Variational Convex Analysis

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(ABSTRACT)

This work develops theoretical and applied results for variational convex analysis. First we present the basic tools of analysis necessary to develop the core theory and applications. New results concerning duality principles for systems originally modeled by non-linear differential equations are shown in chapters 9 to 17. A key aspect of this work is that although the original problems are non-linear with corresponding non-convex variational formulations, the dual formulations obtained are almost always concave and amenable to numerical computations. When the primal problem has no solution in the classical sense, the solution of dual problem is a weak limit of minimizing sequences, and the evaluation of such average behavior is important in many practical applications. Among the results we highlight the dual formulations for micro-magnetism, phase transition models, composites in elasticity and conductivity and others. To summarize, in the present work we introduce convex analysis as an interesting alternative approach for the understanding and computation of some important problems in the modern calculus of variations.

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Chapter 1

Introduction

The main objective of this work is to present recent results of the author about applications of duality to non-convex problems in the calculus of variations. The text is divided into chapters described in the next page, and chapters 2 to 8 present the basic concepts on standard analysis necessary to develop the applications.

Of course, the material presented in the first 8 chapters is not new, with exception of the section on relaxation for the scalar case, where we show different proofs of some theorems presented in Ekeland and Témam's book *Convex Analysis and Variational Problems* (indeed such a book is the theoretical base of the present work), and the section about relaxation for vectorial case. The applications, presented in chapters 9 to 17, correspond to the work of the present author along the last years, and almost all results including the applications of duality for micro-magnetism, composites in elasticity and conductivity and phase transitions, were obtained during the PhD program at Virginia Tech.

The key feature of this work is that while all problems studied here are non-linear with corresponding non-convex variational formulation, it has been almost always possible to develop convex (in fact concave) dual variational formulations, which in general are more amenable to numerical computations.

The section on relaxation for the vectorial case, as its title suggests, presents duality principles that are valid even for vectorial problems. It is worth noting that such results were used within the text to develop concave dual variational formulations in situations such as for conductivity in composites, vectorial examples in phase transitions, etc.

1.1 Summary of Main Results

The main results of this work are summarized as follows.

1.1.1 Duality Applied to a Plate Model

Chapter 9 develops dual variational formulations for the two dimensional equations of the nonlinear elastic Kirchhoff-Love plate model. The first duality principle presented is the classical one and may be found in similar format in Telega [33], Gao [18]. It is worth noting that such results are valid only for positive definite membrane forces. However, we obtain new dual variational formulations which relax or even remove such constraints. In particular we exhibit a convex dual variational formulation which allows non positive definite membrane forces. In the last section, similar to the triality criterion introduced in Gao [20], we obtain sufficient conditions of optimality for the present case. The results are based on fundamental tools of Convex Analysis and the Legendre Transform, which can easily be analytically expressed for the model in question.

1.1.2 Duality Applied to Finite Elasticity

Chapter 10 develops duality for a model in finite elasticity. The dual formulations obtained allow the matrix of stresses to be non positive definite. This is in some sense, an extension of earlier results (which establish the complementary energy as a perfect global optimization duality principle only if the stress tensor is positive definite at the equilibrium point). Also, it is important to emphasize that one of the formulations obtained is convex. Again, the results are based on standard tools of convex analysis and on the concept of Legendre Transform.

1.1.3 Duality Applied to a Shell Model

The main focus of Chapter 11 is the development of dual variational formulations for a nonlinear elastic membrane shell model. In the present literature, the concept of complementary energy can be established only if the external load produces a critical point with positive definite membrane forces matrix. Our idea is to obtain dual variational formulations for which the mentioned constraint is relaxed or even eliminated. It is important to emphasize that one of the dual (in fact, primal dual) formulations here presented is convex. Again, the results are obtained through basic tools of convex analysis and the concept of Legendre Transform, which can be analytically established for the concerned shell model.

1.1.4 Duality Applied to Phase transitions

The first part of Chapter 12 is concerned with the development of dual variational formulations for Ginzburg-Landau type equations. Since the primal formulations are non-convex, we use specific results for distance between two convex functions to obtain the dual approaches. Note that we obtain two different convex dual formulations (in fact one of them is a kind of primal-dual formulation). As a second objective we present duality as an alternative perspective to address the multi-well and related phase transition problems. In these latter cases the dual formulations are concave and the solution of the dual problems reflect the average behavior of minimizing sequences, as weak cluster points, considering that we may not have minimizers in the classical sense through the primal approaches.

1.1.5 Duality Applied to Conductivity in Composites

The main focus of Chapter 13 is the development of a dual variational formulation for a two-phase optimization problem in conductivity. The primal formulation may not have minimizers in the classical sense. In this case, the solution through the dual formulation may be a weak limit of minimizing sequences for the original problem.

1.1.6 Duality Applied to the Optimal Design in Elasticity

The first part of Chapter 14 develops a dual variational formulation for the optimal design of a plate of variable thickness. The design variable, namely the plate thickness, is supposed to minimize the plate deformation work due to a given external load. The second part is concerned with the optimal design for a two-phase problem in elasticity. In this case, we are looking for the mixture of two constituents that minimizes the structural internal work. In both applications the dual formulations were obtained through basic tools of convex analysis.

1.1.7 Duality Applied to Micro-Magnetism

The main focus of Chapter 15 is the development of dual variational formulations for functionals related to ferromagnetism models. We develop duality principles for the so-called hard and full (semi-linear) uniaxial and cubic cases. It is important to emphasize that the new dual formulations here presented are convex and are useful to compute the average behavior of minimizing sequences, specially as the primal formulation has no minimizers in the classical sense. The results are obtained through standard tools of convex analysis.

1.1.8 Duality Applied to Fluid Mechanics

In Chapter 16 we use the concept of Legendre Transform to obtain dual variational formulations for the Navier-Stokes system.

1.1.9 Duality Applied to a Beam Model

Chapter 17 develops existence, duality and numerical results for a non-linear beam model. Our final result is a convex variational formulation for the concerned beam model.

Chapter 2

Topological Vector Spaces

2.1 Introduction

The main objective of this chapter is to present an outline of the basic tools of analysis necessary to develop the subsequent chapters. We assume the reader has a background in linear algebra and elementary real analysis at an undergraduate level. Some short proofs are given but many are left to other sources. More details on this subject may be found in Chapter 1 of "Functional Analysis" by W. Rudin (reference [31]).

2.2 Vector Spaces

We denote by $\mathbb{F}$ a scalar field. In practice this is either $\mathbb{R}$ or $\mathbb{C}$, the set of real or complex numbers.

Definition 2.2.1 (Vector Spaces). *A vector space over $\mathbb{F}$ is a set denoted by U, whose elements are called vectors, for which are defined two operations, namely, addition denoted by $(+) : U \times U \to U$, and, scalar multiplication, denoted by $(.) : \mathbb{F} \times U \to U$, so that the following relations are valid*

1. $u + v = v + u, \forall u, v \in U$,
2. $u + (v + w) = (u + v) + w, \forall u, v, w \in U$,
3. *there exists a vector denoted by θ such that* $u + \theta = u, \forall u \in U$,
4. *for each $u \in U$, there exists a unique vector denoted by $-u$ such that* $u + (-u) = \theta$,
5. $\alpha.(\beta.u) = (\alpha.\beta).u, \forall \alpha, \beta \in \mathbb{F}, \quad u \in U$,

6. $\alpha.(u+v) = \alpha.u + \beta.u, \forall \alpha \in \mathbb{F}, \quad u, v \in U,$

7. $(\alpha + \beta).u = \alpha.u + \beta.u, \forall \alpha, \beta \in \mathbb{F}, \quad u \in U,$

8. $1.u = u, \forall u \in U.$

Remark 2.2.2. *Now and on we will drop the dot (.) in scalar multiplication and denote $\alpha.u$ as αu.*

Definition 2.2.3 (Vector Subspace)**.** *Let U be a vector space. A set $V \subset U$ is said to be a vector subspace of U if V is also a vector space with the same operations as those of U. If $V \neq U$ we say that V is a proper subspace of U.*

Definition 2.2.4 (Finite dimensional Space)**.** *A vector space is said to be of finite dimension if there exists fixed $u_1, u_2, ..., u_n \in U$ such that for each $u \in U$ there are corresponding $\alpha_1, ..., \alpha_n \in \mathbb{F}$ for which*

$$u = \sum_{i=1}^{n} \alpha_i u_i. \tag{2.1}$$

Definition 2.2.5 (Topological Spaces)**.** *A set U is said to be a topological space if it is possible to define a collection σ of subsets of U called a topology in U, for which are valid the following properties:*

1. *$U \in \sigma$,*

2. *$\emptyset \in \sigma$,*

3. *if $A \in \sigma$ and $B \in \sigma$ then $A \cap B \in \sigma$, and*

4. *arbitrary unions of elements in σ also belong to σ.*

Any $A \in \sigma$ is said to be an open set.

Remark 2.2.6. *When necessary, to clarify the notation, we shall denote the vector space U endowed with the topology σ by (U, σ).*

Definition 2.2.7 (Closed Sets)**.** *Let U be a topological space. A set $A \subset U$ is said to be closed if $U - A$ is open. We also denote $U - A = A^c$.*

Proposition 2.2.8. *For closed sets we have the following properties:*

1. *U and $\emptyset$ are closed,*

2. *If A and B are closed sets then $A \cup B$ is closed,*

3. *Arbitrary intersections of closed sets are closed.*

Proof:

1. Since $\emptyset$ is open and $U = \emptyset^c$, by Definition 2.2.7 U is closed. Similarly, since U is open and $\emptyset = U - U = U^c$, $\emptyset$ is closed.
2. A, B closed implies that A^c and B^c are open, and by Definition 2.2.5, $A^c \cup B^c$ is open, so that $A \cap B = (A^c \cup B^c)^c$ is closed.
3. Consider $A = \cap_{\lambda \in L} A_\lambda$, where L is a collection of indices and A_λ is open, $\forall \lambda \in L$. We may write $A = (\cup_{\lambda \in L} A_\lambda^c)^c$ and since A_λ^c is open $\forall \lambda \in L$ we have, by Definition 2.2.5, that A is closed. $\square$

Definition 2.2.9 (Closure). *Given $A \subset U$ we define the closure of A, denoted by $\bar{A}$, as the intersection of all closed sets that contain A.*

Remark 2.2.10. *From Proposition 2.2.8 Item 3 we have that $\bar{A}$ is the smallest closed set that contains A, in the sense that, if C is closed and $A \subset C$ then $\bar{A} \subset C$.*

Definition 2.2.11 (Interior). *Given $A \subset U$ we define its interior, denoted by $\mathring{A}$, as the union of all open sets contained in A.*

Remark 2.2.12. *It is not difficult to prove that if A is open then $A = \mathring{A}$.*

Definition 2.2.13 (Neighborhood). *Given $u_0 \in U$ we say that $\mathcal{V}$ is a neighborhood of u_0 if such a set is open and contains u_0. We denote such neighborhoods by $\mathcal{V}_{u_0}$.*

Proposition 2.2.14. *If $A \subset U$ is a set such that for each $u \in A$ there exists a neighborhood $\mathcal{V}_u \ni u$ such that $\mathcal{V}_u \subset A$, then A is open.*

Proof: This follows from the fact that $A = \cup_{u \in U} \mathcal{V}_u$ and arbitrary union of open sets are open. $\square$

Definition 2.2.15 (Function). *Let U and V be two topological spaces. We say that $f : U \to V$ is a function if f is a collection of pairs $(u, v) \in U \times V$ such that for each $u \in U$ there exists only one $v \in V$ such that $(u, v) \in f$.*

Definition 2.2.16 (Continuity at a Point). *A function $f : U \to V$ is continuous at $u \in U$ if for each neighborhood $\mathcal{V}_2(f(u))$ there exists a neighborhood $\mathcal{V}_1(u)$ such that $f(\mathcal{V}_1(u)) \subset \mathcal{V}_2(f(u))$.*

Definition 2.2.17 (Continuous Function). *A function $f : U \to V$ is continuous if it is continuous at each $u \in U$.*

Proposition 2.2.18. *A function $f : U \to V$ is continuous if and only if $f^{-1}(\mathcal{V})$ is open for each open $\mathcal{V} \subset V$, where*

$$f^{-1}(\mathcal{V}) = \{u \in U \quad | \quad f(u) \in \mathcal{V}\}. \tag{2.2}$$

Proof: Suppose $f^{-1}(\mathcal{V})$ is open whenever $\mathcal{V} \subset V$ is open. Pick $u \in U$ and any $\mathcal{V}$ such that $f(u) \in \mathcal{V}$. Since $u \in f^{-1}(\mathcal{V})$ and $f(f^{-1}(\mathcal{V})) = \mathcal{V}$, we have that f is continuous at $u \in U$. Since $u \in U$ is arbitrary we have that f is continuous. Conversely, suppose f is continuous and pick $\mathcal{V} \subset V$ open. If $f^{-1}(\mathcal{V}) = \emptyset$ we are done, since $\emptyset$ is open,. Thus, suppose $u \in f^{-1}(\mathcal{V})$, since f is continuous, there exists $\mathcal{V}_u$ a neighborhood of u such that $f(\mathcal{V}_u) \subset \mathcal{V}$. This means $\mathcal{V}_u \subset f^{-1}(\mathcal{V})$ and therefore, from Proposition 2.2.14, $f^{-1}(\mathcal{V})$ is open. □

Definition 2.2.19. *We say that (U, σ) is a Hausdorff topological space if, given u_1, $u_2 \in U$, $u_1 \neq u_2$, there exists $\mathcal{V}_1$, $\mathcal{V}_2 \in \sigma$ such that*

$$u_1 \in \mathcal{V}_1 \quad , \quad u_2 \in \mathcal{V}_2 \text{ and } \mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset. \tag{2.3}$$

Definition 2.2.20 (Base)**.** *A collection $\sigma' \subset \sigma$ is said to be a base for σ if every element of σ may be represented as a union of elements of σ'.*

Definition 2.2.21 (Local Base)**.** *A collection $\hat{\sigma}$ of neighborhoods of a point $u \in U$ is said to be a local base at u if each neighborhood of u contains a member of $\hat{\sigma}$.*

Definition 2.2.22 (Topological Vector Space)**.** *A vector space endowed with a topology, denoted by (U, σ), is said to be a topological vector space if and only if*

1. *Every single point of U is a closed set,*
2. *The vector space operations (addition and scalar multiplication) are continuous with respect to σ.*

More specifically, addition is continuous if, given $u, v \in U$ and $\mathcal{V} \in \sigma$ such that $u + v \in \mathcal{V}$ then there exists $\mathcal{V}_u \ni u$ and $\mathcal{V}_v \ni v$ such that $\mathcal{V}_u + \mathcal{V}_v \subset \mathcal{V}$. On the other hand, scalar multiplication is continuous if given $\alpha \in \mathbb{F}$, $u \in U$ and $\mathcal{V} \ni \alpha.u$, there exists $\delta > 0$ and $\mathcal{V}_u \ni u$ such that, $\forall \beta \in \mathbb{F}$ satisfying $|\beta - \alpha| < \delta$ we have $\beta \mathcal{V}_u \subset \mathcal{V}$.

Given (U, σ), let us associate with each $u_0 \in U$ and $\alpha_0 \in \mathbb{F}$ ($\alpha_0 \neq 0$) the functions $T_{u_0} : U \to U$ and $M_{\alpha_0} : U \to U$ defined by

$$T_{u_0}(u) = u_0 + u \tag{2.4}$$

and

$$M_{\alpha_0}(u) = \alpha_0.u. \tag{2.5}$$

The continuity of such functions is a straightforward consequence of the continuity of vector space operations (addition and scalar multiplication). It is clear that the respective inverse maps, namely T_{-u_0} and M_{1/α_0} are also continuous. So if $\mathcal{V}$ is open then $u_0 + \mathcal{V}$, that is $(T_{-u_0})^{-1}(\mathcal{V}) = T_{u_0}(\mathcal{V}) = u_0 + \mathcal{V}$ is open. By analogy $\alpha_0 \mathcal{V}$ is open. Thus σ is completely determined by a local base, so that the term local base will be understood henceforth as a local base at 0. So to summarize, a local base of a topological vector space is a collection Ω of neighborhoods of 0, such that each neighborhood of 0 contains a member of Ω.

Now we present some simple results, namely:

Proposition 2.2.23. *If $A \subset U$ is open, then $\forall u \in A$ there exists a neighborhood $\mathcal{V}$ of θ such that $u + \mathcal{V} \subset A$*

Proof: Just take $\mathcal{V} = A - u$. $\square$

Proposition 2.2.24. *Given a topological vector space (U, σ), any element of σ may be expressed as a union of translates of members of Ω, so that the local base Ω generates the topology σ.*

Proof: Let $A \subset U$ open and $u \in U$. $\mathcal{V} = A - u$ is a neighborhood of θ and by definition of local base, there exists a set $\mathcal{V}_{\Omega u} \subset \mathcal{V}$ such that $\mathcal{V}_{\Omega_u} \in \Omega$. Thus, we may write

$$A = \cup_{u \in A}(u + \mathcal{V}_{\Omega_u}). \quad \square \tag{2.6}$$

2.3 Some Properties of Topological Vector Spaces

In this section we study some fundamental properties of topological vector spaces. We start with the following proposition:

Proposition 2.3.1. *Any topological vector space U is a Hausdorff space.*

Proof: Pick $u_0, u_1 \in U$ such that $u_0 \neq u_1$. Thus $\mathcal{V} = U - \{u_1 - u_0\}$ is an open neighborhood of zero. As $\theta + \theta = \theta$, by the continuity of addition, there exist $\mathcal{V}_1$ and $\mathcal{V}_2$ neighborhoods of θ such that

$$\mathcal{V}_1 + \mathcal{V}_2 \subset \mathcal{V} \tag{2.7}$$

define $\mathcal{U} = \mathcal{V}_1 \cap \mathcal{V}_2 \cap (-\mathcal{V}_1) \cap (-\mathcal{V}_2)$, thus $\mathcal{U} = -\mathcal{U}$ (symmetric) and $\mathcal{U} + \mathcal{U} \subset \mathcal{V}$ and hence

$$u_0 + \mathcal{U} + \mathcal{U} \subset u_0 + \mathcal{V} \subset U - \{u_1\} \tag{2.8}$$

so that

$$u_0 + v_1 + v_2 \neq u_1, \quad \forall v_1, v_2 \in \mathcal{U}, \tag{2.9}$$

or

$$u_0 + v_1 \neq u_1 - v_2, \quad \forall v_1, v_2 \in \mathcal{U}, \tag{2.10}$$

and since $\mathcal{U} = -\mathcal{U}$

$$(u_0 + \mathcal{U}) \cap (u_1 + \mathcal{U}) = \emptyset. \quad \square \tag{2.11}$$

Definition 2.3.2 (Bounded Sets). *A set $A \subset U$ is said to be bounded if to each neighborhood of zero $\mathcal{V}$ there corresponds a number $s > 0$ such that $A \subset t\mathcal{V}$ for each $t > s$.*

Definition 2.3.3 (Convex Sets). *A set $A \subset U$ such that*

$$\textit{if } u, v \in A \textit{ then } \lambda u + (1 - \lambda)v \in A, \quad \forall \lambda \in [0, 1], \tag{2.12}$$

is said to be convex.

Definition 2.3.4 (Locally Convex Spaces)**.** *A topological vector space U is said to be locally convex if there is a local base Ω whose elements are convex.*

Definition 2.3.5 (Balanced sets)**.** *A set $A \subset U$ is said to be balanced if $\alpha A \subset A$, $\forall \alpha$ such that $0 < |\alpha| < 1$.*

Theorem 2.3.6. *In a topological vector space U we have:*

1. *Every neighborhood of zero contains a balanced neighborhood of zero,*
2. *Every convex neighborhood of zero contains a balanced convex neighborhood of zero.*

Proof:

1. Suppose $\mathcal{U}$ is a neighborhood of zero. From the continuity of scalar multiplication, there exist $\mathcal{V}$ (neighborhood of zero) and $\delta > 0$, such that $\alpha\mathcal{V} \subset \mathcal{U}$ whenever $|\alpha| < \delta$. Define $\mathcal{W} = \cup_{|\alpha|<\delta}\alpha\mathcal{V}$, thus $\mathcal{W} \subset \mathcal{U}$ is a balanced neighborhood of zero.

2. Suppose $\mathcal{U}$ is a convex neighborhood of zero in U. Define

$$A = \{\cap \alpha\mathcal{U} \quad | \quad \alpha \in \mathbb{C}, \quad |\alpha| = 1\}. \tag{2.13}$$

As $0.\theta = \theta$ (where $\theta \in U$ denotes the zero vector) from the continuity of scalar multiplication there exists $\delta > 0$ and there is a neighborhood of zero $\mathcal{V}$ such that if $|\beta| < \delta$ then $\beta\mathcal{V} \subset \mathcal{U}$. Define $\mathcal{W}$ as the union of all such $\beta\mathcal{V}$. Thus $\mathcal{W}$ is balanced and $\alpha^{-1}\mathcal{W} = \mathcal{W}$ as $|\alpha| = 1$, so that $\mathcal{W} = \alpha\mathcal{W} \subset \alpha\mathcal{U}$, and hence $\mathcal{W} \subset A$, which implies that the interior $\mathring{A}$ is a neighborhood of zero. Also $\mathring{A} \subset \mathcal{U}$. Since A is intersection of convex sets, it is convex and so is $\mathring{A}$. Now will show that $\mathring{A}$ is balanced and complete the proof. For this, it suffices to prove that A is balanced. Choose r and β such that $0 \leq r \leq 1$ and $|\beta| = 1$. Then

$$r\beta A = \cap_{|\alpha|=1} r\beta\alpha\mathcal{U} = \cap_{|\alpha|=1} r\alpha\mathcal{U}. \tag{2.14}$$

Since $\alpha\mathcal{U}$ is a convex set that contains zero, we obtain $r\alpha\mathcal{U} \subset \alpha\mathcal{U}$, so that $r\beta A \subset A$, which completes the proof. □

Proposition 2.3.7. *Let U be a topological vector space and $\mathcal{V}$ a neighborhood of zero in U. Given $u \in U$, there exists $r \in \mathbb{R}^+$ such that $\beta u \in \mathcal{V}$, $\forall \beta$ such that $|\beta| < r$.*

Proof: Observe that $u + \mathcal{V}$ is a neighborhood of $1.u$, then by the continuity of scalar multiplication, there exists $\mathcal{W}$ neighborhood of u and $r > 0$ such that

$$\beta\mathcal{W} \subset u + \mathcal{V}, \forall \beta \text{ such that } |\beta - 1| < r, \tag{2.15}$$

so that

$$\beta u \in u + \mathcal{V}, \tag{2.16}$$

or

$$(\beta - 1)u \in \mathcal{V}, \text{ where } |\beta - 1| < r, \tag{2.17}$$

and thus

$$\hat{\beta}u \in \mathcal{V}, \forall \hat{\beta} \text{ such that } |\hat{\beta}| < r, \tag{2.18}$$

which completes the proof. $\square$

Corollary 2.3.8. *Let $\mathcal{V}$ be a neighborhood of zero in U, if $\{r_n\}$ is a sequence such that $r_n > 0$, $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n = \infty$, then $U \subset \cup_{n=1}^{\infty} r_n\mathcal{V}$.*

Proof: Let $u \in U$, then $\alpha u \in \mathcal{V}$ for α sufficiently small, from the last proposition $u \in \frac{1}{\alpha}\mathcal{V}$. As $r_n \to \infty$ we have that $r_n > \frac{1}{\alpha}$ for n sufficiently big, so that $u \in r_n\mathcal{V}$, which completes the proof. $\square$

Proposition 2.3.9. *Suppose $\{\delta_n\}$ is sequence such that $\delta_n \to 0$, $\delta_n < \delta_{n-1}$, $\forall n \in \mathbb{N}$ and $\mathcal{V}$ a bounded neighborhood of zero in U, then $\{\delta_n\mathcal{V}\}$ is a local base for U.*

Proof: Let $\mathcal{U}$ be a neighborhood of zero, as $\mathcal{V}$ is bounded, there exists $t_0 \in \mathbb{R}^+$ such that $\mathcal{V} \subset t\mathcal{U}$ for any $t > t_0$. As $\lim_{n\to\infty} \delta_n = 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ then $\delta_n < \frac{1}{t_0}$, so that $\delta_n\mathcal{V} \subset \mathcal{U}, \forall n$ such that $n \geq n_0$. $\square$

Definition 2.3.10 (Convergence in topological vector spaces)**.** *Let U be a topological vector space. We say $\{u_n\}$ converges to $u_0 \in U$, if for each neighborhood $\mathcal{V}$ of u_0 then, there exists $N \in \mathbb{N}$ such that*

$$u_n \in \mathcal{V}, \forall n \geq N.$$

2.4 Compactness in Topological Vector Spaces

We start this section with the definition of open covering.

Definition 2.4.1 (Open Covering)**.** *Given $B \subset U$ we say that $\{\mathcal{O}_\alpha, \ \ \alpha \in \mathrm{A}\}$ is a covering of B if $B \subset \cup_{\alpha\in\mathrm{A}}\mathcal{O}_\alpha$. If $\mathcal{O}_\alpha$ is open $\forall \alpha \in \mathrm{A}$ then $\{\mathcal{O}_\alpha\}$ is said to be an open covering of B.*

Definition 2.4.2 (Compact Sets)**.** *A set $B \subset U$ is said to be compact if each open covering of B has a finite sub-covering. More explicitly, if $B \subset \cup_{\alpha\in\mathrm{A}}\mathcal{O}_\alpha$, where $\mathcal{O}_\alpha$ is open $\forall \alpha \in \mathrm{A}$, then, there exist $\alpha_1, ..., \alpha_n \in \mathrm{A}$ such that $B \subset \mathcal{O}_{\alpha_1} \cup ... \cup \mathcal{O}_{\alpha_n}$, for some n, a finite positive integer.*

Proposition 2.4.3. *A compact subset of a Hausdorff space is closed.*

Proof: Let U be a Hausdorff space and consider $A \subset U$, A compact. Given $x \in A$ and $y \in A^c$, there exist open sets $\mathcal{O}_x$ and $\mathcal{O}_y^x$ such that $x \in \mathcal{O}_x$, $y \in \mathcal{O}_y^x$ and $\mathcal{O}_x \cap \mathcal{O}_y^x = \emptyset$. It is clear that $A \subset \cup_{x\in A}\mathcal{O}_x$ and since A is compact, we may find $\{x_1, x_2, ..., x_n\}$ such that

$A \subset \cup_{i=1}^n \mathcal{O}_{x_i}$. For the selected $y \in A^c$ we have $y \in \cap_{i=1}^n \mathcal{O}_y^{x_i}$ and $(\cap_{i=1}^n \mathcal{O}_y^{x_i}) \cap (\cup_{i=1}^n \mathcal{O}_{x_i}) = \emptyset$. Since $\cap_{i=1}^n \mathcal{O}_y^{x_i}$ is open, and y is an arbitrary point of A^c we have that A^c is open, so that A is closed, which completes the proof. $\square$

Proposition 2.4.4. *A closed subset of a compact U space is compact.*

Proof: Consider $\{\mathcal{O}_\alpha\}$ an open cover of A. Thus $\{A^c, \ \ \mathcal{O}_\alpha, \ \ \alpha \in A\}$ is a cover of U. As U is compact, there exist $\alpha_1, \alpha_2, ..., \alpha_n$ such that $A^c \cup (\cup_{i=1}^n \mathcal{O}_{\alpha_i}) \supset U$, so that $\{\mathcal{O}_{\alpha_i}, \ \ i \in \{1, ..., n\}\}$ covers A, so that A is compact. The proof is complete. $\square$

Definition 2.4.5 (Countably Compact Sets)**.** *A set A is said to be countably compact if every infinite subset of A has a limit point in A.*

Proposition 2.4.6. *Every compact subset of a topological space U is countably compact.*

Proof: Let B an infinite subset of A compact and suppose B has no limit point. Choose $\{x_1, x_2, ...\} \subset B$ and define $F = \{x_1, x_2, x_3, ...\}$. It is clear that F has no limit point. Thus for each $n \in \mathbb{N}$, there exist $\mathcal{O}_n$ open such that $\mathcal{O}_n \cap F = \{x_n\}$. Also, for each $x \in A - F$, there exist $\mathcal{O}_x$ such that $x \in \mathcal{O}_x$ and $\mathcal{O}_x \cap F = \emptyset$. Thus $\{\mathcal{O}_x, \ \ x \in A - F, \ \ \mathcal{O}_1, \mathcal{O}_2, ...\}$ is an open cover of A without a finite subcover, which contradicts the fact that A is compact. $\square$

2.5 Normed and Metric Spaces

We start with the definition of norm.

Definition 2.5.1 (Norm)**.** *A vector space U is said to be a normed space, if it is possible to define a function $\|.\|_U : U \to \mathbb{R}^+ = [0, +\infty)$, called a norm, which satisfies the following properties:*

1. $\|u\|_U > 0$, *if* $u \neq \theta$ *and* $\|u\|_U = 0 \Leftrightarrow u = \theta$
2. $\|u + v\|_U \leq \|u\|_U + \|v\|_U, \forall \ \ u, v \in U,$
3. $\|\alpha u\|_U = |\alpha|\|u\|_U, \forall u \in U, \alpha \in \mathbb{F}.$

Now we present the definition of metric.

Definition 2.5.2 (Metric Space)**.** *A vector space U is said to be a metric space if it is possible to define a function $d : U \times U \to \mathbb{R}^+$, called a metric on U, such that*

1. $0 \leq d(u, v) < \infty, \quad \forall u, v \in U,$
2. $d(u, v) = 0 \Leftrightarrow u = v,$

3. $d(u,v) = d(v,u), \quad \forall u,v \in U,$
4. $d(u,w) \leq d(u,v) + d(v,w), \forall u,v,w \in U.$

A metric can be defined through a norm, that is

$$d(u,v) = \|u - v\|_U. \tag{2.19}$$

In this case we say that the metric is induced by the norm.

The set $B_r(u) = \{v \in U \mid d(u,v) < r\}$ is called the open ball with center at u and radius r. A metric $d : U \times U \to \mathbb{R}^+$ is said to be invariant if

$$d(u+w, v+w) = d(u,v), \forall u,v,w \in U. \tag{2.20}$$

The following are some basic definitions concerning metric and normed spaces:

Definition 2.5.3 (Convergent Sequences). *Given a metric space U, we say that $\{u_n\} \subset U$ converges to $u_0 \in U$ as $n \to \infty$, if given any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that if $n \geq n_0$ then $d(u_n, u_0) < \varepsilon$. In this case we write $u_n \to u_0$ as $n \to +\infty$.*

Definition 2.5.4 (Cauchy Sequence). *$\{u_n\} \subset U$ is said to be a Cauchy sequence if given $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(u_n, u_m) < \varepsilon, \forall m,n \geq n_0$*

Definition 2.5.5 (Completeness). *A metric space U is said to be complete if each Cauchy sequence related to $d : U \times U \to \mathbb{R}^+$ converges to an element of U.*

Definition 2.5.6 (Banach Spaces). *A normed vector space U is said to be a Banach Space if each Cauchy sequence related to the metric induced by the norm converges to an element of U.*

Remark 2.5.7. *We say that a topology σ is compatible with a metric d if any $A \subset \sigma$ is represented by unions and/or finite intersections of open balls. In this case we say that $d : U \times U \to \mathbb{R}^+$ induces the topology σ.*

Definition 2.5.8 (Metrizable Spaces). *A topological vector space (U, σ) is said to be metrizable if σ is compatible with some metric d.*

Definition 2.5.9 (Normable Spaces). *A topological vector space (U, σ) is said to be normable if the induced metric (by this norm) is compatible with σ.*

2.6 Linear Mappings

Given U, V topological vector spaces, a function (mapping) $f : U \to V$, $A \subset U$ and $B \subset V$, we define:

$$f(A) = \{f(u) \mid u \in A\}, \tag{2.21}$$

and the inverse image of B, denoted $f^{-1}(B)$ as

$$f^{-1}(B) = \{u \in U \ \ | \ \ f(u) \in B\}. \tag{2.22}$$

Definition 2.6.1 (Linear Functions)**.** *A function $f : U \to V$ is said to be linear if*

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v), \forall u, v \in U, \quad \alpha, \beta \in \mathbb{F}. \tag{2.23}$$

Definition 2.6.2 (Null Space and Range)**.** *Given $f : U \to V$, we define the null space and the range of f, denoted by $N(f)$ and $R(f)$ respectively, as*

$$N(f) = \{u \in U \ \ | \ \ f(u) = \theta\} \tag{2.24}$$

and

$$R(f) = \{v \in V \ \ | \ \ \exists u \in U \ \ such \ \ that \ \ f(u) = v\}. \tag{2.25}$$

Note that $N(f)$ and $R(f)$ are subspaces of U.

Proposition 2.6.3. *Let U, V be topological vector spaces. If $f : U \to V$ is linear and continuous at θ, then it is continuous everywhere.*

Proof: Since f is linear we have $f(\theta) = \theta$. Since f is continuous at θ, given $\mathcal{V} \subset V$ a neighborhood of zero, there exists $\mathcal{U} \subset U$ neighborhood of zero, such that

$$f(\mathcal{U}) \subset \mathcal{V}. \tag{2.26}$$

Thus

$$v - u \in \mathcal{U} \Rightarrow f(v - u) = f(v) - f(u) \in \mathcal{V}, \tag{2.27}$$

or

$$v \in u + \mathcal{U} \Rightarrow f(v) \in f(u) + \mathcal{V}, \tag{2.28}$$

which means that f is continuous at u. Since u is arbitrary, f is continuous everywhere. $\square$

2.7 Linearity and Continuity

Definition 2.7.1 (Bounded Functions)**.** *A function $f : U \to V$ is said to be bounded if it maps bounded sets into bounded sets.*

Proposition 2.7.2. *A set E is bounded if and only if the following condition is satisfied: whenever $\{u_n\} \subset E$ and $\{\alpha_n\} \subset \mathbb{F}$ are such that $\alpha_n \to 0$ as $n \to \infty$ we have $\alpha_n u_n \to \theta$ as $n \to \infty$.*

Proof: Suppose E is bounded. Let $\mathcal{U}$ be a balanced neighborhood of θ in U, then $E \subset t\mathcal{U}$ for some t. For $\{u_n\} \subset E$, as $\alpha_n \to 0$, there exists N such that if $n > N$ then $t < \frac{1}{|\alpha_n|}$. Since $t^{-1}E \subset \mathcal{U}$ and $\mathcal{U}$ is balanced, we have that $\alpha_n u_n \in \mathcal{U}$, $\forall n > N$, and thus $\alpha_n u_n \to \theta$. Conversely, if E is not bounded, there is a neighborhood $\mathcal{V}$ of θ and $\{r_n\}$ such that $r_n \to \infty$ and $E \not\subseteq r_n\mathcal{V}$, that is, we can choose u_n such that $r_n^{-1}u_n$ is not in $\mathcal{V}$, $\forall n \in \mathbb{N}$, so that $\{r_n^{-1}u_n\}$ does not converge to θ. $\square$

Proposition 2.7.3. *Let $f : U \to V$ be a linear function. Consider the following statements*

1. *f is continuous,*
2. *f is bounded,*
3. *If $u_n \to \theta$ then $\{f(u_n)\}$ is bounded,*
4. *If $u_n \to \theta$ then $f(u_n) \to \theta$.*

Then,

- 1 *implies* 2*,*
- 2 *implies* 3*,*
- *if U is metrizable then* 3 *implies* 4*, which implies* 1*.*

Proof:

1. 1 implies 2: Suppose f is continuous, for $\mathcal{W} \subset V$ neighborhood of zero, there exists a neighborhood of zero in U, denoted by $\mathcal{V}$, such that
$$f(\mathcal{V}) \subset \mathcal{W}. \tag{2.29}$$
If E is bounded, there exists $t_0 \in \mathbb{R}^+$ such that $E \subset t\mathcal{V}$, $\forall t \geq t_0$, so that
$$f(E) \subset f(t\mathcal{V}) = tf(\mathcal{V}) \subset t\mathcal{W}, \quad \forall t \geq t_0, \tag{2.30}$$
and thus f is bounded.

2. 2 implies 3: Suppose $u_n \to \theta$ and let $\mathcal{W}$ be a neighborhood of zero. Then there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $u_n \in \mathcal{V} \subset \mathcal{W}$ where $\mathcal{V}$ is a balanced neighborhood of zero. On the other hand, for $n < N$, there exists K_n such that $u_n \in K_n\mathcal{V}$. Define $K = \max\{1, K_1, ..., K_n\}$. Then $u_n \in K\mathcal{V}, \forall n \in \mathbb{N}$ and hence u_n is bounded, a finally from 2, we have that $\{f(u_n)\}$ is bounded.

3. 3 implies 4: Suppose U is metrizable and let $u_n \to \theta$. Given $K \in \mathbb{N}$, there exists $n_K \in \mathbb{N}$ such that if $n > n_K$ then $d(u_n, \theta) < \frac{1}{K^2}$. Define $\gamma_n = 1$ if $n < n_1$ and $\gamma_n = K$, if $n_K \leq n < n_{K+1}$ so that
$$d(\gamma_n u_n, \theta) = d(K u_n, \theta) \leq K d(u_n, \theta) < K^{-1}. \tag{2.31}$$
Thus since 2 implies 3 we have $\{f(\gamma_n u_n)\}$ is bounded so that, by Proposition 2.7.2 $f(u_n) = \gamma_n^{-1} f(\gamma_n u_n) \to \theta$ as $n \to \infty$.

4. 4 implies 1: suppose 1 fails. Thus there exists a neighborhood of zero $\mathcal{W} \subset V$ such that $f^{-1}(\mathcal{W})$ contains no neighborhood of zero in U. Particularly, we can select $\{u_n\}$ such that $u_n \in B_{1/n}(\theta)$ and $f(u_n)$ not in $\mathcal{W}$ so that $\{f(u_n)\}$ does not converge to zero. Thus 4 fails. $\square$

2.8 Some Classical Results in Banach Spaces

Now we state some very important theorems in Banach spaces, which the proofs may be found in [27]. We start with the definition of nowhere dense set.

Definition 2.8.1 (Nowhere Dense Sets). *A set S in a metric space M is called nowhere dense if $\bar{S}$ has an empty interior.*

Theorem 2.8.2 (Baire Category Theorem). *A complete metric space is never the union of a countable number of nowhere dense sets.*

Theorem 2.8.3 (Principle of Uniform Boundedness). *Let U be a Banach space. Let $\mathcal{F}$ be a family of bounded linear transformations from U to a normed linear space V. Suppose for each $u \in U$, $\{\|Tu\|_V \mid T \in \mathcal{F}\}$ is bounded. Then $\{\|T\| \mid T \in \mathcal{F}\}$ is bounded.*

Theorem 2.8.4 (Open Mapping Theorem). *Let $T : U \to V$ be a bounded linear transformation of the Banach space U onto the Banach space V. Then if M is open in U then $T(M)$ is open in V.*

Theorem 2.8.5 (Inverse Mapping Theorem). *A continuous bijection of one Banach space onto another has a continuous inverse.*

Here we introduce the definition of graph and finish this chapter by stating the Closed Graph Theorem.

Definition 2.8.6 (Graph of a Mapping). *Let T be a mapping of a normed linear space U into a normed linear space V. The **graph** of T, denoted by $\Gamma(T)$, is defined as*
$$\Gamma(T) = \{(u, v) \in U \times V \mid v = T(u)\}.$$

Theorem 2.8.7 (The closed Graph Theorem). *Let U and V be Banach spaces and T a linear map of U into V. Then T is bounded if and only if its graph is closed.*

Chapter 3

The Hahn-Banach Theorems and Weak Topologies

3.1 Introduction

The notion of weak topologies and weak convergence is fundamental in the modern variational analysis. Many important problems are non-convex and have no minimizers in the classical sense. However the minimizing sequences in reflexive spaces may be weakly convergent, and it is important to evaluate the average behavior of such sequences in many practical applications.

3.2 The Hahn-Banach Theorem

In this chapter U denotes a Banach space, unless otherwise indicated. We start this section by stating and proving the Hahn-Banach theorem for real vector spaces, which is sufficient for our purposes.

