflection*) and $(x, \phi, \eta) \to (-x, -\phi, \eta)$ (*reversibility*).

Now, we reformulate (2.2.5) into a dynamical system which was first given by Dias and Iooss [36].

If let

$$u = \phi_x, \qquad \xi = \frac{\eta_x}{(1+\eta_x^2 + P^2\eta_y^2/(P+\alpha)^2)^{1/2}}, \qquad \lambda = F^{-2}, \qquad \varrho = \frac{\tilde{\varrho}_1}{b},$$

then (2.2.5) can be rewritten as follows:

$$\begin{array}{l}
\left.\begin{array}{lcl}
\phi_x & = & u, \\
u_x & = & -\frac{P^2}{(P+\alpha)^2}\phi_{yy} - \phi_{zz}
\end{array}\right\} \text{ in } 0 < z < 1+\eta(x,y), \\
\eta_x = \xi\Big(\frac{1+P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\Big)^{1/2}, \\
\xi_x = \frac{1}{b}u + \frac{\lambda}{b}\eta + \frac{1}{2b}\Big(u^2 + \frac{P^2}{(P+\alpha)^2}\phi_y^2 + \phi_z^2\Big) - \frac{P^2}{(P+\alpha)^2}\Big[\eta_y\Big(\frac{1-\xi^2}{1+P^2\eta_y^2/(P+\alpha)^2}\Big)^{1/2}\Big]_y + \varrho
\end{array} \tag{2.2.6}$$

with boundary conditions

$$\begin{aligned} \phi_z &= 0 && \text{on } z = 0, \\ \phi_z &= \xi(1+u)\left(\frac{1+P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\right)^{1/2} + \frac{P^2}{(P+\alpha)^2}\eta_y\phi_y && \text{on } z = 1+\eta(x,y). \end{aligned} \tag{2.2.7}$$

Next, the unknown domain is transformed to a fixed domain by flattening the upper boundary with

$$\tilde{z} = \frac{z}{1+\eta(x,y)},$$

which has been used in many other research fields, such as oceanography and atmospheric sciences (called σ-coordinates, see Doyle & Durran [42]). Clearly, if $\eta > -1$, then $\tilde{z} \in (0,1)$ when $0 < z < 1+\eta$. In the new system $(x, y, \tilde{z})$, (2.2.6) and (2.2.7) are changed to (dropping tildes)

$$\begin{aligned} &\phi_x = u + \frac{z\phi_z\xi}{1+\eta}\left(\frac{1+P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\right)^{1/2}, \\ &u_x = -\frac{P^2}{(P+\alpha)^2}\left(\phi_y - \frac{z\eta_y}{1+\eta}\phi_z\right)_y + \frac{P^2}{(P+\alpha)^2}\frac{z\eta_y}{1+\eta}\left(\phi_y - \frac{z\eta_y}{1+\eta}\phi_z\right)_z - \frac{\phi_{zz}}{(1+\eta)^2} \\ &\qquad + \frac{zu_z\xi}{1+\eta}\left(\frac{1+P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\right)^{1/2}, \\ &\eta_x = \xi\left(\frac{1+P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\right)^{1/2}, \\ &\xi_x = \frac{1}{b}(u|_{z=1} + \lambda\eta) + \frac{1}{2b}\left[u^2 + \frac{P^2}{(P+\alpha)^2}\left(\phi_y - \frac{z\eta_y}{1+\eta}\phi_z\right)^2 + \frac{\phi_z^2}{(1+\eta)^2}\right]_{z=1} \\ &\qquad - \frac{P^2}{(P+\alpha)^2}\left[\eta_y\left(\frac{1-\xi^2}{1+P^2\eta_y^2/(P+\alpha)^2}\right)^{1/2}\right]_y + \varrho \end{aligned} \tag{2.2.8}$$

with the boundary conditions

$$\begin{aligned} &\phi_z|_{z=0} = 0, \\ &\phi_z|_{z=1} = \frac{1+\eta}{1+P^2\eta_y^2/(P+\alpha)^2}\left[\xi(1+u|_{z=1})\left(\frac{1+P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\right)^{1/2} + \frac{P^2}{(P+\alpha)^2}\eta_y\phi_y|_{z=1}\right]. \end{aligned} \tag{2.2.9}$$

In (2.2.8), the x-variable is considered as the time variable in the classical theory of dynamical systems. The equations are invariant under the reflection $R : y \to -y$ and are reversible with a reverser S defined by

$$S(\phi, u, \eta, \xi) = (-\phi, u, \eta, -\xi),$$

that is, $S(\phi, u, \eta, \xi)(-x)$ is also a solution whenever $(\phi, u, \eta, \xi)(x)$ is. A solution (ϕ, u, η, ξ) is *reversible* if $S(\phi, u, \eta, \xi)(-x) = (\phi, u, \eta, \xi)(x)$.

Now, we introduce an identity. The third equation in (2.2.5) is equivalent to

$$\frac{\partial}{\partial x}\left(\eta + \int_0^{1+\eta(x,y)} \phi_x(x,y,z)dz\right) + \frac{P^2}{(P+\alpha)^2}\frac{\partial}{\partial y}\int_0^{1+\eta(x,y)} \phi_y(x,y,z)dz = 0$$

where we use the first two equations in (2.2.5). Integrating the above equation y from 0 to P and using the fact that η and ϕ are periodic in y, we have

$$\frac{d}{dx}\int_0^P\left(\eta+\int_0^{1+\eta(x,y)}\phi_x(x,y,z)dz\right)dy=0.$$

Using the relationship between z and z' and the first and third equations of (2.2.8), we have the following equation

$$\int_0^P \eta(x,y)dy+\int_0^P\int_0^1 u(x,y,z)(1+\eta(x,y))dzdy=\text{constant} \tag{2.2.10}$$

which plays an important role to reduce the dimensions. The conserved quantity in (2.2.10) is related to the conservation of mass for water-wave problems discussed by Benjamin and Olver in [16].

The main function spaces are defined as follows. Since we are only interested in solutions that are periodic in the y-direction, define

$$\begin{aligned} H_P^s(I) &= \left\{f\in H_{loc}^s(\mathbf{R})\Big|\ f(y) \text{ is periodic with period } P\right\},\\ H_P^s(\Sigma) &= \left\{f\in H_{loc}^s(\mathbf{R}\times(0,1))\Big|\ f(y,z) \text{ is periodic in } y \text{ with period } P\right\},\end{aligned}$$

where H^s is the classical Sobolev space with $\|f\|_{H^s}=\|f\|_s$, $I=(0,P)$, and $\Sigma=(0,P)\times(0,1)$. Let

$$\mathcal{H}_s=H_P^{s+1}(\Sigma)\times H_P^s(\Sigma)\times H_P^{s+1}(I)\times H_P^s(I).$$

Thus, (2.2.8) can be written as

$$\frac{dv}{dx}=F^{(b,\lambda)}(\varrho,\alpha,v) \tag{2.2.11}$$

where $v=(\phi,u,\eta,\xi)$, and $F^{(b,\lambda)}(\varrho,\alpha,\cdot):\mathcal{H}_{s+1}\to\mathcal{H}_s$ is a smooth function. (2.2.9) defines a set in $\mathcal{H}_{s+1}$ with

$$\begin{aligned}\mathcal{S}=\Big\{&(\phi,u,\eta,\xi)\in\mathcal{H}_{s+1};|\xi|<1,\ \eta>-1,\phi_z|_{z=0}=0,\\ &\phi_z|_{z=1}=\frac{1+\eta}{1+P^2\eta_y^2/(P+\alpha)^2}\Big[\xi(1+u|_{z=1})\Big(\frac{1+P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\Big)^{1/2}\\ &+\frac{P^2}{(P+\alpha)^2}\eta_y\phi_y|_{z=1}\Big]\Big\}.\end{aligned}$$

Note that $H_P^s(I)$ and $H_P^t(\Sigma)$ are Banach algebra for $s>1/2, t>1$. Thus, for $0<s<\frac{1}{2}$, if $f\in H_P^{s+1}(I)$ and $g\in H_P^s(I)$, then $fg\in H_P^s(I)$ (similar one holds if I is replaced by Σ) (See Lemma A.1 and Lemma A.2 in Appendix A by Groves & Mielke [55]).

To define the domain of $F^{(b,\lambda)}(\varrho, \alpha, \cdot)$, we note that v must be in $\mathcal{S}$, i.e., v satisfies the nonlinear boundary condition at $z = 1$. Hence, we use the method introduced in Groves & Mielke [55] to transform the nonlinear boundary condition to a linear boundary condition. Define a smooth function $H : \mathcal{H}_{s+1} \to H_P^{s+1}(\Sigma)$ in a neighborhood of 0 by

$$H(\phi, u, \eta, \xi) = z\Big(\frac{1+\eta}{1+P^2\eta_y^2/(P^2+\alpha)^2}\Big)\Big[\xi(1+u)\Big(\frac{1+P^2\eta_y^2/(P^2+\alpha)^2}{1-\xi^2}\Big)^{1/2} + \frac{P^2}{(P^2+\alpha)^2}\eta_y\phi_y\Big].$$

Then (2.2.9) becomes

$$\phi_z = H(\phi, u, \eta, \xi), \qquad \text{at } z = 0, 1.$$

Next, we make a change of dependent variables near 0 using $G : \mathcal{H}_{s+1} \to \mathcal{H}_{s+1}$ given by

$$w = (\psi, u, \eta, \xi) = G(v) = (\phi - \varphi_z, u, \eta, \xi),$$

where $\varphi \in H_P^{s+3}(\Sigma)$ is a unique solution of the linear boundary value problem

$$\begin{aligned} &\Delta\varphi = H(\phi, u, \eta, \xi) \qquad (y, z) \in \Sigma, \\ &\varphi = 0 \qquad \text{on } z = 0, 1. \end{aligned}$$

Thus,

$$\begin{aligned} \psi_z &= \phi_z - \varphi_{zz} \\ &= \phi_z - \big(H(\phi, u, \eta, \xi) - \varphi_{yy}\big). \end{aligned}$$

Since $\varphi_{yy} = 0$ on $z = 0, 1$, it follows that the boundary conditions (2.2.9) are changed to

$$\psi_z = 0 \qquad \text{on } z = 0, 1.$$

The mapping G has the following properties.

Lemma 2.1

(i) *The mapping G is a smooth diffeomorphism from the neighborhood V of 0 in $\mathcal{H}_{s+1}$ onto a neighborhood W of 0 in $\mathcal{H}_{s+1}$. The mapping G, G^{-1}, and their derivatives depend smoothly upon $(b, \lambda, \varrho, \alpha)$ in the neighborhood Λ of $(\hat{b}, \hat{\lambda}, \hat{\varrho}, \hat{\alpha})$ where $\hat{b} > 0, \hat{\lambda} > 0$, $\hat{\varrho}$ and $\hat{\alpha}$ are fixed.*

(ii) *For any fixed $v \in V$, the operator $dG[v] : \mathcal{H}_{s+1} \to \mathcal{H}_{s+1}$ extends to an isomorphism $\hat{dG}[v] : \mathcal{H}_s \to \mathcal{H}_s$. The operators $\hat{dG}[v], (\hat{dG}[v])^{-1} \in \mathcal{L}(\mathcal{H}_s, \mathcal{H}_s)$ depend smoothly upon $(v, b, \lambda, \varrho, \alpha) \in V \times \Lambda$.*

Proof. The proof is similar to one of Lemma 3.3 by Groves & Mielke [55]. Note that the mapping $G : V \to \mathcal{H}_{s+1}$ is smooth and depends smoothly upon $(b, \lambda, \varrho, \alpha)$ since the map $H : V \to H_P^{s+1}(\Sigma)$ is smooth and G is a linear function with respect to H. Clearly, $G(0) = 0$ and

$$dG[v](\phi_1, u_1, \eta_1, \xi_1) = (\phi_1 - \tilde{\varphi}_z, u_1, \eta_1, \xi_1)$$

in which $\tilde{\varphi} \in H_P^{s+3}(\Sigma)$ is a unique solution of the elliptic boundary-value problem

$$\Delta\tilde{\varphi} = d_1 H[v](\phi_1, u_1, \eta_1, \xi_1) \quad \text{in } \Sigma, \qquad \tilde{\varphi} = 0 \quad \text{on } z = 0, 1.$$

A direct calculation shows that

$$\begin{aligned} d_1H[v](\phi_1, u_1, \eta_1, \xi_1) &= \frac{z\eta_1}{1 + P^2\eta_y^2/(P+\alpha)^2}\Big[\xi(1+u)\Big(\frac{1 + P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\Big)^{1/2} \\ &+\frac{P^2}{(P+\alpha)^2}\eta_y\phi_y\Big] - \frac{P^2}{(P+\alpha)^2}\frac{2z(1+\eta)\eta_y\eta_{1y}}{(1 + P^2\eta_y^2/(P+\alpha)^2)^2}\Big[\xi(1+u)\Big(\frac{1 + P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\Big)^{1/2} \\ &+\frac{P^2}{(P+\alpha)^2}\eta_y\phi_y\Big] + \frac{z(1+\eta)}{1 + P^2\eta_y^2/(P+\alpha)^2}\Big[\xi_1(1+u)\Big(\frac{1 + P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\Big)^{1/2} \\ &+\xi u_1\Big(\frac{1 + P^2\eta_y^2/(P+\alpha)^2}{1-\xi^2}\Big)^{1/2} + \frac{P^2}{(P+\alpha)^2}\frac{\xi\eta_y\eta_{1y}(1+u)}{(1-\xi^2)^{1/2}(1 + P^2\eta_y^2/(P+\alpha)^2)^{1/2}} \\ &+\frac{\xi^2\xi_1(1+u)(1 + P^2\eta_y^2/(P+\alpha)^2)^{1/2}}{(1-\xi^2)^{3/2}} + \frac{P^2}{(P+\alpha)^2}\phi_y\eta_{1y} + \frac{P^2}{(P+\alpha)^2}\phi_{1y}\eta_y\Big], \end{aligned} \tag{2.2.12}$$

which gives $d_1H[0](\phi_1, u_1, \eta_1, \xi_1) = z\xi_1$. It is straightforward to see that the mapping $dG[0] : \mathcal{H}_{s+1} \to \mathcal{H}_{s+1}$ is a bijection and its inverse is given by

$$\big(dG[0]\big)^{-1}(\psi_1, u_1, \eta_1, \xi_1) = (\psi_1 + \zeta_z, u_1, \eta_1, \xi_1),$$

where ζ is a unique solution of the elliptic boundary-value problem

$$\Delta\zeta = z\xi_1 \quad \text{in } \Sigma, \qquad \zeta = 0 \quad \text{on} \quad z = 0, 1.$$

Replacing V with a smaller neighborhood of 0 in $\mathcal{H}_{s+1}$ if necessary, one obtains the result stated in the first part of the lemma using the inverse function theorem and the fact that in the present context all estimates used in the proof hold uniformly in $(b, \lambda, \varrho, \alpha) \in \Lambda$.

Next, we prove the second part. Let $v \in V$. If $(\phi_1, u_1, \eta_1, \xi_1) \in \mathcal{H}_s$, from Lemmas A.1 and A.2 by Groves & Mielke [55], we obtain that each term of the right side in (2.2.12) is in $H_P^s(\Sigma)$ and it is a bounded linear mapping from $\mathcal{H}_s$ into $H_P^s(\Sigma)$ which depends smoothly upon $(v, b, \lambda, \varrho, \alpha) \in V \times \Lambda$. It follows that $d_1H[v] : \mathcal{H}_{s+1} \to H_P^{s+1}(\Sigma)$ can be extended as a bounded linear operator $\widehat{d_1H}[v] : \mathcal{H}_s \to H_P^s(\Sigma)$ that depends smoothly on $(v, b, \lambda, \varrho, \alpha) \in V \times \Lambda$. The corresponding result holds for dG since G depends linearly on H, where the extension $\widehat{dG}$ is unique since $\mathcal{H}_{s+1}$ is a dense subspace of $\mathcal{H}_s$.

If T is a bounded linear mapping from $\mathcal{H}_s$ to $\mathcal{H}_s$, define $H_1, H_2 : \mathcal{L}(\mathcal{H}_s, \mathcal{H}_s) \times V \times \Lambda \to \mathcal{L}(\mathcal{H}_s, \mathcal{H}_s)$ by

$$H_1(T, v, b, \lambda, \varrho, \alpha) = d\hat{G}[v]T - I, \qquad H_2(T, v, b, \lambda, \varrho, \alpha) = Td\hat{G}[v] - I,$$

where I denotes the identity operator in $\mathcal{H}_s$. The above calculations show that $d\hat{G}[0] : \mathcal{H}_s \to \mathcal{H}_s$ is an isomorphism and obviously

$$H_j\big((d\hat{G}[0])^{-1}, 0\big) = 0,\ d_1H_j\big((d\hat{G}[0])^{-1}, 0\big) = I,\ j = 1, 2.$$

Replacing V and Λ with smaller neighborhoods of 0 in $\mathcal{H}_{s+1}$ and of $(b_0, \lambda_0, \varrho_0, \alpha_0)$ in $\mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}$ if necessary, one finds from the implicit function theorem that the equations $H_1(T, v, b, \lambda, \varrho, \alpha) = 0$, $H_2(T, v, b, \lambda, \varrho, \alpha) = 0$ have a unique solution $(T_1(v, b, \lambda, \varrho, \alpha), T_2(v, b, \lambda, \varrho, \alpha))$ for each $(v, b, \lambda, \varrho, \alpha) \in V \times \Lambda$ which depends smoothly on $(v, b, \lambda, \varrho, \alpha)$. It then follows that $d\hat{G}[v]$ is a bijection with $\big(d\hat{G}[v]\big)^{-1} = T_1 = T_2$ for each $(v, b, \lambda, \varrho, \alpha) \in V \times \Lambda$. □

Therefore, by Lemma 2.1, the diffeomorphism G transforms (2.2.11) into

$$\dot{w} = g^{(b,\lambda)}(\varrho, \alpha, w), \tag{2.2.13}$$

where $g^{(b,\lambda)}(\varrho, \alpha, \cdot) : W \to \mathcal{H}_s$ is a smooth vector field defined by

$$g^{(b,\lambda)}(\varrho, \alpha, w) = d\hat{G}[G^{-1}(w)]\big(F^{(b,\lambda)}(\varrho, \alpha, G^{-1}(w))\big),$$

and ψ satisfies $\psi_z = 0$ at $z = 0, 1$. Moreover, (2.2.13) can be written as

$$\dot{w} = K_s w + N(b, \lambda, \varrho, \alpha, w), \tag{2.2.14}$$

where $K_s = dg^{(b,\lambda)}[0]$ is a linear operator with domain $\mathcal{D}(K_s) = \big\{w = (\psi, u, \eta, \xi) \in \mathcal{H}_{s+1} : \psi_z|_{z=0} = \psi_z|_{z=1} = 0\big\}$ and $N(b, \lambda, \varrho, \alpha, w) = g^{(b,\lambda)}(\varrho, \alpha, w) - K_s w$.

Chapter 3

Linear Operators and Their Properties

In this chapter, we study the linear operator K_s in (2.2.14) and obtain its various properties.

3.1 Resolvent Estimates

Define an operator $L_s : \mathcal{D}(L_s) \subset \mathcal{H}_s \to \mathcal{H}_s$ given by

$$L_s \begin{pmatrix} \phi \\ u \\ \eta \\ \xi \end{pmatrix} = \begin{pmatrix} u \\ -\phi_{yy} - \phi_{zz} \\ \xi \\ \frac{1}{b}(u|_{z=1} + \lambda\eta) - \eta_{yy} \end{pmatrix} \tag{3.1.1}$$

with domain $\mathcal{D}(L_s)$, a subspace of $\mathcal{H}_{s+1}$, satisfying the boundary conditions

$$\phi_z = 0 \text{ on } z = 0, \qquad \phi_z = \xi \text{ on } z = 1. \tag{3.1.2}$$

The operator L_s is the formal linearization of $F^{(b,\lambda)}(\varrho, \alpha, \phi, u, \eta, \xi)$ (defined in (2.2.11)) at 0 when the small parameter $\alpha = 0$. It is easy to see that $d\hat{G}[0]$ is a homeomorphism of $\mathcal{D}(L_s)$ onto $\mathcal{D}(K_s)$ and that

$$K_s = d\hat{G}[0] L_s \big(d\hat{G}[0]\big)^{-1} \tag{3.1.3}$$

for $s \in (0, \frac{1}{2})$, where $K_s : \mathcal{D}(K_s) \subset \mathcal{H}_s \to \mathcal{H}_s$ is the linear operator in (2.2.14). (3.1.3) shows that the topological properties of L_s and K_s are identical. Therefore, in order to study K_s, it suffices to consider L_s. In particular, their spectra coincide and L_s is densely defined since K_s is densely defined. Define densely-defined operators $L_0 : \mathcal{D}(L_0) \subset \mathcal{H}_0 \to \mathcal{H}_0$

and $K_0 : \mathcal{D}(K_0) \subset \mathcal{H}_0 \to \mathcal{H}_0$ using (3.1.1)-(3.1.3), which are valid since $\left(d\hat{G}[0]\right)^{-1}$ is an isomorphism of $\mathcal{D}(K_0) = \left\{(\psi, u, \eta, \xi) \in \mathcal{H}_1 \mid \psi_z|_{z=0} = \psi_z|_{z=1} = 0\right\}$ onto $\mathcal{D}(L_0)$ and $d\hat{G}[0]$ is an isomorphism of $\mathcal{H}_0$ onto itself.

Lemma 3.1.1 *There exist constants C and $\sigma_0 > 0$ such that each solution $v \in \mathcal{D}(L_0)$ of the resolvent equation*

$$(L_0 - i\sigma I)v = f^*, \tag{3.1.4}$$

in which f^* belongs to $\mathcal{H}_0$ and σ is a real number with $|\sigma| > \sigma_0$, satisfies

$$\|v\|_{\mathcal{H}_1} \leq C\|f^*\|_{\mathcal{H}_0}, \tag{3.1.5}$$

$$\|v\|_{\mathcal{H}_0} \leq \frac{C}{|\sigma|}\|f^*\|_{\mathcal{H}_0}. \tag{3.1.6}$$

Proof. The proof is similar to one of Lemma 3.4 by Groves & Mielke [55]. Let $f^* = (\phi^*, u^*, \eta^*, \xi^*)$. By $v = (\phi, u, \eta, \xi)$, we write $(L_0 - i\sigma I)v = f^*$ as

$$u - i\sigma\phi = \phi^*, \tag{3.1.7}$$

$$-\Delta\phi - i\sigma u = u^*, \tag{3.1.8}$$

$$\xi - i\sigma\eta = \eta^*, \tag{3.1.9}$$

$$\frac{1}{b}(u|_{z=1} + \lambda\eta) - \eta_{yy} - i\sigma\xi = \xi^* \tag{3.1.10}$$

with boundary conditions

$$\phi_z|_{z=0} = 0, \quad \phi_z|_{z=1} = \xi. \tag{3.1.11}$$

From (3.1.9), (3.1.10) and the identity

$$|a_1 + i\sigma b_1|^2 = |a_1|^2 + \sigma^2|b_1|^2 + 2\sigma \mathrm{Im}(a_1\bar{b}_1), \ a_1, b_1 \in \mathbf{C},$$

it is obtained that

$$\begin{aligned}\|\xi^* - (1/b)(u|_{z=1} + \lambda\eta)\|_0^2 + \|\eta^*\|_1^2 &= \| - \eta_{yy} - i\sigma\xi\|_0^2 + \|\xi - i\sigma\eta\|_1^2 \\ &= \|\eta_{yy}\|_0^2 + \sigma^2\|\xi\|_0^2 + \|\xi\|_1^2 + \sigma^2\|\eta\|_1^2 + 2\sigma\mathrm{Im}\int_0^P (\eta_{yy}\bar{\xi} - \xi\bar{\eta} - \xi_y\bar{\eta}_y)dy.\end{aligned}$$

Therefore, an integration by parts

$$\mathrm{Im}\int_0^P (\eta_{yy}\bar{\xi} - \xi_y\bar{\eta}_y)dy \quad = \quad \mathrm{Im}\Big[\int_0^P \eta_{yy}\bar{\xi}dy - (\xi\bar{\eta}_y)\big|_0^P + \int_0^P \xi\bar{\eta}_{yy}dy\Big]$$

$$= \operatorname{Im}\Big[\int_0^P \eta_{yy}\bar{\xi}dy + \int_0^P \xi\bar{\eta}_{yy}dy\Big] = 0$$

and the inequality

$$\Big|2\operatorname{Im}\int_0^P \xi\bar{\eta}dy\Big| \le \|\xi\|_0^2 + \|\eta\|_0^2$$

yield

$$\begin{aligned}&\|\eta_{yy}\|_0^2 + \|\xi\|_1^2 + \sigma^2(\|\eta\|_1^2 + \|\xi\|_0^2)\\ &\quad \le c_1(\|\xi^*\|_0^2 + \frac{1}{b^2}\|u|_{z=1}\|_0^2 + \frac{\lambda^2}{b^2}\|\eta\|_0^2 + \|\eta^*\|_1^2 + |\sigma|\|\xi\|_0^2 + |\sigma|\|\eta\|_0^2)\,,\end{aligned}$$

which gives

$$\|\eta_{yy}\|_0^2 + \|\xi\|_1^2 + \sigma^2(\|\eta\|_1^2 + \|\xi\|_0^2) \le c_2(\|\xi^*\|_0^2 + \|u|_{z=1}\|_0^2 + \|\eta^*\|_1^2)$$

if $|\sigma|$ is large. By (3.1.10), we have

$$\|\eta\|_2^2 \le c_3(\|u|_{z=1}\|_0^2 + \|\eta\|_0^2 + \sigma^2\|\xi\|_0^2 + \|\xi^*\|_0^2)$$

and

$$\|\eta\|_2^2 + \|\xi\|_1^2 + \sigma^2(\|\eta\|_1^2 + \|\xi\|_0^2) \le c_4(\|\xi^*\|_0^2 + \|u\|_1^2 + \|\eta^*\|_1^2) \tag{3.1.12}$$

for $|\sigma|$ large, where the Sobolev embedding theorem has been used.

By (3.1.7) and (3.1.8), we obtain

$$\begin{aligned}&\|\phi^*\|_1^2 + \|u^*\|_0^2 = \|u - i\sigma\phi\|_1^2 + \|\Delta\phi + i\sigma u\|_0^2\\ &\quad = \|u\|_1^2 + \sigma^2\|\phi\|_1^2 + \|\Delta\phi\|_0^2 + \sigma^2\|u\|_0^2 + 2\sigma\operatorname{Im}\int_\Sigma(-u\bar{\phi} - \nabla u\nabla\bar{\phi} + \Delta\phi\bar{u})dydz.\end{aligned}$$

Note that

$$\begin{aligned}&\operatorname{Im}\Big[\int_0^1\int_0^P \Delta\phi\bar{u}dydz - \int_0^1\int_0^P \nabla u\nabla\bar{\phi}dydz\Big]\\ &\quad = \operatorname{Im}\Big[\int_0^1\int_0^P \Delta\phi\bar{u}dydz - \int_0^1\int_0^P (u_y\bar{\phi}_y + u_z\bar{\phi}_z)dydz\Big]\\ &\quad = \operatorname{Im}\Big[\int_0^1\int_0^P \Delta\phi\bar{u}dydz - \Big(\int_0^1 (u\bar{\phi}_y|_{y=0}^{y=P} - \int_0^P u\bar{\phi}_{yy}dy)dz\\ &\qquad + \int_0^P (u\bar{\phi}_z|_{z=0}^{z=1} - \int_0^1 u\bar{\phi}_{zz}dz)dy\Big)\Big] = -\operatorname{Im}\Big[\int_0^P \bar{\xi}u|_{z=1}dy\Big],\end{aligned}$$

which implies

$$\begin{aligned}\left|2\sigma\mathrm{Im}\int_\Sigma(-u\bar{\phi}-\nabla u\nabla\bar{\phi}+\Delta\phi\bar{u})dydz\right| &= \left|2\sigma\mathrm{Im}\int_\Sigma -u\bar{\phi}dydz-2\sigma\mathrm{Im}\int_0^P\bar{\xi}u|_{z=1}dy\right|\\ &\le |\sigma|(\|u\|_0^2+\|\phi\|_0^2)+|\sigma|(\delta^{-1}\|\xi\|_0^2+\delta\|u\|_1^2)\end{aligned}$$

for any $\delta>0$. Therefore, by choosing $\delta>0$ sufficiently small,

$$\|\Delta\phi\|_0^2+\|u\|_1^2+\sigma^2(\|\phi\|_1^2+\|u\|_0^2)\le c_5(\|\phi^*\|_1^2+\|u^*\|_0^2+|\sigma|\|\xi\|_0^2) \tag{3.1.13}$$

for large $|\sigma|$.

Since ϕ is a solution of (3.1.8) and satisfies (3.1.11), standard results in the elliptic regularity theory (Theorem 5.1 by Liona & Magenes [77]) give that

$$\begin{aligned}\|\phi\|_2^2 &\le c_6(\|u^*\|_0^2+\sigma^2\|u\|_0^2+\|\xi\|_{\frac12}^2+\|\phi\|_1^2)\\ &\le c_7(\|u^*\|_0^2+\sigma^2\|u\|_0^2+\|\xi\|_0\|\xi\|_1+\|\phi\|_1^2)\\ &\le c_8(\|u^*\|_0^2+\sigma^2\|u\|_0^2+\epsilon\|\xi\|_1^2+\epsilon^{-1}\|\xi\|_0^2+\|\phi\|_1^2)\end{aligned}$$

where $\epsilon>0$ is arbitrary and $\|\xi\|_{\frac12}^2\le\|\xi\|_0\|\xi\|_1$ is used. Then, by (3.1.13), for σ large,

$$\begin{aligned}\|\phi\|_2^2+\|u\|_1^2+\sigma^2(\|\phi\|_1^2+\|u\|_0^2)\le &c_9(\|\phi^*\|_1^2+\|u^*\|_0^2+\epsilon\|\xi\|_1^2\\ &+\epsilon^{-1}\|\xi\|_0^2+\sigma\|\xi\|_0^2).\end{aligned} \tag{3.1.14}$$

From (3.1.12), we may choose ϵ small enough in (3.1.14) to obtain that for σ large

$$\begin{aligned}\|\phi\|_2^2+\|u\|_1^2+\sigma^2(\|\phi\|_1^2+\|u\|_0^2)\le &c_{10}(\|\phi^*\|_1^2+\|u^*\|_0^2+\|\eta^*\|_1^2+\|\xi^*\|_0^2+\epsilon\|u\|_1^2)\\ \le &c_{11}(\|\phi^*\|_1^2+\|u^*\|_0^2+\|\eta^*\|_1^2+\|\xi^*\|_0^2).\end{aligned} \tag{3.1.15}$$

Therefore, (3.1.12) and (3.1.15) imply

$$\|\eta\|_2^2+\|\xi\|_1^2+\sigma^2(\|\eta\|_1^2+\|\xi\|_0^2)\le c_{12}(\|\phi^*\|_1^2+\|u^*\|_0^2+\|\eta^*\|_1^2+\|\xi^*\|_0^2). \tag{3.1.16}$$

Here, the constants $c_j, j=1,\dots,12$, are independent of σ. Obviously, (3.1.15) and (3.1.16) yield (3.1.5) and (3.1.6). The proof is completed. □

Lemma 3.1.2 *The complex number $\sigma\neq0$ is an eigenvalue of L_s for $s\in[0,\frac12)$ if and only if there exists an integer $k\in\mathbf{Z}$ such that*

$$(\lambda-b\tau_k^2)\tau_k\sin\tau_k-\sigma^2\cos\tau_k=0 \tag{3.1.17}$$

with $\tau_k^2=\sigma^2-\frac{4\pi^2k^2}{P^2}$. 0 is always an eigenvalue of L_s for $s\in[0,\frac12)$ and the corresponding eigenvector is $e_{00}=(1,0,0,0)^T$. Furthermore, the spectrum $\tilde{\sigma}(L_0)$ of L_0 consists entirely of isolated eigenvalues of finite algebraic multiplicity and $\tilde{\sigma}(L_0)\cap i\mathbf{R}$ is a finite set.

Proof. To find the eigenvalues of L_s, we solve the following equation

$$(L_s - \sigma I)v = f^* \tag{3.1.18}$$

where $v = (\phi, u, \eta, \xi) \in \mathcal{D}(L_s)$ and $f^* = (\phi^*, u^*, \eta^*, \xi^*) \in \mathcal{H}_s$. Since all functions are in L^2, we expand v and f^* into their Fourier series with

$$v = \sum_{k\in\mathbf{Z}} v_k \exp(2\pi iky/P), \quad f^* = \sum_{k\in\mathbf{Z}} f_k^* \exp(2\pi iky/P).$$

Then, (3.1.18) is changed to

$$\begin{aligned} u_k(z) - \sigma\phi_k(z) &= \phi_k^*(z), && (3.1.19)\\ \frac{4\pi^2k^2}{P^2}\phi_k(z) - \phi_k''(z) - \sigma u_k(z) &= u_k^*(z), && (3.1.20)\\ \xi_k - \sigma\eta_k &= \eta_k^*, && (3.1.21)\\ \frac{1}{b}(u_k(1) + \lambda\eta_k) + \frac{4\pi^2k^2}{P^2}\eta_k - \sigma\xi_k &= \xi_k^* && (3.1.22) \end{aligned}$$

with $\phi_k'(0) = 0$ and $\phi_k'(1) = \xi_k$. Let $f^* = 0$ and $\sigma \neq 0$. From (3.1.19) and (3.1.20), it is deduced that

$$\phi_k''(z) + \left(\sigma^2 - \frac{4\pi^2k^2}{P^2}\right)\phi_k = 0. \tag{3.1.23}$$

Standard argument shows that to have nontrivial solutions of (3.1.23) with boundary conditions $\phi_k'(0) = 0$ and $\phi_k'(1) = \xi_k$, σ^2 must satisfy (3.1.17).

Now let $s = 0$ and $\sigma \neq 0$. It is straightforward to see that (3.1.19)-(3.1.22) have a unique solution

$$v_k = (\phi_k, u_k, \eta_k, \xi_k) \in H^2(0,1) \times H^1(0,1) \times \mathbf{C} \times \mathbf{C}$$

for each $f^* \in \mathcal{H}_0$ if and only if σ does not satisfy (3.1.17). Lemma 3.1.1 shows that (3.1.17) has no solution if σ is replaced by $i\sigma$ with $|\sigma| > \sigma_0$. By

$$(L_0 - i\sigma I)v_k \exp(2\pi iky/P) = f_k^* \exp(2\pi iky/P),$$

and Lemma 3.1.1, we find

$$\|v_k \exp(2\pi iky/P)\|_{\mathcal{H}_1} \leq C\|f_k^* \exp(2\pi iky/P)\|_{\mathcal{H}_0}.$$

Summing above inequality over all $k \in \mathbf{Z}$, we obtain that $v \in \mathcal{H}_1$ and that $(L_0 - i\sigma I)^{-1} : \mathcal{H}_0 \to \mathcal{H}_1$ is continuous. Thus, (3.1.4) has a unique solution $v \in \mathcal{D}(L_0)$ for each $f^* \in \mathcal{H}_0$, and (3.1.5) indicates that $i\sigma$ belongs to the resolvent set of L_0. Since the resolvent operator $(L_0 - i\sigma I)^{-1} : \mathcal{H}_0 \to \mathcal{H}_0$ is compact and L_0 is closed, the spectrum of L_0 consists of isolated eigenvalues of finite algebraic multiplicity (See Theorem III 6.29 in the book by Kato [69]). Since $i\sigma$ is not in $\tilde{\sigma}(L_0)$ for σ real and large, $\tilde{\sigma}(L_0) \cap i\mathbf{R}$ is finite. □

Because the spectra of K_s and L_s are identical, it follows from Lemmas 3.1.1 and 3.1.2 that there is a constant C_1 such that

$$\|(K_0 - i\sigma I)^{-1}\|_{\mathcal{H}_0\to\mathcal{H}_0} \le \frac{C_1}{|\sigma|}$$

for each real number σ with $|\sigma| \ge \sigma_0$. Here, we discuss the spectrum of K_s for $s \in (0, \frac{1}{2})$.

Theorem 3.1.1 *For each $s \in (0, \frac{1}{2})$, the spectrum of K_s consists of isolated eigenvalues of finite algebraic multiplicity and these eigenvalues are precisely the solutions of* (3.1.17)*. Moreover, there exists a constant C_2 (which depends on s) such that*

$$\|(K_s - i\sigma I)^{-1}\|_{\mathcal{H}_s\to\mathcal{H}_s} \le \frac{C_2}{|\sigma|} \tag{3.1.24}$$

for any real number σ with $|\sigma| > \sigma_0$.

Proof. Note that $\mathcal{D}(K_s)$ and $\mathcal{D}(L_s)$ are closed subsets of $\mathcal{H}_{s+1}$ and equipped with $\mathcal{H}_{s+1}$-norm. By (3.1.5), we know that $(K_0 - i\sigma I)^{-1} : \mathcal{D}(K_0) \to \mathcal{D}(K_0)$ is bounded. From the boundedness of $(K_0 - i\sigma I)^{-1} : \mathcal{H}_0 \to \mathcal{H}_1$, $(K_0 - i\sigma I)$ is closed. Thus, the closed graph theorem implies that $(K_0 - i\sigma I)$ is bounded from $\mathcal{D}(K_0)$ to $\mathcal{H}_0$.

Let $c_j, j = 1, 2, 3, 4$ denote the constants which are independent of σ. It is obtained that

$$\begin{aligned}
\|(K_0 - i\sigma I)^{-1} w\|_{\mathcal{H}_1} &= \|(K_0 - i\sigma_1 I)^{-1}(K_0 - i\sigma_1 I)(K_0 - i\sigma I)^{-1} w\|_{\mathcal{H}_1} \\
&\le c_1\|(K_0 - i\sigma_1 I)(K_0 - i\sigma I)^{-1} w\|_{\mathcal{H}_0} \\
&= c_1\|(K_0 - i\sigma I)^{-1}(K_0 - i\sigma_1 I) w\|_{\mathcal{H}_0} \\
&\le c_2\|(K_0 - i\sigma I)^{-1}\|_{\mathcal{H}_0\to\mathcal{H}_0}\|(K_0 - i\sigma_1 I) w\|_{\mathcal{H}_0} \le \frac{c_3}{|\sigma|}\|w\|_{\mathcal{H}_1}
\end{aligned}$$

for each $w \in \mathcal{D}(K_0)$, where σ_1 is a fixed real number with $\sigma_1 > \sigma_0$. Hence, $(K_0 - i\sigma I)^{-1}$ is a bounded operator from $\mathcal{H}_0$ to $\mathcal{H}_0$ and from $\mathcal{D}(K_0)$ to $\mathcal{D}(K_0)$ with the operator norms less than or equal to $\frac{c_3}{|\sigma|}$. By Theorem 51 in the book by Lions & Magenes [77] and the fact that $\mathcal{D}(K_0)$ is dense in $\mathcal{H}_0$, it is found that $(K_0 - i\sigma I)^{-1}$ is a bounded operator from $[\mathcal{D}(K_0), \mathcal{H}_0]_{1-s}$ into itself satisfying

$$\|(K_0 - i\sigma I)^{-1} v\|_{[\mathcal{D}(K_0),\mathcal{H}_0]_{1-s}} \le \frac{c_4}{|\sigma|}\|v\|_{[\mathcal{D}(K_0),\mathcal{H}_0]_{1-s}}.$$

Using the result in §18.5 on page 107 in the book by Lions & Magenes [77] or Grisvard [49], we obtain

$$[\mathcal{D}(K_0), \mathcal{H}_0]_{1-s} = \big\{w = (\psi, u, \eta, \xi) \in \mathcal{H}_s \mid \psi_z|_{z=0} = \psi_z|_{z=1} = 0\big\}.$$

Since $(K_0 - i\sigma I)^{-1}$ is bounded and $[\mathcal{D}(K_0), \mathcal{H}_0]_{1-s}$ is a closed subspace in $\mathcal{H}_s$, $(K_0 - i\sigma I)^{-1}$ can be extended to a bounded operator from $\mathcal{H}_s$ to $\mathcal{H}_s$, where the fact that the norms on $[\mathcal{D}(K_0), \mathcal{H}_0]_{1-s}$ and $\mathcal{H}_s$ are equivalent is used. Moreover, since $(K_0 - i\sigma I)^{-1}$ and $(K_s - i\sigma I)^{-1}$ are same on $\mathcal{D}(K_0)$ and $\mathcal{D}(K_0)$ is dense in $[\mathcal{D}(K_0), \mathcal{H}_0]_{1-s}$, we may regard $(K_s - i\sigma I)^{-1}$ as $(K_0 - i\sigma I)^{-1}$ by the uniqueness of extension. Thus, (3.1.24) holds and $(K_s - i\sigma I) : \mathcal{D}(K_s) \to \mathcal{H}_s$ is a bijection, which implies that $i\sigma$ is in the resolvent set of K_s. By the boundedness of $(K_s - i\sigma I)^{-1}$ and the closed graph theorem, $(K_s - i\sigma I) : \mathcal{D}(K_s) \to \mathcal{H}_s$ is continuous. It then follows that $(K_s - i\sigma I)^{-1} : \mathcal{H}_s \to \mathcal{D}(K_s)$ is continuous and $(K_s - i\sigma I)^{-1} : \mathcal{H}_s \to \mathcal{H}_s$ is compact. The spectrum of K_s therefore consists of isolated eigenvalues of finite algebraic multiplicity, which are precisely the solutions of (3.1.17). □

3.2 Center Manifold Reduction

First, we introduce an important theorem given by Mielke [82].

Theorem 3.2.1 *Consider the differential equation*

$$\dot{u} = Ku + N(\tilde{\alpha}, u), \tag{3.2.1}$$

in which u belongs to a Hilbert space $\mathcal{X}$, $\tilde{\alpha} \in \mathbf{R}^n$ is a parameter and $K : \mathcal{D}(K) \subset \mathcal{X} \to \mathcal{X}$ is a densely defined closed linear operator. Regard $\mathcal{D}(K)$ as a Hilbert space equipped with the graph norm, and suppose that 0 is an equilibrium point of (3.2.1) when $\tilde{\alpha} = 0$ and that the following hypotheses hold.

(H1) *$\mathcal{X}$ admits a direct-sum decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, where $\mathcal{X}_1, \mathcal{X}_2$ are closed K-invariant subspaces, so that (3.2.1) can be written as the linearly decoupled system*

$$\begin{aligned} \dot{u}_1 &= K_1 u_1 + N_1(\tilde{\alpha}, u_1 + u_2), && (3.2.2)\\ \dot{u}_2 &= K_2 u_2 + N_2(\tilde{\alpha}, u_1 + u_2), && (3.2.3) \end{aligned}$$

in which $K_j = K|_{\mathcal{D}(K)\cap\mathcal{X}_j} : \mathcal{D}(K) \cap \mathcal{X}_j \subset \mathcal{X}_j \to \mathcal{X}_j, j = 1, 2$.

(H2) *$\mathcal{X}_1$ is finite dimensional and the spectrum of K_1 lies on the imaginary axis.*

(H3) *The imaginary axis lies in the resolvent set of K_2 and*

$$\|(K_2 - iaI)^{-1}\| \le \frac{C}{1 + |a|}, \ a \in \mathbf{R},$$

for some constant C that is independent of a.

(H4) *There exists a natural number k and neighborhoods $\Lambda \subset \mathbf{R}^n$ of 0 and $U \subset \mathcal{D}(K)$ of 0 such that N is $(k+1)$-times continuously differentiable on $\Lambda \times U$, its derivatives are bounded and uniformly continuous on $\Lambda \times U$ and $N(0, 0) = 0, d_2N[0, 0] = 0$.*

Under these hypotheses there exist neighborhoods $\tilde{\Lambda} \subset \Lambda$ *of* 0 *and* $\tilde{U}_1 \subset U \cap \mathcal{X}_1$, $\tilde{U}_2 \subset U \cap \mathcal{X}_2$ *of* 0 *and a reduction function* $h : \tilde{\Lambda} \times \tilde{U}_1 \to \tilde{U}_2 \subset \mathcal{X}_2$ *with the following properties. The reduction function* h *is continuously differentiable on* $\tilde{\Lambda} \times \tilde{U}_1$ *up to order* k*, its derivatives are bounded and uniformly continuous on* $\tilde{\Lambda} \times \tilde{U}_1$ *and*

$$h(0,0) = 0, \qquad d_2h[0,0] = 0. \tag{3.2.4}$$

The graph

$$M_C^{\tilde{\alpha}} = \left\{u_1 + h(\tilde{\alpha}, u_1) \in \tilde{U}_1 \times \tilde{U}_2 : u_1 \in \tilde{U}_1\right\}$$

is a center manifold for (3.2.1)*, so that the following statements hold.*

(1) $M_C^{\tilde{\alpha}}$ *is a locally invariant manifold of* (3.2.3)*, i.e., through every point in* $M_C^{\tilde{\alpha}}$ *there passes a unique solution of* (3.2.1) *that remains on* $M_C^{\tilde{\alpha}}$ *as long as it remains in* $\tilde{U}_1 \times \tilde{U}_2$.

(2) *Every small bounded solution* $u(t)$*,* $t \in \mathbf{R}$ *of* (3.2.1) *such that* $(u_1(t), u_2(t)) \in \tilde{U}_1 \times \tilde{U}_2, t \in \mathbf{R}$*, lies completely in* $M_C^{\tilde{\alpha}}$.

(3) *Every solution* $u_1 : \mathbf{R} \to \tilde{U}_1$ *of the reduced equation*

$$\dot{u}_1 = K_1 u_1 + N_1(\tilde{\alpha}, u_1 + h(\tilde{\alpha}, u_1)) \tag{3.2.5}$$

generates a solution $u(t) = u_1(t) + h(\tilde{\alpha}, u_1(t))$ *of the full equation* (3.2.1).

Now, we apply Theorem 3.2.1 to the system (2.2.14). (3.1.24) shows that $(K_s - i\sigma I)^{-1}$ is bounded for large $|\sigma|$. Then K_s is closed. Since the spectrum of K_s consists of isolated eigenvalues, we define the spectral projection $\tilde{\mathcal{P}}$ corresponding to the subset $\tilde{\sigma}(K_s) \cap i\mathbf{R}$ of $\tilde{\sigma}(K_s)$ by

$$\tilde{\mathcal{P}}w = \frac{1}{2\pi i} \int_C (K_s - \tilde{\lambda})^{-1} w \, d\tilde{\lambda}$$

where C is a closed curve in the resolvent set of K_s which contains $\tilde{\sigma}(K_s) \cap i\mathbf{R}$ and no other points of $\tilde{\sigma}(K_s)$. Write $\mathcal{X}_1 = \tilde{\mathcal{P}}(\mathcal{H}_s), \mathcal{X}_2 = (I - \tilde{\mathcal{P}})(\mathcal{H}_s)$. (H1) and (H2) in Theorem 3.2.1 are straightforward (see the book by Kato [69]). (3.1.24) implies (H3) (see the paper by Vanderbauwhede & Iooss [111]) and the analyticity of N gives (H4). Thus, Theorem 3.2.1 and the general theory by Mielke [82] imply the following theorem.

Theorem 3.2.2 *For any natural number* r, (2.2.14) *has a finite dimensional center manifold* $M_C^{\tilde{\alpha}}$ *of class* C^r *in the sense of* Theorem 3.2.1. *The reduced system on* $M_C^{\tilde{\alpha}}$ *also preserves reversibility and symmetries, which* (2.2.14) *has.*

By Theorem 3.2.2, (2.2.14) has a finite dimensional center manifold $M_C^{\tilde{\alpha}}$ where $M_C^{\tilde{\alpha}}$ is equipped with the single coordinate chart $\tilde{W}_1 \subset \mathcal{X}_1$ and the coordinate map $\chi : M_C^{\tilde{\alpha}} \to \tilde{W}_1$ defined by $\chi^{-1}(w_1) = w_1 + h(\tilde{\alpha}, w_1)$ where $\tilde{\alpha} = (b, \lambda, \varrho, \alpha)$.