Theorem 3.2.1 (The Hahn-Banach Theorem)**.** *Consider a functional $p : U \to \mathbb{R}$ satisfying*

$$p(\lambda u) = \lambda p(u), \forall u \in U, \lambda > 0, \tag{3.1}$$

$$p(u+v) \leq p(u) + p(v), \forall u, v \in U. \tag{3.2}$$

Let $V \subset U$ a vector subspace and let $g : V \to \mathbb{R}$ be a linear functional such that

$$g(u) \leq p(u), \forall u \in V. \tag{3.3}$$

Then there exists a linear functional $f : U \to \mathbb{R}$ such that

$$g(u) = f(u), \forall u \in V, \tag{3.4}$$

and

$$f(u) \leq p(u), \forall u \in U. \tag{3.5}$$

Proof: Pick $z \in U - V$. Denote by $\tilde{V}$ the space spanned by V and z, that is

$$\tilde{V} = \{v + \alpha z \mid v \in V \quad and \quad \alpha \in \mathbb{R}\}. \tag{3.6}$$

We may define an extension of g to $\tilde{V}$, denoted by $\tilde{g}$, as

$$\tilde{g}(\alpha z + v) = \alpha \tilde{g}(z) + g(v), \tag{3.7}$$

where $\tilde{g}(z)$ will be appropriately defined. Suppose given $v_1, v_2 \in V$, $\alpha > 0$, $\beta > 0$. Then

$$\begin{aligned} \beta g(v_1) + \alpha g(v_2) &= g(\beta v_1 + \alpha v_2) \\ &= (\alpha+\beta) g(\frac{\beta}{\alpha+\beta} v_1 + \frac{\alpha}{\alpha+\beta} v_2) \\ &\leq (\alpha+\beta) p(\frac{\beta}{\alpha+\beta}(v_1 - \alpha z) + \frac{\alpha}{\alpha+\beta}(v_2 + \alpha z)) \\ &\leq \beta p(v_1 - \alpha z) + \alpha p(v_2 + \beta z) \end{aligned} \tag{3.8}$$

and therefore

$$\frac{1}{\alpha}[-p(v_1 - \alpha z) + g(v_1)] \leq \frac{1}{\beta}[p(v_2 + \beta z) - g(v_2)], \forall v_1, v_2 \in V, \quad \alpha, \beta > 0. \tag{3.9}$$

Thus, there exists $a \in \mathbb{R}$ such that

$$\sup_{v \in V, \alpha > 0} [\frac{1}{\alpha}(-p(v - \alpha z) + g(v))] \leq a \leq \inf_{v \in V, \alpha > 0} [\frac{1}{\alpha}(p(v + \alpha z) - g(v))]. \tag{3.10}$$

If we define $\tilde{g}(z) = a$ we obtain $\tilde{g}(u) \leq p(u), \forall u \in \tilde{V}$. Define by $\mathcal{E}$ the set of extensions e of g, which satisfy $e(u) \leq p(u)$ on the subspace where e is defined. We define a partial order in $\mathcal{E}$ by setting $e_1 \prec e_2$ if e_2 is defined in a larger set than e_1 and $e_1 = e_2$ where both are defined. Let $\{e_\alpha\}_{\alpha \in A}$ be a linearly ordered subset of $\mathcal{E}$. Let V_α be the subspace on which e_α is defined. Define e on $\cup_{\alpha \in A} V_\alpha$ by setting $e(u) = e_\alpha$ on V_α. Clearly $e_\alpha \prec e$ so each linearly ordered set of $\mathcal{E}$ has an upper bound. By the Zorn's lemma, $\mathcal{E}$ has a maximal element f defined on some set $\tilde{U}$ such that $f(u) \leq p(u), \forall u \in \tilde{U}$. We can conclude that $\tilde{U} = U$, otherwise if there was an $z_1 \in U - \tilde{U}$, as above we could have a new extension f_1 to the subspace spanned by z_1 and $\tilde{U}$, contradicting the maximality of f. $\square$

Definition 3.2.2 (Topological Dual Space)**.** *For a Banach space U, we define its Topological Dual Space, as the set of all linear continuous functionals defined on U. We suppose that such dual space of U, may be identified with a space denoted by U^* through a bilinear form*

$\langle .,. \rangle_U : U \times U^* \to \mathbb{R}$. *That is, given* $f : U \to \mathbb{R}$ *linear continuous functional, there exists* $u^* \in U^*$ *such that*

$$f(u) = \langle u, u^* \rangle_U, \forall u \in U. \tag{3.11}$$

The norm of f *, denoted by* $\|f\|_{U^*}$*, is defined as*

$$\|f\|_{U^*} = \sup_{u \in U}\{|\langle u, u^* \rangle_U| \ \ | \ \ \|u\|_U \leq 1\}. \tag{3.12}$$

Corollary 3.2.3. *Let* $V \subset U$ *a vector subspace of* U *and let* $g : V \to \mathbb{R}$ *a linear continuous functional of norm*

$$\|g\|_{V^*} = \sup_{u \in U}\{|g(u)| \ \ | \ \ \|u\|_V \leq 1\}. \tag{3.13}$$

Then, there exists an u^* *in* U^* *such that*

$$\langle u, u^* \rangle_U = g(u), \forall u \in V, \tag{3.14}$$

and

$$\|u^*\|_{U^*} = \|g\|_{V^*}. \tag{3.15}$$

Proof: Apply Theorem 3.2.1 with $p(x) = \|g\|_{V^*}\|u\|_V$. $\square$

Corollary 3.2.4. *Given* $u_0 \in U$ *there exists* $u_0^* \in U^*$ *such that*

$$\|u_0^*\|_{U^*} = \|u_0\|_U \ \ \textit{and} \ \ \langle u_0, u_0^* \rangle_U = \|u_0\|_U^2. \tag{3.16}$$

Proof: Apply Corollary 3.2.3 with $V = \{\alpha u_0 \ \ | \ \ \alpha \in \mathbb{R}\}$ and $g(tu_0) = t\|u_0\|_U^2$ so that $\|g\|_{V^*} = \|u_0\|_U$. $\square$

Corollary 3.2.5. *Given* $u \in U$ *we have*

$$\|u\|_U = \sup_{u^* \in U^*}\{|\langle u, u^* \rangle_U| \ \ | \ \ \|u^*\|_{U^*} \leq 1\}. \tag{3.17}$$

Proof: Suppose $u \neq \theta$. Since

$$|\langle u, u^* \rangle_U| \leq \|u\|_U\|u^*\|_{U^*}, \forall u \in U, u^* \in U^*$$

we have

$$\sup_{u^* \in U^*}\{|\langle u, u^* \rangle_U| \ \ | \ \ \|u^*\|_{U^*} \leq 1\} \leq \|u\|_U. \tag{3.18}$$

However, from last corollary we have that there exists $u_0^* \in U^*$ such that $\|u_0^*\|_{U^*} = \|u\|_U$ and $\langle u, u_0^* \rangle_U = \|u\|_U^2$. Define $u_1^* = \|u\|_U^{-1}u_0^*$. Then $\|u_1^*\|_U = 1$ and $\langle u, u_1^* \rangle_U = \|u\|_U$. $\square$

Definition 3.2.6 (Affine Hyper-Plane)**.** *Let* U *be a Banach space. An affine hyper-plane* H *is a set of the form*

$$H = \{u \in U \ \ | \ \ \langle u, u^* \rangle_U = \alpha\} \tag{3.19}$$

for some $u^* \in U^*$ *and* $\alpha \in \mathbb{R}$.

Proposition 3.2.7. *A hyper-plane H defined as above is closed.*

Proof: The result follows from the continuity of $\langle u, u^* \rangle_U$ as a functional defined in U. $\square$

Definition 3.2.8 (Separation)**.** *Given $A, B \subset U$ we say that a hyper-plane H, defined as above separates A and B if*

$$\langle u, u^* \rangle_U \leq \alpha, \forall u \in A, \quad \textit{and} \quad \langle u, u^* \rangle_U \geq \alpha, \forall u \in B. \tag{3.20}$$

We say that H separates A and B strictly if there exists $\varepsilon > 0$ such that

$$\langle u, u^* \rangle_U \leq \alpha - \varepsilon, \forall u \in A, \quad \textit{and} \quad \langle u, u^* \rangle_U \geq \alpha + \varepsilon, \forall u \in B, \tag{3.21}$$

Theorem 3.2.9 (Hahn-Banach theorem, geometric form)**.** *Consider $A, B \subset U$ two convex disjoint non-empty sets, where A is open. Then there exists a closed hyper-plane that separates A and B.*

We need the following Lemma.

Lemma 3.2.10. *Consider $C \subset U$ a convex open set such that $\theta \in C$. Given $u \in U$, define*

$$p(u) = \inf\{\alpha > 0, \quad \alpha^{-1} u \in C\}. \tag{3.22}$$

Thus, p is such that there exists $M \in \mathbb{R}^+$ satisfying

$$0 \leq p(u) \leq M\|u\|_U, \forall u \in U, \tag{3.23}$$

and

$$C = \{u \in U \;\;|\;\; p(u) < 1\}. \tag{3.24}$$

Also

$$p(u+v) \leq p(u) + p(v), \forall u, v \in U.$$

Proof: Let $r > 0$ be such that $B(\theta, r) \subset C$, thus

$$p(u) \leq \frac{\|u\|_U}{r}, \forall u \in U \tag{3.25}$$

which proves (3.23). Now suppose $u \in C$. Since C is open $(1+\varepsilon)u \in C$ for ε sufficiently small. Therefore $p(u) \leq \frac{1}{1+\varepsilon} < 1$. Conversely, if $p(u) < 1$ there exists $0 < \alpha < 1$ such that $\alpha^{-1}u \in C$ and therefore, since C is convex, $u = \alpha(\alpha^{-1}u) + (1-\alpha)\theta \in C$.

Also, let $u, v \in C$ and $\varepsilon > 0$. Thus $\frac{u}{p(u)+\varepsilon} \in C$ and $\frac{v}{p(v)+\varepsilon} \in C$ so that $\frac{tu}{p(u)+\varepsilon} + \frac{(1-t)v}{p(v)+\varepsilon} \in C, \forall t \in [0,1]$. Particularly for $t = \frac{p(u)+\varepsilon}{p(u)+p(v)+2\varepsilon}$ we obtain $\frac{u+v}{p(u)+p(v)+2\varepsilon} \in C$, which means $p(u+v) \leq p(u) + p(v) + 2\varepsilon, \forall \varepsilon > 0$ $\square$

Lemma 3.2.11. *Consider $C \subset U$ a convex open set and let $u_0 \in U$ be a vector not in C. Then there exists $u^* \in U^*$ such that $\langle u, u^* \rangle_U < \langle u_0, u^* \rangle_U, \forall u \in C$*

Proof: By a translation, we may assume $\theta \in C$. Consider the functional p as in the last lemma. Define $V = \{\alpha u_0 \mid \alpha \in \mathbb{R}\}$. Define g on V, by

$$g(tu_0) = t, \quad t \in \mathbb{R}. \tag{3.26}$$

We have that $g(u) \leq p(u), \forall u \in V$. From the Hahn-Banach theorem, there exist a linear functional f on U which extends g such that

$$f(u) \leq p(u) \leq M\|u\|_U. \tag{3.27}$$

Here we have used lemma 3.2.10. In Particular, $f(u_0) = 1$, and (also from the last lemma) $f(u) < 1, \forall u \in C$. The existence of u^* satisfying the theorem follows from the continuity of f indicated in (3.27). $\square$

Proof of Theorem 3.2.9 Define $C = A + (-B)$ so that C is convex and θ is not in C. From Lemma 3.2.11, there exists $u^* \in U^*$ such that $\langle w, u^* \rangle_U < 0, \forall w \in C$, which means

$$\langle u, u^* \rangle_U < \langle v, u^* \rangle_U, \forall u \in A, \quad v \in B. \tag{3.28}$$

Thus, there exists $\alpha \in \mathbb{R}$ such that

$$\sup_{u \in A} \langle u, u^* \rangle_U \leq \alpha \leq \inf_{v \in B} \langle v, u^* \rangle_U, \tag{3.29}$$

which completes the proof. $\square$

Theorem 3.2.12 (Hahn-Banach theorem, second geometric form)**.** *Consider $A, B \subset U$ two convex disjoint non-empty sets. Suppose A is closed and B is compact. Then there exists an hyper-plane which separates A and B strictly.*

Proof: There exists $\varepsilon > 0$ sufficiently small such that $A_\varepsilon = A + B(0, \varepsilon)$ and $B_\varepsilon = B + B(0, \varepsilon)$ are convex disjoint sets. From Theorem 3.2.9, there exists $u^* \in U^*$ such that $u^* \neq \theta$ and

$$\langle u + \varepsilon w_1, u^* \rangle_U \leq \langle u + \varepsilon w_2, u^* \rangle_U, \forall u \in A, \quad v \in B, \quad w_1, w_2 \in B(0, 1). \tag{3.30}$$

Thus, there exists $\alpha \in \mathbb{R}$ such that

$$\langle u, u^* \rangle_U + \varepsilon \|u^*\|_{U^*} \leq \alpha \leq \langle v, u^* \rangle_U - \varepsilon \|u^*\|_{U^*}, \forall u \in A, \quad v \in B. \ \square \tag{3.31}$$

Corollary 3.2.13. *Suppose $V \subset U$ is a vector subspace such that $\bar{V} \neq U$. Then there exists $u^* \in U^*$ such that $u^* \neq \theta$ and*

$$\langle u, u^* \rangle_U = 0, \forall u \in V. \tag{3.32}$$

Proof: Consider $u_0 \in U$ such that u_0 does not belong to $\bar{V}$. Applying Theorem 3.2.9 to $A = \bar{V}$ and $B = \{u_0\}$ we obtain $u^* \in U^*$ and $\alpha \in \mathbb{R}$ such that $u^* \neq \theta$ and

$$\langle u, u^* \rangle_U < \alpha < \langle u_0, u^* \rangle_U, \forall u \in V. \tag{3.33}$$

Since V is a subspace we must have $\langle u, u^* \rangle_U = 0, \forall u \in V$. $\square$.

3.3 Weak Topologies

Definition 3.3.1 (Weak Neighborhoods and Weak Topologies)**.** *For the topological space U and $u_0 \in U$, we define a weak neighborhood of u_0, denoted by $\mathcal{V}_w$ as*

$$\mathcal{V}_w = \{u \in U \ \ | \ \ |\langle u - u_0, u_i^* \rangle_U| < \varepsilon, \forall i \in \{1, ..., m\}\}, \tag{3.34}$$

for some $m \in \mathbb{N}$, $\varepsilon > 0$, and $u_i^* \in U^*$, $\forall i \in \{1, ..., m\}$. Also, we define the weak topology for U, denoted by $\sigma(U, U^*)$ as the set of arbitrary unions and finite intersections of weak neighborhoods in U.

Proposition 3.3.2. *Consider Z a topological vector space and ψ a function of Z into U. Then ψ is continuous on U endowed with the weak topology, if and only if $u^* \circ \psi$ is continuous, for all $u^* \in U^*$.*

Proof: It is clear that if ψ is continuous with U endowed with the weak topology, then $u^* \circ \psi$ is continuous for all $u^* \in U^*$. Conversely, consider $\mathcal{U}$ a weakly open set in U. We have to show that $\psi^{-1}(\mathcal{U})$ is open in Z. But observe that $\mathcal{U} = \cup_{\lambda \in L} \mathcal{V}_\lambda$, where each $\mathcal{V}_\lambda$ is a weak neighborhood. Thus $\psi^{-1}(\mathcal{U}) = \cup_{\lambda \in L} \psi^{-1}(\mathcal{V}_\lambda)$. The result follows considering that $u^* \circ \psi$ is continuous for all $u^* \in U^*$, so that $\psi^{-1}(\mathcal{V}_\lambda)$ is open, for all $\lambda \in L$. $\square$

Proposition 3.3.3. *A Banach space U is Hausdorff as endowed with the weak topology $\sigma(U, U^*)$.*

Proof: Pick $u_1, u_2 \in U$ such that $u_1 \neq u_2$. From the Hahn-Banach theorem, second geometric form, there exists a hyper-plane separating $\{u_1\}$ and $\{u_2\}$. That is, there exist $u^* \in U^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle u_1, u^* \rangle_U < \alpha < \langle u_2, u^* \rangle_U. \tag{3.35}$$

Defining

$$\mathcal{V}_{w1} = \{u \in U \ \ | \ \ |\langle u - u_1, u^* \rangle| < \alpha - \langle u_1, u^* \rangle_U\}, \tag{3.36}$$

and

$$\mathcal{V}_{w2} = \{u \in U \ \ | \ \ |\langle u - u_2, u^* \rangle_U| < \langle u_2, u^* \rangle_U - \alpha\}, \tag{3.37}$$

we obtain $u_1 \in \mathcal{V}_{w1}$, $u_2 \in \mathcal{V}_{w2}$ and $\mathcal{V}_{w1} \cap \mathcal{V}_{w2} = \emptyset$. $\square$

Remark 3.3.4. *if $\{u_n\} \in U$ is such that u_n converges to u in $\sigma(U, U^*)$ then we write $u_n \rightharpoonup u$.*

Proposition 3.3.5. *Let U be a Banach space. Considering $\{u_n\} \subset U$ we have*

1. *$u_n \rightharpoonup u$, for $\sigma(U, U^*) \Leftrightarrow \langle u_n, u^* \rangle_U \to \langle u, u^* \rangle_U, \forall u^* \in U^*$,*

2. *If $u_n \to u$ strongly (in norm) then $u_n \rightharpoonup u$ weakly,*

3. *If $u_n \rightharpoonup u$ weakly, then $\{\|u_n\|_U\}$ is bounded and $\|u\|_U \leq \liminf_{n \to \infty} \|u_n\|_U$,*

4. If $u_n \rightharpoonup u$ weakly and $u_n^* \to u^*$ strongly in U^* then $\langle u_n, u_n^* \rangle_U \to \langle u, u^* \rangle_U$.

Proof:

1. The result follows directly from the definition of topology $\sigma(U, U^*)$.
2. This follows from the inequality
$$|\langle u_n, u^* \rangle_U - \langle u, u^* \rangle_U| \leq \|u^*\|_{U^*} \|u_n - u\|_U. \tag{3.38}$$
3. Since for every $u^* \in U^*$ the sequence $\{\langle u_n, u^* \rangle_U\}$ is bounded, from the uniform boundedness principle we have that there exists $M > 0$ such that $\|u_n\|_U \leq M, \forall n \in \mathbb{N}$. Furthermore, for $u^* \in U^*$ we have
$$|\langle u_n, u^* \rangle_U| \leq \|u^*\|_{U^*} \|u_n\|_U, \tag{3.39}$$
and taking the limit, we obtain
$$|\langle u, u^* \rangle_U| \leq \liminf_{n \to \infty} \|u^*\|_{U^*} \|u_n\|_U. \tag{3.40}$$
Thus
$$\|u\|_U = \sup_{\|u\|_{U^*} \leq 1} |\langle u, u^* \rangle_U| \leq \liminf_{n \to \infty} \|u^*\|_{U^*} \|u_n\|_U. \tag{3.41}$$
4. Just observe that
$$|\langle u_n, u_n^* \rangle_U - \langle u, u^* \rangle_U| \leq |\langle u_n, u_n^* - u^* \rangle_U| + |\langle u - u_n, u^* \rangle_U| \leq$$
$$\|u_n^* - u^*\|_{U^*} \|u_n\|_U + |\langle u_n - u, u^* \rangle_U|. \quad \square$$

Theorem 3.3.6. *Consider $A \subset U$ a convex set. Thus A is weakly closed if and only if it is strongly closed.*

Proof: Suppose A is strongly closed. Consider u_0 not in A. By the Hahn-Banach theorem there exists a closed hyper-plane which separates u_0 and A strictly. Therefore there exists $\alpha \in \mathbb{R}$ and $u^* \in U^*$ such that
$$\langle u_0, u^* \rangle_U < \alpha < \langle v, u^* \rangle_U, \forall v \in A. \tag{3.42}$$
Define
$$\mathcal{V} = \{u \in U \mid \langle u, u^* \rangle_U < \alpha\}, \tag{3.43}$$
so that $u_0 \in \mathcal{V}$, $\mathcal{V} \subset U - A$. Since $\mathcal{V}$ is open for $\sigma(U, U^*)$ we have that $U - A$ is weakly open, hence A is weakly closed. The converse is obvious. $\square$

3.4 The Weak-star Topology

Definition 3.4.1 (Reflexive Spaces). *Let U be a Banach space. We say that U is reflexive if the canonical injection $J : U \to U^{**}$ defined by*

$$\langle u, u^* \rangle_U = \langle u^*, J(u) \rangle_{U^*}, \forall u \in U, \quad u^* \in U^*, \tag{3.44}$$

is onto.

The weak topology for U^* is denoted by $\sigma(U^*, U^{**})$. By analogy, we can define the topology $\sigma(U^*, U)$, which is called the weak-star topology. A standard neighborhood of $u_0^* \in U^*$ for the weak-star topology, which we denoted by $\mathcal{V}_{w^*}$, is given by

$$\mathcal{V}_{w^*} = \{u^* \in U^* \ \mid \ |\langle u_i, u^* - u_0^* \rangle_U| < \varepsilon, \forall i \in \{1, ..., m\}\} \tag{3.45}$$

for some $\varepsilon > 0$, $m \in \mathbb{N}$, $u_i \in U, \forall i \in \{1, ..., m\}$. It is clear that the weak topology for U^* and the weak-star topology coincide if U is reflexive.

Proposition 3.4.2. *Let U be a Banach space. U^* as endowed with the weak-star topology is a Hausdorff space.*

Proof: The proof similar to that of Proposition 3.3.3. $\square$

3.5 Weak-star Compactness

We start with an important theorem about weak-* compactness.

Theorem 3.5.1 (Banach Alaoglu Theorem). *The set $B_{U^*} = \{f \in U^* \ \mid \ \|f\|_{U^*} \leq 1\}$ is compact for the topology $\sigma(U^*, U)$ (the weak-star topology).*

Proof: For each $u \in U$, we will associate a real number ω_u and denote $\omega = \prod_{u \in U} \omega_u$. We have that $\omega \in \mathbb{R}^U$ and let us consider the projections $P_u : \mathbb{R}^U \to \mathbb{R}$, where $P_u(\omega) = \omega_u$. Consider the weakest topology σ for which the functions P_u $(u \in U)$ are continuous. For U^*, with the topology $\sigma(U^*, U)$ define $\phi : U^* \to \mathbb{R}^U$, by

$$\phi(u^*) = \prod_{u \in U} \langle u, u^* \rangle_U, \forall u^* \in U^*. \tag{3.46}$$

Since for each fixed u the mapping $u^* \to \langle u, u^* \rangle_U$ is weakly-star continuous, we see that, ϕ is σ continuous, since weak-star convergence and convergence in σ are equivalent in U^*. To prove that ϕ^{-1} is continuous, from Proposition 3.3.2, it suffices to show that the function

$\omega \to \langle u, \phi^{-1}(\omega)\rangle_U$ is continuous on $\phi(U^*)$. This is true because $\langle u, \phi^{-1}(\omega)\rangle_U = \omega_u$ on $\phi(U^*)$. On the other hand, it is also clear that $\phi(B_{U^*}) = K$ where

$$K = \{\omega \in \mathbb{R}^U \mid |\omega_u| \leq \|u\|_U, \ \ \omega_{u+v} = \omega_u + \omega_v, \ \ \omega_{\lambda u} = \lambda \omega_u, \forall u, v \in U, \ \ \lambda \in \mathbb{R}\}. \quad (3.47)$$

To finish the proof, it is sufficient, from the continuity of ϕ^{-1}, to show that K is compact in $\mathbb{R}^U$, concerning the topology σ. Observe that $K = K_1 \cap K_2$ where

$$K_1 = \{\omega \in \mathbb{R}^U \mid |\omega_u| \leq \|u\|_U, \forall u \in U\}, \quad (3.48)$$

and

$$K_2 = \{\omega \in \mathbb{R}^U \mid \omega_{u+v} = \omega_u + \omega_v, \ \ \omega_{\lambda u} = \lambda \omega_u, \forall u, v \in U, \ \ \lambda \in \mathbb{R}\}. \quad (3.49)$$

The set $\prod_{u \in U}[-\|u\|_U, \|u\|_U]$ is compact as a Cartesian product of compact intervals. Since $K_1 \subset K$ and K_1 is closed, we have that K_1 is compact (for the topology in question) . On the other hand, K_2 is closed, because defining the closed sets $A_{u,v}$ and $B_{\lambda,u}$ as

$$A_{u,v} = \{\omega \in \mathbb{R}^U \mid \omega_{u+v} - \omega_u - \omega_v = 0\}, \quad (3.50)$$

and

$$B_{\lambda,u} = \{\omega \in \mathbb{R}^U \ \ \omega_{\lambda u} - \lambda \omega_u = 0\} \quad (3.51)$$

we may write

$$K_2 = (\cap_{u,v \in U} A_{u,v}) \cap (\cap_{(\lambda,u) \in \mathbb{R} \times U} B_{\lambda,u}). \quad (3.52)$$

We recall that the K_2 is closed because arbitrary intersections of closed sets are closed. Finally, we have that $K_1 \cap K_2$ is compact, which completes the proof. $\square$

Theorem 3.5.2 (Kakutani). *Let U be a Banach space. Then U is reflexive if and only if*

$$B_U = \{u \in U \mid \|u\|_U \leq 1\} \quad (3.53)$$

is compact for the weak topology $\sigma(U, U^*)$.

Proof: Suppose U is reflexive, then $J(B_U) = B_{U^{**}}$. From the last theorem $B_{U^{**}}$ is compact for the topology $\sigma(U^{**}, U^*)$. Therefore it suffices to verify that $J^{-1} : U^{**} \to U$ is continuous from U^{**} with the topology $\sigma(U^{**}, U^*)$ to U, with the topology $\sigma(U, U^*)$.

From Proposition 3.3.2 it is sufficient to show that the function $u \mapsto \langle f, J^{-1}u\rangle_U$ is continuous for the topology $\sigma(U^{**}, U^*)$, for each $f \in U^*$. Since $\langle f, J^{-1}u\rangle_U = \langle u, f\rangle_{U^*}$ we have completed the first part of the proof. For the second we need two lemmas.

Lemma 3.5.3 (Helly). *Let U be a Banach space, $f_1, ..., f_n \in U^*$ and $\alpha_1, ..., \alpha_n \in \mathbb{R}$, then* 1 *and* 2 *are equivalent, where:*

1.
$$\text{Given } \varepsilon > 0, \text{ there exists } u_\varepsilon \in U \text{ such that } \|u_\varepsilon\|_U \leq 1 \text{ and}$$
$$|\langle u_\varepsilon, f_i\rangle_U - \alpha_i| < \varepsilon, \forall i \in \{1, ..., n\}.$$

2.

$$\left|\sum_{i=1}^n \beta_i \alpha_i\right| \leq \left\|\sum_{i=1}^n \beta_i f_i\right\|_{U^*}, \forall \beta_1, ..., \beta_n \in \mathbb{R}. \tag{3.54}$$

Proof: $1 \Rightarrow 2$: *Fix* $\beta_1, ..., \beta_n \in \mathbb{R}$, $\varepsilon > 0$ *and define* $S = \sum_{i=1}^n |\beta_i|$. *From 1, we have*

$$\left|\sum_{i=1}^n \beta_i \langle u_\varepsilon, f_i \rangle_U - \sum_{i=1}^n \beta_i \alpha_i\right| < \varepsilon S \tag{3.55}$$

and therefore

$$\left|\sum_{i=1}^n \beta_i \alpha_i\right| - \left|\sum_{i=1}^n \beta_i \langle u_\varepsilon, f_i \rangle_U\right| < \varepsilon S \tag{3.56}$$

or

$$\left|\sum_{i=1}^n \beta_i \alpha_i\right| < \left\|\sum_{i=1}^n \beta_i f_i\right\|_{U^*} \|u_\varepsilon\|_U + \varepsilon S \leq \left\|\sum_{i=1}^n \beta_i f_i\right\|_{U^*} + \varepsilon S \tag{3.57}$$

so that

$$\left|\sum_{i=1}^n \beta_i \alpha_i\right| \leq \left\|\sum_{i=1}^n \beta_i f_i\right\|_{U^*} \tag{3.58}$$

since ε is arbitrary.

Now let us show that $2 \Rightarrow 1$. Define $\vec{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ and consider the function $\varphi(u) = (\langle f_1, u \rangle_U, ..., \langle f_n, u \rangle_U)$. Item 1 implies that $\vec{\alpha}$ belongs to the closure of $\varphi(B_U)$. Let us suppose that $\vec{\alpha}$ does not belong to the closure of $\varphi(B_u)$ and obtain a contradiction. Thus we can separate $\vec{\alpha}$ and the closure of $\varphi(B_u)$ strictly, that is there exists $\vec{\beta} = (\beta_1, ..., \beta_n) \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ such that

$$\varphi(u).\vec{\beta} < \gamma < \vec{\alpha}.\vec{\beta}, \forall u \in B_U \tag{3.59}$$

Taking the supremum in u we contradict 2.

Also we need the lemma.

Lemma 3.5.4. *Let U be a Banach space. Then $J(B_U)$ is dense in $B_{U^{**}}$ for the topology $\sigma(U^{**}, U^*)$.*

Proof: Let $u^{**} \in B_{U^{**}}$ and consider $\mathcal{V}_{u^{**}}$ a neighborhood of u^{**} for the topology $\sigma(U^{**}, U^*)$. It suffices to show that $J(B_U) \cap \mathcal{V}_{u^{**}} \neq \emptyset$. As $\mathcal{V}_{u^{**}}$ is a weak neighborhood, there exists $f_1, ..., f_n \in U^*$ and $\varepsilon > 0$ such that

$$\mathcal{V}_{u^{**}} = \{\eta \in U^{**} \ \mid \ \langle f_i, \eta - u^{**} \rangle_{U^*}| < \varepsilon, \forall i \in \{1, ..., n\}\}. \tag{3.60}$$

Define $\alpha_i = \langle f_i, u^{**}\rangle_{U^*}$ and thus for any given $\beta_1, ..., \beta_n \in \mathbb{R}$ we have

$$\left|\sum_{i=1}^{n} \beta_i \alpha_i\right| = \left|\langle \sum_{i=1}^{n} \beta_i f_i, u^{**}\rangle_{U^*}\right| \leq \left\|\sum_{i=1}^{n} \beta_i f_i\right\|_{U^*}, \tag{3.61}$$

so that from Helly lemma, there exists $u_\varepsilon \in U$ such that $\|u_\varepsilon\|_U \leq 1$ and

$$|\langle u_\varepsilon, f_i\rangle_U - \alpha_i| < \varepsilon, \forall i \in \{1, ..., n\} \tag{3.62}$$

or,

$$|\langle f_i, J(u_\varepsilon) - u^{**}\rangle_{U^*}| < \varepsilon, \forall i \in \{1, ..., n\} \tag{3.63}$$

and hence

$$J(u_\varepsilon) \in \mathcal{V}_{u^{**}}. \ \Box \tag{3.64}$$

Now we will complete the proof of Kakutani Theorem. Suppose B_U is weakly compact (that is, compact for the topology $\sigma(U, U^*)$). Observe that $J : U \to U^{**}$ is weakly continuous, that is, it is continuous with U endowed with the topology $\sigma(U, U^*)$ and U^{**} endowed with the topology $\sigma(U^{**}, U^*)$. Thus as B_U is weakly compact, we have that $J(B_U)$ is compact for the topology $\sigma(U^{**}, U^*)$. From the last lemma, $J(B_U)$ is dense $B_{U^{**}}$ for the topology $\sigma(U^{**}, U^*)$. Hence $J(B_U) = B_{U^{**}}$, or $J(U) = U^{**}$, which completes the proof. $\Box$

Proposition 3.5.5. *Let U be a reflexive Banach space. Let $K \subset U$ be a convex closed bounded set. Then K is weakly compact.*

Proof: From Theorem 3.3.6, K is weakly closed (closed for the topology $\sigma(U, U^*)$). Since K is bounded, there exists $\alpha \in \mathbb{R}^+$ such that $K \subset \alpha B_U$. Since K is weakly closed and $K = K \cap \alpha B_U$, we have that it is weakly compact. $\Box$

Proposition 3.5.6. *Let U be a reflexive Banach space and $M \subset U$ a closed subspace. Then M with the norm induced by U is reflexive.*

Proof: We can identify two weak topologies in M, namely:

$$\sigma(M, M^*) \text{ and the trace of } \sigma(U, U^*). \tag{3.65}$$

It can be easily verified that these two topologies coincide (through restrictions and extensions of linear forms). From theorem 2.4.2, it suffices to show that B_M is compact for the topology $\sigma(M, M^*)$. But B_U is compact for $\sigma(U, U^*)$ and $M \subset U$ is closed (strongly) and convex so that it is weakly closed, thus from last proposition, B_M is compact for the topology $\sigma(U, U^*)$, and therefore it is compact for $\sigma(M, M^*)$. $\Box$

3.6 Separable Sets

Definition 3.6.1 (Separable Spaces)**.** *A metric space U is said to be separable if there exist a set $K \subset U$ such that K is countable and dense in U.*

The next Proposition is proved in Brezis [6].

Proposition 3.6.2. *Let U be a separable metric space. If $V \subset U$ then V is separable.*

Theorem 3.6.3. *Let U be a Banach space such that U^* is separable. Then U is separable.*

Proof: Consider $\{u_n^*\}$ a countable dense set in U^*. Observe that

$$\|u_n^*\|_{U^*} = \sup\{|\langle u_n^*, u\rangle_U| \;\mid\; u \in U \;\; and \;\; \|u\|_U = 1\} \tag{3.66}$$

so that for each $n \in \mathbb{N}$, there exists $u_n \in U$ such that $\|u_n\|_U = 1$ and $\langle u_n^*, u_n\rangle_U \geq \frac{1}{2}\|u_n^*\|_{U^*}$.

Define U_0 as the vector space on $\mathbb{Q}$ spanned by $\{u_n\}$, and U_1 as the vector space on $\mathbb{R}$ spanned by $\{u_n\}$. It is clear that U_0 is dense in U_1 and we will show that U_1 is dense in U, so that U_0 is a dense set in U. For, suppose u^* is such that $\langle u^*, u\rangle_U = 0, \forall u \in U_1$. Since $\{u_n^*\}$ is dense in U^*, given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\|u_n^* - u^*\|_{U^*} < \varepsilon$, so that

$$\frac{1}{2}\|u_n^*\|_{U^*} \leq \langle u_n, u_n^*\rangle_U = \langle u_n, u_n^* - u^*\rangle_U + \langle u_n, u^*\rangle_U \leq \|u_n^* - u^*\|_{U^*}\|u_n\|_U + 0 < \varepsilon \tag{3.67}$$

or

$$\|u^*\|_{U^*} \leq \|u_n^* - u^*\|_{U^*} + \|u_n^*\|_{U^*} < \varepsilon + 2\varepsilon = 3\varepsilon. \tag{3.68}$$

Therefore, since ε is arbitrary, $\|u^*\|_{U^*} = \theta$, that is $u^* = 0$. By Corollary 3.2.13 this completes the proof. □

Proposition 3.6.4. *U is reflexive if and only if U^* is reflexive.*

Proof: Suppose U is reflexive, as B_{U^*} is compact for $\sigma(U^*, U)$ and $\sigma(U^*, U) = \sigma(U^*, U^{**})$ we have that B_{U^*} is compact for $\sigma(U^*, U^{**})$, which means that U^* is reflexive.

Suppose U^* is reflexive, from above U^{**} is reflexive. Since $J(U)$ is a closed subspace of U^{**}, from Proposition 3.5.6, J(U) is reflexive. Thus, U is reflexive, since J is a isometry.

Proposition 3.6.5. *Let U be a Banach space. Then U is reflexive and separable if and only if U^* is reflexive and separable.*

Chapter 4

Measure and Integration

4.1 Basic Concepts

In this chapter U denotes a topological space.

Definition 4.1.1 (σ-Algebra). *A collection $\mathcal{M}$ of subsets of U is said to be a σ-Algebra if $\mathcal{M}$ has the following properties:*

1. *$U \in \mathcal{M}$,*
2. *if $A \in \mathcal{M}$ then $U - A \in \mathcal{M}$,*
3. *if $A_n \in \mathcal{M}, \forall n \in \mathbb{N}$, then $\cup_{n=0}^{\infty} A_n \in \mathcal{M}$.*

Definition 4.1.2 (Measurable Spaces). *If $\mathcal{M}$ is a σ-algebra in U we say that U is a measurable space. The elements of $\mathcal{M}$ are called the measurable sets of U.*

Definition 4.1.3 (Measurable Function). *If U is a measurable space and V is a topological space, we say that $f : U \to V$ is a measurable function if $f^{-1}(\mathcal{V})$ is measurable whenever $\mathcal{V} \subset V$ is an open set.*

Remark 4.1.4.
1. *Observe that $\emptyset = U - U$ so that from 1 and 2 in Definition 4.1.1, we have that $\emptyset \in \mathcal{M}$.*
2. *From 1 and 3 from Definition 4.1.1, it is clear that $\cup_{i=1}^{n} A_i \in \mathcal{M}$ whenever $A_i \in \mathcal{M}, \forall i \in \{1, ..., n\}$.*
3. *Since $\cap_{i=1}^{\infty} A_i = (\cup_{i=1}^{\infty} A_i^c)^c$ also from Definition 4.1.1, it is clear that $\mathcal{M}$ is closed under countable intersections.*
4. *Since $A - B = B^c \cap A$ we obtain: if $A, B \in \mathcal{M}$ then $A - B \in \mathcal{M}$.*

Theorem 4.1.5. *Let $\mathcal{F}$ be any collection of subsets of U. Then there exists a smallest σ-algebra $\mathcal{M}_0$ in U such that $\mathcal{F} \subset M_0$.*

Proof: Let Ω be the family of all σ-Algebras that contain $\mathcal{F}$. Since the set of all subsets in U is a σ-algebra, Ω is non-empty.

Let $\mathcal{M}_0 = \cap_{\mathcal{M}_\lambda \subset \Omega} \mathcal{M}_\lambda$, it is clear that $\mathcal{M}_0 \supset \mathcal{F}$, it remains to prove that in fact $\mathcal{M}_0$ is a σ-algebra. Observe that:

1. $U \in \mathcal{M}_\lambda$, $\forall \mathcal{M}_\lambda \in \Omega$, so that, $U \in \mathcal{M}_0$,
2. $A \in \mathcal{M}_0$ implies $A \in \mathcal{M}_\lambda, \forall \mathcal{M}_\lambda \in \Omega$, so that $A^c \in \mathcal{M}_\lambda, \forall \mathcal{M}_\lambda \in \Omega$, which means $A^c \in \mathcal{M}_0$,
3. $\{A_n\} \subset \mathcal{M}_0$ implies $\{A_n\} \subset \mathcal{M}_\lambda$, $\forall \mathcal{M}_\lambda \in \Omega$, so that $\cup_{n=1}^\infty A_n \in \mathcal{M}_\lambda$, $\forall \mathcal{M}_\lambda \in \Omega$, which means $\cup_{n=1}^\infty A_n \in \mathcal{M}_0$.

From Definition 4.1.1 the proof is complete. $\square$

Definition 4.1.6 (Borel Sets)**.** *Let U be a topological space, considering the last theorem there exists a smallest σ-algebra in U, denoted by $\mathcal{B}$, which contains the open sets of U. The elements of $\mathcal{B}$ are called the Borel sets.*

Theorem 4.1.7. *Suppose $\mathcal{M}$ is a σ-algebra in U and V is a topological space. For $f : U \to V$, we have:*

1. *If $\Omega = \{E \subset V \mid f^{-1}(E) \in \mathcal{M}\}$, then Ω is a σ-algebra.*
2. *If $V = [-\infty, \infty]$, and $f^{-1}((\alpha, \infty]) \in \mathcal{M}$, for each $\alpha \in \mathbb{R}$, then f is measurable.*

Proof:

1. (a) $V \in \Omega$ since $f^{-1}(V) = U$ and $U \in \mathcal{M}$.
   (b) $E \in \Omega \Rightarrow f^{-1}(E) \in \mathcal{M} \Rightarrow U - f^{-1}(E) \in \mathcal{M} \Rightarrow f^{-1}(V - E) \in \mathcal{M} \Rightarrow V - E \in \Omega$.
   (c) $\{E_i\} \subset \Omega \Rightarrow f^{-1}(E_i) \in \mathcal{M}, \forall i \in \mathbb{N} \Rightarrow \cup_{i=1}^\infty f^{-1}(E_i) \in \mathcal{M} \Rightarrow f^{-1}(\cup_{i=1}^\infty E_i) \in \mathcal{M} \Rightarrow \cup_{i=1}^\infty E_i \in \Omega$.
   Thus Ω is a σ-algebra.
2. Define $\Omega = \{E \subset [-\infty, \infty] \mid f^{-1}(E) \in \mathcal{M}\}$ from above Ω is a σ- algebra. Given $\alpha \in \mathbb{R}$, let $\{\alpha_n\}$ be a real sequence such that $\alpha_n \to \alpha$ as $n \to \infty$, $\alpha_n < \alpha, \forall n \in \mathbb{N}$. Since $(\alpha_n, \infty] \in \Omega$ for each n and

$$[-\infty, \alpha) = \cup_{n=1}^\infty [-\infty, \alpha_n] = \cup_{n=1}^\infty (\alpha_n, \infty]^C, \tag{4.1}$$

we obtain, $[-\infty, \alpha) \in \Omega$. Furthermore, we have $(\alpha, \beta) = [-\infty, \beta) \cap (\alpha, \infty] \in \Omega$. Since every open set in $[-\infty, \infty]$ may be expressed as a countable union of intervals (α, β) we have that Ω contains all the open sets. Thus, $f^{-1}(E) \in \mathcal{M}$ whenever E is open, so that f is measurable. $\square$

Proposition 4.1.8. *If $\{f_n : U \to [-\infty, \infty]\}$ is a sequence of measurable functions and $g = \sup_{n \geq 1} f_n$ and $h = \limsup_{n \to \infty} f_n$ then g and h are measurable.*

Proof: Observe that $g^{-1}((\alpha, \infty]) = \cup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty])$. From last theorem g is measurable. By analogy $h = \inf_{k \geq 1}\{\sup_{i \geq k} f_i\}$ is measurable. $\square$

4.2 Simple Functions

Definition 4.2.1 (Simple Functions). *A function $f : U \to \mathbb{C}$ is said to be a simple function if its range $(R(f))$ has only finitely many points. If $\{\alpha_1, ..., \alpha_n\} = R(f)$ and we set $A_i = \{u \in U \mid f(u) = \alpha_i\}$, clearly we have: $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, where*

$$\chi_{A_i}(u) = \begin{cases} 1, & if \quad u \in A_i, \\ 0, & otherwise. \end{cases} \tag{4.2}$$

Theorem 4.2.2. *Let $f : U \to [0, \infty]$ be a measurable function. Thus there exists a sequence of simple functions $\{s_n : U \to [0, \infty]\}$ such that*

1. $0 \leq s_1 \leq s_2 \leq ... \leq f$,
2. $s_n(u) \to f(u)$ *as* $n \to \infty, \forall u \in U$.

Proof: Define $\delta_n = 2^{-n}$. To each $n \in \mathbb{N}$ and each $t \in \mathbb{R}^+$, there corresponds a unique integer $K = K_n(t)$ such that

$$K\delta_n \leq t \leq (K+1)\delta_n. \tag{4.3}$$

Defining

$$\varphi_n(t) = \begin{cases} K_n(t)\delta_n, & if \quad 0 \leq t < n, \\ n, \quad if \quad t \geq n, \end{cases} \tag{4.4}$$

we have that each φ_n is a Borel function on $[0, \infty]$, such that

1. $t - \delta_n < \varphi_n(t) \leq t$ if $0 \leq t \leq n$,
2. $0 \leq \varphi_1 \leq ... \leq t$,
3. $\varphi_n(t) \to t$ as $n \to \infty, \forall t \in [0, \infty]$.

It follows that the sequence $\{s_n = \varphi_n \circ f\}$ corresponds to the results indicated above. $\square$

4.3 Measures

Definition 4.3.1 (Measure). *Let $\mathcal{M}$ be a σ-algebra on a topological space U. A function $\mu : \mathcal{M} \to [0, \infty]$ is said to be a measure if $\mu(\emptyset) = 0$ and μ is countably additive, that is, given $\{A_i\} \subset U$, a sequence of pairwise disjoint sets then*

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i). \tag{4.5}$$

In this case $(U, \mathcal{M}, \mu)$ is called a measure space.

Proposition 4.3.2. *Let $\mu : \mathcal{M} \to [0, \infty]$, where $\mathcal{M}$ is a σ-algebra of U. Then we have the following.*

1. *$\mu(A_1 \cup ... \cup A_n) = \mu(A_1) + ... + \mu(A_n)$ for any given $\{A_i\}$ of pairwise disjoint measurable sets of $\mathcal{M}$.*
2. *If $A, B \in \mathcal{M}$ and $A \subset B$ then $\mu(A) \leq \mu(B)$.*
3. *If $\{A_n\} \subset \mathcal{M}$, $A = \cup_{n=1}^{\infty} A_n$ and*
$$A_1 \subset A_2 \subset A_3 \subset ... \tag{4.6}$$
then, $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.
4. *If $\{A_n\} \subset \mathcal{M}$, $A = \cap_{n=1}^{\infty} A_n$, $A_1 \supset A_2 \supset A_3 \supset ....$ and $\mu(A_1)$ is finite then,*
$$\lim_{n\to\infty} \mu(A_n) = \mu(A). \tag{4.7}$$

Proof:

1. Take $A_{n+1} = A_{n+2} = ... = \emptyset$ in Definition 4.1.1 item 1,
2. Observe that $B = A \cup (B - A)$ and $A \cap (B - A) = \emptyset$ so that by above $\mu(A \cup (B - A)) = \mu(A) + \mu(B - A) \geq \mu(A)$,
3. Let $B_1 = A_1$ and let $B_n = A_n - A_{n-1}$ then $B_n \in \mathcal{M}$, $B_i \cap B_j = \emptyset$ if $i \neq j$, $A_n = B_1 \cup ... \cup B_n$ and $A = \cup_{i=1}^{\infty} B_i$. Thus
$$\mu(A) = \mu(\cup_{i=1}^{\infty} B_i) = \sum_{n=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} B_i = \lim_{n\to\infty} \mu(A_n) \tag{4.8}$$
4. Let $C_n = A_1 - A_n$. Then $C_1 \subset C_2 \subset ...$, $\mu(C_n) = \mu(A_1) - \mu(A_n)$, $A_1 - A = \cup_{n=1}^{\infty} C_n$. Thus by 3 we have
$$\mu(A_1) - \mu(A) = \mu(A_1 - A) = \lim_{n\to\infty} \mu(C_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n). \quad \square \tag{4.9}$$

4.4 Integration of Simple Functions

Definition 4.4.1 (Integral for Simple Functions)**.** *For $s : U \to [0, \infty]$, a measurable simple function, that is,*

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \tag{4.10}$$

where

$$\chi_{A_i}(u) = \begin{cases} 1, & if \quad u \in A_i, \\ 0, & otherwise, \end{cases} \tag{4.11}$$

we define the integral of s over $E \subset \mathcal{M}$, denoted by $\int_E s\, d\mu$ as

$$\int_E s\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E). \tag{4.12}$$

The convention $0.\infty = 0$ is used here.

Definition 4.4.2 (Integral for Non-negative Measurable Functions)**.** *If $f : U \to [0, \infty]$ is measurable, for $E \in \mathcal{M}$, we define the integral of f on E, denoted by $\int_E f d\mu$, as*

$$\int_E f d\mu = \sup_{s \in A} \{ \int_E s d\mu \}, \tag{4.13}$$

where

$$A = \{ s \text{ simple and measurable} \mid \; 0 \leq s \leq f \}. \tag{4.14}$$

Definition 4.4.3 (Integral for Measurable Functions)**.** *For a measurable $f : U \to [-\infty, \infty]$ and $E \in \mathcal{M}$, we define $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ and the integral of f on E, denoted by $\int_E f\, d\mu$, as*

$$\int_E f\, d\mu = \int_E f^+\, d\mu - \int_E f^-\, d\mu.$$

Theorem 4.4.4 (Lebesgue's Monotone Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of real measurable functions on U and suppose that*

1. *$0 \leq f_1(u) \leq f_2(u) \leq ... \leq \infty, \forall u \in U$,*
2. *$f_n(u) \to f(u)$ as $n \to \infty, \forall u \in U$.*

   Then,

   (a) f is measurable,

   (b) $\int_U f_n d\mu \to \int_U f d\mu$ as $n \to \infty$.

Proof: Since $\int_U f_n d\mu \leq \int_U f_{n+1} d\mu, \forall n \in \mathbb{N}$, there exists $\alpha \in [0, \infty]$ such that

$$\int_U f_n d\mu \to \alpha, \quad as \quad n \to \infty, \tag{4.15}$$

By Proposition 4.1.8, f is measurable, and since $f_n \leq f$ we have

$$\int_U f_n d\mu \leq \int_U f d\mu. \tag{4.16}$$

From (4.15) and (4.16), we obtain

$$\alpha \leq \int_U f d\mu. \tag{4.17}$$

Let s be any simple function such that $0 \leq s \leq f$, and let $c \in \mathbb{R}$ such that $0 < c < 1$. For each $n \in \mathbb{N}$ we define

$$E_n = \{u \in U \mid f_n(u) \geq cs(u)\}. \tag{4.18}$$

Clearly E_n is measurable and $E_1 \subset E_2 \subset ...$ and $U = \cup_{n \in \mathbb{N}} E_n$. Observe that

$$\int_U f_n d\mu \geq \int_{E_n} f_n d\mu \geq c \int_{E_n} s d\mu. \tag{4.19}$$

Letting $n \to \infty$ and applying Proposition 4.3.2, we obtain

$$\alpha = \lim_{n \to \infty} \int_U f_n d\mu \geq c \int_U s d\mu, \tag{4.20}$$

so that

$$\alpha \geq \int_U s d\mu, \forall s \ \ simple \ \ such \ \ that \ \ 0 \leq s \leq f. \tag{4.21}$$

This implies

$$\alpha \geq \int_U f d\mu. \tag{4.22}$$

From (4.17) and (4.22) the proof is complete. $\square$.

Theorem 4.4.5 (Fatou's Lemma). *If $\{f_n : U \to [0, \infty]\}$ is a sequence of measurable functions, then*

$$\int_U \liminf_{n \to \infty} f_n d\mu \leq \liminf_{n \to \infty} \int_U f_n d\mu. \tag{4.23}$$

Proof: For each $k \in \mathbb{N}$ define $g_k : U \to [0, \infty]$ by

$$g_k(u) = \inf_{i \geq k} \{f_i(u)\}. \tag{4.24}$$

Then

$$g_k \leq f_k \tag{4.25}$$

so that

$$\int_U g_k d\mu \leq \int_U f_k d\mu, \forall k \in \mathbb{N}. \tag{4.26}$$

Also $0 \leq g_1 \leq g_2 \leq ...$, each g_k is measurable, and

$$\lim_{k\to\infty} g_k(u) = \liminf_{n\to\infty} f_n(u), \forall u \in U. \tag{4.27}$$

From the monotone convergence theorem

$$\liminf_{k\to\infty} \int_U g_k d\mu = \lim_{k\to\infty} \int_U g_k d\mu = \int_U \liminf_{n\to\infty} f_n d\mu. \tag{4.28}$$

From (4.26) we have

$$\liminf_{k\to\infty} \int_U g_k d\mu \leq \liminf_{k\to\infty} \{\int_U f_k d\mu\}. \tag{4.29}$$

Thus, from (4.28) and (4.29) we obtain

$$\int_U \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_U f_n d\mu. \quad \square \tag{4.30}$$

Theorem 4.4.6 (Lebesgue's Dominated Convergence Theorem)**.** *Suppose $\{f_n\}$ is sequence of complex measurable functions on U such that*

$$\lim_{n\to\infty} f_n(u) = f(u), \forall u \in U. \tag{4.31}$$

If there exists a measurable function $g : U \to \mathbb{R}^+$ such that $\int_U g d\mu < \infty$ and $|f_n(u)| \leq g(u), \forall u \in U, \quad n \in \mathbb{N}$, then

1. $\int_U |f| d\mu < \infty$,
2. $\lim_{n\to\infty} \int_U |f_n - f| d\mu = 0$.

Proof:

1. This inequality holds since f is measurable and $|f| \leq g$.
2. Since $2g - |f_n - f| \geq 0$ we may apply the Fatou's Lemma and obtain:

$$\int_U 2g d\mu \leq \liminf_{n\to\infty} \int_U (2g - |f_n - f|) d\mu, \tag{4.32}$$

so that

$$\limsup_{n\to\infty} \int_U |f_n - f| d\mu \leq 0. \tag{4.33}$$

Hence

$$\lim_{n\to\infty} \int_U |f_n - f| d\mu = 0. \tag{4.34}$$

This completes the proof. $\square$

We finish this chapter with an important remark:

Remark 4.4.7. *In a measurable space U we say that a property holds almost everywhere (a.e.) if it holds on U except for a set of measure zero.*

Chapter 5

Distributions

5.1 Basic Definitions and Results

Definition 5.1.1 (Test Functions, the Space $\mathcal{D}(\Omega)$)**.** *Let $\Omega \in \mathbb{R}^n$ be a nonempty open set. For each $K \subset \Omega$ compact, consider the space $\mathcal{D}_K$, the set of all $C^\infty(\Omega)$ functions with support in K. We define the space of test functions, denoted by $\mathcal{D}(\Omega)$ as*

$$\mathcal{D}(\Omega) = \cup_{K \subset \Omega} \mathcal{D}_K, \quad K \quad \text{compact}. \tag{5.1}$$

Thus $\phi \in \mathcal{D}(\Omega)$ if and only if $\phi \in C^\infty(\Omega)$ and the support of ϕ is a compact subset of Ω.

Definition 5.1.2 (Topology for $\mathcal{D}(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set.*

1. *For every $K \subset \Omega$ compact, σ_K denotes the topology which a local base is defined by $\{\mathcal{V}_{N,n}\}$, where $N, n \in \mathbb{N}$,*

$$\mathcal{V}_{N,n} = \{\phi \in \mathcal{D}_K \quad | \quad \|\phi\|_N < 1/n\} \tag{5.2}$$

and

$$\|\phi\|_N = \max\{|D^\alpha \phi(x)| \quad | \quad x \in \Omega, |\alpha| \leq N\}. \tag{5.3}$$

2. *$\hat{\sigma}$ denotes the collection of all convex balanced sets $\mathcal{W} \in \mathcal{D}(\Omega)$ such that $\mathcal{W} \cap \mathcal{D}_K \subset \sigma_K$ for every compact $K \subset \Omega$.*

3. *We define σ in $\mathcal{D}(\Omega)$ as the collection of all unions of sets of the form $\phi + \mathcal{W}$, for $\phi \in \mathcal{D}(\Omega)$ and $\mathcal{W} \in \hat{\sigma}$.*

Theorem 5.1.3. *Concerning the last definition we have the following.*

1. *σ is a topology in $\mathcal{D}(\Omega)$.*

2. Through σ, $\mathcal{D}(\Omega)$ is made into a locally convex topological vector space.

Proof:

1. From item 3 of Definition 5.1.2, it is clear that arbitrary unions of elements of σ are elements of σ. Let us now show that finite intersections of elements of σ also belongs to σ. Suppose $\mathcal{V}_1 \in \sigma$ and $\mathcal{V}_2 \in \sigma$, if $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ we are done. Thus, suppose $\phi \in \mathcal{V}_1 \cap \mathcal{V}_2$. By the definition of σ there exist two sets of indices L_1 and L_2, such that

$$\mathcal{V}_i = \cup_{\lambda \in L_i}(\phi_{i\lambda} + \mathcal{W}_{i\lambda}), \quad for \;\; i = 1, 2, \tag{5.4}$$

and as $\phi \in \mathcal{V}_1 \cap \mathcal{V}_2$ there exist $\phi_i \in \mathcal{D}(\Omega)$ and $\mathcal{W}_i \in \hat{\sigma}$ such that

$$\phi \in \phi_i + \mathcal{W}_i, \quad for \;\; i = 1, 2. \tag{5.5}$$

Thus there exists $K \in \Omega$ such that $\phi_i \subset \mathcal{D}_K$ for $i \in \{1, 2\}$. Since $\mathcal{D}_K \cap \mathcal{W}_i \in \sigma_K$, $\mathcal{D}_K \cap \mathcal{W}_i$ is open in $\mathcal{D}_K$ so that from (5.5) there exists $0 < \delta_i < 1$ such that

$$\phi - \phi_i \in (1 - \delta_i)\mathcal{W}_i, \quad for \;\; i \in \{1, 2\}. \tag{5.6}$$

From (5.6) and from the convexity of $\mathcal{W}_i$ we have

$$\phi - \phi_i + \delta_i \mathcal{W}_i \subset (1 - \delta_i)\mathcal{W}_i + \delta_i \mathcal{W}_i = \mathcal{W}_i \tag{5.7}$$

so that

$$\phi + \delta_i \mathcal{W}_i \subset \phi_i + \mathcal{W}_i \subset \mathcal{V}_i, \quad for \;\; i \in \{1, 2\}. \tag{5.8}$$

Define $\mathcal{W}_\phi = (\delta_1 \mathcal{W}_1) \cap (\delta_2 \mathcal{W}_2)$ so that

$$\phi + \mathcal{W}_\phi \subset \mathcal{V}_i, \tag{5.9}$$

and therefore we may write

$$\mathcal{V}_1 \cap \mathcal{V}_2 = \cup_{\phi \in \mathcal{V}_1 \cap \mathcal{V}_2}(\phi + \mathcal{W}_\phi) \in \sigma. \tag{5.10}$$

This completes the proof.

2. It suffices to show that single points are closed sets in $\mathcal{D}(\Omega)$ and the vector space operations are continuous.

   (a) Pick $\phi_1, \;\; \phi_2 \in \mathcal{D}(\Omega)$ such that $\phi_1 \neq \phi_2$ and define

   $$\mathcal{V} = \{\phi \in \mathcal{D}(\Omega) \;\;|\;\; \|\phi\|_0 < \|\phi_1 - \phi_2\|_0\}. \tag{5.11}$$

   Thus $\mathcal{V} \in \hat{\sigma}$ and $\{\phi_1\} \not\subseteq \phi_2 + \mathcal{V}$. As $\phi_2 + \mathcal{V}$ is open and belongs to $\mathcal{D}(\Omega) - \{\phi_1\}$ and $\phi_2 \neq \phi_1$ is arbitrary, it follows that $\mathcal{D}(\Omega) - \{\phi_1\}$ is open, so that $\{\phi_1\}$ is closed.

(b) The proof that addition is σ-continuous follows from the convexity of any element of $\hat{\sigma}$. Thus given $\phi_1, \phi_2 \in \mathcal{D}(\Omega)$ and $\mathcal{V} \in \hat{\sigma}$ we have

$$\phi_1 + \frac{1}{2}\mathcal{V} + \phi_2 + \frac{1}{2}\mathcal{V} = \phi_1 + \phi_2 + \mathcal{V}. \tag{5.12}$$

(c) To prove the continuity of scalar multiplication, first consider $\phi_0 \in \mathcal{D}(\Omega)$ and $\alpha_0 \in \mathbb{R}$. Then:

$$\alpha\phi - \alpha_0\phi_0 = \alpha(\phi - \phi_0) + (\alpha - \alpha_0)\phi_0. \tag{5.13}$$

For $\mathcal{V} \in \hat{\sigma}$ there exists $\delta > 0$ such that $\delta\phi_0 \in \frac{1}{2}\mathcal{V}$. Let us define $c = \frac{1}{2}(|\alpha_0| + \delta)$. Thus if $|\alpha - \alpha_0| < \delta$ then $(\alpha - \alpha_0)\phi_0 \in \frac{1}{2}\mathcal{V}$. Let $\phi \in \mathcal{D}(\Omega)$ such that

$$\phi - \phi_0 \in c\mathcal{V} = \frac{1}{2(|\alpha_0| + \delta)}\mathcal{V}, \tag{5.14}$$

so that

$$(|\alpha_0| + \delta)(\phi - \phi_0) \in \frac{1}{2}\mathcal{V}. \tag{5.15}$$

This means

$$\alpha(\phi - \phi_0) + (\alpha - \alpha_0)\phi_0 \in \frac{1}{2}\mathcal{V} + \frac{1}{2}\mathcal{V} = \mathcal{V}. \tag{5.16}$$

Therefore $\alpha\phi - \alpha_0\phi_0 \in \mathcal{V}$ whenever $|\alpha - \alpha_0| < \delta$ and $\phi - \phi_0 \in c\mathcal{V}$. $\square$

For the next result the proof may be found in Rudin [31].

Proposition 5.1.4. *A convex balanced set $\mathcal{V} \subset \Omega$ is open if and only if $\mathcal{V} \in \sigma$.*

Proposition 5.1.5. *The topology σ_K of $\mathcal{D}_K \subset \mathcal{D}(\Omega)$ coincides with the topology that $\mathcal{D}_K$ inherits from $\mathcal{D}(\Omega)$.*

Proof: From Proposition 5.1.4 we have

$$\mathcal{V} \in \sigma \;\; \textit{implies} \;\; \mathcal{D}_K \cap \mathcal{V} \in \sigma_K. \tag{5.17}$$

Now suppose $\mathcal{V} \in \sigma_K$, we must show that there exists $A \in \sigma$ such that $\mathcal{V} = A \cap \mathcal{D}_K$. The definition of σ_K implies that for every $\phi \in \mathcal{V}$, there exist $N \in \mathbb{N}$ and $\delta_\phi > 0$ such that

$$\{\varphi \in \mathcal{D}_K \;\mid\; \|\varphi - \phi\|_N < \delta_\phi\} \subset \mathcal{V}. \tag{5.18}$$

Define

$$\mathcal{U}_\phi = \{\varphi \in \mathcal{D}(\Omega) \;\mid\; \|\varphi\|_N < \delta_\phi\}. \tag{5.19}$$

Then $\mathcal{U}_\phi \in \hat{\sigma}$ and

$$\mathcal{D}_K \cap (\phi + \mathcal{U}_\phi) = \phi + (\mathcal{D}_K \cap \mathcal{U}_\phi) \subset \mathcal{V}. \tag{5.20}$$

Defining $A = \cup_{\phi\in\mathcal{V}}(\phi + \mathcal{U}_\phi)$, we have completed the proof. $\square$

The proof for the next result may also be found in Rudin [31].

Proposition 5.1.6. *If A is a bounded set of $\mathcal{D}(\Omega)$ then $A \subset \mathcal{D}_K$ for some $K \subset \Omega$, and there are $M_N < \infty$ such that $\|\phi\|_N \leq M_N, \forall \phi \in A, \quad N \in \mathbb{N}$.*

Proposition 5.1.7. *If $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{D}(\Omega)$, then $\{\phi_n\} \subset \mathcal{D}_K$ for some $K \subset \Omega$ compact, and*

$$\lim_{i,j \to \infty} \|\phi_i - \phi_j\|_N = 0, \forall N \in \mathbb{N}. \tag{5.21}$$

Proof: Since Cauchy sequences are bounded, we have that $\{\phi_n\} \subset \mathcal{D}_K$ for some $K \subset \Omega$ compact. The result indicated in (5.21) follows from the fact that $\{\phi_n\}$ is also a Cauchy sequence in σ_K. $\square$

Proposition 5.1.8. *If $\phi_n \to 0$ in $\mathcal{D}(\Omega)$, then there exists a compact $K \subset \Omega$ which contains the support of $\phi_n, \forall n \in \mathbb{N}$ and $D^\alpha \phi_n \to 0$ uniformly, for each multi-index α.*

The proof follows directly from last proposition.

Theorem 5.1.9. *Suppose $T : \mathcal{D}(\Omega) \to V$ is linear, where V is a locally convex space. Then the following statements are equivalent.*

1. *T is continuous.*
2. *T is bounded.*
3. *If $\phi_n \to 0$ in $\mathcal{D}(\Omega)$ then $T(\phi_n) \to 0$ as $n \to \infty$.*
4. *The restrictions of T to each $\mathcal{D}_K$ are continuous.*

Proof:
$1 \Rightarrow 2$. This follows from Proposition 2.7.3 .

$2 \Rightarrow 3$. Suppose T is bounded and $\phi_n \to 0$ in $\mathcal{D}(\Omega)$, by last proposition $\phi_n \to 0$ in some $\mathcal{D}_K$ so that $\{\phi_n\}$ is bounded and $\{T(\phi_n)\}$ is also bounded. Hence by Proposition 2.7.3, $T(\phi_n) \to 0$ in V.

$3 \Rightarrow 4$. Assume 3 holds and consider $\{\phi_n\} \subset \mathcal{D}_K$. If $\phi_n \to 0$ then, by Proposition 5.1.5, $\phi_n \to 0$ in $\mathcal{D}(\Omega)$, so that, by above $T(\phi_n) \to 0$ in V. Since $\mathcal{D}_K$ is metrizable, also by proposition 2.7.3 we have that 4 follows.

$4 \Rightarrow 1$. Assume 4 holds and let $\mathcal{V}$ be a convex balanced neighborhood of zero in V. Define $\mathcal{U} = T^{-1}(\mathcal{V})$. Thus $\mathcal{U}$ is balanced and convex. By Proposition 5.1.5, $\mathcal{U}$ is open in $\mathcal{D}(\Omega)$ if and only if $\mathcal{D}_K \cap \mathcal{U}$ is open in $\mathcal{D}_K$ for each compact $K \subset \Omega$, thus if the restrictions of T to each $\mathcal{D}_K$ are continuous at 0, then T is continuous at 0, hence 4 implies 1. $\square$

Definition 5.1.10 (Distribution). *A linear functional in $\mathcal{D}(\Omega)$ which is continuous with respect to σ is said to be a Distribution.*

Proposition 5.1.11. *Every differential operator is a continuous mapping from $\mathcal{D}(\Omega)$ into $\mathcal{D}(\Omega)$.*

Proof: Since $\|D^\alpha \phi\|_N \leq \|\phi\|_{|\alpha|+N}, \forall N \in \mathbb{N}$, D^α is continuous on each $\mathcal{D}_K$, so that by Theorem 5.1.9, D^α is continuous on $\mathcal{D}(\Omega)$. $\square$

Theorem 5.1.12. *Denoting by $\mathcal{D}'(\Omega)$ the dual space of $\mathcal{D}(\Omega)$ we have that $T : \mathcal{D}(\Omega) \to \mathbb{R} \in \mathcal{D}'(\Omega)$ if and only if for each compact set $K \subset \Omega$ there exists an $N \in \mathbb{N}$ and $c \in \mathbb{R}^+$ such that*

$$|T(\phi)| \leq c\|\phi\|_N, \forall \phi \in \mathcal{D}_K. \tag{5.22}$$

The proof follows from the equivalence of 1 and 4 in Theorem 5.1.9. $\square$

5.2 Differentiation of Distributions

Definition 5.2.1 (Derivatives for Distributions)**.** *Given $T \in \mathcal{D}'(\Omega)$ and a multi-index α, we define the D^α derivative of T as*

$$D^\alpha T(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi), \forall \phi \in \mathcal{D}(\Omega). \tag{5.23}$$

Remark 5.2.2. *Observe that if $|T(\phi)| \leq c\|\phi\|_N, \forall \phi \in \mathcal{D}(\Omega)$ for some $c \in \mathbb{R}^+$, then*

$$|D^\alpha T(\phi)| \leq c\|D^\alpha \phi\|_N \leq c\|\phi\|_{N+|\alpha|}, \forall \phi \in \mathcal{D}(\Omega), \tag{5.24}$$

thus $D^\alpha T \in \mathcal{D}'(\Omega)$. Therefore, derivatives of distributions are also distributions.

Theorem 5.2.3. *Suppose $\{T_n\} \subset \mathcal{D}'(\Omega)$. Let $T : \mathcal{D}(\Omega) \to \mathbb{R}$ be defined by*

$$T(\phi) = \lim_{n\to\infty} T_n(\phi), \forall \phi \in \mathcal{D}(\Omega). \tag{5.25}$$

Then $T \in \mathcal{D}'(\Omega)$, and

$$D^\alpha T_n \to D^\alpha T \quad in \quad \mathcal{D}'(\Omega). \tag{5.26}$$

Proof: Let K be an arbitrary compact subset of Ω. Since (5.25) holds for every $\phi \in \mathcal{D}_K$, and since $\mathcal{D}_K$ is a *Fréchet* space, the Banach-Steinhaus theorem implies that the restriction of T to $\mathcal{D}_K$ is continuous. It follows from Theorem 5.1.9 that T is continuous in $\mathcal{D}(\Omega)$, that is, $T \in \mathcal{D}'(\Omega)$. On the other hand

$$(D^\alpha T)(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi) = (-1)^{|\alpha|} \lim_{n\to\infty} T_n(D^\alpha \phi) = \lim_{n\to\infty} (D^\alpha T_n(\phi)), \forall \phi \in \mathcal{D}(\Omega). \quad \square \tag{5.27}$$

Chapter 6

Lebesgue and Sobolev Spaces

We start with the definition of Lebesgue spaces, denoted by $L^p(\Omega)$, where $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^n$ is an open set.

6.1 Definition and Properties of L^p Spaces

Definition 6.1.1 (L^p Spaces)**.** *For $1 \leq p < \infty$, we say that $u \in L^p(\Omega)$ if $u : \Omega \to \mathbb{R}$ is measurable and*

$$\int_\Omega |u|^p dx < \infty. \tag{6.1}$$

We also denote $\|u\|_p = [\int_\Omega |u|^p dx]^{1/p}$ and will show that $\|.\|_p$ is a norm.

Definition 6.1.2 (L^∞ Spaces)**.** *We say that $u \in L^\infty(\Omega)$ if u is measurable and there exists $M \in \mathbb{R}^+$, such that $|u(x)| \leq M, \quad a.e. \ \ in \ \ \Omega$. We define*

$$\|u\|_\infty = \inf\{M > 0 \ \ | \ \ |u(x)| \leq M, \quad a.e. \ \ in \ \ \Omega\}. \tag{6.2}$$

We will show that $\|.\|_\infty$ is a norm. For $1 \leq p \leq \infty$, we define q by the relations

$$q = \begin{cases} +\infty, & if \ \ p = 1, \\ \frac{p}{p-1}, & if \ \ 1 < p < +\infty, \\ 1, & if \ \ p = +\infty, \end{cases}$$

so that symbolically we have

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The next result is fundamental in the proof of the Sobolev Imbedding Theorem.

Theorem 6.1.3 (Hölder Inequality). *Consider $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, with $1 \le p \le \infty$. Then $uv \in L^1(\Omega)$ and*

$$\int_\Omega |uv|dx \le \|u\|_p \|v\|_q. \tag{6.3}$$

Proof: The result is clear if $p = 1$ or $p = \infty$. You may assume $\|u\|_p, \|v\|_q > 0$, otherwise the result is also obvious. Thus suppose $1 < p < \infty$. From the concavity of log function on $(0, \infty)$ we obtain

$$\log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \ge \frac{1}{p}\log a^p + \frac{1}{q}\log b^q = \log(ab). \tag{6.4}$$

Thus,

$$ab \le \frac{1}{p}(a^p) + \frac{1}{q}(b^q), \quad \forall a \ge 0, b \ge 0. \tag{6.5}$$

Therefore

$$|u(x)||v(x)| \le \frac{1}{p}|u(x)|^p + \frac{1}{q}|v(x)|^q, \quad a.e. \;\; in \;\; \Omega. \tag{6.6}$$

Hence $|uv| \in L^1(\Omega)$ and

$$\int_\Omega |uv|dx \le \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q. \tag{6.7}$$

Replacing u by λu in (6.7) $\lambda > 0$, we obtain

$$\int_\Omega |uv|dx \le \frac{\lambda^{p-1}}{p}\|u\|_p^p + \frac{1}{\lambda q}\|v\|_q^q. \tag{6.8}$$

For $\lambda = \|u\|_p^{-1}\|v\|_q^{q/p}$ we obtain the Hölder inequality. $\square$

The next step is to prove that $\|.\|_p$ is a norm.

Theorem 6.1.4. *$L^p(\Omega)$ is a vector space and $\|.\|_p$ is norm $\forall p$ such that $1 \le p \le \infty$.*

Proof: If $p = 1$ or $p = \infty$ the result is clear. Thus, suppose $1 < p < \infty$. For $u, v \in L^p(\Omega)$ we have

$$|u(x) + v(x)|^p \le (|u(x)| + |v(x)|)^p \le 2^p(|u(x)|^p + |v(x)|^p), \tag{6.9}$$

so that $u + v \in L^p(\Omega)$. On the other hand

$$\|u + v\|_p^p = \int_\Omega |u + v|^{p-1}|u + v|dx \le \int_\Omega |u + v|^{p-1}|u|dx + \int_\Omega |u + v|^{p-1}|v|dx, \tag{6.10}$$

and hence, from Hölder's inequality

$$\|u + v\|_p^p \le \|u + v\|_p^{p-1}\|u\|_p + \|u + v\|_p^{p-1}\|v\|_p, \tag{6.11}$$

that is,

$$\|u + v\|_p \le \|u\|_p + \|v\|_p, \forall u, v \in L^p(\Omega). \;\; \square \tag{6.12}$$

Theorem 6.1.5. *$L^p(\Omega)$ is a Banach space for any p such that $1 \leq p \leq \infty$.*

Proof: Suppose $p = \infty$. Suppose $\{u_n\}$ is Cauchy sequence in $L^\infty(\Omega)$. Thus, given $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that, if $m, n \geq N_k$ then

$$\|u_m - u_n\|_\infty < \frac{1}{k}. \tag{6.13}$$

Therefore, for each k, there exist a set E_k such that $m(E_k) = 0$, and

$$|u_m(x) - u_n(x)| < \frac{1}{k}, \quad \forall x \in \Omega - E_k, \quad \forall m, n \geq N_k. \tag{6.14}$$

Observe that $E = \cup_{k=1}^{\infty} E_k$ is such that $m(E) = 0$. Thus $\{u_n(x)\}$ is a real Cauchy sequence at each $x \in \Omega - E$. Define $u(x) = \lim_{n\to\infty} u_n(x)$ on $\Omega - E$. Letting $m \to \infty$ in (6.14) we obtain

$$|u(x) - u_n(x)| < \frac{1}{k}, \quad \forall x \in \Omega - E, \quad \forall n \geq N_k. \tag{6.15}$$

Thus $u \in L^\infty(\Omega)$ and $\|u_n - u\|_\infty \to 0$ as $n \to \infty$.

Now suppose $1 \leq p < \infty$. Let $\{u_n\}$ a Cauchy sequence in $L^p(\Omega)$. We can extract a subsequence $\{u_{n_k}\}$ such that

$$\|u_{n_{k+1}} - u_{n_k}\|_p \leq \frac{1}{2^k}, \forall k \in \mathbb{N}. \tag{6.16}$$

To simplify the notation we write u_k in place of u_{n_k}, so that

$$\|u_{k+1} - u_k\|_p \leq \frac{1}{2^k}, \forall k \in \mathbb{N}. \tag{6.17}$$

Defining

$$g_n(x) = \sum_{k=1}^{n} |u_{k+1}(x) - u_k(x)|, \tag{6.18}$$

we obtain

$$\|g_n\|_p \leq 1, \quad \forall n \in \mathbb{N}. \tag{6.19}$$

From the monotone convergence theorem and (6.19), $g_n(x)$ converges to a limit $g(x)$ with $g \in L^p(\Omega)$. On the the other hand, for $m \geq n \geq 2$ we have

$$|u_m(x) - u_n(x)| \leq |u_m(x) - u_{m-1}(x)| + ... + |u_{n+1}(x) - u_n(x)| \leq g(x) - g_{n-1}(x), \quad a.e. \;\; in \;\; \Omega. \tag{6.20}$$

Hence $\{u_n(x)\}$ is Cauchy *a.e.* in Ω and converges to a limit $u(x)$ so that

$$|u(x) - u_n(x)| \leq g(x), \quad a.e. \;\in\; \Omega, \quad for \;\; n \geq 2, \tag{6.21}$$

which means $u \in L^p(\Omega)$. Finally from $|u_n(x) - u(x)| \to 0$, *a.e. in* Ω, $|u_n(x) - u(x)|^p \leq |g(x)|^p$ and the Lebesgue dominated convergence theorem implies

$$\|u_n - u\|_p \to 0 \;\; as \;\; n \to \infty. \;\; \square \tag{6.22}$$

Theorem 6.1.6. *Let $\{u_n\} \subset L^p(\Omega)$ and $u \in L^p(\Omega)$ such that $\|u_n - u\|_p \to 0$. Then there exists a subsequence $\{u_{n_k}\}$ such that*

1. $u_{n_k}(x) \to u(x)$, *a.e. in* Ω,
2. $|u_{n_k}(x)| \leq h(x)$, *a.e. in* $\Omega, \forall k \in \mathbb{N}$, *for some* $h \in L^p(\Omega)$.

Proof: the result is clear for $p = \infty$. Suppose $1 \leq p < \infty$. From the last theorem we can easily obtain that $|u_{n_k}(x) - u(x)| \to 0$ as $k \to \infty$, a.e. in Ω. To complete the proof, just take $h = u + g$, where is defined in the proof of the last theorem. $\square$

Theorem 6.1.7. *$L^p(\Omega)$ is reflexive for all p such that $1 < p < \infty$.*

Proof: We divide the proof into 3 parts.

1. For $2 \leq p < \infty$ we have that

$$\left\|\frac{u+v}{2}\right\|_{L^p(\Omega)} + \left\|\frac{u-v}{2}\right\|_{L^p(\Omega)} \leq \frac{1}{2}(\|u\|^p_{L^p(\Omega)} + \|v\|^p_{L^p(\Omega)}), \quad \forall u, v \in L^p(\Omega). \tag{6.23}$$

   Proof: Observe that

$$\alpha^p + \beta^p \leq (\alpha^2 + \beta^2)^{p/2}, \forall \alpha, \beta \geq 0. \tag{6.24}$$

   Now taking $\alpha = \left|\frac{a+b}{2}\right|$ and $\beta = \left|\frac{a-b}{2}\right|$ in (6.24), we obtain (using the convexity of $t^{p/2}$),

$$|\frac{a+b}{2}|^p + |\frac{a-b}{2}|^p \leq (|\frac{a+b}{2}|^2 + |\frac{a-b}{2}|^2)^{p/2} = (\frac{a^2}{2} + \frac{b^2}{2})^{p/2} \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p. \tag{6.25}$$

   The inequality (6.23) follows immediately.

2. $L^p(\Omega)$ is uniformly convex, and therefore reflexive for $2 \leq p < \infty$.

   Proof: Suppose $\varepsilon > 0$ and suppose that

$$\|u\|_p \leq 1, \quad \|v\|_p \leq 1 \quad and \quad \|u - v\|_p > \varepsilon. \tag{6.26}$$

   From part 1, we obtain

$$\left\|\frac{u+v}{2}\right\|_p^p < 1 - \left(\frac{\varepsilon}{2}\right)^p, \tag{6.27}$$

   and therefore

$$\left\|\frac{u+v}{2}\right\|_p < 1 - \delta, \tag{6.28}$$

   for $\delta = 1 - (1 - (\varepsilon/2)^p)^p > 0$. Thus $L^p(\Omega)$ is uniformly convex and reflexive (Theorem III.29, Brézis [6]).

3. $L^p(\Omega)$ is reflexive for $1 < p \leq 2$. From 2 we can conclude that L^q is reflexive. We will define $T : L^p(\Omega) \to (L^q)^*$ by
$$\langle Tu, f\rangle_{L^q(\Omega)} = \int_\Omega uf dx, \forall u \in L^p(\Omega), \quad f \in L^q(\Omega). \tag{6.29}$$
From the Hölder inequality, we obtain
$$|\langle Tu, f\rangle_{L^q(\Omega)}| \leq \|u\|_p \|f\|_q, \tag{6.30}$$
so that
$$\|Tu\|_{(L^q(\Omega))^*} \leq \|u\|_p. \tag{6.31}$$
Pick $u \in L^p(\Omega)$ and define $f_0(x) = |u(x)|^{p-2}u(x)$ ($f_0(x) = 0$ if $u(x) = 0$). Thus, we have that $f_0 \in L^q(\Omega)$, $\|f_0\|_q = \|u\|_p^{p-1}$ and $\langle Tu, f_0\rangle_{L^q(\Omega)} = \|u\|_p^p$. Therefore,
$$\|Tu\|_{(L^q(\Omega))^*} \geq \frac{\langle Tu, f_0\rangle_{L^q(\Omega)}}{\|f_0\|_q} = \|u\|_p \tag{6.32}$$
Hence from (6.31) and (6.32) we have
$$\|Tu\|_{(L^q(\Omega))^*} = \|u\|_p, \forall u \in L^p(\Omega). \tag{6.33}$$
Thus T is an isometry from $L^p(\Omega)$ to a closed subspace of $(L^q(\Omega))^*$. Since from the first part $L^q(\Omega)$ is reflexive, we have that $(L^q(\Omega))^*$ is reflexive. From proposition III.17 in Brezis [6], $T(L^p(\Omega))$ and $L^p(\Omega)$ are reflexive. □

Theorem 6.1.8 (Riesz Representation Theorem). *Let $1 < p < \infty$ and let f be a continuous linear functional on $L^p(\Omega)$. Then there exists a unique $u_0 \in L^q$ such that*
$$f(v) = \int_\Omega vu_0 \, dx, \ \forall v \in L^p(\Omega). \tag{6.34}$$
Furthermore
$$\|f\|_{(L^p)^*} = \|u_0\|_q. \tag{6.35}$$

Proof: First we define the operator $T : L^q(\Omega) \to (L^p(\Omega))^*$ by
$$\langle Tu, v\rangle_{L^p(\Omega)} = \int_\Omega uv \, dx, \forall v \in L^p(\Omega). \tag{6.36}$$
Similarly to last theorem, we obtain
$$\|Tu\|_{(L^p(\Omega))^*} = \|u\|_q. \tag{6.37}$$
We have to show that T is onto. Define $E = T(L^q(\Omega))$. As E is a closed subspace, it suffices to show that E is dense in $(L^p(\Omega))^*$. Suppose $h \in (L^p)^{**} = L^p$ is such that
$$\langle Tu, h\rangle_{L^p(\Omega)} = 0, \forall u \in L^q(\Omega). \tag{6.38}$$
Choosing $u = |h|^{p-2}h$ we may conclude that $h = 0$, which completes the proof. □

Definition 6.1.9. *Let $1 \leq p \leq \infty$. We say that $u \in L^p_{loc}(\Omega)$ if $u\chi_K \in L^p(\Omega)$ for all compact $K \subset \Omega$.*

6.1.1 Spaces of Continuous Functions

We introduce some definitions and properties concerning spaces of continuous functions. First, we recall that by a domain we mean an open set in $\mathbb{R}^n$. Thus for a domain $\Omega \subset \mathbb{R}^n$ and for any nonnegative integer m we define by $C^m(\Omega)$ the set of all functions u which the partial derivatives $D^\alpha u$ are continuous on Ω for any α such that $|\alpha| \leq m$. We define $C^\infty(\Omega) = \cap_{m=0}^{\infty} C^m(\Omega)$ and denote $C^0(\Omega) = C(\Omega)$. The sets $C_0(\Omega)$ and $C_0^\infty(\Omega)$ consist of functions in $C(\Omega)$ and $C^\infty(\Omega)$ respectively, with compact support in Ω. On the other hand, $C_B^m(\Omega)$ denotes the set of functions $u \in C^m(\Omega)$ for which $D^\alpha u$ is bounded on Ω for $0 \leq |\alpha| \leq m$. Observe that $C_B^m(\Omega)$ is a Banach space with the norm denoted by $\|.\|_{B,m}$ given by

$$\|u\|_{B,m} = \max_{0\leq|\alpha|\leq m} \sup_{x\in\Omega} \{|D^\alpha u(x)|\} .$$

Also, we define $C^m(\bar{\Omega})$ as the set of functions $u \in C^m(\Omega)$ for which $D^\alpha u$ is bounded and uniformly continuous on Ω for $0 \leq |\alpha| \leq m$. Observe that $C^m(\bar{\Omega})$ is a closed subspace of $C_B^m(\Omega)$ and is also a Banach space with the norm inherited from $C_B^m(\Omega)$. Finally we define the spaces of Hölder continuous functions.

Definition 6.1.10 (Spaces of Hölder Continuous Functions)**.** *If $0 < \lambda < 1$, for a nonnegative integer m we define the space of Hölder continuous functions denoted by $C^{m,\lambda}(\bar{\Omega})$, as the subspace of $C^m(\bar{\Omega})$ consisting of those functions u for which, for $0 \leq |\alpha| \leq m$, there exists a constant K such that*

$$|D^\alpha u(x) - D^\alpha u(y)| \leq K|x-y|^\lambda, \forall x, y \in \Omega.$$

$C^{m,\lambda}(\bar{\Omega})$ is a Banach space with the norm denoted by $\|.\|_{m,\lambda}$ given by

$$\|u\|_{m,\lambda} = \|u\|_{B,m} + \max_{0\leq|\alpha|\leq m} \sup_{x,y\in\Omega} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda}, \quad x \neq y \right\} .$$

Theorem 6.1.11. *The space $C_0(\Omega)$ is dense in $L^p(\Omega)$, for $1 \leq p < \infty$.*

Proof: For the proof we need the following lemma:

Lemma 6.1.12. *Let $f \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega fu \, dx = 0, \forall u \in C_0(\Omega). \tag{6.39}$$

Then $f = 0$ a.e. in Ω.

Suppose $f \in L^1(\Omega)$ and $m(\Omega) < \infty$. Given $\varepsilon > 0$, there exists $f_1 \in C_0(\Omega)$ such that $\|f - f_1\|_1 < \varepsilon$ (the proof can be found in [6]) and thus, from (6.39) we obtain

$$|\int_\Omega f_1 u \, dx| \leq \varepsilon \|u\|_\infty, \forall u \in C_0(\Omega). \tag{6.40}$$

Defining

$$K_1 = \{x \in \Omega \ \mid \ f_1(x) \geq \varepsilon\}, \tag{6.41}$$

and

$$K_2 = \{x \in \Omega \ \mid \ f_1(x) \leq -\varepsilon\}. \tag{6.42}$$

As K_1 and K_2 are disjoint compact sets, by the Urysohn Theorem there exists $u_0 \in C_0(\Omega)$ such that

$$u_0(x) = \begin{cases} +1, & if \ \ x \ \in \ K_1, \\ -1, & if \ \ x \in K_2 \end{cases} \tag{6.43}$$

and

$$|u_0(x)| \leq 1, \forall x \in \Omega. \tag{6.44}$$

Also defining $K = K_1 \cup K_2$, we may write

$$\int_\Omega f_1 u_0 \; dx = \int_{\Omega - K} f_1 u_0 \; dx + \int_K f_1 u_0 \; dx. \tag{6.45}$$

Observe that, from (6.40)

$$\int_K |f_1| \; dx \leq \int_\Omega |f_1 u_0| \; dx \leq \varepsilon \tag{6.46}$$

so that

$$\int_\Omega |f_1| \; dx = \int_K |f_1| \; dx + \int_{\Omega - K} |f_1| \; dx \leq \varepsilon + \varepsilon m(\Omega). \tag{6.47}$$

Hence

$$\|f\|_1 \leq \|f - f_1\|_1 + \|f_1\|_1 \leq 2\varepsilon + \varepsilon m(\Omega). \tag{6.48}$$

Since $\varepsilon > 0$ is arbitrary, we have that $f = 0$ a.e. in Ω. Finally, if $m(\Omega) = \infty$, define

$$\Omega_n = \{x \in \Omega \ \mid \ \text{dist}(x, \Omega^c) > 1/n \ \ and \ \ |x| < n\}. \tag{6.49}$$

It is clear that $\Omega = \cup_{n=1}^\infty \Omega_n$ and from above $f = 0$ a.e. on $\Omega_n, \forall n \in \mathbb{N}$, so that $f = 0$ a.e. in Ω.