In the following calculation, it is more convenient to use the coordinate chart $\tilde{V}_1 = (dG[0])^{-1}(\tilde{W}_1)$ so that $\tilde{V}_1$ is the center subspace of the linear operator L_s. The coordinate map $\zeta : M_C^{\tilde{\alpha}} \to \tilde{V}_1$ can be written as $\zeta^{-1}(v_1) = v_1 + k_1(\tilde{\alpha}, v_1)$ where $k_1 : \tilde{\Lambda} \times \tilde{V}_1 \to V$ is given by

$$k(\tilde{\alpha}, v_1) = G^{-1}\Big(dG[0](v_1) + h(\tilde{\alpha}, dG[0](v_1))\Big) - v_1.$$

In this coordinate system, $M_C^{\tilde{\alpha}}$ is changed to $\{v_1 + k(\tilde{\alpha}, v_1) : v_1 \in \tilde{V}_1\}$, which lies in $\mathcal{D}(F^{(b,\lambda)}(\varrho, \alpha, \cdot))$ and defines a center manifold for the system (2.2.11).

In order to obtain the equations for the center manifold, we derive the adjoint operator L_s^* of L_s. Define an inner product $(\cdot, \cdot)$ in $\mathcal{H}_s$ as follows:

$$\begin{aligned} (v, v_1) \;&=\; \int_0^1 \int_0^P (\phi_y \bar{\phi}_{1y} + \phi_z \bar{\phi}_{1z}) dy dz + \int_0^1 \int_0^P u \bar{u}_1 dy dz \\ &\quad + \int_0^P (\eta_y \bar{\eta}_{1y} + \frac{\lambda}{b} \eta \bar{\eta}_1) dy + \int_0^P \xi \bar{\xi}_1 dy, \end{aligned} \tag{3.2.6}$$

where $v = (\phi, u, \eta, \xi), v_1 = (\phi_1, u_1, \eta_1, \xi_1) \in \mathcal{H}_s$. Note that L_s is closed and densely defined.

Theorem 3.2.3 *The adjoint operator L_s^* of L_s is given by*

$$L_s^*(\phi^*, u^*, \eta^*, \xi^*) = \left(u^*, -\phi^*_{yy} - \phi^*_{zz}, \xi^*, -u^*|_{z=1} + \frac{\lambda}{b}\eta^* - \eta^*_{yy} \right) \tag{3.2.7}$$

where $(\phi^*, u^*, \eta^*, \xi^*) \in \mathcal{H}_{s+1}$ with $\phi^*_z|_{z=0} = 0, \phi^*_z|_{z=1} = -\frac{1}{b}\xi^*$. Moreover, σ is in the spectrum of L_s if and only if $\bar{\sigma}$ is in the spectrum of L_s^*.

Proof. First notice that the expression of L_s is not changed by replacing ϕ with $\phi + C$ for any constant C. A direct calculation $(L_s v, v_1) = (v, L_s^* v_1)$ with integration by parts gives the result. □

3.3 Eigenpairs of L_s and its Adjoint L_s^*

The distribution of eigenvalues of L_s is the same as that discussed by Groves & Mielke [55], although the eigenvectors are different. For the sake of completeness, we list the eigenpairs of L_s without the proof.

L_s has a zero eigenvalue with eigenvector $e_{00} = (1, 0, 0, 0)^T$ and generalized eigenvector $e_{01} = (0, 1, -1/\lambda, 0)^T$, which has a Jordan chain of length 2 for $\lambda \neq 1$, length 4 for $\lambda = 1, b \neq \frac{1}{3}$, and length 6 for $\lambda = 1, b = \frac{1}{3}$, respectively. The generalized eigenspace E_0 has a basis $\{e_{00}, e_{01}, \cdots, e_{0n}\}$ for n either 1 or 3 or 5 as appropriate. If the coordinates in the e_{00} and e_{0j} directions are (q, p_j), the action of the reverser S on this generalized eigenspace is given by either $S(q,p_1)=(-q,p_1)$, or $S(q,p_1,p_2,p_3)=(-q,p_1,-p_2,p_3)$, or $S(q,p_1,p_2,p_3,p_4,p_5)=(-q,p_1,-p_2,p_3,-p_4,p_5)$. The eigenspace is invariant under the reflector $R: y\to -y$ and the translator $T_a: y\to y+a, a\in \mathbf{R}$. Here, we remark that the eigenvector e_{00} can be removed by using the translation invariance of q (i.e., $q\to q+c$).

Any other purely imaginary eigenvalues, whose eigenvectors are independent of y, are called two-dimensional and geometrically simple. They appear in pairs $\pm iq$ with an eigenvector e satisfying $L_s e = iqe$, $L_s\bar{e}=-iq\bar{e}$. Such a eigenvalue has a Jordan chain of length 2 if and only if λ, b and q satisfy

$$b=\frac{-1}{2\sinh^2 q}+\frac{1}{2q\tanh q},\quad \lambda=\frac{q^2}{2\sinh^2 q}+\frac{q}{2\tanh q}, \tag{3.3.1}$$

which defines a curve C_0 parameterized by q in (b,λ)-space (See Figure 1.2). When $(b,\lambda)\notin C_0$, choose $\{e,\bar{e}\}$ as the basis of $E_{iq}\oplus E_{-iq}$ and $C,\bar{C}$ as the coordinates in e and $\bar{e}$ directions. The action of the reverser S on $E_{iq}\oplus E_{-iq}$ is $S(C)=\bar{C}$. In fact, since $L_s e = iqe$ and $SL_s e=-L_s Se = iqSe$, we can choose e such that $Se=\bar{e}$. When $(b,\lambda)\in C_0$, there is a generalized eigenvector f such that $(L_s-iqI)f=e, (L_s+iqI)\bar{f}=\bar{e}$. Thus, $\{e,f,\bar{e},\bar{f}\}$ is a basis for $E_{iq}\oplus E_{-iq}$. If A,B are the coordinates in e and f directions, the action of the reverser S on this space is $S(A,B)=(\bar{A},-\bar{B})$ since $S(L_s-iqI)f=Se$ yields $Sf=-\bar{f}$. Again, it is invariant under the reflector R and the translator T_a because of independence of y.

The purely imaginary eigenvalues, whose eigenvectors depend upon y-variable, are geometrically double and appear in pairs $\pm is_k$ with $L_s e^k_{\cos}=is_k e^k_{\cos}$, $L_s e^k_{\sin}=is_k e^k_{\sin}$, $L_s\bar{e}^k_{\cos}=-is_k\bar{e}^k_{\cos}$, $L_s\bar{e}^k_{\sin}=-is_k\bar{e}^k_{\sin}$, where

$$e^k_{\cos}=v_{kc}(z)\cos(2k\pi y/P),\qquad e^k_{\sin}=v_{ks}(z)\sin(2k\pi y/P),$$

and $v_{kc}(z)$ and $v_{ks}(z)$ depend only upon z. Jordan chains exist only if b,λ and s_k are related by the equations

$$b=\frac{-s_k^2+2p_k^2}{2p_k^3\tanh p_k}-\frac{s_k^2}{2p_k^2\sinh^2 p_k},\quad \lambda=\frac{3s_k^2-2p_k^2}{2p_k\tanh p_k}+\frac{s_k^2}{2\sinh^2 p_k} \tag{3.3.2}$$

where $p_k^2=s_k^2+4\pi^2k^2/P^2$, which define a curve C_k parameterized by s_k in (b,λ)-space (See Figure 1.2). If $(b,\lambda)\notin C_k$, $\{e^k_{\cos},\bar{e}^k_{\cos},e^k_{\sin},\bar{e}^k_{\sin}\}$ can be chosen as a basis for $E_{is_k}\oplus E_{-is_k}$. If $(C^k_{\cos},C^k_{\sin})$ is the coordinate in $e^k_{\cos},e^k_{\sin}$ directions, then actions of R,S and T_a on $E_{is_k}\oplus E_{-is_k}$ are given by

$$\begin{aligned} &R(C^k_{\cos},C^k_{\sin})=(C^k_{\cos},-C^k_{\sin}),\quad S(C^k_{\cos},C^k_{\sin})=(\bar{C}^k_{\cos},\bar{C}^k_{\sin})\,,\\ &T_a(C^k_{\cos},C^k_{\sin})=(C^k_{\cos},C^k_{\sin})\mathcal{R}_{2k\pi a/P}\,, \end{aligned}$$

where $\mathcal{R}_\theta$ is a 2×2 matrix representing a rotation through an angle θ. If $(b,\lambda)\in C_k$, the length of Jordan chain for each eigenvalue is 2 and there are generalized eigenvectors $f^k_{\cos}$ and $f^k_{\sin}$ such that

$$(L_s-is_kI)f^k_{\cos}=e^k_{\cos},\quad (L_s-is_kI)f^k_{\sin}=e^k_{\sin},$$

$$(L_s + is_k I)\bar{f}^k_{\cos} = \bar{e}^k_{\cos}, \quad (L_s + is_k I)\bar{f}^k_{\sin} = \bar{e}^k_{\sin},$$

with

$$f^k_{\cos} = w_k(z)\cos(2k\pi y/P), \quad f^k_{\sin} = w_k(z)\sin(2k\pi y/P)$$

for some smooth vector $w_k(z)$. Here, $\{e^k_{\cos}, \bar{e}^k_{\cos}, f^k_{\cos}, \bar{f}^k_{\cos}, e^k_{\sin}, \bar{e}^k_{\sin}, f^k_{\sin}, \bar{f}^k_{\sin}\}$ is a basis for $E_{is_k} \oplus E_{-is_k}$. If $(A^k_{\cos}, A^k_{\sin}, B^k_{\cos}, B^k_{\sin})$ is the coordinate in $e^k_{\cos}, e^k_{\sin}, f^k_{\cos}, f^k_{\sin}$ directions, the actions of R, S, T_a on this space are given by

$$\begin{aligned} R(A^k_{\cos}, A^k_{\sin}, B^k_{\cos}, B^k_{\sin}) &= (A^k_{\cos}, -A^k_{\sin}, B^k_{\cos}, -B^k_{\sin}), \\ S(A^k_{\cos}, A^k_{\sin}, B^k_{\cos}, B^k_{\sin}) &= (\bar{A}^k_{\cos}, \bar{A}^k_{\sin}, -\bar{B}^k_{\cos}, -\bar{B}^k_{\sin}), \\ T_a(A^k_{\cos}, A^k_{\sin}, B^k_{\cos}, B^k_{\sin}) &= ((A^k_{\cos}, A^k_{\sin})\mathcal{R}_{2k\pi a/P}, (B^k_{\cos}, B^k_{\sin})\mathcal{R}_{2k\pi a/P}). \end{aligned}$$

The subspace

$$\mathcal{T} = \{A^k_{\cos} = A^k_{\sin} = B^k_{\cos} = B^k_{\sin} = C^k_{\cos} = C^k_{\sin} = 0 \text{ for all } k\}$$

consists of two-dimensional vectors (which do not depend on y).

Since the system (2.2.14) is invariant under the reflection $y \to -y$, we may restrict to the waves which are even in the y-direction and have only Fourier cosine series expansions. In the following, we are interested in the cases: C_1^+ and $(b_0, 1)$ where $(b_0, 1)$ is the intersection point between the curve C_1 and the line $\lambda = 1$ (see Figure 1.2).

Chapter 4

Existence of Solutions for (b, F^{-2}) near C_1^+

In this chapter, we consider the existence of generalized solitary waves for (b, F^{-2}) near C_1^+ and the parameter $\alpha = 0$ ($P + \alpha$ is the period in the y direction). For the sake of simplicity, we consider the case $b > 0$ and $F^{-2} = \lambda + \mu$ where (b, λ) is on the curve C_1^+ and $\mu > 0$. Note that the same proof holds if $b = \tilde{b}_0 + b_1(\mu)$, $F^{-2} = \lambda + \lambda_1(\mu)$ where $(\tilde{b}_0, \lambda)$ is on C_1^+ and the curve $(\tilde{b}_0 + b_1(\mu), \lambda + \lambda_1(\mu))$ approaches $(\tilde{b}_0, \lambda)$ from the right side of C_1^+ non-tangentially with respect to C_1^+ as $\mu \to 0^+$. The purely imaginary part of the spectrum $\tilde{\sigma}(L_s)$ for (b, λ) on C_1^+ consists of two-dimensional eigenvalues $0, \pm is_{00}, \pm is_{01}$ and three-dimensional eigenvalues $\pm is_1$. Here, $\pm is_{00}$, $\pm is_{01}$ are simple and $\pm is_1$ are double. The existence result can be described as follows.

Theorem 4.1 *If s_{00}, s_{01}, s_1 satisfy a non-resonant condition, i.e., there is no nonzero integer vector (k_0, k_1, k_2) such that $s_{00}k_0 + s_{01}k_1 + s_1 k_2 = 0$, then there exist a $\mu_0 > 0$ and a continuous function $\varrho_1(\mu)$ such that for each μ with $0 < \mu < \mu_0$ and $\varrho = \varrho_1(\mu)\mu^5$, the system* (2.2.5) *for $\alpha = 0$ has a generalized solitary wave solution so that*

$$\eta(x,y) = \Big[A_0 \mu^{1/2} \mathrm{sech}((q_1\mu)^{1/2} x)\cos(s_1 x) + R_0(x;\mu)\Big]\cos(2\pi y/P) + \varsigma(x) S^{(p)}(x) + R_1(x,y;\mu)$$

where A_0 is a nonzero constant dependent on (b, λ, P), R_1 is even and periodic in y with period P, the cut-off function $\varsigma(x)$ is in $C^\infty(\mathbf{R}, \mathbf{R})$ satisfying $0 \le \varsigma(x) \le 1$ and

$$\varsigma(x) = \begin{cases} 1, & |x| \ge 2, \\ 0, & |x| \le 1, \end{cases} , \tag{4.0.1}$$

and R_0 and R_1 satisfy uniformly that

$$|R_0(x;\mu)| \le A_1 \mu e^{-(q_1\mu)^{1/2}|x|}, \quad |R_1(x,y;\mu)| \le A_2 \mu^{5/2} e^{-r\mu^{1/2}|x|}$$

for $x \in \mathbf{R}$ and some fixed constants $A_1, A_2, r > 0$. Here $q_1 = P/(2bs_1r_{10}^2)$ where $\sigma_1^2 = s_1^2 + (4\pi^2/P^2)$ and

$$r_{10}^2 = \frac{Ps_1}{4b\sigma_1^5\sinh^3(\sigma_1)}\big[(4\sigma_1^2 - 3s_1^2)\big(\sinh^2(\sigma_1)\cosh(\sigma_1) + \sigma_1\sinh(\sigma_1)\big) - 2s_1^2\sigma_1^2\cosh(\sigma_1)\big] > 0\,.$$

Also, $S^{(p)}(x)$ is a periodic function with period $2\pi/\tilde{k}$, and if $x \geq 0$,

$$S^{(p)}(x) = \hat{J}_2\mu^{9/2}\cos\tilde{k}(x+\delta) + R_3(x;\mu)$$

for some fixed amplitude $\hat{J}_2 > 0$, wave number $\tilde{k}$ and phase shift δ, with

$$|\delta| \leq A_3\sqrt{\mu}\,, \quad |\tilde{k} - s_{00}| \leq A_4\mu\,, \quad |R_3(x;\mu)| \leq A_5\mu^5\,,$$

where $A_j, j = 3, 4, 5$ are fixed constants. Moreover, the same result holds if s_{00} is replaced by s_{01}.

The expression for ϕ can be given, which is similar to that for the surface height. The graph for η is given in Figure 1.3.

4.1 Normal Form Analysis

Since the spectrum of L_0 for such (b, λ) consists entirely of isolated eigenvalues of finite algebraic multiplicity, we can write

$$\begin{aligned} v \;=\; & a_0e_{00} + a_1e_{01} + AU_{10} + BU_{11} + CU_{00} + DU_{01} + \bar{A}\bar{U}_{10} + \bar{B}\bar{U}_{11} \\ & + \bar{C}\bar{U}_{00} + \bar{D}\bar{U}_{01} + v_2 \end{aligned}$$

where e_{00} and e_{01} are the eigenvector and the generalized eigenvector of the eigenvalue 0, U_{10}, U_{11}, $\bar{U}_{10}$ and $\bar{U}_{11}$ are the eigenvector and the generalized eigenvector of the eigenvalues $\pm is_1$, U_{00}, $\bar{U}_{00}$, U_{01} and $\bar{U}_{01}$ are the eigenvectors of the eigenvalues $\pm is_{00}$, and $\pm is_{01}$, respectively, and v_2 is a linear combination of eigenvectors and generalized eigenvectors corresponding to the rest of eigenvalues. Since $e_{00} = (1, 0, 0, 0)^T$ and (2.2.11) only have terms with e_{00} that always have y- or z-derivatives of e_{00}, the right side of (2.2.11) is independent of a_0. Thus, a_1, A, B, C, D, their complex conjugates and v_2 are independent of a_0. Now we apply the center manifold reduction theorem 3.2.2, and obtain that all small bounded solutions of (2.2.11) are of the form

$$\begin{aligned} v \;=\; & a_0e_{00} + a_1e_{01} + AU_{10} + BU_{11} + CU_{00} + DU_{01} + \bar{A}\bar{U}_{10} + \bar{B}\bar{U}_{11} \\ & + \bar{C}\bar{U}_{00} + \bar{D}\bar{U}_{01} + \Phi(\mu, \varrho, a_0, a_1, A, B, C, D, \bar{A}, \bar{B}, \bar{C}, \bar{D}) \end{aligned} \tag{4.1.1}$$

where a_0, a_1 are real and Φ is independent of a_0 and contains terms which are at least quadratic in its arguments with $v_2 = \Phi$. Moreover, the amplitudes satisfy the reduced system

$$\frac{d}{dx}X = LX + F_0(\mu, \varrho, X), \tag{4.1.2}$$

where $X = (a_0, a_1, A, B, C, D, \bar{A}, \bar{B}, \bar{C}, \bar{D})^T$, L is given by

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & is_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & is_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & is_{00} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & is_{01} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -is_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -is_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -is_{00} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -is_{01} \end{pmatrix}$$

and $F_0(\mu, 0, 0) = 0$, $D_X F_0(0, 0, 0) = 0$ and $|F_0(0, 0, X)| = O(|X|^2)$. Here, $F_0(\mu, \varrho, X)$ is independent of a_0.

By the identity (2.2.10), a_1 can be solved as a function of A, B, C, D and their conjugates. Therefore, we can drop the equations of a_0 and a_1 in (4.1.2) and only focus on the equations for A, B, C, D and their complex conjugates. For the sake of simplicity, we still use (4.1.2) to denote the reduced systems with $X = (A, B, C, D, \bar{A}, \bar{B}, \bar{C}, \bar{D})^T$.

To compute the normal form of (4.1.2), let us first concentrate on the equations for A, B, C, D. Note that the reverser S is given by

$$S : \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \rightarrow \begin{pmatrix} \bar{A} \\ -\bar{B} \\ \bar{C} \\ \bar{D} \end{pmatrix},$$

and that $SF_0 = -F_0 S, SL = -LS$. Write F_0 as the part with ϱ and the part without ϱ,

$$F_0 = F_1(\mu, A, B, C, D, \bar{A}, \bar{B}, \bar{C}, \bar{D}) + \varrho F_2(\mu, \varrho, A, B, C, D, \bar{A}, \bar{B}, \bar{C}, \bar{D}).$$

For the sake of convenience, we let $\mu = 0$ and only consider F_1 at this moment. From the general theory of normal forms (see Theorem 2 in the paper by Elphick *et al.* [44] for a characterization at any order, or Section I.1.3 in the book by Iooss & Adelmeyer [64]), it is found that there exists a change of variables from X to Y, which is close to an identity and transforms the system (4.1.2) with F_1 only into

$$\frac{d}{dx}Y = LY + \mathcal{P}(Y) + o(|Y|^n), \tag{4.1.3}$$

where $\mathcal{P}$ is a polynomial of degree $\leq n$ (n is arbitrary but fixed), with $\mathcal{P}(0) = 0$ and $D\mathcal{P}(0) = 0$. For notational simplicity, we still use $X = (A, B, C, D, \bar{A}, \bar{B}, \bar{C}, \bar{D})^T$ for Y. Here, $\mathcal{P}$ satisfies $S\mathcal{P}(X) = -\mathcal{P}(SX)$ and

$$D\mathcal{P}(X)L^*X = L^*\mathcal{P}(X), \tag{4.1.4}$$

for any X where $L^* = \bar{L}^T$ (See Theorem I.11 on page 23 in the book by Iooss & Adelmeyer [64]).

Let $\mathcal{P} = (P_0, Q_0, P_1, P_2, \bar{P}_0, \bar{Q}_0, \bar{P}_1, \bar{P}_2)^T$ and define a differential operator

$$D^* = -is_1 A\frac{\partial}{\partial A} + (A - is_1 B)\frac{\partial}{\partial B} - is_{00}C\frac{\partial}{\partial C} - is_{01}D\frac{\partial}{\partial D} + c.c. \tag{4.1.5}$$

where *c.c.* denotes the complex conjugates. Then, (4.1.4) is equivalent to $D^*\mathcal{P} = L^*\mathcal{P}$, which gives

$$D^*P_0 = -is_1 P_0, \qquad D^*Q_0 = -is_1 Q_0 + P_0, \tag{4.1.6}$$

$$D^*P_1 = -is_{00}P_1, \qquad D^*P_2 = -is_{01}P_2 \tag{4.1.7}$$

and their complex conjugates.

To determine $\mathcal{P}$, seven independent first integrals of $D^* = 0$ are needed and can be found as follows,

$$\begin{array}{l} u_1 = A\bar{A},\ u_2 = \frac{i}{2}(A\bar{B} - \bar{A}B),\ u_3 = C\bar{C},\ u_4 = D\bar{D}, \\ u_5 = \frac{B}{A} + \frac{1}{is_1}\ln A,\ u_6 = \frac{B}{A} + \frac{1}{is_{00}}\ln C,\ u_7 = \frac{B}{A} + \frac{1}{is_{01}}\ln D. \end{array} \tag{4.1.8}$$

Then we have an important lemma and its proof is given in Section 6.1.

Lemma 4.1.1 *Assume H is a polynomial of X with degree n and $D^*H = 0$. Then $H(X) = Q(u_1, u_2, u_3, u_4)$, where Q is a polynomial of its arguments, provided that there is no nonzero integer vector (k_0, k_1, k_2) such that $s_{00}k_0 + s_{01}k_1 + s_1k_2 = 0$.*

Now, we calculate the components of the polynomial $\mathcal{P}$ in (4.1.3). From (4.1.6), it is found that

$$D^*(\bar{A}P_0) = (D^*\bar{A})P_0 + \bar{A}D^*P_0 = is_1\bar{A}P_0 - is_1\bar{A}P_0 = 0.$$

By Lemma 4.1.1, $\bar{A}P_0 = P_0^*(u_1, u_2, u_3, u_4)$ is a polynomial of its arguments. Since $P_0 = P_0^*/\bar{A}$ is also a polynomial, by the forms of u_1, u_2, u_3, u_4 in terms of $A, \bar{A}$, it is deduced that $P_0 = A\tilde{P}_0(u_1, u_2, u_3, u_4)$ where $\tilde{P}_0$ is a polynomial of its arguments. Moreover, let $Q_0^* = B\tilde{P}_0(u_1, u_2, u_3, u_4)$. Then, by using $P_0 = A\tilde{P}_0 = \frac{A}{B}Q_0^*$ and $D^*(\bar{A}P_0) = 0$, we obtain

$$0 = D^*\left(\frac{A\bar{A}}{B}Q_0^*\right) = -is_1\frac{A\bar{A}}{B}Q_0^* + is_1\frac{A\bar{A}}{B}Q_0^* + (A - is_1B)\frac{A\bar{A}}{-B^2}Q_0^* + \frac{A\bar{A}}{B}D^*Q_0^*$$

$$= (A - is_1B)\frac{A\bar{A}}{-B^2}Q_0^* + \frac{A\bar{A}}{B}D^*Q_0^*,$$

and $D^*Q_0^* = \frac{A-is_1B}{B}Q_0^* = \frac{A}{B}Q_0^* - is_1Q_0^* = -is_1Q_0^* + P_0$, which, together with the Q_0-equation in (4.1.6), yields

$$D^*(Q_0 - Q_0^*) = -is_1(Q_0 - Q_0^*) \quad \text{or} \quad D^*(\bar{A}(Q_0 - Q_0^*)) = 0\,.$$

Therefore, again, $Q_0 = Q_0^* + A\tilde{Q}_0 = B\tilde{P}_0 + A\tilde{Q}_0$, where $\tilde{Q}_0$ is a polynomial of $u_j, j = 1, 2, 3, 4$. A similar argument gives

$$P_1 = C\tilde{P}_1(u_1, u_2, u_3, u_4), \qquad P_2 = D\tilde{P}_2(u_1, u_2, u_3, u_4)\,,$$

where $\tilde{P}_1, \tilde{P}_2$ are polynomials of their arguments.

Hence, we can write (4.1.2) as

$$\begin{aligned}
\dot{A} &= is_1A + B + A\tilde{P}_0(u_1, u_2, u_3, u_4) + h.o.t., \\
\dot{B} &= is_1B + B\tilde{P}_0(u_1, u_2, u_3, u_4) + A\tilde{Q}_0(u_1, u_2, u_3, u_4) + h.o.t., \\
\dot{C} &= is_{00}C + C\tilde{P}_1(u_1, u_2, u_3, u_4) + h.o.t., \\
\dot{D} &= is_{01}D + D\tilde{P}_2(u_1, u_2, u_3, u_4) + h.o.t.
\end{aligned} \tag{4.1.9}$$

and their complex conjugates, where $h.o.t.$ stands for the terms of A, B, C, D and their complex conjugates with orders equal to and higher than $n + 1$. Since the reversibility is preserved, we find that

$$\begin{aligned}
\tilde{P}_0(u_1, u_2, u_3, u_4) &= -\overline{\tilde{P}_0(u_1, u_2, u_3, u_4)}, \qquad \tilde{Q}_0(u_1, u_2, u_3, u_4) = \overline{\tilde{Q}_0(u_1, u_2, u_3, u_4)}, \\
\tilde{P}_1(u_1, u_2, u_3, u_4) &= -\overline{\tilde{P}_1(u_1, u_2, u_3, u_4)}, \qquad \tilde{P}_2(u_1, u_2, u_3, u_4) = -\overline{\tilde{P}_2(u_1, u_2, u_3, u_4)}.
\end{aligned}$$

Therefore, if let $\tilde{P}_0 = i\hat{P}_0, \tilde{Q}_0 = \hat{Q}_0, \tilde{P}_1 = i\hat{P}_1$ and $\tilde{P}_2 = i\hat{P}_2$ where $\hat{P}_0, \hat{Q}_0, \hat{P}_1$ and $\hat{P}_2$ are real, then we rewrite (4.1.9) as (for notational simplicity, drop hats)

$$\begin{aligned}
\dot{A} &= is_1A + B + iAP_0(A\bar{A}, \tfrac{i}{2}(A\bar{B} - \bar{A}B), C\bar{C}, D\bar{D}) + h.o.t., \\
\dot{B} &= is_1B + iBP_0(A\bar{A}, \tfrac{i}{2}(A\bar{B} - \bar{A}B), C\bar{C}, D\bar{D}) \\
&\quad + AQ_0(A\bar{A}, \tfrac{i}{2}(A\bar{B} - \bar{A}B), C\bar{C}, D\bar{D}) + h.o.t., \\
\dot{C} &= is_{00}C + iCP_1(A\bar{A}, \tfrac{i}{2}(A\bar{B} - \bar{A}B), C\bar{C}, D\bar{D}) + h.o.t., \\
\dot{D} &= is_{01}D + iDP_2(A\bar{A}, \tfrac{i}{2}(A\bar{B} - \bar{A}B), C\bar{C}, D\bar{D}) + h.o.t.
\end{aligned} \tag{4.1.10}$$

and their complex conjugates, where P_0, Q_0, P_1, P_2 are real polynomials with respect to their arguments.

A similar analysis holds if μ is not zero (more details can be found in Section I.20 on page 35 in the book by Iooss & Adelmeyer [64]). Thus, (4.1.10) becomes

$$\begin{aligned}
\dot{A} &= is_1A + B + iAP_0(\mu, A\bar{A}, i(A\bar{B} - \bar{A}B), C\bar{C}, D\bar{D}) \\
&\quad +O(|(A,B,C,D)||(\mu,A,B,C,D)|^n), \\
\dot{B} &= is_1B + iBP_0(\mu, A\bar{A}, i(A\bar{B} - \bar{A}B), C\bar{C}, D\bar{D}) \\
&\quad +AQ_0(\mu, A\bar{A}, i(A\bar{B} - \bar{A}B), C\bar{C}, D\bar{D}) \\
&\quad +O(|(A,B,C,D)||(\mu,A,B,C,D)|^n), \\
\dot{C} &= is_{00}C + iCP_1(\mu, A\bar{A}, i(A\bar{B} - \bar{A}B), C\bar{C}, D\bar{D}) \\
&\quad +O(|(A,B,C,D)||(\mu,A,B,C,D)|^n), \\
\dot{D} &= is_{01}D + iDP_2(\mu, A\bar{A}, i(A\bar{B} - \bar{A}B), C\bar{C}, D\bar{D}) \\
&\quad +O(|(A,B,C,D)||(\mu,A,B,C,D)|^n)
\end{aligned} \tag{4.1.11}$$

and their complex conjugates, where P_0, Q_0, P_1, P_2 are real polynomials of their arguments such that $P_0(0) = Q_0(0) = P_1(0) = P_2(0) = 0$ satisfying

$$\begin{aligned}
P_0(\mu, u_1, u_2, u_3, u_4) &= p_1\mu + p_2u_1 + p_3u_2 + p_4u_3 + p_5u_4 \\
&\quad +O(|(\mu, u_1, u_2, u_3, u_4)|^2), \\
Q_0(\mu, u_1, u_2, u_3, u_4) &= q_1\mu + q_2u_1 + q_3u_2 + q_4u_3 + q_5u_4 \\
&\quad +O(|(\mu, u_1, u_2, u_3, u_4)|^2), \\
P_2(\mu, u_1, u_2, u_3, u_4) &= p_{21}\mu + p_{22}u_1 + p_{23}u_2 + p_{24}u_3 + p_{25}u_4 \\
&\quad +O(|(\mu, u_1, u_2, u_3, u_4)|^2),
\end{aligned} \tag{4.1.12}$$

and $q_1 > 0$, $q_2 < 0$ and $p_{21} \neq 0$ (which are to be verified later in (6.3.14), (6.3.13) and (6.3.15)). If the term ϱF_2 is included, the system (4.1.11) can be written as

$$\dot{X} = F(\mu, X) + \mathcal{R}(\mu, X) + \varrho\wp(\mu, \varrho, X), \tag{4.1.13}$$

where the remainder $\mathcal{R}$ is of order $O(|(A,B,C,D)||(\mu,A,B,C,D)|^n)$, $\varrho\wp$ is the term of ϱF_2 after transformation, and the fourth component $\wp_{01}$ of $\wp$ is written as

$$\begin{aligned}
\wp_{01}(\mu, \varrho, A, B, C, D, \bar{A}, \bar{B}, \bar{C}, \bar{D}) &= d_{20} + d_{21}\mu + d_{22}A + d_{23}\bar{A} + d_{24}A^2 \\
&\quad +d_{25}A\bar{A} + d_{26}\bar{A}^2 + d_{27}B + d_{28}\bar{B} + \cdots .
\end{aligned} \tag{4.1.14}$$

Here, $d_{20} = -i\frac{P}{r_{01}^2}$, $d_{21} = 0$ (which are to be verified later in (6.3.16) and (6.3.17), and r_{01} is defined in (6.3.3)), constants d_{2j} must be pure imaginary or zero for $j = 2, \cdots, 6$ and d_{27}, d_{28} must be real since $S\wp_{01} = -\wp_{01}S$. By choosing k in (H4) of Theorem 3.2.1 sufficiently large, one may take n arbitrarily large and let $\mathcal{R}$ and $\wp$ have arbitrarily high order derivatives with respect to their arguments. In the following, we say that a function is smooth if it has arbitrarily high order derivatives with respect to its arguments.

To find the solutions of (4.1.13) for small μ, we let $X = \sqrt{\mu}\hat{X}$, which changes (4.1.13) to

$$\dot{\hat{X}} = \hat{F}(\mu, \hat{X}) + \epsilon\hat{\mathcal{R}}(\mu, \hat{X}) + \hat{\varrho}\hat{\wp}(\mu, \sqrt{\mu}\hat{\varrho}, \hat{X}) \tag{4.1.15}$$

where $\hat{F}(\mu,\hat{X}) = \frac{1}{\sqrt{\mu}}F(\mu,X)$, $\hat{\mathcal{R}}(\mu,\hat{X}) = \frac{1}{\sqrt{\mu^n}}\mathcal{R}(\mu,X)$, $\hat{\wp}(\mu,\varrho,\hat{X}) = \wp(\mu,\varrho,X)$, $\epsilon = \sqrt{\mu^{n-1}}$, and $\varrho = \sqrt{\mu}\hat{\varrho}$. Then the problem of the existence of generalized solitary wave solutions of (4.1.13) is equivalent to that of (4.1.15). In the following, we concentrate on (4.1.15). Now we adopt the method introduced by Groves &d Mielke [55], that is, ϵ will be used to activate or deactivate the remainder terms which are neglected in the dominant system $\dot{\hat{X}} = \hat{F}(\mu,\hat{X})$. It is treated as a parameter which is independent of μ, so that $\mu \in (0,\mu_0]$ and $\epsilon \in [0,\epsilon_0]$ for some positive constants μ_0 and ϵ_0. The compatibility of the results thus obtained with the relationship $\epsilon = \sqrt{\mu^{n-1}}$ will be checked.

Now we are ready to give a proof of the existence of generalized solitary wave solutions of (4.1.15). Here we note that two-dimensional solutions of (4.1.15) correspond to the case with $\{\hat{A} = \hat{B} = 0\}$. Therefore, the space $\{\hat{A} = \hat{B} = 0\}$ is an invariant subspace of (4.1.15).

4.2 Two-Dimensional Periodic Solutions

Firstly, we apply the classical Lyapunov-Schmidt method to show that (4.1.15) has two-dimensional periodic solutions which will determine the forms of the generalized solitary wave solutions at infinity. The construction is very standard and the general theory for reversible systems has been discussed by Kielhöfer [71]. Here, we need to have a family of periodic solutions with appropriate properties, where $\mu, \epsilon > 0$ are arbitrary small and fixed. The amplitude of the periodic solution and ϱ are parameters that vary. Since we need the estimates of the periodic solutions in terms of μ and other parameters, which are not explicitly given in [71], a sketch of the proof for the existence of periodic solutions and their estimates will be given in this section.

Let $\tilde{x} = (s_{00} + r_1)x$ where r_1 is a real and small constant to be determined. Denote $H^m([0,2\pi])$ a space of periodic functions of $\tilde{x}$ with a period 2π such that their derivatives up to order m are in $L^2([0,2\pi])$ with a norm defined by

$$\|f\|_m^2 = \sum_{n\in\mathbf{Z}}(1+n^{2m})|f_n|^2$$

where $f = \sum_n f_n e^{in\tilde{x}} \in H^m([0,2\pi])$. Therefore, we consider periodic solutions of (4.1.15) with period $2\pi/(s_{00}+r_1)$.

Let $\hat{A} = \hat{B} = 0$ and $\hat{C}_p$ and $\hat{D}_p$ denote the third and fourth components in (4.1.15), which, from (4.1.11) and (4.1.15), satisfy the equations

$$\begin{aligned}\hat{C}_p' =& \frac{1}{s_{00}+r_1}[is_{00}\hat{C}_p + \tilde{H}(\mu,\epsilon,\hat{\varrho},r_1,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)],\\ \hat{D}_p' =& \frac{1}{s_{00}+r_1}[is_{01}\hat{D}_p + \tilde{G}(\mu,\epsilon,\hat{\varrho},r_1,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)]\end{aligned} \quad (4.2.1)$$

and their complex conjugates, where $'$ means taking derivatives with respect to $\tilde{x}$,

$$\begin{aligned}
&\tilde{H}(\mu,\epsilon,\hat{\varrho},r_1,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p) = i\hat{C}_p P_1(\mu,0,0,\mu|\hat{C}_p|^2,\mu|\hat{D}_p|^2)\\
&\quad +\epsilon\check{\mathcal{R}}_{00}(\mu,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p) + \hat{\varrho}\check{\wp}_{00}(\mu,\sqrt{\mu}\hat{\varrho},\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p),\\
&\tilde{G}(\mu,\epsilon,\hat{\varrho},r_1,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p) = i\hat{D}_p P_2(\mu,0,0,\mu|\hat{C}_p|^2,\mu|\hat{D}_p|^2)\\
&\quad +\epsilon\check{\mathcal{R}}_{01}(\mu,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p) + \hat{\varrho}\check{\wp}_{01}(\mu,\sqrt{\mu}\hat{\varrho},\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p),
\end{aligned} \tag{4.2.2}$$

and $\check{\mathcal{R}}_{00}$, $\check{\wp}_{00}$, $\check{\mathcal{R}}_{01}$ and $\check{\wp}_{01}$ are the 3rd and 4th components of $\hat{\mathcal{R}}$ and $\hat{\wp}$ in (4.1.15) when $\hat{A} = \hat{B} = 0$, respectively.

The system (4.2.1) corresponds to four purely imaginary eigenvalues $\pm i$ and $\pm i s_{01}/s_{00}$. Without loss of generality, assume that $s_{00} > s_{01}$. Since the solution is periodic with period 2π in $\tilde{x}$, we write

$$\hat{C}_p(\tilde{x}) = \sum_n C_n \mathrm{e}^{in\tilde{x}}, \quad \hat{D}_p(\tilde{x}) = \sum_n D_n \mathrm{e}^{in\tilde{x}} \tag{4.2.3}$$

where $\hat{C}_p, \hat{D}_p$ are in $H^1([0,2\pi])$, and C_n and D_n are complex numbers. Substitute (4.2.3) into (4.2.1) and make the coefficient of each term in the Fourier series equal, which yields

$$C_n = \frac{1}{i((n-1)s_{00}+nr_1)}\left[\tilde{H}(\mu,\epsilon,\hat{\varrho},r_1,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)\right]_n, \qquad \text{for } n \neq 1, \tag{4.2.4}$$

$$D_n = \frac{1}{i(ns_{00}-s_{01}+nr_1)}\left[\tilde{G}(\mu,\epsilon,\hat{\varrho},r_1,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)\right]_n, \tag{4.2.5}$$

and

$$ir_1 C_1 = \left[\tilde{H}(\mu,\epsilon,\hat{\varrho},r_1,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)\right]_1, \tag{4.2.6}$$

together with their complex conjugates, where $n \in \mathbf{Z}$ and $[f]_n$ is the n-th Fourier coefficient of f.

In the following, we will first solve (4.2.4) and (4.2.5) for $C_n, n \neq 1$ and D_n, and then solve (4.2.6) for r_1 by making C_1 as a free constant to be chosen later.

Fix C_1, and define two spaces

$$\begin{aligned}
H_1^1([0,2\pi]) &= \{f(\tilde{x}) = \sum_n f_n e^{in\tilde{x}} \in H^1([0,2\pi]) \mid f_1 = 0\ \},\\
H_{-1}^1([0,2\pi]) &= \{f(\tilde{x}) = \sum_n f_n e^{in\tilde{x}} \in H^1([0,2\pi]) \mid f_{-1} = 0\ \}.
\end{aligned}$$

For $\tilde{U} \in H_1^1([0,2\pi])$ and $\tilde{V} \in H^1([0,2\pi])$, define a mapping $\Theta(\tilde{U},\tilde{V},\bar{\tilde{U}},\bar{\tilde{V}};\varpi)$ from $H_1^1([0,2\pi]) \times H^1([0,2\pi]) \times H_{-1}^1([0,2\pi]) \times H^1([0,2\pi])$ to itself by

$$\Theta(\tilde{U},\tilde{V},\bar{\tilde{U}},\bar{\tilde{V}};\varpi) =$$

$$\begin{pmatrix} \sum_{n\neq 1} \frac{1}{i((n-1)s_{00}+nr_1)} \left[\tilde{H}(\mu,\epsilon,\hat{\varrho},r_1,\tilde{U}+C_1e^{i\tilde{x}},\tilde{V},\bar{\tilde{U}}+\bar{C}_1e^{-i\tilde{x}},\bar{\tilde{V}})\right]_n e^{in\tilde{x}} \\ \sum_{n} \frac{1}{i(ns_{00}-s_{01}+nr_1)} \left[\tilde{G}(\mu,\epsilon,\hat{\varrho},r_1,\tilde{U}+C_1e^{i\tilde{x}},\tilde{V},\bar{\tilde{U}}+\bar{C}_1e^{-i\tilde{x}},\bar{\tilde{V}})\right]_n e^{in\tilde{x}} \\ \sum_{n\neq 1} \frac{1}{i((-n+1)s_{00}-nr_1)} \left[\bar{\tilde{H}}(\mu,\epsilon,\hat{\varrho},r_1,\tilde{U}+C_1e^{i\tilde{x}},\tilde{V},\bar{\tilde{U}}+\bar{C}_1e^{-i\tilde{x}},\bar{\tilde{V}})\right]_{-n} e^{-in\tilde{x}} \\ \sum_{n} \frac{1}{i(-ns_{00}+s_{01}-nr_1)} \left[\bar{\tilde{G}}(\mu,\epsilon,\hat{\varrho},r_1,\tilde{U}+C_1e^{i\tilde{x}},\tilde{V},\bar{\tilde{U}}+\bar{C}_1e^{-i\tilde{x}},\bar{\tilde{V}})\right]_{-n} e^{-in\tilde{x}} \end{pmatrix} \tag{4.2.7}$$

where $\varpi = (\mu, \epsilon, \hat{\varrho}, r_1, C_1, \bar{C}_1)$. Assume that $B_r(0)$ is a ball with a radius r in the space $H_1^1([0,2\pi]) \times H^1([0,2\pi]) \times H_{-1}^1([0,2\pi]) \times H^1([0,2\pi])$.

Lemma 4.2.1 *For* $(\tilde{U},\tilde{V},\bar{\tilde{U}},\bar{\tilde{V}}),(\tilde{U}_1,\tilde{V}_1,\bar{\tilde{U}}_1,\bar{\tilde{V}}_1),(\tilde{U}_2,\tilde{V}_2,\bar{\tilde{U}}_2,\bar{\tilde{V}}_2) \in \bar{B}_r(0)$ *and small* ϖ,

$$\begin{aligned} &\|\Theta(\tilde{U},\tilde{V},\bar{\tilde{U}},\bar{\tilde{V}};\varpi)\|_1 \le M\left((\mu+\epsilon\sqrt{\mu}+|\hat{\varrho}|)(\|\tilde{U}\|_1+\|\tilde{V}\|_1+|C_1|)+|\hat{\varrho}|\right), \\ &\|\Theta(\tilde{U}_1,\tilde{V}_1,\bar{\tilde{U}}_1,\bar{\tilde{V}}_1;\varpi)-\Theta(\tilde{U}_2,\tilde{V}_2,\bar{\tilde{U}}_2,\bar{\tilde{V}}_2;\varpi)\|_1 \\ &\qquad \le M(\mu+\epsilon\sqrt{\mu}+|\hat{\varrho}|)(\|\tilde{U}_1-\tilde{U}_2\|_1+\|\tilde{V}_1-\tilde{V}_2\|_1) \end{aligned}$$

where $M>0$ *is a constant bounded uniformly for any bounded* ϖ *and* r.

Proof. For the simplicity, we use $\tilde{H}$ and $\tilde{H}_j$ to denote $\tilde{H}(\mu,\epsilon,\hat{\varrho},r_1,\tilde{U}+C_1e^{i\tilde{x}},\tilde{V},\bar{\tilde{U}}+\bar{C}_1e^{-i\tilde{x}},\bar{\tilde{V}})$ and $\tilde{H}(\mu,\epsilon,\hat{\varrho},r_1,\tilde{U}_j+C_1e^{i\tilde{x}},\tilde{V}_j,\bar{\tilde{U}}_j+\bar{C}_1e^{-i\tilde{x}},\bar{\tilde{V}}_j)$ for $j=1,2$, respectively. Note that $P_1, P_2, \check{\mathcal{R}}_{00}, \check{\mathcal{R}}_{01}, \check{\wp}_{00}$ and $\check{\wp}_{01}$ in (4.2.2) are smooth enough in their arguments and $H^1([0,2\pi])$ is embedded in $L^\infty([0,2\pi])$. Thus, $\tilde{H}, \tilde{G}$ map $H_1^1([0,2\pi]) \times H^1([0,2\pi]) \times H_{-1}^1([0,2\pi]) \times H^1([0,2\pi])$ into $L^2([0,2\pi])$. From the expression of $\tilde{H}$ in (4.2.2), it is straightforward to show

$$\begin{aligned} &\|\tilde{H}\|_0 \le M\left((\mu+\epsilon\sqrt{\mu}+|\hat{\varrho}|)(\|\tilde{U}\|_1+\|\tilde{V}\|_1+|C_1|)+|\hat{\varrho}|\right), \\ &\|\tilde{H}_1-\tilde{H}_2\|_0 \le M(\mu+\epsilon\sqrt{\mu}+|\hat{\varrho}|)(\|\tilde{U}_1-\tilde{U}_2\|_1+\|\tilde{V}_1-\tilde{V}_2\|_1) \end{aligned} \tag{4.2.8}$$

for small ϖ where the facts that P_1 has a factor μ and $\check{\mathcal{R}}_{00}$ has a factor $\sqrt{\mu}$ have been used. The similar estimates hold for $\tilde{G}$. From the definition of $\Theta(\tilde{U},\tilde{V},\bar{\tilde{U}},\bar{\tilde{V}};\varpi)$ in (4.2.7), we can easily see that the n-th Fourier coefficient of $\Theta(\tilde{U},\tilde{V},\bar{\tilde{U}},\bar{\tilde{V}};\varpi)$ has a factor of $1/n$, which gives that Θ is in $H^1([0,2\pi])$ if $\tilde{H}$ and $\tilde{G}$ are in $L^2([0,2\pi])$. The lemma is then just an easy consequence of (4.2.8). □

Now, we take $r = 2M((\mu+\epsilon\sqrt{\mu}+|\hat{\varrho}|)|C_1|+|\hat{\varrho}|)$ with $M(\mu+\epsilon\sqrt{\mu}+|\hat{\varrho}|) \le 1/2$ for small enough μ and $\hat{\varrho}$. Then Lemma 4.2.1 shows that Θ is a contraction mapping on $\bar{B}_r(0)$ for small ϖ. Therefore, the fixed point theorem yields that Θ has a unique fixed point which is a smooth function of ϖ. Write this fixed point as

$$(\hat{C}_p^0, \hat{D}_p^0, \bar{\hat{C}}_p^0, \bar{\hat{D}}_p^0)(\mu,\epsilon,\hat{\varrho},r_1,C_1,\bar{C}_1)(\tilde{x}) \tag{4.2.9}$$

which satisfies

$$
\begin{aligned}
&(\hat{C}_p^0, \hat{D}_p^0, \bar{\hat{C}}_p^0, \bar{\hat{D}}_p^0)(\mu, \epsilon, 0, r_1, 0, 0)(\tilde{x}) = 0, \text{ for all } \tilde{x} \in [0, 2\pi], \\
&\|\hat{C}_p^0\|_1 + \|\hat{D}_p^0\|_1 + \|\bar{\hat{C}}_p^0\|_1 + \|\bar{\hat{D}}_p^0\|_1 \leq 2M((\mu + \epsilon\sqrt{\mu} + |\hat{\varrho}|)|C_1| + |\hat{\varrho}|). \quad (4.2.10)
\end{aligned}
$$

Moreover, the same argument holds for the solution in $H^m([0, 2\pi])$ with any integer $m > 0$, which also satisfies (4.2.10). Thus, by the uniqueness, the fixed point in (4.2.9) is in $H^m([0, 2\pi])$ and satisfies (4.2.10) with $H^m([0, 2\pi])$-norm for any $m > 0$. For notational simplicity, we use $(\hat{C}_p, \hat{D}_p, \bar{\hat{C}}_p, \bar{\hat{D}}_p)(\tilde{x})$ to denote

$$
\left(\hat{C}_p^0(\tilde{x}) + C_1 e^{i\tilde{x}}, \hat{D}_p^0(\tilde{x}), \bar{\hat{C}}_p^0(\tilde{x}) + \bar{C}_1 e^{-i\tilde{x}}, \bar{\hat{D}}_p^0(\tilde{x})\right).
$$

Now we solve (4.2.6) for r_1. Substitute (4.2.9) into (4.2.6) and obtain

$$
-ir_1 C_1 + g_1(\mu, \epsilon, \hat{\varrho}, r_1, C_1, \bar{C}_1) = 0, \quad (4.2.11)
$$

where

$$
\begin{aligned}
g_1(\mu, \epsilon, \hat{\varrho}, r_1, C_1, \bar{C}_1) &= i[\hat{C}_p P_1(\mu, 0, 0, \mu|\hat{C}_p|^2, \mu|\hat{D}_p|^2)]_1 \\
&+\epsilon\left[\check{\mathcal{R}}_{00}(\mu, \hat{C}_p, \hat{D}_p, \bar{\hat{C}}_p, \bar{\hat{D}}_p)\right]_1 + \hat{\varrho}\left[\check{\wp}_{00}(\mu, \sqrt{\mu}\hat{\varrho}, \hat{C}_p, \hat{D}_p, \bar{\hat{C}}_p, \bar{\hat{D}}_p)\right]_1
\end{aligned}
$$

is smooth when $\mu, \epsilon, \hat{\varrho}, r_1, C_1, \bar{C}_1$ are near 0.

Define two operators $\hat{S}$ and τ_θ by

$$
\hat{S}\left(\hat{C}_p, \hat{D}_p\right)(\tilde{x}) = S\left(\hat{C}_p, \hat{D}_p\right)(-\tilde{x}), \quad (4.2.12)
$$

$$
\tau_\theta\left(\hat{C}_p, \hat{D}_p\right)(\tilde{x}) = \left(\hat{C}_p, \hat{D}_p\right)(\tilde{x} + \theta) \quad (4.2.13)
$$

for any real number θ. If $\left(\hat{C}_p, \hat{D}_p\right)(\tilde{x})$ is a solution of (4.2.1), so are (4.2.12) and (4.2.13) since (4.2.1) has the reversibility property and the translation invariance. Using $S(\hat{C}_p, \hat{D}_p) = (\bar{\hat{C}}_p, \bar{\hat{D}}_p)$ and the fact that

$$
S\left(\hat{C}_p, \hat{D}_p\right)(-\tilde{x}) = \left(\bar{\hat{C}}_p, \bar{\hat{D}}_p\right)(-\tilde{x}) = \left(\sum_n \bar{C}_n \mathrm{e}^{in\tilde{x}}, \sum_n \bar{D}_n \mathrm{e}^{in\tilde{x}}\right) \quad (4.2.14)
$$

is a solution, we can replace C_n, D_n by $\bar{C}_n, \bar{D}_n$ to obtain a similar equation of (4.2.11), which is

$$
-ir_1 \bar{C}_1 + g_1(\mu, \epsilon, \hat{\varrho}, r_1, \bar{C}_1, C_1) = 0. \quad (4.2.15)
$$

(4.2.11) and (4.2.15) imply that

$$
g_1(\mu, \epsilon, \hat{\varrho}, r_1, \bar{C}_1, C_1) = -\overline{g_1(\mu, \epsilon, \hat{\varrho}, r_1, C_1, \bar{C}_1)}. \quad (4.2.16)
$$

From (4.2.13), we have

$$\begin{aligned}\left(\hat{C}_p, \hat{D}_p\right)(\tilde{x}+\theta) &= \left(\sum_n C_n \mathrm{e}^{in(\tilde{x}+\theta)}, \sum_n D_n \mathrm{e}^{in(\tilde{x}+\theta)}\right) \\ &= \left(\sum_n C_n \mathrm{e}^{in\theta}\mathrm{e}^{in\tilde{x}}, \sum_n D_n \mathrm{e}^{in\theta}\mathrm{e}^{in\tilde{x}}\right). \end{aligned} \tag{4.2.17}$$

By a similar argument as that for (4.2.15), it is obtained that

$$-ir_1 C_1 \mathrm{e}^{i\theta} + g_1(\mu, \epsilon, \hat{\varrho}, r_1, C_1\mathrm{e}^{i\theta}, \bar{C}_1\mathrm{e}^{-i\theta}) = 0,$$

which together with (4.2.11) yields

$$g_1(\mu, \epsilon, \hat{\varrho}, r_1, \mathrm{e}^{i\theta}C_1, \mathrm{e}^{-i\theta}\bar{C}_1) = \mathrm{e}^{i\theta} g_1(\mu, \epsilon, \hat{\varrho}, r_1, C_1, \bar{C}_1). \tag{4.2.18}$$

By choosing appropriate θ in (4.2.18), we have that for $C_1 = \mathrm{e}^{i\alpha_1}|C_1|$

$$g_1(\mu, \epsilon, \hat{\varrho}, r_1, C_1, \bar{C}_1) = \mathrm{e}^{i\alpha_1}\tilde{g}_0(\mu, \epsilon, \hat{\varrho}, r_1, |C_1|)\,.$$

(4.2.16) and the definition of g_1 in (4.2.11) show that $\tilde{g}_0$ is purely imaginary and has at least a factor of $|C_1|$. Thus, we can write

$$\tilde{g}_0(\mu, \epsilon, \hat{\varrho}, r_1, |C_1|) = i|C_1|\tilde{g}_1(\mu, \epsilon, \hat{\varrho}, r_1, |C_1|)\,.$$

Hence, $g_1(\mu, \epsilon, \hat{\varrho}, r_1, C_1, \bar{C}_1) = iC_1\tilde{g}_1(\mu, \epsilon, \hat{\varrho}, r_1, |C_1|)$, where $\tilde{g}_1$ is real and smooth in its arguments. Thus, (4.2.11) gives $r_1 = \tilde{g}_1(\mu, \epsilon, \hat{\varrho}, r_1, |C_1|)$. From the definition of g_1 in (4.2.11) and (4.2.6) and the fact that P_1 in (4.2.2) has a factor μ and $\check{\mathcal{R}}_{00}$ has a factor $\sqrt{\mu}$, we can use a similar proof of Lemma 4.2.1 to show that $\tilde{g}_1$ is a contraction mapping satisfying $|\tilde{g}_1| \le 2M((\mu+\epsilon\sqrt{\mu}+|\hat{\varrho}|)|C_1|+|\hat{\varrho}|)$. Thus, by the fixed point theorem, $\tilde{g}_1$ has a unique fixed point

$$r_1 = r_1(\mu, \epsilon, \hat{\varrho}, |C_1|) \tag{4.2.19}$$

as a smooth real-valued function for small $(\mu, \epsilon, \hat{\varrho}, |C_1|)$ satisfying

$$|r_1| \le 2M(\mu + \epsilon\sqrt{\mu} + |\hat{\varrho}|). \tag{4.2.20}$$

Thus, (4.2.1) has a periodic solution $\hat{C}_p(\mu, \epsilon, \hat{\varrho}, C_1, \bar{C}_1)(\tilde{x})$ and $\hat{D}_p(\mu, \epsilon, \hat{\varrho}, C_1, \bar{C}_1)(\tilde{x})$ which belong to $H^m[(0, 2\pi)]$ if $\mu \in (0, \mu_0]$, $\epsilon \in [0, \epsilon_0]$, $|\hat{\varrho}| \in [0, \mu^{1/4}\varrho_0]$, $|C_1| \in [0, J_1]$ where μ_0, ϵ_0, ϱ_0, and J_1 are fixed positive and small constants.

In the following, we choose

$$C_1 = \hat{J}_1 > 0 \tag{4.2.21}$$

and, by the relations $\tilde{x} = (s_{00} + r_1)x$ and $\varrho = \mu^{1/2}\hat{\varrho}$, write the periodic solution $\hat{C}_p$ and $\hat{D}_p$ as $\hat{C}_p(\mu, \epsilon, \varrho, \hat{J}_1)(x)$ and $\hat{D}_p(\mu, \epsilon, \varrho, \hat{J}_1)(x)$ with the frequency

$$\omega_1(\mu, \epsilon, \varrho, \hat{J}_1) = s_{00} + r_1(\mu, \epsilon, \varrho, \hat{J}_1) \tag{4.2.22}$$

for $\mu \in (0, \mu_0], \epsilon \in [0, \epsilon_0], |\varrho| \in [0, \mu^{3/4}\varrho_0], \hat{J}_1 \in [0, J_1]$. Moreover, the solution is reversible since C_1 is real.

Define

$$\begin{aligned} \hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(x) \;\; = \;\; & \big(0, 0, \hat{C}_p(\mu, \epsilon, \varrho, \hat{J}_1)(x), \hat{D}_p(\mu, \epsilon, \varrho, \hat{J}_1)(x), \\ & 0, 0, \overline{\hat{C}_p(\mu, \epsilon, \varrho, \hat{J}_1)}(x), \overline{\hat{D}_p(\mu, \epsilon, \varrho, \hat{J}_1)}(x)\big)^T \end{aligned} \tag{4.2.23}$$

for $\hat{J}_1 > 0$ small, which is smooth for x and small $(\mu, \epsilon, \varrho, \hat{J}_1)$ and $\hat{X}_{\mu,\epsilon,0,0}(x) = 0$. Then, $\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(x)$ is a periodic solution of (4.1.15) with frequency $\omega_1(\mu, \epsilon, \varrho, \hat{J}_1)$, which from (4.2.10) satisfies that for any $m > 0$

$$\left\|\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(x)\right\|_{H^m([0,2\pi])} \leq M_1 \left(\hat{J}_1 + \frac{|\varrho|}{\mu^{1/2}}\right) \tag{4.2.24}$$

where

$$\varrho \in [0, \mu^{3/4}\varrho_0]\,. \tag{4.2.25}$$

The Sobolev embedding theorem gives that (4.2.24) holds also in $C_B^m(\mathbf{R})$-norm, which is a space of continuously differentiable functions up to order m with a supreme norm.

4.3 Existence of Solitary Wave Solutions

In this section, we focus on the existence of generalized solitary wave solutions of (4.1.15) which tend to the periodic solutions $\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}$ obtained in Section 4.2 as $x \to \pm\infty$. The following theorem will be proved, which implies Theorem 4.1.

Theorem 4.3.1 *There exist two positive constants μ_0 and $\hat{J}_2$. For $\mu \in (0, \mu_0]$, $\epsilon \in [0, \mu^{11/2}]$ and $\hat{J}_1 = \frac{\epsilon}{\mu^2}\hat{J}_2$, there exist two continuous functions θ_1 and ϱ_1 of μ with $\varrho = \frac{\epsilon}{\mu}\varrho_1$ so that* (4.1.15) *has a generalized solitary wave solution which is reversible and approaches the periodic solution $\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(x + \theta_1)$ as $x \to +\infty$, provided that*

(1) *there is no nonzero integer vector (k_0, k_1, k_2) such that $s_{00}k_0 + s_{01}k_1 + s_1k_2 = 0$,*

(2) *$2q_1p_{22} - q_2p_{21} \neq 0$, where q_1, q_2, p_{21}, p_{22} are defined in* (4.1.12).

Remark 4.3.1 *If take $P = 2\pi$, $b = 0.101765$ and $\lambda = 2.25737$, then $s_1 = 5$, $s_{00} = 6.31271$, $s_{01} = 3.49137$, and $2q_1p_{22} - q_2p_{21} = 238.705 \neq 0$.*

Note that the condition (4.2.25) is automatically true under the assumption of Theorem 4.3.1.

The basic idea for the proof of this theorem is similar to the one given by Groves & Mielke [55]. We divide it into three steps. Firstly, we show that the system $\dot{\hat{X}} = \hat{F}(\mu, \hat{X})$ has a homoclinic orbit approaching 0 as $x \to \pm\infty$ when $\hat{C} = \hat{D} = 0$. Then we prove that (4.1.15) has a solution exponentially approaching the periodic solution $\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(x + \theta_1)$ for some phase shift θ_1 as $x \to +\infty$. Finally, using the reversibility and adjusting the Bernoulli constant ϱ, we show that this solution can be extended to $x \in (-\infty, 0)$ and approaches $S\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(-x + \theta_1)$ as $x \to -\infty$ as well. In the following, we use M to denote a positive constant.

Step 1: Solutions of (4.1.15) with $\epsilon = \varrho = 0$

In this case, the system of (4.1.15) is

$$\dot{\hat{X}} = \hat{F}(\mu, \hat{X}), \tag{4.3.1}$$

which is reversible in the same way as (4.1.15) and has an extra symmetry

$$R_{\theta_0}\hat{F}(\mu, \hat{X}) = \hat{F}(\mu, R_{\theta_0}\hat{X}), \tag{4.3.2}$$

where $R_{\theta_0} : (\hat{A}, \hat{B}, \hat{C}, \hat{D}) \to (\mathrm{e}^{i\theta_0}\hat{A}, \mathrm{e}^{i\theta_0}\hat{B}, \hat{C}, \hat{D})$.

If $\hat{C} = \hat{D} = 0$, (4.3.1) is changed to

$$\begin{aligned} \dot{\hat{A}} &= is_1\hat{A} + \hat{B} + i\hat{A}P_0(\mu, \mu\hat{A}\bar{\hat{A}}, i\mu(\hat{A}\bar{\hat{B}} - \bar{\hat{A}}\hat{B}), 0, 0), \\ \dot{\hat{B}} &= is_1\hat{B} + i\hat{B}P_0(\mu, \mu\hat{A}\bar{\hat{A}}, i\mu(\hat{A}\bar{\hat{B}} - \bar{\hat{A}}\hat{B}), 0, 0) \\ &\quad + \hat{A}Q_0(\mu, \mu\hat{A}\bar{\hat{A}}, i\mu(\hat{A}\bar{\hat{B}} - \bar{\hat{A}}\hat{B}), 0, 0) \end{aligned} \tag{4.3.3}$$

together with their complex conjugates. By (4.1.12),

$$Q_0(\mu, 0, 0, 0, 0) = \gamma^2 > 0 \tag{4.3.4}$$

with $\gamma > 0$ if $\mu > 0$. From Iooss & Pérouème [68], (4.3.3) has a homoclinic orbit $R_{\Omega_0 x+\theta_0}\hat{\mathcal{H}}_0(x)$ for $\theta_0 \in [0, 2\pi)$ and μ small positive, which is reversible for $\theta_0 \in \{0, \pi\}$, where $\Omega_0 = s_1 + P_0(\mu, 0, 0, 0, 0)$. Here, $\hat{\mathcal{H}}_0 = (\hat{\mathcal{H}}_A, \hat{\mathcal{H}}_B, \bar{\hat{\mathcal{H}}}_A, \bar{\hat{\mathcal{H}}}_B)^T$ is a solution of the equations

$$\begin{aligned} \dot{\hat{A}} &= \hat{B} + i\hat{A}[P_0(\mu, \mu\hat{A}\bar{\hat{A}}, i\mu(\hat{A}\bar{\hat{B}} - \bar{\hat{A}}\hat{B}), 0, 0) - P_0(\mu, 0, 0, 0, 0)], \\ \dot{\hat{B}} &= i\hat{B}[P_0(\mu, \mu\hat{A}\bar{\hat{A}}, i\mu(\hat{A}\bar{\hat{B}} - \bar{\hat{A}}\hat{B}), 0, 0) - P_0(\mu, 0, 0, 0, 0)] \\ &\quad + \hat{A}Q_0(\mu, \mu\hat{A}\bar{\hat{A}}, i\mu(\hat{A}\bar{\hat{B}} - \bar{\hat{A}}\hat{B}), 0, 0) \end{aligned} \tag{4.3.5}$$

and their complex conjugates, satisfying

$$\bar{\hat{\mathcal{H}}}_A(x) = \hat{\mathcal{H}}_A(-x),\ \bar{\hat{\mathcal{H}}}_B(x) = -\hat{\mathcal{H}}_B(-x) \tag{4.3.6}$$

and

$$\begin{aligned} \hat{\mathcal{H}}_A &= \frac{\sqrt{q_1}}{\sqrt{-q_2/2}}\mathrm{sech}\big(\gamma x\big) + O(\sqrt{\mu}), \\ \hat{\mathcal{H}}_B &= -\frac{\sqrt{\mu}q_1}{\sqrt{-q_2/2}}\mathrm{sech}\big(\gamma x\big)\tanh\big(\gamma x\big) + O(\mu) \end{aligned} \tag{4.3.7}$$

with

$$|\hat{\mathcal{H}}_0(x)| \le M\mathrm{e}^{-|\nu||x|},\ x \in \mathbf{R} \tag{4.3.8}$$

for every $|\nu| \le \gamma$. Thus, $R_{\Omega_0 x+\theta_0}\hat{\mathcal{H}}(x)$ is a reversible homoclinic orbit of (4.3.1) for $\theta_0 \in \{0, \pi\}$ which approaches the origin as $x \to \pm\infty$ where

$$\hat{\mathcal{H}} = (\hat{\mathcal{H}}_A, \hat{\mathcal{H}}_B, 0, 0, \bar{\hat{\mathcal{H}}}_A, \bar{\hat{\mathcal{H}}}_B, 0, 0)^T. \tag{4.3.9}$$

Step 2: Solutions of (4.1.15) with $\epsilon \neq 0, \varrho \neq 0$ for $x \in [0, +\infty)$

To find the solutions of (4.1.15) near $R_{\Omega_0 x+\theta_0}\hat{\mathcal{H}}(x)$ with $\epsilon \neq 0, \varrho \neq 0$, we assume that the solution of (4.1.15) has a form

$$\hat{X}(x; \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1) = R_{\Omega_0 x+\theta_0}\big(\hat{\mathcal{H}}(x) + \hat{Z}(x)\big) + \varsigma(x)\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(x + \theta_1) \tag{4.3.10}$$

for $x \in [0, \infty)$ and $\theta_0, \theta_1 \in [0, 2\pi)$ where ς is given in (4.0.1) and $\hat{Z}(x)$ is a perturbation term to be determined, which tends exponentially to 0 as $x \to +\infty$ so that $\hat{X}$ is a solution of (4.1.15) that approaches the periodic orbit $\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(x + \theta_1)$.

Since $R_{\Omega_0 x+\theta_0}\hat{\mathcal{H}}(x)$ is a solution of (4.1.15) with $\epsilon = \varrho = 0$ and $\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}$ is a solution of (4.1.15), after plugging (4.3.10) into (4.1.15), we obtain

$$\frac{d\hat{Z}}{dx} = \mathcal{L}(x)(\hat{Z}) + N_1(\hat{Z}, x, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1) + \epsilon N_2(\hat{Z}, x, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1) \tag{4.3.11}$$

where $\mathcal{L}(x) = d_2\hat{F}[\mu, \hat{\mathcal{H}}(x)] - L_{\Omega_0}$, $L_{\Omega_0}(A, B, C, D) = (i\Omega_0 A, i\Omega_0 B, 0, 0)$, d_2 means taking the Fréchet derivative with respect to the second variable, and

$$\begin{aligned} &N_1(\hat{Z}, x, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1) = \hat{F}(\mu, \hat{\mathcal{H}} + \hat{Z} + R_{-\Omega_0 x-\theta_0}\varsigma(x)\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}) \\ &\qquad -\hat{F}(\mu, \hat{\mathcal{H}}) - d_2\hat{F}[\mu, \hat{\mathcal{H}}](\hat{Z}) - \varsigma(x)\hat{F}(\mu, R_{-\Omega_0 x-\theta_0}\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}), \\ &N_2(\hat{Z}, x, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1) = R_{-\Omega_0 x-\theta_0}\tilde{\mathcal{R}}(\mu, \varrho, R_{\Omega_0 x+\theta_0}(\hat{\mathcal{H}} + \hat{Z}) + \varsigma(x)\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}) \\ &\qquad\qquad -\varsigma(x)R_{-\Omega_0 x-\theta_0}\tilde{\mathcal{R}}(\mu, \varrho, \hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}) - \tfrac{\varsigma'(x)}{\epsilon}R_{-\Omega_0 x-\theta_0}\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}, \\ &\tilde{\mathcal{R}}(\mu, \varrho, \hat{X}) = \hat{\mathcal{R}}(\mu, \hat{X}) + \tfrac{\varrho}{\epsilon\sqrt{\mu}}\wp(\mu, \varrho, \hat{X}). \end{aligned} \tag{4.3.12}$$

Here, the fact that $R_{\theta_0}\hat{F}(\mu, X) = \hat{F}(\mu, R_{\theta_0}X)$ is used.

The functions N_1 and N_2 have the following estimates whose proofs are similar ones in Proposition 4.4 by Groves & Mielke [55].

Lemma 4.3.1 *For $x \geq 0$, if $|\hat{Z}| \leq M_0$, then the functions N_1 and N_2 satisfy*

$$\begin{array}{l} |N_1(\hat{Z}, x, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1)| \leq M\sqrt{\mu}(\hat{J}_1(\mathrm{e}^{-\nu x} + |\hat{Z}|) + |\hat{Z}|^2), \\ |N_2(\hat{Z}, x, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1)| \leq M\left((\sqrt{\mu} + \frac{|\varrho_1|}{\mu})(\mathrm{e}^{-\nu x} + |\hat{Z}|) + \frac{\hat{J}_1}{\epsilon}e^{-\nu x}\right), \\ |N_1(\hat{Z}_1, x, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1) - N_1(\hat{Z}_2, x, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1)| \\ \qquad \leq M\sqrt{\mu}(\hat{J}_1 + |\hat{Z}_1| + |\hat{Z}_2|)|\hat{Z}_1 - \hat{Z}_2|, \\ |N_2(\hat{Z}_1, x, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1) - N_2(\hat{Z}_2, x, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1)| \\ \qquad \leq M(\sqrt{\mu} + \frac{|\varrho_1|}{\mu})|\hat{Z}_1 - \hat{Z}_2|. \end{array} \tag{4.3.13}$$

Proof. Since $\hat{F}$ in (4.3.12) is a polynomial, we write

$$\hat{F}(\mu, \hat{Z}) = L\hat{Z} + M_2(\hat{Z}, \hat{Z}) + \cdots + M_n(\hat{Z}, \cdots, \hat{Z})$$

where M_j is a polynomial of degree j. The terms of degree 1 in N_1 are canceled. Now, consider the terms of degree k in N_1 with $2 \leq k \leq n$. Here, we only check the case for $x \geq 2$. The case for $0 \leq x \leq 2$ is easier. For $x \geq 2$, we denote

$$\begin{array}{l} \mathcal{M}_k(\hat{Z}) = M_k(\hat{\mathcal{H}} + \hat{Z} + R_{-\Omega_0 x - \theta_0}\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}, \cdots, \hat{\mathcal{H}} + \hat{Z} + R_{-\Omega_0 x - \theta_0}\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}) \\ - M_k(\hat{\mathcal{H}}, \cdots, \hat{\mathcal{H}}) - kM_k(\hat{\mathcal{H}}, \cdots, \hat{\mathcal{H}}, \hat{Z}) - M_k(R_{-\Omega_0 x - \theta_0}\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}, \cdots, R_{-\Omega_0 x - \theta_0}\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}) . \end{array}$$

By $\hat{F}(\mu, \hat{X}) = \frac{1}{\sqrt{\mu}}F(\mu, \sqrt{\mu}\hat{X})$, we obtain that the terms in $\mathcal{M}_k(\hat{Z})$ can be divided into two types. One only consists of $\hat{\mathcal{H}}$ and $\hat{Z}$, which must have a factor $\hat{Z}^m$ with $m \geq 2$, since the terms with $m = 0, 1$ are canceled. Another one has at least a factor of $R_{-\Omega_0 x - \theta_0}\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}$. Therefore, by (4.2.24), (4.3.8) and $|\hat{Z}| \leq M_0$,

$$|\mathcal{M}_k(\hat{Z})| \leq M\sqrt{\mu}\Big(|\hat{Z}|^2 + (\hat{J}_1 + \frac{|\varrho|}{\sqrt{\mu}})(\mathrm{e}^{-\nu x} + |\hat{Z}|)\Big) \leq M\sqrt{\mu}\Big(|\hat{Z}|^2 + \hat{J}_1(\mathrm{e}^{-\nu x} + |\hat{Z}|)\Big) .$$

Here, we have used the fact that $\hat{J}_1 \geq \frac{|\varrho|}{\sqrt{\mu}}$ for small μ, which is from the assumptions $\hat{J}_1 = \epsilon\hat{J}_2/\mu^2$ and $\varrho = \epsilon\varrho_1/\mu$ with finite $\hat{J}_2 > 0$ and ϱ_1 stated in the Theorem 4.3.1. Thus, the first inequality of (4.3.13) is obtained. The smoothness of $\tilde{\mathcal{R}}$ and $\varrho = \frac{\epsilon}{\mu}\varrho_1$ directly imply the second inequality of (4.3.13).

To check the third inequality of (4.3.13), consider $\mathcal{M}_k(\hat{Z}_1) - \mathcal{M}_k(\hat{Z}_2)$. From the above argument, we know that $\mathcal{M}_k(\hat{Z})$ has terms either with a factor $\hat{Z}^m$ and $m \geq 2$ or with a factor $R_{-\Omega_0 x - \theta_0}\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}$. Therefore, by (4.2.24), (4.3.8) and $|\hat{Z}| \leq M_0$ again,

$$|\mathcal{M}_k(\hat{Z}_1) - \mathcal{M}_k(\hat{Z}_2)| \leq M\sqrt{\mu}\Big((|\hat{Z}_1| + |\hat{Z}_2|)|\hat{Z}_1 - \hat{Z}_2| + (\hat{J}_1 + \frac{|\varrho|}{\sqrt{\mu}})|\hat{Z}_1 - \hat{Z}_2|\Big)$$

$$\leq M\sqrt{\mu}\Big(|\hat{Z}_1| + |\hat{Z}_2| + \hat{J}_1\Big)|\hat{Z}_1 - \hat{Z}_2|.$$

The proof of the last one in (4.3.13) is similar. □

Obviously, the solution $\hat{Z}$ of (4.3.11) exists if x is in a finite interval and an initial condition is given. The idea for proving the existence of solutions for $x \geq 0$ is to change (4.3.11) to integral equations and then use the fixed point theorem to prove the existence of a fixed point of the integral equations.

The linear equation of (4.3.11) can be rewritten as

$$\frac{d\hat{Z}}{dx} = \mathcal{L}_\infty \hat{Z} + (\mathcal{L}(x) - \mathcal{L}_\infty)(\hat{Z}) \tag{4.3.14}$$

where $\mathcal{L}_\infty$ is given by

$$\mathcal{L}_\infty = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\Omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\Omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i\Omega_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\Omega_2 \end{pmatrix},$$

$\Omega_1 = s_{00} + P_1(\mu, 0, 0, 0, 0)$, $\Omega_2 = s_{01} + P_2(\mu, 0, 0, 0, 0)$ and γ is defined in (4.3.4). The spectrum of $\mathcal{L}_\infty$ can be calculated directly. It consists of six eigenvalues

$$\lambda_1 = \gamma,\ \lambda_2 = -\gamma,\ \lambda_3 = i\Omega_1,\ \lambda_4 = -\lambda_3,\ \lambda_5 = i\Omega_2,\ \lambda_6 = -\lambda_5.$$

To find a fundamental matrix of (4.3.14), we note that

$$\frac{d}{dx} R_\theta \hat{\mathcal{H}}(x) = \hat{F}(\mu, R_\theta \hat{\mathcal{H}}(x)) - L_{\Omega_0} R_\theta \hat{\mathcal{H}}(x). \tag{4.3.15}$$

Differentiating (4.3.15) with respect to x or θ at $\theta = 0$ yields that

$$s_1(x) = \frac{1}{\sqrt{\mu}} \frac{d}{dx} \hat{\mathcal{H}}(x)\,, \quad s_2(x) = (i\hat{\mathcal{H}}_A, i\hat{\mathcal{H}}_B, 0, 0, -i\bar{\hat{\mathcal{H}}}_A, -i\bar{\hat{\mathcal{H}}}_B, 0, 0)^T$$

are two linearly independent solutions of (4.3.14) that decay exponentially as $x \to +\infty$. It is easy to check that

$$Ss_j(0) = -s_j(0),\ j = 1, 2, \tag{4.3.16}$$

satisfying

$$|s_j(x)| \leq M e^{-\gamma x} \qquad \text{for} \qquad x \in [0, +\infty)\,. \tag{4.3.17}$$

Since $\{\hat{C} = \hat{D} = 0\}$ is an invariant space of (4.3.14), if u_j^∞, $j = 1, 2$, are the eigenvectors of λ_1 for $\mathcal{L}_\infty$, then by Problem 29 in Chapter 3 of [29], we can find solutions $u_1(x), u_2(x)$ of (4.3.14) with forms $u_j = (\hat{A}_j, \hat{B}_j, 0, 0, \bar{\hat{A}}_j, \bar{\hat{B}}_j, 0, 0)^T, j = 1, 2$, such that

$$u_j(x)\mathrm{e}^{-\gamma x} \to \mathrm{Span}\{u_j^\infty, j = 1, 2\} \text{ as } x \to \infty,\ |u_j(x)| \le M\mathrm{e}^{\gamma x} \text{ for } x \in [0, \infty). \quad (4.3.18)$$

Without loss of generality, we can assume

$$Su_j(0) = u_j(0),\ j = 1, 2, \quad (4.3.19)$$

since otherwise we can add linear combinations of $s_1(x)$ and $s_2(x)$ to u_j so that (4.3.19) holds. Moreover, if d_k^∞ $(k = 3, \cdots, 6)$ are the eigenvectors of λ_k for $\mathcal{L}_\infty$, we can apply Problem 29 in Chapter 3 of [29] again to find that (4.3.14) has solutions $d_k(x)$ of forms $d_k = (0, 0, \hat{C}_k, \hat{D}_k, 0, 0, \bar{\hat{C}}_k, \bar{\hat{D}}_k)^T, k = 3, \cdots, 6$ (again $\{\hat{A} = \hat{B} = 0\}$ is an invariant space of (4.3.14)) such that as $x \to \infty$,

$$d_k(x) \to \mathrm{Span}\{d_k^\infty \mathrm{e}^{\lambda_k x}, k = 3, \cdots, 6\}, \qquad |d_k(x)| \le M \text{ for } x \in [0, \infty) \quad (4.3.20)$$

if $k = 3, \cdots, 6$. It is obvious that

$$Sd_k(0) = d_k(0) \quad (4.3.21)$$

holds for the first two components of $d_k(x)$ as well as the fifth and sixth components of $d_k(x)$. Now, the matrix

$$\mathcal{B}(x) = \big(s_1(x)|s_2(x)|u_1(x)|u_2(x)|d_3(x)|d_4(x)|d_5(x)|d_6(x)\big)$$

is a fundamental matrix for (4.3.14). By a direct calculation of $\mathcal{L}(x)$ and using the property that P_0, Q_0, P_1 and P_2 defined in (4.1.11) are all real, it is straightforward to show that the sum of first four entries on the diagonal line of $\mathcal{L}(x)$ is purely imaginary. It is also known that the last four rows of $\mathcal{L}(x)$ are complex conjugates of the first four rows, which implies that the trace of $\mathcal{L}(x)$ is equal to 0 and $\det(\mathcal{B}(x))$ is independent of x. Moreover, by taking $x \to \infty$,

$$\det(\mathcal{B}(x)) \;=\; \det\big(s_1(x)|s_2(x)|u_1(x)|u_2(x)|d_3(x)|d_4(x)|d_5(x)|d_6(x)\big) = M\gamma^2. \quad (4.3.22)$$

Let the fundamental set of solutions for the adjoint equation of (4.3.14) be

$$\{s_1^*(x), s_2^*(x), u_1^*(x), u_2^*(x), d_3^*(x), d_4^*(x), d_5^*(x), d_6^*(x)\}$$

which is the dual of $\{s_1(x), s_2(x), u_1(x), u_2(x), d_3(x), d_4(x), d_5(x), d_6(x)\}$ in the sense of the Euclidean inner product on $\mathbf{C}^8$ for each fixed x. It follows from (4.3.17), (4.3.18) and (4.3.20) that for $x \in [0, \infty)$

$$\begin{array}{l} |u_1^*(x)| + |u_2^*(x)| \le \frac{M}{\mu}\mathrm{e}^{-\gamma x}, \qquad |s_1^*(x)| + |s_2^*(x)| \le \frac{M}{\mu}\mathrm{e}^{\gamma x}, \\ |d_3^*(x)| + |d_4^*(x)| + |d_5^*(x)| + |d_6^*(x)| \le \frac{M}{\mu} \end{array} \quad (4.3.23)$$

where the factor $1/\mu$ is from the fact that (4.3.22) is used as a denominator when the Cramer's rule is applied to find solutions of the adjoint equation of (4.3.14).

The solution of (4.3.11) that decays to zero at infinity can be found as

$$\begin{aligned}\hat{Z} =& \sum_{j=1}^{2}\int_0^x < N_1(\hat{Z}, s, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1) + \epsilon N_2(\hat{Z}, s, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1), s_j^*(s) > ds\ s_j(x)\\ &-\sum_{j=1}^{2}\int_x^\infty < N_1(\hat{Z}, s, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1) + \epsilon N_2(\hat{Z}, s, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1), u_j^*(s) > ds\ u_j(x)\\ &-\sum_{j=3}^{6}\int_x^\infty < N_1(\hat{Z}, s, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1) + \epsilon N_2(\hat{Z}, s, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1), d_j^*(s) > ds\ d_j(x)\end{aligned}$$

or

$$\hat{Z} = \mathcal{F}(\epsilon, \hat{Z}), \tag{4.3.24}$$

where $< \cdot >$ denotes the Euclidean inner product on $\mathbf{C}^8$. Here, we note that because of special structures of N_1, N_2 and s_j^*, u_j^*, d_j^* (i.e., the first four components are complex conjugates of last four components), the inner products in (4.3.24) are real numbers.

Choose ν such that $\nu/\gamma \in (\frac{1}{2}, 1)$ is fixed, where γ is given in (4.3.4). Consider (4.3.24) as a fixed point problem in a Banach space

$$E_\nu = \big\{\hat{Z} \in C([0,\infty) \times S^1 \times S^1) \mid \sup_{x\in[0,+\infty)} \{|\hat{Z}(x, \theta_0, \theta_1)|\mathrm{e}^{\nu x}\} < \infty\big\}$$

with the norm

$$\|\hat{Z}\|_\nu = \sup\big\{|\hat{Z}(x, \theta_0, \theta_1)|\mathrm{e}^{\nu x} \mid x \in [0,\infty)\big\}.$$

The following lemma is needed.

Lemma 4.3.2 *For $x \geq 0$, the function $\mathcal{F}$ satisfies*

$$\begin{aligned}&\|\mathcal{F}(\epsilon, \hat{Z})\|_\nu \leq \tfrac{M}{\mu^{3/2}}\big[\sqrt{\mu}\|\hat{Z}\|_\nu^2 + \sqrt{\mu}\hat{J}_1\|\hat{Z}\|_\nu + \hat{J}_1\big],\\ &\|\mathcal{F}(\epsilon, \hat{Z}_1) - \mathcal{F}(\epsilon, \hat{Z}_2)\|_\nu \leq \tfrac{M}{\mu}\big[\hat{J}_1 + \|\hat{Z}_1\|_\nu + \|\hat{Z}_2\|_\nu\big]\|\hat{Z}_1 - \hat{Z}_2\|_\nu\end{aligned} \tag{4.3.25}$$

for $\hat{Z}, \hat{Z}_1, \hat{Z}_2 \in E_v$.

Proof. For simplicity, $N_1(\hat{Z}, s), N_2(\hat{Z}, s)$ are used to denote

$$N_1(\hat{Z}, s, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1), \quad N_2(\hat{Z}, s, \mu, \epsilon, \varrho, \theta_0, \theta_1, \hat{J}_1),$$

respectively. By (4.3.13), we know that

$$|\epsilon N_2(\hat{Z}, s)| \ \leq\ M\left(\epsilon\big(\sqrt{\mu} + \frac{|\varrho_1|}{\mu}\big)(\mathrm{e}^{-\nu x} + |\hat{Z}|) + \hat{J}_1 e^{-\nu x}\right)$$

$$\leq \ M\left(\sqrt{\mu}\big(\hat{J}_1|\hat{Z}|+|\hat{Z}|^2\big)+\hat{J}_1 e^{-\nu x}\right)$$

since $\hat{J}_1=\frac{\epsilon}{\mu^2}\hat{J}_2$ and $\mu>0$ is small. By (4.3.17), (4.3.18), (4.3.20) and (4.3.23), it is obtained that

$$\begin{aligned}
&\Big|\int_0^x < N_1(\hat{Z},s)+\epsilon N_2(\hat{Z},s), s_1^*(s) > ds\ s_1(x)\Big| \mathrm{e}^{\nu x}\\
&\leq \int_0^x \frac{M}{\mu}\left(\sqrt{\mu}\big(\hat{J}_1\|\hat{Z}\|_\nu \mathrm{e}^{-\nu s}+\|\hat{Z}\|_\nu^2\mathrm{e}^{-2\nu s}\big)+\hat{J}_1\mathrm{e}^{-\nu s}\right)\mathrm{e}^{\gamma s}ds\mathrm{e}^{-(\gamma-\nu)x}\\
&\leq \frac{M}{\mu}\frac{1}{\gamma}\left[\frac{\hat{J}_1}{(1-\nu/\gamma)}(1+\sqrt{\mu}\|\hat{Z}\|_\nu)+\frac{\sqrt{\mu}}{2\nu/\gamma-1}\|\hat{Z}\|_\nu^2\right]\\
&\leq \frac{M}{\mu^{3/2}}\big[\hat{J}_1(1+\sqrt{\mu}\|\hat{Z}\|_\nu)+\sqrt{\mu}\|\hat{Z}\|_\nu^2\big]\,,
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_x^\infty < N_1(\hat{Z},s)+\epsilon N_2(\hat{Z},s), u_1^*(s) > ds\ u_1(x)\Big| \mathrm{e}^{\nu x}\\
&\leq \int_x^\infty \frac{M}{\mu}\left(\sqrt{\mu}\big(\hat{J}_1\|\hat{Z}\|_\nu \mathrm{e}^{-\nu s}+\|\hat{Z}\|_\nu^2\mathrm{e}^{-2\nu s}\big)+\hat{J}_1\mathrm{e}^{-\nu s}\right)\mathrm{e}^{-\gamma s}ds\mathrm{e}^{(\gamma+\nu)x}\\
&\leq \frac{M}{\mu\gamma}\left[\frac{\hat{J}_1}{(1+\nu/\gamma)}(1+\sqrt{\mu}\|\hat{Z}\|_\nu)+\frac{\sqrt{\mu}}{2\nu/\gamma+1}\|\hat{Z}\|_\nu^2\right]\\
&\leq \frac{M}{\mu^{3/2}}\big[\hat{J}_1(1+\sqrt{\mu}\|\hat{Z}\|_\nu)+\sqrt{\mu}\|\hat{Z}\|_\nu^2\big]\,,
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_x^\infty < N_1(\hat{Z},s)+\epsilon N_2(\hat{Z},s), d_3^*(s) > ds\ d_3(x)\Big| \mathrm{e}^{\nu x}\\
&\leq \int_x^\infty \frac{M}{\mu}\left(\sqrt{\mu}\big(\hat{J}_1\|\hat{Z}\|_\nu \mathrm{e}^{-\nu s}+\|\hat{Z}\|_\nu^2\mathrm{e}^{-2\nu s}\big)+\hat{J}_1\mathrm{e}^{-\nu s}\right)ds\mathrm{e}^{\nu x}\\
&\leq \frac{M}{\mu}\frac{1}{\gamma}\left[\frac{\hat{J}_1}{\nu/\gamma}(1+\sqrt{\mu}\|\hat{Z}\|_\nu)+\frac{\sqrt{\mu}}{2\nu/\gamma}\|\hat{Z}\|_\nu^2\right]\leq \frac{M}{\mu^{3/2}}\big[\hat{J}_1(1+\sqrt{\mu}\|\hat{Z}\|_\nu)+\sqrt{\mu}\|\hat{Z}\|_\nu^2\big]\,,
\end{aligned}$$

since $\nu/\gamma\in(\frac{1}{2},1)$ with $\gamma=\sqrt{Q_0(\mu,0,0,0,0)}$ and $\frac{\sqrt{\mu}}{\gamma}$ is bounded. Similarly, other terms can be estimated so that the first inequality of (4.3.25) is proved.

For the second inequality of (4.3.25), from (4.3.13), it is found that

$$\begin{aligned}
&\big|\epsilon N_2(\hat{Z}_1,x,\mu,\epsilon,\theta_0,\theta_1,\theta_2,\hat{J}_1)-\epsilon N_2(\hat{Z}_2,x,\mu,\epsilon,\theta_0,\theta_1,\theta_2,\hat{J}_1)\big|\\
&\quad\leq \epsilon M\left(\sqrt{\mu}+\frac{|\varrho_1|}{\mu}\right)|\hat{Z}_1-\hat{Z}_2|\leq M\sqrt{\mu}\Big(\hat{J}_1+|\hat{Z}_1|+|\hat{Z}_2|\Big)|\hat{Z}_1-\hat{Z}_2|,
\end{aligned}$$

which gives

$$\begin{aligned}
&\Big|\int_0^x < N_1(\hat{Z}_1,s)-N_1(\hat{Z}_2,s)+\epsilon(N_2(\hat{Z}_1,s)-N_2(\hat{Z}_2,s)),s_1^*(s)> ds\ s_1(x)\Big|e^{\nu x}\\
&\le \int_0^x \frac{M}{\mu}\sqrt{\mu}\big[\hat{J}_1e^{\nu s}+\|\hat{Z}_1\|_\nu+\|\hat{Z}_2\|_\nu\big]\|\hat{Z}_1-\hat{Z}_2\|_\nu e^{(\gamma-2\nu)s}ds e^{-(\gamma-\nu)x}\\
&\le \frac{M}{\mu}\big[\hat{J}_1+\|\hat{Z}_1\|_\nu+\|\hat{Z}_2\|_\nu\big]\|\hat{Z}_1-\hat{Z}_2\|_\nu.
\end{aligned}$$

Similar argument holds for other terms, which completes the proof. □

If we let $r=\hat{J}_1/\mu^2=\epsilon\hat{J}_2/\mu^4$, then we can show from Lemma 4.3.2 that $\mathcal{F}$ is a contraction on $\bar{B}_r(0)\subset E_\nu$ if

$$\frac{M}{\mu^{3/2}}[(\epsilon\hat{J}_2/\mu^{7/2})(1+\mu^2)+\mu^2]\le 1\,,\quad \frac{M}{\mu}[(\hat{J}_2\epsilon/\mu^2+2(\epsilon\hat{J}_2/\mu^4)]\le 1/2\,. \tag{4.3.26}$$

Therefore, if $\mu>0$ is small and $\epsilon\le\mu^{11/2}$, the condition (4.3.26) is satisfied, which implies that (4.3.11) has a unique solution $\hat{Z}(x;\mu,\epsilon,\varrho,\theta_0,\theta_1,\hat{J}_1)$ for each $\theta_0,\theta_1\in S^1$ satisfying

$$\|\hat{Z}(x;\mu,\epsilon,\varrho,\theta_0,\theta_1,\hat{J}_1)\|_\nu\le r=\frac{\hat{J}_1}{\mu^2}. \tag{4.3.27}$$

By differentiating (4.3.24) with respect to θ_0 and using the same argument as that for (4.3.27) and an extension of a contraction mapping principle by Walter [115], we can show that $\hat{Z}_{\theta_0}(x;\mu,\epsilon,\varrho,\theta_0,\theta_1,\hat{J}_1)$ exists. Similarly, we can prove that $\hat{Z}$ is smooth in its arguments.

Step 3: Construction of the solutions for $x\in(-\infty,+\infty)$

Now, it is known that $\hat{X}(x;\mu,\epsilon,\varrho,\theta_0,\theta_1,\hat{J}_1)$ defined in (4.3.10) exists for $x\ge 0$. Next step is to construct a reversible $\hat{X}$. The idea is to solve the following equation

$$(I-S)\hat{X}(0;\mu,\epsilon,\varrho,0,\theta_1,\hat{J}_1)=0 \tag{4.3.28}$$

for θ_1,ϱ. Then, define the solution of (4.1.15) as $\hat{X}_f(x)=\hat{X}(x;\mu,\epsilon,\varrho,0,\theta_1,\hat{J}_1)$ for $x\ge 0$ and $\hat{X}_f(x)=S\Big(\hat{X}(-x;\mu,\epsilon,\varrho,0,\theta_1,\hat{J}_1)\Big)$ for $x\le 0$, which gives a reversible solution $\hat{X}_f(x)$ of (4.1.15).

Assume that $\hat{Z}(x)=(\hat{A},\hat{B},\hat{C},\hat{D},\bar{\hat{A}},\bar{\hat{B}},\bar{\hat{C}},\bar{\hat{D}})^T(x)$. Then (4.3.28) is equivalent to

$$\big(\hat{\mathcal{H}}_A(0)+\hat{A}(0)\big)-\big(\bar{\hat{\mathcal{H}}}_A(0)+\bar{\hat{A}}(0)\big)=0, \tag{4.3.29}$$

$$\big(\hat{\mathcal{H}}_B(0)+\hat{B}(0)\big)-\big(-\bar{\hat{\mathcal{H}}}_B(0)-\bar{\hat{B}}(0)\big)=0, \tag{4.3.30}$$

$$\mathrm{Im}\hat{C}(0) = 0, \tag{4.3.31}$$
$$\mathrm{Im}\hat{D}(0) = 0. \tag{4.3.32}$$

Using (4.3.6), (4.3.16), (4.3.19), (4.3.21) and (4.3.24), it is easy to check that (4.3.29) and (4.3.30) hold automatically. Thus, we only have to study (4.3.31) and (4.3.32).

Lemma 4.3.3 *If the assumption* (2) *in* Theorem 4.3.1 *is satisfied and the cut-off function* ς *is chosen so that* $\int_1^2 \sin(s_{01}s)\varsigma'(s)ds \neq 0$*, then for small* $\mu > 0$ *and* $\epsilon \in [0, \mu^{11/2}]$*, the equations* (4.3.31) and (4.3.32) *can be transformed to a system*

$$\gamma_1 + \mathcal{F}_1(\gamma_1; \mu, \epsilon) = 0 \tag{4.3.33}$$

where $\gamma_1 = (\theta_1, \varrho_1)^T$,

$$\mathcal{F}_1(\gamma_1; \mu, \epsilon) = \begin{pmatrix} \sqrt{\mu}\tilde{R}_C(\theta_1, \varrho_1, \mu, \epsilon) \\ \sqrt{\mu}\tilde{R}_D(\theta_1, \varrho_1, \mu, \epsilon) \end{pmatrix},$$

$\varrho = \frac{\epsilon}{\mu}\varrho_1$*, and* $\tilde{R}_C$ *and* $\tilde{R}_D$ *are real, differentiable with respect to their arguments, and bounded.*

The proof is given in Section 6.2.