Finally, to finish the proof of Theorem 6.1.11, suppose $h \in L^q(\Omega)$ is such that

$$\int_\Omega hu \; dx = 0, \forall u \in C_0(\Omega). \tag{6.50}$$

Observe that $h \in L^1_{loc}(\Omega)$ since $\int_K |h| \; dx \leq \|h\|_q m(K)^{1/p} < \infty$. From last lemma $h = 0$ a.e. in Ω, which completes the proof. $\square$

Theorem 6.1.13. *$L^p(\Omega)$ is separable for any $1 \leq p < \infty$.*

Proof: The result follows from last theorem and from the fact that $C_0(K)$ is separable for each $K \subset \Omega$ compact (from the Weierstrass theorem, polynomials with rational coefficients are dense $C_0(K)$). Observe that $\Omega = \cup_{n=1}^\infty \Omega_n$, Ω_n defined as in (6.49), where $\bar{\Omega}_n$ is compact, $\forall n \in \mathbb{N}$.

Theorem 6.1.14. *We denote*

$$\tilde{u}(x) = \begin{cases} u(x), & if x \in \Omega, \\ 0, & otherwise. \end{cases}$$

Let $1 \leq p < \infty$. A bounded set $K \leq L^p(\Omega)$ is pre-compact in $L^p(\Omega)$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ and a subset $A \subset\subset \Omega$ such that for every $u \in K$ and every $h \in \mathbb{R}^n$ with $|h| < \delta$ we have

$$\int_\Omega |\tilde{u}(x+h) - \tilde{u}(x)|\, dx < \varepsilon^p, \tag{6.51}$$

and

$$\int_{\Omega - A} |u(x)|^p\, dx < \varepsilon^p. \tag{6.52}$$

Proof: See Adams [1] Theorem 2.21.

6.2 The Sobolev Spaces

Now we define the Sobolev spaces, denoted by $W^{m,p}(\Omega)$.

Definition 6.2.1 (Sobolev Spaces)**.** *We say that $u \in W^{m,p}(\Omega)$ if $u \in L^p(\Omega)$ and $D^\alpha u \in L^p(\Omega)$, for all α such that $0 \leq |\alpha| \leq m$, where the derivatives are understood in the distributional sense.*

Definition 6.2.2. *We define the norm $\|.\|_{m,p}$ for $W^{m,p}(\Omega)$, where $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, as*

$$\|u\|_{m,p} = \left\{ \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p^p \right\}^{1/p}, \quad if \quad 1 \leq p < \infty, \tag{6.53}$$

and

$$\|u\|_{m,\infty} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_\infty . \tag{6.54}$$

Theorem 6.2.3. *$W^{m,p}(\Omega)$ is a Banach space.*

Proof: Consider $\{u_n\}$ a Cauchy sequence in $W^{m,p}(\Omega)$. Then $\{D^\alpha u_n\}$ is a Cauchy sequence for each $0 \leq |\alpha| \leq m$. Since $L^p(\Omega)$ is complete there exist functions u and u_α, for $0 \leq |\alpha| \leq m$, in $L^p(\Omega)$ such that $u_n \to u$ and $D^\alpha u_n \to u_\alpha$ in $L^p(\Omega)$ as $n \to \infty$. From above $L^p(\Omega) \subset L^1_{loc}(\Omega)$ and so u_n determines a distribution $T_{u_n} \in \mathcal{D}'(\Omega)$. For any $\phi \in \mathcal{D}(\Omega)$ we have, by Hölder's inequality

$$|T_{u_n}(\phi) - T_u(\phi)| \leq \int_\Omega |u_n(x) - u(x)||\phi(x)| dx \leq \|\phi\|_q \|u_n - u\|_p. \tag{6.55}$$

Hence $T_{u_n}(\phi) \to T_u(\phi)$ for every $\phi \in \mathcal{D}(\Omega)$ as $n \to \infty$. Similarly $T_{D^\alpha u_n}(\phi) \to T_{u_\alpha}(\phi)$ for every $\phi \in \mathcal{D}(\Omega)$. We have that

$$T_{u_\alpha}(\phi) = \lim_{n\to\infty} T_{D^\alpha u_n}(\alpha) = \lim_{n\to\infty}(-1)^{|\alpha|}T_{u_n}(D^\alpha\phi) = (-1)T_u(D^\alpha\phi) = T_{D^\alpha u}(\phi), \tag{6.56}$$

for every $\phi \in \mathcal{D}(\Omega)$. Thus $u_\alpha = D^\alpha u$ in the sense of distributions, for $0 \leq |\alpha| \leq m$, and $u \in W^{m,p}(\Omega)$. As $\lim_{n\to\infty} \|u - u_n\|_{m,p} = 0$, $W^{m,p}(\Omega)$ is complete. □

Remark 6.2.4. *Observe that distributional and classical derivatives coincide when the latter exist and are continuous. We define $S \subset W^{m,p}(\Omega)$ by*

$$S = \{\phi \in C^m(\Omega) \ \mid \ \|\phi\|_{m,p} < \infty\} \tag{6.57}$$

Thus, the completion of S concerning the norm $\|.\|_{m,p}$ is denoted by $H^{m,p}(\Omega)$.

Corollary 6.2.5. $H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$

Proof: Since $W^{m,p}(\Omega)$ is complete we have that $H^{m,p}(\Omega) \subset W^{m,p}(\Omega)$. □

Theorem 6.2.6. *$W^{m,p}(\Omega)$ is separable if $1 \leq p < \infty$, and is reflexive and uniformly convex if $1 < p < \infty$. Particularly, $W^{m,2}(\Omega)$ is a separable Hilbert space with the inner product*

$$(u, v)_m = \sum_{0\leq|\alpha|\leq m} \langle D^\alpha u, D^\alpha v\rangle_{L^2(\Omega)}. \tag{6.58}$$

Proof: We can see $W^{m,p}(\Omega)$ as a subspace of $L^p(\Omega, \mathbb{R}^N)$, where $N = \sum_{0\leq|\alpha|\leq m} 1$. From the relevant properties for $L^p(\Omega)$, we have that $L^p(\Omega; \mathbb{R}^N)$ is a reflexive and uniformly convex for $1 < p < \infty$ and separable for $1 \leq p < \infty$. Given $u \in W^{m,p}(\Omega)$, we may associate the vector $Pu \in L^p(\Omega; \mathbb{R}^N)$ defined by

$$Pu = \{D^\alpha u\}_{0\leq|\alpha|\leq m}. \tag{6.59}$$

Since $\|Pu\|_{p^N} = \|u\|_{m,p}$, we have that $W^{m,p}$ is closed subspace of $L^p(\Omega; \mathbb{R}^N)$. Thus from theorem 1.21 in Adams [1], we have that $W^{m,p}(\Omega)$ is separable if $1 \leq p < \infty$ and, reflexive and uniformly convex, if $1 < p < \infty$. □

Lemma 6.2.7. *Let $1 \leq p < \infty$ and define $U = L^p(\Omega; \mathbb{R}^N)$. For every continuous linear functional f on U, there exists a unique $v \in L^q(\Omega; \mathbb{R}^N) = U^*$ such that*

$$f(u) = \sum_{i=1}^N \langle u_i, v_i\rangle, \forall u \in U. \tag{6.60}$$

Moreover,

$$\|f\|_{U^*} = \|v\|_{q^N}, \tag{6.61}$$

where $\|.\|_{q^N} = \|.\|_{L^q(\Omega,\mathbb{R}^N)}$.

Proof: For $u = (u_1, ..., u_n) \in L^p(\Omega; \mathbb{R}^N)$ we may write

$$f(u) = f((u_1, 0, ..., 0)) + ... + f((0, ..., 0, u_j, 0, ..., 0)) + ... + f((0, ..., 0, u_n)), \tag{6.62}$$

and since $f((0, ..., 0, u_j, 0, ..., 0))$ is continuous linear functional on $u_j \in L^p(\Omega)$, there exists a unique $v_j \in L^q(\Omega)$ such that $f(0, ..., 0, u_j, 0, ..., 0) = \langle u_j, v_j\rangle_{L^2(\Omega)}, \forall u_j \in L^p(\Omega)$, $\forall\ 1 \leq j \leq N$, so that

$$f(u) = \sum_{i=1}^{N} \langle u_i, v_i\rangle, \forall u \in U. \tag{6.63}$$

From Hölder's inequality we obtain

$$|f(u)| \leq \sum_{j=1}^{N} \|u_j\|_p \|v_j\|_q \leq \|u\|_{p^N} \|v\|_{q^N}, \tag{6.64}$$

and hence $\|f\|_{U^*} \leq \|v\|_{q^N}$. The equality in (6.64) is achieved for $u \in L^p(\Omega, \mathbb{R}^N)$, $1 < p < \infty$ such that

$$u_j(x) = \begin{cases} |v_j|^{q-2}\bar{v}_j, & if \quad v_j \neq 0 \\ 0, & if \quad v_j = 0. \end{cases} \tag{6.65}$$

If $p = 1$ choose k such that $\|v_k\|_\infty = \max_{1 \leq j \leq N} \|v_j\|_\infty$. Given $\varepsilon > 0$, there is a measurable set A such that $m(A) > 0$ and $|v_k(x)| \geq \|v_k\|_\infty - \varepsilon, \forall x \in A$. Defining $u(x)$ as

$$u_i(x) = \begin{cases} \bar{v}_k/v_k, \quad if \quad i = k, \quad x \in A \quad and \quad v_k(x) \neq 0 \\ 0, \qquad\qquad\qquad\qquad\qquad\qquad otherwise, \end{cases} \tag{6.66}$$

we have

$$f(u_k) = \langle u, v_k\rangle_{L^2(\Omega)} = \int_A |v_k| dx \geq (\|(v_k\|_\infty - \varepsilon)\|u\|_1 = (\|v\|_{\infty^N} - \varepsilon)\|u_k\|_{1^N}. \tag{6.67}$$

Since ε is arbitrary, the proof is complete. $\square$

Theorem 6.2.8. *Let $1 \leq p < \infty$. Given a continuous linear functional f on $W^{m,p}(\Omega)$, there exists $v \in L^q(\Omega, \mathbb{R}^N)$ such that*

$$f(u) = \sum_{0 \leq |\alpha| \leq m} \langle D^\alpha u, v_\alpha\rangle_{L^2(\Omega)}. \tag{6.68}$$

Proof: Consider f a continuous linear operator on $U = W^{m,p}(\Omega)$. By the Hahn Banach Theorem, we can extend f to $\tilde{f}$, on $L^p(\Omega; \mathbb{R}^N)$, so that $\|\tilde{f}\|_{q^N} = \|f\|_{U^*}$ and by the last theorem, there exists $\{v_\alpha\} \in L^q(\Omega; \mathbb{R}^N)$ such that

$$\tilde{f}(\hat{u}) = \sum_{0 \leq |\alpha| \leq m} \langle \hat{u}_\alpha, v_\alpha\rangle_{L^2(\Omega)}, \forall v \in L^p(\Omega; \mathbb{R}^N). \tag{6.69}$$

In particular for $u \in W^{m,p}(\Omega)$, defining $\hat{u} = \{D^\alpha u\} \in L^p(\Omega; \mathbb{R}^N)$ we obtain

$$f(u) = \tilde{f}(\hat{u}) = \sum_{1 \leq |\alpha| \leq m} \langle D^\alpha u, v_\alpha \rangle_{L^2(\Omega)}. \tag{6.70}$$

Finally, observe that, also from the Hahn-Banach theorem $\|f\|_{U^*} = \|\tilde{f}\|_{q^N} = \|v\|_{q^N}$. $\square$
Now we state some density results. The proofs and more details may be found in Adams [1] Chapters 2 and 3.

Proposition 6.2.9. *$C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ if $1 \leq p < \infty$.*

Theorem 6.2.10 (Meyers and Serrin)**.** *If $1 \leq p \leq \infty$, then $H^{m,p}(\Omega) = W^{m,p}(\Omega)$.*

6.3 The Sobolev Imbedding Theorem

We start with some preliminary definitions.

Definition 6.3.1 (The Cone Condition)**.** *The set $\Omega \subset \mathbb{R}^n$ has the cone property if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C.*

Definition 6.3.2 (The Strong Local Lipschitz Property)**.** *Ω has the strong local Lipschitz property if there exist positive numbers δ and M, a locally finite open cover $\{\mathcal{U}_j\}$ of bdryΩ, and for each $\mathcal{U}_j$ a real valued function f_j of $n-1$ variables, such that the following conditions hold.*

1. *For some $N \in \mathbb{N}$, every collection of $N+1$ of the sets $\mathcal{U}_j$ has empty intersection.*
2. *For every $x, y \in \Omega_\delta = \{x \in \Omega \mid dist(x, bdry\Omega) < \delta\}$ such that $|x-y| < \delta$ there exists j such that*
$$x, y \in \mathcal{V}_j = \{z \in \mathcal{U}_j \mid dist(z, bdry\mathcal{U}_j) > \delta\}. \tag{6.71}$$
3. *Each function f_j satisfies a Lipschitz condition with constant M, that is*
$$|f(\xi_1, ..., \xi_{n-1}) - f(\eta_1, ..., \eta_{n-1})| \leq M|(\xi_1 - \eta_1, ..., \xi_{n-1} - \eta_{n-1})|. \tag{6.72}$$
4. *For some Cartesian coordinate system $(\xi_{j,1}, ..., \xi_{j,n})$ in $\mathcal{U}_j$ the set $\Omega \cap \mathcal{U}_j$ is represented by the inequality $\xi_{j,n} < f_j(\xi_{j,1}, ..., \xi_{j,n-1})$.*

6.3.1 The Statement of Sobolev Imbedding Theorem

Now we present the Sobolev Imbedding Theorem. For a proof see Adams [1], Chapter 3. We recall that for normed spaces X, Y the notation

$$X \to Y$$

means that $X \subset Y$ and there exists a constant $K > 0$ such that

$$\|u\|_Y \leq K\|u\|_X, \forall u \in X.$$

If in addition the imbedding is compact then for any bounded sequence $\{u_n\} \subset X$ there exists a convergent subsequence $\{u_{n_k}\}$, which converges to some u in the norm $\|.\|_Y$.

Theorem 6.3.3 (The Sobolev Imbedding Theorem)**.** *Let Ω be a domain in $\mathbb{R}^n$ and, for $1 \leq k \leq n$, let Ω_k be the intersection of Ω with a plane of dimension k in $\mathbb{R}^n$ (if k=n, then $\Omega_k = \Omega$). Let $j \geq 0$ and $m \geq 1$ be integers and let $1 \leq p < \infty$.*

1. ***Part I.*** *Suppose Ω satisfies the cone condition.*

   (a) ***Case A*** *If either $mp > n$ or $m = n$ and $p = 1$ then*

   $$W^{j+m,p}(\Omega) \to C_B^j(\Omega). \tag{6.73}$$

   Moreover, if $1 \leq k \leq n$, then

   $$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \quad for \quad p \leq q \leq \infty, \tag{6.74}$$

   and, in particular

   $$W^{m,p}(\Omega) \to L^q(\Omega), \quad for \quad p \leq q \leq \infty. \tag{6.75}$$

   (b) ***Case B*** *If $1 \leq k \leq n$ and $mp = n$, then*

   $$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \quad for \quad p \leq q < \infty, \tag{6.76}$$

   and, in particular

   $$W^{m,p}(\Omega) \to L^q(\Omega), \quad for \quad p \leq q < \infty. \tag{6.77}$$

   (c) ***Case C*** *If $mp < n$ and either $n - mp < k \leq n$ or $p = 1$ and $n - m \leq k \leq n$, then*

   $$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega_k), \quad for \quad p \leq q \leq p^* = \frac{kp}{n - mp}, \tag{6.78}$$

   and, in particular

   $$W^{m,p}(\Omega) \to L^q(\Omega), \quad for \quad p \leq q \leq p^* = \frac{kp}{n - mp}. \tag{6.79}$$

   The imbedding constants depend only on n, m, p, q, j, k and the dimensions of the C in the cone condition.

2. ***Part II.*** *Suppose* Ω *satisfies the strong local Lipschitz condition. Then the target* C_B^j *in the first imbedding above may be replaced by* $C^j(\bar{\Omega})$, *and the imbedding can be further refined as follows:*
   If $mp > n > (m-1)p$, *then*
$$W^{j+m,p} \to C^{j,\lambda}(\bar{\Omega}), \quad for \quad 0 < \lambda \leq m - (n/p), \tag{6.80}$$
   and if $n = (m-1)p$, *then*
$$W^{j+m,p} \to C^{j,\lambda}(\bar{\Omega}), \quad for \quad 0 < \lambda \leq 1. \tag{6.81}$$
   Also, if $n = m-1$ *and* $p = 1$, *then (6.81) holds for* $\lambda = 1$ *as well.*

3. ***Part III.*** *All imbeddings in Parts A and B are valid for arbitrary domains* Ω *if the* $W-$ *space undergoing the imbedding is replaced with the corresponding* W_0- *space.*

6.4 The Rellich-Kondrachov Theorem

In this section we present the Rellich-Kondrachov theorem. We start with the following result which is proved in [1].

Theorem 6.4.1. *Let* m *be a non-negative integer and let* $0 < \nu < \lambda \leq 1$. *Then following imbeddings exist:*

1. $C^{m+1}(\bar{\Omega}) \to C^m(\bar{\Omega})$,
2. $C^{m,\lambda}(\bar{\Omega}) \to C^m(\bar{\Omega})$,
3. $C^{m,\lambda}(\bar{\Omega}) \to C^{m,\nu}(\bar{\Omega})$.

If Ω *is bounded, then imbeddings* 2 *and* 3 *are compact.*

Theorem 6.4.2 (Rellich-Kondrachov)**.** *Let* Ω *be a bounded domain. Let* j, m *be integers,* $j \geq 0, m \geq 1$, *and let* $1 \leq p < \infty$.

1. ***Part I***- *If* Ω *has the cone property and* $mp \leq n$, *then the following imbeddings are compact:*
$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega), \quad 0 < n - mp < n \quad and \quad 1 \leq np/(n-mp), \tag{6.82}$$
$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega), \quad if \quad n = mp, \quad 1 \leq q < \infty. \tag{6.83}$$

2. ***Part II***- *If Ω has the cone property and $mp > n$, then the following imbeddings are compact:*

$$W^{j+m,p} \to C^j_B(\Omega), \tag{6.84}$$

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega), \quad if \quad 1 \le q \le \infty. \tag{6.85}$$

3. ***Part III***- *If Ω has the strong Lipschitz property, then the following imbeddings are compact:*

$$W^{j+m,p}(\Omega) \to C^j(\Omega), \quad if \quad mp > n, \tag{6.86}$$

$$W^{j+m,p}(\Omega) \to C^{j,\lambda}(\Omega), \quad if \quad mp > n \ge (m-1)p \quad and \quad 0 < \lambda < m - n/p. \tag{6.87}$$

4. ***Part IV***- *All the above imbeddings are compact if we replace $W^{j+m,p}(\Omega)$ by $W_0^{j+m,p}(\Omega)$.*

Remark 6.4.3. *Given X, Y, Z spaces, for which we have the imbeddings $X \to Y$ and $Y \to Z$ and if one of these imbeddings is compact then the composite imbedding $X \to Z$ is compact. Since the extension operator $u \to \tilde{u}$ where $\tilde{u}(x) = u(x)$ if $x \in \Omega$ and $\tilde{u}(x) = 0$ otherwise, defines an imbedding $W_0^{j+m,p}(\Omega) \to W^{j+m,p}(\mathbb{R}^n)$ we have that Part-IV of above theorem follows from the application of Parts I-III to $\mathbb{R}^n$ (despite the fact we are assuming Ω bounded, the general results may be found in Adams [1]).*

Remark 6.4.4. *To prove the compactness of any of above imbeddings it is sufficient to consider the case $j = 0$. Suppose, for example, that the first imbedding has been proved for $j = 0$. For $j \ge 1$ and $\{u_i\}$ bounded sequence in $W^{j+m,p}(\Omega)$ we have that $\{D^\alpha u_i\}$ is bounded in $W^{m,p}(\Omega)$ for each α such that $|\alpha| \le j$. From the case $j = 0$ it is possible to extract a subsequence (similarly to a diagonal process) $\{u_{i_k}\}$ for which $\{D^\alpha u_{i_k}\}$ converges in $L^q(\Omega)$ for each α such that $|\alpha| \le j$, so that $\{u_{i_k}\}$ converges in $W^{j,q}(\Omega)$.*

Remark 6.4.5. *Since Ω is bounded, $C_B^0(\Omega) \to L^q(\Omega)$ for $1 \le q \le \infty$. In fact*

$$\|u\|_{0,q,\Omega} \le \|u\|_{C_B^0}[vol(\Omega)]^{1/q}. \tag{6.88}$$

Thus the compactness of (6.85) (for $j = 0$) follows from that of (6.84).

Proof of Part III : If $mp > n > (m-1)p$ and $0 < \lambda < (m-n)/p$, then there exists μ such that $\lambda < \mu < m - (n/p)$. Since Ω is bounded, the imbedding $C^{0,\mu}(\Omega) \to C^{0,\lambda}(\Omega)$ is compact by Theorem 1.31 in Adams [1]. Since by the Sobolev Imbedding Theorem we have $W^{m,p}(\Omega) \to C^{0,\mu}(\Omega)$, we have that imbedding (6.87) is compact.

If $mp > n$, let j^* be the non-negative integer satisfying $(m-j^*)p > n \ge (m-j^*-1)p$. Thus we have the chain of imbeddings

$$W^{m,p}(\Omega) \to W^{m-j^*,p}(\Omega) \to C^{0,\mu}(\Omega) \to C(\Omega), \tag{6.89}$$

where $0 < \mu < m - j^* - (n/p)$. The last imbedding in (6.89) is compact by Theorem 1.31 in Adams [1], so that (6.86) is compact for $j = 0$.

Proof of Part II: Supposing that Ω has the cone property, we may write $\Omega = \cup_{k=1}^{M} \Omega_k$, where each Ω_k has the strong local Lipschitz property. If $mp > n$, then $W^{m,p}(\Omega) \to W^{m,p}(\Omega_k) \to C(\Omega_k)$, the latter imbedding being compact as proved above. If $\{u_i\}$ is a bounded sequence in $W^{m,p}(\Omega)$, we may select a subsequence $\{u_{il}\}$ whose restriction to Ω_k converges in $C(\Omega_k)$, for all k such that $1 \leq k \leq M$. Thus, $\{u_{il}\}$ converges in $C_B^0(\Omega)$ proving that (6.84) is compact for $j = 0$. Therefore from the above remarks, (6.85) is also compact. For the proof of Part I, we need the following lemma:

Lemma 6.4.6. *Let Ω be an bounded domain in $\mathbb{R}^n$. Let $1 \leq q_1 \leq q_0$ and suppose*

$$W^{m,p}(\Omega) \to L^{q_0}(\Omega), \tag{6.90}$$

$$W^{m,p}(\Omega) \to L^{q_1}. \tag{6.91}$$

Suppose also that (6.91) is compact. If $q_1 \leq q < q_0$, then the imbedding

$$W^{m,p} \to L^q(\Omega) \tag{6.92}$$

is compact.

Proof: Define $\lambda = q_1(q_0 - q)/(q(q_0 - q_1))$ and $\mu = q_0(q - q_1)/(q(q_0 - q_1))$. We have that $\lambda > 0$ and $\mu \geq 0$. From Hölder's inequality and (6.90) there exists $K \in \mathbb{R}^+$ such that,

$$\|u\|_{0,q,\Omega} \leq \|u\|_{0.q_1,\Omega}^{\lambda} \|u\|_{0,q_0,\Omega}^{\mu} \leq K \|u\|_{0,q_1,\Omega} \|u\|_{m,p,\Omega}^{\mu}, \forall u \in W^{m,p}(\Omega). \tag{6.93}$$

Thus considering a sequence $\{u_i\}$ bounded in $W^{m,p}(\Omega)$, since (6.91) is compact there exists a subsequence $\{u_{nk}\}$ that converges, and is therefore a Cauchy sequence in $L^{q_1}(\Omega)$. From (6.93), $\{u_{nk}\}$ is also a Cauchy sequence in $L^q(\Omega)$, so that (6.92) is compact.

Proof of Part I: Consider $j = 0$. Define $q_0 = np/(n - mp)$. To prove the imbedding

$$W^{m,p}(\Omega) \to L^q(\Omega), \quad 1 \leq q < q_0, \tag{6.94}$$

is compact, by last lemma it suffices to do so only for $q = 1$. For $k \in \mathbb{N}$, define

$$\Omega_k = \{x \in \Omega \mid dist(x, \partial\Omega) > 2/k\}. \tag{6.95}$$

Suppose A is set of functions bounded in $W^{m,p}(\Omega)$. Also, suppose given $\varepsilon > 0$, and define, for $u \in W^{m,p}$, $\tilde{u}(x) = u(x)$ if $x \in \Omega$, $\tilde{u}(x) = 0$, otherwise. From Hölder's inequality and considering that $W^{m,p}(\Omega) \to L^{q_0}(\Omega)$, we have

$$\int_{\Omega - \Omega_k} |u(x)| dx \leq \left\{ \int_{\Omega - \Omega_k} |u(x)|^{q_0} dx \right\}^{1/q_0} \left\{ \int_{\Omega - \Omega_k} 1 dx \right\}^{1-1/q_0} \leq K_1 \|u\|_{m,p,\Omega} [vol(\Omega - \Omega_k)]^{1-1/q_0}, \tag{6.96}$$

Thus, since u is bounded in $W^{m,p}$, there exists $K_0 \in \mathbb{N}$ such that if $k \geq K_0$ then

$$\int_{\Omega - \Omega_k} |u(x)| dx < \varepsilon, \tag{6.97}$$

and, for every $h \in \mathbb{R}^n$,

$$\int_{\Omega-\Omega_k} |\tilde{u}(x+h) - \tilde{u}(x)|dx < 2\varepsilon. \tag{6.98}$$

Observe that if $|h| < 1/k$, then $x + th \in \Omega_{2k}$ provided $x \in \Omega_k$ and $0 \leq t \leq 1$. If $u \in C^\infty(\Omega)$ we have that

$$\begin{aligned}
\int_{\Omega_k} |u(x+h) - u(x)| &\leq \int_{\Omega_k} dx \int_0^1 |\frac{du(x+th)}{dt}|dt \\
&\leq |h| \int_0^1 dt \int_{\Omega_{2k}} |\nabla u(y)|dy \leq |h| \|u\|_{1,1,\Omega} \\
&\leq K_2 |h| \|u\|_{m,p,\Omega}.
\end{aligned} \tag{6.99}$$

Since $C^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$, for $|h|$ sufficiently small

$$\int_\Omega |\tilde{u}(x+h) - \tilde{u}(x)|dx < 3\varepsilon, \forall u \in A, \tag{6.100}$$

which means that A is pre-compact in $L^1(\Omega)$ and therefore from Theorem 2.21 in Adams [1], the imbedding indicated (6.94) is compact for $q = 1$, which completes the proof.

Chapter 7

Basic Concepts on Convex Analysis

7.1 Convex Sets and Convex Functions

Let S be a subset of a vector space U. We recall that S is convex if given $u, v \in S$ then

$$\lambda u + (1-\lambda)v \in S, \forall \lambda \in [0,1]. \tag{7.1}$$

Definition 7.1.1 (Convex hull)**.** *Let S be a subset of a vector space U, we define the convex hull of S, denoted by $Co(S)$ as*

$$Co(S) = \left\{ \sum_{i=1}^{n} \lambda_i u_i \ \mid \ n \in \mathbb{N}, \ \ \sum_{i=1}^{n} \lambda_i = 1, \ \ \lambda_i \geq 0, \ \ u_i \in S, \forall i \in \{1, ..., n\} \right\}. \tag{7.2}$$

Definition 7.1.2 (Convex Functional)**.** *Let S be convex subset of the vector space U. A functional $F : S \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ is said to be convex if*

$$F(\lambda u + (1-\lambda)v) \leq \lambda F(u) + (1-\lambda)F(v), \forall u, v \in S, \lambda \in [0,1]. \tag{7.3}$$

Definition 7.1.3 (Lower Semi-continuity)**.** *Let U be Banach space. We say that $F : U \to \bar{\mathbb{R}}$ is lower semi-continuous (l.s.c.) at $u \in U$, if*

$$\liminf_{n \to +\infty} F(u_n) \geq F(u), \tag{7.4}$$

whenever

$$u_n \to u \ \ \textit{strongly} \ \ (\textit{in} \ \ \textit{norm}). \tag{7.5}$$

Definition 7.1.4 (Weak Lower Semi-Continuity)**.** *Let U be Banach space. We say that $F : U \to \bar{\mathbb{R}}$ is weakly lower semi-continuous (w.l.s.c.) at $u \in U$, if*

$$\liminf_{n \to +\infty} F(u_n) \geq F(u), \tag{7.6}$$

whenever

$$u_n \rightharpoonup u, \ \ \textit{weakly}. \tag{7.7}$$

Remark 7.1.5. *We say that F is a (weak) lower semi-continuous function, if $F : U \to \bar{\mathbb{R}}$ is (weak) lower semi-continuous $\forall u \in U$.*

Definition 7.1.6 (Epigraph)**.** *Given $F : U \to \bar{\mathbb{R}}$ we define its Epigraph, denoted by $Epi(F)$ as*

$$Epi(F) = \{(u, a) \in U \times \mathbb{R} \ \ | \ \ a \geq F(u)\}.$$

Now we present a very important result but which we do not prove.

Proposition 7.1.7. *A function $F : U \to \bar{\mathbb{R}}$ is l.s.c. (lower semi-continuous) if and only if its epigraph is closed.*

Corollary 7.1.8. *Every convex l.s.c. function $F : U \to \bar{\mathbb{R}}$ is also w.l.s.c. (weakly lower semi-continuous).*

Proof: The result follows from the fact that the epigraph of F is convex and closed convex sets are weakly closed. $\square$

Definition 7.1.9 (Affine Continuous Function)**.** *Let U be a Banach space. A functional $F : U \to \mathbb{R}$ is said to be affine continuous if there exist $u^* \in U^*$ and $\alpha \in \mathbb{R}$ such that*

$$F(u) = \langle u, u^* \rangle_U + \alpha, \forall u \in U. \tag{7.8}$$

Definition 7.1.10 ($\Gamma(U)$)**.** *Let U be a Banach space, we say that $F : U \to \bar{\mathbb{R}}$ belongs to $\Gamma(U)$ and write $F \in \Gamma(U)$ if F can be represented as the point-wise supremum of a family of affine continuous functions. If $F \in \Gamma(U)$, $F \neq +\infty$ and $F \neq -\infty$ for some $u \in U$ then we write $F \in \Gamma_0(U)$.*

Proposition 7.1.11. *Let U be a Banach space, then $F \in \Gamma(U)$ if and only if F is convex and l.s.c., and if F takes the value $-\infty$ then $F \equiv -\infty$.*

Definition 7.1.12 (Convex Envelope)**.** *Let U be a Banach space. Given $F : U \to \bar{\mathbb{R}}$, we define its convex envelope, denoted by $CF : U \to \bar{\mathbb{R}}$ as*

$$CF(u) = \sup_{(u^*, \alpha) \in A^*} \{\langle u, u^* \rangle + \alpha\}, \tag{7.9}$$

where

$$A^* = \{(u^*, \alpha) \in U^* \times \mathbb{R} \ \ | \ \ \langle v, u^* \rangle_U + \alpha \leq F(v), \forall v \in U\} \tag{7.10}$$

Definition 7.1.13 (Polar Functionals)**.** *Given $F : U \to \bar{\mathbb{R}}$, we define the related polar functional, denoted by $F^* : U^* \to \bar{\mathbb{R}}$, as*

$$F^*(u^*) = \sup_{u \in U} \{\langle u, u^* \rangle_U - F(u)\}, \forall u^* \in U^*. \tag{7.11}$$

Definition 7.1.14 (Bipolar Functional)**.** *Given $F : U \to \bar{\mathbb{R}}$, we define the related bipolar functional, denoted by $F^{**} : U \to \bar{\mathbb{R}}$, as*

$$F^{**}(u) = \sup_{u^* \in U^*} \{\langle u, u^* \rangle_U - F^*(u^*)\}, \forall u \in U. \tag{7.12}$$

Proposition 7.1.15. *Given $F : U \to \bar{\mathbb{R}}$, then $F^{**}(u) = CF(u)$ and in particular if $F \in \Gamma(U)$ then $F^{**}(u) = F(u)$.*

Proof: By definition, the convex envelope of F is the supremum of all affine continuous minorants of F. We can consider only the maximal minorants, that functions of the form

$$u \mapsto \langle u, u^* \rangle_U - F^*(u^*). \tag{7.13}$$

Thus,

$$CF(u) = \sup_{u^* \in U^*} \{\langle u, u^* \rangle_U - F^*(u^*)\} = F^{**}(u). \quad \square \tag{7.14}$$

Corollary 7.1.16. *Given $F : U \to \bar{\mathbb{R}}$, we have $F^* = F^{***}$.*

Proof Since $F^{**} \leq F$ we obtain

$$F^* \leq F^{***}. \tag{7.15}$$

On the other hand, we have

$$F^{**}(u) \geq \langle u, u^* \rangle_U - F^*(u^*), \tag{7.16}$$

so that

$$F^{***}(u^*) = \sup_{u \in U} \{\langle u, u^* \rangle_U - F^{**}(u)\} \leq F^*(u^*). \tag{7.17}$$

From (7.15) and (7.17) we obtain $F^*(u^*) = F^{***}(u^*)$. $\square$

Definition 7.1.17 (Gâteaux Differentiability)**.** *A functional $F : U \to \bar{\mathbb{R}}$ is said to be Gâteaux differentiable at $u \in U$ if there exists $u^* \in U^*$ such that:*

$$\lim_{\lambda \to 0} \frac{F(u + \lambda h) - F(u)}{\lambda} = \langle h, u^* \rangle_U, \quad \forall h \in U. \tag{7.18}$$

The vector u^* is said to be the Gâteaux derivative of $F : U \to \mathbb{R}$ at u and may be denoted as follows:

$$u^* = \frac{\partial F(u)}{\partial u} \text{ or } u^* = \delta F(u) \tag{7.19}$$

Definition 7.1.18 (Sub-gradients)**.** *Given $F : U \to \bar{\mathbb{R}}$, we define the set of sub-gradients of F at u, denoted by $\partial F(u)$ as:*

$$\partial F(u) = \{u^* \in U^*, \text{ such that } \langle v - u, u^* \rangle_U + F(u) \leq F(v), \quad \forall v \in U\}. \tag{7.20}$$

Definition 7.1.19 (Adjoint Operator)**.** *Let U and Y be Banach spaces and $\Lambda : U \to Y$ a continuous linear operator. The Adjoint Operator related to Λ, denoted by $\Lambda^* : Y^* \to U^*$ is defined through the equation:*

$$\langle u, \Lambda^* v^* \rangle_U = \langle \Lambda u, v^* \rangle_Y, \quad \forall u \in U, \quad v^* \in Y^*. \tag{7.21}$$

Lemma 7.1.20 (Continuity of Convex Functions)**.** *If in a neighborhood of a point $u \in U$, a convex function F is bounded above by a finite constant, then F is continuous at u.*

Proof:*By translation, we may reduce the problem to the case where $u = \theta$ and $F(u) = 0$. Let $\mathcal{V}$ be a neighborhood of origin such that $F(v) \leq a < +\infty, \forall v \in \mathcal{V}$. Define $\mathcal{W} = \mathcal{V} \cap (-\mathcal{V})$ (which is a symmetric neighborhood of origin). Pick $\varepsilon \in (0,1)$. If $v \in \varepsilon\mathcal{W}$, since F is convex and*

$$\frac{v}{\varepsilon} \in \mathcal{V} \tag{7.22}$$

we may infer that

$$F(v) \leq (1-\varepsilon)F(0) + \varepsilon F(v/\varepsilon) \leq \varepsilon a. \tag{7.23}$$

Also

$$\frac{-v}{\varepsilon} \in \mathcal{V}. \tag{7.24}$$

Thus

$$F(v) \geq (1+\varepsilon)F(0) - \varepsilon F(-v/\varepsilon) \geq -\varepsilon a. \tag{7.25}$$

Therefore

$$|F(v)| \leq \varepsilon a, \forall v \in \varepsilon\mathcal{W}, \tag{7.26}$$

that is, F is continuous at $u = \theta$. $\square$

Proposition 7.1.21. *Let $F : U \to \bar{\mathbb{R}}$ be a convex function finite and continuous at $u \in U$. Then $\partial F(u) \neq \emptyset$.*

Proof: *Since F is convex, $Epi(F)$ is convex, as F is continuous at u, we have that $Epi(F)$ is non-empty. Observe that $(u, F(u))$ belongs to the boundary of $Epi(F)$, so that denoting $A = Epi(F)$, we may separate $(u, F(u))$ from $\mathring{A}$ by a closed hyper-plane H, which may be written as*

$$H = \{(v, a) \in U \times \mathbb{R} \mid \langle v, u^* \rangle_U + \alpha a = \beta\}, \tag{7.27}$$

for some fixed $\alpha, \beta \in \mathbb{R}$ and $u^* \in U^*$, so that

$$\langle v, u^* \rangle_U + \alpha a \geq \beta, \forall (v, a) \in Epi(F), \tag{7.28}$$

and

$$\langle u, u^* \rangle_U + \alpha F(u) = \beta, \tag{7.29}$$

where $(\alpha, \beta, u^*) \neq (0, 0, \theta)$. Suppose $\alpha = 0$, thus we have

$$\langle v - u, u^* \rangle_U \geq 0, \forall v \in U, \tag{7.30}$$

and thus we obtain $u^* = \theta$, and $\beta = 0$. Therefore we may assume $\alpha > 0$ (considering (7.28)) so that $\forall v \in U$ we have

$$\frac{\beta}{\alpha} - \langle v, u^*/\alpha \rangle_U \leq F(v), \tag{7.31}$$

and

$$\frac{\beta}{\alpha} - \langle u, u^*/\alpha \rangle_U = F(u), \tag{7.32}$$

or

$$\langle v - u, -u^*/\alpha \rangle_U + F(u) \leq F(v), \forall v \in U, \tag{7.33}$$

so that

$$-u^*/\alpha \in \partial F(u), \quad \square \tag{7.34}$$

Definition 7.1.22 (Carathéodory Mapping)**.** *Let $S \subset \mathbb{R}^n$ be an open set, we say that that $g : S \times \mathbb{R}^l \to \mathbb{R}$ is a Carathéodory mapping if:*

$$\forall \xi \in \mathbb{R}^l, \quad x \mapsto g(x, \xi) \quad is \quad a \quad measurable \quad function,$$

and

$$for \quad almost \quad all \quad x \in S, \quad \xi \mapsto g(x, \xi) \quad is \quad a \quad continuous \quad function.$$

The proof of next results may be found in Ekeland and Temam [14].

Proposition 7.1.23. *Let E and F be two Banach spaces, S a Borel subset of $\mathbb{R}^n$, and $g : S \times E \to F$ a Carathéodory mapping. For each measurable function $u : S \to E$, let $G_1(u)$ be the measurable function $x \mapsto g(x, u(x)) \in F$.*

If G_1 maps $L^p(S, E)$ into $L^r(S, F)$ for $1 \leq p, r < \infty$, then G_1 is continuous in the norm topology.

For the functional $G : U \to \mathbb{R}$, defined by $G(u) = \int_S g(x, u(x)) dS$, where $U = U^* = [L^2(S)]^l$ (this is a especial case of the more general hypothesis presented in [14]) we have the following result.

Proposition 7.1.24. *Considering the last proposition we can express $G^* : U^* \to \bar{\mathbb{R}}$ as :*

$$G^*(u^*) = \int_S g^*(x, u^*(x)) dS, \tag{7.35}$$

where $g^*(x, y) = \sup_{\eta \in \mathbb{R}^l} (y \cdot \eta - g(x, \eta))$, almost everywhere in S.

For non-convex functionals it may be sometimes difficult to express analytically conditions for a global extremum. This fact motivates the definition of Legendre Transform, which is established through a local extremum.

Definition 7.1.25 (Legendre's Transform and Associated Functional). *Consider a differentiable function $g : \mathbb{R}^n \to \mathbb{R}$. Its Legendre Transform, denoted by $g_L^* : R_L^n \to \mathbb{R}$ is expressed as:*

$$g_L^*(y^*) = x_{0^i} \cdot y_i^* - g(x_0), \tag{7.36}$$

where x_0 is the solution of the system:

$$y_i^* = \frac{\partial g(x_0)}{\partial x_i}, \tag{7.37}$$

and $R_L^n = \{y^* \in \mathbb{R}^n$ such that equation (7.37) has a unique solution$\}$.

Furthermore, considering the functional $G : Y \to \mathbb{R}$ defined as $G(v) = \int_S g(v)dS$, we define the Associated Legendre Transform Functional, denoted by $G_L^* : Y_L^* \to \mathbb{R}$ as:

$$G_L^*(v^*) = \int_S g_L^*(v^*)dS, \tag{7.38}$$

where $Y_L^* = \{v^* \in Y^* \mid v^*(x) \in R_L^n, \quad a.e. \quad in \quad S\}$.