Since $\tilde{R}_C$ and $\tilde{R}_D$ are differentiable, we can use a similar argument as that in the proof of Lemma 5.2 to show that the derivatives of $\tilde{R}_C$ and $\tilde{R}_D$ with respect to θ_1 or ϱ_1 are bounded, which implies that $\mathcal{F}_1$ is Lipschitz continuous with respect to γ_1 with a Lipschitz constant strictly less than 1 for small $\mu > 0$ and ϵ satisfying the condition in Lemma 4.3.3.

To apply the fixed point theorem, we choose a ball $\bar{B}_r(0) \subset \mathbf{R}^2$ with a radius $r = M\mu^{1/4}$. Then, it is easy to see that $-\mathcal{F}_1$ is a contraction in $\bar{B}_r(0)$ and has a fixed point $(\theta_1, \varrho_1)^T \in \bar{B}_r(0)$ for small $\mu > 0$ and $\epsilon \in [0, \mu^{11/2}]$. Thus, we have solved (4.3.28) for

$$(\theta_1, \varrho) = (\theta_1, \frac{\epsilon}{\mu}\varrho_1)$$

when μ is small and $\epsilon \in [0, \mu^{11/2}]$. By the relationship between ϵ and μ, i.e., $\epsilon = \mu^{(n-1)/2}$, we can choose a suitable n such that $\epsilon \in [0, \mu^{11/2}]$ is satisfied. For example, choose $n = 13$ with $\epsilon = \mu^6$.

To construct the solution for $x < 0$, we know from the reversibility of the system (4.1.15) that both $\hat{X}(x; \mu, \epsilon, \frac{\epsilon}{\mu}\varrho_1, 0, \theta_1, \frac{\epsilon}{\mu^2}\hat{J}_2)$ and $S\left(\hat{X}(-x; \mu, \epsilon, \frac{\epsilon}{\mu}\varrho_1, 0, \theta_1, \frac{\epsilon}{\mu^2}\hat{J}_2)\right)$ are solutions of (4.1.15) and at $x = 0$

$$S\left(\hat{X}(0; \mu, \epsilon, \frac{\epsilon}{\mu}\varrho_1, 0, \theta_1, \frac{\epsilon}{\mu^2}\hat{J}_2)\right) = \hat{X}(0; \mu, \epsilon, \frac{\epsilon}{\mu}\varrho_1, 0, \theta_1, \frac{\epsilon}{\mu^2}\hat{J}_2).$$

Thus, by the uniqueness of the solution for an initial value problem, we can define a solution of (4.1.15) as

$$\hat{X}_f(x) = \begin{cases} \hat{X}(x; \mu, \epsilon, \frac{\epsilon}{\mu}\varrho_1, 0, \theta_1, \frac{\epsilon}{\mu^2}\hat{J}_2) & \text{for } x \geq 0, \\ S\left(\hat{X}(-x; \mu, \epsilon, \frac{\epsilon}{\mu}\varrho_1, 0, \theta_1, \frac{\epsilon}{\mu^2}\hat{J}_2)\right) & \text{for } x \leq 0. \end{cases}$$

Then $S\hat{X}_f(-x) = \hat{X}_f(x)$. Thus, the solution $\hat{X}_f(x)$ of (4.1.15) is a reversible homoclinic connection to the periodic orbit $\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(x + \theta_1)$ as $x \to +\infty$ and the periodic orbit $S\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(-x + \theta_1)$ as $x \to -\infty$. This completes the proof of Theorem 4.3.1.

Chapter 5

Existence of Solutions for (b, F^{-2}) near $(b_0, 1)$

In this chapter, we follow the idea in Chapter 4 and consider the case: (b, F^{-2}) is close to $(b_0, 1)$ from the region D_1^- (see Figure 1.2). For $(b_0, 1)$, L_s has a zero eigenvalue with Jordan chain of length 4, two-dimensional eigenvalues $\pm is_{20}$ which are simple and three-dimensional eigenvalues $\pm is_{10}$ which are double. Suppose that $b = b_0 + \mu_1, F^{-2} = 1 - \mu_2$ where μ_1, μ_2 are positive and small. The existence result can be described as follows.

Theorem 5.1 *Let $\mu_1 = \mu, \mu_2 = k_2\mu, \alpha = k_3\mu$ where $k_2 > 0, k_3$ are constants. If s_{10}, s_{20} satisfy a non-resonant condition, i.e., there is no nonzero integer vector (k_0, k_1) such that $s_{10}k_0 + s_{20}k_1 = 0$, then there exist a $\mu_0 > 0$ and a continuous function $\varrho_1(\mu)$ such that for each μ with $0 < \mu < \mu_0$ and $\varrho = \varrho_1(\mu)\mu^{21/8}$, the system (2.2.5) has a generalized solitary wave solution so that*

$$\begin{aligned}\eta(x,y) =& \mu d + \mu\tilde{d}_{01} + A_0\mu \text{sech}^2((c\mu)^{1/2}x) + R_0(x;\mu) \\ &+ \Big[A_1\mu\text{sech}((c\mu)^{1/2}x)\tanh((c\mu)^{1/2}x)\cos(s_{10}x) + R_1(x;\mu)\Big]\cos(2\pi y/(P+\alpha)) \\ &+ \varsigma(\mu^{1/2}x)S^{(p)}(\mu^{1/2}x) + R_2(x,y;\mu)\end{aligned}$$

where $d, \tilde{d}_{01}, c, A_0, A_1$ are nonzero constants dependent on (b_0, P, k_2, k_3), R_2 is even and periodic in y with period $P+\alpha$, $\varsigma(x)$ is given in (4.0.1), and $\tilde{d}_{01}$, R_0, R_1, and R_2 satisfy uniformly that

$$|\tilde{d}_{01}| \le A_2\mu^{1/2},\ |R_0(x;\mu)| + |R_1(x;\mu)| \le A_3\mu^{3/2}e^{-(c\mu)^{1/2}|x|},\ |R_2(x,y;\mu)| \le A_4\mu^{23/16}e^{-r\mu^{1/2}|x|}$$

for $x \in \mathbf{R}$ and some fixed constants $A_j, j = 2,3,4, r > 0$. Also, $S^{(p)}(\mu^{1/2}x)$ is a periodic function with period $2\pi/\tilde{k}$, and if $x \ge 0$,

$$S^{(p)}(\mu^{1/2}x) = \tilde{I}_2\mu^{11/4}\cos\tilde{k}(x+\delta) + R_3(x;\mu)$$

for some fixed amplitude $\tilde{I}_2 > 0$, wave number $\tilde{k}$ and phase shift δ, with

$$|\delta| \leq A_5\sqrt{\mu}\,, \quad |\tilde{k} - s_{20}| \leq A_6\mu\,, \quad |R_3(x;\mu)| \leq A_7\mu^{13/4}\,,$$

where $A_j, j = 5, 6, 7$ are fixed constants.

The expression for ϕ can be given, which is similar to that for the surface height. The graph for η is given in Figure 1.4.

5.1 Normal Form Analysis

Suppose that the eigenvector of the zero eigenvalue is e_{00} and generalized eigenvectors are e_{01}, e_{02}, e_{03}; the eigenvector of is_{20} $(-is_{20})$ is U_{20} $(\bar{U}_{20})$, the eigenvector of is_{10} $(-is_{10})$ is U_{10} $(\bar{U}_{11})$ and its generalized eigenvector is U_{11} $(\bar{U}_{11})$, respectively (see Section 6.6). Since the spectrum of L_s for such $(b_0, 1)$ consists entirely of isolated eigenvalues of finite algebraic multiplicity, we can write

$$\begin{aligned} v &= A_{00}e_{00} + A_{01}e_{01} + A_{02}e_{02} + A_{03}e_{03} + AU_{10} + BU_{11} + CU_{20} \\ &\quad + \bar{A}\bar{U}_{10} + \bar{B}\bar{U}_{11} + \bar{C}\bar{U}_{20} + v_2 \end{aligned}$$

where A_{0j} are real for $j = 0, 1, 2, 3$, and v_2 is a linear combination of eigenvectors and generalized eigenvectors corresponding to the rest of eigenvalues. Note that the right side of (2.2.8) is independent of A_{00} because of e_{00}. Thus, A_{0j}, A, B, C, their complex conjugates and v_2 are independent of A_{00} for $j = 1, 2, 3$. Then the center manifold reduction theorem 3.2.2 yields that all small bounded solutions of (2.2.8) are of the form

$$\begin{aligned} v &= A_{00}e_{00} + A_{01}e_{01} + A_{02}e_{02} + A_{03}e_{03} + AU_{10} + BU_{11} + CU_{20} + \bar{A}\bar{U}_{10} \\ &\quad + \bar{B}\bar{U}_{11} + \bar{C}\bar{U}_{20} + \Phi(\underline{\mu}, \varrho, A_{00}, A_{01}, A_{02}, A_{03}, A, B, C, \bar{A}, \bar{B}, \bar{C}) \end{aligned} \tag{5.1.1}$$

where $\underline{\mu} = (\mu_1, \mu_2, \alpha)$ and Φ is independent of A_{00} and contains terms which are at least quadratic in its arguments with $v_2 = \Phi$. Moreover, the amplitudes satisfy the reduced system

$$\frac{d}{dx}X = LX + F_0(\underline{\mu}, \varrho, X), \tag{5.1.2}$$

where $X = (A_{00}, A_{01}, A_{02}, A_{03}, A, B, C, \bar{A}, \bar{B}, \bar{C})^T$, L is given by

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & is_{10} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & is_{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & is_{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -is_{10} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -is_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -is_{20} \end{pmatrix}$$

and $F_0(\underline{\mu}, 0, 0) = 0$, $D_X F_0(0,0,0) = 0$ and $|F_0(0,0,X)| = O(|X|^2)$. Note that $F_0(\underline{\mu}, \varrho, X)$ is independent of A_{00}.

From the identity (2.2.10), A_{03} can be solved as a function of $\underline{\mu}, \varrho, A_{01}, A_{02}, A, B, C$ and their complex conjugates. Therefore, we can drop the equations of A_{00} and A_{03} in (5.1.2) and only focus on the equations for A_{01}, A_{02}, A, B, C and their complex conjugates. For the sake of simplicity, we still use (5.1.2) to denote the reduced systems.

To compute the normal form of (5.1.2), let us first concentrate on the equations for A_{01}, A_{02}, A, B, C. Note that the reverser S is given by

$$S : \begin{pmatrix} A_{01} \\ A_{02} \\ A \\ B \\ C \end{pmatrix} \to \begin{pmatrix} A_{01} \\ -A_{02} \\ \bar{A} \\ -\bar{B} \\ \bar{C} \end{pmatrix},$$

and

$$SF_0 = -F_0 S, \qquad SL = -LS.$$

Write F_0 as

$$F_0 = F_{0,1}(\underline{\mu}, A_{01}, A_{02}, A, B, C, \bar{A}, \bar{B}, \bar{C}) + \varrho F_{0,2}(\underline{\mu}, \varrho, A_{01}, A_{02}, A, B, C, \bar{A}, \bar{B}, \bar{C}). \quad (5.1.3)$$

For the sake of convenience, we let $\underline{\mu} = 0$ and only consider $F_{0,1}$ at this moment. There exists a change of variables from X to Y which is close to an identity, and transforms the system (5.1.2) with $F_{0,1}$ only into

$$\frac{d}{dx} Y = LY + \mathcal{P}(Y) + o(|Y|^n), \quad (5.1.4)$$

where $\mathcal{P}$ is a polynomial of degree $\le n$ (n is arbitrary but fixed), with $\mathcal{P}(0) = 0$ and $D\mathcal{P}(0) = 0$. In the following we still use $X = (A_{01}, A_{02}, A, B, C, \bar{A}, \bar{B}, \bar{C})^T$ to replace Y. Here, $\mathcal{P}$ satisfies

$$S\mathcal{P}(X) = -\mathcal{P}(SX)$$

and

$$D\mathcal{P}(X)L^*X = L^*\mathcal{P}(X), \tag{5.1.5}$$

for any X where $L^* = \bar{L}^T$.

Let $\mathcal{P} = (P_{01}, P_{02}, P_{10}, P_{11}, P_{20}, \bar{P}_{10}, \bar{P}_{11}, \bar{P}_{20})^T$ and define a differential operator

$$D^* \;=\; A_{01}\frac{\partial}{\partial A_{02}} - is_{10}A\frac{\partial}{\partial A} + (A - is_{10}B)\frac{\partial}{\partial B} - is_{20}C\frac{\partial}{\partial C} + c.c. \tag{5.1.6}$$

where *c.c.* denotes the complex conjugates. Then, (5.1.5) is equivalent to

$$D^*\mathcal{P} = L^*\mathcal{P},$$

which gives

$$\begin{aligned} &D^*P_{01} = 0, \qquad D^*P_{02} = P_{01}, \qquad D^*P_{10} = -is_{10}P_{10}, \\ &D^*P_{11} = -is_{10}P_{11} + P_{10}, \qquad D^*P_{20} = -is_{20}P_{20} \end{aligned} \tag{5.1.7}$$

and their complex conjugates.

To determine $\mathcal{P}$, seven independent first integrals of $D^* = 0$ are needed and can be found as follows,

$$\begin{aligned} &u_1 = A_{01}, u_2 = A\bar{A},\ u_3 = \tfrac{i}{2}(A\bar{B} - \bar{A}B),\ u_4 = C\bar{C}, \\ &u_5 = \tfrac{B}{A} + \tfrac{1}{is_{10}}\ln A,\ u_6 = \tfrac{B}{A} + \tfrac{1}{is_{20}}\ln C,\ u_7 = A_{02} - A_{01}\tfrac{B}{A}. \end{aligned} \tag{5.1.8}$$

Then we have an important lemma and its proof is given in Section 6.4.

Lemma 5.1.1 *Assume that H is a polynomial of X with degree n and $D^*H = 0$. Then*

$$H(X) = Q(u_1, u_2, u_3, u_4, u_2u_7),$$

where Q is a polynomial of its arguments, provided that there is no nonzero integer vector (l_1, l_2) such that $s_{10}l_1 + s_{20}l_2 = 0$.

Now we calculate the components of the polynomial $\mathcal{P}$ in (5.1.4). From (5.1.7), we have $D^*P_{01} = 0$. Lemma 5.1.1 yields that P_{01} is a polynomial in $u_1, u_2, u_3, u_4, u_2u_7$. Note that $D^*(A_{01}P_{02} - A_{02}P_{01}) = 0$ shows $A_{01}P_{02} - A_{02}P_{01} = \tilde{P}_{02}(u_1, u_2, u_3, u_4, u_2u_7)$ where $\tilde{P}_{02}$ is a polynomial of its arguments, i.e., $P_{02} = \frac{1}{A_{01}}[A_{02}P_{01} + \tilde{P}_{02}]$. It is easy to check that $D^*(\bar{A}P_{10}) = 0$. By Lemma 5.1.1, $\bar{A}P_{10} = P^*_{10}(u_1, u_2, u_3, u_4, u_2u_7)$ where P^*_{10} is a polynomial of its arguments. Since $P_{10} = P^*_{10}/\bar{A}$ is also a polynomial, by the forms of u_1, u_2, u_3, u_4, u_7 in terms of $A, \bar{A}$, it is deduced that $P_{10} = A\tilde{P}_{10}(u_1, u_2, u_3, u_4, u_7)$ where

$$\tilde{P}_{10} = \tilde{P}_{10,1}(u_1, u_2, u_3, u_4, u_2u_7) + u_7\tilde{P}_{10,2}(u_1, u_2, u_3, u_4, u_2u_7), \tag{5.1.9}$$

and $\tilde{P}_{10,1}$ and $\tilde{P}_{10,2}$ are polynomials of their arguments. Moreover, it is also easy to see that

$$D^*(\bar{A}^2(AP_{11} - BP_{10})) = 0$$

which yields $\bar{A}^2(AP_{11} - BP_{10}) = P_{11}^*$ by Lemma 5.1.1 where P_{11}^* is a polynomial of u_1, u_2, u_3, u_4,u_2u_7, that is, from the forms of u_1, u_2, u_3, u_4, u_7,

$$AP_{11} = BP_{10} + A^2\tilde{P}_{11,1} + A^2u_7^2\tilde{P}_{11,2} + A^2u_7\tilde{P}_{11,3}$$

where $\tilde{P}_{11,1}, \tilde{P}_{11,2}, \tilde{P}_{11,3}$ are polynomials of $u_1, u_2, u_3, u_4, u_2u_7$. Let

$$\tilde{P}_{11} = \tilde{P}_{11,1} + u_7^2\tilde{P}_{11,2} + u_7\tilde{P}_{11,3} \tag{5.1.10}$$

and then from (5.1.9)

$$\begin{aligned} P_{11} &= B\tilde{P}_{10}(u_1, u_2, u_3, u_4, u_7) + A\tilde{P}_{11}(u_1, u_2, u_3, u_4, u_7) \\ &= B\tilde{P}_{10,1}(u_1, u_2, u_3, u_4, u_2u_7) + Bu_7\tilde{P}_{10,2}(u_1, u_2, u_3, u_4, u_2u_7) \\ &\quad +A\tilde{P}_{11,1}(u_1, u_2, u_3, u_4, u_2u_7) + Au_7^2\tilde{P}_{11,2}(u_1, u_2, u_3, u_4, u_2u_7) \\ &\quad +Au_7\tilde{P}_{11,3}(u_1, u_2, u_3, u_4, u_2u_7) \end{aligned}$$

where $Bu_7\tilde{P}_{10,2} + Au_7^2\tilde{P}_{11,2}$ is a polynomial of A_{01}, A_{02}, A, B, C and their complex conjugates since P_{11} is a polynomial of its arguments and $u_7 = A_{02} - A_{01}B/A$. A similar argument gives

$$P_{20} = C\tilde{P}_{20}(u_1, u_2, u_3, u_4, u_2u_7),$$

where $\tilde{P}_{20}$ is a polynomial of its arguments.

Now we can write (5.1.2) for $\underline{\mu} = \varrho = 0$ as

$$\begin{aligned} \dot{A}_{01} &= A_{02} + P_{01}(u_1, u_2, u_3, u_4, u_2u_7) + h.o.t., \\ \dot{A}_{02} &= \tfrac{1}{A_{01}}\Big[A_{02}P_{01}(u_1, u_2, u_3, u_4, u_2u_7) + \tilde{P}_{02}(u_1, u_2, u_3, u_4, u_2u_7)\Big] + h.o.t., \\ \dot{A} &= is_{10}A + B + A\tilde{P}_{10}(u_1, u_2, u_3, u_4, u_7) + h.o.t., \\ \dot{B} &= is_{10}B + B\tilde{P}_{10}(u_1, u_2, u_3, u_4, u_7) + A\tilde{P}_{11}(u_1, u_2, u_3, u_4, u_7) + h.o.t., \\ \dot{C} &= is_{20}C + C\tilde{P}_{20}(u_1, u_2, u_3, u_4, u_2u_7) + h.o.t. \end{aligned} \tag{5.1.11}$$

and their complex conjugates, where $h.o.t.$ stands for the terms of A_{01}, A_{02}, A, B, C and their complex conjugates with orders equal to and higher than $n + 1$.

A similar analysis holds if $\underline{\mu}$ is not zero. Thus, (5.1.11) for $\varrho = 0$ becomes (after dropping tildes)

$$\begin{aligned} \dot{A}_{01} &= A_{02} + P_{01}(\underline{\mu}, u_1, u_2, u_3, u_4, u_2u_7) \\ &\qquad +O(|(A_{01}, A_{02}, A, B, C)||(\underline{\mu}, A_{01}, A_{02}, A, B, C)|^n), \\ \dot{A}_{02} &= \frac{1}{A_{01}}\Big[A_{02}P_{01}(\underline{\mu}, u_1, u_2, u_3, u_4, u_2u_7) + P_{02}(\underline{\mu}, u_1, u_2, u_3, u_4, u_2u_7)\Big] \end{aligned}$$

$$
\begin{aligned}
& +O(|(A_{01},A_{02},A,B,C)||(\underline{\mu},A_{01},A_{02},A,B,C)|^n),\\
\dot{A} &= is_{10}A+B+AP_{10}(\underline{\mu},u_1,u_2,u_3,u_4,u_7)\\
& +O(|(A_{01},A_{02},A,B,C)||(\underline{\mu},A_{01},A_{02},A,B,C)|^n),\\
\dot{B} &= is_{10}B+BP_{10}(\underline{\mu},u_1,u_2,u_3,u_4,u_7)+AP_{11}(\underline{\mu},u_1,u_2,u_3,u_4,u_7)\\
& +O(|(A_{01},A_{02},A,B,C)||(\underline{\mu},A_{01},A_{02},A,B,C)|^n),\\
\dot{C} &= is_{20}C+CP_{20}(\underline{\mu},u_1,u_2,u_3,u_4,u_2u_7)\\
& +O(|(A_{01},A_{02},A,B,C)||(\underline{\mu},A_{01},A_{02},A,B,C)|^n) \qquad (5.1.12)
\end{aligned}
$$

and their complex conjugates, where P_{01}, P_{02} are real, $P_{01}, P_{02}, P_{10}, P_{11}, P_{20}$ are polynomials of their arguments such that $P_{01}(0) = P_{02}(0) = P_{10}(0) = P_{11}(0) = P_{20}(0) = 0$. Moreover, from (5.1.9) and (5.1.10),

$$
\begin{aligned}
P_{10} &= P_{10,1}(\underline{\mu},u_1,u_2,u_3,u_4,u_2u_7)+u_7P_{10,2}(\underline{\mu},u_1,u_2,u_3,u_4,u_2u_7),\\
P_{11} &= P_{11,1}(\underline{\mu},u_1,u_2,u_3,u_4,u_2u_7)+u_7^2P_{11,2}(\underline{\mu},u_1,u_2,u_3,u_4,u_2u_7)\\
& +u_7P_{11,3}(\underline{\mu},u_1,u_2,u_3,u_4,u_2u_7) \qquad (5.1.13)
\end{aligned}
$$

where $P_{10,1}$, $P_{10,2}$, $P_{11,1}$, $P_{11,2}$ and $P_{11,3}$ are polynomials of their arguments, and $Bu_7P_{10,2}$ $+Au_7^2P_{11,2}$ is a polynomial of A_{01}, A_{02}, A, B, C and their complex conjugates since $BP_{10} + AP_{11}$ is a polynomial. The reversibility gives that

$$
\begin{aligned}
& P_{01}(\underline{\mu},u_1,u_2,u_3,u_4,u_2u_7) = -P_{01}(\underline{\mu},u_1,u_2,u_3,u_4,u_2\tilde{u}_7),\\
& P_{02}(\underline{\mu},u_1,u_2,u_3,u_4,u_2u_7) = P_{02}(\underline{\mu},u_1,u_2,u_3,u_4,u_2\tilde{u}_7),\\
& P_{10}(\underline{\mu},u_1,u_2,u_3,u_4,u_7) = -\overline{P_{10}(\underline{\mu},u_1,u_2,u_3,u_4,\tilde{u}_7)},\\
& P_{11}(\underline{\mu},u_1,u_2,u_3,u_4,u_7) = \overline{P_{11}(\underline{\mu},u_1,u_2,u_3,u_4,\tilde{u}_7)},\\
& P_{20}(\underline{\mu},u_1,u_2,u_3,u_4,u_2u_7) = -\overline{P_{20}(\underline{\mu},u_1,u_2,u_3,u_4,u_2\tilde{u}_7)} \qquad (5.1.14)
\end{aligned}
$$

where $\tilde{u}_7 = -\bar{u}_7$. Note that P_{01}, P_{10}, P_{11} and P_{20} have the following forms

$$
\begin{aligned}
P_{01} &= (p_{01}^{(1)}\mu_1+p_{01}^{(2)}\mu_2+p_{01}^{(3)}\alpha)A_{01}+p_{01}^{(4)}A_{01}^2+p_{01}^{(5)}A\bar{A}\\
& +p_{01}^{(6)}\frac{i}{2}(A\bar{B}-\bar{A}B)+p_{01}^{(7)}C\bar{C}+P_{01}^*(\underline{\mu},X),\\
P_{10} &= p_{10}^{(1)}\mu_1+p_{10}^{(2)}\mu_2+p_{10}^{(3)}\alpha+p_{10}^{(4)}A_{01}+P_{10}^*(\underline{\mu},X),\\
P_{11} &= p_{11}^{(1)}\mu_1+p_{11}^{(2)}\mu_2+p_{11}^{(3)}\alpha+p_{11}^{(4)}A_{01}+P_{11}^*(\underline{\mu},X),\\
P_{20} &= p_{20}^{(1)}\mu_1+p_{20}^{(2)}\mu_2+p_{20}^{(3)}\alpha+p_{20}^{(4)}A_{01}+P_{20}^*(\underline{\mu},X) \qquad (5.1.15)
\end{aligned}
$$

where

$$
\begin{aligned}
& P_{01}^*(\underline{\mu},X) = O(|(\underline{\mu},A_{01},A_{02},A,B,C)|^3),\\
& P_{10}^*(\underline{\mu},X) = P_{11}^*(\underline{\mu},X) = P_{20}^*(\underline{\mu},X) = O(|(\underline{\mu},A_{01},A_{02},A,B,C)|^2). \qquad (5.1.16)
\end{aligned}
$$

By the first and third equations of (5.1.14), we know that $p_{10}^{(1)}, p_{10}^{(2)}, p_{10}^{(3)}, p_{10}^{(4)}$ are purely imaginary and

$$p_{01}^{(1)} = p_{01}^{(2)} = p_{01}^{(3)} = p_{01}^{(4)} = p_{01}^{(5)} = p_{01}^{(6)} = p_{01}^{(7)} = 0. \tag{5.1.17}$$

If let $P_{02}^* = \frac{1}{A_{01}}[A_{02}P_{01} + P_{02}]$, then P_{02}^* is a polynomial of $\underline{\mu}, A_{01}, A_{02}, A, B, C$ and their complex conjugates, and can be written as

$$P_{02}^* = (p_{02}^{(1)}\mu_1 + p_{02}^{(2)}\mu_2 + p_{02}^{(3)}\alpha)A_{01} + p_{02}^{(4)}A_{01}^2 + p_{02}^{(5)}A\bar{A} + p_{02}^{(6)}\frac{i}{2}(A\bar{B} - \bar{A}B)$$
$$+p_{02}^{(7)}C\bar{C} + O(|(\underline{\mu}, A_{01}, A_{02}, A, B, C)|^3) + O(|A_{02}||(\underline{\mu}, A_{01}, A_{02}, A, B, C)|^3) \tag{5.1.18}$$

since $A_{02}P_{01}$ has the factor A_{02}. Some coefficients in (5.1.15) and (5.1.18) are given by (6.6.12).

If the term $\varrho F_{0,2}$ is included, the system (5.1.12) becomes

$$\dot{X} = F(\underline{\mu}, X) + \mathcal{R}(\underline{\mu}, X) + \varrho\wp(\underline{\mu}, \varrho, X), \tag{5.1.19}$$

where the remainder $\mathcal{R}$ is of order $O(|(A_{01}, A_{02}, A, B, C)||(\underline{\mu}, A_{01}, A_{02}, A, B, C)|^n)$, $\wp$ corresponds to the term $F_{0,2}$ in (5.1.3) after the transformations, and $\mathcal{R}$ and $\wp$ have arbitrarily high order derivatives with respect to their arguments. F can be written as

$$F(\underline{\mu}, X) = F_{\deg\le 2}(\underline{\mu}, X) + F_{\deg\ge 3}(\underline{\mu}, X) \tag{5.1.20}$$

where $F_{\deg\le 2}$ is a polynomial of $(\underline{\mu}, X)$ with degree ≤ 2 and $F_{\deg\ge 3}$ is a polynomial of $(\underline{\mu}, X)$ with degree ≥ 3. Obviously, $F(\underline{\mu}, X)$ has an symmetry

$$R_{\theta_0}F(\underline{\mu}, X) = F(\underline{\mu}, R_{\theta_0}X) \tag{5.1.21}$$

for any $\theta_0 \in [0, 2\pi)$ where

$$R_{\theta_0}(A_{01}, A_{02}, A, B, C) = (A_{01}, A_{02}, \mathrm{e}^{i\theta_0}A, \mathrm{e}^{i\theta_0}B, C). \tag{5.1.22}$$

Assume that l_{01}, l_{02}, l_{10}, l_{11} and l_{20} are the coefficients of ϱ in A_{01}, A_{02}, A, B and C equations, respectively, which are from (6.6.13)

$$l_{01} = l_{10} = l_{11} = 0, \quad l_{02} = \frac{b_0 P}{r_{00}^2}, \quad l_{20} = -i\frac{P}{r_{20}^2} \tag{5.1.23}$$

where r_{00} and r_{20} are given by (6.6.1) and (6.6.4) respectively.

Since $b = b_0 + \mu_1$, $\lambda = 1 - \mu_2$ and α is a small parameter in the period $P + \alpha$, we just consider the case

$$\mu_1 = \mu, \quad \mu_2 = k_2\mu, \quad \alpha = k_3\mu \tag{5.1.24}$$

where $k_2 > 0, k_3$ are constants and μ is positive. For simplicity, we still use (5.1.19) to denote the system by replacing $\underline{\mu}$ with μ. Thus, we just prove the existence of homoclinic orbits of the system (5.1.19), approaching periodic solutions, in order to look for the existence of generalized solitary waves of the original system (2.2.11). In the following, we will first find its equilibrium after the scaling for (5.1.19) and then only focus on this system near the equilibrium—the system (5.3.6).

5.2 Scaling

Let

$$
\begin{aligned}
&x = \mu^{-1/2}\tilde{x}, \quad A_{01} = \mu\tilde{A}_{01}, \quad A_{02} = \mu^{3/2}\tilde{A}_{02}, \quad A = \mu e^{i(s_{10}\mu^{-1/2}\tilde{x}+\theta_0)}\tilde{A},\\
&B = \mu^{3/2}e^{i(s_{10}\mu^{-1/2}\tilde{x}+\theta_0)}\tilde{B}, \quad C = \mu^{3/2}\tilde{C},
\end{aligned} \tag{5.2.1}
$$

where $\theta_0 \in [0, 2\pi)$. Then (5.1.19) is changed to (dropping the tildes)

$$
\begin{aligned}
\dot{A}_{01} &= A_{02} + \frac{1}{\mu^{3/2}}P_{01}(\mu, X_1) + \frac{1}{\mu^{3/2}}\left[\mathcal{R}_{01}(\mu, X_2) + \varrho\wp_{01}(\mu, \varrho, X_2)\right],\\
\dot{A}_{02} &= \frac{1}{\mu^3 A_{01}}\left[\mu^{3/2}A_{02}P_{01}(\mu, X_1) + P_{02}(\mu, X_1)\right] + \frac{1}{\mu^2}\left[\mathcal{R}_{02}(\mu, X_2) + \varrho\wp_{02}(\mu, \varrho, X_2)\right],\\
\dot{A} &= B + \frac{1}{\mu^{1/2}}AP_{10}(\mu, \tilde{X}_1) + \frac{e^{-i(s_{10}\mu^{-1/2}x+\theta_0)}}{\mu^{3/2}}\left[\mathcal{R}_{10}(\mu, X_2) + \varrho\wp_{10}(\mu, \varrho, X_2)\right],\\
\dot{B} &= \frac{1}{\mu^{1/2}}BP_{10}(\mu, \tilde{X}_1) + \frac{1}{\mu}AP_{11}(\mu, \tilde{X}_1) + \frac{e^{-i(s_{10}\mu^{-1/2}x+\theta_0)}}{\mu^2}\left[\mathcal{R}_{11}(\mu, X_2) + \varrho\wp_{11}(\mu, \varrho, X_2)\right],\\
\dot{C} &= \frac{1}{\sqrt{\mu}}\left[is_{20}C + CP_{20}(\mu, X_1)\right] + \frac{1}{\mu^2}\left[\mathcal{R}_{20}(\mu, X_2) + \varrho\wp_{20}(\mu, \varrho, X_2)\right]
\end{aligned} \tag{5.2.2}
$$

and their complex conjugates where

$$
\begin{aligned}
X_1 &= \Big(\mu A_{01}, \mu^2 A\bar{A}, \mu^{5/2}\frac{i}{2}(A\bar{B} - \bar{A}B), \mu^3 C\bar{C}, \mu^{7/2}(A\bar{A}A_{02} - A_{01}\bar{A}B)\Big),\\
\tilde{X}_1 &= \Big(\mu A_{01}, \mu^2 A\bar{A}, \mu^{5/2}\frac{i}{2}(A\bar{B} - \bar{A}B), \mu^3 C\bar{C}, \mu^{3/2}(A_{02} - A_{01}B/A)\Big),\\
X_2 &= \Big(\mu A_{01}, \mu^{3/2}A_{02}, \mu e^{i(s_{10}\mu^{-1/2}x+\theta_0)}A, \mu^{3/2}e^{i(s_{10}\mu^{-1/2}x+\theta_0)}B, \mu^{3/2}C,\\
&\qquad \mu e^{-i(s_{10}\mu^{-1/2}x+\theta_0)}\bar{A}, \mu^{3/2}e^{-i(s_{10}\mu^{-1/2}x+\theta_0)}\bar{B}, \mu^{3/2}\bar{C}\Big),\\
\mathcal{R} &= (\mathcal{R}_{01}, \mathcal{R}_{02}, \mathcal{R}_{10}, \mathcal{R}_{11}, \mathcal{R}_{20}, \bar{\mathcal{R}}_{10}, \bar{\mathcal{R}}_{11}, \bar{\mathcal{R}}_{20})^T,\\
\wp &= (\wp_{01}, \wp_{02}, \wp_{10}, \wp_{11}, \wp_{20}, \bar{\wp}_{10}, \bar{\wp}_{11}, \bar{\wp}_{20})^T.
\end{aligned}
$$

For convenience, we write (5.2.2) as

$$
\dot{X} = \mathcal{F}_1(\mu, X) + \mathcal{R}_1(x, \mu, \theta_0, X) + \varrho\wp_1(x, \mu, \varrho, \theta_0, X) \tag{5.2.3}
$$

where $X = (A_{01}, A_{02}, A, B, C, \bar{A}, \bar{B}, \bar{C})^T$, $\mathcal{R}_1$ and $\wp_1$ correspond to $\mathcal{R}$ and $\wp$ after the scaling respectively. Moreover, $\mathcal{F}_1$ can be written as from (5.1.15)-(5.1.18), (5.1.24) and (6.6.12)

$$
\mathcal{F}_1(\mu, X) = F_2(X) + \check{F}_2(\mu, X) + F_3(\mu, X) \tag{5.2.4}
$$

where

$$F_2(X) = \begin{pmatrix} A_{02} \\ p_{02}^{(2)} k_2 A_{01} + p_{02}^{(4)} A_{01}^2 + p_{02}^{(5)} A\bar{A} \\ B \\ A(p_{11}^{(1)} + p_{11}^{(2)} k_2 + p_{11}^{(3)} k_3 + p_{11}^{(4)} A_{01}) \\ \frac{1}{\sqrt{\mu}} i s_{20} C \\ \bar{B} \\ \bar{A}(p_{11}^{(1)} + p_{11}^{(2)} k_2 + p_{11}^{(3)} k_3 + p_{11}^{(4)} A_{01}) \\ -\frac{1}{\sqrt{\mu}} i s_{20} \bar{C} \end{pmatrix}, \tag{5.2.5}$$

$$\check{F}_2(\mu, X) = \begin{pmatrix} 0 \\ \sqrt{\mu} p_{02}^{(6)} \frac{i}{2}(A\bar{B} - \bar{A}B) + \mu p_{02}^{(7)} C\bar{C} \\ \sqrt{\mu} A(p_{10}^{(1)} + p_{10}^{(2)} k_2 + p_{10}^{(3)} k_3 + p_{10}^{(4)} A_{01}) \\ \sqrt{\mu} B(p_{10}^{(1)} + p_{10}^{(2)} k_2 + p_{10}^{(3)} k_3 + p_{10}^{(4)} A_{01}) \\ \sqrt{\mu} C(p_{20}^{(1)} + p_{20}^{(2)} k_2 + p_{20}^{(3)} k_3 + p_{20}^{(4)} A_{01}) \\ \sqrt{\mu} \bar{A}(\bar{p}_{10}^{(1)} + \bar{p}_{10}^{(2)} k_2 + \bar{p}_{10}^{(3)} k_3 + \bar{p}_{10}^{(4)} A_{01}) \\ \sqrt{\mu} \bar{B}(\bar{p}_{10}^{(1)} + \bar{p}_{10}^{(2)} k_2 + \bar{p}_{10}^{(3)} k_3 + \bar{p}_{10}^{(4)} A_{01}) \\ \sqrt{\mu} \bar{C}(\bar{p}_{20}^{(1)} + \bar{p}_{20}^{(2)} k_2 + \bar{p}_{20}^{(3)} k_3 + \bar{p}_{20}^{(4)} A_{01}) \end{pmatrix}. \tag{5.2.6}$$

Here, $F_2 + \check{F}_2$ and F_3 correspond to F in (5.1.20) with polynomials of degree ≤ 2 and degree ≥ 3, respectively.

Remark 5.2.1 *$\check{F}_2(\mu, X)$ has a factor $\sqrt{\mu}$ and $F_3(\mu, X)$ has a factor μ. Furthermore, the second component of $\check{F}_2(\mu, X)$ has a factor μ if $A = B = 0$.*

Note that $\mathcal{R}$ in (5.1.19) is of order $O(|(A_{01}, A_{02}, A, B, C)||(\mu, A_{01}, A_{02}, A, B, C)|^n)$. From (5.1.23), the scaling (5.2.1) and the relation between $\mathcal{R} + \varrho\wp$ and $\mathcal{R}_1 + \varrho\wp_1$, we easily get the following lemma.

Lemma 5.2.1 *For $A = B = 0$, $\mathcal{R}_1 + \varrho\wp_1$ in (5.2.3) does not explicitly include x and θ_0. Furthermore,*

$$\mathcal{R}_1[1] + \varrho\wp_1[1] = O(\mu^{n-1/2}) + \frac{1}{\mu^{3/2}}\left(O(|\varrho\mu|) + O(|\varrho^2|)\right),$$
$$\mathcal{R}_1[2] + \varrho\wp_1[2] = O(\mu^{n-1}) + \frac{1}{\mu^2}\left(\frac{b_0 P}{r_{00}^2}\varrho + O(|\varrho\mu|) + O(|\varrho^2|)\right),$$
$$\mathcal{R}_1[3] + \varrho\wp_1[3] = O(\mu^{n-1/2}) + \frac{1}{\mu^{3/2}}\left(O(|\varrho\mu|) + O(|\varrho^2|)\right),$$
$$\mathcal{R}_1[4] + \varrho\wp_1[4] = O(\mu^{n-1}) + \frac{1}{\mu^2}\left(O(|\varrho\mu|) + O(|\varrho^2|)\right),$$

$$\mathcal{R}_1[5] + \varrho\wp_1[5] = O(\mu^{n-1}) + \frac{1}{\mu^2}\left(-i\frac{P}{r_{20}^2}\varrho + O(|\varrho\mu|) + O(|\varrho^2|)\right) \tag{5.2.7}$$

for any bounded X_2 where $F[i]$ means the i-th component of F.

5.3 Equilibrium

In the following, we consider the equilibrium of (5.2.3). The dominant system of (5.2.3) is

$$\dot{X} = F_2(X). \tag{5.3.1}$$

It is easy to check that $\underline{d} = (d, 0, 0, 0, 0, 0, 0, 0)^T$ is an equilibrium of (5.3.1) where

$$d = -p_{02}^{(2)}k_2/p_{02}^{(4)}. \tag{5.3.2}$$

Let $\underline{\tilde{d}} = \underline{d} + \underline{d_1}$ be an equilibrium of (5.2.3) where

$$\underline{d_1} = (\tilde{d}_{01}, \tilde{d}_{02}, \tilde{d}_{10}, \tilde{d}_{11}, \tilde{d}_{20}, \bar{\tilde{d}}_{10}, \bar{\tilde{d}}_{11}, \bar{\tilde{d}}_{20})^T$$

is to be determined. Since the two-dimensional solutions correspond to the case $\{A = B = 0\}$, we may take $\tilde{d}_{10} = \tilde{d}_{11} = 0$ which yields that $P_{01} = 0$ by the first equation of (5.1.14), and from Lemma 5.2.1 the right side of (5.2.3) does not explicitly contain x and θ_0. Replacing X by $\underline{\tilde{d}}$ in the right side of (5.2.3) and setting it equal to 0, it obtained that

$$\gamma_1 = \mathcal{M}_1(\gamma_1; \mu, \varrho) \tag{5.3.3}$$

where $\gamma_1 = (\tilde{d}_{02}, \tilde{d}_{01}, \tilde{d}_{20}, \bar{\tilde{d}}_{20})^T$,

$$\mathcal{M}_1(\gamma_1; \mu, \varrho) = \begin{pmatrix} -f_{01}(\mu, \varrho, \underline{d_1}) \\ -\frac{p_{02}^{(4)}}{p_{02}^{(2)}k_2}\tilde{d}_{01}^2 - \frac{b_0P}{p_{02}^{(2)}k_2r_{00}^2}\frac{\varrho}{\mu^2} - \frac{1}{p_{02}^{(2)}k_2}f_{02}(\mu, \varrho, \underline{d_1}) \\ i\frac{\sqrt{\mu}}{s_{20}}f_{20}(\mu, \varrho, \underline{d_1}) \\ -i\frac{\sqrt{\mu}}{s_{20}}\overline{f_{20}(\mu, \varrho, \underline{d_1})} \end{pmatrix}$$

and f_{01}, f_{02}, f_{20} are the first, second, and fifth components of the remainder terms of $\mathcal{F}_1 + \mathcal{R}_1 + \varrho\wp_1$, respectively. Here we use the coefficients of ϱ given by (5.1.23).

Lemma 5.3.1 *For a ball $B_r(0) \subset \mathbf{R}^2 \times \mathbf{C}^2$ with a radius $r = O(\mu^{1/4})$, the map $\mathcal{M}_1$ in (5.3.3) has a fixed point $\gamma_1 \in \bar{B}_r(0)$ for small $\mu > 0$ provided that*

$$\varrho = O(\mu^{5/2}). \tag{5.3.4}$$

Moreover,

$$|\underline{d_1}| \le M\sqrt{\mu}, \quad \tilde{d}_{01} = -\frac{b_0P}{p_{02}^{(2)}k_2r_{00}^2}\frac{\varrho}{\mu^2} + O(\mu), \quad |\tilde{d}_{20}| \le M\left(\mu^{n-1/2} + \frac{|\varrho|}{\mu^{3/2}}\right). \tag{5.3.5}$$

Proof. From (5.2.6), (5.2.7) and Remark 5.2.1, we obtain that

$$\begin{aligned} f_{01} &= \check{F}_2[1] + F_3[1] + \mathcal{R}_1[1] + \varrho\wp_1[1] = 0 + O(\mu) + O(\mu^{n-1/2}) + \frac{1}{\mu^{3/2}}O(|\varrho\mu| + |\varrho^2|) \\ &= O(\mu + \frac{|\varrho|}{\sqrt{\mu}} + \frac{\varrho^2}{\mu^{3/2}}), \\ f_{02} &= \check{F}_2[2] + F_3[2] + \mathcal{R}_1[2] + \varrho\wp_1[2] - \frac{b_0 P}{r_{00}^2}\frac{\varrho}{\mu^2} \\ &= O(\mu) + O(\mu) + O(\mu^{n-1}) + \frac{1}{\mu^2}O(|\varrho\mu| + |\varrho^2|) = O(\mu + \frac{|\varrho|}{\mu} + \frac{\varrho^2}{\mu^2}), \\ f_{20} &= \check{F}_2[5] + F_3[5] + \mathcal{R}_1[5] + \varrho\wp_1[5] = O(\sqrt{\mu}) + O(\mu) + O(\mu^{n-1}) + \frac{1}{\mu^2}O(|\varrho|) \\ &= O(\sqrt{\mu} + \frac{|\varrho|}{\mu^2}) \end{aligned}$$

for large n and any bounded $\underline{d}_1$ since $\tilde{d}_{10} = \tilde{d}_{11} = 0$ where $F[i]$ means the i-th component of F. It is easy to show that $\mathcal{M}_1$ is a contraction map on $\bar{B}_r(0)$ for small $\mu > 0$ under the assumption (5.3.4). Thus, $\mathcal{M}_1$ has a unique fixed point γ_1, i.e., $\underline{\tilde{d}}$ has been obtained as a function of μ and ϱ. From the second equation of (5.3.3) and the assumption (5.3.4), we get

$$\tilde{d}_{01} = -\frac{b_0 P}{p_{02}^{(2)} k_2 r_{00}^2}\frac{\varrho}{\mu^2} + O(\mu)$$

which implies $|\underline{d_1}| \leq M\sqrt{\mu}$. Moreover, since the C-component of $\mathcal{F}_1$ in (5.2.3) has a factor C, the last equations of (5.2.7) and (5.3.3) yield

$$|\tilde{d}_{20}| \leq M\left(\mu^{n-1/2} + \frac{|\varrho|}{\mu^{3/2}}\right)$$

which completes the proof. □

Replacing X by $\tilde{X} + \underline{\tilde{d}}$ in (5.2.3) with $\tilde{X} = (\tilde{A}_{01}, \tilde{A}_{02}, \tilde{A}, \tilde{B}, \tilde{C}, \bar{\tilde{A}}, \bar{\tilde{B}}, \bar{\tilde{C}})^T$, we have

$$\dot{\tilde{X}} = \tilde{\mathcal{F}}_1(\mu, \tilde{X}) + \tilde{\mathcal{R}}_1(x, \mu, \theta_0, \tilde{X}) + \varrho\tilde{\wp}_1(x, \mu, \varrho, \theta_0, \tilde{X}) \tag{5.3.6}$$

where from (5.2.4)

$$\begin{aligned} &\tilde{\mathcal{F}}_1(\mu, \tilde{X}) = \mathcal{F}_1(\mu, \tilde{X} + \underline{\tilde{d}}) - \mathcal{F}_1(\mu, \underline{\tilde{d}}) = F_2(\tilde{X} + \underline{\tilde{d}}) - F_2(\underline{\tilde{d}}) \\ &\qquad\qquad + \check{F}_2(\mu, \tilde{X} + \underline{\tilde{d}}) - \check{F}_2(\mu, \underline{\tilde{d}}) + F_3(\mu, \tilde{X} + \underline{\tilde{d}}) - F_3(\mu, \underline{\tilde{d}}), \\ &\tilde{\mathcal{R}}_1(x, \mu, \theta_0, \tilde{X}) = \mathcal{R}_1(x, \mu, \theta_0, \tilde{X} + \underline{\tilde{d}}) - \mathcal{R}_1(x, \mu, \theta_0, \underline{\tilde{d}}), \\ &\tilde{\wp}_1(x, \mu, \varrho, \theta_0, \tilde{X}) = \wp_1(x, \mu, \varrho, \theta_0, \tilde{X} + \underline{\tilde{d}}) - \wp_1(x, \mu, \varrho, \theta_0, \underline{\tilde{d}}). \end{aligned} \tag{5.3.7}$$

Then, we have the following

$$F_2(\tilde{X} + \underline{\tilde{d}}) - F_2(\underline{\tilde{d}}) = \tilde{F}_2(\tilde{X}) + \hat{F}_2(\mu, \tilde{X}) \tag{5.3.8}$$

where

$$\tilde{F}_2(\tilde{X}) = \begin{pmatrix} \tilde{A}_{02} \\ -p_{02}^{(2)}k_2\tilde{A}_{01} + p_{02}^{(4)}\tilde{A}_{01}^2 + p_{02}^{(5)}\tilde{A}\bar{\tilde{A}} \\ \tilde{B} \\ \tilde{A}(p_{11}^{(1)} + p_{11}^{(2)}k_2 + p_{11}^{(3)}k_3 + p_{11}^{(4)}d + p_{11}^{(4)}\tilde{A}_{01}) \\ \frac{1}{\sqrt{\mu}}is_{20}\tilde{C} \\ \bar{\tilde{B}} \\ \bar{\tilde{A}}(p_{11}^{(1)} + p_{11}^{(2)}k_2 + p_{11}^{(3)}k_3 + p_{11}^{(4)}d + p_{11}^{(4)}\tilde{A}_{01}) \\ -\frac{1}{\sqrt{\mu}}is_{20}\bar{\tilde{C}} \end{pmatrix}, \tag{5.3.9}$$

$$\hat{F}_2(\mu, \tilde{X}) = \begin{pmatrix} 0 \\ 2p_{02}^{(4)}\tilde{d}_{01}\tilde{A}_{01} \\ p_{11}^{(4)}\tilde{d}_{01}\tilde{A} \\ 0 \\ 0 \\ p_{11}^{(4)}\tilde{d}_{01}\bar{\tilde{A}} \\ 0 \\ 0 \end{pmatrix}. \tag{5.3.10}$$

Remark 5.3.1 (1) *$\tilde{\mathcal{R}}_1 + \varrho\tilde{\wp}_1$ does not explicitly include x and θ_0 for $\tilde{A} = \tilde{B} = 0$.*

(2) *Note that $S\underline{d} = \underline{d}$. Since the right side of (5.2.3) does not explicitly contain x, θ_0 and then is reversible for $A = B = 0$, the uniqueness implies that $S\underline{d}_1 = \underline{d}_1$ and $S\underline{\tilde{d}} = \underline{\tilde{d}}$.*

The problem of the existence of homoclinic orbits for the system (5.1.19) is equivalent to the one for the system (5.3.6). In the following, we concentrate on the system (5.3.6). Here we note that two-dimensional solutions of (5.3.6) correspond to the case with $\{\tilde{A} = \tilde{B} = 0\}$. Therefore, the space $\{\tilde{A} = \tilde{B} = 0\}$ is an invariant subspace of (5.3.6).

5.4 Homoclinic Solutions of Dominant System

In this section, we look for a homoclinic orbit of the dominant system of (5.3.6), which is

$$\dot{\tilde{X}} = \tilde{F}_2(\tilde{X}). \tag{5.4.1}$$

In Section 5.6, we will prove that this homoclinic orbit persists for the system (5.3.6).

The direction calculations yield the following lemma.

Lemma 5.4.1 *Let*

$$a = \frac{3p_{02}^{(2)}k_2}{3p_{02}^{(4)} - p_{11}^{(4)}}, \qquad c = \sqrt{\frac{-p_{02}^{(2)}p_{11}^{(4)}k_2}{2(3p_{02}^{(4)} - p_{11}^{(4)})}}, \qquad e = \sqrt{\frac{6ac^2 + a^2p_{02}^{(4)}}{p_{02}^{(5)}}}$$

where some constants are given in (6.6.12). *If k_2, k_3 in* (5.1.24) *satisfy the equation*

$$c^2 - (p_{11}^{(1)} + p_{11}^{(2)}k_2 + p_{11}^{(3)}k_3 + dp_{11}^{(4)}) = 0, \tag{5.4.2}$$

then $\mathcal{H}(x)$ is a solution of (5.4.1)*, which exponentially approaches* 0 *as $x \to \pm\infty$. Here,*

$$\mathcal{H} = (H_{01}, H_{02}, H_{10}, H_{11}, H_{20}, \bar{H}_{10}, \bar{H}_{11}, \bar{H}_{20})^T \tag{5.4.3}$$

and

$$\begin{aligned} &H_{01} = a\text{sech}^2(cx),\ H_{02} = -2ac\text{sech}^2(cx)\tanh(cx),\ H_{10} = e\text{sech}(cx)\tanh(cx), \\ &H_{11} = ec\left(\text{sech}^3(cx) - \text{sech}(cx)\tanh^2(cx)\right),\ H_{20} = 0 \end{aligned} \tag{5.4.4}$$

and their complex conjugates. Moreover,

$$\begin{aligned} &H_{01}(-x) = H_{01}(x), \quad H_{02}(-x) = -H_{02}(x), \\ &\bar{H}_{10}(-x) = H_{10}(-x) = -H_{10}(x), \quad \bar{H}_{11}(-x) = H_{11}(-x) = H_{11}(x) \end{aligned} \tag{5.4.5}$$

and

$$|H_{01}(x)| + |H_{02}(x)| + |H_{10}(x)| + |H_{11}(x)| + |H_{20}(x)| \le M\mathrm{e}^{-|\nu||x|} \tag{5.4.6}$$

for $x \in \mathbf{R}$ and every $|\nu| \le c$.

Remark 5.4.1 (1) *The system* (5.4.1) *without $C, \bar{C}$-components can be written as*

$$\begin{aligned} \ddot{\tilde{A}}_{01} &= -p_{02}^{(2)}k_2\tilde{A}_{01} + p_{02}^{(4)}\tilde{A}_{01}^2 + p_{02}^{(5)}\tilde{A}\bar{\tilde{A}}, \\ \ddot{\tilde{A}} &= \tilde{A}(p_{11}^{(1)} + p_{11}^{(2)}k_2 + p_{11}^{(3)}k_3 + p_{11}^{(4)}d + p_{11}^{(4)}\tilde{A}_{01}) \end{aligned}$$

with the complex conjugate of the last one, which corresponds to a system of coupled Schrödinger-KdV equations while the dominant system of (4.3.5) *in* Section 4.3 *and* Groves & Mielke [55]'s *paper is the Schrödinger equations.*

(2) *If let $P = 2\pi$, then we have*

$$b = 0.186783,\ a = 0.345547k_2,\ c = 1.28195\sqrt{k_2},\ d = -0.6667k_2,\ e = 1.39342k_2.$$

Thus, the equation (5.4.2) *can be computed as follows*

$$k_2 = -0.0834502(46.23 - 0.59036k_3).$$

If we choose $k_3 \in [79, 252]$, then the above equation can be solved for the positive k_2 such that $(b_0 + \mu, 1 - k_2\mu)$ is in the given region D_1^- (see Figure 1.2).

5.5 Two-Dimensional Periodic Solutions

In this section, we use the classical Lyapunov-Schmidt method to show that (5.3.6) has two-dimensional periodic solutions. The idea is similar to one in Section 4.2.

Let $\hat{x} = (s_{20} + r_1)\mu^{-1/2}x$ where r_1 is a small real constant to be determined. Therefore, we consider periodic solutions of (5.3.6) with period $2\pi\sqrt{\mu}/(s_{20} + r_1)$.

Since the space $\{\tilde{A} = \tilde{B} = 0\}$ is invariant, let $\tilde{A} = \tilde{B} = 0$ and $\tilde{A}^p_{01}, \tilde{A}^p_{02}$ and $\tilde{C}^p$ denote the first, second and fifth components in (5.3.6) which satisfy the equations

$$\frac{d}{d\hat{x}}\tilde{A}^p_{01} = \frac{\sqrt{\mu}}{s_{20} + r_1}[\tilde{A}^p_{02} + h_{01}(\mu, \varrho, \tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p)], \tag{5.5.1}$$
$$\frac{d}{d\hat{x}}\tilde{A}^p_{02} = \frac{\sqrt{\mu}}{s_{20} + r_1}[-p^{(2)}_{02}k_2\tilde{A}^p_{01} + h_{02}(\mu, \varrho, \tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p)], \tag{5.5.2}$$
$$\frac{d}{d\hat{x}}\tilde{C}^p = \frac{1}{s_{20} + r_1}[is_{20}\tilde{C}^p + h_{20}(\mu, \varrho, \tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p)], \tag{5.5.3}$$
$$\frac{d}{d\hat{x}}\bar{\tilde{C}}^p = \frac{1}{s_{20} + r_1}[-is_{20}\bar{\tilde{C}}^p + \overline{h_{20}(\mu, \varrho, \tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p)}], \tag{5.5.4}$$

where h_{01}, h_{02}, h_{20} are the first, second and fifth components of the remainder terms of $\tilde{\mathcal{F}}_1 + \tilde{\mathcal{R}}_1 + \varrho\tilde{\wp}_1$, which from (1) of Remark 5.3.1 do not explicitly contain x and θ_0 for $\tilde{A} = \tilde{B} = 0$.

Since we look for periodic solutions with period 2π, write

$$\tilde{A}^p_{01}(\hat{x}) = \sum_n A_{01,n}\mathrm{e}^{in\hat{x}}, \quad \tilde{A}^p_{02}(\hat{x}) = \sum_n A_{02,n}\mathrm{e}^{in\hat{x}},$$
$$\tilde{C}^p(\hat{x}) = \sum_n C_n\mathrm{e}^{in\hat{x}}, \quad A_{01,n} = \overline{A_{01,-n}}, \quad A_{02,n} = \overline{A_{02,-n}}. \tag{5.5.5}$$

Substitute (5.5.5) into (5.5.1)-(5.5.4) and make the coefficient of each term in the Fourier series equal,

$$A_{01,0} = \tilde{H}_{01,0}(\mu, \varrho, \tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p), \tag{5.5.6}$$
$$A_{01,n} = \frac{1}{in(s_{20} + r_1)}\tilde{H}_{01,n}(\mu, \varrho, \tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p), \quad n \neq 0, \tag{5.5.7}$$
$$A_{02,0} = \tilde{H}_{02,0}(\mu, \varrho, \tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p), \tag{5.5.8}$$
$$A_{02,n} = \frac{1}{in(s_{20} + r_1)}\tilde{H}_{02,n}(\mu, \varrho, \tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p), \quad n \neq 0, \tag{5.5.9}$$
$$C_n = \frac{1}{i((n-1)s_{20} + nr_1)}\tilde{H}_{20,n}(\mu, \varrho, \tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p), \quad n \neq 1, \tag{5.5.10}$$
$$ir_1C_1 = \Big[h_{20}(\mu, \varrho, \tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p)\Big]_1 \tag{5.5.11}$$

and the complex conjugates of (5.5.10) and (5.5.11) for $n \in \mathbf{Z}$, where $[f]_k$ is the k-th Fourier coefficient of f, and

$$
\begin{aligned}
\tilde{H}_{01,0}(\mu,\varrho,\tilde{A}^p_{01},\tilde{A}^p_{02},\tilde{C}^p,\bar{\tilde{C}}^p) &= \frac{1}{p^{(2)}_{02}k_2}\Big[h_{02}(\mu,\varrho,\tilde{A}^p_{01},\tilde{A}^p_{02},\tilde{C}^p,\bar{\tilde{C}}^p)\Big]_0,\\
\tilde{H}_{01,n}(\mu,\varrho,\tilde{A}^p_{01},\tilde{A}^p_{02},\tilde{C}^p,\bar{\tilde{C}}^p) &= \sqrt{\mu}\Big[A_{02,n}+\Big[h_{01}(\mu,\varrho,\tilde{A}^p_{01},\tilde{A}^p_{02},\tilde{C}^p,\bar{\tilde{C}}^p)\Big]_n\Big], n\neq 0,\\
\tilde{H}_{02,0}(\mu,\varrho,\tilde{A}^p_{01},\tilde{A}^p_{02},\tilde{C}^p,\bar{\tilde{C}}^p) &= -\Big[h_{01}(\mu,\varrho,\tilde{A}^p_{01},\tilde{A}^p_{02},\tilde{C}^p,\bar{\tilde{C}}^p)\Big]_0,\\
\tilde{H}_{02,n}(\mu,\varrho,\tilde{A}^p_{01},\tilde{A}^p_{02},\tilde{C}^p,\bar{\tilde{C}}^p) &= \sqrt{\mu}\Big[-p^{(2)}_{02}k_2A_{01,n}\\
&\quad+\Big[h_{02}(\mu,\varrho,\tilde{A}^p_{01},\tilde{A}^p_{02},\tilde{C}^p,\bar{\tilde{C}}^p)\Big]_n\Big],\quad n\neq 0,\\
\tilde{H}_{20,n}(\mu,\varrho,\tilde{A}^p_{01},\tilde{A}^p_{02},\tilde{C}^p,\bar{\tilde{C}}^p) &= \Big[h_{20}(\mu,\varrho,\tilde{A}^p_{01},\tilde{A}^p_{02},\tilde{C}^p,\bar{\tilde{C}}^p)\Big]_n,\quad n\neq 1.
\end{aligned}
$$

In the following, we will first solve (5.5.6)-(5.5.10) for $A_{01,n}, A_{02,n}$ and $C_n, n\neq 1$, and then solve (5.5.11) for r_1 by making C_1 as a free constant to be chosen later.

Fix C_1, and define two spaces

$$
\begin{aligned}
H^m_1([0,2\pi]) &= \{f(\hat{x}) = \sum_n f_n e^{in\hat{x}} \in H^m([0,2\pi]) \mid f_1 = 0\ \},\\
H^m_{-1}([0,2\pi]) &= \{f(\hat{x}) = \sum_n f_n e^{in\hat{x}} \in H^m([0,2\pi]) \mid f_{-1} = 0\ \}
\end{aligned}
$$

where $H^m([0,2\pi])$ is defined in Section 4.2. For $\tilde{U}_{01}, \tilde{U}_{02} \in H^m([0,2\pi])$ and $\tilde{U}_{20} \in H^m_1([0,2\pi])$, we define a mapping Θ from $H^m([0,2\pi]) \times H^m([0,2\pi]) \times H^m_1([0,2\pi]) \times H^m_{-1}([0,2\pi])$ to itself by

$$
\Theta(\tilde{U}_{01},\tilde{U}_{02},\tilde{U}_{20},\bar{\tilde{U}}_{20};\varpi) = (\Theta_{01},\Theta_{02},\Theta_{20},\bar{\Theta}_{20})^T(\tilde{U}_{01},\tilde{U}_{02},\tilde{U}_{20},\bar{\tilde{U}}_{20};\varpi) \tag{5.5.12}
$$

where $\varpi = (\mu,\varrho,r_1,C_1,\bar{C}_1)$ and

$$
\begin{aligned}
\Theta_{01}(\tilde{U}_{01},\tilde{U}_{02},\tilde{U}_{20},\bar{\tilde{U}}_{20};\varpi) &= \tilde{H}_{01,0}(\mu,\varrho,\tilde{U}_{01},\tilde{U}_{02},\tilde{U}_{20}+C_1e^{i\hat{x}},\bar{\tilde{U}}_{20}+\bar{C}_1e^{-i\hat{x}})\\
&\quad+\sum_{n\neq 0}\frac{1}{in(s_{20}+r_1)}\tilde{H}_{01,n}(\mu,\varrho,\tilde{U}_{01},\tilde{U}_{02},\tilde{U}_{20}+C_1e^{i\hat{x}},\bar{\tilde{U}}_{20}+\bar{C}_1e^{-i\hat{x}})e^{in\hat{x}},\\
\Theta_{02}(\tilde{U}_{01},\tilde{U}_{02},\tilde{U}_{20},\bar{\tilde{U}}_{20};\varpi) &= \tilde{H}_{02,0}(\mu,\varrho,\tilde{U}_{01},\tilde{U}_{02},\tilde{U}_{20}+C_1e^{i\hat{x}},\bar{\tilde{U}}_{20}+\bar{C}_1e^{-i\hat{x}})\\
&\quad+\sum_{n\neq 0}\frac{1}{in(s_{20}+r_1)}\tilde{H}_{02,n}(\mu,\varrho,\tilde{U}_{01},\tilde{U}_{02},\tilde{U}_{20}+C_1e^{i\hat{x}},\bar{\tilde{U}}_{20}+\bar{C}_1e^{-i\hat{x}})e^{in\hat{x}},
\end{aligned}
$$

$$
\Theta_{20}(\tilde{U}_{01},\tilde{U}_{02},\tilde{U}_{20},\bar{\tilde{U}}_{20};\varpi)
$$

$$= \sum_{n\neq 1} \frac{1}{i((n-1)s_{20}+nr_1)} \tilde{H}_{20,n}(\mu, \varrho, \tilde{U}_{01}, \tilde{U}_{02}, \tilde{U}_{20}+C_1e^{i\hat{x}}, \bar{\tilde{U}}_{20}+\bar{C}_1e^{-i\hat{x}})e^{in\hat{x}}.$$

Assume that $B_r(0)$ is a ball in the space $H^m([0,2\pi])\times H^m([0,2\pi])\times H_1^m([0,2\pi])\times H_{-1}^m([0,2\pi])$ with a radius $r \le M_1\sqrt{\mu}$ where M_1 is a constant.

Lemma 5.5.1 *Under the assumption* (5.3.4),

$$\|\Theta(\tilde{U}_{01}, \tilde{U}_{02}, \tilde{U}_{20}, \bar{\tilde{U}}_{20}; \varpi)\|_m \le M\sqrt{\mu}\left(\|\tilde{U}_{01}\|_m + \|\tilde{U}_{02}\|_m + \|\tilde{U}_{20}\|_m + |C_1|\right),$$
$$\|\Theta(\tilde{V}_{01}, \tilde{V}_{02}, \tilde{V}_{20}, \bar{\tilde{V}}_{20}; \varpi) - \Theta(\tilde{W}_{01}, \tilde{W}_{02}, \tilde{W}_{20}, \bar{\tilde{W}}_{20}; \varpi)\|_m$$
$$\le M\sqrt{\mu}(\|\tilde{V}_{01}-\tilde{W}_{01}\|_m + \|\tilde{V}_{02}-\tilde{W}_{02}\|_m + \|\tilde{V}_{20}-\tilde{W}_{20}\|_m)$$

for $(\tilde{U}_{01}, \tilde{U}_{02}, \tilde{U}_{20}, \bar{\tilde{U}}_{20}), (\tilde{V}_{01}, \tilde{V}_{02}, \tilde{V}_{20}, \bar{\tilde{V}}_{20}), (\tilde{W}_{01}, \tilde{W}_{02}, \tilde{W}_{20}, \bar{\tilde{W}}_{20}) \in \bar{B}_r(0)$ *and small* ϖ *where* $M > 0$ *is a constant bounded uniformly for any bounded* ϖ *and* M_1*, and* $\|\cdot\|_m$ *is defined in* Section 4.2.

Proof. From the assumption (5.3.4) and (5.3.5), $\tilde{d}_{01}$ is of order at least $O(\sqrt{\mu})$. Then, (5.2.7), Remark 5.2.1 and the expressions of $\tilde{F}_2$ and $\hat{F}_2$ in (5.3.9) and (5.3.10) yield that h_{01} and h_{20} have a factor μ, and h_{02} has a term $2p_{02}^{(4)}\tilde{d}_{01}\tilde{U}_{01}+p_{02}^{(4)}(\tilde{U}_{01})^2$ bounded by $M_2\sqrt{\mu}\|\tilde{U}_{01}\|_m$ for some constant M_2 and other terms with a factor $\sqrt{\mu}$. The rest of the proof is similar to the proof of Lemma 4.2.1 in Section 4.2. □

Now, we take $r = \frac{M\sqrt{\mu}}{1-3M\sqrt{\mu}}|C_1|$ with $M\sqrt{\mu} < \frac{1}{3}$ for small enough μ and $|C_1|$ (In Section 5.6, we will choose $|C_1| = O(\mu^{5/4})$ such that $r \le M_1\sqrt{\mu}$). Then Lemma 5.5.1 shows that Θ is a contraction mapping on $\bar{B}_r(0)$ for small ϖ. Therefore, the fixed point theorem yields that Θ has a unique fixed point which is a smooth function of ϖ. Write this fixed point as

$$(\hat{A}_{01}^p, \hat{A}_{02}^p, \hat{C}^p, \bar{\hat{C}}^p)(\mu, \varrho, r_1, C_1, \bar{C}_1)(\hat{x}) \tag{5.5.13}$$

which satisfies

$$(\hat{A}_{01}^p, \hat{A}_{02}^p, \hat{C}^p, \bar{\hat{C}}^p)(\mu, \varrho, r_1, 0, 0)(\hat{x}) = 0 \text{ for all } \hat{x} \in [0, 2\pi],$$
$$\|\hat{A}_{01}^p\|_m + \|\hat{A}_{02}^p\|_m + \|\hat{C}^p\|_m \le \frac{M\sqrt{\mu}}{1-3M\sqrt{\mu}}|C_1|. \tag{5.5.14}$$

For notational simplicity, we use $(\tilde{A}_{01}^p, \tilde{A}_{02}^p, \tilde{C}^p, \bar{\tilde{C}}^p)(\hat{x})$ to denote

$$\left(\hat{A}_{01}^p(\hat{x}), \hat{A}_{02}^p(\hat{x}), \hat{C}^p(\hat{x})+C_1e^{i\hat{x}}, \bar{\hat{C}}^p(\hat{x})+\bar{C}_1e^{-i\hat{x}}\right).$$

Substituting (5.5.13) into (5.5.11) and using the reversibility and the translation invariance, the fixed point theorem yields r_1 as a smooth real-valued function

$$r_1 = r_1(\mu, \varrho, |C_1|) \tag{5.5.15}$$

for small $(\mu, \varrho, |C_1|)$ since h_{20} has a factor μ under the assumption (5.3.4). Furthermore, r_1 satisfies

$$|r_1| \le M\mu \tag{5.5.16}$$

for bounded μ, ϱ and $|C_1|$ where M is a constant. More details can be found in Section 4.2.

Thus, by (5.3.4), (5.3.6) has a periodic solution $\tilde{A}^p_{01}(\hat{x})$, $\tilde{A}^p_{02}(\hat{x})$, $\tilde{C}^p(\hat{x})$, and $\bar{\tilde{C}}^p(\hat{x})$ with $\tilde{A} = \tilde{B} = 0$, which belongs to $(H^m[(0, 2\pi)])^4$ if $\mu \in (0, \mu_0]$, $|\varrho| \in [0, \mu^{5/2}\varrho_0]$, $|C_1| \in [0, I_1]$ where μ_0, ϱ_0 and I_1 are fixed positive and small constants.

In the following, we choose

$$C_1 = \tilde{I}_1 > 0 \tag{5.5.17}$$

and, by the relation $\hat{x} = (s_{20} + r_1)\mu^{-1/2}x$, write the solution $(\tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p)^T(\hat{x})$ as $(\tilde{A}^p_{01}, \tilde{A}^p_{02}, \tilde{C}^p, \bar{\tilde{C}}^p)^T(\mu, \varrho, \tilde{I}_1)(x)$ with the frequency

$$\omega_1(\mu, \varrho, \tilde{I}_1) = \mu^{-1/2}(s_{20} + r_1(\mu, \varrho, \tilde{I}_1)) \tag{5.5.18}$$

for $\mu \in (0, \mu_0], |\varrho| \in [0, \mu^{5/2}\varrho_0], \tilde{I}_1 \in [0, I_1]$. Define

$$\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x) \;=\; \left(\tilde{A}^p_{01}, \tilde{A}^p_{02}, 0, 0, \tilde{C}^p, 0, 0, \bar{\tilde{C}}^p\right)(\mu, \varrho, \tilde{I}_1)(x) \tag{5.5.19}$$

for $\tilde{I}_1 > 0$ small, which is smooth for x and small $(\mu, \varrho, \tilde{I}_1)$ and $\tilde{X}_{\mu,\varrho,0}(x) = 0$. Then, $\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x)$ is a periodic solution of (5.3.6) with frequency $\omega_1(\mu, \varrho, \tilde{I}_1)$, which from (5.5.14) satisfies

$$\left\|\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x)\right\|_m \le M\tilde{I}_1 \tag{5.5.20}$$

for any $m > 0$ where $M > 0$ is a constant and

$$\varrho \in [0, \mu^{5/2}\varrho_0]\,. \tag{5.5.21}$$

Sobolev embedding theorem gives that (5.5.20) holds also in $C^m_B(\mathbf{R})$-norm, which is a space of continuously differentiable functions up to order m with a supreme norm.

5.6 Existence of Solitary Wave Solutions

In this section, we first apply the fixed point theorem to prove the existence of a homoclinic orbit of (5.3.6) approaching the periodic solution $\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1)$ for some phase shift θ_1 as $x \in [0, \infty)$ (see Step 1). Then, we go back to the reversible system (5.1.19) and obtain that this homoclinic orbit can be extended to $(-\infty, 0]$ by using the reversibility and adjusting the Bernoulli constant ϱ (see Step 2). The basic idea is is similar to one in Section 4.3.

By the scaling (5.2.1), the equilibrium $\underline{\tilde{d}}_1$ given in Section 5.3 and (5.5.19), define

$$\hat{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1) = \Big(\mu(\tilde{A}^p_{01}(\sqrt{\mu}x + \theta_1) + d + \tilde{d}_{01}), \mu^{3/2}(\tilde{A}^p_{02}(\sqrt{\mu}x + \theta_1) + \tilde{d}_{02}), 0, 0,$$

$$\mu^{3/2}(\tilde{C}^p(\sqrt{\mu}x+\theta_1)+\tilde{d}_{20}),0,0,\mu^{3/2}(\bar{\tilde{C}}^p(\sqrt{\mu}x+\theta_1)+\bar{\tilde{d}}_{20})\Big)^T \tag{5.6.1}$$

and

$$\tilde{a}=-\int_0^\infty\Big[(-p_{02}^{(2)}k_2+2p_{02}^{(4)}H_{01}(s))A_{01}(s)+p_{02}^{(5)}H_{10}(s)(A(s)+\bar{A}(s))\Big]\,ds-\frac{2ap_{02}^{(4)}}{c}, \tag{5.6.2}$$

where $\theta_1\in[0,2\pi)$, A_{01}, A_{02}, A, B, $\bar{A}$ and $\bar{B}$ are the solutions of the following equations, exponentially approaching zero as $x\to\infty$,

$$\begin{aligned}&\dot{A}_{01}=A_{02},\\&\dot{A}_{02}=(-p_{02}^{(2)}k_2+2p_{02}^{(4)}H_{01})A_{01}+p_{02}^{(5)}H_{10}(A+\bar{A})+2p_{02}^{(4)}H_{01},\\&\dot{A}=B,\\&\dot{B}=p_{11}^{(4)}H_{10}A_{01}+A(c^2+p_{11}^{(4)}H_{01})+p_{11}^{(4)}H_{10}\end{aligned}$$

and their complex conjugates, and other coefficients and functions are given in (5.1.15), (5.1.18), (5.1.24) and Lemma 5.4.1 respectively.

It is easy to check that $\hat{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)$ is a periodic solution of (5.1.19) by the relation between (5.1.19) and (5.3.6). Then we have the following theorem.

Theorem 5.6.1 *There exist two positive constants μ_0 and $\tilde{I}_2$. For $\mu\in(0,\mu_0]$, $\tilde{I}_1=\mu^{5/4}\tilde{I}_2$, there are two continuous functions $\tilde{\theta}_1$ and ϱ_1 of μ with $\theta_1=\sqrt{\mu}\tilde{\theta}_1$ and $\varrho=\mu^{21/8}\varrho_1$ so that* (5.1.19) *has a homoclinic orbit which is reversible and approaches the periodic solution $\hat{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)$ as $x\to\infty$ and the periodic solution $S\hat{X}_{\mu,\varrho,\tilde{I}_1}(-x+\theta_1)$ as $x\to-\infty$ provided that*

(1) *there is no nonzero integer vector (l_1,l_2) such that $s_{10}l_1+s_{20}l_2=0$;*

(2) $\tilde{a}\neq 0$ (*The definition of $\tilde{a}$ is given in* (5.6.2)).

Remark 5.6.1 *Let $P=2\pi$ and $k_3=80$. From* (5.4.2)*, we can solve for k_2 and then numerically $\tilde{a}=-40.8862$, which is not equal to* 0.

Under the assumption of Theorem 5.6.1, the condition (5.5.21) holds automatically.

Back to the original system (2.2.11), Theorem 5.6.1 implies Theorem 5.1. Thus, the system (2.2.11) has a three-dimensional generalized solitary water wave, periodic in the y-direction, approaching a two-dimensional water wave which is periodic in the propagation direction x and independent of y as $x\to\infty$ (see Figure 1.4).

Step 1: Homoclinic solutions of (5.3.6) for $x\in[0,+\infty)$

Let a solution of (5.3.6) be

$$\tilde{\mathcal{S}}(x;\mu,\varrho,\theta_0,\theta_1,\tilde{I}_1)=\mathcal{H}(x)+\tilde{Z}(x)+\varsigma(x)\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1) \tag{5.6.3}$$

for $x \in [0, \infty)$ and $\theta_0, \theta_1 \in [0, 2\pi)$ where the cut-off function $\varsigma(x) \in C^\infty(\mathbf{R}, \mathbf{R})$ is defined in (4.0.1) and $\tilde{Z}(x)$ is a function to be determined which tends exponentially to 0 as $x \to \infty$ so that $\tilde{\mathcal{S}}(x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1)$ is a solution of (5.3.6) that approaches the periodic orbit $\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1)$ as $x \to \infty$.

Since $\mathcal{H}(x)$ is a solution of (5.4.1), and $\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)$ and $\tilde{\mathcal{S}}(x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1)$ are solutions of (5.3.6), after plugging (5.6.3) into (5.3.6), we obtain

$$\frac{d\tilde{Z}}{dx} - \tilde{F}_2'[\mathcal{H}(x)](\tilde{Z}) = \tilde{N}(\tilde{Z}, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1) \tag{5.6.4}$$

where $'$ means the Fréchet derivative, and

$$\begin{aligned}
&\tilde{N}(\tilde{Z}, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1) = N_1(\tilde{Z}, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1) + N_2(\tilde{Z}, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1),\\
&N_1(\tilde{Z}, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1) = -\tilde{F}_2(\mathcal{H}(x)) - \tilde{F}_2'[\mathcal{H}(x)](\tilde{Z})\\
&\qquad -\varsigma(x)\tilde{\mathcal{F}}_1(\mu, \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1)) + \tilde{\mathcal{F}}_1(\mu, \mathcal{H}(x) + \tilde{Z}(x) + \varsigma(x)\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1)),\\
&N_2(\tilde{Z}, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1) = -\varsigma(x)\tilde{\mathcal{R}}_2(x, \mu, \varrho, \theta_0, \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1))\\
&\qquad +\tilde{\mathcal{R}}_2(x, \mu, \varrho, \theta_0, \mathcal{H}(x) + \tilde{Z}(x) + \varsigma(x)\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1)) - \varsigma'(x)\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1),\\
&\tilde{\mathcal{R}}_2(x, \mu, \varrho, \theta_0, \tilde{X}) = \tilde{\mathcal{R}}_1(x, \mu, \theta_0, \tilde{X}) + \varrho\tilde{\wp}_1(x, \mu, \varrho, \theta_0, \tilde{X}).
\end{aligned}$$

Let M denote a positive constant and fix ν such that $\nu/c \in (\frac{1}{2}, 1)$ where c is defined in Lemma 5.4.1. The function $\tilde{N}$ has the following estimates.

Lemma 5.6.1 *If $|\hat{Z}| \leq M_0$ for some positive constant M_0, then for $x \geq 0$,*

$$\begin{aligned}
&|\tilde{N}(\tilde{Z}, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1)| \leq M[\sqrt{\mu}(\mathrm{e}^{-\nu x} + |\tilde{Z}|) + |\tilde{Z}|^2 + \tilde{I}_1\mathrm{e}^{-\nu x}],\\
&|\tilde{N}(\tilde{Z}_1, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1) - \tilde{N}(\tilde{Z}_2, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1)|\\
&\qquad \leq M(\sqrt{\mu} + |\tilde{Z}_1| + |\tilde{Z}_2|)|\tilde{Z}_1 - \tilde{Z}_2|.
\end{aligned} \tag{5.6.5}$$

Proof. Here, we only check the case for $x \geq 2$ so $\varsigma(x) = 1$ and $\varsigma'(x) = 0$. The case for $0 \leq x \leq 2$ is easier. From (5.3.7) and (5.3.8), $\tilde{\mathcal{F}}_1$ can be written as

$$\begin{aligned}
\tilde{\mathcal{F}}_1(\mu, Z) &= LZ + M_2(Z, Z) + \hat{F}_2(\mu, Z) + \check{F}_2(\mu, Z + \underline{\tilde{d}}) - \check{F}_2(\mu, \underline{\tilde{d}})\\
&\quad +F_3(\mu, Z + \underline{\tilde{d}}) - F_3(\mu, \underline{\tilde{d}})
\end{aligned}$$

where M_2 is a polynomial of degree 2 and $\tilde{F}_2(Z) = LZ + M_2(Z, Z)$. Thus, N_1 is changed to

$$\begin{aligned}
&N_1(\tilde{Z}, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1) = -L\mathcal{H}(x) - M_2(\mathcal{H}(x), \mathcal{H}(x)) - L\tilde{Z}(x)\\
&\quad -2M_2(\mathcal{H}(x), \tilde{Z}(x)) - L\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1) - M_2(\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1), \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1))\\
&\quad -\hat{F}_2(\mu, \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1)) - \check{F}_2(\mu, \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1) + \underline{\tilde{d}}) + \check{F}_2(\mu, \underline{\tilde{d}})
\end{aligned}$$

$$
\begin{aligned}
&-F_3(\mu, \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}}) + F_3(\mu, \underline{\tilde{d}}) + L(\mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)) \\
&+M_2(\mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1), \mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)) \\
&+\hat{F}_2(\mu, \mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)) \\
&+\check{F}_2(\mu, \mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}}) - \check{F}_2(\mu, \underline{\tilde{d}}) \\
&+F_3(\mu, \mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}}) - F_3(\mu, \underline{\tilde{d}}) \\
=\ & 2M_2(\mathcal{H}(x), \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)) + 2M_2(\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1), \tilde{Z}(x)) + M_2(\tilde{Z}(x), \tilde{Z}(x)) \\
&+\hat{F}_2(\mu, \mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)) - \hat{F}_2(\mu, \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)) \\
&+\check{F}_2(\mu, \mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}}) - \check{F}_2(\mu, \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}}) \\
&+F_3(\mu, \mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}}) - F_3(\mu, \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}}). \quad (5.6.6)
\end{aligned}
$$

From (5.3.5) and (5.3.10), we know that $\hat{F}_2$ has a factor $\tilde{d}_{01}$ with order $O(\mu + \frac{|\varrho|}{\mu^2})$, which implies that by (5.4.6)

$$
\begin{aligned}
&|\hat{F}_2(\mu, \mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)) - \hat{F}_2(\mu, X_{\mu,\varrho,\tilde{I}_1}(x+\theta_1))| \\
&\quad \le M(\mu + \frac{|\varrho|}{\mu^2})(\mathrm{e}^{-\nu x} + |\tilde{Z}|) \\
&\quad \le M\sqrt{\mu}(\mathrm{e}^{-\nu x} + |\tilde{Z}|) \quad (5.6.7)
\end{aligned}
$$

since $\varrho = \mu^{21/8}\varrho_1$ from the assumption in Theorem 5.6.1. Remark 5.2.1 shows that $\check{F}_2$ has a factor $\sqrt{\mu}$ and F_3 has a factor μ so that

$$
\begin{aligned}
&|\check{F}_2(\mu, \mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}}) - \check{F}_2(\mu, \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}})| \\
&\quad \le M\sqrt{\mu}(\mathrm{e}^{-\nu x} + |\tilde{Z}|). \quad (5.6.8) \\
&|F_3(\mu, \mathcal{H}(x)+\tilde{Z}(x)+\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}}) - F_3(\mu, \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)+\underline{\tilde{d}})| \\
&\quad \le M\mu(\mathrm{e}^{-\nu x} + |\tilde{Z}|). \quad (5.6.9)
\end{aligned}
$$

By (5.4.6), (5.5.20), and (5.6.6)-(5.6.9), we get

$$
|N_1(\tilde{Z}, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1)| \le M[\sqrt{\mu}(\mathrm{e}^{-\nu x} + |\tilde{Z}|) + |\tilde{Z}|^2] \quad (5.6.10)
$$

where we use the assumption $\tilde{I}_1 = \mu^{5/4}\tilde{I}_2$ in Theorem 5.6.1. (5.2.7) and $\varrho = \mu^{21/8}\varrho_1$ yield

$$
\begin{aligned}
|N_2(\tilde{Z}, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1)| &\le M\left(\mu^{n-1} + \frac{|\varrho|}{\mu^2}\right)(\mathrm{e}^{-\nu x} + |\tilde{Z}|) \\
&\le M\sqrt{\mu}(\mathrm{e}^{-\nu x} + |\tilde{Z}|) \quad (5.6.11)
\end{aligned}
$$

for large n. (5.6.10) and (5.6.11) give the first inequality of (5.6.5).

For the second inequality of (5.6.5), we have

$$
|N_1(\tilde{Z}_1, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1) - N_1(\tilde{Z}_2, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1)|
$$

$$
\begin{aligned}
&\le |2M_2(\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1), \tilde{Z}_1(x)) - 2M_2(\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1), \tilde{Z}_2(x))| \\
&\quad + |M_2(\tilde{Z}_1(x), \tilde{Z}_1(x)) - M_2(\tilde{Z}_2(x), \tilde{Z}_2(x))| \\
&\quad + |\hat{F}_2(\mu, \mathcal{H}(x) + \tilde{Z}_1(x) + \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1)) \\
&\qquad - \hat{F}_2(\mu, \mathcal{H}(x) + \tilde{Z}_2(x) + \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1))| \\
&\quad + |\check{F}_2(\mu, \mathcal{H}(x) + \tilde{Z}_1(x) + \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1) + \underline{\tilde{d}}) \\
&\qquad - \check{F}_2(\mu, \mathcal{H}(x) + \tilde{Z}_2(x) + \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1) + \underline{\tilde{d}})| \\
&\quad + |F_3(\mu, \mathcal{H}(x) + \tilde{Z}_1(x) + \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1) + \underline{\tilde{d}}) \\
&\qquad - F_3(\mu, \mathcal{H}(x) + \tilde{Z}_2(x) + \tilde{X}_{\mu,\varrho,\tilde{I}_1}(x+\theta_1) + \underline{\tilde{d}})| \\
&\le M\tilde{I}_1|\tilde{Z}_1 - \tilde{Z}_2| + M(|\tilde{Z}_1| + |\tilde{Z}_2|)|\tilde{Z}_1 - \tilde{Z}_2| + M\left(\mu + \frac{|\varrho|}{\mu^2}\right)|\tilde{Z}_1 - \tilde{Z}_2| \\
&\qquad + M\sqrt{\mu}|\tilde{Z}_1 - \tilde{Z}_2| + M\mu|\tilde{Z}_1 - \tilde{Z}_2| \\
&\le M(\sqrt{\mu} + |\tilde{Z}_1| + |\tilde{Z}_2|)|\tilde{Z}_1 - \tilde{Z}_2|,
\end{aligned}
$$

$$
\begin{aligned}
|N_2(\tilde{Z}_1, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1) - N_2(\tilde{Z}_2, x, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1)| &\le M\left(\mu^{n-1} + \frac{|\varrho|}{\mu^2}\right)|\tilde{Z}_1 - \tilde{Z}_2| \\
&\le M(\sqrt{\mu} + |\tilde{Z}_1| + |\tilde{Z}_2|)|\tilde{Z}_1 - \tilde{Z}_2|
\end{aligned}
$$

which yield the second inequality. □

Obviously, the solution $\hat{Z}$ of (5.6.4) exists if x is in a finite interval and an initial condition is given. The idea for proving the existence of solutions for $x \ge 0$ is to change (5.6.4) to integral equations and then use the fixed point theorem to prove the existence of a fixed point of the integral equations.

First, consider the linear equation of (5.6.4)

$$
\frac{d\tilde{Z}}{dx} = \mathcal{L}(x)(\tilde{Z})
$$

with $\mathcal{L}(x) = \tilde{F}_2'[\mathcal{H}(x)]$. From (5.3.9), (5.4.2) and (5.4.3), we can write this equation as

$$
\frac{d\tilde{Z}}{dx} = \mathcal{L}_\infty \tilde{Z} + (\mathcal{L}(x) - \mathcal{L}_\infty)(\tilde{Z}) \tag{5.6.12}
$$

where $\mathcal{L}(x)$ and $\mathcal{L}_\infty$ are given by

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-p_{02}^{(2)}k_2 + 2p_{02}^{(4)}H_{01} & 0 & p_{02}^{(5)}H_{10} & 0 & 0 & p_{02}^{(5)}H_{10} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
p_{11}^{(4)}H_{10} & 0 & c^2 + p_{11}^{(4)}H_{01} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{\mu}}is_{20} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
p_{11}^{(4)}H_{10} & 0 & 0 & 0 & 0 & c^2 + p_{11}^{(4)}H_{01} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{\mu}}is_{20}
\end{pmatrix}
$$

and

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p_{02}^{(2)}k_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{\mu}}is_{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{\mu}}is_{20} \end{pmatrix}$$

respectively. The spectrum of $\mathcal{L}_\infty$ can be calculated directly. It consists of six eigenvalues

$$\lambda_1 = c,\ \lambda_2 = -c,\ \lambda_3 = i\sqrt{p_{02}^{(2)}k_2},\ \lambda_4 = -\lambda_3,\ \lambda_5 = i\frac{1}{\sqrt{\mu}}s_{20},\ \lambda_6 = -\lambda_5.$$

Now we look for the fundamental matrix for the system (5.6.12). Since $\mathcal{H}(x)$ is a solution of the system (5.4.1), we get the following identity

$$\frac{d}{dx}R_\theta\mathcal{H}(x) = \tilde{F}_2(R_\theta\mathcal{H}(x)) = R_\theta\tilde{F}_2(\mathcal{H}(x)) \tag{5.6.13}$$

for any $\theta \in [0, 2\pi)$ where the operator R_θ is defined in (5.1.22). Differentiating it with respect to x or θ at $\theta = 0$, we obtain that

$$s_1(x) = \frac{d}{dx}\mathcal{H}(x), \quad s_2(x) = (H_{01}, H_{02}, iH_{10}, iH_{11}, 0, -iH_{10}, -iH_{11}, 0)^T(x) \tag{5.6.14}$$

are two linearly independent solutions of the system (5.6.12) that decay exponentially as $x \to \infty$, which satisfy by (5.4.6)

$$|s_1(x)| + |s_2(x)| \le M\mathrm{e}^{-cx} \text{ for } x \in [0, \infty). \tag{5.6.15}$$

It is clear that

$$d_5(x) = (0, 0, 0, 0, \mathrm{e}^{\frac{1}{\sqrt{\mu}}is_{20}x}, 0, 0, \mathrm{e}^{-\frac{1}{\sqrt{\mu}}is_{20}x})^T, \tag{5.6.16}$$

$$d_6(x) = (0, 0, 0, 0, i\mathrm{e}^{\frac{1}{\sqrt{\mu}}is_{20}x}, 0, 0, -i\mathrm{e}^{-\frac{1}{\sqrt{\mu}}is_{20}x})^T \tag{5.6.17}$$

are the solutions of (5.6.12), which salsify

$$|d_5(x)| + |d_6(x)| \le M \qquad \text{for } x \in [0, \infty) \tag{5.6.18}$$

and, if only $\tilde{A}, \tilde{B}, \tilde{\bar{A}}$ and $\tilde{\bar{B}}$ components are considered,

$$Sd_k(0) = -d_k(0), \quad k = 5, 6. \tag{5.6.19}$$

Suppose that u_1^∞ and u_2^∞ are the eigenvectors of λ_1, d_3^∞ is the eigenvector of λ_3 and d_4^∞ is the eigenvector of λ_4 for $\mathcal{L}_\infty$. Since $\{\tilde{C} = \bar{\tilde{C}} = 0\}$ is an invariant space of (5.6.12), by Problem 29 in Chapter 3 in the book by Coddington & Levinson [29], we can choose solutions $u_1(x)$, $u_2(x)$, $d_3(x)$ and $d_4(x)$ of (5.6.12) with forms $(\tilde{A}_{01}, \tilde{A}_{02}, \tilde{A}, \tilde{B}, 0, \bar{\tilde{A}}, \bar{\tilde{B}}, 0)^T$ such that

$$\begin{aligned} &u_j(x)\mathrm{e}^{-cx} \to \mathrm{Span}\{u_j^\infty\} \text{ as } x \to \infty,\ |u_j(x)| \le M\mathrm{e}^{cx} \text{ for } x \in [0, \infty), \\ &d_k(x) \to \mathrm{Span}\{d_k^\infty \mathrm{e}^{\lambda_k x}\} \text{ as } x \to \infty,\ |d_k(x)| \le M \text{ for } x \in [0, \infty), \end{aligned} \tag{5.6.20}$$

where $j = 1, 2$ and $k = 3, 4$.

If we consider only the $\tilde{A}, \tilde{B}, \bar{\tilde{A}}$ and $\bar{\tilde{B}}$ components, without loss of generality, we can assume

$$Su_j(0) = -u_j(0), \quad Sd_k(0) = -d_k(0), \quad j = 1, 2, k = 3, 4 \tag{5.6.21}$$

since otherwise we can add the linear combinations of $s_1(x)$ and $s_2(x)$ to $u_j(x)$ or $d_k(x)$ so that (5.6.21) holds.

The matrix

$$\mathcal{B}(x) = \big(s_1(x)|s_2(x)|u_1(x)|u_2(x)|d_3(x)|d_4(x)|d_5(x)|d_6(x)\big)$$

is a fundamental matrix for (5.6.12). From the expression of $\mathcal{L}(x)$ in (5.6.12), it is easy to check that the trace of $\mathcal{L}(x)$ is equal to 0, which gives $\det(\mathcal{B}(x))$ is independent of x. Moreover, by taking $x \to \infty$, $\det(\mathcal{B}(x)) = M$. Let the fundamental set of solutions for the adjoint equation of (5.6.12) be

$$\{s_1^*(x), s_2^*(x), u_1^*(x), u_2^*(x), d_3^*(x), d_4^*(x), d_5^*(x), d_6^*(x)\}$$

which is the dual of $\{s_1(x), s_2(x), u_1(x), u_2(x), d_3(x), d_4(x), d_5(x), d_6(x)\}$ in the sense of the Euclidean inner product on $\mathbf{C}^8$ for each fixed x.

It follows from (5.6.15), (5.6.18) and (5.6.20) that for $x \in [0, \infty)$

$$\begin{aligned} &|u_1^*(x)| + |u_2^*(x)| \le M\mathrm{e}^{-cx}, \qquad |s_1^*(x)| + |s_2^*(x)| \le M\mathrm{e}^{cx}, \\ &|d_3^*(x)| + |d_4^*(x)| + |d_5^*(x)| + |d_6^*(x)| \le M\,. \end{aligned} \tag{5.6.22}$$

The solution of (5.6.4) that decays to zero at infinity can be found as

$$\begin{aligned} \tilde{Z} =& \sum_{j=1}^{2} \int_0^x < \tilde{N}(\tilde{Z}, s, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1), s_j^*(s) > ds\ s_j(x) \\ &- \sum_{j=1}^{2} \int_x^\infty < \tilde{N}(\tilde{Z}, s, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1), u_j^*(s) > ds\ u_j(x) \end{aligned}$$

$$-\sum_{j=3}^{6}\int_x^{\infty} < \tilde{N}(\tilde{Z}, s, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1), d_j^*(s) > ds\ d_j(x)$$

or

$$\tilde{Z} = \mathcal{F}(\tilde{Z}), \tag{5.6.23}$$

where $< \cdot >$ denotes the Euclidean inner product on $\mathbf{C}^8$. Here, we note that because of special structures of $\tilde{N}$ and s_j^*, u_j^*, d_j^* (i.e., the first two components are real, and the third, fourth and fifth components are complex conjugates of the rest components), the inner products in (5.6.23) are real numbers.

Consider (5.6.23) as a fixed point problem in a Banach space

$$E_\nu = \left\{\tilde{Z} \in C([0,\infty) \times S^1 \times S^1) \mid \sup_{x\in[0,\infty)} \{|\tilde{Z}(x,\theta_0,\theta_1)|\mathrm{e}^{\nu x}\} < \infty\right\}$$

with the norm

$$\|\tilde{Z}\|_\nu = \sup\left\{|\tilde{Z}(x,\theta_0,\theta_1)|\mathrm{e}^{\nu x} \mid x \in [0,\infty), \theta_0, \theta_1 \in S^1\right\}.$$

The following lemma is needed.