About the Legendre transform we still have the following results:

Proposition 7.1.26. *Considering the last definitions, suppose that for each $y^* \in R_L^n$ at least in a neighborhood (of y^*) it is possible to define a differentiable function by the expression*

$$x_0(y^*) = [\frac{\partial g}{\partial x}]^{-1}(y^*). \tag{7.39}$$

Then, $\forall \quad i \in \{1, ..., n\}$we may write:

$$y_i^* = \frac{\partial g(x_0)}{\partial x_i} \Leftrightarrow x_{0^i} = \frac{\partial g_L^*(y^*)}{\partial y_i^*} \tag{7.40}$$

Proof: Suppose firstly that:

$$y_i^* = \frac{\partial g(x_0)}{\partial x_i}, \quad \forall \quad i \in \{1, ..., n\}, \tag{7.41}$$

thus:

$$g_L^*(y^*) = y_i^* x_{0^i} - g(x_0) \tag{7.42}$$

and taking derivatives for this expression we have:

$$\frac{\partial g_L^*(y^*)}{\partial y_i^*} = y_j^* \frac{\partial x_{0^j}}{\partial y_i^*} + x_{0^i} - \frac{\partial g(x_0)}{\partial x_j} \frac{\partial x_{0^j}}{\partial y_i^*}, \tag{7.43}$$

or

$$\frac{\partial g_L^*(y^*)}{\partial y_i^*} = (y_j^* - \frac{\partial g(x_0)}{\partial x_j}) \frac{\partial x_{0^j}}{\partial y_i^*} + x_{0^i} \tag{7.44}$$

which from (7.41) implies that:

$$\frac{\partial g_L^*(y^*)}{\partial y_i^*} = x_{0^i}, \quad \forall \ \ i \quad \in \{1, ..., n\}. \tag{7.45}$$

This completes the first half of the proof. Conversely, suppose now that:

$$x_{0^i} = \frac{\partial g_L^*(y^*)}{\partial y_i^*}, \quad \forall \ \ i \in \{1, ..., n\}. \tag{7.46}$$

As $y^* \in R_L^n$ there exists $\bar{x}_0 \in \mathbb{R}^n$ such that:

$$y_i^* = \frac{\partial g(\bar{x}_0)}{\partial x_i} \quad \forall \ \ i \in \{1, ..., n\}, \tag{7.47}$$

and,

$$g_L^*(y^*) = y_i^* \bar{x}_{0^i} - g(\bar{x}_0) \tag{7.48}$$

and therefore taking derivatives for this expression we can obtain:

$$\frac{\partial g_L^*(y^*)}{\partial y_i^*} = y_j^* \frac{\partial \bar{x}_{0^j}}{\partial y_i^*} + \bar{x}_{0^i} - \frac{\partial g(\bar{x}_0)}{\partial x_j} \frac{\partial \bar{x}_{0^j}}{\partial y_i^*}, \tag{7.49}$$

$\forall \ i \in \{1, ..., n\}$, so that:

$$\frac{\partial g_L^*(y^*)}{\partial y_i^*} = (y_j^* - \frac{\partial g(\bar{x}_0)}{\partial x_j}) \frac{\partial \bar{x}_{0^j}}{\partial y_i^*} + \bar{x}_{0^i} \tag{7.50}$$

$\forall \ i \in \{1, ..., n\}$, which from (7.46) and (7.47), implies that:

$$\bar{x}_{0^i} = \frac{\partial g_L^*(y^*)}{\partial y_i^*} = x_{0^i}, \quad \forall \ \ i \in \{1, ..., n\}, \tag{7.51}$$

from this and (7.47) we have:

$$y_i^* = \frac{\partial g(\bar{x}_0)}{\partial x_i} = \frac{\partial g(x_0)}{\partial x_i} \quad \forall \ \ i \in \{1, ..., n\}. \ \ \square \tag{7.52}$$

Theorem 7.1.27. *Consider the functional $J : U \to \bar{\mathbb{R}}$ defined as $J(u) = (G \circ \Lambda)(u) - \langle u, f \rangle_U$ where $\Lambda (= \{\Lambda_i\}) : U \to Y$ $(i \in \{1, ..., n\})$ is a continuous linear operator and, $G : Y \to \mathbb{R}$ is a functional that can be expressed as $G(v) = \int_S g(v) dS, \quad \forall v \in Y$ (here $g : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function that admits Legendre Transform denoted by $g_L^* : R_L^n \to \mathbb{R}$. That is, the hypothesis mentioned at Proposition 7.1.26 are satisfied).*

Under these assumptions we have:

$$\delta J(u_0) = \theta \Leftrightarrow \delta(-G_L^*(v_0^*) + \langle u_0, \Lambda^* v_0^* - f \rangle_U) = \theta, \tag{7.53}$$

where $v_0^* = \frac{\partial G(\Lambda(u_0))}{\partial v}$ is supposed to be such that $v_0^*(x) \in R_L^n$, a.e. in S and in this case:

$$J(u_0) = -G_L^*(v_0^*). \tag{7.54}$$

Proof: Suppose first that $\delta J(u_0) = \theta$, that is:

$$\Lambda^* \frac{\partial G(\Lambda u_0)}{\partial v} - f = \theta \tag{7.55}$$

which, as $v_0^* = \frac{\partial G(\Lambda u_0)}{\partial v}$ implies that:

$$\Lambda^* v_0^* - f = \theta, \tag{7.56}$$

and

$$v_{0^i}^* = \frac{\partial g(\Lambda u_0)}{\partial x_i}. \tag{7.57}$$

Thus from the last proposition we can write:

$$\Lambda_i(u_0) = \frac{\partial g_L^*(v_0^*)}{\partial y_i^*}, \quad for \quad i \in \{1, .., n\} \tag{7.58}$$

which means:

$$\Lambda u_0 = \frac{\partial G_L^*(v_0^*)}{\partial v^*}. \tag{7.59}$$

Therefore from (7.56) and (7.59) we have:

$$\delta(-G_L^*(v_0^*) + \langle u_0, \Lambda^* v_0^* - f \rangle_U) = \theta. \tag{7.60}$$

This completes the first part of the proof.

Conversely, suppose now that:

$$\delta(-G_L^*(v_0^*) + \langle u_0, \Lambda^* v_0^* - f \rangle_U) = \theta, \tag{7.61}$$

that is:

$$\Lambda^* v_0^* - f = \theta \tag{7.62}$$

and

$$\Lambda u_0 = \frac{\partial G_L^*(v_0^*)}{\partial v^*}. \tag{7.63}$$

Clearly, from (7.63), the last proposition and (7.62) we can write:

$$v_0^* = \frac{\partial G(\Lambda(u_0))}{\partial v} \tag{7.64}$$

and

$$\Lambda^* \frac{\partial G(\Lambda u_0)}{\partial v} - f = \theta, \tag{7.65}$$

which implies:

$$\delta J(u_0) = \theta. \tag{7.66}$$

Finally, we have:

$$J(u_0) = G(\Lambda u_0) - \langle u_0, f \rangle_U \tag{7.67}$$

From this, (7.62) and (7.64) we have

$$J(u_0) = G(\Lambda u_0) - \langle u_0, \Lambda^* v_0^* \rangle_U = G(\Lambda u_0) - \langle \Lambda u_0, v_0^* \rangle_Y = -G_L^*(v_0^*). \quad \square \tag{7.68}$$

7.2 Duality in Convex Optimization

Let U be a Banach space. Given $F : U \to \bar{\mathbb{R}}$ ($F \in \Gamma_0(U)$) we define the problem $\mathcal{P}$ as

$$\mathcal{P}: \text{ minimize } F(u) \text{ on } U. \tag{7.69}$$

We say that $u_0 \in U$ is a solution of problem $\mathcal{P}$ if $F(u_0) = \inf_{u \in U} F(u)$. Consider a function $\phi(u, p)$ ($\phi : U \times Y \to \bar{\mathbb{R}}$) such that

$$\phi(u, 0) = F(u), \tag{7.70}$$

we define the problem $\mathcal{P}^*$, as

$$\mathcal{P}^*: \text{ maximize } -\phi^*(0, p^*) \text{ on } Y^*. \tag{7.71}$$

Observe that

$$\phi^*(0, p^*) = \sup_{(u,p) \in U \times Y} \{\langle 0, u\rangle_U + \langle p, p^*\rangle_Y - \phi(u, p)\} \geq -\phi(u, 0), \tag{7.72}$$

or

$$\inf_{u \in U}\{\phi(u, 0)\} \geq \sup_{p^* \in Y^*}\{-\phi^*(0, p^*)\}. \tag{7.73}$$

Proposition 7.2.1. *Consider $\phi \in \Gamma_0(U \times Y)$. If we define*

$$h(p) = \inf_{u \in U}\{\phi(u, p)\}, \tag{7.74}$$

then h is convex.

Proof: We have to show that given $p, q \in Y$ and $\lambda \in (0, 1)$, we have

$$h(\lambda p + (1 - \lambda)q) \leq \lambda h(p) + (1 - \lambda)h(q). \tag{7.75}$$

If $h(p) = +\infty$ or $h(q) = +\infty$ we are done. Thus let us assume $h(p) < +\infty$ and $h(q) < +\infty$. For each $a > h(p)$ there exists $u \in U$ such that

$$h(p) \leq \phi(u, p) \leq a, \tag{7.76}$$

and, if $b > h(q)$, there exists $v \in U$ such that

$$h(q) \leq \phi(v, q) \leq b. \tag{7.77}$$

Thus

$$h(\lambda p + (1 - \lambda)q) \leq \inf_{w \in U}\{\phi(w, \lambda p + (1 - \lambda)q)\}$$

$$\leq \phi(\lambda u + (1 - \lambda)v, \lambda p + (1 - \lambda)q) \leq \lambda\phi(u, p) + (1 - \lambda)\phi(v, q) \leq \lambda a + (1 - \lambda)b. \tag{7.78}$$

Letting $a \to h(p)$ and $b \to h(q)$ we obtain

$$h(\lambda p + (1 - \lambda)q) \leq \lambda h(p) + (1 - \lambda)h(q). \quad \Box \tag{7.79}$$

Proposition 7.2.2. *For h as above, we have $h^*(p^*) = \phi^*(0,p^*), \forall p^* \in Y^*$, so that*

$$h^{**}(0) = \sup_{p^* \in Y^*} \{-\phi^*(0,p^*)\}. \tag{7.80}$$

Proof: Observe that

$$h^*(p^*) = \sup_{p \in Y}\{\langle p, p^*\rangle_Y - h(p)\} = \sup_{p \in Y}\{\langle p, p^*\rangle_Y - \inf_{u \in U}\{\phi(u,p)\}\}, \tag{7.81}$$

so that

$$h^*(p^*) = \sup_{(u,p) \in U \times Y} \{\langle p, p^*\rangle_Y - \phi(u,p)\} = \phi^*(0,p^*). \quad \Box \tag{7.82}$$

Proposition 7.2.3. *The set of solutions of the problem $\mathcal{P}^*$ (the dual problem) is identical to $\partial h^{**}(0)$.*

Proof: Consider $p_0^* \in Y^*$ a solution of Problem $\mathcal{P}^*$, that is,

$$-\phi^*(0,p_0^*) \geq -\phi^*(0,p^*), \forall p^* \in Y^*, \tag{7.83}$$

which is equivalent to

$$-h^*(p_0^*) \geq -h^*(p^*), \forall p^* \in Y^*, \tag{7.84}$$

which is equivalent to

$$-h(p_0^*) = \sup_{p^* \in Y^*} \{\langle 0, p^*\rangle_Y - h^*(p^*)\} \Leftrightarrow -h^*(p_0^*) = h^{**}(0) \Leftrightarrow p_0^* \in \partial h^{**}(0). \quad \Box \tag{7.85}$$

Theorem 7.2.4. *Consider $\phi : U \times Y \to \bar{\mathbb{R}}$ convex. Assume $\inf_{u \in U}\{\phi(u,0)\} \in \mathbb{R}$ and there exists $u_0 \in U$ such that $p \mapsto \phi(u_0,p)$ is finite and continuous at $0 \in Y$, then*

$$\inf_{u \in U}\{\phi(u,0)\} = \sup_{p^* \in Y^*} \{-\phi^*(0,p^*)\}, \tag{7.86}$$

and the dual problem has at least one solution.

Proof: By hypothesis $h(0) \in \mathbb{R}$ and as was shown above, h is convex. As the function $p \mapsto \phi(u_0,p)$ is convex and continuous at $0 \in Y$, there exists a neighborhood $\mathcal{V}$ of zero in Y such that

$$\phi(u_0,p) \leq M < +\infty, \forall p \in \mathcal{V}, \tag{7.87}$$

for some $M \in \mathbb{R}$. Thus, we may write

$$h(p) = \inf_{u \in U}\{\phi(u,p)\} \leq \phi(u_0,p) \leq M, \forall p \in \mathcal{V}. \tag{7.88}$$

Hence, from Lemma 7.1.20, h is continuous at 0. Thus by Proposition 7.1.21, h is sub-differentiable at 0, which means $h(0) = h^{**}(0)$. Therefore by Proposition 7.2.3, the dual problem has solutions and

$$h(0) = \inf_{u \in U}\{\phi(u,0)\} = \sup_{p^* \in Y^*}\{-\phi^*(0,p^*)\} = h^{**}(0). \quad \Box \tag{7.89}$$

Now we apply the last results to $\phi(u,p) = G(\Lambda u + p) + F(u)$, where $\Lambda : U \to Y$ is a continuous linear operator whose adjoint operator is denoted by $\Lambda^* : Y^* \to U^*$. We may enunciate the following theorem.

Theorem 7.2.5. *Suppose U is a reflexive Banach space and define $J : U \to \mathbb{R}$ by*

$$J(u) = G(\Lambda u) + F(u) = \phi(u,0), \tag{7.90}$$

where $\lim J(u) = +\infty$ as $\|u\|_U \to \infty$ and $F \in \Gamma_0(U)$, $G \in \Gamma_0(Y)$. Also suppose there exists $\hat{u} \in U$ such that $J(\hat{u}) < +\infty$ with the function $p \mapsto G(p)$ continuous at $\Lambda\hat{u}$. Under such hypothesis, there exist $u_0 \in U$ and $p_0^* \in Y^*$ such that

$$J(u_0) = \min_{u \in U}\{J(u)\} = \max_{p^* \in Y^*}\{-G^*(p^*) - F^*(-\Lambda^* p^*)\} = -G^*(p_0^*) - F^*(-\Lambda^* p_0^*). \tag{7.91}$$

Proof: The existence of solutions for the primal problem follows from the direct method of calculus of variations. That is, considering a minimizing sequence, from above (coercivity hypothesis), such a sequence is bounded and has a weakly convergent subsequence to some $u_0 \in U$. Finally, from the lower semi-continuity of primal formulation, we may conclude that u_0 is a minimizer. The other conclusions follow from Theorem 7.2.4 just observing that

$$\phi^*(0,p^*) = \sup_{u \in U, p \in Y}\{\langle p, p^*\rangle_Y - G(\Lambda u + p) - F(u)\} = \sup_{u \in U, q \in Y}\{\langle q, p^*\rangle - G(q) - \langle \Lambda u, p^*\rangle - F(u)\}, \tag{7.92}$$

so that

$$\phi^*(0,p^*) = G^*(p^*) + \sup_{u \in U}\{-\langle u, \Lambda^* p^*\rangle_U - F(u)\} = G^*(p^*) + F^*(-\Lambda^* p^*). \tag{7.93}$$

Thus,

$$\inf_{u \in U}\{\phi(u,0)\} = \sup_{p^* \in Y^*}\{-\phi^*(0,p^*)\} \tag{7.94}$$

and solutions u_0 and p_0^* for the primal and dual problems, respectively, imply that

$$J(u_0) = \min_{u \in U}\{J(u)\} = \max_{p^* \in Y^*}\{-G^*(p^*) - F^*(-\Lambda^* p^*)\} = -G^*(p_0^*) - F^*(-\Lambda^* p_0^*). \quad \Box \tag{7.95}$$

7.3 Relaxation for the Scalar Case

In this section, $\Omega \subset \mathbb{R}^N$ denotes a bounded open Lipschitz set. The proof of next result is found in [14].

Theorem 7.3.1. *Let $r \in \mathbb{N}$ and let u_k, $1 \leq k \leq r$ be piecewise affine functions from Ω into $\mathbb{R}$ and $\{\alpha_k\}_{1 \geq k \leq r}$ such that $\alpha_k > 0, \forall k \in \{1, ..., r\}$ and $\sum_{k=1}^r \alpha_k = 1$. Given $\varepsilon > 0$, there exists a locally Lipschitz function $u : \Omega \to \mathbb{R}$ and r disjoint open sets Ω_k, $1 \leq k \leq r$, such that*

$$|m(\Omega_k) - \alpha_k m(\Omega)| < \alpha_k \varepsilon, \quad \forall k \in \{1, ..., r\}, \tag{7.96}$$

$$\nabla u(x) = \nabla u_k(x), \quad a.e. \ \ on \ \ \Omega_k, \tag{7.97}$$

$$|\nabla u(x)| \leq \max_{1 \leq k \leq r}\{|\nabla u_k(x)|\}, \quad a.e. \ \ on \ \ \Omega, \tag{7.98}$$

$$\left| u(x) - \sum_{k=1}^r \alpha_k u_k \right| < \varepsilon, \quad \forall x \in \Omega, \tag{7.99}$$

$$u(x) = \sum_{k=1}^r \alpha_k u_k(x), \forall x \in \partial\Omega. \tag{7.100}$$

The next result is also found in [14].

Proposition 7.3.2. *Let $r \in \mathbb{N}$ and let u_k, $1 \leq k \leq r$ be piecewise affine functions from Ω into $\mathbb{R}$. We take a finite family $\mathcal{F}$ of normal integrands of $\Omega \times \mathbb{R}^N$ and a positive function $c \in L^1(\Omega)$ which satisfy*

$$c(x) \geq \sup\{f(x, \xi) \ \ | \ \ f \in \mathcal{F}, \ \ |\xi| \leq \max_{1 \leq k \leq r}\{\|\nabla u_k\|_\infty\}\}. \tag{7.101}$$

Given $\varepsilon > 0$, there exists a locally Lipschitz function $u : \Omega \to \mathbb{R}$ such that

$$\left| \int_\Omega f(x, \nabla u) dx - \sum_{k=1}^r \alpha_k \int_\Omega f(x, \nabla u_k) dx \right| < \varepsilon, \tag{7.102}$$

$$|\nabla u(x)| \leq \max_{1 \leq k \leq r}\{|\nabla u_k(x)|\}, \quad a.e. \ \ in \ \ \Omega, \tag{7.103}$$

$$|u(x) - \sum_{k=1}^r \alpha_k u_k(x)| < \varepsilon, \forall x \in \Omega \tag{7.104}$$

$$u(x) = \sum_{k=1}^r \alpha_k u_k(x), \forall x \in \partial\Omega. \tag{7.105}$$

Proof: It is sufficient to establish the result for functions u_k affine over Ω, since Ω can be divided into pieces on which u_k are affine, and such pieces can be put together through (7.105). Let $\varepsilon > 0$ be given. We know that simple functions are dense in $L^1(\Omega)$, concerning the L^1 norm. Thus there exists a partition of Ω into a finite number of open sets $\mathcal{O}_i$, $1 \leq i \leq N_1$ and a negligible set, and there exists $\bar{f}_k$ constant functions over each $\mathcal{O}_i$ such that

$$\int_\Omega |f(x, \nabla u_k(x)) - \bar{f}_k(x)| dx < \varepsilon, \forall f \in \mathcal{F}, \quad 1 \leq k \leq r. \tag{7.106}$$

Now choose $\delta > 0$ such that

$$\delta \leq \frac{\varepsilon}{N_1(1 + \max_{1\leq k\leq r}\{\|\bar{f}_k\|_\infty\})} \tag{7.107}$$

and if B is a measurable set

$$m(B) < \delta \Rightarrow \int_B c(x) dx \leq \varepsilon / N_1. \tag{7.108}$$

Now we apply Theorem 7.3.1, to each of the open sets $\mathcal{O}_i$, therefore there exists a locally Lipschitz function $u : \mathcal{O}_i \to \mathbb{R}$ and there exist r open disjoints spaces Ω_k^i, $1 \leq k \leq r$, such that

$$|m(\Omega_k^i) - \alpha_k m(\mathcal{O}_i)| \leq \alpha_k \delta, \quad for \quad 1 \leq k \leq r, \tag{7.109}$$

$$|\nabla u(x)| \leq \max_{1\leq k\leq r}\{|\nabla u_k(x)|\}, \quad a.e. \;\; \Omega_i, \tag{7.110}$$

$$\left| u(x) - \sum_{k=1}^r \alpha_k u_k(x) \right| \leq \delta, \forall x \in \mathcal{O}_i \tag{7.111}$$

$$u(x) = \sum_{k=1}^r \alpha_k u_k(x), \forall x \in \partial \mathcal{O}_i. \tag{7.112}$$

We can define $u = \sum_{k=1}^r \alpha_k u_k$ on $\Omega - \cup_{i=1}^{N_1} \mathcal{O}_i$. Therefore u is continuous and locally Lipschitz. Now observe that

$$\int_{\mathcal{O}_i} f(x, \nabla u(x)) dx - \sum_{k=1}^r \int_{\Omega_k^i} f(x, \nabla u_k(x)) dx = \int_{\mathcal{O}_i - \cup_{k=1}^r \Omega_k^i} f(x, \nabla u(x)) dx. \tag{7.113}$$

From $|f(x, \nabla u(x))| \leq c(x)$, $m(\mathcal{O}_i - \cup_{k=1}^r \Omega_k^i) \leq \delta$ and (7.108) we obtain

$$\left| \int_{\mathcal{O}_i} f(x, \nabla u(x)) dx - \sum_{k=1}^r \int_{\Omega_k^i} f(x, \nabla u_k(x) dx \right| = \left| \int_{\mathcal{O}_i - \cup_{k=1}^r \Omega_k^i} f(x, \nabla u(x)) dx \right| \leq \varepsilon / N_1. \tag{7.114}$$

Considering that $\bar{f}_k$ is constant in $\mathcal{O}_i$, from (7.106), (7.108), (7.107) and (7.109) we obtain

$$\sum_{k=1}^r | \int_{\Omega_k^i} \bar{f}_k(x) dx - \alpha_k \int_{\mathcal{O}_i} \bar{f}_k(x) dx | < \varepsilon / N_1. \tag{7.115}$$

We recall that $\Omega_k = \cup_{i=1}^{N_1} \Omega_k^i$ so that

$$\left| \int_\Omega f(x, \nabla u(x))dx - \sum_{k=1}^r \alpha_k \int_\Omega f(x, \nabla u_k(x))dx \right| \leq$$

$$\left| \int_\Omega f(x, \nabla u(x))dx - \sum_{k=1}^r \int_{\Omega_k} f(x, \nabla u_k(x))dx \right| + \sum_{k=1}^r \int_{\Omega_k} |f(x, \nabla u_k(x) - \bar{f}_k(x)|dx +$$

$$+ \sum_{k=1}^p \left| \int_{\Omega_k} \bar{f}_k(x)dx - \alpha_k \int_\Omega \bar{f}_k(x)dx \right| + \sum_{k=1}^r \alpha_k \int_\Omega |\bar{f}_k(x) - f(x, \nabla u_k(x))|dx. \tag{7.116}$$

From (7.114), (7.106),(7.115) and (7.106) again, we obtain

$$\left| \int_\Omega f(x, \nabla u(x))dx - \sum_{k=1}^r \alpha_k \int_\Omega f(x, \nabla u_k)dx \right| < 4\varepsilon. \quad \square \tag{7.117}$$

The next result we do not prove it.

Proposition 7.3.3. *If $u \in W_0^{1,p}(\Omega)$ there exists a sequence $\{u_n\}$ of piecewise affine functions over Ω, null on $\partial\Omega$, such that*

$$u_n \to u, \quad in \quad L^p(\Omega) \tag{7.118}$$

and

$$\nabla u_n \to \nabla u, \quad in \quad L^p(\Omega; \mathbb{R}^N). \tag{7.119}$$

Proposition 7.3.4. *For p such that $1 < p < \infty$, suppose that $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodore function , for which there exist $a_1, a_2 \in L^1(\Omega)$ and constants $c_1 \geq c_2 > 0$ such that*

$$a_2(x) + c_2|\xi|^p \leq f(x, \xi) \leq a_1(x) + c_1|\xi|^p, \forall x \in \Omega, \quad \xi \in \mathbb{R}^N. \tag{7.120}$$

Then, given $u \in W^{1,p}(\Omega)$ piecewise affine, $\varepsilon > 0$ and a neighborhood $\mathcal{V}$ of zero in the topology $\sigma(L^p(\Omega, \mathbb{R}^N), L^q(\Omega, \mathbb{R}^N))$ there exists a function $v \in W^{1,p}(\Omega)$ such that

$$\nabla v - \nabla u \in \mathcal{V}, \tag{7.121}$$

$$u = v \text{ on } \partial\Omega,$$

$$\|v - u\|_\infty < \varepsilon, \tag{7.122}$$

and

$$\left| \int_\Omega f(x, \nabla v(x))dx - \int_\Omega f^{**}(x, \nabla u(x))dx \right| < \varepsilon. \tag{7.123}$$

Proof: Suppose given $\varepsilon > 0$, $u \in W^{1,p}(\Omega)$ piecewise affine continuous, and a neighborhood $\mathcal{V}$ of zero, which may be expressed as

$$\mathcal{V} = \{w \in L^p(\Omega, \mathbb{R}^N) \ \mid \ \left|\int_\Omega h_m \cdot w dx\right| < \eta, \forall m \in \{1, ..., M\}\}, \tag{7.124}$$

where $M \in \mathbb{N}$, $h_m \in L^q(\Omega, \mathbb{R}^N)$, $\eta \in \mathbb{R}^+$. By hypothesis, there exists a partition of Ω into a negligible set Ω_0 and open subspaces Δ_i, $1 \leq i \leq r$, over which $\nabla u(x)$ is constant. From standard results of convex analysis in $\mathbb{R}^N$, for each $i \in \{1, ..., r\}$ we can obtain $\{\alpha_k \geq 0\}_{1 \leq k \leq N+1}$, and ξ_k such that $\sum_{k=1}^{N+1} \alpha_k = 1$ and

$$\sum_{k=1}^{N+1} \alpha_k \xi_k = \nabla u, \forall x \in \Delta_i, \tag{7.125}$$

and

$$\sum_{k=1}^{N+1} \alpha_k f(x, \xi_k) = f^{**}(x, \nabla u(x)). \tag{7.126}$$

Define $\beta_i = \max_{k \in \{1,...,N+1\}}\{|\xi_k| \ \ on \ \ \Delta_i\}$, and $\rho_1 = \max_{i \in \{1,...,r\}}\{\beta_i\}$, and $\rho = \max\{\rho_1, \|\nabla u\|_\infty\}$. Now, observe that we can obtain functions $\hat{h}_m \in C_0^\infty(\Omega; \mathbb{R}^N)$ such that

$$\max_{m \in \{1,...,M\}} \|\hat{h}_m - h_m\|_{L^q(\Omega, \mathbb{R}^N)} < \frac{\eta}{4\rho m(\Omega)}. \tag{7.127}$$

Define $C = \max_{m \in \{1,...,M\}} \|div(\hat{h}_m)\|_{L^q(\Omega)}$ and we can also define

$$\varepsilon_1 = \min\{\varepsilon, 1/m(\Omega), \eta/(2Cm(\Omega))\} \tag{7.128}$$

We recall that ρ does not depend on ε. Furthermore, for each $i \in \{1, ..., r\}$ there exists a compact subset $K_i \subset \Delta_i$ such that

$$\int_{\Delta_i - K_i} [a_1(x) + c_1(x) \max_{|\xi| \leq \rho}\{|\xi|^p\}] dx < \frac{\varepsilon_1}{r}. \tag{7.129}$$

Also, observe that the restrictions of f and f^{**} to $K_i \times \rho B$ are continuous, so that from this and from the compactness of ρB, for all $x \in K_i$, we can find an open ball ω_x with center in x and contained in Ω, such that

$$|f^{**}(y, \nabla u(x)) - f^{**}(x, \nabla u(x))| < \frac{\varepsilon_1}{m(\Omega)}, \forall y \in \omega_x \cap K_i, \tag{7.130}$$

and

$$|f(y, \xi) - f(x, \xi)| < \frac{\varepsilon_1}{m(\Omega)}, \forall y \in \omega_x \cap K_i, \forall \xi \in \rho B. \tag{7.131}$$

Therefore, we may write

$$\left|f^{**}(y, \nabla u(x)) - \sum_{k=1}^{N+1} \alpha_k f(y, \xi_k)\right| < \frac{2\varepsilon_1}{m(\Omega)}, \forall y \in \omega_x \cap K_i. \tag{7.132}$$

We can cover the compact set K_i with a finite number of those open balls ω_x, denoted by ω_j, $1 \leq j \leq l$. Consider the open sets $\omega'_j = \omega_j - \cup_{i=1}^{j-1}\bar{\omega}_i$, we have that $\cup_{j=1}^{l}\bar{\omega}'_j = \cup_{j=1}^{l}\bar{\omega}_j$. Defining functions u_k, for $1 \leq k \leq n+1$ such that $\nabla u_k = \xi_k$ and $u = \sum_{k=1}^{N+1}\alpha_k u_k$ we may apply Proposition 7.3.2 to each of the open sets ω'_j, so that we obtain functions $v_i \in W^{1,p}(\Omega)$ such that

$$\left|\int_{\omega'_j} f(x, \nabla v_i(x))dx - \sum_{k=1}^{n+1}\alpha_k \int_{\omega'_j} f(x, \xi_k)dx\right| < \frac{\varepsilon_1}{rl}, \tag{7.133}$$

$$|\nabla v_i| < \rho, \forall x \in \omega'_j, \tag{7.134}$$

$$|v_i(x) - u(x)| < \varepsilon_1, \forall x \in \omega'_j, \tag{7.135}$$

and

$$v_i(x) = u(x), \forall x \in \partial\omega'_j. \tag{7.136}$$

Finally we set

$$v_i = u \quad on \quad \Delta_i - \cup_{j=1}^{l}\omega_j. \tag{7.137}$$

We may define a continuous mapping $v : \Omega \to \mathbb{R}$ by

$$v(x) = v_i(x), \quad if \quad x \in \Delta_i, \tag{7.138}$$

$$v(x) = u(x), \quad if \quad x \in \Omega_0. \tag{7.139}$$

We have that $v(x) = u(x), \forall x \in \partial\Omega$ and $\|\nabla v\|_\infty < \rho$. Also, from (7.129)

$$\int_{\Delta_i - K_i} |f^{**}(x, \nabla u(x)|dx < \frac{\varepsilon_1}{r} \tag{7.140}$$

and

$$\int_{\Delta_i - K_i} |f(x, \nabla v(x)|dx < \frac{\varepsilon_1}{r}. \tag{7.141}$$

On the other hand, from (7.132) and (7.133)

$$\left|\int_{K_i \cap \omega'_j} f(x, \nabla v(x))dx - \int_{K_i \cap \omega'_j} f^{**}(x, \nabla u(x))dx\right| \leq \frac{\varepsilon_1}{rl} + \frac{\varepsilon_1 m(\omega'_j \cap K_i)}{m(\Omega)} \tag{7.142}$$

so that

$$|\int_{K_i} f(x, \nabla v(x))dx - \int_{K_i} f^{**}(x, \nabla u(x))dx| \leq \frac{\varepsilon_1}{r} + \frac{\varepsilon_1 m(K_i)}{m(\Omega)}. \ \Box \tag{7.143}$$

Now summing up in i and considering (7.140) and (7.141) we obtain (7.123). Also, observe that from above, we have

$$\|v - u\|_\infty < \varepsilon_1, \tag{7.144}$$

and thus

$$\begin{aligned}\left|\int_\Omega \hat{h}_m \cdot (\nabla v(x) - \nabla u(x))dx\right| &= \left|-\int_\Omega div(\hat{h}_m)(v(x)-u(x))dx\right| \\ &\leq \|div(\hat{h}_m)\|_{L^q(\Omega)}\|v-u\|_{L^p(S)} \\ &\leq C\varepsilon_1 m(\Omega) \\ &< \frac{\eta}{2}. \end{aligned} \tag{7.145}$$

Also we have that

$$\left|\int_\Omega (\hat{h}_m - h_m)\cdot(\nabla v - \nabla u)dx\right| \leq \|\hat{h}_m - h_m\|_{L^q(\Omega,\mathbb{R}^N)}\|\nabla v - \nabla u\|_{L^p(\Omega,\mathbb{R}^N)} \leq \frac{\eta}{2}. \tag{7.146}$$

Thus

$$\left|\int_\Omega h_m \cdot (\nabla v - \nabla u)dx\right| < \eta, \forall m \in \{1,...,M\}. \ \square \tag{7.147}$$

Theorem 7.3.5. *Assuming the hypothesis of last theorem, given a function $u \in W_0^{1,p}(\Omega)$, given $\varepsilon > 0$ and a neighborhood of zero $\mathcal{V}$ in $\sigma(L^p(\Omega,\mathbb{R}^N), L^q(\Omega,\mathbb{R}^N))$, we have that there exists a function $v \in W_0^{1,p}(\Omega)$ such that*

$$\nabla v - \nabla u \in \mathcal{V}, \tag{7.148}$$

and

$$\left|\int_\Omega f(x,\nabla v(x))dx - \int_\Omega f^{**}(x,\nabla u(x))dx\right| < \varepsilon. \tag{7.149}$$

Proof: We can approximate u by a function w which is piecewise affine and null on the boundary. Thus, there exists $\delta > 0$ such that we can obtain $w \in W_0^{1,p}(\Omega)$ piecewise affine such that

$$\|u - w\|_{1,p} < \delta \tag{7.150}$$

so that

$$\nabla w - \nabla u \in \frac{1}{2}\mathcal{V}, \tag{7.151}$$

and

$$\left|\int_\Omega f^{**}(x,\nabla w(x))dx - \int_\Omega f^{**}(x,\nabla u(x))dx\right| < \frac{\varepsilon}{2}. \tag{7.152}$$

From Proposition we may obtain $v \in W^{1,p}(\Omega)_0$ such that

$$\nabla v - \nabla w \in \frac{1}{2}\mathcal{V}, \tag{7.153}$$

and

$$\left|\int_\Omega f^{**}(x,\nabla w(x))dx - \int_\Omega f(x,\nabla v(x))dx\right| < \frac{\varepsilon}{2}. \tag{7.154}$$

From (7.152) and (7.154)

$$\left|\int_\Omega f^{**}(x, \nabla u(x))dx - \int_\Omega f(x, \nabla v(x))dx\right| < \varepsilon. \tag{7.155}$$

and from (7.151) and (7.153) we have

$$\nabla v - \nabla u \in \mathcal{V}. \quad \square \tag{7.156}$$

To finish this chapter, we present two theorems which summarize the last results.

Theorem 7.3.6. *Let f be a Carathéodory function from $\Omega \times \mathbb{R}^N$ into $\mathbb{R}$ which satisfies*

$$a_2(x) + c_2|\xi|^p \leq f(x, \xi) \leq a_1(x) + c_1|\xi|^p \tag{7.157}$$

where a_1, $a_2 \in L^1(\Omega)$, $1 < p < +\infty$, $b \geq 0$ and $c_1 \geq c_2 > 0$. Under such assumptions, defining $\hat{U} = W_0^{1,p}(\Omega)$, we have

$$\inf_{u \in \hat{U}} \left\{\int_\Omega f(x, \nabla u)dx\right\} = \min_{u \in \hat{U}} \left\{\int_\Omega f^{**}(x, \nabla u)dx\right\} \tag{7.158}$$

The solutions of relaxed problem are weak cluster points in $W_0^{1,p}(\Omega)$ of the minimizing sequences of primal problem.

Proof: The existence of solutions for the convex relaxed formulation is a consequence of the reflexivity of U and coercivity hypothesis, which allows an application of the direct method of calculus of variations. That is, considering a minimizing sequence, from above (coercivity hypothesis), such a sequence is bounded and has a weakly convergent subsequence to some $\hat{u} \in W^{1,p}(\Omega)$. Finally, from the lower semi-continuity of relaxed formulation, we may conclude that $\hat{u}$ is a minimizer. The relation (7.158) follows from last theorem. $\square$

Theorem 7.3.7. *Let f be a Carathéodory function from $\Omega \times \mathbb{R}^N$ into $\mathbb{R}$ which satisfies*

$$a_2(x) + c_2|\xi|^p \leq f(x, \xi) \leq a_1(x) + c_1|\xi|^p \tag{7.159}$$

where a_1, $a_2 \in L^1(\Omega)$, $1 < p < +\infty$, $b \geq 0$ and $c_1 \geq c_2 > 0$. Let $u_0 \in W^{1,p}(\Omega)$. Under such assumptions, defining $\hat{U} = \{u \mid u - u_0 \in W_0^{1,p}(\Omega)\}$, we have

$$\inf_{u \in \hat{U}} \left\{\int_\Omega f(x, \nabla u)dx\right\} = \min_{u \in \hat{U}} \left\{\int_\Omega f^{**}(x, \nabla u)dx\right\} \tag{7.160}$$

The solutions of relaxed problem are weak cluster points in $W^{1,p}(\Omega)$ of the minimizing sequences of primal problem.

Proof: Just apply the last theorem to the integrand $g(x, \xi) = f(x, \xi + \nabla u_0)$. For details see [14]. $\square$

7.4 Duality Suitable for the Vectorial Case

Definition 7.4.1 (A Cone and its Partial Order Relation)**.** *Let U be a Banach space and $m > 0$. We define $\mathcal{C}(m)$ as*

$$\mathcal{C}(m) = \{(u, a) \in U \times \mathbb{R} \ \mid \ a + m\|u\|_U \leq 0\}. \tag{7.161}$$

Also, we define an order relation for the cone $\mathcal{C}(m)$, namely

$$(u, a) \leq (v, b) \Leftrightarrow (v - u, b - a) \in \mathcal{C}(m). \tag{7.162}$$

Proposition 7.4.2. *Let $S \subset U \times \mathbb{R}$ be a closed set such that*

$$\inf\{a \ \mid \ (u, a) \in S\} > -\infty. \tag{7.163}$$

Then S has a maximal element under the order relation of last definition.

Proof: See Ekeland and Témam [14], page 28.

The next result is particularly relevant for non-convex functionals.

Theorem 7.4.3. *Let $F : U \to \bar{\mathbb{R}}$ be lower semi-continuous functional such that $-\infty < \inf_{u \in U}\{F(u)\} < +\infty$. Given $\varepsilon > 0$, suppose $u \in U$ is such that*

$$F(u) \leq \inf_{u \in U}\{F(u)\} + \varepsilon, \tag{7.164}$$

then, for each $\lambda > 0$, there exists $u_\lambda \in U$ such that

$$\|u - u_\lambda\|_U \leq \lambda \ \text{ and } \ F(u_\lambda) \leq F(u), \tag{7.165}$$

and

$$Epi(F) \cap \{(u_\lambda, F(u_\lambda)) + \mathcal{C}(\varepsilon/\lambda)\} = (u_\lambda, F(u_\lambda)). \tag{7.166}$$

Proof: We will apply the last proposition to $S = Epi(F)$, which is a closed set. For the order relation associated with $\mathcal{C}(\varepsilon/\lambda)$, there exists a maximal element, which we denote by (u_λ, a_λ). Thus $(u_\lambda, a_\lambda) \geq (u, F(u))$. Since $(u_\lambda, a_\lambda,)$ is maximal, we have $a_\lambda = F(u_\lambda)$ and hence (7.166) is satisfied. Also observe that

$$(u, F(u)) \leq (u_\lambda, F(u_\lambda)), \tag{7.167}$$

so that

$$\frac{\varepsilon}{\lambda}\|u - u_\lambda\|_U \leq F(u) - F(u_\lambda). \tag{7.168}$$

From this and (7.164) we obtain

$$0 \leq F(u) - F(u_\lambda) \leq \varepsilon, \tag{7.169}$$

and therefore

$$\|u - u_\lambda\|_U \leq \lambda. \ \square \tag{7.170}$$

Remark 7.4.4. *Observe that*

$$F(u_\lambda) - \frac{\varepsilon}{\lambda} t\|v\|_U \leq F(u_\lambda + tv), \forall t \in [0,1], \quad v \in U, \tag{7.171}$$

so that, if F is Gâteaux differentiable, we obtain

$$-\frac{\varepsilon}{\lambda}\|v\|_U \leq \langle \delta F(u_\lambda), v\rangle_U. \tag{7.172}$$

Thus

$$\|\delta F(u_\lambda)\|_{U^*} \leq \varepsilon/\lambda. \tag{7.173}$$

Now, for $\lambda = \sqrt{\varepsilon}$ we obtain the following result.

Theorem 7.4.5. *Let $F : U \to \mathbb{R}$ be a Gâteaux differentiable functional. Given $\varepsilon > 0$ suppose that $u \in U$ is such that*

$$F(u) \leq \inf_{u \in U}\{F(u)\} + \varepsilon. \tag{7.174}$$

Then there exists $v \in U$ such that

$$F(v) \leq F(u), \tag{7.175}$$

$$\|u - v\|_U \leq \sqrt{\varepsilon}, \tag{7.176}$$

and

$$\|\delta F(v)\|_{U^*} \leq \sqrt{\varepsilon}. \quad \Box \tag{7.177}$$

The next theorem easily follows from above results.

Theorem 7.4.6. *Let $J : U \to \mathbb{R}$, be defined by*

$$J(u) = G(\nabla u) - \langle f, u\rangle_{L^2(S;\mathbb{R}^N)}, \tag{7.178}$$

where

$$U = W_0^{1,2}(S;\mathbb{R}^N), \tag{7.179}$$

We suppose G is Gâteaux-differentiable and J bounded from below. Then, given $\varepsilon > 0$, there exists $u_\varepsilon \in U$ such that

$$J(u_\varepsilon) - \inf_{u \in U}\{J(u)\} < \varepsilon, \tag{7.180}$$

and

$$\|\delta J(u_\varepsilon)\|_{U^*} < \sqrt{\varepsilon}. \quad \Box \tag{7.181}$$

Now we establish a general duality principle that is applicable to more complex situations concerning vectorial problems in the calculus of variations. In fact, it is very simple result, given by the following theorem:

Theorem 7.4.7. *Consider $(G \circ \Lambda) : U \to \mathbb{R}$ (not necessarily convex) such that $J : U \to \mathbb{R}$ defined by*

$$J(u) = G(\Lambda u) - \langle u, f\rangle_U, \forall u \in U,$$

is bounded from below (here as usual $\Lambda : U \to Y$ is a continuous linear operator). Under such assumptions, we have

$$\inf_{u\in U}\{J(u)\} = \sup_{v^*\in A^*}\{-(G\circ\Lambda)^*(\Lambda^* v^*)\}$$

where

$$A^* = \{v^* \in Y^* \mid \Lambda^* v^* - f = 0\}.$$

Proof: The proof is simple, just observe that

$$-(G\circ\Lambda)^*(\Lambda^* v^*) = -(G\circ\Lambda)^*(f) = -\sup_{u\in U}\{\langle u, f\rangle_U - G(\Lambda u)\}, \forall v^* \in A^*. \quad \square$$

Remark 7.4.8. *What seems to be relevant is that, when computing $(G\circ\Lambda)^*(\Lambda^* v^*)$, we obtain a duality which is perfect concerning the convex envelope of the primal formulation, that is, no duality gap, as may be seen in the next chapters.*

We finish this Chapter with the most important result we have obtained for vectorial problems in the Calculus of Variations, namely:

Theorem 7.4.9. *Consider $(G \circ \Lambda) : U \to \mathbb{R}$ and $(F \circ \Lambda_1) : U \to \mathbb{R}$ convex l.s.c. functionals such that $J : U \to \mathbb{R}$ defined as*

$$J(u) = (G\circ\Lambda)(u) - (F\circ\Lambda_1)(u) - \langle u, f\rangle_U$$

is below bounded. (Here $\Lambda : U \to Y$ and $\Lambda_1 : U \to Y_1$ are continuous linear operators whose adjoint operators are denoted by $\Lambda^* : Y^* \to U^*$ and $\Lambda_1^* : Y^* \to U^*$, respectively). Also we suppose the existence of $L : Y_1 \to Y$ continuous and linear operator such that L^* is onto and

$$\Lambda(u) = L(\Lambda_1(u)), \forall u \in U.$$

Under such assumptions, we have

$$\inf_{u\in U}\{J(u)\} \geq \sup_{v^*\in A^*}\{\inf_{z^*\in Y_1^*}\{F^*(L^* z^*) - G^*(v^* + z^*)\}\},$$

where

$$A^* = \{v^* \in Y^* \mid \Lambda^* v^* = f\}.$$

Proof: Observe that

$$G^*(v^* + z^*) \geq \langle \Lambda u, v^*\rangle_Y + \langle \Lambda u, z^*\rangle_Y - G(\Lambda u), \forall u \in U,$$

that is,

$$-F^*(L^*z^*) + G^*(v^* + z^*) \geq \langle u, f\rangle_U - F^*(L^*z^*) + \langle \Lambda_1 u, L^*z^*\rangle_{Y_1} - G(\Lambda u), \forall u \in U, v^* \in A^*$$

so that

$$\sup_{z^* \in Y_1^*}\{-F^*(L^*z^*) + G^*(v^* + z^*)\} \geq \sup_{z^* \in Y_1^*}\{\langle u, f\rangle_U - F^*(L^*z^*) + \langle \Lambda_1 u, L^*z^*\rangle_{Y_1} - G(\Lambda u)\},$$

for $v^* \in A^*$, and therefore

$$G(\Lambda u) - F(\Lambda_1 u) - \langle u, f\rangle_U \geq \inf_{z^* \in Y_1^*}\{F^*(L^*z^*) - G^*(v^* + z^*)\}, \quad if \quad v^* \in A^*,$$

which means

$$\inf_{u \in U}\{J(u)\} \geq \sup_{v^* \in A^*}\{\inf_{z^* \in Y_1^*}\{F^*(L^*z^*) - G^*(v^* + z^*)\}\},$$

where

$$A^* = \{v^* \in Y^* \mid \Lambda^* v^* = f\}. \quad \square$$

Chapter 8

Constrained Variational Optimization

8.1 Basic Concepts

We start with the definition of cone:

Definition 8.1.1 (Cone)**.** *Given U a Banach space, we say that $C \subset U$ is a cone with vertex at origin, if given $u \in C$, we have that $\lambda u \in C$, $\forall \lambda \geq 0$. By analogy we define a cone with vertex at $p \in U$ as $P = p + C$, where C is any cone with vertex at origin.*

Definition 8.1.2. *Let P be a convex cone in U. For $u, v \in U$ we write $u \geq v$ (with respect to P) if $u - v \in P$. In particular $u \geq \theta$ if and only if $u \in C$. Also*

$$P^+ = \{u^* \in U^* \mid \langle u, u^* \rangle_U \geq 0, \forall u \in P\}. \tag{8.1}$$

If $u^* \in P^+$ we write $u^* \geq \theta^*$.