Lemma 5.6.2 *The function $\mathcal{F}$ satisfies*

$$\|\mathcal{F}(\tilde{Z})\|_\nu \leq M\left[\|\tilde{Z}\|_\nu^2 + \sqrt{\mu}(1 + \|\tilde{Z}\|_\nu)\right],$$
$$\|\mathcal{F}(\tilde{Z}_1) - \mathcal{F}(\tilde{Z}_2)\|_\nu \leq M\left[\sqrt{\mu} + \|\tilde{Z}_1\|_\nu + \|\tilde{Z}_2\|_\nu\right]\|\tilde{Z}_1 - \tilde{Z}_2\|_\nu$$

for $\tilde{Z}, \tilde{Z}_1, \tilde{Z}_2 \in E_v$.

Proof. By (5.6.15), (5.6.18), (5.6.20), (5.6.22) and Lemma 5.6.1, it is obtained that

$$\begin{aligned}
&\Big|\int_0^x < \tilde{N}(\tilde{Z}, s, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1), s_1^*(s) > ds\ s_1(x)\Big|\mathrm{e}^{\nu x} \\
&\leq \int_0^x M[\sqrt{\mu}(1 + \|\tilde{Z}\|_\nu)\mathrm{e}^{-\nu s} + \|\tilde{Z}\|_\nu^2\mathrm{e}^{-2\nu s} + \tilde{I}_1\mathrm{e}^{-\nu s}]\mathrm{e}^{cs}ds\mathrm{e}^{-(c-\nu)x} \\
&\leq M\left[\frac{\sqrt{\mu}}{(c-\nu)}(1 + \|\tilde{Z}\|_\nu) + \frac{1}{2\nu - c}\|\tilde{Z}\|_\nu^2 + \frac{\tilde{I}_1}{c-\nu}\right] \\
&\leq M\left[\sqrt{\mu}(1 + \|\tilde{Z}\|_\nu) + \|\tilde{Z}\|_\nu^2\right],
\end{aligned}$$

$$\Big|\int_x^{\infty} < \tilde{N}(\tilde{Z}, s, \mu, \varrho, \theta_0, \theta_1, \tilde{I}_1), u_1^*(s) > ds\ u_1(x)\Big|\mathrm{e}^{\nu x}$$

$$
\begin{aligned}
&\le \int_x^\infty M\big[\sqrt{\mu}(1+\|\tilde Z\|_\nu)\mathrm{e}^{-\nu s} + \|\tilde Z\|_\nu^2 \mathrm{e}^{-2\nu s} + \tilde I_1 \mathrm{e}^{-\nu s}\big]\mathrm{e}^{-cs} ds \mathrm{e}^{(c+\nu)x} \\
&\le M\left[\frac{\sqrt{\mu}}{(c+\nu)}(1+\|\tilde Z\|_\nu) + \frac{1}{2\nu+c}\|\tilde Z\|_\nu^2 + \frac{\tilde I_1}{c+\nu}\right] \\
&\le M\big[\sqrt{\mu}(1+\|\tilde Z\|_\nu) + \|\tilde Z\|_\nu^2\big],
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\int_x^\infty < \tilde N(\tilde Z, s, \mu, \varrho, \theta_0, \theta_1, \tilde I_1), d_3^*(s) > ds\ d_3(x)\Big|\mathrm{e}^{\nu x} \\
&\quad\le \int_x^\infty M\big[\sqrt{\mu}(1+\|\tilde Z\|_\nu)\mathrm{e}^{-\nu s} + \|\tilde Z\|_\nu^2 \mathrm{e}^{-2\nu s} + \tilde I_1 \mathrm{e}^{-\nu s}\big] ds \mathrm{e}^{\nu x} \\
&\quad\le M\left[\frac{\sqrt{\mu}}{\nu}(1+\|\tilde Z\|_\nu) + \frac{1}{2\nu}\|\tilde Z\|_\nu^2 + \frac{\tilde I_1}{\nu}\right] \\
&\quad\le M\big[\sqrt{\mu}(1+\|\tilde Z\|_\nu) + \|\tilde Z\|_\nu^2\big],
\end{aligned}
$$

since $\nu/c \in (\frac{1}{2}, 1)$ and $\tilde I_1 = \mu^{5/4}\tilde I_2$ from the assumption in Theorem 5.6.1. Similarly, other terms can be estimated so that the first inequality of the lemma is proved.

For the second inequality, it is found that

$$
\begin{aligned}
&\Big|\int_0^x < \tilde N(\tilde Z_1, s, \mu, \varrho, \theta_0, \theta_1, \tilde I_1) - \tilde N(\tilde Z_2, s, \mu, \varrho, \theta_0, \theta_1, \tilde I_1), s_1^*(s) > ds\ s_1(x)\Big|\mathrm{e}^{\nu x} \\
&\le \int_0^x M\big[\sqrt{\mu}\mathrm{e}^{\nu s} + \|\tilde Z_1\|_\nu + \|\tilde Z_2\|_\nu\big]\|\tilde Z_1 - \tilde Z_2\|_\nu \mathrm{e}^{(c-2\nu)s} ds \mathrm{e}^{-(c-\nu)x} \\
&\le M\big[\sqrt{\mu} + \|\tilde Z_1\|_\nu + \|\tilde Z_2\|_\nu\big]\|\tilde Z_1 - \tilde Z_2\|_\nu.
\end{aligned}
$$

Similar argument holds for other terms, which completes the proof. □

If we let $r = M\mu^s$ for any $s \in (0, \frac{1}{2})$, then we can show from Lemma 5.6.2 that $\mathcal{F}$ is a contraction on $\bar B_r(0) \subset E_\nu$ if μ is small. Therefore, (5.6.4) has a unique solution $\tilde Z(x; \mu, \varrho, \theta_0, \theta_1, \tilde I_1)$ for each $\theta_0, \theta_1 \in S^1$ satisfying

$$
|\tilde Z(x; \mu, \varrho, \theta_0, \theta_1, \tilde I_1)| \le M\mu^{7/16}, \quad x \in [0, \infty) \tag{5.6.24}
$$

where we take $s = 7/16$ in order to use this result to find the reversible solution in Step 2.

By differentiating (5.6.23) with respect to other arguments and using the same argument as that for (5.6.24) and an extension of a contraction mapping principle in the book by Walter [115], we can show that $\hat Z$ is smooth in its arguments.

Thus, we have showed that $\tilde{\mathcal{S}}(x; \mu, \varrho, \theta_0, \theta_1, \tilde I_1)$ defined in (5.6.3) exists for $x \ge \tilde x_0$ with any fixed $\tilde x_0 \in (-\infty, +\infty)$, which will be used to obtain a reversible homoclinic orbit of (5.1.19) for $x \in (-\infty, \infty)$ in the following.

Step 2: Reversible homoclinic orbits of (5.1.19) for $x \in (-\infty, \infty)$

By the definition of $\tilde{\mathcal{S}}(x;\mu,\varrho,\theta_0,\theta_1,\tilde{I}_1)$ in (5.6.3), the periodic solution $\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x)$ in (5.5.19) and the equilibrium $\underline{\tilde{d}}$ in Section 5.3, let

$$\begin{aligned}\tilde{\mathcal{S}}_1(x;\mu,\varrho,\theta_0,\theta_1,\tilde{I}_1) = \Big(&\mu\big(H_{01}(\sqrt{\mu}x) + \tilde{A}_{01}(\sqrt{\mu}x) + \varsigma(\sqrt{\mu}x)\tilde{A}^p_{01}(\sqrt{\mu}x+\theta_1) + d + \tilde{d}_{01}\big),\\ &\mu^{3/2}\big(H_{02}(\sqrt{\mu}x) + \tilde{A}_{02}(\sqrt{\mu}x) + \varsigma(\sqrt{\mu}x)\tilde{A}^p_{02}(\sqrt{\mu}x+\theta_1) + \tilde{d}_{02}\big),\\ &\mu e^{i(s_{10}x+\theta_0)}\big(H_{10}(\sqrt{\mu}x) + \tilde{A}_{10}(\sqrt{\mu}x)\big), \mu^{3/2}e^{i(s_{10}x+\theta_0)}\big(H_{11}(\sqrt{\mu}x) + \tilde{A}_{11}(\sqrt{\mu}x)\big),\\ &\mu^{3/2}\big(\tilde{C}(\sqrt{\mu}x) + \varsigma(\sqrt{\mu}x)\tilde{C}^p(\sqrt{\mu}x+\theta_1) + \tilde{d}_{20}\big), \mu e^{-i(s_{10}x+\theta_0)}\big(H_{10}(\sqrt{\mu}x) + \bar{\tilde{A}}_{10}(\sqrt{\mu}x)\big),\\ &\mu^{3/2}e^{-i(s_{10}x+\theta_0)}\big(H_{11}(\sqrt{\mu}x) + \bar{\tilde{A}}_{11}(\sqrt{\mu}x)\big),\\ &\mu^{3/2}\big(\bar{\tilde{C}}(\sqrt{\mu}x) + \varsigma(\sqrt{\mu}x)\bar{\tilde{C}}^p(\sqrt{\mu}x+\theta_1) + \bar{\tilde{d}}_{20}\big)\Big)^T\end{aligned}$$

where $\tilde{Z}(x) = (\tilde{A}_{01}, \tilde{A}_{02}, \tilde{A}, \tilde{B}, \tilde{C}, \bar{\tilde{A}}, \bar{\tilde{B}}, \bar{\tilde{C}})^T(x)$. From the relation between (5.1.19) and (5.3.6), it is easy to check that $\tilde{\mathcal{S}}_1(x;\mu,\varrho,\theta_0,\theta_1,\tilde{I}_1)$ is a solution of (5.1.19) for $x \in [\tilde{x}_0, \infty)$ with any number $\tilde{x}_0$.

To construct a reversible solution of (5.1.19), the idea is to solve the following equation

$$(I - S)\tilde{\mathcal{S}}_1(x_1;\mu,\varrho,\theta_0,\theta_1,\tilde{I}_1) = 0$$

for $x_1, \theta_0, \theta_1, \varrho$. Then, define the solution of (5.1.19) as $\tilde{\mathcal{S}}_2(x) = \tilde{\mathcal{S}}_1(x;\mu,\varrho,\theta_0,\theta_1,\tilde{I}_1)$ for $x \geq 0$ and $\tilde{\mathcal{S}}_2(x) = S\left(\tilde{\mathcal{S}}_1(-x;\mu,\varrho,\theta_0,\theta_1,\tilde{I}_1)\right)$ for $x \leq 0$, which gives a reversible solution $\tilde{\mathcal{S}}_2(x)$ of (5.1.19).

This equation is equivalent to by the relation between $\tilde{\mathcal{S}}_1$ and $\tilde{\mathcal{S}}$

$$(I - S)R_{s_{10}\mu^{-1/2}\tilde{x}_1+\theta_0}\left(\tilde{\mathcal{S}}(\tilde{x}_1;\mu,\varrho,\theta_0,\theta_1,\tilde{I}_1) + \underline{\tilde{d}}\right) = 0$$

where $x_1 = \mu^{-1/2}\tilde{x}_1$ and R_{θ_0} is defined in (5.1.22). (2) of Remark 5.3.1 changes the above equation to

$$(I - S)R_{s_{10}\mu^{-1/2}\tilde{x}_1+\theta_0}\tilde{\mathcal{S}}(\tilde{x}_1;\mu,\varrho,\theta_0,\theta_1,\tilde{I}_1) = 0. \tag{5.6.25}$$

Taking $\tilde{x}_1 = 0$ and $\theta_0 = \frac{\pi}{2}$, it obtained that from (5.6.3) and (5.6.25)

$$\begin{aligned}&H_{02}(0) + \tilde{A}_{02}(0) = 0,\\ &H_{10}(0)\sin(\frac{\pi}{2}) + \mathrm{Im}[\tilde{A}(0)e^{i\frac{\pi}{2}}] = 0\,,\\ &H_{11}(0)\cos(\frac{\pi}{2}) + \mathrm{Re}[\tilde{B}(0)e^{i\frac{\pi}{2}}] = 0\,,\\ &\mathrm{Im}[\tilde{C}(0)] = 0,\end{aligned}$$

where we use the fact that $\varsigma(0) = 0$. By (5.4.4), we know that $H_{02}(0) = H_{10}(0) = 0$ so the above equations are equivalent to

$$\tilde{A}_{02}(0) = 0, \tag{5.6.26}$$

$$\tilde{A}(0) + \bar{\tilde{A}}(0) = 0\,, \tag{5.6.27}$$

$$\tilde{B}(0) - \bar{\tilde{B}}(0) = 0\,, \tag{5.6.28}$$

$$\mathrm{Im}[\tilde{C}(0)] = 0. \tag{5.6.29}$$

(5.6.19), (5.6.21) and (5.6.23) show that the equations (5.6.27) and (5.6.28) hold automatically. In the following, we just focus on the equations (5.6.26) and (5.6.29) and solve for ϱ and θ_1.

Lemma 5.6.3 *Assume that the assumption* (2) *in Theorem* 5.6.1 *holds. Then* (5.6.26) *and* (5.6.29) *are equivalent to the following equations*

$$\gamma_2 = \mathcal{F}_2(\gamma_2; \mu) \tag{5.6.30}$$

where $\gamma_2 = (\varrho_1, \tilde{\theta}_1)^T$, $\varrho = \mu^{21/8}\varrho_1$, $\theta_1 = \sqrt{\mu}\tilde{\theta}_1$,

$$\mathcal{F}_2(\gamma_2; \mu) = \begin{pmatrix} \mu^{1/4}\hat{f}_{A02}(\mu, \varrho_1, \tilde{\theta}_1) \\ \mu^{3/8}\hat{f}_C(\mu, \varrho_1, \tilde{\theta}_1) \end{pmatrix}$$

$\hat{f}_{A_{02}}$ *and* $\hat{f}_C$ *are smooth in their arguments and bounded.*

The proof is given in Section 6.5.

Choose a ball $B_r(0) \subset \mathbf{R}^2$ with a radius $r = O(\mu^{3/16})$. It is easy to show that $\mathcal{F}_2$ is a contraction on $\bar{B}_r(0)$ if μ is small. Therefore, (5.6.30) has a unique solution γ_2, i.e., (5.6.25) has a solution

$$(x_1, \theta_0, \theta_1, \varrho) = (0, \frac{\pi}{2}, \sqrt{\mu}\tilde{\theta}_1, \mu^{21/8}\varrho_1)$$

for $\mu > 0$ small enough.

Using the reversibility and the translation invariance, we know that $\tilde{\mathcal{S}}_1(x + x_1; \mu, \mu^{21/8}\varrho_1, \theta_0, \theta_1, \mu^{5/4}\tilde{I}_2)$ and $S\tilde{\mathcal{S}}_1(-x + x_1; \mu, \mu^{21/8}\varrho_1, \theta_0, \theta_1, \mu^{5/4}\tilde{I}_2)$ are solutions of (5.1.19) and

$$S\tilde{\mathcal{S}}_1(x_1; \mu, \mu^{21/8}\varrho_1, \theta_0, \theta_1, \mu^{5/4}\tilde{I}_2) = \tilde{\mathcal{S}}_1(x_1; \mu, \mu^{21/8}\varrho_1, \theta_0, \theta_1, \mu^{5/4}\tilde{I}_2)$$

so we can define

$$\tilde{\mathcal{S}}_2(x) = \begin{cases} \tilde{\mathcal{S}}_1(x + x_1; \mu, \mu^{21/8}\varrho_1, \theta_0, \theta_1, \mu^{5/4}\tilde{I}_2) & \text{for } x \ge 0, \\ S\tilde{\mathcal{S}}_1(-x + x_1; \mu, \mu^{21/8}\varrho_1, \theta_0, \theta_1, \mu^{5/4}\tilde{I}_2) & \text{for } x \le 0. \end{cases}$$

Then $S\tilde{\mathcal{S}}_2(-x) = \tilde{\mathcal{S}}_2(x)$. Thus, the solution $\tilde{\mathcal{S}}_2(x)$ of (5.1.19) is a reversible homoclinic connection to the periodic orbit $\hat{X}_{\mu,\varrho,\tilde{I}_1}(x + \theta_1)$ defined in (5.6.1) as $x \to \infty$ and the periodic orbit $S\hat{X}_{\mu,\varrho,\tilde{I}_1}(-x + \theta_1)$ as $x \to -\infty$. This completes the proof of Theorem 5.6.1.

Chapter 6

Appendices

In this chapter, the proofs of some lemmas and calculations of some coefficients in Chapter 4 and Chapter 5 are given.

6.1 Proof of Lemma 4.1.1

The following lemma is needed for the proof.

Lemma 6.1.1 *Given real* k_j, $j = 1, 2, \cdots, n$, *and* $c_1 \mathrm{e}^{ik_1 x} + \cdots + c_n \mathrm{e}^{ik_n x} \equiv \text{constant}$ *for all* x *in some interval, then for* $j = 1, 2, \ldots n$, *either* $c_j = 0$ *or* $k_j = 0$ *or* $c_j \mathrm{e}^{ik_j x}$ *is canceled with other terms.*

The proof is straightforward.

To prove Lemma 4.1.1, from the general theory of differential operators it follows that $H(X) = Q(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ since $\{u_j\}_{j=1}^{7}$ are independent.

From (4.1.8), we obtain the following (The basic idea comes from the unpublished manuscript of A. Ilichev and K. Kirchgässner in 1998)

$$
\begin{aligned}
&A = u_1^{\frac{1}{2}} \mathrm{e}^{i\alpha_A}, \quad B = u_5 u_1^{\frac{1}{2}} \mathrm{e}^{i\alpha_A} - \frac{u_1^{\frac{1}{2}} \mathrm{e}^{i\alpha_A}}{is_1} (\frac{1}{2} \ln u_1 + i\alpha_A), \\
&C = u_3^{\frac{1}{2}} \mathrm{e}^{i\alpha_C}, \qquad D = u_4^{\frac{1}{2}} \mathrm{e}^{i\alpha_D}, \\
&\alpha_C = s_{00}[u_6 - u_5 + \frac{1}{is_1} (\frac{1}{2} \ln u_1 + i\alpha_A)] + \frac{i}{2} \ln u_3 - 2k_1\pi, \\
&\alpha_D = s_{01}[u_7 - u_5 + \frac{1}{is_1} (\frac{1}{2} \ln u_1 + i\alpha_A)] + \frac{i}{2} \ln u_4 - 2k_2\pi
\end{aligned}
$$

and their complex conjugates, where $\alpha_A, \alpha_C, \alpha_D \in [0, 2\pi)$ and the integers k_1, k_2 are chosen so that $\alpha_C, \alpha_D \in [0, 2\pi)$.

Using the above equalities, the components of $X = (A, B, C, D, \bar{A}, \bar{B}, \bar{C}, \bar{D})$ in

$$\mathcal{G} = \big\{X \in \mathbf{C}^8 \big| |A|, |B|, |C|, |D| > 0\big\}$$

can be reconstructed using vector $\underline{u} = (u_1, \cdots, u_7) \in \mathcal{S}$ where

$$\mathcal{S} = \big\{\underline{u} \in \mathbf{R}^4 \times \mathbf{C}^3 \big| u_1, u_3, u_4 > 0, \quad \mathrm{Im}(u_5) = \frac{u_2}{u_1} - \frac{1}{2s_1}\ln u_1, \\ \mathrm{Im}(u_6) = \frac{u_2}{u_1} - \frac{1}{2s_{00}}\ln u_3,\ \mathrm{Im}(u_7) = \frac{u_2}{u_1} - \frac{1}{2s_{01}}\ln u_4\big\}.$$

It is straightforward to see that for each fixed $\alpha_A \in [0, 2\pi)$ there exists a unique $X \in \mathcal{G}$ defined by a $\underline{u} \in \mathcal{S}$, which is analytic in terms of α_A. Since $H(X) = Q(u_j, j = 1, 2, \dots 7)$ where $u_j, j = 1, \dots 7$ are independent of α_A, Q is independent of α_A, which implies that $H = Q$ is a constant when α_A changes.

The polynomial $H(X)$ reads

$$H(X) = \sum_{j_1+\cdots+j_8=n} H_{j_1\cdots j_8} A^{j_1} B^{j_2} C^{j_3} D^{j_4} \bar{A}^{j_5} \bar{B}^{j_6} \bar{C}^{j_7} \bar{D}^{j_8}$$

where $j_1, \cdots, j_8$ are nonnegative integers. Thus, by the relations between X and $u_j, j = 1, 2, \dots, 7$, we can write $H(X)$ in terms of $u_j, j = 1, 2, \dots, 7$ as follows,

$$\begin{aligned} H(X) &= \sum_{j_1+\cdots+j_8=n} H_{j_1\cdots j_8} u_1^{\frac{1}{2}(j_1+j_5)} \mathrm{e}^{i(j_1-j_5)\alpha_A} \Big[u_5 u_1^{\frac{1}{2}} \mathrm{e}^{i\alpha_A} - \frac{u_1^{\frac{1}{2}}\mathrm{e}^{i\alpha_A}}{is_1}\Big(\frac{1}{2}\ln u_1 + i\alpha_A\Big)\Big]^{j_2} \\ &\qquad \cdot u_3^{\frac{1}{2}(j_3+j_7)} \mathrm{e}^{i(j_3-j_7)\big[s_{00}\big(u_6-u_5+\frac{1}{is_1}(\frac{1}{2}\ln u_1+i\alpha_A)\big)+\frac{1}{2}i\ln u_3\big]} \\ &\qquad \cdot u_4^{\frac{1}{2}(j_4+j_8)} \mathrm{e}^{i(j_4-j_8)\big[s_{01}\big(u_7-u_5+\frac{1}{is_1}(\frac{1}{2}\ln u_1+i\alpha_A)\big)+\frac{1}{2}i\ln u_4\big]} \\ &\qquad \cdot\Big[\bar{u}_5 u_1^{\frac{1}{2}} \mathrm{e}^{-i\alpha_A} - \frac{u_1^{\frac{1}{2}}\mathrm{e}^{-i\alpha_A}}{-is_1}\Big(\frac{1}{2}\ln u_1 - i\alpha_A\Big)\Big]^{j_6} \\ &= \sum_{j_1+\cdots+j_8=n} H_{j_1\cdots j_8} u_1^{\frac{1}{2}(j_1+j_2+j_5+j_6+\frac{s_{00}}{s_1}(j_3-j_7)+\frac{s_{01}}{s_1}(j_4-j_8))} \\ &\qquad \cdot\sum_{k_1}^{j_2} C_{j_2}^{k_1}[u_5 - \frac{1}{i2s_1}\ln u_1]^{k_1}\Big(-\frac{1}{s_1}\Big)^{j_2-k_1} {\alpha_A}^{j_2-k_1} u_3^{j_7} \mathrm{e}^{is_{00}(j_3-j_7)(u_6-u_5)} \\ &\qquad \cdot u_4^{j_8} \mathrm{e}^{is_{01}(j_4-j_8)(u_7-u_5)} \sum_{k_2}^{j_6} C_{j_6}^{k_2}\Big[\bar{u}_5 - \frac{1}{-i2s_1}\ln u_1\Big]^{k_2}\Big(-\frac{1}{s_1}\Big)^{j_6-k_2} {\alpha_A}^{j_6-k_2} \\ &\qquad \cdot\mathrm{e}^{i\big[(j_1-j_5)+(j_2-j_6)+\frac{(j_3-j_7)s_{00}}{s_1}+\frac{(j_4-j_8)s_{01}}{s_1}\big]\alpha_A} \end{aligned}$$

$$= \sum_{j=0}^{\beta} H_j(\alpha_A)\alpha_A^j \tag{6.1.1}$$

for all $\alpha_A \in [0, 2\pi)$. Here $H_j(\alpha_A)$ is a finite sum of the form $\mathrm{e}^{ir\alpha_A}$ with r real.

Lemma 6.1.2 *$H_0(\alpha_A) \equiv H(X)$ and $H_j(\alpha_A) \equiv 0$ for all $\alpha_A \in [0, 2\pi)$ and $j = 1, \cdots, \beta$.*

Proof. We use the mathematical induction. Let $x = \alpha_A$ and $\tilde{C} = H(X)$ be independent of x. First, consider the equation

$$\tilde{C} \equiv H_0(x) + H_1(x)x. \tag{6.1.2}$$

If $H_1(x) \equiv 0$, then the lemma is true. If not, suppose that $H_1(x_0) \neq 0$. Thus there exists a constant $\delta > 0$ such that $H_1(x) \neq 0$ for $x \in (x_0 - \delta, x_0 + \delta)$. Then, assume that $H_0(x) = \sum_{j=0}^{n_0} d_j^0 \mathrm{e}^{ir_j^0 x}$ and $H_1(x) = \sum_{j=0}^{n_1} d_j^1 \mathrm{e}^{ir_j^1 x}$, where $r_{n_0}^0$ and $r_{n_1}^1$ are positive and $r_{n_0}^0 > r_k^0 \geq 0$, $r_{n_1}^1 > r_j^1 \geq 0$ for $k = 0, \cdots, n_0 - 1$, $j = 0, \cdots, n_1 - 1$. Note that (6.1.2) is equivalent to $x = (\tilde{C} - H_0(x))/H_1(x)$ for $x \in (x_0 - \delta, x_0 + \delta)$. Differentiate it with respect to x and obtain

$$H_1^2(x) = -H_0'(x)H_1(x) - H_1'(x)(\tilde{C} - H_0(x)), \tag{6.1.3}$$

where the leading term of the left side is $(d_{n_1}^1)^2 \mathrm{e}^{i2r_{n_1}^1 x}$ and the leading term of the right side is $id_{n_0}^0 d_{n_1}^1 (r_{n_1}^1 - r_{n_0}^0)\mathrm{e}^{i(r_{n_0}^0 + r_{n_1}^1)x}$.

Case 1: $r_{n_0}^0 > r_{n_1}^1$. Compare the leading terms on both sides and obtain $d_{n_0}^0 d_{n_1}^1 = 0$ by Lemma 6.1.1, which implies that either H_0 or H_1 is constant with respect to x.

(i) If H_0 is constant and $H_0(x) = \tilde{C}$, (6.1.3) implies $H_1(x) = 0$, which contradicts to $H_1(x) \neq 0$ for $x \in (x_0 - \delta, x_0 + \delta)$. If H_0 is constant and $H_0(x) \neq \tilde{C}$, (6.1.3) gives

$$-H_1'(x) = H_1^2(x)/(\tilde{C} - H_0(x)).$$

Since the leading term of the left side is $-id_{n_1}^1 r_{n_1}^1 \mathrm{e}^{ir_{n_1}^1 x}$ and the leading term of the right side is $\frac{(d_{n_1}^1)^2}{\tilde{C} - H_0(x)} \mathrm{e}^{2ir_{n_1}^1 x}$, by Lemma 6.1.1, we obtain $d_{n_1}^1 = 0$, i.e., $H_1(x)$ is constant and by the above equation $H_1(x) = 0$, which is a contradiction.

(ii) If $H_1(x)$ is constant, (6.1.3) yields $H_0(x) = -H_1(x)x + C_1$ for some constant C_1, which contradicts to $H_0(x)$ as the sum of form e^{irx}.

Case 2: $r_{n_0}^0 \leq r_{n_1}^1$. Then, $d_{n_1}^1 = 0$, which implies that $H_1(x)$ is constant and yields a contradiction.

Hence, $H_1(x)$ has to be zero.

If some r_i^0 or r_j^1 is negative, we can multiply (6.1.2) by e^{irx} and consider the new equation $0 \equiv \tilde{H}_0(x) + \tilde{H}_1(x)x$, which is (6.1.2) with $\tilde{C} = 0$, where $\tilde{H}_0(x) = (H_0(x) - \tilde{C})e^{irx}$ and $\tilde{H}_1(x) = H_1(x)e^{irx}$. By choosing large $r > 0$ so that all r_j^0 in $\tilde{H}_0(x)$ and r_j^1 in $\tilde{H}_1(x)$ are nonnegative, the above argument shows that $\tilde{H}_1(x) = 0$ or $H_1(x) = 0$, which contradicts $H_1(x) \neq 0$ for $x \in (x_0 - \delta, x_0 + \delta)$.

Therefore, the lemma is proved for $n = 1$.

Next, assume that the lemma is true for n and consider

$$\tilde{C} = H_0(x) + H_1(x)x + \cdots + H_n(x)x^n + H_{n+1}(x)x^{n+1}. \tag{6.1.4}$$

If $H_{n+1}(x) \equiv 0$, by the assumption, the lemma is true. If $H_{n+1}(x_0) \neq 0$, there exists a constant $\delta > 0$ such that $H_{n+1}(x) \neq 0$ for $x \in (x_0 - \delta, x_0 + \delta)$. Solve (6.1.4) for x^{n+1},

$$x^{n+1} = [\tilde{C} - (H_0(x) + H_1(x)x + \cdots + H_n(x)x^n)]/H_{n+1}(x)$$

with $x \in (x_0 - \delta, x_0 + \delta)$. Differentiate it with respect to x,

$$\begin{aligned}(n+1)H_{n+1}^2(x)x^n &= -H_{n+1}(x)[H_0(x) + H_1(x)x + \cdots + H_n(x)x^n]' \\ &\quad -H_{n+1}'(x)[\tilde{C} - (H_0(x) + H_1(x)x + \cdots + H_n(x)x^n)],\end{aligned}$$

which is of the form (6.1.4) with $n+1$ replaced by n. By the induction hypothesis, the coefficient of x^n is

$$H_{n+1}(x)H_n'(x) - H_{n+1}'(x)H_n(x) + (n+1)H_{n+1}^2(x) \equiv 0. \tag{6.1.5}$$

(6.1.5) is same as (6.1.3) except for some different coefficients. Thus, by the same argument for the case $n = 1$ using (6.1.3), we obtain $H_{n+1}(x) \equiv 0$, which contradicts to $H_{n+1}(x) \neq 0$. Hence, the proof of the lemma is completed by induction. □

We note that $H_0(\alpha_A)$ is the sum of the form

$$e^{i[(j_1-j_5)+(j_2-j_6)+\frac{(j_3-j_7)s_{00}}{s_1}+\frac{(j_4-j_8)s_{01}}{s_1}]\alpha_A}.$$

Lemma 6.1.1 implies that from the assumption on the relation of s_{00}, s_{01}, s_1, $H_0(\alpha_A)$ is a constant for any $\alpha_A \in [0, 2\pi)$ if and only if $(j_1 - j_5) + (j_2 - j_6) = 0$, $j_3 - j_7 = 0$ and $j_4 - j_8 = 0$. Therefore, Q must be a polynomial in u_3 and u_4 (since $C, \bar{C}$ appear in pairs in H) and independent of u_6 and u_7 (since the coefficients of the form e^{irx} in (6.1.1) do not contain u_6, u_7 by using the fact that $j_3 = j_7$ and $j_4 = j_8$).

By the relations between the partial derivatives of H and Q, we find

$$\frac{\partial H}{\partial \bar{A}} = A\frac{\partial Q}{\partial u_1} - \frac{i}{2}B\frac{\partial Q}{\partial u_2}, \tag{6.1.6}$$

$$\frac{\partial H}{\partial B} = -\frac{i}{2}\bar{A}\frac{\partial Q}{\partial u_2} + \frac{1}{A}\frac{\partial Q}{\partial u_5}, \tag{6.1.7}$$

$$\frac{\partial H}{\partial \bar{B}} = \frac{i}{2} A \frac{\partial Q}{\partial u_2}. \tag{6.1.8}$$

From (6.1.8), since H is a polynomial in $\bar{B}$, we can keep taking the derivatives with respect to $\bar{B}$ to make the left side of (6.1.8) equal to zero, which implies $\partial^m Q/\partial u_2^m = 0$ for some $m > 0$ and Q is a polynomial in u_2. By (6.1.6) and (6.1.8), we obtain

$$\frac{\partial Q}{\partial u_1} = \frac{1}{A}\Big(\frac{\partial}{\partial \bar{A}} + \frac{B}{A}\frac{\partial}{\partial \bar{B}}\Big) H.$$

Again, the same argument gives that Q is a polynomial in u_1. Similarly, (6.1.7) yields that Q is a polynomial in u_5.

Since H is polynomial in A, we obtain that Q is independent of u_5 since otherwise H, which has a form $|A|^k$, cannot be equal to Q that has a form $|A|^l |\ln A|^m$ as $|A| \to \infty$, where k, l, m are nonnegative integers. Hence, Q is a polynomial in u_1, u_2, u_3, u_4 and is independent of u_5, u_6, u_7. The proof of this lemma is completed. □

6.2 Proof of Lemma 4.3.3

The idea to prove the lemma is to find θ_1 and ϱ such that (4.3.31) and (4.3.32) hold. In order to do that, we compute the leading-order terms in $\hat{C}$ and $\hat{D}$ defined in (4.3.11) when ϵ and μ are small. Since the leading order terms in (4.3.11) may involve $\hat{C}_p$ and $\hat{D}_p$, we first derive the forms of $\hat{C}_p$ and $\hat{D}_p$.

Note that the third and fourth components $\hat{C}_p, \hat{D}_p$ of the periodic function $\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1}(x)$ defined in (4.2.23) are solutions of

$$\begin{aligned}\dot{\hat{C}}_p &= is_{00}\hat{C}_p + i\hat{C}_p P_1(\mu,0,0,\mu|\hat{C}_p|^2,\mu|\hat{C}_p|^2) \\ &\quad +\epsilon\check{\mathcal{R}}_{00}(\mu,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p) - i\frac{P}{r_{00}^2}\frac{\varrho}{\sqrt{\mu}} + \frac{\varrho}{\sqrt{\mu}}\tilde{\wp}_{00}(\mu,\varrho,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{C}}_p), \quad (6.2.1)\\ \dot{\hat{D}}_p &= is_{01}\hat{D}_p + i\hat{D}_p P_2(\mu,0,0,\mu|\hat{C}_p|^2,\mu|\hat{D}_p|^2) \\ &\quad +\epsilon\check{\mathcal{R}}_{01}(\mu,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{C}}_p) - i\frac{P}{r_{01}^2}\frac{\varrho}{\sqrt{\mu}} + \frac{\varrho}{\sqrt{\mu}}\tilde{\wp}_{01}(\mu,\varrho,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p) \quad (6.2.2)\end{aligned}$$

where $\tilde{\wp}_{00} = \check{\wp}_{00} + i\frac{P}{r_{00}^2}$, $\tilde{\wp}_{01} = \check{\wp}_{01} + i\frac{P}{r_{01}^2}$, $\check{\wp}_{00}$ and $\check{\wp}_{01}$ are defined in (4.2.2), r_{00} and r_{01} are given in (6.3.3), and the coefficient d_{20} is defined in (4.1.14).

Integrating (6.2.1) and (6.2.2) yields

$$\hat{C}_p(x) \;=\; e^{is_{00}x}\hat{C}_p(0) + \int_0^x e^{is_{00}(x-s)}\big[i\hat{C}_p P_1(\mu,0,0,\mu|\hat{C}_p|^2,\mu|\hat{D}_p|^2)$$

$$
\begin{aligned}
& +\epsilon\check{\mathcal{R}}_{00}(\mu,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)-i\frac{P}{r_{00}^2}\frac{\varrho}{\sqrt{\mu}}+\frac{\varrho}{\sqrt{\mu}}\tilde{\wp}_{00}(\mu,\varrho,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)\big]ds \\
= {} & \mathrm{e}^{is_{00}x}\left(\hat{C}_p(0)-\frac{P}{r_{00}^2 s_{00}}\frac{\varrho}{\sqrt{\mu}}\right)+\frac{P}{r_{00}^2 s_{00}}\frac{\varrho}{\sqrt{\mu}}+\vartheta_0(x), \qquad (6.2.3)\\
\hat{D}_p(x) = {} & \mathrm{e}^{is_{01}x}\hat{D}_p(0)+\int_0^x \mathrm{e}^{is_{01}(x-s)}\big[i\hat{D}_p P_2(\mu,0,0,\mu|\hat{C}_p|^2,\mu|\hat{D}_p|^2) \\
& +\epsilon\check{\mathcal{R}}_{01}(\mu,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)-i\frac{P}{r_{01}^2}\frac{\varrho}{\sqrt{\mu}}+\frac{\varrho}{\sqrt{\mu}}\tilde{\wp}_{01}(\mu,\varrho,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)\big]ds \\
= {} & \mathrm{e}^{is_{01}x}\left(\hat{D}_p(0)-\frac{P}{r_{01}^2 s_{01}}\frac{\varrho}{\sqrt{\mu}}\right)+\frac{P}{r_{01}^2 s_{01}}\frac{\varrho}{\sqrt{\mu}}+\vartheta_1(x) \qquad (6.2.4)
\end{aligned}
$$

where

$$
\begin{aligned}
\vartheta_0(x) = {} & \int_0^x \mathrm{e}^{is_{00}(x-s)}\big[i\hat{C}_p P_1(\mu,0,0,\mu|\hat{C}_p|^2,\mu|\hat{D}_p|^2) \\
& +\epsilon\check{\mathcal{R}}_{00}(\mu,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)+\frac{\varrho}{\sqrt{\mu}}\tilde{\wp}_{00}(\mu,\varrho,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)\big]ds,\\
\vartheta_1(x) = {} & \int_0^x \mathrm{e}^{is_{01}(x-s)}\big[i\hat{D}_p P_2(\mu,0,0,\mu|\hat{C}_p|^2,\mu|\hat{D}_p|^2) \\
& +\epsilon\check{\mathcal{R}}_{01}(\mu,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)+\frac{\varrho}{\sqrt{\mu}}\tilde{\wp}_{01}(\mu,\varrho,\hat{C}_p,\hat{D}_p,\bar{\hat{C}}_p,\bar{\hat{D}}_p)\big]ds.
\end{aligned}
$$

Let $\tilde{x}=(s_{00}+r_1)x$ where r_1 is defined in (4.2.19). Then, (6.2.3) is changed to

$$
\hat{C}_p(\tilde{x})=\mathrm{e}^{i\frac{s_{00}}{s_{00}+r_1}\tilde{x}}\left(\hat{C}_p(0)-\frac{P}{r_{00}^2 s_{00}}\frac{\varrho}{\sqrt{\mu}}\right)+\frac{P}{r_{00}^2 s_{00}}\frac{\varrho}{\sqrt{\mu}}+\vartheta_0\left(\frac{\tilde{x}}{s_{00}+r_1}\right),
$$

which is a 2π-period function. To find $\hat{C}_p(0)$, we know that the coefficient of $\mathrm{e}^{i(s_{00}+r_1)x}$ in $\hat{C}_p(x)$ is $C_1=\hat{J}_1$ (see the assumption in (4.2.21)). Thus,

$$
\begin{aligned}
\hat{J}_1 = {} & \frac{1}{2\pi}\int_0^{2\pi}\hat{C}_p(\tilde{x})\mathrm{e}^{-i\tilde{x}}d\tilde{x} \\
= {} & \frac{1}{2\pi}\left(\hat{C}_p(0)-\frac{P}{r_{00}^2 s_{00}}\frac{\varrho}{\sqrt{\mu}}\right)\int_0^{2\pi}\mathrm{e}^{-i\frac{r_1}{s_{00}+r_1}\tilde{x}}d\tilde{x}+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-i\tilde{x}}\vartheta_0\left(\frac{\tilde{x}}{s_{00}+r_1}\right)d\tilde{x} \\
= {} & \frac{1}{2\pi}\left(\hat{C}_p(0)-\frac{P}{r_{00}^2 s_{00}}\frac{\varrho}{\sqrt{\mu}}\right)(2\pi+\zeta)+\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-i\tilde{x}}\vartheta_0\left(\frac{\tilde{x}}{s_{00}+r_1}\right)d\tilde{x},
\end{aligned}
$$

where $\zeta(r_1)=\big(\mathrm{e}^{i\frac{-2\pi r_1}{s_{00}+r_1}}-1\big)/(i\frac{-r_1}{s_{00}+r_1})-2\pi=O(r_1)$ and $\zeta(0)=0$. Thus,

$$
\hat{C}_p(0)-\frac{P}{r_{00}^2 s_{00}}\frac{\varrho}{\sqrt{\mu}} = \frac{1}{1+\frac{1}{2\pi}\zeta}\left[\hat{J}_1-\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-i\tilde{x}}\vartheta_0\left(\frac{\tilde{x}}{s_{00}+r_1}\right)d\tilde{x}\right],
$$

which gives

$$\begin{aligned}\hat{C}_p(x) &= \frac{e^{is_{00}x}}{1+\frac{1}{2\pi}\zeta}\hat{J}_1 + \frac{P}{r_{00}^2 s_{00}}\frac{\varrho}{\sqrt{\mu}} \\ &\quad -\frac{e^{is_{00}x}}{1+\frac{1}{2\pi}\zeta}\frac{1}{2\pi}\int_0^{2\pi} e^{-i\tilde{x}}\vartheta_0\left(\frac{\tilde{x}}{s_{00}+r_1}\right)d\tilde{x} + \vartheta_0(x). \qquad (6.2.5)\end{aligned}$$

Since by (4.2.1) and (4.2.20),

$$\epsilon\check{\mathcal{R}}_{00} = \epsilon O(\sqrt{\mu}), \qquad \frac{\varrho}{\sqrt{\mu}}\tilde{\wp}_{00} = \varrho\frac{O(\mu)}{\sqrt{\mu}} = \varrho O(\sqrt{\mu}), \qquad P_1 = O(\mu),$$

and the assumptions $\hat{J}_1 = \frac{\epsilon}{\mu^2}\hat{J}_2$, $\epsilon \in [0, \mu^{11/2}]$ and $\varrho = \frac{\epsilon}{\mu}\varrho_1$, we know that $\hat{C}_p(x) = O(\frac{\epsilon}{\mu^2})$ and $\vartheta_0 = O(\frac{\epsilon}{\mu})$.

Moreover, $\hat{D}_p(x)$ is a $\frac{2\pi}{s_{00}+r_1}$-period solution with $\hat{D}_p(0) = \hat{D}_p(\frac{2\pi}{s_{00}+r_1})$, which, from (6.2.4), gives that

$$\hat{D}_p(0) = \frac{P}{r_{01}^2 s_{01}}\frac{\varrho}{\sqrt{\mu}} + \frac{1}{1-e^{is_{01}\frac{2\pi}{s_{00}+r_1}}}\vartheta_1\left(\frac{2\pi}{s_{00}+r_1}\right)$$

and

$$\hat{D}_p(x) = \frac{P}{r_{01}^2 s_{01}}\frac{\varrho}{\sqrt{\mu}} + \frac{1}{1-e^{is_{01}\frac{2\pi}{s_{00}+r_1}}}\vartheta_1\left(\frac{2\pi}{s_{00}+r_1}\right)e^{is_{01}x} + \vartheta_1(x)$$

where ϑ_1 contains terms which are of order $O(\frac{\epsilon}{\sqrt{\mu}})$. Since $d_{21} = 0$ (see (4.1.14)) and $\varrho = \frac{\epsilon}{\mu}\varrho_1$, replace $\hat{D}_p(x)$ by $\frac{P}{r_{01}^2 s_{01}}\frac{\varrho}{\sqrt{\mu}}$ in $\vartheta_1(x)$ to obtain from (4.1.12) and (4.1.14) that

$$\vartheta_1(x) = \int_0^x e^{is_{01}(x-s)}\frac{iPp_{21}}{r_{01}^2 s_{01}}\varrho\sqrt{\mu} + O(\epsilon\sqrt{\mu}) = \frac{Pp_{21}}{r_{01}^2 s_{01}}\varrho\sqrt{\mu}\frac{e^{is_{01}x}-1}{s_{01}} + O(\epsilon\sqrt{\mu})$$

and

$$\vartheta_1\left(\frac{2\pi}{s_{00}+r_1}\right) = \frac{Pp_{21}}{r_{01}^2 s_{01}}\varrho\sqrt{\mu}\frac{e^{is_{01}\frac{2\pi}{s_{00}+r_1}}-1}{s_{01}} + O(\epsilon\sqrt{\mu}).$$

Therefore, we find that

$$\hat{D}_p(x) = \frac{P}{r_{01}^2 s_{01}}\frac{\varrho}{\sqrt{\mu}} - \frac{Pp_{21}}{r_{01}^2 s_{01}^2}\varrho\sqrt{\mu} + O(\epsilon\sqrt{\mu}). \qquad (6.2.6)$$

Next, consider the equations in (4.3.11) for $\hat{C}$ and $\hat{D}$. Rewrite the equations in (4.3.11) for $\hat{C}$ and $\hat{D}$ as

$$\dot{\hat{C}}(x) - is_{00}\hat{C} = i\hat{C}\tilde{P}_1 + i\varsigma(x)\hat{C}_p(x+\theta_1)[\tilde{P}_1 - P_1(\mu, 0, 0, \mu\hat{C}_p\bar{\hat{C}}_p, \mu\hat{D}_p\bar{\hat{D}}_p)]$$

$$\begin{aligned}
&\quad +\epsilon\tilde{\mathcal{R}}_{00}(\mu,\varrho,R_{\Omega_0x+\theta_0}(\hat{\mathcal{H}}+\hat{Z})+\varsigma(x)\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1})\\
&\quad -\epsilon\varsigma(x)\tilde{\mathcal{R}}_{00}(\mu,\varrho,\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1})-\varsigma'(x)\hat{C}_p(x+\theta_1),\\
\dot{\hat{D}}(x)-is_{01}\hat{D} &= i\hat{D}\tilde{P}_2+i\varsigma(x)\hat{D}_p(x+\theta_1)[\tilde{P}_2-P_2(\mu,0,0,\mu\hat{C}_p\bar{\hat{C}}_p,\mu\hat{D}_p\bar{\hat{D}}_p)]\\
&\quad +\epsilon\tilde{\mathcal{R}}_{01}(\mu,\varrho,R_{\Omega_0x+\theta_0}(\hat{\mathcal{H}}+\hat{Z})+\varsigma(x)\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1})\\
&\quad -\epsilon\varsigma(x)\tilde{\mathcal{R}}_{01}(\mu,\varrho,\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1})-\varsigma'(x)\hat{D}_p(x+\theta_1)
\end{aligned}$$

where $\tilde{\mathcal{R}}_{00}$ and $\tilde{\mathcal{R}}_{01}$ are the third and the fourth components of $\tilde{\mathcal{R}}(\mu,\varrho,\hat{X})$ in (4.3.12), and

$$\begin{aligned}
\tilde{P}_j &= P_j\big(\mu,\mu(\hat{\mathcal{H}}_A+\hat{A})\overline{(\hat{\mathcal{H}}_A+\hat{A})},\\
&\quad i\mu((\hat{\mathcal{H}}_A+\hat{A})\overline{(\hat{\mathcal{H}}_B+\hat{B})}-\overline{(\hat{\mathcal{H}}_A+\hat{A})}(\hat{\mathcal{H}}_B+\hat{B})),\\
&\quad \mu(\hat{C}+\varsigma(x)\hat{C}_p(x+\theta_1))\overline{(\hat{C}+\varsigma(x)\hat{C}_p(x+\theta_1))},\\
&\quad \mu(\hat{D}+\varsigma(x)\hat{D}_p(x+\theta_1))\overline{(\hat{D}+\varsigma(x)\hat{D}_p(x+\theta_1))}\big),\quad j=1,2.
\end{aligned}$$

Since $\hat{C}(x),\hat{D}(x)\to 0$ as $x\to\infty$, we obtain from the last equation of (4.3.12)

$$\begin{aligned}
\hat{C}(x) &= -\int_x^\infty e^{is_{00}(x-s)}\big[i\hat{C}\tilde{P}_1+i\varsigma(s)\hat{C}_p(s+\theta_1)(\tilde{P}_1-P_1(\mu,0,0,\mu\hat{C}_p\bar{\hat{C}}_p,\mu\hat{D}_p\bar{\hat{D}}_p))\\
&\quad +\epsilon\tilde{\mathcal{R}}_{00}(\mu,\varrho,R_{\Omega_0s+\theta_0}(\hat{\mathcal{H}}+\hat{Z})+\varsigma(s)\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1})\\
&\quad -\epsilon\varsigma(s)\tilde{\mathcal{R}}_{00}(\mu,\varrho,\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1})-\varsigma'(s)\hat{C}_p(s+\theta_1)\big]ds,\\
\hat{D}(x) &= -\int_x^\infty e^{is_{01}(x-s)}\big[i\hat{D}\tilde{P}_2+i\varsigma(s)\hat{D}_p(s+\theta_1)(\tilde{P}_2-P_2(\mu,0,0,\mu\hat{C}_p\bar{\hat{C}}_p,\mu\hat{D}_p\bar{\hat{D}}_p))\\
&\quad +\epsilon\hat{\mathcal{R}}_{01}(\mu,R_{\Omega_0s+\theta_0}(\hat{\mathcal{H}}+\hat{Z})+\varsigma(s)\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1})-\epsilon\varsigma(s)\hat{\mathcal{R}}_{01}(\mu,\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1})\\
&\quad +\frac{\varrho}{\sqrt{\mu}}\wp_{01}(\mu,\varrho,R_{\Omega_0s+\theta_0}(\hat{\mathcal{H}}+\hat{Z})+\varsigma(s)\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1})\\
&\quad -\frac{\varrho}{\sqrt{\mu}}\varsigma(s)\wp_{01}(\mu,\varrho,\hat{X}_{\mu,\epsilon,\varrho,\hat{J}_1})-\varsigma'(s)\hat{D}_p(s+\theta_1)\big]ds
\end{aligned}$$

where $\hat{\mathcal{R}}_{01}$ and $\wp_{01}$ are the fourth components of $\hat{\mathcal{R}}$ and $\wp$ respectively.