Proposition 8.1.3. *Let U be a Banach space and P be a closed cone in U. If $u \in U$ satisfies $\langle u, u^* \rangle_U \geq 0$, $\forall u^* \geq \theta^*$, then $u \geq \theta$.*

Proof: We repeat here the proof found in Luenberger [23], page 215. Assume u is not in P. Then by the separating hyperplane theorem there is an $u^* \in U^*$ such that $\langle u, u^* \rangle_U < \langle p, u^* \rangle_U, \forall p \in P$. Since P is cone we must have $\langle p, u^* \rangle_U \geq 0$, otherwise we would have $\langle u, u^* \rangle > \langle \alpha p, u^* \rangle_U$ for some $\alpha > 0$. Thus $u^* \in P^+$. Finally, since $\inf_{p \in P}\{\langle p, u^* \rangle_U\} = 0$, we obtain $\langle u, u^* \rangle_U < 0$ which completes the proof. $\square$

Definition 8.1.4 (Convex Mapping)**.** *Let U, Z be vector spaces. Let $P \subset Z$ be a cone. A mapping $G : U \to Z$ is said to be convex if the domain of G is convex and*

$$G(\alpha u_1 + (1 - \alpha)u_2) \leq \alpha G(u_1) + (1 - \alpha)G(u_2), \forall u_1, u_2 \in U, \alpha \in [0, 1]. \tag{8.2}$$

Consider the problem $\mathcal{P}$, defined as

$$\textit{Problem } \mathcal{P}: \ \text{Minimize } F : U \to \mathbb{R} \text{ subject to } u \in \Omega, \ \textit{and} \ G(u) \leq \theta$$

Define

$$\omega(z) = \inf\{F(u) \ | \ u \in \Omega \ \textit{and} \ G(u) \leq z\}. \tag{8.3}$$

For such a functional we have the following result.

Proposition 8.1.5. *If F is a real convex functional and G is convex, then ω is convex.*

Proof: Observe that

$$\begin{aligned}
\omega(\alpha z_1 + (1-\alpha)z_2) &= \inf\{F(u) \ | \ u \in \Omega \ \textit{and} \ G(u) \leq \alpha z_1 + (1-\alpha)z_2\} \qquad (8.4)\\
&\leq \inf\{F(u) \ | \ u = \alpha u_1 + (1-\alpha)u_2 \ \ u_1, u_2 \in \Omega \\
&\quad \textit{and} \ G(u_1) \leq z_1, \ \ G(u_2) \leq z_2\} \qquad (8.5)\\
&\leq \alpha \inf\{F(u_1) \ | \ u_1 \in \Omega, \ \ G(u_1) \leq z_1\} \\
&\quad +(1-\alpha)\inf\{F(u_2) \ | \ u_2 \in \Omega, \ \ G(u_2) \leq z_2\} \qquad (8.6)\\
&\leq \alpha\omega(z_1) + (1-\alpha)\omega(z_2). \ \square \qquad (8.7)
\end{aligned}$$

Now we establish the Lagrange multiplier theorem for convex global optimization.

Theorem 8.1.6. *Let U be a vector space, Z a Banach space, Ω a convex subset of U, P a positive cone of Z. Assume that P contains a interior point. Let F be a real convex functional on Ω and G a convex mapping from Ω into Z. Assume the existence of $u_1 \in \Omega$ such that $G(u_1) < \theta$. Defining*

$$\mu_0 = \inf_{u \in \Omega}\{F(u) \ | \ G(u) \leq \theta\}, \tag{8.8}$$

then there exists $z_0^* \geq \theta$, $z_0^* \in Z^*$ such that

$$\mu_0 = \inf_{u \in \Omega}\{F(u) + \langle G(u), z_0^* \rangle_Z\}. \tag{8.9}$$

Furthermore, if the infimum in (8.8) is attained by $u_0 \in U$ such that $G(u_0) \leq \theta$, it is also attained in (8.9) by the same u_0 and also $\langle G(u_0), z_0^* \rangle_V = 0$. We refer to z_0^* as the Lagrangian Multiplier.

Proof: Consider the space $W = \mathbb{R} \times Z$ and the sets A, B where

$$A = \{(r, z) \in (\mathbb{R}, Z) \ | \ r \geq F(u), \ \ z \geq G(u) \ \ \textit{for} \ \ \textit{some} \ \ u \in \Omega\}, \tag{8.10}$$

and

$$B = \{(r, z) \in (\mathbb{R}, Z) \mid r \leq \mu_0, \quad z \leq \theta\}, \tag{8.11}$$

where $\mu_0 = \inf_{u \in \Omega}\{F(u) \mid G(u) \leq \theta\}$. Since F and G are convex, A and B are convex sets. It is clear that A contains no interior point of B, and since $N = -P$ contains an interior point , the set B contains an interior point. Thus, from the separating hyperplane theorem, there is a non-zero element $w_0^* = (r_0, z_0^*) \in W^*$ such that

$$r_0 r_1 + \langle z_1, z_0^* \rangle_Z \geq r_0 r_2 + \langle z_2, z_0^* \rangle_Z, \forall (r_1, z_1) \in A, \quad (r_2, z_2) \in B. \tag{8.12}$$

From the nature of B it is clear that $w_0^* \geq \theta$. That is, $r_0 \geq 0$ and $z_0^* \geq \theta$. We will show that $r_0 > 0$. The point $(\mu_0, \theta) \in B$, hence

$$r_0 r + \langle z, z_0^* \rangle_Z \geq r_0 \mu_0, \forall (r, z) \in A. \tag{8.13}$$

If $r_0 = 0$ then $\langle G(u_1), z_0^* \rangle_Z \geq 0$ and $z_0^* \neq \theta$. Since $G(u_1) < \theta$ and $z^* \geq \theta$ we have a contradiction. Therefore $r_0 > 0$ and, without loss of generality we may assume $r_0 = 1$. Since the point (μ_0, θ) is arbitrarily close to A and B, we have

$$\mu_0 = \inf_{(r,z) \in A}\{r + \langle z, z_0^* \rangle_Z\} \leq \inf_{u \in \Omega}\{F(u) + \langle G(u), z_0^* \rangle_Z\} \leq \inf\{F(u) \mid u \in \Omega, \quad G(u) \leq \theta\} = \mu_0. \tag{8.14}$$

Also, if there exists u_0 such that $G(u_0) \leq \theta$, $\mu_0 = F(u_0)$, then

$$\mu_0 \leq F(u_0) + \langle G(u_0), z_0^* \rangle_Z \leq F(u_0) = \mu_0. \tag{8.15}$$

Hence

$$\langle G(u_0), z_0^* \rangle_Z = 0. \quad \square \tag{8.16}$$

Corollary 8.1.7. *Let the hypothesis of the last theorem hold. Suppose*

$$F(u_0) = \inf_{u \in \Omega}\{F(u) \mid G(u) \leq \theta\}. \tag{8.17}$$

Then there exists $z_0^* \geq \theta$ such that the Lagrangian $L : U \times Z^* \to \mathbb{R}$ defined by

$$L(u, z^*) = F(u) + \langle G(u), z^* \rangle_Z \tag{8.18}$$

has a saddle point at (u_0, z_0^*). That is

$$L(u_0, z^*) \leq L(u_0, z_0^*) \leq L(u, z_0^*), \forall u \in \Omega, z^* \geq \theta. \tag{8.19}$$

Proof: For z_0^* obtained in the last theorem, we have

$$L(u_0, z_0^*) \leq L(u, z_0^*), \forall u \in \Omega. \tag{8.20}$$

As $G(u_0, z_0^*) = 0$, we have

$$L(u_0, z^*) - L(u_0, z_0^*) = \langle G(u_0), z^* \rangle_Z - \langle G(u_0), z_0^* \rangle_Z = \langle G(u_0), z^* \rangle_Z \leq 0. \quad \square \tag{8.21}$$

We now prove two theorems relevant to develop the subsequent section.

Theorem 8.1.8. *Let $F : \Omega \subset U \to \mathbb{R}$ and $G : \Omega \to Z$. Let $P \subset Z$ be a cone. Suppose there exist $(u_0, z_0^*) \in U \times Z^*$ where $z_0^* \geq \theta$ and $u_0 \in \Omega$ such that*

$$F(u_0) + \langle G(u_0), z_0^* \rangle_Z \leq F(u) + \langle G(u), z_0^* \rangle_Z, \forall u \in \Omega. \tag{8.22}$$

Then

$$F(u_0) + \langle G(u_0), z_0^* \rangle_Z = \inf\{F(u) \ \mid \ u \in \Omega \ \ and \ \ G(u) \leq G(u_0)\}. \tag{8.23}$$

Proof: Suppose there is a $u_1 \in \Omega$ such that $F(u_1) < F(u_0)$ and $G(u_1) \leq G(u_0)$. Thus

$$\langle G(u_1), z_0^* \rangle_Z \leq \langle G(u_0), z_0^* \rangle_Z \tag{8.24}$$

so that

$$F(u_1) + \langle G(u_1), z_0^* \rangle_Z < F(u_0) + \langle G(u_0), z_0^* \rangle_Z, \tag{8.25}$$

which contradicts the hypothesis of the theorem. $\square$

Theorem 8.1.9. *Let F be a convex real functional and $G : \Omega \to Z$ convex and let u_0 and u_1 be solutions to the problems $\mathcal{P}_0$ and $\mathcal{P}_1$ respectively, where*

$$\mathcal{P}_0 : \ minimize\ F(u)\ subject\ to\ u \in \Omega\ and\ G(u) \leq z_0, \tag{8.26}$$

and

$$\mathcal{P}_1 : \ minimize\ F(u)\ subject\ to\ u \in \Omega\ and\ G(u) \leq z_1. \tag{8.27}$$

Suppose z_0^* and z_1^* are the Lagrange multipliers related to these problems. Then

$$\langle z_1 - z_0, z_1^* \rangle_Z \leq F(u_0) - F(u_1) \leq \langle z_1 - z_0, z_0^* \rangle_Z. \tag{8.28}$$

Proof: For u_0, z_0^* we have

$$F(u_0) + \langle G(u_0) - z_0, z_0^* \rangle_Z \leq F(u) + \langle G(u) - z_0, z_0^* \rangle_Z, \forall u \in \Omega, \tag{8.29}$$

and, particularly for $u = u_1$ and considering that $\langle G(u_0) - z_0, z_0^* \rangle_Z = 0$, we obtain

$$F(u_0) - F(u_1) \leq \langle G(u_1) - z_0, z_0^* \rangle_Z \leq \langle z_1 - z_0, z_0^* \rangle_Z. \tag{8.30}$$

A similar argument applied to u_1, z_1^* provides us the other inequality. $\square$

8.2 Duality

Consider the basic convex programming problem:

$$Mininize \ \ F(u) \ \ subject \ \ to \ \ G(u) \leq \theta, \ \ u \in \Omega, \tag{8.31}$$

where $F : U \to \mathbb{R}$ is a convex functional, $G : U \to Z$ is convex mapping, and Ω is a convex set. We define $\varphi : Z^* \to \mathbb{R}$ by

$$\varphi(z^*) = \inf_{u \in \Omega}\{F(u) + \langle G(u), z^* \rangle_Z\}. \tag{8.32}$$

Proposition 8.2.1. *φ is concave and*

$$\varphi(z^*) = \inf_{z \in \Gamma}\{\omega(z) + \langle z, z^* \rangle_Z\}, \tag{8.33}$$

where

$$\omega(z) = \inf_{u \in \Omega}\{F(u) \;\mid\; G(u) \leq z\}, \tag{8.34}$$

and

$$\Gamma = Range(G).$$

Proof: Observe that

$$\begin{aligned}
\varphi(z^*) &= \inf_{u \in \Omega}\{F(u) + \langle G(u), z^* \rangle_Z\} \\
&\leq \inf_{u \in \Omega}\{F(u) + \langle z, z^* \rangle_Z \;\mid\; G(u) \leq z\} \\
&= \omega(z) + \langle z, z^* \rangle_Z, \forall z^* \geq \theta, z \in \Gamma.
\end{aligned} \tag{8.35}$$

On the other hand, for any $u_1 \in \Omega$, defining $z_1 = G(u_1)$, we obtain

$$F(u_1) + \langle G(u_1), z^* \rangle_Z \geq \inf_{u \in \Omega}\{F(u) + \langle z_1, z^* \rangle_Z \;\mid\; G(u) \leq z_1\} = \omega(z_1) + \langle z_1, z^* \rangle_Z, \tag{8.36}$$

so that

$$\varphi(z^*) \geq \inf_{z \in \Gamma}\{\omega(z) + \langle z, z^* \rangle_Z\}. \quad \Box \tag{8.37}$$

Theorem 8.2.2 (Lagrange Duality)**.** *Consider $F : \Omega \subset U \to \mathbb{R}$ a convex functional, Ω a convex set, and $G : U \to Z$ a convex mapping. Suppose there exists a u_1 such that $G(u_1) < \theta$ and that $\inf_{u \in \Omega}\{F(u) \;\mid\; G(u) \leq \theta\} < \infty$. Under such assumptions, we have*

$$\inf_{u \in \Omega}\{F(u) \;\mid\; G(u) \leq \theta\} = \max_{z^* \geq \theta}\{\varphi(z^*)\}. \tag{8.38}$$

If the infimum on the left side in (12.50) is achieved at some $u_0 \in U$ and the max on the right side at $z_0^* \in Z^*$, then

$$\langle G(u_0), z_0^* \rangle_Z = 0 \tag{8.39}$$

and u_0 minimizes $F(u) + \langle G(u), z_0^* \rangle_Z$ on Ω.

Proof: For $z_0^* \geq \theta$ we have

$$\inf_{u \in \Omega}\{F(u) + \langle G(u), z^* \rangle_Z\} \leq \inf_{u \in \Omega, G(u) \leq \theta}\{F(u) + \langle G(u), z^* \rangle_Z\} \leq \inf_{u \in \Omega, G(u) \leq \theta} F(u) \leq \mu_0. \tag{8.40}$$

or

$$\varphi(z^*) \leq \mu_0. \tag{8.41}$$

The result follows from Theorem 8.1.6. $\Box$

8.3 Lagrange Multiplier Theorems

Definition 8.3.1 (Regular Point). *Let U, V be Banach spaces and consider $T : D \subset U \to V$ (D open) a continuously Fréchet differentiable mapping. We say that $u_0 \in U$ is a regular point of T if $T'(u_0) : U \to V$ is onto.*

Now we present the version for Banach spaces of Inverse Function Theorem. For a proof see Luenberger [23].

Theorem 8.3.2 (Generalized Inverse Function Theorem). *Let U, V be Banach spaces and consider $T : D \subset U \to V$ (D open) a continuously Fréchet differentiable mapping. Let $u_0 \in U$ be a regular point of T. Then there is a neighborhood $\mathcal{V}_{v_0}$ of $v_0 = T(u_0)$ and $K \in \mathbb{R}^+$, such that for each $v \in \mathcal{V}_{v_0}$, there exists $u \in U$ such that*

$$T(u) = v \text{ and } \|u - u_0\|_U \leq K\|v - v_0\|_V. \tag{8.42}$$

Before the final result, we need the lemma:

Lemma 8.3.3. *Suppose the functional $F : U \to \mathbb{R}$ achieves a local extremum under the constraint $H(u) = \theta$ at the point u_0. Also assume that F and H are continuously Fréchet differentiable in an open set containing u_0 and that u_0 is a regular point of H. Then $\langle h, F'(u_0)\rangle_U = 0$ for all h satisfying $H'(u_0)h = \theta$.*

Proof: Without loss of generality, suppose the local extremum is a minimum. Consider the transformation $T : U \to \mathbb{R} \times Z$ defined by $T(u) = (F(u), H(u))$. Suppose there exists a h such that $H'(u_0)h = \theta$, $F'(u_0)h \neq 0$, then $T'(u_0) = (F'(u_0), H'(u_0)) : U \to \mathbb{R} \times Z$ is onto since $H'(u_0)$ is onto Z. By the inverse function theorem, given $\varepsilon > 0$ there exists $u \in U$ and $\delta > 0$ with $\|u - u_0\|_U < \varepsilon$ such that $T(u) = (F(u_0) - \delta, \theta)$, which contradicts the assumption that u_0 is a local minimum. $\square$

Theorem 8.3.4 (Lagrange Multiplier). *Suppose F is a continuously Fréchet differentiable functional which has a local extremum under the constraint $H(u) = \theta$ at the regular point u_0, then there exists a Lagrangian multiplier $z_0^* \in Z^*$ such that the Lagrangian functional*

$$L(u) = F(u) + \langle H(u), z_0^*\rangle_Z \tag{8.43}$$

is stationary at u_0, that is, $F'(u_0) + H'(u_0)^* z_0^* = \theta$.

Proof: From last lemma we have that $F'(u_0)$ is orthogonal to the null space of $H'(u_0)$. Since the the range of $H'(u_0)$ is closed, it follows that

$$F'(u_0) \in \mathcal{R}[H'(u_0)]^*, \tag{8.44}$$

therefore there exists $z_0^* \in Z^*$ such that

$$F'(u_0) = -H'(u_0)^* z_0^*. \ \square \tag{8.45}$$

Chapter 9

Duality Applied to a Plate Model

9.1 Introduction

The main objective of the present chapter is to develop systematic approaches for obtaining dual variational formulations for systems originally modeled by non-linear differential equations.

Duality for linear systems is well established and is the main subject of classical convex analysis, since in case of linearity, both primal and dual formulations are generally convex. In case of non-linear differential equations, some complications occur and the standard models of duality for convex analysis must be modified and extended.

In particular in the case of Kirchhoff-Love plate model, there is a non-linearity concerning the strain tensor (that is, a geometric non-linearity). To apply the classical results of convex analysis and obtain the complementary formulation is possible only for a special class of external loads. This leads to non-compressed plates, please see Telega [33], Gao [18] and other references therein.

We now describe the primal formulation and related duality principles. Consider a plate whose middle surface is represented by an open bounded set $S \subset \mathbb{R}^2$, whose boundary is denoted by Γ, subjected to a load to be specified. We denote by $u_\alpha : S \to \mathbb{R}$ $(\alpha = 1, 2)$ the horizontal displacements and by $w : S \to \mathbb{R}$, the vertical displacement field. The boundary value form of the Kirchhoff-Love model can be expressed by the equations:

$$\left\{ \begin{array}{l} N_{\alpha\beta,\beta} = 0, \\ Q_{\alpha,\alpha} + M_{\alpha\beta,\alpha\beta} + P = 0, \quad a.e. \;\; in \;\; S \end{array} \right. \tag{9.1}$$

and

$$\begin{cases} N_{\alpha\beta}.n_\beta - \bar{P}_\alpha = 0, \\ \\ (Q_\alpha + M_{\alpha\beta,\beta})n_\alpha + \frac{\partial(M_{\alpha\beta}t_\alpha n_\beta)}{\partial s} - \bar{P} = 0, \\ \\ M_{\alpha\beta}n_\alpha n_\beta - M_n = 0, \quad on \quad \Gamma_t, \end{cases} \tag{9.2}$$

where,

$$N_{\alpha\beta} = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu},$$

$$M_{\alpha\beta} = h_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}$$

and,

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + w_{,\alpha}w_{,\beta}),$$

$$\kappa_{\alpha\beta}(u) = -w_{,\alpha\beta},$$

with the boundary conditions

$$u_\alpha = w = \frac{\partial w}{\partial \mathbf{n}} = 0, \quad on \quad \Gamma_u.$$

Here, $\{N_{\alpha\beta}\}$ denote the membrane forces, $\{M_{\alpha\beta}\}$ denote the moments and $\{Q_\alpha\} = \{N_{\alpha\beta}w_{,\beta}\}$ stand for functions related to the rotation work of membrane forces, $P \in L^2(S)$ is a field of vertical distributed forces applied on S, $(\bar{P}_\alpha, \bar{P}) \in (L^2(\Gamma_t))^3$ denote forces applied to Γ_t concerning the horizontal directions defined by $\alpha = 1, 2$ and vertical direction respectively. M_n are distributed moments applied also to Γ_t, where Γ is such that $\Gamma_u \subset \Gamma$, $\Gamma = \Gamma_u \cup \Gamma_t$ and $\Gamma_u \cap \Gamma_t = \emptyset$. Finally, the matrices $\{H_{\alpha\beta\lambda\mu}\}$ and $\{h_{\alpha\beta\lambda\mu}\}$ are related to the coefficients of Hooke's Law.

The corresponding primal variational formulation to this boundary value model is represented by the functional $J : U \to \mathbb{R}$, where

$$J(u) = \frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu}dS - \int_S PwdS - \int_{\Gamma_t}(\bar{P}w + \bar{P}_\alpha u_\alpha - M_n\frac{\partial w}{\partial \mathbf{n}})d\Gamma$$

and

$$U = \{(u_\alpha, w) \in W^{1,2}(S) \times W^{1,2}(S) \times W^{2,2}(S), \quad u_\alpha = w = \frac{\partial w}{\partial \mathbf{n}} = 0 \ \ on \ \ \Gamma_u\}.$$

The first duality principle presented is the classical one (again we mention the earlier similar results in Telega [33], Gao [18]) , and is obtained by applying a little change of Rockafellar's approach for convex analysis. We have developed a different proof from the one found in [33], by using the definition of Legendre Transform and related properties. Such a result may be summarized as

$$\inf_{u \in U}\{J(u)\} = \sup_{v^* \in \mathcal{A}^* \cap C^*}\{-G_L^*(v^*)\} \tag{9.3}$$

The dual functional, denoted by $-G_L^*: \mathcal{A}^* \cap C^* \to \bar{\mathbb{R}}$ is expressed as

$$G_L^*(v^*) = \left\{ \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{N}_{\alpha\beta} Q_\alpha Q_\beta dS \right\},$$

where C^* is defined by equations (9.1) and (9.2) and,

$$\mathcal{A}^* = \{v^* \in Y^* | \ \ N_{11} > 0, N_{22} > 0, \ \ and \ \ N_{11}N_{22} - N_{12}^2 > 0, \ \ a.e. \ \ in \ \ S\}, \tag{9.4}$$

here $v^* = \{N_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha\} \in Y^* = L^2(S;\mathbb{R}^{10}) \equiv \mathbf{L}^2(S)$

Therefore, since the functional $G_L^*(v^*)$ is convex in $\mathcal{A}^*$, the duality is perfect if the optimum solution for the primal formulation satisfies the constraints indicated in (9.4), however it is important to emphasize that such constraints imply no compression along the plate.

For the second and third principles, we emphasize that our dual formulations remove or relax the constraints on the external load, and are valid even for compressed plates.

Still for these two principles, we use a theorem (Toland [35]) which does not require convexity of primal functionals. Such a result can be summarized as:

$$\inf_{u \in U}\{G(u) - F(u)\} = \inf_{u^* \in U^*}\{F^*(u^*) - G^*(u^*)\}$$

Here $G: U \to \mathbb{R}$ and $F: U \to \mathbb{R}$ and, $F^*: U^* \to \mathbb{R}$ and $G^*: U^* \to \mathbb{R}$ denote the primal and dual functionals respectively.

In particular for the second principle, we modify the above result by applying it to a not one to one relation between primal and dual variables, obtaining the final duality principle expressed as follows

$$\inf_{(u,p) \in U \times Y}\{J_K(u,p)\} \leq \inf_{(\hat{u},v^*) \in U \times Y^*}\{J_K^*(\hat{u},v^*)\}$$

where

$$J_K(u,p) = G(\Lambda u + p) - F(u) + \frac{K}{2}\langle p,p\rangle_{\mathbf{L}^2(S)}$$

and

$$J_K^*(\hat{u},v^*) = F^*(\Lambda^* v^*) - G_L(v^*) + K\left\|\Lambda\hat{u} - \frac{\partial g_L^*(v^*)}{\partial y^*}\right\|_{\mathbf{L}^2(S)}^2 + \frac{1}{2K}\langle v^*,v^*\rangle_{\mathbf{L}^2(S)}.$$

Here $K \in \mathbb{R}$ is a positive constant and we are particularly concerned with the fact that

$$J_K(u_K,p_K) \to J(u_0), \ \ as \ \ K \to +\infty$$

and

$$J_K^*(\hat{u}_K,v_K^*) \to J(u_0), \ \ as \ \ K \to +\infty$$

where

$$J_K(u_K, p_K) = \inf_{(u,p)\in U\times Y}\{J_K(u,p)\},$$

$$J_K^*(\hat{u}_K, v_K^*) = \inf_{(\hat{u},v^*)\in U\times Y^*}\{J_K^*(\hat{u}, v^*)\}$$

and

$$J(u_0) = \inf_{u\in U}\{J(u) = G(\Lambda u) - F(u)\}.$$

Even though we do not prove it in the present article, postponing a more rigorous analysis concerning the behavior of u_K indicated above as $K \to +\infty$, for a future work.

For the third duality principle, the dual variables must satisfy the following constraints :

$$N_{11} + K > 0, \;\; N_{22} + K > 0 \;\; and \;\; (N_{11} + K)(N_{22} + K) - N_{12}^2 > 0, \;\; a.e. \;\; in \;\; S. \tag{9.5}$$

Such a principle may be summarized by the following result,

$$\inf_{u\in U}\{G(\Lambda u) - F(\Lambda_1 u) - \langle u, p\rangle_U\} \le \inf_{z^*\in Y^*}\{\sup_{v^*\in B^*(z^*)}\{F^*(z^*) - G_L^*(v^*)\}\},$$

where

$$B^*(z^*) = \{v^* \in Y^* \;\;|\;\; \Lambda^* v^* - \Lambda_1^* z^* - p = 0\}$$

Therefore the constant $K > 0$ must be chosen so that the optimum point of the primal formulation satisfies the constraints indicated in (9.5). This is because these relations also define an enlarged region in which the analytical expression of the functional $G_L^* : Y^* \to \mathbb{R}$ is convex, so that, in this case, negative membrane forces are allowed.

In Section 9.7, we present a convex dual variational formulation which may be expressed through the following duality principle:

$$\inf_{u\in U}\{J(u)\} = \sup_{(v^*,z^*,w)\in E^*\cap B^*}\{-G^*(v^*) + \langle z_\alpha^*, z_\alpha^*\rangle_{L^2(S)}/(2K)\}$$

where,

$$G^*(v^*) = G_L^*(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu}dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu}dS + \frac{1}{2}\int_S \bar{N}_{\alpha\beta}^K Q_{,\alpha}Q_{,\beta}dS$$

if $v^* \in E^*$, where

$v^* = \{\{N_{\alpha\beta}\}, \{M_{\alpha\beta}\}, \{Q_\alpha\}\} \in E^* \Leftrightarrow v^* \in L^2(S, \mathbb{R}^{10})$ and

$$N_{11} + K > 0 \quad N_{22} + K > 0 \; and \;\; (N_{11} + K)(N_{22} + K) - N_{12}^2 > 0, \;\; a.e. \;\; in \;\; S,$$

where

$$\{\bar{N}_{\alpha\beta}^K\} = \left\{\begin{array}{cc} N_{11} + K & N_{12} \\ N_{12} & N_{22} + K \end{array}\right\}^{-1} \tag{9.6}$$

and

$$(v^*, z^*) \in B^* \Leftrightarrow \begin{cases} N_{\alpha\beta,\beta} + P_\alpha = 0, \\ \\ Q_{\alpha,\alpha} + M_{\alpha\beta,\alpha\beta} - z^*_{\alpha,\alpha} + P = 0, \\ \\ \bar{h}_{1212} M_{12} + z^*_{1,2}/K = 0, \\ \\ z^*_{1,2} = z^*_{2,1}, \quad a.e. \;\; in \;\; S, \quad and, \quad z^* = \theta \;\; on \;\; \Gamma. \end{cases}$$

Here we are assuming the existence of $u_0 \in U$ such that $\delta J(u_0) = \theta$, and so that there exists $K > 0$ for which $N_{11}(u_0) + K > 0$, $N_{22}(u_0) + K > 0$, $(N_{11}(u_0) + K)(N_{22}(u_0) + K) - N_{12}(u_0)^2 > 0$ (a.e in S) and $h_{1212}/(2K_0) > K$ where K_0 is the constant related to Poincaré Inequality and,

$$N_{\alpha\beta}(u_0) = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u_0).$$

Finally, in the last section, we prove a result similar to those obtained through the triality criterion introduced in Gao [20] and establish sufficient conditions for the existence of a minimizer for the primal formulation. Such conditions may be summarized by $\delta J(u_0) = \theta$ and

$$\frac{1}{2}\int_S N_{\alpha\beta}(u_0) w_{,\alpha} w_{,\beta} dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu} w_{,\alpha\beta} w_{,\lambda\mu} dS \geq 0, \forall w \in W_0^{2,2}(S). \quad \square$$

For this last result, our proof is new. The statement of results themselves follows those of Gao [20].

We are now ready to state the result of Toland [35], through which will be constructed three duality principles.

Theorem 9.1.1. *Let $J : U \longrightarrow \overline{\mathbb{R}}$ be a functional defined as $J(u) = G(u) - F(u)$, $\forall u \in U$, where there exists $u_0 \in U$ such that $J(u_0) = \inf_{u \in U}\{J(u)\}$ and $\partial F(u_0) \neq \emptyset$, then*

$$\inf_{u \in U}\{G(u) - F(u)\} = \inf_{u^* \in U^*}\{F^*(u^*) - G^*(u^*)\}$$

and for $u_0^* \in \partial F(u_0)$ we have,

$$F^*(u_0^*) - G^*(u_0^*) = \inf_{u^* \in U^*}\{F^*(u^*) - G^*(u^*)\}.$$

Furthermore $u_0^* \in \partial G(u_0)$.

9.2 The Primal Variational Formulation

Let $S \subset \mathbb{R}^2$ be an open bounded set (with a boundary denoted by Γ) which represents the middle surface of a plate of thickness h. The vectorial basis related to the Cartesian system

$\{x_1, x_2, x_3\}$ is denoted by $(\mathbf{a}_\alpha, \mathbf{a}_3)$, where $\alpha = 1, 2$ (in general Greek indices stand for 1 or 2), $\mathbf{a}_3$ denotes the vector normal to S, $\mathbf{t}$ is the vector tangent to Γ and $\mathbf{n}$ is the outer normal to S. The displacements will be denoted by:

$$\hat{\mathbf{u}} = \{\hat{u}_\alpha, \hat{u}_3\} = \hat{u}_\alpha \mathbf{a}_\alpha + \hat{u}_3 \mathbf{a}_3,$$

The Kirchhoff-Love relations are

$$\hat{u}_\alpha(x_1, x_2, x_3) = u_\alpha(x_1, x_2) - x_3 w(x_1, x_2)_{,\alpha} \text{ and } \hat{u}_3(x_1, x_2, x_3) = w(x_1, x_2),$$

where $-h/2 \leq x_3 \leq h/2$ so that we have $u = (u_\alpha, w) \in U$ where

$$U = \left\{ (u_\alpha, w) \in W^{1,2}(S) \times W^{1,2}(S) \times W^{2,2}(S), \quad u_\alpha = w = \frac{\partial w}{\partial \mathbf{n}} = 0 \ \ on \ \ \Gamma_u \right\}.$$

We divide the boundary into two parts, so that $\Gamma_u \subset \Gamma$, $\Gamma = \Gamma_u \cup \Gamma_t$ and $\Gamma_u \cap \Gamma_t = \emptyset$. The strain tensors are denoted by

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}[\Lambda_{1\alpha\beta}(u) + \Lambda_{2\alpha}(u)\Lambda_{2\beta}(u)] \tag{9.7}$$

and

$$\kappa_{\alpha\beta}(u) = \Lambda_{3\alpha\beta}(u) \tag{9.8}$$

where: $\Lambda = \{\{\Lambda_{1\alpha\beta}\}, \{\Lambda_{2\alpha}\}, \{\Lambda_{3\alpha\beta}\}\} : U \to Y = Y^* = L^2(S; \mathbb{R}^{10}) \equiv \mathbf{L}^2(S)$ is defined by:

$$\Lambda_{1\alpha\beta}(u) = u_{\alpha,\beta} + u_{\beta,\alpha}, \tag{9.9}$$

$$\Lambda_{2\alpha}(u) = w_{,\alpha} \tag{9.10}$$

and

$$\Lambda_{3\alpha\beta}(u) = -w_{,\alpha\beta}. \tag{9.11}$$

The constitutive relations are expressed as

$$N_{\alpha\beta} = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}, \tag{9.12}$$

$$M_{\alpha\beta} = h_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} \tag{9.13}$$

where: $\{H_{\alpha\beta\lambda\mu}\}$ and $\{h_{\alpha\beta\lambda\mu} = \frac{h^2}{12} H_{\alpha\beta\lambda\mu}\}$, are positive definite matrices and such that $H_{\alpha\beta\lambda\mu} = H_{\alpha\beta\mu\lambda} = H_{\beta\alpha\lambda\mu} = H_{\beta\alpha\mu\lambda}$. Furthermore $\{N_{\alpha\beta}\}$ denote the membrane forces and $\{M_{\alpha\beta}\}$ the moments. The plate stored energy, denoted by $(G \circ \Lambda) : U \to \mathbb{R}$ is expressed as

$$(G \circ \Lambda)(u) = \frac{1}{2}\int_S N_{\alpha\beta}\gamma_{\alpha\beta} dS + \frac{1}{2}\int_S M_{\alpha\beta}\kappa_{\alpha\beta} dS \tag{9.14}$$

and the external work, denoted as $F : U \to \mathbb{R}$, is given by

$$F(u) = \int_S PwdS + \int_{\Gamma_t} (\bar{P}w + \bar{P}_\alpha u_\alpha - M_n \frac{\partial w}{\partial \mathbf{n}})d\Gamma, \tag{9.15}$$

where P denotes a vertical distributed load applied in S and $\bar{P}, \bar{P}_\alpha$ are forces applied on $\Gamma_t \subset \Gamma$ related to directions defined by $\mathbf{a}_3$ and $\mathbf{a}_\alpha$ respectively, and, M_n denote moments also applied on Γ_t. The potential energy, denoted by $J : U \to \mathbb{R}$ is expressed as:

$$J(u) = (G \circ \Lambda)(u) - F(u)$$

It is important to emphasize that conditions for the existence of a minimizer (here denoted by u_0) related to $G(\Lambda u) - F(u)$ were presented in Ciarlet [11]. Such $u_0 \in U$ satisfies the equation:

$$\delta(G(\Lambda u_0) - F(u_0)) = \theta$$

and we should expect at least one minimizer if $\|\bar{P}_\alpha\|_{L_2(\Gamma_t)}$ is small enough and $m(\Gamma_u) > 0$ (here m stands for the Lebesgue measure) and with no restrictions concerning the magnitude of $\|P_\alpha\|_{L_2(S)}$ if $m(\Gamma) = m(\Gamma_u)$, so that in the latter case, we consider a field of distributed forces $\{P_\alpha\}$ applied on S.

Some inequalities of Sobolev type are necessary to prove the above result, and in this work we assume some regularity hypothesis concerning S and its boundary, namely: in addition to S being open and bounded, also we assume it is connected with a Lipschitz continuous boundary Γ, so that S is locally on one side of Γ.

The formal proof of existence of a minimizer for $J(u) = G(\Lambda u) - F(u)$ is obtained through the Direct Method of Calculus of variations. We do not repeat this procedure here, we just refer to Ciarlet [11] for details.

9.3 The Legendre Transform

In this section we determine the Legendre Transform related to the function $g : \mathbb{R}^{10} \to \mathbb{R}$ where:

$$g(y) = \frac{1}{2} H_{\alpha\beta\lambda\mu}[(y_{1^{\alpha\beta}} + y_{1^{\beta\alpha}} + y_{2^\alpha} y_{2^\beta})/2][(y_{1^{\lambda\mu}} + y_{1^{\mu\lambda}} + y_{2^\lambda} y_{2^\mu})/2] + \frac{1}{2} h_{\alpha\beta\lambda\mu} y_{3^{\alpha\beta}} y_{3^{\lambda\mu}} \tag{9.16}$$

and we recall that

$$G(\Lambda u) = \int_S g(\Lambda u) dS.$$

From Definition 7.1.25 we may write

$$g_L^*(y^*) = \langle y_0, y^* \rangle_{\mathbb{R}^{10}} - g(y_0)$$

where y_0 is the unique solution of the system,

$$y_i^* = \frac{\partial g(y_0)}{\partial y_i}$$

which for the above function g, implies:

$$y_{1\alpha\beta}^* = H_{\alpha\beta\lambda\mu}(y_{1\lambda\mu} + y_{2\lambda}y_{2\mu}/2)$$

$$y_{2\alpha}^* = H_{\alpha\beta\lambda\mu}(y_{1\lambda\mu}, +y_{2\lambda}y_{2\mu}/2)y_{2\beta} = y_{1\alpha\beta}^* y_{2\beta},$$

and

$$y_{3\alpha\beta}^* = h_{\alpha\beta\lambda\mu}y_{3\lambda\mu}.$$

Inverting this system we obtain

$$y_{02^1} = (y_{1^{22}}^*.y_{2^1}^* - y_{1^{12}}^*.y_{2^2}^*)/\Delta,$$

$$y_{02^2} = (-y_{1^{12}}^*.y_{2^1}^* + y_{1^{11}}^*.y_{2^2}^*)/\Delta,$$

and

$$y_{01^{\alpha\beta}} = \bar{H}_{\alpha\beta\lambda\mu}y_{1^{\lambda\mu}}^* - y_{02^\alpha}.y_{02^\beta}/2$$

where

$$\{\bar{H}_{\alpha\beta\lambda\mu}\} = \{H_{\alpha\beta\lambda\mu}\}^{-1},$$

$\Delta = y_{1^{11}}^* y_{1^{22}}^* - (y_{1^{12}}^*)^2$ (we recall that $y_{1^{12}}^* = y_{1^{21}}^*$, as a result of the symmetries of $\{H_{\alpha\beta\lambda\mu}\}$). By analogy,

$$y_{03^{\alpha\beta}} = \bar{h}_{\alpha\beta\lambda\mu}v_{3^{\lambda\mu}}^*$$

where:

$$\{\bar{h}_{\alpha\beta\lambda\mu}\} = \{h_{\alpha\beta\lambda\mu}\}^{-1}.$$

Thus we can define the set R_L^n, concerning Definition 7.1.25 as

$$R_L^n = \{y^* \in \mathbb{R}^{10} \quad | \quad \Delta \neq 0\}. \tag{9.17}$$

After some simple algebraic manipulations we obtain the expression for $g_L^* : R_L^n \to \mathbb{R}$, that is,

$$g_L^*(y^*) = \frac{1}{2}\bar{H}_{\alpha\beta\lambda\mu}y_{1^{\alpha\beta}}^* y_{1^{\lambda\mu}}^* dS + \frac{1}{2}\bar{h}_{\alpha\beta\lambda\mu}y_{3^{\alpha\beta}}^* y_{3^{\lambda\mu}}^* dS + \frac{1}{2}y_{1^{\alpha\beta}}^* y_{02^\alpha}y_{02^\beta}dS. \tag{9.18}$$

Also from Definition 7.1.25, we have

$$Y_L^* = \{v^* \in Y^* = L^2(S;\mathbb{R}^{10}) \equiv \mathbf{L}^2(S) \quad | \quad v^*(x) \in R_L^n \quad a.e. \quad in \quad S\}$$

so that $G_L^* : Y_L^* \to \mathbb{R}$ may be expressed as

$$G_L^*(v^*) = \int_S g_L^*(v^*)dS.$$

Or, from (9.18),

$$G_L^*(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} v^*_{1\alpha\beta} v^*_{1\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} v^*_{3\alpha\beta} v^*_{3\lambda\mu} dS + \frac{1}{2}\int_S v^*_{1\alpha\beta} v_{02^\alpha} v_{02^\beta} dS \quad \Box$$

Changing the notation, as indicated below,

$$v^*_{1\alpha\beta} = N_{\alpha\beta}, \quad v^*_{2^\alpha} = Q_\alpha = v^*_{1\alpha\beta} v_{02^\beta} = N_{\alpha\beta} v_{02^\beta}, \quad v^*_{3\alpha\beta} = M_{\alpha\beta}$$

we could express $G_L^* : Y_L^* \to \mathbb{R}$ as

$$G_L^*(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{N}_{\alpha\beta} Q_\alpha Q_\beta dS,$$

where

$$\bar{N}_{\alpha\beta} = \{N_{\alpha\beta}\}^{-1}.$$

Remark 9.3.1. *Also we can use the transformation*

$$Q_\alpha = N_{\alpha\beta} w_{,\beta}$$

and obtain

$$G_L^*(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \frac{1}{2}\int_S N_{\alpha\beta} w_{,\alpha} w_{,\beta} dS.$$

The term denoted by $G_p : Y^* \times U \to \mathbb{R}$ and expressed as

$$G_p(v^*, w) = \frac{1}{2}\int_S N_{\alpha\beta} w_{,\alpha} w_{,\beta} dS$$

is known as the ***gap function****.*

9.4 The Classical Dual Formulation

In this section we establish the dual variational formulation in the classical sense.

We recall that $J : U \to \mathbb{R}$ is expressed by

$$J(u) = (G \circ \Lambda)(u) - F(u),$$

where $(G \circ \Lambda) : U \to \mathbb{R}$ and $F : U \to \mathbb{R}$ were defined by equations (9.14) and (9.15) respectively. It is known and easy to see that

$$\inf_{u \in U}\{G(\Lambda u) + F(u)\} \geq \sup_{v^* \in Y^*}\{-G^*(v^*) - F^*(-\Lambda^* v^*)\}. \tag{9.19}$$

Now we prove a result concerning the representation of the polar functional, namely:

Proposition 9.4.1. *Considering the earlier definitions and assumptions on $G : Y \to \mathbb{R}$ (see section 9.2), expressed by $G(v) = \int_S g(v)dS$, where $g : \mathbb{R}^{10} \to \mathbb{R}$ is indicated in (9.16), we have*

$$v^* \in \mathcal{A}^* \Rightarrow G^*(v^*) = G^*_L(v^*)$$

where

$$G^*_L(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{N}_{\alpha\beta} Q_\alpha Q_\beta dS \tag{9.20}$$

and

$$\mathcal{A}^* = \{v^* = \{N_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha\} \in Y^* |\ \ N_{11} > 0, N_{22} > 0, \ \ and \ \ N_{11}N_{22} - N_{12}^2 > 0, \ \ a.e. \ \ in \ \ S\} \tag{9.21}$$

Proof: First, consider the quadratic inequality in x as indicated below,

$$\bar{a}x^2 + \bar{b}x + \bar{c} \leq 0, \forall x \in \mathbb{R},$$

which is equivalent to

$$(\bar{a} < 0 \text{ and } \bar{b}^2 - 4\bar{a}\bar{c} \leq 0) \ \ or \ \ (\bar{a} = 0, \bar{b} = 0 \text{ and } \bar{c} \leq 0). \tag{9.22}$$

Consider now the inequality

$$a_1x^2 + b_1xy + c_1y^2 + d_1x + e_1y + f_1 \leq 0, \forall x, y \in \mathbb{R}^2 \tag{9.23}$$

and the quadratic equation related to the variable x, for

$$\bar{a} = a_1, \ \ \bar{b} = b_1y + d_1 \ \ and \ \ \bar{c} = c_1y^2 + e_1y + f_1,$$

and for $a_1 < 0$, from (9.22) the inequality (9.23) is equivalent to

$$(b_1^2 - 4a_1c_1)y^2 + (2b_1d_1 - 4a_1e_1)y + d_1^2 - 4a_1f_1 \leq 0, \ \ \forall y \in \mathbb{R}.$$

Finally, for

$$\bar{a} = b_1^2 - 4a_1c_1 < 0, \ \ \bar{b} = 2b_1d_1 - 4a_1e_1 \text{ and } \bar{c} = d_1^2 - 4a_1f_1,$$

also from (9.22), the last inequality is equivalent to

$$-c_1d_1^2 - a_1e_1^2 + b_1d_1e_1 - (b_1^2 - 4a_1c_1)f_1 \leq 0. \tag{9.24}$$

In order to represent the polar functional related to the plate stored energy, we first consider the polar functional related to $g_1(y)$, where

$$g_1(y) = \frac{1}{2}H_{\alpha\beta\lambda\mu}(y_{1^{\alpha\beta}} + \frac{1}{2}y_{2^\alpha}y_{2^\beta})(y_{1^{\lambda\mu}} + \frac{1}{2}y_{2^\lambda}y_{2^\mu}),$$

$$g(y) = g_1(y) + g_2(y)$$

and

$$g_2(y) = \frac{1}{2} h_{\alpha\beta\lambda\mu} y_{3\alpha\beta} y_{3\lambda\mu}.$$

In fact we determine a set in which the polar functional is represented by the Legendre Transform $g^*_{1L}(y^*)$, where, from (9.18),

$$g^*_{1L}(y^*) = \frac{1}{2}\bar{H}_{\alpha\beta\lambda\mu} y^*_{1\alpha\beta} y^*_{1\lambda\mu} + \frac{y^*_{111}(y^*_{22})^2 - 2.y^*_{112} y^*_{21} y^*_{22} + y^*_{122}(y^*_{21})^2}{2[y^*_{111} y^*_{122} - (y^*_{112})^2]}. \tag{9.25}$$

Thus, since

$$g^*_1(y^*) = \sup_{y \in \mathbb{R}^6} \{y^*_{1\alpha\beta} y_{1\alpha\beta} + y^*_{2\alpha} y_{2\alpha} - g_1(y)\}$$

we can write

$$g^*_{1L}(y^*) = g^*_1(y^*) \Leftrightarrow g^*_{1L}(y^*) \geq y^*_{1\alpha\beta} y_{1\alpha\beta} + y^*_{2\alpha} y_{2\alpha} - g_1(y), \forall y \in \mathbb{R}^6.$$

Or

$$y^*_{1\alpha\beta} y_{1\alpha\beta} + y^*_{2\alpha} y_{2\alpha} - \frac{1}{2} H_{\alpha\beta\lambda\mu}(y_{1\alpha\beta} + \frac{1}{2} y_{2\alpha} y_{2\beta})(y_{1\lambda\mu} + \frac{1}{2} y_{2\lambda} y_{2\mu}) - g^*_{1L}(y^*) \leq 0, \forall y \in \mathbb{R}^6. \tag{9.26}$$

However, considering the transformation

$$\bar{y}_{1\alpha\beta} = y_{1\alpha\beta} + \frac{1}{2} y_{2\alpha} y_{2\beta},$$

$$y_{1\alpha\beta} = \bar{y}_{1\alpha\beta} - \frac{1}{2} y_{2\alpha} y_{2\beta}, \tag{9.27}$$

and substituting such relations into (9.26), we obtain

$$g^*_{1L}(y^*) = g^*_1(y^*) \Leftrightarrow$$

$$y^*_{1\alpha\beta}(\bar{y}_{1\alpha\beta} - \frac{1}{2} y_{2\alpha} y_{2\beta}) + y^*_{2\alpha} y_{2\alpha} - \frac{1}{2} H_{\alpha\beta\lambda\mu} \bar{y}_{1\alpha\beta} \bar{y}_{1\lambda\mu} - g^*_{1L}(y^*) \leq 0, \forall \{\bar{y}_{1\alpha\beta}, y_{2\alpha}\} \in \mathbb{R}^6. \tag{9.28}$$

On the other hand, since $\{H_{\alpha\beta\lambda\mu}\}$ is a positive definite matrix we have

$$\sup_{\{\bar{y}_{1\alpha\beta}\} \in \mathbb{R}^4} \{y^*_{1\alpha\beta} \bar{y}_{1\alpha\beta} - \frac{1}{2} H_{\alpha\beta\lambda\mu} \bar{y}_{1\alpha\beta} \bar{y}_{1\lambda\mu}\} = \frac{1}{2} \bar{H}_{\alpha\beta\lambda\mu} y^*_{1\alpha\beta} y^*_{1\lambda\mu}. \tag{9.29}$$

Thus considering (9.29) and the expression of $g^*_L(y^*)$ indicated in (9.25) , inequality (9.28) is satisfied if

$$-\frac{1}{2} y^*_{1\alpha\beta} y_{2\alpha} y_{2\beta} + y^*_{2\alpha} y_{2\alpha} - \frac{y^*_{111}.(y^*_{22})^2 - 2.y^*_{112}.y^*_{21} y^*_{22} + y^*_{122}.(y^*_{21})^2}{2[y^*_{111} y^*_{122} - (y^*_{112})^2]} \leq 0, \forall \{y_{2\alpha}\} \in \mathbb{R}^2. \tag{9.30}$$

So, for

$$a_1 = -\frac{1}{2}y^*_{1^{11}}, \quad b_1 = -y^*_{1^{12}}, \quad c_1 = -\frac{1}{2}y^*_{1^{22}}, \quad , d_1 = y^*_{2^1}, \quad e_1 = y^*_{2^2}$$

and

$$f_1 = -\frac{y^*_{1^{11}}.(y^*_{2^2})^2 - 2.y^*_{1^{12}}.y^*_{2^1}y^*_{2^2} + y^*_{1^{22}}.(y^*_{2^1})^2}{2[y^*_{1^{11}}y^*_{1^{22}} - (y^*_{1^{12}})^2]}$$

we obtain

$$-c_1d_1^2 - a_1e_1^2 + b_1d_1e_1 - (b_1^2 - 4a_1c_1)f_1 = 0$$

Therefore from (9.24), the inequality (9.26) is satisfied if $a_1 < 0$ ($y^*_{1^{11}} > 0$) and $b_1^2 - 4a_1c_1 < 0$ ($y^*_{1^{11}}y^*_{1^{22}} - (y^*_{1^{12}})^2 > 0$ which implies $y^*_{1^{22}} > 0$).

Thus we have shown that

$$y^* \in A^* \Rightarrow g_1^*(y^*) = g^*_{1L}(y^*), \tag{9.31}$$

where

$$A^* = \{y^* \in \mathbb{R}^6 | \quad y^*_{1^{11}} > 0, \quad y^*_{1^{22}} > 0, \quad y^*_{1^{11}}y^*_{1^{22}} - (y^*_{1^{12}})^2 > 0\}.$$

On the other hand, by analogy to above results, it can easily be proved that

$$g_2^*(y^*) = g^*_{2L}(y^*), \forall \{y^*_{3\alpha\beta}\} \in \mathbb{R}^3 \tag{9.32}$$

where

$$g^*_{2L}(y^*) = \frac{1}{2}\bar{h}_{\alpha\beta\lambda\mu}y^*_{3\alpha\beta}y^*_{3\lambda\mu} \tag{9.33}$$

and

$$g_2^*(y^*) = \sup_{y \in \mathbb{R}^3}\{y^*_{3\alpha\beta}y_{3\alpha\beta} - \frac{1}{2}h_{\alpha\beta\lambda\mu}y_{3\alpha\beta}y_{3\lambda\mu}\}.$$

From (9.31) and (9.32), we can write

$$if \quad y^* \in A^* \quad then \quad g_1^*(y^*) + g_2^*(y^*) = g^*_{1L}(y^*) + g^*_{2L}(y^*) \leq (g_1 + g_2)^*(y^*).$$

As $(g_1 + g_2)^*(y^*) \leq g_1^*(y^*) + g_2^*(y^*)$ we have

$$if \quad y^* \in A^* \quad then \quad g_L^*(y^*) = g^*_{1L}(y^*) + g^*_{2L}(y^*) = (g_1 + g_2)^*(y^*) = g^*(y^*). \tag{9.34}$$

However, from Proposition 7.1.24

$$G^*(v^*) = \int_S g^*(v^*)dS \tag{9.35}$$

so that from (9.34) and (9.35) we obtain

$$v^* \in \mathcal{A}^* \Rightarrow G^*(v^*) = \int_S g_L^*(v^*)dS = G_L^*(v^*)$$

where,

$$\mathcal{A}^* = \{v^* \in Y^* \quad | \quad v^*(x) \in A^*, \quad a.e. \quad in \quad S\}.$$

Alternatively,

$$\mathcal{A}^* = \{v^* \in Y^* \ \mid \ v^*_{111} > 0, \ \ v^*_{122} > 0, \ \ and \ \ v^*_{111}v^*_{122} - (v^*_{112})^2 > 0, \ \ a.e. \ \ in \ \ S\}. \tag{9.36}$$

Thus, through the notation

$$v^*_{1\alpha\beta} = N_{\alpha\beta}, \ \ v^*_{2\alpha} = Q_\alpha = v^*_{1\alpha\beta}v_{02^\beta} = N_{\alpha\beta}v_{02^\beta}, \ \ v^*_{3\alpha\beta} = M_{\alpha\beta}$$

we have

$$\mathcal{A}^* = \{v^* = \{N_{\alpha\beta}, M_{\alpha\beta}, Q_\alpha\} \in Y^* | \ N_{11} > 0, N_{22} > 0, \ and \ N_{11}N_{22} - N^2_{12} > 0, \ a.e. \ in \ S\} \ \square \tag{9.37}$$

9.4.1 The Polar Functional Related to $F : U \to \bar{\mathbb{R}}$

We are concerned with the evaluation of the extremum,

$$F^*(-\Lambda^* v^*) = \sup_{u \in U}\{\langle u, -\Lambda^* v^*\rangle_U - F(u)\},$$

or

$$F^*(-\Lambda^* v^*) = \sup_{u \in U}\{\langle \Lambda u, -v^*\rangle_Y - F(u)\}.$$

Considering

$$F(u) = -\left(\int_S PwdS + \int_{\Gamma_t} (\bar{P}w + \bar{P}_\alpha u_\alpha - M_n \frac{\partial w}{\partial n}) d\Gamma\right) = \langle u, f\rangle_U$$

we have

$$F^*(-\Lambda^* v^*) = \begin{cases} 0, & if \ \ v^* \in C^*, \\ +\infty, & \text{otherwise}, \end{cases} \tag{9.38}$$

where $v^* \in C^* \Leftrightarrow v^* \in Y^*$ and

$$\begin{cases} v^*_{1\alpha\beta,\beta} = 0, \\ v^*_{2\alpha,\alpha} + v^*_{3\alpha\beta,\alpha\beta} + P = 0, \ \ a.e. \ \ in \ \ S, \end{cases} \tag{9.39}$$

and

$$\begin{cases} v^*_{1\alpha\beta}.n_\beta - \bar{P}_\alpha = 0, \\ \\ (v^*_{2\alpha} + v^*_{3\alpha\beta,\beta}).n_\alpha + \frac{\partial(v^*_{3\alpha\beta}t_\alpha n_\beta)}{\partial s} - \bar{P} = 0, \\ \\ v^*_{3\alpha\beta}n_\alpha n_\beta - M_n = 0, \ \ on \ \ \Gamma_t. \ \ \square \end{cases} \tag{9.40}$$

Remark 9.4.2. *We can also denote*

$$C^* = \{v^* \in Y^* \ \mid \ \Lambda^* v^* = f\}, \tag{9.41}$$

where the relation $\Lambda^* v^* = f$ is defined by (9.39) and (9.40).

9.4.2 The First Duality Principle

Considering inequality (9.19), the expression of $G^*(v^*)$, and the set C^* above defined, we can write

$$\inf_{u\in U}\{(G\circ\Lambda)(u)-F(u)\}\geq \sup_{v^*\in\mathcal{A}^*\cap C^*}\{-G_L^*(v^*)\} \tag{9.42}$$

so that the final form of the concerned duality principle results from the following theorem.

Theorem 9.4.3. *Let $(G\circ\Lambda):U\to\mathbb{R}$ and $F:U\to\mathbb{R}$ be defined by (9.14) and (9.15) respectively (and here we express F as $F(u)=\langle u,f\rangle_U$). If $-G_L^*:Y_L^*\to\mathbb{R}$ attains a local extremum at $v_0^*\in\mathcal{A}^*$ under the constraint $\Lambda^*v^*-f=0$, then*

$$\inf_{u\in U}\{(G\circ\Lambda)(u)+F(u)\}=\sup_{v^*\in\mathcal{A}^*\cap C^*}\{-G_L^*(v^*)\}$$

and $u_0\in U$ and $v_0^*\in Y^*$ such that:

$$\delta\{-G_L^*(v_0^*)+\langle u_0,\Lambda^*v_0^*-f\rangle_U\}=\theta$$

are also such that

$$J(u_0)=-G_L^*(v_0^*)\ \ \text{and}\ \ \delta J(u_0)=\theta.$$

The proof of above theorem is consequence of the standard necessary conditions for a local extremum for $-G_L^*:Y_L^*\to\mathbb{R}$ under the constraint $\Lambda^*v^*-f=\theta$, the inequality (9.42) plus an application of Theorem 7.1.27.

Therefore, in a more explicit format we would have

$$\begin{aligned}
&\inf_{u\in U}\left\{\frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}dS+\frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu}dS\right.\\
&\left.-\left(\int_S PwdS+\int_{\Gamma_t}\bar{P}wdS+\int_{\Gamma_t}\bar{P}_\alpha u_\alpha d\Gamma-\int_{\Gamma_t}M_n\frac{\partial w}{\partial\mathbf{n}}d\Gamma\right)\right\}\\
=&\sup_{v^*\in\mathcal{A}^*\cap C^*}\left\{-\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu}dS-\frac{1}{2}\int_S\bar{h}_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu}dS\right.\\
&\left.-\frac{1}{2}\int_S\bar{N}_{\alpha\beta}Q_\alpha Q_\beta dS\right\}
\end{aligned} \tag{9.43}$$

where $v^*\in C^*\Leftrightarrow v^*\in Y^*$ and,

$$\begin{cases} N_{\alpha\beta,\beta}=0,\\ Q_{\alpha,\alpha}+M_{\alpha\beta,\alpha\beta}+P=0, & a.e.\ \ in\ \ S\end{cases}$$

and

$$\begin{cases} N_{\alpha\beta}.n_\beta - \bar{P}_\alpha = 0, \\ \\ (Q_\alpha + M_{\alpha\beta,\beta})n_\alpha + \frac{\partial(M_{\alpha\beta}t_\alpha n_\beta)}{\partial s} - \bar{P} = 0, \\ \\ M_{\alpha\beta}n_\alpha n_\beta - M_n = 0, \quad on \quad \Gamma_t, \end{cases}$$

with the set $\mathcal{A}^*$ defined by (9.21) and

$$\{\bar{N}_{\alpha\beta}\} = \{N_{\alpha\beta}\}^{-1}. \quad \square$$

9.5 The Second Duality Principle

The next result is a extension of Theorem 9.1.1 and, instead of calculating the polar functional related to the main part of primal formulation, it is determined its Legendre Transform and associated functional.

Theorem 9.5.1. *Consider Gâteaux differentiable functionals $G \circ \Lambda : U \to \bar{\mathbb{R}}$ and $F \circ \Lambda_1 : U \to \bar{\mathbb{R}}$ where only the second one is necessarily convex, through which is defined the functional $J_K : U \times Y \to \bar{\mathbb{R}}$ expressed as*

$$J_K(u,p) = G(\Lambda u + p) + K\langle p, p\rangle_{\boldsymbol{L}^2(S)} - F(\Lambda_1 u) - \frac{K\langle p, p\rangle_{\boldsymbol{L}^2(S)}}{2} - \langle u, u_0^*\rangle_U.$$

Suppose there exists $(u_0, p_0) \in U \times Y$ such that

$$J_K(u_0, p_0) = \inf_{(u,p)\in U\times Y}\{J_K(u,p)\}$$

and $\delta J_K(u_0, p_0) = \theta$. Here $\Lambda = \{\Lambda_i\} : U \to Y$ and $\Lambda_1 : U \to Y$ are continuous linear operators whose adjoint operators are denoted by $\Lambda^* : Y^* \to U^*$ and $\Lambda_1^* : Y^* \to U^*$ respectively.

Furthermore assume there exists a differentiable function denoted by $g : \mathbb{R}^n \to \mathbb{R}$ so that $G : Y \to \bar{\mathbb{R}}$ may be expressed as $G(v) = \int_\Omega g(v)dS, \quad \forall v \in Y$ where g admits differentiable Legendre transform denoted by $g_L^* : R_L^n \to \mathbb{R}$.

Under these assumptions we have

$$\inf_{(u,p)\in U\times Y}\{J_K(u,p)\} \leq \inf_{(z^*,v^*,\hat{u})\in E^*}\{J_K^*(z^*,v^*,\hat{u})\},$$

where

$$J_K^*(z^*,v^*,\hat{u}) = F^*(z^*) + (1/2K)\langle v^*, v^*\rangle_{\boldsymbol{L}^2(S)} - G_L^*(v^*) + K\sum_{i=1}^n \left\|\Lambda_i\hat{u} - \frac{\partial g_L^*(v^*)}{\partial y_i^*}\right\|^2_{L^2(S)},$$

and

$$E^* = \{(z^*, v^*, \hat{u}) \in Y^* \times Y_L^* \times U \quad | \quad -\Lambda_1^* z^* + \Lambda^* v^* - u_0^* = \theta\}.$$

Also, the functions z_0^*, v_0^*, *and* $\hat{u}_0$, *defined by*

$$z_0^* = \frac{\partial F(\Lambda_1 u_0)}{\partial v},$$

$$v_0^* = \frac{\partial G(\Lambda u_0 + p_0)}{\partial v},$$

and

$$\hat{u}_0 = u_0$$

are such that

$$-\Lambda_1^* z_0^* + \Lambda^* v_0^* - u_0^* = \theta,$$

and thus

$$J_K(u_0, p_0) \leq \inf_{(z^*, v^*, \hat{u}) \in E^*} \{J_K^*(z^*, v^*, \hat{u})\} \leq J_K(u_0, p_0) + 2K\langle p_0, p_0\rangle_{\mathbf{L}^2(S)} \tag{9.44}$$

where we are assuming that $v_0^* \in Y_L^*$.

Proof: Defining $\alpha = \inf_{(u,p) \in U \times Y}\{J_K(u,p)\}$, $G_1(u,p) = G(\Lambda u + p) + K\langle p, p\rangle_{\mathbf{L}^2(S)}$ and $G_2(u,p) = F(\Lambda_1 u) + (K/2)\langle p, p\rangle_{\mathbf{L}^2(S)} + \langle u, u_0^*\rangle_U$ we have:

$$G_1(u,p) \geq G_2(u,p) + \alpha, \forall (u,p) \in U \times Y.$$

Thus, $\forall \quad v^* \in Y_L^*$, we have

$$\sup_{(u,p) \in U \times Y} \{\langle v^*, \Lambda u + p\rangle_{\mathbf{L}^2(S)} - G_2(u,p)\} \geq \langle v^*, \Lambda u + p\rangle_{\mathbf{L}^2(S)} - G_1(u,p) + \alpha, \forall (u,p) \in U \times Y. \tag{9.45}$$

From Theorem 7.2.5:

$$\sup_{(u,p) \in U \times Y}\{\langle v^*, \Lambda u + p\rangle_{\mathbf{L}^2(S)} - G_2(u,p)\} = \inf_{z^* \in C^*(v^*)}\{F(z^*) + (1/2K)\langle v^*, v^*\rangle_{\mathbf{L}^2(\Omega)}\} \tag{9.46}$$

where

$$C^*(v^*) = \{z^* \in Y^* \quad | \quad -\Lambda_1^* z^* + \Lambda^* v^* - u_0^* = \theta\}.$$

Furthermore

$$\langle v^*, \Lambda u + p\rangle_{\mathbf{L}^2(S)} - G_1(u,p) = \langle v^*, \Lambda u + p\rangle_{\mathbf{L}^2(S)} - G(\Lambda u + p) - K\langle p, p\rangle_{\mathbf{L}^2(S)}.$$

Choosing $u = \hat{u}$ and p satisfying the equations

$$v_i^* = \frac{\partial G(\Lambda\hat{u} + p)}{\partial v_i},$$

from a well known Legendre Transform property, we obtain

$$p_i = \frac{\partial G_L(v^*)}{\partial v_i^*} - \Lambda_i \hat{u}$$

so that

$$\langle v^*, \Lambda u + p \rangle_{\mathbf{L}^2(S)} - G_1(u,p) = G_L^*(v^*) - K \sum_{i=1}^n \left\| \Lambda_i \hat{u} - \frac{\partial g_L^*(v^*)}{\partial y_i^*} \right\|^2_{L^2(S)}.$$

From last results and inequality (9.45) we have

$$\begin{aligned} \inf_{z^* \in C^*(v^*)} \{F^*(z^*) &+ (1/2K)\langle v^*, v^* \rangle_{\mathbf{L}^2(S)}\} - G_L^*(v^*) + K \sum_{i=1}^n \left\| \Lambda_i \hat{u} - \frac{\partial g_L^*(v^*)}{\partial y_i^*} \right\|^2_{L^2(S)} \\ &\geq \alpha \\ &= \inf_{(u,p) \in U \times Y} \{J_K(u,p)\} \qquad (9.47) \end{aligned}$$

that is,

$$\begin{aligned} F^*(z^*) &+ \frac{1}{2K}\langle v^*, v^* \rangle_{\mathbf{L}^2(S)} - G_L^*(v^*) + K \sum_{i=1}^n \left\| \Lambda_i \hat{u} - \frac{\partial g_L^*(v^*)}{\partial y_i^*} \right\|^2_{L^2(S)} \\ &\geq \alpha \\ &= \inf_{(u,p) \in U \times Y} \{J_K(u,p)\}, \quad if \quad z^* \in C^*(v^*). \qquad (9.48) \end{aligned}$$

Hence

$$\begin{aligned} \inf_{(z^*, v^*, \hat{u}) \in E^*} \{F^*(z^*) &+ (1/2K)\langle v^*, v^* \rangle_{\mathbf{L}^2(S)} - G_L^*(v^*) + K \sum_{i=1}^n \left\| \Lambda_i \hat{u} - \frac{\partial g_L^*(v^*)}{\partial y_i^*} \right\|^2_{L^2(S)}\} \\ &\geq \alpha \\ &= \inf_{(u,p) \in U \times Y} \{J_K(u,p)\} \qquad (9.49) \end{aligned}$$

so that:

$$\inf_{(z^*, v^*, \hat{u}) \in E^*} \{J_K^*(z^*, v^*, \hat{u})\} \geq \inf_{(u,p) \in U \times Y} \{J_K(u,p)\}$$

where $E^* = C^*(v^*) \times Y_L^* \times U$, and the remaining conclusions follow from the expressions of $J_K(u_0, p_0)$ and $J_K^*(z_0^*, v_0^*, \hat{u}_0)$. $\square$

Remark 9.5.2. *We conjecture that the duality gap between the primal and dual formulations, namely* $2K\langle p_0, p_0\rangle_{\mathbf{L}^2(S)}$, *goes to zero as* $K \to +\infty$, *since* $p_0 \in Y$ *satisfies the extremal condition:*

$$\frac{1}{K}\frac{\partial G(\Lambda u_0 + p_0)}{\partial v} + p_0 = 0,$$

and $J_K(u,p)$ *is bounded from below. We do not prove it in the present work.*

In the application of last theorem to the Kirchhoff-Love plate model, we would have $F(\Lambda_1 u) = \theta$, and therefore the variable z^* is not present in the dual formulation. Also,

$$\langle u, u_0^*\rangle_U = \int_S P w dS + \int_{\Gamma_t}\left(\bar{P}_\alpha u_\alpha + \bar{P} w - M_n \frac{\partial w}{\partial \mathbf{n}}\right) d\Gamma \tag{9.50}$$

and thus the relevant duality principle could be expressed as

$$\inf_{u\in U}\left\{G(\Lambda u + p) + K\langle p, p\rangle_{\mathbf{L}^2(S)} - \langle u, u_0^*\rangle_U - \frac{K}{2}\langle p,p\rangle_{\mathbf{L}^2(S)}\right\}$$

$$\leq$$

$$\begin{aligned}\inf_{(v^*,\hat{u})\in E^*}\Bigg\{&-\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu}dS - \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu}M_{\alpha\beta}M_{\lambda\mu}dS - \frac{1}{2}\int_S \bar{N}_{\alpha\beta}Q_\alpha Q_\beta dS + \\ &\frac{1}{2K}\int_S N_{\alpha\beta}N_{\alpha\beta}dS + \frac{1}{2K}\int_S M_{\alpha\beta}M_{\alpha\beta}dS \\ &+K\sum_{\alpha,\beta=1}^{2}\left\|\frac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha}) - \bar{H}_{\alpha\beta\lambda\mu}N_{\lambda\mu} + \frac{1}{2}v_{02^\alpha}v_{02^\beta}\right\|^2_{L^2(S)} \\ &+K\sum_{\alpha=1}^{2}\|w_{,\alpha} - v_{02^\alpha}\|^2_{L^2(S)} + K\sum_{\alpha,\beta=1}^{2}\|-w_{,\alpha\beta} - \bar{h}_{\alpha\beta\lambda\mu}M_{\lambda\mu}\|^2_{L^2(S)}\Bigg\}\end{aligned} \tag{9.51}$$

where $(v^*, \hat{u}) \in E^* = C^* \times U \Leftrightarrow (v^*, \hat{u}) \in Y_L^* \times U$ and,

$$\begin{cases} N_{\alpha\beta,\beta} = 0, \\ Q_{\alpha,\alpha} + M_{\alpha\beta,\alpha\beta} + P = 0, \quad a.e. \;\; in \;\; S \end{cases}$$

and

$$\begin{cases} N_{\alpha\beta}.n_\beta - \bar{P}_\alpha = 0, \\ \\ (Q_\alpha + M_{\alpha\beta,\beta})n_\alpha + \frac{\partial(M_{\alpha\beta}t_\alpha n_\beta)}{\partial s} - \bar{P} = 0, \\ \\ M_{\alpha\beta}n_\alpha n_\beta - M_n = 0, \quad on \;\; \Gamma_t, \end{cases}$$

where $\{v_{02^\alpha}\}$ is defined through the equations

$$Q_\alpha = N_{\alpha\beta}v_{02^\beta}$$

and,

$$\{\bar{N}_{\alpha\beta}\} = \{N_{\alpha\beta}\}^{-1}.$$

Finally, we recall that

$$Y_L^* = \{v^* \in Y^* \ \ | \ \ \Delta = N_{11}N_{22} - (N_{12})^2 \neq 0, \ \ a.e \ \ in \ \ S\}. \ \square$$

9.6 The Third Duality Principle

Now we establish the third result, which may be summarized by the following theorem:

Theorem 9.6.1. *Let U be a reflexive Banach space, $(G \circ \Lambda) : U \to \bar{\mathbb{R}}$ a convex Gâteaux differentiable functional and $(F \circ \Lambda_1) : U \to \bar{\mathbb{R}}$ convex, coercive and lower semi-continuous (l.s.c.) such that the functional*

$$J(u) = (G \circ \Lambda)(u) - F(\Lambda_1 u) - \langle u, p \rangle_U$$

is bounded from below , where $\Lambda : U \to Y$ and $\Lambda_1 : U \to Y$ are continuous linear operators.

Then we may write:

$$\inf_{z^* \in Y^*} \sup_{v^* \in B^*(z^*)} \{F^*(z^*) - G^*(v^*)\} \geq \inf_{u \in U}\{J(u)\}$$

where $B^*(z^*) = \{v^* \in Y^*$ such that $\Lambda^* v^* - \Lambda_1^* z^* - p = 0\}$

Proof: By hypothesis there exists $\alpha \in \mathbb{R}$ $(\alpha = \inf_{u \in U}\{J(u)\})$ so that $J(u) \geq \alpha$, $\forall u \in U$.

That is,

$$(G \circ \Lambda)(u) \geq F(\Lambda_1 u) + \langle u, p \rangle_U + \alpha, \forall u \in U.$$

The above inequality clearly implies that

$$\sup_{u \in U}\{\langle u, u^* \rangle_U - F(\Lambda_1 u) - \langle u, p \rangle_U\} \geq \sup_{u \in U}\{\langle u, u^* \rangle_U - (G \circ \Lambda)(u)\} + \alpha$$

$\forall u^* \in U^*$. Since F is convex, coercive and l.s.c., by Theorem 7.2.5 we may write

$$\sup_{u \in U}\{\langle u, u^* \rangle_U - F(\Lambda_1 u) - \langle u, p \rangle_U\} = \inf_{z^* \in A^*(u^*)}\{F^*(z^*)\},$$

where,

$$A^*(u^*) = \{z^* \in Y^* \ \ | \ \ \Lambda_1^* z^* + p = u^*\}.$$

Since G also satisfies the hypothesis of Theorem 7.2.5, we have

$$\sup_{u \in U}\{\langle u, u^* \rangle_U - (G \circ \Lambda)(u)\} = \inf_{v^* \in D^*(u^*)}\{G^*(v^*)\},$$

where
$$D^*(u^*) = \{v^* \in Y^* \quad | \quad \Lambda^* v^* = u^*\}.$$

Therefore we may summarize the last results as

$$F(z^*) + \sup_{v^* \in D^*(u^*)} \{-G^*(v^*)\} \geq \alpha, \quad \forall z^* \in A^*(u^*).$$

This inequality implies
$$F(z^*) + \sup_{v^* \in B^*(z^*)} \{-G^*(v^*)\} \geq \alpha,$$

so that we can write

$$\inf_{z^* \in Y^*} \sup_{v^* \in B^*(z^*)} \{F^*(z^*) - G^*(v^*)\} \geq \inf_{u \in U} \{J(u)\}$$

where $B^*(z^*) = \{v^* \in Y^* \quad | \quad \Lambda^* v^* - \Lambda_1^* z^* - p = 0\}$. $\square$

We will apply the last theorem to a changed functional concerning the primal formulation related to the Kirchhoff-Love plate model. We redefine $(G \circ \Lambda) : U \to \bar{\mathbb{R}}$ and $(F \circ \Lambda_1) : U \to \bar{\mathbb{R}}$ as

$$(G \circ \Lambda)(u) = \frac{1}{2} \int_S H_{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}(u) \gamma_{\lambda\mu}(u) dS + \frac{1}{2} \int_S h_{\alpha\beta\lambda\mu} \kappa_{\alpha\beta}(u) \kappa_{\lambda\mu}(u) dS + \frac{1}{2} K \int_S w_{,\alpha} \, w_{,\alpha} \, dS$$

if $N_{11}(u) + K > 0$, $N_{22}(u) + K > 0$ and $(N_{11}(u) + K)(N_{22}(u) + K) - N_{12}(u)^2 > 0$ and, $+\infty$ otherwise.

Remark 9.6.2. *Notice that $(G \circ \Lambda) : U \to \bar{\mathbb{R}}$ is convex and Gâteaux differentiable on its effective domain, which is sufficient for our purposes, since the concerned Fenchel conjugate may be easily expressed through the region of interest.*

Also, we define
$$F(\Lambda_1 u) = \frac{1}{2} K \int_S w_{,\alpha} \, w_{,\alpha} \, dS$$

$$\langle u, p \rangle_U = \int_S P w dS + \int_S P_\alpha u_\alpha dS$$

where
$$u = (u_\alpha, w) \in U = W_0^{1,2}(S) \times W_0^{1,2}(S) \times W_0^{2,2}(S).$$

These boundary conditions refer to a clamped plate. Furthermore,

$$\Lambda_1(u) = \{w_{,1} \, , w_{,2} \, \}$$

and

$$\Lambda = \{\Lambda_{1\alpha\beta}, \Lambda_{2\alpha}, \Lambda_{3\alpha\beta}\}$$

as indicated in (9.9), (9.10) and (9.11).

Calculating $G^* : Y^* \to \bar{\mathbb{R}}$ and $F^* : Y^* \to \bar{\mathbb{R}}$ we would obtain

$$\begin{aligned} G^*(v^*) = &\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \\ &+\frac{1}{2}\int_S N_{\alpha\beta} w_{,\alpha}\, w_{,\beta}\, dS + \frac{1}{2}K\int_S w_{,\alpha}\, w_{,\alpha}\, dS \end{aligned} \tag{9.52}$$

if $v^* \in E^*$. Here

$$v^* = \{N_{\alpha\beta}, M_{\alpha\beta}, w_{,\alpha}\},$$

$E^* = \{v^* \in Y^* \mid N_{11}+K > 0,\ N_{22}+K > 0 \ \textit{and}\ (N_{11}+K)(N_{22}+K) - N_{12}^2 > 0,\ \textit{a.e. in}\ S\}$

and,

$$F^*(z^*) = \frac{1}{2K}\int_S (z_1^*)^2 dS + \frac{1}{2K}\int_S (z_2^*)^2 dS.$$

Furthermore, $v^* \in B^*(z^*) \Leftrightarrow v^* \in Y^*$ and,

$$\begin{cases} N_{\alpha\beta,\beta} + P_\alpha = 0, \\ -(z_\alpha^*)_{,\alpha} + (N_{\alpha\beta} w_{,\beta})_{,\alpha} + M_{\alpha\beta,\alpha\beta} + K w_{,\alpha\alpha} + P = 0, \quad a.e.\ \ in\ \ S. \end{cases}$$

Finally, we can express the application of last theorem as:

$$\begin{aligned} \inf_{z^* \in Y^*} \sup_{v^* \in B^*(z^*) \bigcap E^*} &\{\frac{1}{2K}\int_S (z_1^*)^2 dS + \frac{1}{2K}\int_S (z_2^*)^2 dS - \\ &-\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS - \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \\ &-\frac{1}{2}\int_S N_{\alpha\beta} w_{,\alpha}\, w_{,\beta}\, dS - \frac{1}{2}K\int_S w_{,\alpha}\, w_{,\alpha}\, dS\} \\ &\geq \inf_{u \in U}\{J(u)\} \quad \square \end{aligned} \tag{9.53}$$

The above inequality can in fact represents an equality if the positive real constant K is chosen so that the point of local extremum $v_0^* = \frac{\partial G(\Lambda u_0)}{\partial v} \in E^*$ (which means $N_{11}(u_0)+K > 0$, $N_{22}(u_0)+K > 0$, and $(N_{11}(u_0)+K)(N_{22}(u_0)+K) - N_{12}(u_0)^2 > 0$). The mentioned equality is a result of a little change concerning Theorem 7.1.27.

Remark 9.6.3. *For the determination of $G^*(v^*)$ in (9.52) we have used the transformation*

$$Q_\alpha = N_{\alpha\beta} w_{,\beta} + K w_{,\alpha},$$

similarly as indicated in Remark 9.3.1.

9.7 A Convex Dual Formulation

Remark 9.7.1. *In this section we assume*

$$H_{\alpha\beta\lambda\mu} = h\{\frac{4\lambda_0\mu_0}{\lambda_0 + 2\mu_0}\delta_{\alpha\beta}\delta_{\lambda\mu} + 2\mu_0(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\lambda})\},$$

and

$$h_{\alpha\beta\lambda\mu} = \frac{h^2 H_{\alpha\beta\lambda\mu}}{12},$$

where $\delta_{\alpha\beta}$ denotes the Kronecker delta and λ_0, μ_0 are appropriate constants.