The following lemma is needed.

Lemma 6.2.1

$$\begin{aligned}
&\int_0^\infty \cos(s_{01}s)\mathrm{sech}(\sqrt{q_1\mu}s)ds=\frac{\pi}{2\sqrt{q_1\mu}\cosh(\frac{\pi s_{01}}{2\sqrt{q_1\mu}})},\\
&\int_0^\infty \cos(s_{01}s)\mathrm{sech}^2(\sqrt{q_1\mu}s)ds=\frac{\pi s_{01}}{2q_1\mu\sinh(\frac{\pi s_{01}}{2\sqrt{q_1\mu}})},\\
&\int_0^\infty \sin(s_{01}s)\mathrm{sech}(\sqrt{q_1\mu}s)\tanh(\sqrt{q_1\mu}s)ds=\frac{\pi s_{01}}{2q_1\mu\cosh(\frac{\pi s_{01}}{2\sqrt{q_1\mu}})}.
\end{aligned}$$

Since the cut-off function $\varsigma(s)$ does not appear in the terms of $\hat{\wp}_{01}$ which contain $\hat{\mathcal{H}}_A(x)$, $\bar{\hat{\mathcal{H}}}_A(x)$, $\hat{\mathcal{H}}_B(x)$, and $\bar{\hat{\mathcal{H}}}_B(x)$, by the fact that $d_{2j}, j = 2, \cdots, 6$ in (4.1.14) are purely imaginary or zero and the expressions of $\hat{\mathcal{H}}_A(x)$ and $\bar{\hat{\mathcal{H}}}_A(x)$, the imaginary part of

$$\int_0^\infty \mathrm{e}^{-is_{01}s} M_2(\hat{\mathcal{H}}_A(s), \bar{\hat{\mathcal{H}}}_A(s))ds,$$

where M_2 is any polynomial of degree 2 in its arguments with purely imaginary coefficients and $M_2(0,0) = 0$, has a leading order term similar to the form in Lemma 6.2.1, which is exponentially small as $\mu \to 0^+$. Similarly, the leading order terms of the imaginary parts in $\int_0^\infty \mathrm{e}^{-is_{01}s}\hat{\mathcal{H}}_B(s)ds$, $\int_0^\infty \mathrm{e}^{-is_{01}s}\bar{\hat{\mathcal{H}}}_B(s)ds$ exponentially approach zero as $\mu \to 0^+$. Using these properties and the fact that $d_{21} = 0$ and d_{27}, d_{28} are real, we deduce from (4.3.31), (4.3.32), (6.2.5), (6.2.6) and the estimate (4.3.27) that

$$\begin{aligned}
0 &= \mathrm{Im}\hat{C}(0) = \mathrm{Im}\Big[-\int_0^\infty \mathrm{e}^{-is_{00}s}\big(-\varsigma'(s)\hat{C}_p(s+\theta_1)\big)ds\Big] + R_1(\theta_1, \varrho, \mu, \epsilon) \\
&= \mathrm{Im}\Big[\int_0^\infty \mathrm{e}^{-is_{00}s}\varsigma'(s)\mathrm{e}^{is_{00}(s+\theta_1)}\hat{J}_1 ds\Big] + R_{11}(\theta_1, \varrho, \mu, \epsilon), \\
0 &= \mathrm{Im}\hat{D}(0) = \mathrm{Im}\Big[-\int_0^\infty \mathrm{e}^{-is_{01}s}\big(-\varsigma'(s)\hat{D}_p(s+\theta_1) - i\frac{P}{r_{01}^2}\frac{\varrho}{\sqrt{\mu}}(1-\varsigma(s))\big)ds \\
&\quad -\int_0^\infty \mathrm{e}^{-is_{01}s}\big(i\varsigma(s)\hat{D}_p(s+\theta_1)p_{22}\mu\hat{\mathcal{H}}_A(s)\bar{\hat{\mathcal{H}}}_A(s)\big)ds\Big] + R_2(\theta_1, \varrho, \mu, \epsilon) \\
&= \mathrm{Im}\Big[\frac{P}{r_{01}^2 s_{01}}\frac{\varrho}{\sqrt{\mu}}\int_0^\infty \mathrm{e}^{-is_{01}s}\big(\varsigma'(s) + is_{01}(1-\varsigma(s)) - \frac{p_{21}}{s_{01}}\mu\varsigma'(s)\big)ds \\
&\quad -i\frac{Pp_{22}}{r_{01}^2 s_{01}}\varrho\sqrt{\mu}\int_0^\infty \mathrm{e}^{-is_{01}s}\varsigma(s)\hat{\mathcal{H}}_A(s)\bar{\hat{\mathcal{H}}}_A(s)ds\Big] + R_{21}(\theta_1, \varrho, \mu, \epsilon),
\end{aligned}$$

where R_1 and R_{11} are of order $O(\frac{\epsilon}{\mu^{3/2}})$, and R_2 and R_{21} are of order $O(\epsilon)$. Moreover, it is easy to check that R_{11} and R_{21} are differentiable with respect to their arguments. Thus, the above equations yield

$$\begin{aligned}
&\int_0^\infty \varsigma'(s)\sin(s_{00}\theta_1)\hat{J}_1 ds + R_{12}(\theta_1, \varrho, \mu, \epsilon) = 0, \\
&\frac{P}{r_{01}^2 s_{01}}\frac{\varrho}{\sqrt{\mu}}\int_0^\infty \big[-\sin(s_{01}s)\varsigma'(s) + \cos(s_{01}s)s_{01}(1-\varsigma(s)) + \frac{p_{21}}{s_{01}}\mu\sin(s_{01}s)\varsigma'(s)\big]ds \\
&\quad + \frac{2q_1 Pp_{22}}{q_2 r_{01}^2 s_{01}}\varrho\sqrt{\mu}\int_0^\infty \cos(s_{01}s)\varsigma(s)\mathrm{sech}^2(\sqrt{q_1\mu}s)ds + R_{22}(\theta_1, \varrho, \mu, \epsilon) = 0,
\end{aligned}$$

where R_{12} is of order $O(\frac{\epsilon}{\mu^{3/2}})$ and R_{22} is of order $O(\epsilon)$.

By the definition of the cut-off function $\varsigma(x)$, it is found that

$$\int_1^2 \varsigma'(s)\sin(s_{00}\theta_1)\hat{J}_1 ds + R_{12}(\theta_1, \varrho, \mu, \epsilon) = 0, \qquad (6.2.7)$$

$$\frac{2q_1Pp_{22}}{q_2r_{01}^2s_{01}}\varrho\sqrt{\mu}\Big[\int_0^\infty \cos(s_{01}s)\text{sech}^2(\sqrt{q_1\mu}s)ds - \int_0^2 \cos(s_{01}s)(1-\varsigma(s))\text{sech}^2(\sqrt{q_1\mu}s)ds\Big] + \frac{Pp_{21}}{r_{01}^2s_{01}^2}\varrho\sqrt{\mu}\int_1^2 \sin(s_{01}s)\varsigma'(s)ds + R_{22}(\theta_1,\varrho,\mu,\epsilon) = 0. \tag{6.2.8}$$

Using $\text{sech}^2(\sqrt{q_1\mu}x) = 1 + O(x^2\mu)$ for small $\mu > 0$, we know that

$$\int_0^2 \cos(s_{01}s)(1-\varsigma(s))\text{sech}^2(\sqrt{q_1\mu}s)ds = \int_0^2 \cos(s_{01}s)(1-\varsigma(s))ds + g_1(\mu)$$
$$= \frac{1}{s_{01}}\int_0^2 \sin(s_{01}s)\varsigma'(s)ds + g_1(\mu) = \frac{1}{s_{01}}\int_1^2 \sin(s_{01}s)\varsigma'(s)ds + g_1(\mu)$$

where g_1 is of order $O(\mu)$. From the above equation, Lemma 6.2.1, $\hat{J}_1 = \frac{\epsilon}{\mu^2}\hat{J}_2$ and $\varrho = \frac{\epsilon}{\mu}\varrho_1$, we see that (6.2.7) and (6.2.8) are equivalent to

$$\hat{J}_2\sin(s_{00}\theta_1) + \sqrt{\mu}R_{13}(\theta_1,\varrho_1,\mu,\epsilon) = 0, \tag{6.2.9}$$
$$\frac{-2q_1p_{22}+q_2p_{21}}{q_2r_{01}^2s_{01}^2}P\int_1^2 \sin(s_{01}s)\varsigma'(s)ds\varrho_1 + \sqrt{\mu}R_{23}(\theta_1,\varrho_1,\mu,\epsilon) = 0, \tag{6.2.10}$$

where $\hat{J}_2 \neq 0$ is a fixed positive constant, R_{13} and R_{23} are uniformly bounded with respect to the arguments under the condition on ϵ. If $2q_1p_{22} - q_2p_{21} \neq 0$, $\varsigma(x)$ can be chosen such that

$$\Big(2q_1p_{22} - q_2p_{21}\Big)\int_1^2 \sin(s_{01}s)\varsigma'(s)ds \neq 0. \tag{6.2.11}$$

Thus, from (6.2.11), (6.2.9) and (6.2.10) can be changed to

$$\theta_1 + \sqrt{\mu}\tilde{R}_C(\theta_1,\varrho_1,\mu,\epsilon) = 0, \tag{6.2.12}$$
$$\varrho_1 + \sqrt{\mu}\tilde{R}_D(\theta_1,\varrho_1,\mu,\epsilon) = 0\,, \tag{6.2.13}$$

where

$$\tilde{R}_C = (s_{00}\sqrt{\mu})^{-1}\arcsin\Big(\hat{J}_2^{-1}\sqrt{\mu}R_{13}(\theta_1,\varrho_1,\mu,\epsilon)\Big)\,,$$
$$\tilde{R}_D = \frac{q_2r_{01}^2s_{01}^2}{(-2q_1p_{22}+q_2p_{21})P\int_1^2 \sin(s_{01}s)\varsigma'(s)ds}R_{23}(\theta_1,\varrho_1,\mu,\epsilon)$$

are bounded and differentiable with respect to their arguments under the condition on ϵ, which yields (4.3.33). The lemma is proved. □

6.3 Some Calculations for (b, F^{-2}) near C_1^+

By Chapter 2 and some straightforward calculations, we obtain that the eigenvector e_{00} of L_s corresponding to 0 and the generalized eigenvector e_{01} are given by

$$e_{00} = (1,0,0,0)^T, \qquad e_{01} = (0,1,-1/\lambda,0)^T .$$

For $j = 0,1$, the eigenvector U_{0j} of L_s corresponding to is_{0j} and the eigenvector V_{0j}^* of L_s^* corresponding to $-is_{0j}$ are given by

$$U_{0j} = \Big(i\frac{\cosh(s_{0j}z)}{s_{0j}\sinh(s_{0j})}, -\frac{\cosh(s_{0j}z)}{\sinh(s_{0j})}, \frac{1}{s_{0j}}, i\Big)^T, \tag{6.3.1}$$

$$V_{0j}^* = \frac{1}{r_{0j}^2}\Big(-i\frac{\cosh(s_{0j}z)}{bs_{0j}\sinh(s_{0j})}, -\frac{\cosh(s_{0j}z)}{b\sinh(s_{0j})}, -\frac{1}{s_{0j}}, i\Big)^T \tag{6.3.2}$$

where

$$r_{0j}^2 = P\Big(\frac{1}{b\sinh^2(s_{0j})} + 1 - \frac{\lambda}{bs_{0j}^2}\Big) > 0. \tag{6.3.3}$$

They satisfy $(U_{0j}, V_{0j}^*) = 1$ and $SU_{0j} = \bar{U}_{0j}$. The eigenvector U_{10} and the generalized eigenvector U_{11} of L_s corresponding to is_1 and the eigenvector V_{11}^* and the generalized eigenvector V_{10}^* of L_s^* corresponding to $-is_1$ are given by

$$U_{10} = \Big(i\frac{\cosh(\sigma_1 z)}{\sigma_1\sinh(\sigma_1)},\ -\frac{s_1\cosh(\sigma_1 z)}{\sigma_1\sinh(\sigma_1)},\ \frac{1}{s_1},\ i\Big)^T\cos(\frac{2\pi y}{P}), \tag{6.3.4}$$

$$\begin{aligned} U_{11} = \Big(& -\frac{s_1(\sinh(\sigma_1)+\sigma_1\cosh(\sigma_1))}{\sigma_1^3\sinh^2(\sigma_1)}\cosh(\sigma_1 z) + \frac{s_1}{\sigma_1^2\sinh(\sigma_1)}z\sinh(\sigma_1 z), \\ & -i\frac{s_1^2(\sinh(\sigma_1)+\sigma_1\cosh(\sigma_1))}{\sigma_1^3\sinh^2(\sigma_1)}\cosh(\sigma_1 z) + i\frac{s_1^2}{\sigma_1^2\sinh(\sigma_1)}z\sinh(\sigma_1 z) \\ & +i\frac{\cosh(\sigma_1 z)}{\sigma_1\sinh(\sigma_1)},\ i\frac{1}{s_1^2},\ 0\Big)^T\cos(\frac{2\pi y}{P}), \end{aligned} \tag{6.3.5}$$

$$V_{11}^* = \frac{1}{r_{10}^2}\Big(-\frac{\cosh(\sigma_1 z)}{b\sigma_1\sinh(\sigma_1)},\ i\frac{s_1\cosh(\sigma_1 z)}{b\sigma_1\sinh(\sigma_1)},\ i\frac{1}{s_1},\ 1\Big)^T\cos(\frac{2\pi y}{P}), \tag{6.3.6}$$

$$\begin{aligned} V_{100}^* = \frac{1}{r_{10}^2}\Big(& i\frac{s_1(\sinh(\sigma_1)+\sigma_1\cosh(\sigma_1))}{b\sigma_1^3\sinh^2(\sigma_1)}\cosh(\sigma_1 z) - i\frac{s_1}{b\sigma_1^2\sinh(\sigma_1)}z\sinh(\sigma_1 z), \\ & \frac{s_1^2(\sinh(\sigma_1)+\sigma_1\cosh(\sigma_1))}{b\sigma_1^3\sinh^2(\sigma_1)}\cosh(\sigma_1 z) - \frac{s_1^2}{b\sigma_1^2\sinh(\sigma_1)}z\sinh(\sigma_1 z) \\ & -\frac{\cosh(\sigma_1 z)}{b\sigma_1\sinh(\sigma_1)},\ \frac{1}{s_1^2},\ 0\Big)^T\cos(\frac{2\pi y}{P}), \end{aligned}$$

$$V_{10}^* = V_{100}^* - \frac{(U_{11}, V_{100}^*)}{(U_{11}, V_{11}^*)}V_{11}^*, \tag{6.3.7}$$

where

$$\begin{aligned}\sigma_1^2 &= s_1^2 + \frac{4\pi^2}{P^2}, \\ r_{10}^2 &= \frac{Ps_1}{4b\sigma_1^5 \sinh^3(\sigma_1)}\big[(4\sigma_1^2 - 3s_1^2)\big(\sinh^2(\sigma_1)\cosh(\sigma_1) + \sigma_1\sinh(\sigma_1)\big) \\ &\quad -2s_1^2\sigma_1^2\cosh(\sigma_1)\big]. \end{aligned} \tag{6.3.8}$$

They satisfy

$$(U_{10}, V_{10}^*) = 1, \qquad (U_{11}, V_{11}^*) = 1, \qquad SU_{10} = \bar{U}_{10}, \qquad SU_{11} = -\bar{U}_{11}.$$

Lemma 6.3.1 *The right side of* (6.3.8) *is positive.*

Proof. Let $\tilde{P} = 4\pi^2 P^{-2}$. Then,

$$\begin{aligned} f(\sigma_1) &:= (4\sigma_1^2 - 3s_1^2)\big(\sinh^2(\sigma_1)\cosh(\sigma_1) + \sigma_1\sinh(\sigma_1)\big) - 2s_1^2\sigma_1^2\cosh(\sigma_1) \\ &= (\sigma_1^2 + 3\tilde{P})\big[\sinh^2(\sigma_1)\cosh(\sigma_1) + \sigma_1\sinh(\sigma_1)\big] - 2\sigma_1^2(\sigma_1^2 - \tilde{P})\cosh(\sigma_1) \\ &= (\sigma_1^2 + 3\tilde{P})\cosh(\sigma_1)\big[g(\sigma_1) + (8\sigma_1^2\tilde{P})(\sigma_1^2 + 3\tilde{P})^{-1}\big], \end{aligned}$$

where $g(\sigma_1) = \sinh^2(\sigma_1) + \sigma_1\tanh(\sigma_1) - 2\sigma_1^2$. It is straightforward to show that

$$\sinh^2(\sigma_1) > \sigma_1^2 + \frac{1}{3}\sigma_1^4, \quad \tanh(\sigma_1) > \sigma_1 - \frac{1}{3}\sigma_1^3$$

for $\sigma_1 > 0$. Thus, $g(\sigma_1) > \sigma_1^2 + \frac{1}{3}\sigma_1^4 + \sigma_1^2 - \frac{1}{3}\sigma_1^4 - 2\sigma_1^2 = 0$, which gives $f(\sigma_1) > 0$ or the right side of (6.3.8) is positive. □

Let $\mu = 0$ and $\varrho = 0$. From the system (2.2.8), we obtain

$$\dot{v} = L_s v + N_2(v,v) + N_3(v,v,v) + h.o.t. \tag{6.3.9}$$

where N_2 denotes the sum of all quadratic terms, N_3 denotes the sum of all cubic terms, and $h.o.t.$ denotes the higher order terms. Assume

$$\begin{aligned}&\Phi(0,0,A,B,C,D,\bar{A},\bar{B},\bar{C},\bar{D}) = \Phi_{20000000}A^2 + \Phi_{10001000}A\bar{A} \\ &\quad +\Phi_{10010000}AD + \Phi_{00011000}\bar{A}D + \Phi_{20001000}A|A|^2 + \Phi_{10011000}A\bar{A}D + \cdots .\end{aligned}$$

Under these assumptions, (4.1.1) becomes

$$\begin{aligned} v &= AU_{10} + BU_{11} + CU_{00} + DU_{01} + \bar{A}\bar{U}_{10} + \bar{B}\bar{U}_{11} + \bar{C}\bar{U}_{00} + \bar{D}\bar{U}_{01} \\ &\quad +\Phi(0,0,A,B,C,D,\bar{A},\bar{B},\bar{C},\bar{D}). \end{aligned} \tag{6.3.10}$$

Substituting (6.3.10) into (6.3.9) yields

$$\dot{v} = \dot{A}U_{10} + \dot{B}U_{11} + \dot{C}U_{00} + \dot{D}U_{01} + \cdots + 2A\dot{A}\Phi_{20000000}$$

$$
\begin{aligned}
&+(\dot{A}\bar{A}+A\dot{\bar{A}})\Phi_{10001000}+\Phi_{10010000}(\dot{A}D+A\dot{D})+\Phi_{00011000}(\dot{\bar{A}}D+\bar{A}\dot{D})\\
&+(2\dot{A}A\bar{A}+A^2\dot{\bar{A}})\Phi_{20001000}+\Phi_{10011000}(\dot{A}\bar{A}D+A\dot{\bar{A}}D+A\bar{A}\dot{D})+\cdots\\
= \;& L_s v+N_2(v,v)+N_3(v,v,v)+h.o.t..
\end{aligned}
$$

Comparing the coefficients for $\mu=0$, one finds that

$$
\begin{aligned}
A:&\ L_sU_{10}=is_1U_{10},\qquad B:\ L_sU_{11}=is_1U_{11}+U_{10},\\
A^2:&\ L_s\Phi_{20000000}+N_2(U_{10},U_{10})=2is_1\Phi_{20000000},\\
A\bar{A}:&\ L_s\Phi_{10001000}+2N_2(U_{10},\bar{U}_{10})=0,\\
AD:&\ L_s\Phi_{10010000}+2N_2(U_{10},U_{01})=(is_1+is_{01})\Phi_{10010000},\\
A\bar{D}:&\ L_s\Phi_{10000001}+2N_2(U_{10},\bar{U}_{01})=(is_1-is_{01})\Phi_{10000001},\\
A^2\bar{A}:&\ L_s\Phi_{20001000}+2N_2(U_{10},\Phi_{10001000})+2N_2(\bar{U}_{10},\Phi_{20000000})\\
&+3N_3(U_{10},U_{10},\bar{U}_{10})=is_1\Phi_{20001000}+ip_2U_{10}+q_2U_{11},\\
A\bar{A}D:&\ L_s\Phi_{10011000}+2N_2(U_{10},\Phi_{00011000})+2N_2(\bar{U}_{10},\Phi_{10010000})\\
&+2N_2(U_{01},\Phi_{10001000})+6N_3(U_{10},\bar{U}_{10},U_{01})=is_{01}\Phi_{10011000}+ip_{22}U_{01}.
\end{aligned}
$$

Thus, the coefficients q_2 and p_{22} in (4.1.12) are

$$
\begin{aligned}
q_2 &= \big((L_s-is_1)\Phi_{20001000}+2N_2(U_{10},\Phi_{10001000})+2N_2(\bar{U}_{10},\Phi_{20000000})\\
&\quad +3N_3(U_{10},U_{10},\bar{U}_{10}),V_{11}^*\big),\\
p_{22} &= -i\big((L_s-is_{01})\Phi_{10011000}+2N_2(U_{10},\Phi_{00011000})+2N_2(\bar{U}_{10},\Phi_{10010000})\\
&\quad +2N_2(U_{01},\Phi_{10001000})+6N_3(U_{10},\bar{U}_{10},U_{01}),V_{01}^*\big).
\end{aligned}
$$

Using the definition of the inner product in (3.2.6) and integration by parts, we have

$$
\begin{aligned}
&\big((L_s-is_1)\Phi_{20001000},V_{11}^*\big)=-\int_0^P\big[2H_2(U_{10},\Phi_{10001000})+2H_2(\bar{U}_{10},\Phi_{20000000})\\
&\quad +3H_3(U_{10},U_{10},\bar{U}_{10})\big]\bar{V}_{11}^*[2]\big|_{z=1}dy,\\
&\big((L_s-is_{01})\Phi_{10011000},V_{01}^*\big)=-\int_0^P\big[2H_2(U_{10},\Phi_{00011000})+2H_2(\bar{U}_{10},\Phi_{10010000})\\
&\quad +2H_2(U_{01},\Phi_{10001000})+6H_3(U_{10},\bar{U}_{10},U_{01})\big]\bar{V}_{01}^*[2]\big|_{z=1}dy
\end{aligned}
$$

where we use the fact that $L_s^*V_{11}^*=-is_1V_{11}^*$, $L_s^*V_{01}^*=-is_{01}V_{01}^*$, and $\Phi_{20001000}$ satisfies boundary conditions

$$
\begin{aligned}
&(\Phi_{20001000}[1])_z=0\qquad \text{on } z=0,\\
&(\Phi_{20001000}[1])_z=\Phi_{20001000}[4]+2H_2(U_{10},\Phi_{10001000})+2H_2(\bar{U}_{10},\Phi_{20000000})\\
&\quad +3H_3(U_{10},U_{10},\bar{U}_{10})\qquad \text{on } z=1
\end{aligned}
$$

and $\Phi_{10011000}$ satisfies the boundary conditions

$$
(\Phi_{10011000}[1])_z=0\qquad \text{on } z=0,
$$

$$(\Phi_{10011000}[1])_z = \Phi_{10011000}[4] + 2H_2(U_{10}, \Phi_{00011000}) + 2H_2(\bar{U}_{10}, \Phi_{10010000})$$
$$+2H_2(U_{01}, \Phi_{10001000}) + 6H_3(U_{10}, \bar{U}_{10}, U_{01}) \qquad \text{on } z = 1.$$

Here, $[j]$ denotes the j-th component and $H_k = \frac{1}{k!}\mathrm{d}^k H[0,0,0,0]$, where d is the Fréchet derivative. Thus,

$$\begin{aligned} q_2 &= \big(2N_2(U_{10}, \Phi_{10001000}) + 2N_2(\bar{U}_{10}, \Phi_{20000000}) + 3N_3(U_{10}, U_{10}, \bar{U}_{10}), V_{11}^*\big) \\ &\quad - \int_0^P \big[2H_2(U_{10}, \Phi_{10001000}) + 2H_2(\bar{U}_{10}, \Phi_{20000000}) \\ &\qquad +3H_3(U_{10}, U_{10}, \bar{U}_{10})\big]\bar{V}_{11}^*[2]\big|_{z=1} dy, && (6.3.11) \\ p_{22} &= -i\big(2N_2(U_{10}, \Phi_{00011000}) + 2N_2(\bar{U}_{10}, \Phi_{10010000}) \\ &\quad +2N_2(U_{01}, \Phi_{10001000}) + 6N_3(U_{10}, \bar{U}_{10}, U_{01}), V_{01}^*\big) \\ &\quad +i\int_0^P \big[2H_2(U_{10}, \Phi_{00011000}) + 2H_2(\bar{U}_{10}, \Phi_{10010000}) \\ &\qquad +2H_2(U_{01}, \Phi_{10001000}) + 6H_3(U_{10}, \bar{U}_{10}, U_{01})\big]\bar{V}_{01}^*[2]\big|_{z=1} dy. && (6.3.12) \end{aligned}$$

We can find q_2 and p_{22} by solving several differential equations. For example, in order to computer q_2, we need $\Phi_{10001000}$. By expanding the right side of (2.2.9) at 0, substituting (6.3.10) into (2.2.9) and comparing the coefficient of $A\bar{A}$, we obtain

$$L_s\Phi_{10001000} + 2N_2(U_{10}, \bar{U}_{10}) = 0$$

with boundary conditions

$$(\Phi_{10001000}[1])_z = 0 \qquad \text{on } z = 0,$$
$$(\Phi_{10001000}[1])_z = \Phi_{10001000}[4] + 2H_2(U_{10}, \bar{U}_{10}) \qquad \text{on } z = 1,$$

which can be easily solved for $\Phi_{10001000}$. Using the same method, we can solve for $\Phi_{20000000}$. If $P = 2\pi$ and write q_2 as a function in terms of s_1, we can use the numerical calculation to show

$$q_2 < 0. \tag{6.3.13}$$

For p_{22}, the computation is more complicated, and if s_1 is given, we can also find the values of p_{22} numerically.

We note that $L_{\lambda+\mu}$ can be written as two parts $L_{\lambda+\mu}U = L_\mu U + L_\lambda U$, where $L_\mu U = (0,0,0,\frac{\mu}{b}\eta)^T$. By using the method discussed above and comparing the coefficient of μA, we have

$$\begin{aligned} q_1 &= \big((0,0,0,\frac{1}{b}U_{10}[3])^T, V_{11}^*\big) = \big((0,0,0,\frac{1}{bs_1}\cos(\frac{2\pi y}{P}))^T, V_{11}^*\big) \\ &= \frac{1}{r_{10}^2}\frac{1}{bs_1}\int_0^P \cos^2(\frac{2\pi y}{P})dy = \frac{P}{2bs_1r_{10}^2} > 0. && (6.3.14) \end{aligned}$$

Similarly,

$$p_{21} = \frac{1}{i}\big((0,0,0,\frac{1}{b}U_{01}[3])^T, V_{01}^*\big) = -\frac{P}{bs_{01}r_{01}^2}. \tag{6.3.15}$$

We note that Φ contains terms which are at least quadratic in its arguments so that the leading coefficients of ϱ in (6.2.1) and (6.2.2) are obtained by

$$\big((0,0,0,1)^T, V_{00}^*\big) = -iPr_{00}^{-2}, \qquad d_{20} = \big((0,0,0,1)^T, V_{01}^*\big) = -iPr_{01}^{-2}. \tag{6.3.16}$$

Moreover, we can choose an appropriate Φ in (4.1.1) such that d_{21}, which is the coefficient of $\frac{\varrho}{\sqrt{\mu}}\mu$ in $\frac{\varrho}{\sqrt{\mu}}\hat{\wp}$, equals zero, i.e.,

$$d_{21} = 0. \tag{6.3.17}$$

Indeed, let the coefficient of $\mu\varrho$ in Φ be $\Phi_{\mu\varrho}$ in (6.3.10) and let the coefficients of $\mu\varrho$ in the equations $\dot{A}, \dot{B}, \dot{C}, \dot{D}$ in (4.1.13) be d_A, d_B, d_C, d_{21}, respectively. Assume that

$$\begin{aligned} v &= AU_{10} + BU_{11} + CU_{00} + DU_{01} + \bar{A}\bar{U}_{10} + \bar{B}\bar{U}_{11} + \bar{C}\bar{U}_{00} + \bar{D}\bar{U}_{01} \\ &\quad + \cdots + \mu A\Phi_{\mu A} + \mu B\Phi_{\mu B} + \mu C\Phi_{\mu C} + \mu D\Phi_{\mu D} \\ &\quad + \mu\bar{A}\Phi_{\mu\bar{A}} + \mu\bar{B}\Phi_{\mu\bar{B}} + \mu\bar{C}\Phi_{\mu\bar{C}} + \mu\bar{D}\Phi_{\mu\bar{D}} + \mu\varrho\Phi_{\mu\varrho} + \cdots . \end{aligned}$$

After substituting v into (6.3.9), the equation for the coefficients with respect to $\mu\varrho$ is

$$\begin{aligned} & d_A U_{10} + d_B U_{11} + d_C U_{00} + d_{21}U_{01} + \bar{d}_A\bar{U}_{10} + \bar{d}_B\bar{U}_{11} + \bar{d}_C\bar{U}_{00} + \bar{d}_{21}\bar{U}_{01} \\ & + \frac{-iP}{r_{00}^2}\Phi_{\mu C} + \frac{-iP}{r_{01}^2}\Phi_{\mu D} + \frac{iP}{r_{00}^2}\Phi_{\mu\bar{C}} + \frac{iP}{r_{01}^2}\Phi_{\mu\bar{D}} = L_s\Phi_{\mu\varrho} \end{aligned} \tag{6.3.18}$$

together with the boundary conditions $(\Phi_{\mu\varrho}[1])_z|_{z=0} = 0, (\Phi_{\mu\varrho}[1])_z|_{z=1} = \Phi_{\mu\varrho}[4]$. By solving the equations of the coefficients of $\mu C, \mu D, \mu\bar{C}$ and $\mu\bar{D}$ for $\Phi_{\mu C}$, $\Phi_{\mu D}$, $\Phi_{\mu\bar{C}}$ and $\Phi_{\mu\bar{D}}$, respectively, it can be shown that the solvability condition for the nonhomogeneous boundary value problem (6.3.18) is automatically satisfied if $d_A = d_B = d_C = d_{21} = 0$. Therefore, there is at least one $\Phi_{\mu\varrho}$ of (6.3.18) such that the terms with $\mu\varrho$ in the equations $\dot{A}, \dot{B}, \dot{C}, \dot{D}$ in (4.1.13) vanish.

6.4 Proof of Lemma 5.1.1

The proof is similar to one of Lemma 4.1.1. From the general theory of differential operators, it follows that $H(X) = Q(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ since $\{u_i\}_{i=1}^7$ are independent. By (5.1.8), we obtain the following

$$A_{01} = u_1, \qquad A_{02} = u_1u_5 + u_7 - \frac{u_1}{is_{10}}(\frac{1}{2}\ln u_2 + i\alpha_A),$$

$$A = u_2^{\frac{1}{2}} \mathrm{e}^{i\alpha_A}, \quad B = u_5 u_2^{\frac{1}{2}} \mathrm{e}^{i\alpha_A} - \frac{u_2^{\frac{1}{2}} \mathrm{e}^{i\alpha_A}}{is_{10}} (\frac{1}{2} \ln u_2 + i\alpha_A),$$
$$C = u_4^{\frac{1}{2}} \mathrm{e}^{i\alpha_C}, \quad \alpha_C = s_{20}[u_6 - u_5 + \frac{1}{is_{10}} (\frac{1}{2} \ln u_2 + i\alpha_A)] + \frac{i}{2} \ln u_4 - 2k_1\pi,$$

and their complex conjugates, where $\alpha_A, \alpha_C \in [0, 2\pi)$ and the integer k_1 is chosen so that $\alpha_C \in [0, 2\pi)$.

Since $H(X) = Q(u_j, j = 1, 2, \ldots 7)$ where $u_j, j = 1, \ldots 7$ are independent of α_A, Q is independent of α_A, which implies that $H = Q$ is a constant when α_A changes.

The polynomial $H(X)$ reads

$$H(X) = \sum_{j_1 + \cdots + j_8 = n} H_{j_1 \cdots j_8} A_{01}^{j_1} A_{02}^{j_2} A^{j_3} B^{j_4} C^{j_5} \bar{A}^{j_6} \bar{B}^{j_7} \bar{C}^{j_8}$$

where $j_1, \cdots, j_8$ are nonnegative integers. Thus, by the relations between X and $u_j, j = 1, 2, \ldots, 7$, we can write $H(X)$ in terms of $u_j, j = 1, 2, \ldots, 7$ as follows,

$$\begin{aligned}
H(X) = & \sum_{j_1 + \cdots + j_8 = n} H_{j_1 \cdots j_8} u_1^{j_1} (u_1 u_5 + u_7 - \frac{u_1}{is_{10}} (\frac{1}{2} \ln u_2 + i\alpha_A))^{j_2} u_2^{\frac{1}{2}(j_3 + j_6)} \\
& \cdot \mathrm{e}^{i(j_3 - j_6)\alpha_A} \Big[u_5 u_2^{\frac{1}{2}} \mathrm{e}^{i\alpha_A} - \frac{u_2^{\frac{1}{2}} \mathrm{e}^{i\alpha_A}}{is_{10}} \Big(\frac{1}{2} \ln u_2 + i\alpha_A \Big) \Big]^{j_4} \\
& \cdot u_4^{\frac{1}{2}(j_5 + j_8)} \mathrm{e}^{i(j_5 - j_8)\left[s_{20} \left(u_6 - u_5 + \frac{1}{is_{10}} (\frac{1}{2} \ln u_2 + i\alpha_A) \right) + \frac{1}{2} i \ln u_4 \right]} \\
& \cdot \Big[\bar{u}_5 u_2^{\frac{1}{2}} \mathrm{e}^{-i\alpha_A} - \frac{u_2^{\frac{1}{2}} \mathrm{e}^{-i\alpha_A}}{-is_{10}} \Big(\frac{1}{2} \ln u_2 - i\alpha_A \Big) \Big]^{j_7} \\
= & \sum_{j_1 + \cdots + j_8 = n} H_{j_1 \cdots j_8} u_1^{j_1} u_2^{\frac{1}{2}(j_3 + j_4 + j_6 + j_7 + \frac{s_{20}}{s_{10}}(j_5 - j_8))} \\
& \cdot \sum_{k_0}^{j_2} C_{j_2}^{k_0} (u_1 u_5 + u_7 - \frac{u_1 \ln u_2}{i2s_{10}})^{k_0} (-\frac{u_1}{s_{10}})^{j_2 - k_0} \alpha_A^{j_2 - k_0} \\
& \cdot \sum_{k_1}^{j_4} C_{j_4}^{k_1} [u_5 - \frac{1}{i2s_{10}} \ln u_2]^{k_1} \Big(- \frac{1}{s_{10}} \Big)^{j_4 - k_1} {\alpha_A}^{j_4 - k_1} u_4^{j_8} \mathrm{e}^{is_{20}(j_5 - j_8)(u_6 - u_5)} \\
& \cdot \sum_{k_2}^{j_7} C_{j_7}^{k_2} \Big[\bar{u}_5 - \frac{1}{-i2s_{10}} \ln u_2 \Big]^{k_2} \Big(- \frac{1}{s_1} \Big)^{j_7 - k_2} {\alpha_A}^{j_7 - k_2} \\
& \cdot \mathrm{e}^{i\left[(j_3 - j_6) + (j_4 - j_7) + \frac{(j_5 - j_8)s_{20}}{s_{10}} \right]\alpha_A} \\
= & \sum_{j=0}^{\beta} H_j(\alpha_A) \alpha_A^j
\end{aligned} \tag{6.4.1}$$

for all $\alpha_A \in [0, 2\pi)$. Here $H_j(\alpha_A)$ is the finite sum of the form $\mathrm{e}^{ir\alpha_A}$ and r is real.

Using Lemma 6.1.2, we obtain that $H_0(\alpha_A) \equiv H(X)$. Note that $H_0(\alpha_A)$ is the sum of the form

$$e^{i[(j_3-j_6)+(j_4-j_7)+\frac{(j_5-j_8)s_{20}}{s_{10}}]\alpha_A}.$$

Lemma 6.1.1 implies that from the assumption on the relation of s_{10}, s_{20}, $H_0(\alpha_A)$ is a constant for any $\alpha_A \in [0, 2\pi)$ if and only if $(j_3 - j_6) + (j_4 - j_7) = 0$ and $j_5 - j_8 = 0$. Therefore, Q must be a polynomial in u_4 (since $C, \bar{C}$ appear pairly in H) and independent of u_6 (since the coefficients of the form e^{irx} in (6.4.1) do not contain u_6 by using the fact that $j_5 = j_8$).

By the relations between the partial derivatives of H and Q, we find

$$\frac{\partial H}{\partial A_{01}} = \frac{\partial Q}{\partial u_1} - \frac{B}{A}\frac{\partial Q}{\partial u_7}, \tag{6.4.2}$$

$$\frac{\partial H}{\partial A_{02}} = \frac{\partial Q}{\partial u_7}, \tag{6.4.3}$$

$$\frac{\partial H}{\partial \bar{A}} = A\frac{\partial Q}{\partial u_2} - \frac{i}{2}B\frac{\partial Q}{\partial u_3}, \tag{6.4.4}$$

$$\frac{\partial H}{\partial B} = -\frac{i}{2}\bar{A}\frac{\partial Q}{\partial u_3} + \frac{1}{A}\frac{\partial Q}{\partial u_5} - \frac{A_{01}}{A}\frac{\partial Q}{\partial u_7}, \tag{6.4.5}$$

$$\frac{\partial H}{\partial \bar{B}} = \frac{i}{2}A\frac{\partial Q}{\partial u_3}. \tag{6.4.6}$$

From (6.4.6), since H is a polynomial in $\bar{B}$, we can keep taking the derivatives with respect to $\bar{B}$ to make the left side of (6.4.6) equal to zero, which implies $\partial^m Q/\partial u_3^m = 0$ for some $m > 0$ and Q is a polynomial in u_3. By (6.4.4) and (6.4.6), we obtain

$$\frac{\partial Q}{\partial u_2} = \frac{1}{A}\Big(\frac{\partial}{\partial \bar{A}} + \frac{B}{A}\frac{\partial}{\partial \bar{B}}\Big)H.$$

Again, same argument gives that Q is a polynomial in u_2. Similarly, (6.4.2), (6.4.3) and (6.4.5) yield that Q is a polynomial in u_1, u_5, u_7.

Since H is a polynomial in A, we obtain that Q is independent of u_5 because otherwise H, which has a form $|A|^k$, cannot be equal to Q that has a form $|A|^l|\ln A|^m$ as $|A| \to \infty$, where k, l, m are integers. Hence, Q is a polynomial in u_1, u_2, u_3, u_4, u_7 and is independent of u_5, u_6. From the expressions of u_1, u_2, u_3, u_4 and u_7, Q has to be a polynomial in $u_1, u_2, u_3, u_4, u_2u_7$. The proof of this lemma is completed. □

6.5 Proof of Lemma 5.6.3

In the following, we divide the proof into two steps. First compute the expressions of $\tilde{A}_{01}, \tilde{A}_{02}, \tilde{A}, \tilde{B}$ and change (5.6.26) to the first equation of (5.6.30). Then compute the expressions of $\tilde{C}^p$ and $\tilde{C}$ and change (5.6.29) to the second equation (5.6.30).

Let

$$
\begin{aligned}
X_3 &= \Big(\mu(\tilde{A}^p_{01}+d+\tilde{d}_{01}),0,0,\mu^3(\tilde{C}^p+\tilde{d}_{20})(\bar{\tilde{C}}^p+\bar{\tilde{d}}_{20}),0\Big)\,,\\
X_4 &= \Big(\mu(d+\tilde{d}_{01}),0,0,\mu^3\tilde{d}_{20}\bar{\tilde{d}}_{20},0\Big)\,,\\
X_5 &= \Big(\mu(\tilde{A}^p_{01}+d+\tilde{d}_{01}),\mu^{3/2}(\tilde{A}^p_{02}+\tilde{d}_{02}),0,0,\mu^{3/2}(\tilde{C}^p+\tilde{d}_{20}),\\
&\qquad 0,0,\mu^{3/2}(\bar{\tilde{C}}^p+\bar{\tilde{d}}_{20})\Big),\\
X_6 &= \Big(\mu(d+\tilde{d}_{01}),\mu^{3/2}\tilde{d}_{02},0,0,\mu^{3/2}\tilde{d}_{20},0,0,\mu^{3/2}\bar{\tilde{d}}_{20}\Big)\,,\\
\hat{X}_3 &= \Big(\mu(H_{01}+\tilde{A}_{01}+\varsigma\tilde{A}^p_{01}+d+\tilde{d}_{01}),\mu^2(H_{10}+\tilde{A})(H_{10}+\bar{\tilde{A}}),\\
&\qquad \mu^{5/2}\frac{i}{2}((H_{10}+\tilde{A})(H_{11}+\bar{\tilde{B}})-(H_{10}+\bar{\tilde{A}})(H_{11}+\tilde{B})),\\
&\qquad \mu^3(\tilde{C}+\varsigma\tilde{C}^p+\tilde{d}_{20})(\bar{\tilde{C}}+\varsigma\bar{\tilde{C}}^p+\bar{\tilde{d}}_{20}),\\
&\qquad \mu^{7/2}((H_{10}+\tilde{A})(H_{10}+\bar{\tilde{A}})(H_{02}+\tilde{A}_{02}+\varsigma\tilde{A}^p_{02}+\tilde{d}_{02})\\
&\qquad -(H_{01}+\tilde{A}_{01}+\varsigma\tilde{A}^p_{01}+d+\tilde{d}_{01})(H_{10}+\bar{\tilde{A}})(H_{11}+\tilde{B}))\Big),\\
\tilde{X}_3 &= \Big(\mu(H_{01}+\tilde{A}_{01}+\varsigma\tilde{A}^p_{01}+d+\tilde{d}_{01}),\mu^2(H_{10}+\tilde{A})(H_{10}+\bar{\tilde{A}}),\\
&\qquad \mu^{5/2}\frac{i}{2}((H_{10}+\tilde{A})(H_{11}+\bar{\tilde{B}})-(H_{10}+\bar{\tilde{A}})(H_{11}+\tilde{B})),\\
&\qquad \mu^3(\tilde{C}+\varsigma\tilde{C}^p+\tilde{d}_{20})(\bar{\tilde{C}}+\varsigma\bar{\tilde{C}}^p+\bar{\tilde{d}}_{20}),\\
&\qquad \mu^{3/2}(H_{02}+\tilde{A}_{02}+\varsigma\tilde{A}^p_{02}+\tilde{d}_{02}\\
&\qquad -(H_{01}+\tilde{A}_{01}+\varsigma\tilde{A}^p_{01}+d+\tilde{d}_{01})(H_{11}+\bar{\tilde{B}})/(H_{10}+\tilde{A})\Big),\\
\hat{X}_5 &= \Big(\mu(H_{01}+\tilde{A}_{01}+\varsigma\tilde{A}^p_{01}+d+\tilde{d}_{01}),\mu^{3/2}(H_{02}+\tilde{A}_{02}+\varsigma\tilde{A}^p_{02}+\tilde{d}_{02}),\\
&\qquad \mu e^{i(\mu^{-1/2}s_{20}x+\theta_0)}(H_{10}+\tilde{A}),\mu^{3/2}e^{i(\mu^{-1/2}s_{20}x+\theta_0)}(H_{11}+\tilde{B})\\
&\qquad \mu^{3/2}(\tilde{C}+\varsigma\tilde{C}^p+\tilde{d}_{20}),\mu e^{-i(\mu^{-1/2}s_{20}x+\theta_0)}(H_{10}+\bar{\tilde{A}}),\\
&\qquad \mu^{3/2}e^{-i(\mu^{-1/2}s_{20}x+\theta_0)}(H_{11}+\bar{\tilde{B}}),\mu^{3/2}(\bar{\tilde{C}}+\varsigma\bar{\tilde{C}}^p+\bar{\tilde{d}}_{20})\Big). \qquad (6.5.1)
\end{aligned}
$$

Step 1: The first equation of (5.6.30)

From (5.6.4), the relation between (5.2.3) and (5.3.6), and the fact that $\mathcal{H}(x)$ is a solution of (5.4.1), the equations for $\tilde{A},\tilde{B}$ are given by

$$
\begin{aligned}
\dot{\tilde{A}} &= -\dot{H}_{10}+H_{11}+\tilde{B}+\frac{1}{\mu^{1/2}}(H_{10}+\tilde{A})P_{10}(\mu,\tilde{X}_3)\\
&\qquad +\frac{e^{-i(s_{10}\mu^{-1/2}x+\theta_0)}}{\mu^{3/2}}[\mathcal{R}_{10}(\mu,\hat{X}_5)+\varrho\wp_{10}(\mu,\varrho,\hat{X}_5)]
\end{aligned}
$$

$$
\begin{aligned}
&= \tilde{B} + \frac{1}{\mu^{1/2}} H_{10}\Big(\big(p_{10}^{(1)} + p_{10}^{(2)} k_2 + p_{10}^{(3)} k_3 + p_{10}^{(4)} (H_{01} + \tilde{A}_{01} + \varsigma \tilde{A}_{01}^p + d + \tilde{d}_{01})\big)\mu \\
&\quad + P_{10}^*(\mu, \tilde{X}_3)\Big) + \frac{1}{\mu^{1/2}} \tilde{A} P_{10}(\mu, \tilde{X}_3) \\
&\quad + \frac{e^{-i(s_{10}\mu^{-1/2}x+\theta_0)}}{\mu^{3/2}} [\mathcal{R}_{10}(\mu, \hat{X}_5) + \varrho \wp_{10}(\mu, \varrho, \hat{X}_5)], \qquad (6.5.2) \\
\dot{\tilde{B}} &= -\dot{H}_{11} + \frac{1}{\mu^{1/2}} (H_{11} + \tilde{B}) P_{10}(\mu, \tilde{X}_3) + \frac{1}{\mu} (H_{10} + \tilde{A}) P_{11}(\mu, \tilde{X}_3) \\
&\quad + \frac{e^{-i(s_{10}\mu^{-1/2}x+\theta_0)}}{\mu^2} [\mathcal{R}_{11}(\mu, \hat{X}_5) + \varrho \wp_{11}(\mu, \varrho, \hat{X}_5)] \\
&= -\dot{H}_{11} + \frac{1}{\mu^{1/2}} (H_{11} + \tilde{B}) P_{10}(\mu, \tilde{X}_3) + \frac{1}{\mu} (H_{10} + \tilde{A}) \Big(\big(p_{11}^{(1)} + p_{11}^{(2)} k_2 + p_{11}^{(3)} k_3 \\
&\quad + p_{11}^{(4)} (H_{01} + \tilde{A}_{01} + \varsigma \tilde{A}_{01}^p + d + \tilde{d}_{01})\big)\mu + P_{11}^*(\mu, \tilde{X}_3)\Big) \\
&\quad + \frac{e^{-i(s_{10}\mu^{-1/2}x+\theta_0)}}{\mu^2} [\mathcal{R}_{11}(\mu, \hat{X}_5) + \varrho \wp_{11}(\mu, \varrho, \hat{X}_5)] \\
&= \frac{1}{\mu^{1/2}} (H_{11} + \tilde{B}) \Big(\big(p_{10}^{(1)} + p_{10}^{(2)} k_2 + p_{10}^{(3)} k_3 + p_{10}^{(4)} (H_{01} + \tilde{A}_{01} + \varsigma \tilde{A}_{01}^p + d + \tilde{d}_{01})\big)\mu \\
&\quad + P_{10}^*(\mu, \tilde{X}_3)\Big) + \frac{1}{\mu} \tilde{A} \Big(\big(p_{11}^{(1)} + p_{11}^{(2)} k_2 + p_{11}^{(3)} k_3 \\
&\quad + p_{11}^{(4)} (H_{01} + \tilde{A}_{01} + \varsigma \tilde{A}_{01}^p + d + \tilde{d}_{01})\big)\mu + P_{11}^*(\mu, \tilde{X}_3)\Big) \\
&\quad + \frac{1}{\mu} H_{10} \left(p_{11}^{(4)} (\tilde{A}_{01} + \varsigma \tilde{A}_{01}^p + \tilde{d}_{01})\mu + P_{11}^*(\mu, \hat{X}_3)\right) \\
&\quad + \frac{e^{-i(s_{10}\mu^{-1/2}x+\theta_0)}}{\mu^2} [\mathcal{R}_{11}(\mu, \hat{X}_5) + \varrho \wp_{11}(\mu, \varrho, \hat{X}_5)] \qquad (6.5.3)
\end{aligned}
$$

where we use (5.1.15), (5.2.2) and the fact that $\mathcal{R}_{10} + \varrho\wp_{10} = \mathcal{R}_{11} + \varrho\wp_{11} = 0$ for $X = \underline{\tilde{d}}$ since $\tilde{A}, \tilde{B}$-components of $\underline{\tilde{d}}$ are zero.

(5.1.16) shows that P_{10}^*, P_{11}^* have a factor μ^2 while (5.3.5) with the assumption $\varrho = \mu^{21/8}\varrho_1$ from Theorem 5.6.1 yield that $\tilde{d}_{01}$ is of order $O(\mu^{5/8})$, which, together with (5.2.7), (5.5.20) and (5.6.24), change (6.5.2) and (6.5.3) to

$$
\begin{aligned}
\dot{\tilde{A}} &= \tilde{B} + \sqrt{\mu} H_{10}(a_1 + p_{10}^{(4)} H_{01}) + f_A, \qquad (6.5.4) \\
\dot{\tilde{B}} &= p_{11}^{(4)} H_{10} \tilde{A}_{01} + \tilde{A}(c^2 + p_{11}^{(4)} H_{01}) + \sqrt{\mu} H_{11}(a_1 + p_{10}^{(4)} H_{01}) + p_{11}^{(4)} \tilde{d}_{01} H_{10} + f_B \qquad (6.5.5)
\end{aligned}
$$

where f_A, f_B are of order at least $O(\mu^{7/8})$ and $a_1 = p_{10}^{(1)} + p_{10}^{(2)} k_2 + p_{10}^{(3)} k_3 + p_{10}^{(4)} d$ is purely imaginary (see the sentence below (5.1.16)). Here we use the definition of c in (5.4.2).

From the sentence above (5.1.18), we know that $P_{02}^* = \frac{1}{A_{01}}[A_{02}P_{01} + P_{02}]$. Then the equation for $\tilde{A}_{02}$ satisfies

$$\begin{aligned}\dot{\tilde{A}}_{02} &= -\dot{H}_{02}(x) - \varsigma(x)\frac{1}{\mu^2}(P_{02}^*(\mu, X_3) - P_{02}^*(\mu, X_4)) + \frac{1}{\mu^2}(P_{02}^*(\mu, \hat{X}_3) - P_{02}^*(\mu, X_4)) \\ &\quad -\varsigma(x)\frac{1}{\mu^2}[\mathcal{R}_{02}(\mu, X_5) - \mathcal{R}_{02}(\mu, \varrho, X_6) + \varrho(\wp_{02}(\mu, \varrho, X_5) - \wp_{02}(\mu, \varrho, X_6))] \\ &\quad +\frac{1}{\mu^2}[\mathcal{R}_{02}(\mu, \hat{X}_5) - \mathcal{R}_{02}(\mu, \varrho, X_6) + \varrho(\wp_{02}(\mu, \varrho, \hat{X}_5) - \wp_{02}(\mu, \varrho, X_6))] \\ &\quad -\varsigma'(x)\tilde{A}_{02}^p(x + \theta_1). \end{aligned} \tag{6.5.6}$$

Because of the assumption $\tilde{I}_1 = \mu^{5/4}\tilde{I}_2$ in Theorem 5.6.1, (5.5.20) gives that $\tilde{X}_{\mu,\varrho,\tilde{I}_1}$ is of order $O(\mu^{5/4})$. From (5.1.18), we know that

$$\begin{aligned}&\frac{1}{\mu^2}(P_{02}^*(\mu, X_3) - P_{02}^*(\mu, X_4)) = p_{02}^{(2)}k_2(\tilde{A}_{01}^p + d + \tilde{d}_{01}) + p_{02}^{(4)}(\tilde{A}_{01}^p + d + \tilde{d}_{01})^2 \\ &\qquad - \left(p_{02}^{(2)}k_2(d + \tilde{d}_{01}) + p_{02}^{(4)}(\tilde{d} + \tilde{d}_{01})^2\right) + O(\mu) = O(\mu), \\ &\frac{1}{\mu^2}(P_{02}^*(\mu, \hat{X}_3) - P_{02}^*(\mu, X_4) = p_{02}^{(2)}k_2(\tilde{A}_{01} + H_{01} + \varsigma\tilde{A}_{01}^p + d + \tilde{d}_{01}) \\ &\qquad +p_{02}^{(4)}(\tilde{A}_{01} + H_{01} + \varsigma\tilde{A}_{01}^p + d + \tilde{d}_{01})^2 + p_{02}^{(5)}(\tilde{A} + H_{10})(\bar{\tilde{A}} + H_{10}) \\ &\qquad - \left(p_{02}^{(2)}k_2(d + \tilde{d}_{01}) + p_{02}^{(4)}(d + \tilde{d}_{01})^2\right) \\ &\qquad +\sqrt{\mu}p_{02}^{(6)}\frac{i}{2}\left((H_{10} + \tilde{A})(H_{11} + \bar{\tilde{B}}) - (H_{10} + \bar{\tilde{A}})(H_{11} + \tilde{B})\right) + O(\mu) \\ &\quad = \dot{H}_{02} + (-p_{02}^{(2)}k_2 + 2p_{02}^{(4)}H_{01})\tilde{A}_{01} + 2p_{02}^{(4)}\tilde{d}_{01}H_{01} + p_{02}^{(5)}H_{10}(\tilde{A} + \bar{\tilde{A}}) + O(\mu^{7/8}).\end{aligned}$$

It is straightforward to check that from (5.2.7)

$$\begin{aligned}&\frac{1}{\mu^2}[\mathcal{R}_{02}(\mu, X_5) - \mathcal{R}_{02}(\mu, X_6) + \varrho(\wp_{02}(\mu, \varrho X_5) - \wp_{02}(\mu, \varrho, X_6))] = O(\mu), \\ &\frac{1}{\mu^2}[\mathcal{R}_{02}(\mu, \hat{X}_5) - \mathcal{R}_{02}(\mu, X_6) + \varrho(\wp_{02}(\mu, \varrho, \hat{X}_5) - \wp_{02}(\mu, \varrho, X_6))] = O(\mu).\end{aligned}$$

(6.5.6) can rewritten as

$$\dot{\tilde{A}}_{02}(x) = (-p_{02}^{(2)}k_2 + 2p_{02}^{(4)}H_{01})\tilde{A}_{01} + p_{02}^{(5)}H_{10}(\tilde{A} + \bar{\tilde{A}}) + 2p_{02}^{(4)}\tilde{d}_{01}H_{01} + f_{A_{02}}, \tag{6.5.7}$$

where $f_{A_{02}}$ is of order at least $O(\mu^{7/8})$.

Similarly, the equation of $\tilde{A}_{01}$ is given by

$$\dot{\tilde{A}}_{01} = \tilde{A}_{02} + f_{A_{01}} \tag{6.5.8}$$

where $f_{A_{01}}$ is of order at least $O(\mu^{7/8})$.

Equations for $\tilde{A}_{02}$ and $\tilde{B}$ have the term including $\tilde{d}_{01}$ which contains $\frac{\varrho}{\mu^2}$ (See (5.3.5)). In order to solve for ϱ to get the reversible solution, we decompose $\tilde{A}_{01}$, $\tilde{A}_{02}$, $\tilde{A}$, $\tilde{B}$ as the following

$$\begin{aligned} &\tilde{A}_{01} = \tilde{A}_{01,0} + \tilde{d}_{01}\tilde{A}_{01,\varrho}, && \tilde{A}_{02} = \tilde{A}_{02,0} + \tilde{d}_{01}\tilde{A}_{02,\varrho}, \\ &\tilde{A} = \tilde{A}_0 + \tilde{d}_{01}\tilde{A}_\varrho, && \tilde{A}_0 = \tilde{A}_0^r + i\tilde{A}_0^i, \\ &\tilde{B} = \tilde{B}_0 + \tilde{d}_{01}\tilde{B}_\varrho, && \tilde{B}_0 = \tilde{B}_0^r + i\tilde{B}_0^i, \end{aligned}$$

where $\tilde{A}_{01,0}, \tilde{A}_{02,0}, \tilde{A}_0, \tilde{B}_0$ do not contain $\tilde{d}_{01}$ (i.e, do not contain ϱ), and $\tilde{A}_0^r$, $\tilde{A}_0^i$, $\tilde{B}_0^r$ and $\tilde{B}_0^i$ are real. Similarly, we decompose $f_{A_{01}}$, $f_{A_{02}}$, f_A and f_B. From (6.5.4)-(6.5.8), we know that $\tilde{A}_{01,0}$, $\tilde{A}_{02,0}$, $\tilde{A}_0^r$ and $\tilde{B}_0^r$ satisfy

$$\begin{aligned} &\dot{\tilde{A}}_{01,0} = \tilde{A}_{02,0} + f_{A_{01},0}, \\ &\dot{\tilde{A}}_{02,0} = (-p_{02}^{(2)}k_2 + 2p_{02}^{(4)}H_{01})\tilde{A}_{01,0} + 2p_{02}^{(5)}H_{10}\tilde{A}_0^r + f_{A_{02},0}, \\ &\dot{\tilde{A}}_0^r = \tilde{B}_0^r + f_{A,0}^r, \\ &\dot{\tilde{B}}_0^r = p_{11}^{(4)}H_{10}\tilde{A}_{01,0} + \tilde{A}_0^r(c^2 + p_{11}^{(4)}H_{01}) + f_{B,0}^r, \end{aligned}$$

which show that $\tilde{A}_{01,0}$, $\tilde{A}_{02,0}$, $\tilde{A}_0^r$ and $\tilde{B}_0^r$ are of order at least $O(\mu^{7/8})$.

Similarly, $\tilde{A}_{01,\varrho}$, $\tilde{A}_{02,\varrho}$, $\tilde{A}_\varrho$ and $\tilde{B}_\varrho$ satisfy

$$\begin{aligned} &\dot{\tilde{A}}_{01,\varrho} = \tilde{A}_{02,\varrho} + f_{A_{01},\varrho}, \\ &\dot{\tilde{A}}_{02,\varrho} = (-p_{02}^{(2)}k_2 + 2p_{02}^{(4)}H_{01})\tilde{A}_{01,\varrho} + p_{02}^{(5)}H_{10}(\tilde{A}_\varrho + \bar{\tilde{A}}_\varrho) + 2p_{02}^{(4)}H_{01} + f_{A_{02},\varrho}, \\ &\dot{\tilde{A}}_\varrho = \tilde{B}_\varrho + f_{A,\varrho}, \\ &\dot{\tilde{B}}_\varrho = p_{11}^{(4)}H_{10}\tilde{A}_{01,\varrho} + \tilde{A}_\varrho(c^2 + p_{11}^{(4)}H_{01}) + p_{11}^{(4)}H_{10} + f_{B,\varrho} \end{aligned}$$

and their complex conjugates where $f_{A_{01},\varrho}$, $f_{A_{02},\varrho}$, $f_{A,\varrho}$ and $f_{B,\varrho}$ are of order $O(\mu^{1/4})$ since $\tilde{d}_{01}$ in (5.3.5) is of order at least $O(\mu^{5/8})$ from the assumption $\varrho = \mu^{21/8}\varrho_1$ in Theorem 5.6.1. Suppose that $\hat{A}_{01,\varrho}$, $\hat{A}_{02,\varrho}$, $\hat{A}_\varrho$, $\hat{B}_\varrho$, $\bar{\hat{A}}_\varrho$ and $\bar{\hat{B}}_\varrho$ are the solutions of the equations

$$\begin{aligned} &\dot{\hat{A}}_{01,\varrho} = \hat{A}_{02,\varrho}, \\ &\dot{\hat{A}}_{02,\varrho} = (-p_{02}^{(2)}k_2 + 2p_{02}^{(4)}H_{01})\hat{A}_{01,\varrho} + p_{02}^{(5)}H_{10}(\hat{A}_\varrho + \bar{\hat{A}}_\varrho) + 2p_{02}^{(4)}H_{01}, \\ &\dot{\hat{A}}_\varrho = \hat{B}_\varrho, \\ &\dot{\hat{B}}_\varrho = p_{11}^{(4)}H_{10}\hat{A}_{01,\varrho} + \hat{A}_\varrho(c^2 + p_{11}^{(4)}H_{01}) + p_{11}^{(4)}H_{10} \end{aligned}$$

and their complex conjugates. Then $\tilde{A}_{01,\varrho} = \hat{A}_{01,\varrho} + O(\mu^{1/4})$, $\tilde{A}_{02,\varrho} = \hat{A}_{02,\varrho} + O(\mu^{1/4})$, and $\tilde{A}_\varrho = \hat{A}_\varrho + O(\mu^{1/4})$ so

$$\tilde{A}_{01} = \tilde{d}_{01}\hat{A}_{01,\varrho} + O(\mu^{7/8}), \quad \tilde{A}_{02} = \tilde{d}_{01}\hat{A}_{02,\varrho} + O(\mu^{7/8}), \quad \mathrm{Re}\tilde{A} = \tilde{d}_{01}\mathrm{Re}\hat{A}_\varrho + O(\mu^{7/8}).$$

By using the above, integrate (6.5.7) from 0 to ∞ and then (5.6.26) is changed to

$$\tilde{a}\tilde{d}_{01} = \tilde{f}_{A02}$$

where $\tilde{a}$ is defined in (5.6.2) and $\tilde{f}_{A02}$ is of order $O(\mu^{7/8})$. From (5.3.5) and the assumption of Theorem 5.6.1, the above equation can be written as

$$\varrho_1 = \mu^{1/4}\hat{f}_{A02}(\mu, \varrho_1, \tilde{\theta}_1)$$

where $\theta_1 = \sqrt{\mu}\tilde{\theta}_1$ and $\hat{f}_{A02}$ is bounded and smooth in its arguments. This is the first equation of (5.6.30).

Step 2: The second equation of (5.6.30)

In order to get the second equation of (5.6.30), we first note that the fifth component $\tilde{C}^p$ of the periodic function $\tilde{X}_{\mu,\varrho,\tilde{I}_1}(x)$ with a period $\frac{2\pi\sqrt{\mu}}{s_{20}+r_1}$ (see (5.5.18)) is a solution of

$$\begin{aligned}\dot{\tilde{C}}^p = {} & \frac{1}{\sqrt{\mu}}\left[is_{20}\tilde{C}^p + i(\tilde{C}^p + \tilde{d}_{20})P_{20}(X_3) - i\tilde{d}_{20}P_{20}(X_4)\right] \\ & + \frac{1}{\mu^2}\left[\check{R}_{20}(\mu, \varrho, X_5) - \check{R}_{20}(\mu, \varrho, X_6)\right],\end{aligned} \tag{6.5.9}$$

where $\check{R}_{20}(\mu, \varrho, X) = \mathcal{R}_{20}(\mu, X) - \varrho\wp_{20}(\mu, \varrho, X)$. Letting $\tilde{x} = (s_{20}+r_1)\mu^{-1/2}x$ and integrating (6.5.9), we have

$$\tilde{C}^p(\tilde{x}) = \mathrm{e}^{i\frac{s_{20}}{s_{20}+r_1}\tilde{x}}\tilde{C}^p(0) + \mathcal{A}(\tilde{x}, \mu, \varrho, \tilde{X}_{\mu,\varrho,\tilde{I}_1}) \tag{6.5.10}$$

where

$$\begin{aligned}\mathcal{A}(\tilde{x}, \mu, \varrho, \tilde{X}_{\mu,\varrho,\tilde{I}_1}) = {} & \int_0^{\tilde{x}} \mathrm{e}^{i\frac{s_{20}}{s_{20}+r_1}(\tilde{x}-s)} \frac{1}{s_{20}+r_1}\Big[i(\tilde{C}^p + \tilde{d}_{20})P_{20}(X_3) - i\tilde{d}_{20}P_{20}(X_4) \\ & + \frac{1}{\mu^{3/2}}\left(\check{R}_{20}(\mu, \varrho, X_5) - \check{R}_{20}(\mu, \varrho, X_6)\right)\Big]ds.\end{aligned}$$

Since the coefficient of $\mathrm{e}^{i(s_{20}+r_1)\mu^{-1/2}x}$ in $\tilde{C}^p(x)$ is $C_1 = \tilde{I}_1$ (see (5.5.17)), we obtain

$$\begin{aligned}\tilde{I}_1 & = \frac{1}{2\pi}\int_0^{2\pi} \tilde{C}^p(\tilde{x})\mathrm{e}^{-i\tilde{x}}d\tilde{x} \\ & = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-i\tilde{x}}\left(\mathrm{e}^{i\frac{s_{20}}{s_{20}+r_1}\tilde{x}}\tilde{C}^p(0) + \mathcal{A}(\tilde{x}, \mu, \varrho, \tilde{X}_{\mu,\varrho,\tilde{I}_1})\right)d\tilde{x} \\ & = \begin{cases} \tilde{C}^p(0) + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-i\tilde{x}}\mathcal{A}(\tilde{x}, \mu, \varrho, \tilde{X}_{\mu,\varrho,\tilde{I}_1})d\tilde{x}, \text{ if } r_1 = 0, \\ \frac{1}{2\pi}\tilde{C}^p(0)(2\pi + \zeta) + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-i\tilde{x}}\mathcal{A}(\tilde{x}, \mu, \varrho, \tilde{X}_{\mu,\varrho,\tilde{I}_1})d\tilde{x}, \text{ if } r_1 \neq 0 \end{cases}\end{aligned}$$

where $\zeta = \left(\mathrm{e}^{i\frac{-2\pi r_1}{s_{20}+r_1}} - 1\right)/\left(i\frac{-r_1}{s_{20}+r_1}\right) - 2\pi = O(r_1)$. Thus,

$$\tilde{C}^p(0) = \begin{cases} \tilde{I}_1 - \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-i\tilde{x}}\mathcal{A}(\tilde{x}, \mu, \varrho, \tilde{X}_{\mu,\varrho,\tilde{I}_1})d\tilde{x}, \text{ if } r_1 = 0, \\ \frac{1}{1+\frac{1}{2\pi}\zeta}\left[\tilde{I}_1 - \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-i\tilde{x}}\mathcal{A}(\tilde{x}, \mu, \varrho, \tilde{X}_{\mu,\varrho,\tilde{I}_1})d\tilde{x}\right], \text{ if } r_1 \neq 0. \end{cases}$$

For $r_1 = 0$,

$$\tilde{C}^p(\tilde{x}) = e^{i\frac{s_{20}}{s_{20}+r_1}\tilde{x}}\tilde{I}_1 - e^{i\frac{s_{20}}{s_{20}+r_1}\tilde{x}}\frac{1}{2\pi}\int_0^{2\pi} e^{-i\tilde{x}}\mathcal{A}(\tilde{x},\mu,\varrho,\tilde{X}_{\mu,\varrho,\tilde{I}_1})d\tilde{x} + \mathcal{A}(\tilde{x},\mu,\varrho,\tilde{X}_{\mu,\varrho,\tilde{I}_1})$$

and for $r_1 \neq 0$,

$$\tilde{C}^p(\tilde{x}) = \frac{e^{i\frac{s_{20}}{s_{20}+r_1}\tilde{x}}}{1+\frac{1}{2\pi}\zeta}\tilde{I}_1 - \frac{e^{i\frac{s_{20}}{s_{20}+r_1}\tilde{x}}}{1+\frac{1}{2\pi}\zeta}\frac{1}{2\pi}\int_0^{2\pi} e^{-i\tilde{x}}\mathcal{A}(\tilde{x},\mu,\varrho,\tilde{X}_{\mu,\varrho,\tilde{I}_1})d\tilde{x} + \mathcal{A}(\tilde{x},\mu,\varrho,\tilde{X}_{\varrho,\tilde{I}_1}).$$

(5.5.16) implies $r_1 = O(\mu)$ and the definition of P_{20} in (5.1.15) gives that $P_{20} = O(\mu)$. From the last estimate of (5.2.7), we note that the term $-i\frac{1}{\mu^2}\frac{P}{r_{20}^2}\varrho$ is cancelled and then by (5.3.5), $\tilde{I}_1 = \mu^{5/4}\tilde{I}_2$ and $\varrho = \mu^{21/8}\varrho_1$ in Theorem 5.6.1, we have

$$\check{R}_{20}(\mu,\varrho,X_5) - \check{R}_{20}(\mu,\varrho,X_6) = O(\mu^{n+1} + \mu|\varrho|),\ \mathcal{A}(\tilde{x},\mu,\varrho,\tilde{X}_{\mu,\varrho,\tilde{I}_1}) = O(\mu^{17/8}), \quad (6.5.11)$$

so $\tilde{C}^p(\tilde{x}) = O(\tilde{I}_1)$ for n large, i.e.,

$$\tilde{C}^p(x) = O(\tilde{I}_1) = O(\mu^{5/4}). \quad (6.5.12)$$

The fifth component $\tilde{C}(x)$ of $\tilde{Z}$ satisfies

$$\begin{aligned}\dot{\tilde{C}}(x) = &-\frac{\varsigma(x)}{\sqrt{\mu}}\left[is_{20}\tilde{C}^p(x+\theta_1) + i\left(\tilde{C}^p(x+\theta_1)+\tilde{d}_{20}\right)P_{20}(X_3) - i\tilde{d}_{20}P_{20}(X_4)\right]\\ &-\frac{\varsigma(x)}{\mu^2}\left[\check{R}_{20}(\mu,\varrho,X_5) - \check{R}_{20}(\mu,\varrho,X_6)\right] + \frac{1}{\sqrt{\mu}}\Big[is_{20}\left(\tilde{C}(x)+\varsigma(x)\tilde{C}^p(x+\theta_1)\right)\\ &+i\left(\tilde{C}(x)+\varsigma(x)\tilde{C}^p(x+\theta_1)+\tilde{d}_{20}\right)P_{20}(\hat{X}_3) - i\tilde{d}_{20}P_{20}(X_4)\Big]\\ &+\frac{1}{\mu^2}\left[\check{R}_{20}(\mu,\varrho,\hat{X}_5) - \check{R}_{20}(\mu,\varrho,X_6)\right] - \varsigma'(x)\tilde{C}^p(x+\theta_1).\end{aligned} \quad (6.5.13)$$

Since $\tilde{C}(x)$ exponentially approaches 0 as $x \to \infty$, we obtain

$$\begin{aligned}\tilde{C}(x) = &-\int_x^\infty e^{is_{20}\mu^{-1/2}(x-s)}\Big[-\frac{\varsigma(s)}{\sqrt{\mu}}\Big(i\left(\tilde{C}^p(s+\theta_1)+\tilde{d}_{20}\right)P_{20}(X_3)\\ &-i\tilde{d}_{20}P_{20}(X_4)\Big) - \frac{\varsigma(s)}{\mu^2}\left(\check{R}_{20}(\mu,\varrho,X_5) - \check{R}_{20}(\mu,\varrho,X_6)\right)\\ &+\frac{1}{\sqrt{\mu}}\left(i\left(\tilde{C}(s)+\varsigma(s)\tilde{C}^p(s+\theta_1)+\tilde{d}_{20}\right)P_{20}(\hat{X}_3) - i\tilde{d}_{20}P_{20}(X_4)\right)\\ &+\frac{1}{\mu^2}\left(\check{R}_{20}(\mu,\varrho,\hat{X}_5) - \check{R}_{20}(\mu,\varrho,X_6)\right) - \varsigma'(s)\tilde{C}^p(s+\theta_1)\Big]ds\\ = &-\int_2^\infty e^{is_{20}\mu^{-1/2}(x-s)}\frac{-i}{\sqrt{\mu}}\tilde{C}^p(s+\theta_1)\left(P_{20}(X_3) - P_{20}(\hat{X}_3)\right)ds\end{aligned}$$

$$
\begin{aligned}
&-\int_x^2 \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{-i}{\sqrt{\mu}}\varsigma(s)\tilde{C}^p(s+\theta_1)\left(P_{20}(X_3)-P_{20}(\hat{X}_3)\right)ds\\
&-\int_x^2 \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{i}{\sqrt{\mu}}\tilde{d}_{20}P_{20}(X_4)(\varsigma(s)-1)ds\\
&-\int_2^\infty \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{-i}{\sqrt{\mu}}\tilde{d}_{20}\left(P_{20}(X_3)-P_{20}(\hat{X}_3)\right)ds\\
&-\int_x^2 \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{-i}{\sqrt{\mu}}\tilde{d}_{20}\left(P_{20}(X_3)\varsigma(s)-P_{20}(\hat{X}_3)\right)ds\\
&-\int_2^\infty \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{-1}{\mu^2}\left(\check{R}_{20}(\mu,\varrho,X_5)-\check{R}_{20}(\mu,\varrho,\hat{X}_5)\right)ds\\
&-\int_x^2 \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{1}{\mu^2}\Big[\check{R}_{20}(\mu,\varrho,X_6)(\varsigma(s)-1)\\
&\qquad\qquad -\check{R}_{20}(\mu,\varrho,X_5)\varsigma(s)+\check{R}_{20}(\mu,\varrho,\hat{X}_5)\Big]ds\\
&-\int_x^\infty \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{i}{\sqrt{\mu}}\tilde{C}(s)P_{20}(\hat{X}_3)ds\\
&+\int_1^2 \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\varsigma'(s)\tilde{C}^p(s+\theta_1)ds
\end{aligned}
\tag{6.5.14}
$$

where the definition of ς is use.

From $\tilde{I}_1=\mu^{5/4}\tilde{I}_2$ in Theorem 5.6.1, (5.5.20) and $P_{20}=O(\mu)$ in (5.1.15), we obtain that

$$
\begin{aligned}
&\left|\int_2^\infty \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{-i}{\sqrt{\mu}}\tilde{C}^p(s+\theta_1)\left(P_{20}(X_3)-P_{20}(\hat{X}_3)\right)ds\right|\\
&\qquad\le M\sqrt{\mu}\tilde{I}_1\int_2^\infty H_{01}(s)ds\le M\mu^{7/4},\\
&\left|\int_x^2 \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{-i}{\sqrt{\mu}}\varsigma(s)\tilde{C}^p(s+\theta_1)\left(P_{20}(X_3)-P_{20}(\hat{X}_3)\right)ds\right|\le M\mu^{7/4}.
\end{aligned}
$$

(5.3.5) and $\varrho=\mu^{21/8}\varrho_1$ yield

$$
\begin{aligned}
&\left|\int_x^2 \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{i}{\sqrt{\mu}}\tilde{d}_{20}P_{20}(X_4)(\varsigma(s)-1)ds\right|\le M\sqrt{\mu}|\tilde{d}_{20}|=M\mu^{13/8},\\
&\left|-\int_2^\infty \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{-i}{\sqrt{\mu}}\tilde{d}_{20}\left(P_{20}(X_3)-P_{20}(\hat{X}_3)\right)ds\right|\\
&\qquad\le M\sqrt{\mu}|\tilde{d}_{20}|\int_2^\infty H_{01}(s)ds\le M\mu^{13/8},\\
&\left|-\int_x^2 \mathrm{e}^{is_{20}\mu^{-1/2}(x-s)}\frac{-i}{\sqrt{\mu}}\tilde{d}_{20}\left(P_{20}(X_3)\varsigma(s)-P_{20}(\hat{X}_3)\right)ds\right|\le M\mu^{13/8}.
\end{aligned}
$$

Using (5.2.7) and the fact that the terms including $-i\frac{P}{r_{20}^2}\varrho$ in $\check{R}_{20}$ are cancelled, we have for n large

$$\left|\int_2^\infty e^{is_{20}\mu^{-1/2}(x-s)}\frac{-1}{\mu^2}\left(\check{R}_{20}(\mu,\varrho,X_5)-\check{R}_{20}(\mu,\varrho,\hat{X}_5)\right)ds\right| \le M\frac{|\varrho|}{\mu}\int_2^\infty H_{01}ds \le M\mu^{13/8},$$

$$\left|\int_x^2 e^{is_{20}\mu^{-1/2}(x-s)}\frac{1}{\mu^2}\Big[\check{R}_{20}(\mu,\varrho,X_6)(\varsigma(s)-1)-\check{R}_{20}(\mu,\varrho,X_5)\varsigma(s)+\check{R}_{20}(\mu,\varrho,\hat{X}_5)\Big]ds\right|$$
$$\le M\frac{|\varrho|}{\mu} = M\mu^{13/8}.$$

The above inequalities yield

$$\tilde{C}(x) = \int_1^2 e^{is_{20}\mu^{-1/2}(x-s)}\varsigma'(s)\tilde{C}^p(s+\theta_1)ds + R_{21}(x,\mu,\varrho,\theta_0,\theta_1)$$

where R_{21} is of order $O(\mu^{13/8})$. Thus, (5.6.29) becomes

$$\int_1^2 \varsigma'(s)\sin(s_{20}\mu^{-1/2}\theta_1)\tilde{I}_2 ds + R_{22}(\mu,\varrho,\theta_1) = 0,$$

where $\theta_0 = \frac{\pi}{2}$, $\tilde{I}_1 = \mu^{5/4}\tilde{I}_2$ and R_{22} is of order $O(\mu^{3/8})$, which yields

$$\sin(s_{20}\mu^{-1/2}\theta_1) + R_{23}(\mu,\varrho,\theta_1) = 0,$$

where R_{23} is of order $O(\mu^{3/8})$. Let $\theta_1 = \sqrt{\mu}\tilde{\theta}_1$, which gives

$$\tilde{\theta}_1 = \mu^{3/8}\hat{f}_C(\mu,\varrho_1,\tilde{\theta}_1),$$

where $\hat{f}_C$ is bounded and smooth in its arguments, which gives the second equation of (5.6.30). Thus, the proof is completed. □

6.6 Some Calculations for (b, F^{-2}) near $(b_0, 1)$

By the straightforward calculation, the eigenvector e_{00} and the generalized eigenvectors e_{0j} of L_s corresponding to 0, the eigenvector e_{00}^* of L_s^* and the generalized eigenvectors e_{0j}^* corresponding to 0, $j = 1, 2, 3$, are given by

$$\begin{aligned}
&e_{00} = (1,0,0,0)^T, && e_{01} = (0,1,-1,0)^T,\\
&e_{02} = (-\tfrac{z^2}{2},0,0,-1)^T, && e_{03} = (0,-\tfrac{z^2}{2},\tfrac{1}{2}-b_0,0)^T,\\
&e_{00}^* = \tfrac{1}{r_{00}^2}(1,0,0,0)^T, && e_{01}^* = \tfrac{1}{r_{00}^2}(0,-\tfrac{z^2}{2}+\alpha, b_0(b_0-\tfrac{1}{2})+b_0\alpha,0)^T,\\
&e_{02}^* = \tfrac{1}{r_{00}^2}(-\tfrac{z^2}{2}+\alpha,0,0,b_0)^T, && e_{03}^* = \tfrac{1}{r_{00}^2}(0,1,b_0,0)^T
\end{aligned}$$

where $\alpha = \frac{b_0^2 - b_0 + \frac{1}{5}}{\frac{1}{3} - b_0}$ and

$$r_{00}^2 = P(\frac{1}{3} - b_0) > 0. \tag{6.6.1}$$

They satisfy $(e_{0j}, e_{0j}^*) = 1$ for $j = 0, 1, 2, 3$; the eigenvector U_{20} of L_s corresponding to is_{20} and the eigenvector V_{20}^* of L_s^* corresponding to $-is_{20}$ are given by

$$\begin{aligned}
U_{20} &= \Big(i\frac{\cosh(s_{20}z)}{s_{20}\sinh(s_{20})}, -\frac{\cosh(s_{20}z)}{\sinh(s_{20})}, \frac{1}{s_{20}}, i\Big)^T, &(6.6.2)\\
V_{20}^* &= \frac{1}{r_{20}^2}\Big(-i\frac{\cosh(s_{20}z)}{b_0 s_{20}\sinh(s_{20})}, -\frac{\cosh(s_{20}z)}{b_0\sinh(s_{20})}, -\frac{1}{s_{20}}, i\Big)^T &(6.6.3)
\end{aligned}$$

where

$$r_{20}^2 = P\Big(\frac{1}{b_0\sinh^2(s_{20})} + 1 - \frac{\lambda}{b_0 s_{20}^2}\Big) > 0. \tag{6.6.4}$$

They satisfy $(U_{20}, V_{20}^*) = 1$ and $SU_{20} = \bar{U}_{20}$; the eigenvector U_{10} and the generalized eigenvector U_{11} of L_s corresponding to is_{10} and the eigenvector V_{11}^* and the generalized eigenvector V_{10}^* of L_s^* corresponding to $-is_{10}$ are given by

$$\begin{aligned}
U_{10} &= \Big(i\frac{\cosh(\sigma_{10}z)}{\sigma_{10}\sinh(\sigma_{10})},\ -\frac{s_{10}\cosh(\sigma_{10}z)}{\sigma_{10}\sinh(\sigma_{10})},\ \frac{1}{s_{10}},\ i\Big)^T\cos(\frac{2\pi y}{P}), &(6.6.5)\\
U_{11} &= \Big(-\frac{s_{10}(\sinh(\sigma_{10}) + \sigma_{10}\cosh(\sigma_{10}))}{\sigma_{10}^3\sinh^2(\sigma_{10})}\cosh(\sigma_{10}z)\\
&\quad + \frac{s_{10}}{\sigma_{10}^2\sinh(\sigma_{10})}z\sinh(\sigma_{10}z), -i\frac{s_{10}^2(\sinh(\sigma_{10}) + \sigma_{10}\cosh(\sigma_{10}))}{\sigma_{10}^3\sinh^2(\sigma_{10})}\cosh(\sigma_{10}z)\\
&\quad + i\frac{s_{10}^2}{\sigma_{10}^2\sinh(\sigma_{10})}z\sinh(\sigma_{10}z) + i\frac{\cosh(\sigma_{10}z)}{\sigma_{10}\sinh(\sigma_{10})},\ i\frac{1}{s_{10}^2},\ 0\Big)^T\cos(\frac{2\pi y}{P}), &(6.6.6)\\
V_{11}^* &= \frac{1}{r_{10}^2}\Big(-\frac{\cosh(\sigma_{10}z)}{b_0\sigma_{10}\sinh(\sigma_{10})},\ i\frac{s_{10}\cosh(\sigma_{10}z)}{b_0\sigma_{10}\sinh(\sigma_{10})},\ i\frac{1}{s_{10}},\ 1\Big)^T\cos(\frac{2\pi y}{P}), &(6.6.7)\\
V_{100}^* &= \frac{1}{r_{10}^2}\Big(i\frac{s_{10}(\sinh(\sigma_{10}) + \sigma_{10}\cosh(\sigma_{10}))}{b_0\sigma_{10}^3\sinh^2(\sigma_{10})}\cosh(\sigma_{10}z)\\
&\quad -i\frac{s_{10}}{b_0\sigma_{10}^2\sinh(\sigma_{10})}z\sinh(\sigma_{10}z), \frac{s_{10}^2(\sinh(\sigma_{10}) + \sigma_{10}\cosh(\sigma_{10}))}{b_0\sigma_{10}^3\sinh^2(\sigma_{10})}\cosh(\sigma_{10}z)\\
&\quad - \frac{s_{10}^2}{b_0\sigma_{10}^2\sinh(\sigma_{10})}z\sinh(\sigma_{10}z) - \frac{\cosh(\sigma_{10}z)}{b_0\sigma_{10}\sinh(\sigma_{10})},\ \frac{1}{s_{10}^2},\ 0\Big)^T\cos(\frac{2\pi y}{P}),\\
V_{10}^* &= V_{100}^* - \frac{(U_{11}, V_{100}^*)}{(U_{11}, V_{11}^*)}V_{11}^*, &(6.6.8)
\end{aligned}$$

where

$$\sigma_{10}^2 = s_{10}^2 + \frac{4\pi^2}{P^2},$$

$$\begin{aligned} r_{10}^2 &= \frac{Ps_{10}}{4b_0\sigma_{10}^5\sinh^3(\sigma_{10})}\big[(4\sigma_{10}^2-3s_{10}^2)\big(\sinh^2(\sigma_{10})\cosh(\sigma_{10})+\sigma_{10}\sinh(\sigma_{10})\big) \\ &\quad -2s_{10}^2\sigma_{10}^2\cosh(\sigma_{10})\big] > 0. \end{aligned} \tag{6.6.9}$$

(See Lemma 6.3.1.) They satisfy

$$(U_{10}, V_{10}^*) = 1, \qquad (U_{11}, V_{11}^*) = 1, \qquad SU_{10} = \bar{U}_{10}, \qquad SU_{11} = -\bar{U}_{11}.$$

From the system (2.2.8), we obtain

$$\dot{v} = (L_s + \mu_1 L_{\mu_1} + \mu_2 L_{\mu_2} + \alpha L_\alpha)v + N_2(v,v) + N_3(v,v,v) + h.o.t. \tag{6.6.10}$$

where N_2 denotes the sum of all quadratic terms, N_3 denotes the sum of all cubic terms, and *h.o.t.* denotes the higher order terms. Here L_{μ_1} is given by

$$L_{\mu_1}\begin{pmatrix}\phi\\ u\\ \eta\\ \xi\end{pmatrix} = -\frac{1}{b_0^2}\begin{pmatrix}0\\ 0\\ 0\\ u|_{z=1}+\eta\end{pmatrix}$$

with boundary conditions $\phi_z|_{z=0} = 0, \phi_z|_{z=1} = 0$, L_{μ_2} is given by

$$L_{\mu_2}\begin{pmatrix}\phi\\ u\\ \eta\\ \xi\end{pmatrix} = -\frac{1}{b_0}\begin{pmatrix}0\\ 0\\ 0\\ \eta\end{pmatrix}$$

with boundary conditions $\phi_z|_{z=0} = 0, \phi_z|_{z=1} = 0$ and L_α is given by

$$L_{\alpha}\begin{pmatrix}\phi\\ u\\ \eta\\ \xi\end{pmatrix} = -\frac{2}{P}\begin{pmatrix}0\\ -\phi_{yy}\\ 0\\ -\eta_{yy}\end{pmatrix}$$

with boundary conditions $\phi_z|_{z=0} = 0, \phi_z|_{z=1} = 0$.

Assume

$$\begin{aligned} &\Phi(\mu_1,\mu_2,\alpha,\varrho,A_{01},A_{02},A,B,C,\bar{A},\bar{B},\bar{C}) = \mu_1(\Phi_{10000000}A_{01} + \Phi_{01000000}A_{02} \\ &\quad +\Phi_{00100000}A + \cdots) + \mu_2(\tilde{\Phi}_{10000000}A_{01} + \tilde{\Phi}_{01000000}A_{02} + \tilde{\Phi}_{00100000}A + \cdots) \\ &\quad +\alpha(\hat{\Phi}_{10000000}A_{01} + \hat{\Phi}_{01000000}A_{02} + \hat{\Phi}_{00100000}A + \cdots) \\ &\quad +\Phi_{20000000}A_{01}^2 + \Phi_{10100000}A_{01}A + \Phi_{00100100}A\bar{A} + \cdots . \end{aligned}$$

Under these assumptions, (5.1.1) becomes

$$v = A_{01}e_{01} + A_{02}e_{02} + AU_{10} + BU_{11} + CU_{20} + \bar{A}\bar{U}_{10} + \bar{B}\bar{U}_{11} + \bar{C}\bar{U}_{20} +$$

$$+\Phi(\mu_1,\mu_2,\alpha,\varrho,A_{01},A_{02},A,B,C,\bar{A},\bar{B},\bar{C}). \tag{6.6.11}$$

Substituting (6.6.11) into (6.6.10) and comparing the coefficients, one finds that

$$\begin{aligned}
\mu_1 A_{01}: &\quad p_{02}^{(1)}e_{02} = L_s\Phi_{10000000} + L_{\mu_1}e_{01},\\
\mu_2 A_{01}: &\quad p_{02}^{(2)}e_{02} = L_s\tilde{\Phi}_{10000000} + L_{\mu_2}e_{01},\\
\alpha A_{01}: &\quad p_{02}^{(3)}e_{02} = L_s\hat{\Phi}_{10000000} + L_{\alpha}e_{01},\\
\mu_1 A_{02}: &\quad \Phi_{10000000} = L_s\Phi_{01000000} + L_{\mu_1}e_{02},\\
\mu_2 A_{02}: &\quad \tilde{\Phi}_{10000000} = L_s\tilde{\Phi}_{01000000} + L_{\mu_2}e_{02},\\
\alpha A_{02}: &\quad \hat{\Phi}_{10000000} = L_s\hat{\Phi}_{01000000} + L_{\alpha}e_{02},\\
A_{01}^2: &\quad p_{02}^{(4)}e_{02} = L_s\Phi_{20000000} + N_2(e_{01},e_{01}),\\
\mu_1 A: &\quad is_{10}\Phi_{00100000} + p_{10}^{(1)}U_{10} + p_{11}^{(1)}U_{11} = L_s\Phi_{00100000} + L_{\mu_1}U_{10},\\
\mu_2 A: &\quad is_{10}\tilde{\Phi}_{00100000} + p_{10}^{(2)}U_{10} + p_{11}^{(2)}U_{11} = L_s\tilde{\Phi}_{00100000} + L_{\mu_2}U_{10},\\
\alpha A: &\quad is_{10}\hat{\Phi}_{00100000} + p_{10}^{(3)}U_{10} + p_{11}^{(3)}U_{11} = L_s\hat{\Phi}_{00100000} + L_{\alpha}U_{10},\\
A_{01}A_{02}: &\quad 2\Phi_{20000000} = L\Phi_{11000000} + 2N_2(e_{01},e_{02}),\\
AA_{01}: &\quad p_{10}^{(4)}U_{10} + p_{11}^{(4)}U_{11} + is_{10}\Phi_{10100000} = L_2\Phi_{10100000} + 2N_2(e_{01},U_{10}),\\
A\bar{A}: &\quad p_{02}^{(5)}e_{02} = L_s\Phi_{00100100} + 2N_2(U_{10},\bar{U}_{10}).
\end{aligned}$$

Adopting the idea in Section 6.3, the coefficients can be found as follows

$$\begin{aligned}
p_{02}^{(1)} &= p_{02}^{(3)} = 0, \quad p_{02}^{(2)} = \frac{1}{\frac{1}{3}-b_0} > 0, \quad p_{02}^{(4)} = \frac{3}{2}\frac{1}{\frac{1}{3}-b_0} > 0,\\
p_{02}^{(5)} &= \frac{6\coth(\sigma_{10}) + 3\sigma_{10}\mathrm{csch}^2(\sigma_{10})}{2(1-3b_0)\sigma_{10}} > 0, \quad p_{11}^{(1)} = \frac{P\sigma_{10}^2}{2b_0 s_{10} r_{10}^2},\\
p_{11}^{(2)} &= \frac{-P}{2b_0 s_{10} r_{10}^2}, \quad p_{11}^{(3)} = -\frac{2\pi^2(2b_0\sigma_{10}^3 + s_{10}^2(\coth(\sigma_{10}) + \sigma_{10}\mathrm{csch}^2(\sigma_{10})))}{b_0 P^2 r_{10}^2 s_{10}\sigma_{10}^3},\\
p_{11}^{(4)} &= -\frac{P\mathrm{csch}^2(\sigma_{10}) s_{10}(\sinh(2\sigma_{10}) + \sigma_{10})}{2b_0\sigma_{10} r_{10}^2}.
\end{aligned} \tag{6.6.12}$$

Section 6.3 yields more details about how to solve these constants.

We note that Φ contains terms which are at least quadratic in its arguments so that the leading coefficients of ϱ in (5.1.19) are

$$\begin{aligned}
l_{01} &= \big((0,0,0,1)^T, e_{01}^*\big) = 0, \quad l_{02} = \big((0,0,0,1)^T, e_{02}^*\big) = \frac{b_0 P}{r_{00}^2},\\
l_{10} &= \big((0,0,0,1)^T, V_{10}^*\big) = 0, \quad l_{11} = \big((0,0,0,1)^T, V_{11}^*\big) = 0,\\
l_{20} &= \big((0,0,0,1)^T, V_{20}^*\big) = -i\frac{P}{r_{20}^2}.
\end{aligned} \tag{6.6.13}$$

Chapter 7

Future Work

In my thesis, I discussed the three-dimensional capillary-gravity water waves with one finite layer of fluid. In the future, I will consider the cases: one infinite layer, two finite layers or two layers with one infinite layer.

In recent years, there are some papers which investigated the water waves with vorticity. Wahlén [112, 113] proved the existence of steady periodic capillary and gravity-capillary water waves on flows with arbitrary distributions by changing the problem to one of an elliptic equation with a nonlinear boundary condition. They are symmetric two-dimensional waves whose profiles are monotone between crest and trough (monotonicity means strictly increasing from trough to crest and vise versa). Then he [114] considered it with constant vorticity and formulated the governing equations as a Hamiltonian system in 2007. At the same year, Groves and Wahlén [58] studied the gravity-capillary water waves with an arbitrary distribution of vorticity. Here the waves are not restricted to be periodic. They applied variational formulations to the problem and obtained a Hamiltonian system. Using the center manifold reduction, they showed the existence of the solitary water waves. I am interested in the following problems:

1. Two-dimensional capillary-gravity water waves with vorticity (one layer of infinite depth, two layers with finite or infinite depth),

2. Three-dimensional capillary-gravity water waves with vorticity (one layer, two layers with finite or infinite depth),

3. Using the direct spatial dynamic approach without a Hamiltonian structure.

I also note that there are some results about the external pressure not equal to 0, i.e. forced water waves. For example, Kirchgässner [72] studied two-dimensional cases and gave the bifurcations such as chaos under a small external pressure. The three-dimensional case is still open. The stability and instability problems stated in Chapter 1 also amaze me.

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