The next result may be summarized by the following Theorem:

Theorem 9.7.2. *Consider the functionals $(G \circ \Lambda) : U \to \bar{\mathbb{R}}$, $(F \circ \Lambda_1) : U \to \bar{\mathbb{R}}$ and $\langle u, p\rangle_U$ defined as*

$$(G \circ \Lambda)(u) = \frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u)dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(u)\kappa_{\lambda\mu}(u)dS + \frac{1}{2}K\int_S w_{,\alpha}\, w_{,\alpha}\, dS,$$

$$F(\Lambda_1 u) = \frac{1}{2}K\int_S w_{,\alpha}\, w_{,\alpha}\, dS,$$

and

$$\langle u, p\rangle_U = \int_S P w dS + \int_S P_\alpha u_\alpha dS,$$

where

$$u = (u_\alpha, w) \in U = W_0^{1,2}(S) \times W_0^{1,2}(S) \times W_0^{2,2}(S).$$

The operators $\{\gamma_{\alpha\beta}\}$ and $\{\kappa_{\alpha\beta}\}$ are defined in (9.7) and (9.8), respectively. Furthermore, we define $J(u) = (G \circ \Lambda)(u) - F(\Lambda_1 u) - \langle u, p\rangle_U$, and

$$\Lambda_1(u) = \{w_{,1}\, , w_{,2}\, \}.$$

Suppose there exists $u_0 \in U$ such that $\delta J(u_0) = 0$, and that there exists $K > 0$ for which $N_{11}(u_0) + K > 0$, $N_{22}(u_0) + K > 0$, $(N_{11}(u_0) + K)(N_{22}(u_0) + K) - N_{12}(u_0)^2 > 0$ (a.e in S) and $h_{1212}/(2K_0) > K$ where K_0 is the constant related to Poincaré Inequality and,

$$N_{\alpha\beta}(u_0) = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u_0).$$

Then,

$$\begin{aligned} J(u_0) = \min_{u\in U}\{J(u)\} &= \max_{(v^*,z^*,w)\in E^*\cap B^*}\{-G^*(v^*) + \langle z^*_\alpha, z^*_\alpha\rangle_{L^2(S)}/(2K)\} \\ &= -G^*(v^*_0) + \langle z^*_{0\alpha}, z^*_{0\alpha}\rangle_{L^2(S)}/(2K) \end{aligned} \quad (9.54)$$

where,

$$v_0^* = \frac{\partial G(\Lambda u_0)}{\partial v} \quad and \quad z_{0^\alpha}^* = K w_{0,\alpha},$$

$$G^*(v^*) = G_L^*(v^*) = \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu} M_{\alpha\beta} M_{\lambda\mu} dS + \frac{1}{2}\int_S \bar{N}^K_{\alpha\beta} Q_{,\alpha} Q_{,\beta} dS$$

if $v^* \in E^*$*, where* $v^* = \{\{N_{\alpha\beta}\}, \{M_{\alpha\beta}\}, \{Q_\alpha\}\} \in E^* \Leftrightarrow v^* \in L^2(S, \mathbb{R}^{10})$ *and*

$$N_{11} + K > 0 \quad N_{22} + K > 0 \text{ and } (N_{11} + K)(N_{22} + K) - N_{12}^2 > 0, \quad a.e. \text{ in } S$$

and,

$$(v^*, z^*) \in B^* \Leftrightarrow \begin{cases} N_{\alpha\beta,\beta} + P_\alpha = 0, \\ Q_{\alpha,\alpha} + M_{\alpha\beta,\alpha\beta} - z^*_{\alpha,\alpha} + P = 0, \\ \bar{h}_{1212} M_{12} + z^*_{1,2}/K = 0, \\ z^*_{1,2} = z^*_{2,1}, \quad a.e. \text{ in } S, \quad and, \quad z^* = \theta \text{ on } \Gamma, \end{cases}$$

being $\{\bar{N}^K_{\alpha\beta}\}$ *as indicated in (9.6).*

Proof: Similarly to Proposition 9.4.1, we may obtain the following result. If $v^* \in E^*$ then

$$G_L^*(v^*) = G^*(v^*) \geq \langle v^*, \Lambda u\rangle_Y - G(\Lambda u), \quad \forall u \in U,$$

so that

$$G_L^*(v^*) - \frac{1}{2K}\langle z^*_\alpha, z^*_\alpha\rangle_{L^2(S)} \geq \langle v^*, \Lambda u\rangle_Y - \frac{1}{2K}\langle z^*_\alpha, z^*_\alpha\rangle_{L^2(S)} - G(\Lambda u), \quad \forall u \in U,$$

and thus, as $\Lambda^* v^* - \Lambda_1^* z^* - p = 0$ (see the definition of B^*) we obtain

$$Q_{\alpha,\alpha} + M_{\alpha\beta,\alpha\beta} - z^*_{\alpha,\alpha} + P = 0 \text{ a.e. in } S.$$

Through this equation we may symbolically write

$$M_{12} = \Lambda_{3^{12}}^{-1}\{(-Q_{\alpha,\alpha} + z^*_{\alpha,\alpha} - \bar{M}_{\alpha\beta,\alpha\beta} - P)/2\}, \tag{9.55}$$

where $\bar{M}_{\alpha\beta,\alpha\beta}$ denotes $M_{11,11} + M_{22,22}$, in S, so that substituting such a relation in the last inequality we have

$$\begin{aligned} \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} dS &+ \frac{1}{2}\int_S \bar{h}_{1111} M_{11}^2 dS + \int_S \bar{h}_{1122} M_{11} M_{22} dS + \frac{1}{2}\int_S \bar{h}_{2222} M_{22}^2 dS \\ &+ 2\int_S \bar{h}_{1212}(\Lambda_{3^{12}}^{-1}(v^*, z^*))^2 dS + \frac{1}{2}\int_S \bar{N}^K_{\alpha\beta} Q_{,\alpha} Q_{,\beta} dS \\ &- \frac{1}{2K}\langle z^*_\alpha, z^*_\alpha\rangle_{L^2(S)} \\ &\geq \langle \Lambda_1 u, z^*\rangle_{L^2(S;\mathbb{R}^2)} - \frac{1}{2K}\langle z^*_\alpha, z^*_\alpha\rangle_{L^2(S)} - G(\Lambda u) \\ &+ \langle u, p\rangle_U, \quad \forall u \in U, \end{aligned} \tag{9.56}$$

where M_{12} is made explicit through equation (9.55). This equation makes z^* an independent variable, so that evaluating the supremum concerning z^*, particularly for the left side of above inequality, the global extremum is achieved through the equation :

$$-([\Lambda_{3_{12}}^{-1}]^*[\bar{h}_{1212}\Lambda_{3_{12}}^{-1}(v^*,z^*)])_{,\alpha} - z^*_\alpha/K = 0, \quad a.e. \ in \ \ S.$$

This means

$$-\bar{h}_{1212}\Lambda_{3_{12}}^{-1}(v^*,z^*) - z^*_{\alpha,\beta}/K = 0, \quad a.e. \ \ in \ S \ \ and \ \ z_1^* = z_2^* = 0 \ \ on \ \ \Gamma$$

or

$$\bar{h}_{1212}M_{12} + z^*_{\alpha,\beta}/K = 0, \ a.e. \ in \ S \ \ and \ \ z_1^* = z_2^* = 0 \ \ on \ \ \Gamma$$

for $(\alpha,\beta) = (1,2)$ and $(2,1)$. Therefore, after evaluating the suprema in both sides of (9.56), we may write

$$G_L^*(v^*) - \frac{1}{2K}\langle z^*_\alpha, z^*_\alpha\rangle_{L^2(S)} \geq F(\Lambda_1 u) - G(\Lambda u) + \langle u,p\rangle_U, \quad \forall u \in U, \quad and \quad (v^*,z^*) \in B^* \cap E^*.$$

and it seems to be clear that the condition $h_{1212}/(2K_0) > K$ guarantees coercivity for the expression of left side in the last inequality (see the next remark), so that the unique local extremum concerning z^* is also a global extremum. The equality and remaining conclusions results from the Gâteaux differentiability of primal and dual formulations and an application (with little changes) of Theorem 7.1.27. □

Remark 9.7.3. *Observe that the dual functional could be expressed as*

$$\begin{aligned} G_L^*(v^*) - \frac{1}{2K}\langle z^*_\alpha, z^*_\alpha\rangle_{L^2(S)} \;=\; & \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}N_{\lambda\mu}dS + \frac{1}{2}\int_S \bar{h}_{1111}M_{11}^2dS + \int_S \bar{h}_{1122}M_{11}M_{22}dS \\ & + \frac{1}{2}\int_S \bar{h}_{2222}M_{22}^2dS + \frac{1}{2}\int_S \bar{N}^K_{\alpha\beta}Q_{,\alpha}Q_{,\beta}dS + \int_S h_{1212}(z^*_{1,2})^2/K^2dS \\ & + \int_S h_{1212}(z^*_{2,1})^2/K^2dS - \frac{1}{2K}\langle z^*_\alpha, z^*_\alpha\rangle_{L^2(S)}. \end{aligned} \tag{9.57}$$

Thus, through the relation $h_{1212}/(2K_0) > K$ (where K_0 is the constant related to Poincaré inequality), it is now clear that the dual formulation is convex on $E^* \cap B^*$.

9.8 A Final Result, Other Sufficient Conditions of Optimality

This final result is developed similarly to the triality criterion introduced in Gao [20], which describes, in some situations, sufficient conditions for optimality.

We prove the following result

Theorem 9.8.1. *Consider $J : U \to \mathbb{R}$ where $J(u) = G(\Lambda u) + F(u)$,*

$$G(\Lambda u) = \frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u)dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}w_{,\alpha\beta}w_{,\lambda\mu}dS.$$

Here the operators $\gamma_{\alpha\beta}$ are defined as in (9.7),

$$F(u) = -\int_S PwdS \equiv -\langle u, f\rangle_U,$$

and,

$$U = W_0^{1,2}(S) \times W_0^{1,2}(S) \times W_0^{2,2}(S).$$

Then, if $u_0 \in U$ is such that $\delta J(u_0) = \theta$ and

$$\frac{1}{2}\int_S N_{\alpha\beta}(u_0)w_{,\alpha}w_{,\beta}dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}w_{,\alpha\beta}w_{,\lambda\mu}dS \geq 0, \forall w \in W_0^{2,2}(S), \tag{9.58}$$

we have

$$J(u_0) = \min_{u\in U}\{J(u)\}.$$

Proof: It is clear that

$$G(\Lambda u) + F(u) \geq -(G\circ\Lambda)^*(u^*) - F^*(-u^*), \forall u \in U, \quad u^* \in U^*,$$

so that

$$G(\Lambda u) + F(u) \geq -(G\circ\Lambda)^*(\Lambda^* v^*) - F^*(-\Lambda^* v^*), \forall u \in U, \quad v^* \in Y^*. \tag{9.59}$$

Consider u_0 for which $\delta J(u_0) = \theta$ and such that (9.58) is satisfied.

Defining

$$v_0^* = \frac{\partial G(\Lambda u_0)}{\partial v},$$

from Theorem 7.1.27 we have that

$$\delta(-G_L^*(v_0^*) + \langle u_0, \Lambda v_0^* - f\rangle_U) = \theta,$$

$$J(u_0) = -G_L(v_0^*),$$

and

$$\Lambda^* v_0^* = f.$$

This means

$$F^*(-\Lambda^* v_0^*) = 0.$$

On the other hand

$$(G\circ\Lambda)^*(\Lambda^* v_0^*) = \sup_{u\in U}\{\langle \Lambda u, v_0^*\rangle_Y - G(\Lambda u)\},$$

or

$$\begin{aligned}(G\circ\Lambda)^*(\Lambda v_0^*) &= \sup_{u\in U}\Big\{\langle\frac{u_{\alpha,\beta}+u_{\beta,\alpha}}{2},N_{\alpha\beta}(u_0)\rangle_{L^2(S)}+\langle -w_{,\alpha\beta},M_{\alpha\beta}(u_0)\rangle_{L^2(S)}\\ &\quad+\langle w_{,\alpha}Q_\alpha(u_0)\rangle_{L^2(S)}-\frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u)dS\\ &\quad-\frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}w_{,\alpha\beta}w_{,\lambda\mu}dS\}. \end{aligned} \tag{9.60}$$

Since

$$\gamma_{\alpha\beta}(u)=\frac{u_{\alpha,\beta}+u_{\beta,\alpha}}{2}+\frac{1}{2}w_{,\alpha}w_{,\beta},$$

from the last equality, we may write

$$\begin{aligned}(G\circ\Lambda)^*(\Lambda^* v_0^*) &= \sup_{u\in U}\Big\{\langle\gamma_{\alpha\beta}(u),N_{\alpha\beta}(u_0)\rangle_{L^2(S)}-\langle\frac{w_{,\alpha}w_{,\beta}}{2},N_{\alpha\beta}(u_0)\rangle_{L^2(S)}\\ &\quad+\langle -w_{,\alpha\beta},M_{\alpha\beta}(u_0)\rangle_{L^2(S)}+\langle w_{,\alpha},Q_\alpha(u_0)\rangle_{L^2(S)}\\ &\quad-\frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u)dS-\frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}w_{,\alpha\beta}w_{,\lambda\mu}dS\Big\}. \end{aligned} \tag{9.61}$$

As $(Q_\alpha(u_0))_{,\alpha}+(M_{\alpha\beta}(u_0))_{,\alpha\beta}+P=0$, we obtain

$$\begin{aligned}(G\circ\Lambda)^*(\Lambda^* v_0^*) &\leq \sup_{u\in U}\Big\{-\Big\langle\frac{w_{,\alpha}w_{,\beta}}{2},N_{\alpha\beta}(u_0)\Big\rangle_{L^2(S)}-\frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}w_{,\alpha\beta}w_{,\lambda\mu}dS\\ &\quad+\int_S PwdS\Big\}+\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}(u_0)N_{\lambda\mu}(u_0)dS. \end{aligned} \tag{9.62}$$

Therefore, from hypothesis (9.58) the extremum indicated in (9.62) is attained for functions satisfying

$$(N_{\alpha\beta}(u_0)\hat{w}_\beta)_{,\alpha}-(h_{\alpha\beta\lambda\mu}\hat{w}_{\lambda\mu})_{,\alpha\beta}+P=0. \tag{9.63}$$

From $\delta J(u_0)=\theta$ and boundary conditions we obtain

$$\hat{w}=w_0,\ a.e.\ in\ S,$$

so that

$$\begin{aligned}(G\circ\Lambda)^*(\Lambda^* v_0^*) \leq \langle\frac{w_{0,\alpha}w_{0,\beta}}{2},N_{\alpha\beta}(u_0)\rangle_{L^2(S)}+\frac{1}{2}\int_S h_{\alpha\beta\lambda\mu}w_{0,\alpha\beta}w_{0,\lambda\mu}dS\\ +\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}(u_0)N_{\lambda\mu}(u_0)dS. \end{aligned} \tag{9.64}$$

However, since

$$Q_\alpha(u_0)=N_{\alpha\beta}(u_0)w_{0,\beta},$$

and

$$M_{\alpha\beta}(u_0)=-h_{\alpha\beta\lambda\mu}w_{0,\lambda\mu}$$

from (9.64) we obtain

$$\begin{aligned}(G \circ \Lambda)^*(\Lambda^* v_0^*) \;\leq\; & \frac{1}{2}\int \bar{N}_{\alpha\beta}(u_0)Q_\alpha(u_0)Q_\beta(u_0)dS + \frac{1}{2}\int_S \bar{h}_{\alpha\beta\lambda\mu}M_{\alpha\beta}(u_0)M_{\lambda\mu}(u_0)dS \\ & + \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}N_{\alpha\beta}(u_0)N_{\lambda\mu}(u_0)dS. \end{aligned} \tag{9.65}$$

Hence

$$(G \circ \Lambda)^*(\Lambda^* v_0^*) \leq G_L(v_0^*) = -J(u_0),$$

and thus as $F^*(-\Lambda^* v_0^*) = 0$, we have that

$$J(u_0) \leq -(G \circ \Lambda)^*(\Lambda^* v_0^*) - F^*(-\Lambda^* v_0^*),$$

which, from (9.59) completes the proof. $\square$

9.9 Final Remarks

In this chapter we presented four different dual variational formulations for the Kirchhoff-Love plate model. Earlier results (see references [33],[18]) present a constraint concerning the gap functional to establish the complementary energy (dual formulation). In the present work the dual formulations are established on the hypothesis of existence of a global extremum for the primal functional and the results are applicable even for compressed plates. In particular the second duality principle is obtained through an extension of a theorem met in [35], and in this case we are concerned with the solution behavior as $K \to +\infty$, even though a rigorous and complete analysis of such behavior has been postponed for a future work. However, what seems to be interesting is that the dual formulation as indicated in (9.51) is represented by a natural extension of the results found in [35] (particularly Theorem 9.1.1), plus a kind of penalization concerning the inversion of constitutive equations.

It is worth noting that the third dual formulation was based on the same theorem, despite the fact such a result had not been directly used, we followed a similar idea to prove the duality principle. For this last result, the membrane forces are allowed to be negative since it is observed the restriction $N_{11} + K > 0$, $N_{22} + K > 0$ and $(N_{11} + K)(N_{22} + K) - N_{12}^2 > 0$, $a.e.\ \ in\ \ S$, where $K \in \mathbb{R}$ is a positive suitable constant.

In section 9.7, we obtained a convex dual variational formulation for the plate model, which allows non positive definite membrane force matrices. In this formulation, the Poincaré inequality plays a fundamental role.

Finally, in the last section, we developed a result similar to Gao's triality criterion presented in [20]. In the plate application this gives sufficient conditions for optimality. We present a new proof of sufficient conditions of existence of a global extremum for the primal problem.

As earlier mentioned, such conditions may be summarized by $\delta J(u_0) = \theta$ and

$$\frac{1}{2}\int_S N_{\alpha\beta}(u_0) w_{,\alpha} w_{,\beta} dS + \frac{1}{2}\int_S h_{\alpha\beta\lambda\mu} w_{,\alpha\beta} w_{,\lambda\mu} dS \geq 0, \forall w \in W_0^{2,2}(S). \quad \square$$

Chapter 10

Duality Applied to Elasticity

10.1 Introduction and Primal Formulation

Our first objective in the present chapter is to establish a dual variational formulation for a finite elasticity model. Even though existence of solutions for this model has been proven in Ciarlet [10], the concept of complementary energy, as a global optimization approach, is possible to be defined only if the stress tensor is positive definite at a critical point. Thus we have the goal of relaxing such constraints and start by describing the primal formulation.

Consider $S \subset \mathbb{R}^3$ an open, bounded, connected set, which represents the reference volume occupied by an elastic solid under the load $f \in L^2(S; \mathbb{R}^3)$. We denote by Γ the boundary of S. The field of displacements under the action of f is denoted by $u \equiv (u_1, u_2, u_3) \in U$, where u_1, u_2, and u_3 denotes the displacements related to directions x, y, and z respectively, on the cartesian basis (x, y, z). Here U is defined as

$$U = \{u = (u_1, u_2, u_3) \in W^{1,4}(S; \mathbb{R}^3) \mid u = (0,0,0) \equiv \theta \ \ on \ \ \Gamma\} \tag{10.1}$$

Denoting the stress tensor by $\{\sigma_{ij}\}$, where

$$\sigma_{ij} = H_{ijkl}\left(\frac{1}{2}(u_{k,l} + u_{k,l} + u_{m,k}u_{m,l})\right) \tag{10.2}$$

and $\{H_{ijkl}\}$ is a positive definite matrix related to the coefficients of Hooke's Law, the boundary value form of the finite elasticity model is given by

$$\begin{cases} \sigma_{ij,j} + (\sigma_{im}u_{m,j})_{,j} + f_i = 0, \quad a.e. \ \ in \ \ S, \\ \\ u = \theta, \quad on \ \ \Gamma. \end{cases} \tag{10.3}$$

The corresponding primal variational formulation is represented by $J : U \to \mathbb{R}$, where

$$J(u) = \frac{1}{2}\int_S H_{ijkl}\left(\frac{1}{2}(u_{i,j} + u_{i,j} + u_{m,i}u_{m,j})\right)\left(\frac{1}{2}(u_{k,l} + u_{k,l} + u_{m,k}u_{m,l})\right)dx - \langle u, f\rangle_{L^2(S;\mathbb{R}^3)} \tag{10.4}$$

10.2 The Duality Principles

In this section we establish the duality principles. We start with the following theorem.

Theorem 10.2.1. *Define $J : U \to \mathbb{R}$ as*

$$J(u) = G^{**}(\Lambda u) - F_1(u) \tag{10.5}$$

where

$$\begin{aligned} G(\Lambda u) &= \frac{1}{2}\int_S H_{ijkl}\left(\frac{1}{2}(u_{i,j} + u_{i,j} + u_{m,i}u_{m,j})\right)\left(\frac{1}{2}(u_{k,l} + u_{k,l} + u_{m,k}u_{m,l})\right)dx \\ &+ \frac{K}{2}\langle u_{m,i}, u_{m,i}\rangle_{L^2(S)}, \end{aligned} \tag{10.6}$$

$$F_1(u) = F(u) - \langle u, f\rangle_{L^2(S;\mathbb{R}^3)}, \tag{10.7}$$

and

$$F(u) = \frac{K}{2}\langle u_{m,i}, u_{m,i}\rangle_{L^2(S)}. \tag{10.8}$$

Here $\Lambda : U \to Y = Y_1 \times Y_2 \equiv L^2(S;\mathbb{R}^9) \times L^2(S,\mathbb{R}^9)$ is given by

$$\Lambda u = \{\Lambda_1 u, \Lambda_2 u\} \equiv \{\{\frac{1}{2}(u_{i,j} + u_{i,j})\}, \{u_{m,i}\}\}. \tag{10.9}$$

Thus we can write

$$\inf_{u\in U}\{J(u)\} \leq \inf_{z^*\in Y_2^*}\ \sup_{v^*\in C^*(z^*)}\{F^*(z^*) - G^*(v^*)\}, \tag{10.10}$$

where

$$v^* \equiv \{\sigma, Q\}, \tag{10.11}$$

$$C^*(z^*) = \{(\sigma, Q) \in Y^* \mid \sigma_{ij,j} + Q_{ij,j} - z^*_{ij,j} + f_i = 0, \quad a.e. \ \ in \ \ S\}, \tag{10.12}$$

$$F^*(z^*) = \frac{1}{2K}\langle z^*_{im}, z^*_{im}\rangle_{L^2(S)} \tag{10.13}$$

and

$$G^*(v^*) = G^*_L(v^*) = \frac{1}{2}\int_S \bar{H}_{ijkl}\sigma_{ij}\sigma_{kl}dx + \frac{1}{2}\int_S \bar{\sigma}_{ij}Q_{mi}Q_{mj}dx \tag{10.14}$$

if σ_K is positive definite in S, where

$$\sigma_K = \left\{ \begin{array}{lll} \sigma_{11} + K & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} + K & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} + K \end{array} \right\} \tag{10.15}$$

and also $\{\bar{\sigma}_{ij}\} = {\sigma_K}^{-1}$. Finally, $\{\bar{H}_{ijkl}\} = \{H_{ijkl}\}^{-1}$.

Proof: Defining $\alpha \equiv \inf_{u \in U}\{J(u)\}$ we have

$$G^{**}(\Lambda u) - F_1(u) \geq \alpha, \quad \forall u \in U, \tag{10.16}$$

or

$$-F_1(u) \geq -G^{**}(\Lambda u) + \alpha, \quad \forall u \in U, \tag{10.17}$$

so that

$$\sup_{u \in U}\{\langle u, u^* \rangle_U - F_1(u)\} \geq \sup_{u \in U}\{\langle u, u^* \rangle_U - G(\Lambda u)\} + \alpha. \tag{10.18}$$

However, from Theorem 7.2.5

$$F_1^*(u^*) = \sup_{u \in U}\{\langle u, u^* \rangle_U - F_1(u)\} = \inf_{z^* \in C^*(u^*)}\{F^*(z^*)\} \tag{10.19}$$

where

$$C^*(u^*) = \{z^* \in Y_2^* \ \mid \ z^*_{ij,j} - u_i^* - f_i = 0, \ \ a.e. \ \ in \ \ S\}. \tag{10.20}$$

On the other hand, also from Theorem 7.2.5

$$(G^{**} \circ \Lambda)^*(u^*) = \sup_{u \in U}\{\langle u, u^* \rangle_U - G^{**}(\Lambda u)\} = \inf_{v^* \in D^*(u^*)}\{G^*(v^*)\} \tag{10.21}$$

where

$$D^*(u^*) = \{v^* \in Y^* \ \mid \ \sigma_{ij,j} + Q_{ij,j} - u_i^* = 0, \ \ a.e. \ \ in \ \ S\}. \tag{10.22}$$

We can summarize the last results by

$$\inf_{u \in U}\{J(u)\} = \alpha \leq F_1^*(u^*) - (G^{**} \circ \Lambda)^*(u^*) \leq F^*(z^*) + \sup_{v^* \in D^*(u^*)}\{-G^*(v^*)\}, \tag{10.23}$$

$\forall z^* \in C^*(u^*)$.

Hence we can write

$$\inf_{u \in U}\{J(u)\} \leq \inf_{z^* \in Y_1^*} \sup_{v^* \in D^*(z^*)}\{F^*(z^*) - G^*(v^*)\} \tag{10.24}$$

where

$$D^*(z^*) = \{v^* \in Y^* \ \mid \ \sigma_{ij,j} + Q_{ij,j} - z^*_{ij,j} + f_i = 0, \ \ a.e. \ \ in \ \ S\}. \tag{10.25}$$

Finally, we have to prove that $G_L^*(v^*) = G^*(v^*)$ if σ_K is positive definite in S. We start by formally calculating $g_L^*(y^*)$, the Legendre transform of $g(y)$, where

$$g(y) = H_{ijkl}\left(y_{1^{ij}} + \frac{1}{2}y_{2^{mi}}y_{2^{mj}}\right)\left(y_{1^{kl}} + \frac{1}{2}y_{2^{mk}}y_{2^{ml}}\right) + \frac{K}{2}y_{2^{mi}}y_{2^{mi}}. \tag{10.26}$$

We recall that

$$g_L^*(y^*) = \langle y, y^*\rangle_{\mathbb{R}^{18}} - g(y) \tag{10.27}$$

where $y \in \mathbb{R}^{18}$ is solution of equation

$$y^* = \frac{\partial g(y)}{\partial y}. \tag{10.28}$$

Thus

$$y^*_{1^{ij}} = \sigma_{ij} = H_{ijkl}\left(y_{1^{kl}} + \frac{1}{2}y_{2^{mk}}y_{2^{ml}}\right) \tag{10.29}$$

and

$$y^*_{2^{mi}} = Q_{mi} = H_{ijkl}\left(y_{1^{kl}} + \frac{1}{2}y_{2^{ok}}y_{2^{ol}}\right)y_{2^{mj}} + Ky_{2^{mi}} \tag{10.30}$$

so that

$$Q_{mi} = \sigma_{ij}y_{2^{mj}} + Ky_{2^{mi}}. \tag{10.31}$$

Inverting these last equations, we have

$$y_{2^{mi}} = \bar{\sigma}_{ij}Q_{mj} \tag{10.32}$$

where

$$\{\bar{\sigma}_{ij}\} = \sigma_K^{-1} = \left\{\begin{array}{lll} \sigma_{11}+K & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}+K & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}+K \end{array}\right\}^{-1} \tag{10.33}$$

and also

$$y_{1^{ij}} = \bar{H}_{ijkl}\sigma_{kl} - \frac{1}{2}y_{2^{mi}}y_{2^{mj}}. \tag{10.34}$$

Finally

$$g_L^*(\sigma, Q) = \frac{1}{2}\bar{H}_{ijkl}\sigma_{ij}\sigma_{kl} + \frac{1}{2}\bar{\sigma}_{ij}Q_{mi}Q_{mj}. \tag{10.35}$$

Now we will prove that $g_L^*(v^*) = g^*(v^*)$ if σ_K is positive definite. First observe that

$$\begin{aligned} g^*(v^*) &= \sup_{y\in\mathbb{R}^{18}}\left\{\langle y_1, \sigma\rangle_{\mathbb{R}^9} + \langle y_2, Q\rangle_{\mathbb{R}^9} - \frac{1}{2}H_{ijkl}\left(y_{1^{ij}} + \frac{1}{2}y_{2^{mi}}y_{2^{mj}}\right)\left(y_{1^{kl}} + \frac{1}{2}y_{2^{mk}}y_{2^{ml}}\right)\right. \\ &\quad \left. -\frac{K}{2}y_{2^{mi}}y_{2^{mi}}\right\} \\ &= \sup_{(\bar{y}_1, y_2)\in\mathbb{R}^9\times\mathbb{R}^9}\left\{\langle \bar{y}_{1^{ij}} - \frac{1}{2}y_{2^{mi}}y_{2^{mj}}, \sigma_{ij}\rangle_{\mathbb{R}} + \langle y_2, Q\rangle_{\mathbb{R}^9} - \frac{1}{2}H_{ijkl}[\bar{y}_{1^{ij}}][\bar{y}_{1^{kl}}] - \frac{K}{2}y_{2^{mi}}y_{2^{mi}}\right\}. \end{aligned}$$

The result follows just observing that

$$\sup_{\bar{y}_1 \in \mathbb{R}^9} \left\{ \langle \bar{y}_{1^{ij}}, \sigma_{ij} \rangle_{\mathbb{R}} - \frac{1}{2} H_{ijkl} [\bar{y}_{1^{ij}}][\bar{y}_{1^{kl}}] \right\} = \frac{1}{2} \bar{H}_{ijkl} \sigma_{ij} \sigma_{kl} \tag{10.36}$$

and

$$\sup_{y_2 \in \mathbb{R}^9} \left\{ \langle -\frac{1}{2} y_{2^{mi}} y_{2^{mj}}, \sigma_{ij} \rangle_{\mathbb{R}} + \langle y_2, Q \rangle_{\mathbb{R}^9} - \frac{K}{2} y_{2^{mi}} y_{2^{mi}} \right\} = \frac{1}{2} \bar{\sigma}_{ij} Q_{mi} Q_{mj} \tag{10.37}$$

if σ_K is positive definite. $\square$

Through the next theorem we obtain a convex primal dual formulation for the finite elasticity model.

Theorem 10.2.2. *The solution of the boundary value problem indicated in (10.3) minimizes the functional $\hat{J} : U \times Y \to \mathbb{R}$ under the constraint*

$$\sigma_{ij,j} + Q_{ij,j} - K u_{i,jj} + f_i = 0, \quad a.e. \quad in \quad S, \tag{10.38}$$

where

$$\hat{J}(u, v^*) = G^*(v^*) - \langle \Lambda u, v^* \rangle_Y + G^{**}(\Lambda u), \tag{10.39}$$

$$\begin{aligned} G(\Lambda u) \quad = \quad & \frac{1}{2} \int_S H_{ijkl} \left(\frac{1}{2}(u_{i,j} + u_{i,j}) + \frac{1}{2} u_{m,i} u_{m,j} \right) \left(\frac{1}{2}(u_{k,l} + u_{k,l}) + \frac{1}{2} u_{m,k} u_{m,l} \right) dx \\ & + \frac{K}{2} \langle u_{m,i}, u_{m,i} \rangle_{L^2(S)} \end{aligned} \tag{10.40}$$

and

$$G^*(v^*) = G^*_L(v^*) = \frac{1}{2} \int_S \bar{H}_{ijkl} \sigma_{ij} \sigma_{kl} dx + \frac{1}{2} \int_S \bar{\sigma}_{ij} Q_{mi} Q_{mj} dx \tag{10.41}$$

if σ_K is positive definite in S. Here

$$\sigma_K = \left\{ \begin{array}{lll} \sigma_{11} + K & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} + K & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} + K \end{array} \right\} \tag{10.42}$$

and also $\{\bar{\sigma}_{ij}\} = {\sigma_K}^{-1}$.

Proof: Consider $u_0 \in U$ a solution for the boundary value problem. If $K > 0$ is big enough, defining

$$v_0^* = \frac{\partial G(\Lambda u_0)}{\partial v},$$

we have that (u_0, v_0^*) minimizes $\hat{J}$ and satisfies (10.38). $\square$

10.3 Final Results, Sufficient Conditions of Optimality

Our final result also establishes a criterium of optimality.

Theorem 10.3.1. *Consider the functionals* $(G \circ \Lambda) : U \to \mathbb{R}$ *and* $(F \circ \Lambda_1) : U \to \mathbb{R}$ *given by*

$$G(\Lambda u) = \frac{1}{2} \int_S H_{ijkl} \left(\frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i} u_{m,j}) \right) \left(\frac{1}{2}(u_{k,l} + u_{l,k} + u_{m,k} u_{m,l}) \right) dx,$$

$$F(\Lambda u) = \frac{K}{2} \int_S \Lambda_{ij}(u) \Lambda_{ij}(u) dx.$$

Here

$$U = W_0^{1,2}(S; \mathbb{R}^3),$$

and $\Lambda : U \to Y = Y^* = L^4(S; \mathbb{R}^9)$ *is expressed as*

$$\Lambda u = \{\Lambda_{ij}(u)\} = \left\{ \frac{1}{2}(u_{i,j} + u_{j,i} + u_{m,i} u_{m,j}) \right\}.$$

Define

$$J(u) = G(\Lambda u) - \langle u, f \rangle_U$$

where $f \in L^2(S; \mathbb{R}^3)$ *is such that that* J *is bounded below.*

Then,

$$\inf_{u \in U} \{J(u)\} \geq \sup_{\sigma \in Y^*} \left\{ \inf_{\hat{\sigma} \in Y^*} \left\{ \tilde{J}(\sigma, \hat{\sigma}) \right\} \right\}, \tag{10.43}$$

where

$$\tilde{J}(\sigma, \hat{\sigma}) = \tilde{F}(\hat{\sigma}) - G^*(\sigma + \hat{\sigma}) - \tilde{F}_f(-\sigma),$$

$$\tilde{F}(\hat{\sigma}) = \frac{1}{2K} \int_S \hat{\sigma}_{ij} \hat{\sigma}_{ij} dx,$$

$$G^*(\sigma + \hat{\sigma}) = \frac{1}{2} \int_S \bar{H}_{ijkl} (\sigma_{ij} + \hat{\sigma}_{ij})(\sigma_{kl} + \hat{\sigma}_{kl}) dx,$$

$$\tilde{F}_f(\hat{\sigma}) = \sup_{u \in U} \{ \langle \Lambda_{ij}(u), \sigma_{ij} \rangle_{L^2(S)} - F(\Lambda u) + \langle u, f \rangle_U \}.$$

Furthermore, if there exists $(\sigma_0, \hat{\sigma}_0) \in Y^* \times Y^*$ *such that* $\delta \tilde{J}(\sigma_0, \hat{\sigma}_0) = \theta$, *and* $K > 0$ *is such that*

$$\hat{J}(\hat{\sigma}) = \tilde{F}(\hat{\sigma}) - G^*(\sigma_0 + \hat{\sigma})$$

is coercive, so that the infimum indicated in right side of (10.43) is attained at $\hat{\sigma}_0$, *we have*

$$\tilde{J}(\sigma_0, \hat{\sigma}_0) = \max_{\sigma \in Y^*} \left\{ \min_{\hat{\sigma} \in Y^*} \left\{ \tilde{J}(\sigma, \hat{\sigma}) \right\} \right\}.$$

Finally, if there exists a corresponding $u_0 \in U$ *such that* $\delta J(u_0) = \theta$,

$$\sigma_0 + \hat{\sigma}_0 = \frac{\partial G(\Lambda u_0)}{\partial v}$$

and

$$\tilde{F}_f(-\sigma_0) = \{\langle \Lambda_{ij}(u_0), \sigma_{0^{ij}} \rangle_{L^2(S)} - F(\Lambda u_0) + \langle u_0, f \rangle_U\},$$

it is also such that

$$J(u_0) = \min_{u \in U} \{J(u)\}.$$

Proof: First observe that

$$\begin{aligned} G^*(\sigma + \hat{\sigma}) + \tilde{F}_f(-\sigma) \;\; &\geq \;\; \langle \Lambda_{ij}(u), \sigma_{ij} \rangle_{L^2(S)} + \langle \Lambda_{ij}(u), \hat{\sigma}_{ij} \rangle_{L^2(S)} \\ &\quad - \langle \Lambda_{ij}(u), \sigma_{ij} \rangle_{L^2(S)} - F(\Lambda u) + \langle u, f \rangle_U - G(\Lambda u), \quad (10.44) \end{aligned}$$

or

$$\begin{aligned} -\tilde{F}(\hat{\sigma}) + G^*(\sigma + \hat{\sigma}) + \tilde{F}_f(-\sigma) \;\; &\geq \;\; -\tilde{F}(\hat{\sigma}) + \langle \Lambda_{ij}(u), \hat{\sigma}_{ij} \rangle_{L^2(S)} \\ &\quad - F(\Lambda u) + \langle u, f \rangle_U - G(\Lambda u). \quad (10.45) \end{aligned}$$

Thus, taking the supremum in $\hat{\sigma}$ in both sides of last inequality, we obtain

$$\begin{aligned} \sup_{\hat{\sigma} \in Y^*} \{-\tilde{F}(\hat{\sigma}) + G^*(\sigma + \hat{\sigma}) + \tilde{F}_f(-\sigma)\} \;\; &\geq \;\; \sup_{\hat{\sigma} \in Y^*} \{-\tilde{F}(\hat{\sigma}) + \langle \Lambda_{ij}(u), \hat{\sigma}_{ij} \rangle_{L^2(S)} \\ &\quad - F(\Lambda u) + \langle u, f \rangle_U - G(\Lambda u)\}. \quad (10.46) \end{aligned}$$

That is,

$$\sup_{\hat{\sigma} \in Y^*} \{-\tilde{F}(\hat{\sigma}) + G^*(\sigma + \hat{\sigma}) + \tilde{F}_f(-\sigma)\} \geq \langle u, f \rangle_U - G(\Lambda u), \forall u \in U.$$

so that

$$\inf_{u \in U} \{G(\Lambda u) - \langle u, f \rangle_U\} \geq \sup_{\sigma \in Y^*} \left\{ \inf_{\hat{\sigma} \in Y^*} \left\{ \tilde{F}(\hat{\sigma}) - G^*(\sigma + \hat{\sigma}) - \tilde{F}_f(-\sigma) \right\} \right\}.$$

Finally, an application of a little change of Theorem 7.1.27, now for the non-linear operator case, completes the proof (we recall that such a theorem establishes a equality between the primal and dual formulations at a critical point). $\square$

Chapter 11

Duality Applied to a Membrane Shell Model

11.1 Introduction and Primal Formulation

In this chapter we establish dual variational formulations for the elastic membrane shell model presented in [12] (P.Ciarlet). Ciarlet proves existence of solutions, however, the complementary energy as developed in [33](J.J.Telega), is possible only for a special class of external loads, that generate a critical point with positive definite membrane tensor. In this chapter we relax such constraints, and in fact our final result is a primal dual formulation which is convex. Now we describe the primal formulation.

Consider a domain $S \subset \mathbb{R}^2$ and a injective mapping $\vec{\theta} : S \times [-\varepsilon, \varepsilon] \to \mathbb{R}^3$ such that $S_0 = \vec{\theta}(\bar{S})$ denotes the middle surface of a shell of thickness 2ε.

The mapping $\vec{\theta}$ may be expressed as:

$$\vec{\theta}(x) = \vec{\theta}_1(y) + x_3 \boldsymbol{a}_3(y)$$

where $\boldsymbol{a}_3 = (\boldsymbol{a}_1 \times \boldsymbol{a}_2)/\|\boldsymbol{a}_1 \times \boldsymbol{a}_2\|$ and $\boldsymbol{a}_\alpha = \partial_\alpha \vec{\theta}$, $y = (y_1, y_2)$ denote the curvilinear coordinates, $x = (y, x_3)$ and $-\varepsilon \leq x_3 \leq \varepsilon$.

The contravariant basis of the tangent plane to S in y, denoted by $\{\boldsymbol{a}^\alpha\}$ is defined through the relations

$$\boldsymbol{a}^\alpha \cdot \boldsymbol{a}_\beta = \delta_\alpha^\beta, \quad and \quad \boldsymbol{a}^3 = \boldsymbol{a}_3$$

The covariant components of the metric tensor, denoted by $\{a_{\alpha\beta}\}$, are defined as

$$a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$$

and the concerned contravariant components are denoted by $\{a^{\alpha\beta}\}$ and expressed as

$$a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$$

It is not difficult to show that $\boldsymbol{a}_\alpha = a_{\alpha\beta}\boldsymbol{a}^\beta$, $\{a^{\alpha\beta}\} = \{a_{\alpha\beta}\}^{-1}$ and $\boldsymbol{a}^\alpha = a^{\alpha\beta}\boldsymbol{a}_\beta$. We also define $\sqrt{a(y)} = \|\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)\|$.

The curvature tensor denoted by $\{b_{\alpha\beta}\}$ is expressed as

$$b_{\alpha\beta} = \boldsymbol{a}_3 \cdot \partial_\alpha \boldsymbol{a}^\beta.$$

Concerning the displacements due to external loads action, we denote them by $\eta = \eta_i \boldsymbol{a}^i$, and the admissible displacements field is denoted by U, where

$$U = \{\eta \in W^{1,4}(S;\mathbb{R}^3) \ \mid \ \eta = (0,0,0) \ \ on \ \ \Gamma_0\}$$

and here $\Gamma = \Gamma_0 \cup \Gamma_1$ ($\Gamma_0 \cap \Gamma_1 = \emptyset$) denotes the boundary of S.

We now state the Theorem 9-1-1 of reference [12] (Mathematical Elasticity , Vol. III -Theory of shells), by P.Ciarlet.

Theorem 11.1.1. *Let $S \subset \mathbb{R}^2$ a domain and $\vec{\theta} \in C^2(\bar{S};\mathbb{R}^3)$ be an injective mapping such that the two vectors $\boldsymbol{a}_\alpha = \partial_\alpha \vec{\theta}$ are linearly independent in all points of $\bar{S}$, let $\boldsymbol{a}_3 = (\boldsymbol{a}_1 \times \boldsymbol{a}_2)/\|\boldsymbol{a}_1 \times \boldsymbol{a}_2\|$, and let the vectors $\boldsymbol{a}^i$ be defined by $\boldsymbol{a}^i.\boldsymbol{a}_j = \delta^i_j$. Given a displacement field $\eta_i \boldsymbol{a}^i$ of the surface $S_0 = \vec{\theta}(\bar{S})$, let the covariant components of the change of metric tensor associated with this displacement field be defined by*

$$G_{\alpha\beta}(\eta) = (a_{\alpha\beta}(\eta) - a_{\alpha\beta}),$$

where $a_{\alpha\beta}$ and $a_{\alpha\beta}(\eta)$ denote the covariant components of the metric tensors of the surfaces $\vec{\theta}(\bar{S})$ and $(\vec{\theta} + \eta_i \boldsymbol{a}^i)(\bar{S})$ respectively. Then

$$G_{\alpha\beta}(\eta) = (\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}\eta_{m\|\alpha}\eta_{n\|\beta})/2$$

where $\eta_{\alpha\|\beta} = \partial_\beta \eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3$ and $\eta_{3\|\beta} = \partial_\beta \eta_3 + b^\sigma_\beta \eta_\sigma$. $\square$

We define the constitutive relations as

$$N^{\alpha\beta} = a_\varepsilon^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\eta) \tag{11.1}$$

where $\{N^{\alpha\beta}\}$ denotes the membrane forces, and

$$a_\varepsilon^{\alpha\beta\sigma\tau} = \frac{\varepsilon}{4}\left(\frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})\right) \tag{11.2}$$

The potential energy (stored energy plus external work) is expressed by the functional $J : U \to \mathbb{R}$ where,

$$J(\eta) = \frac{1}{2}\int_S a_\varepsilon^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\eta) G_{\alpha\beta}(\eta)\sqrt{a}\, dy - \int_S p^i \eta_i \sqrt{a}\, dy$$

where $p^i = \int_{-\varepsilon}^{\varepsilon} f^i \, dx_3$, here $\{f^i\} \in L^2(S \times [-\varepsilon, \varepsilon]; \mathbb{R}^3)$ denotes the external load density. We will define $(G \circ \Lambda) : U \to \mathbb{R}$ and $F : U \to \mathbb{R}$ as

$$(G \circ \Lambda)(\eta) = \frac{1}{2} \int_S a_\varepsilon^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\eta) G_{\alpha\beta}(\eta) \sqrt{a} \, dy$$

and,

$$F(\eta) = \int_S p^i \eta_i \sqrt{a} \, dy, \tag{11.3}$$

where $\{\Lambda\} = \{\Lambda_{1\alpha\beta}, \Lambda_{2^{m\alpha}}\}$, $\Lambda_{1\alpha\beta}(\eta) = (\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha})/2$ and $\Lambda_{2^{m\alpha}}(\eta) = \eta_{m\|\alpha}$. Thus, the primal variational formulation is given by $J : U \to \mathbb{R}$, where

$$J(\eta) = G(\Lambda\eta) - F(\eta). \tag{11.4}$$

11.2 The Legendre Transform

We will be concerned with the Legendre Transform related to the function $g : \mathbb{R}^{10} \to \mathbb{R}$ expressed as

$$g(y) = \frac{1}{2} a_\varepsilon^{\alpha\beta\sigma\tau} (y_{1^{\alpha\beta}} + \frac{1}{2} a^{mn} y_{2^{m\alpha}} y_{2^{n\beta}})(y_{1^{\sigma\tau}} + \frac{1}{2} a^{kl} y_{2^{k\sigma}} y_{2^{l\tau}}) \tag{11.5}$$

Its Legendre Transform, denoted by $g_L^* : R_L^{10} \to \mathbb{R}$ is given by

$$g_L^*(y^*) = \langle y, y^* \rangle_{\mathbb{R}^{10}} - g(y) \tag{11.6}$$

where $y \in \mathbb{R}^{10}$ is solution of the system

$$y^* = \frac{\partial g(y)}{\partial y}. \tag{11.7}$$

That is,

$$y^*_{1^{\alpha\beta}} = a_\varepsilon^{\alpha\beta\sigma\tau} (y_{1^{\sigma\tau}} + \frac{1}{2} a^{kl} y_{2^{k\sigma}} y_{2^{l\tau}}) \tag{11.8}$$

and

$$y^*_{2^{m\alpha}} = a_\varepsilon^{\alpha\beta\sigma\tau} (y_{1^{\sigma\tau}} + \frac{1}{2} a^{kl} y_{2^{k\sigma}} y_{2^{l\tau}}) a^{mn} y_{2^{n\beta}} \tag{11.9}$$

or

$$y^*_{2^{m\alpha}} = y^*_{1^{\alpha\beta}} a^{mn} y^*_{2^{n\beta}}. \tag{11.10}$$

Therefore, after simple algebraic manipulations we would obtain

$$g_L^*(y^*) = \frac{1}{2} \bar{a}_{\alpha\beta\sigma\tau} y^*_{1^{\sigma\tau}} y^*_{1^{\alpha\beta}} + \frac{1}{2} R_{\alpha\beta mn} y^*_{2^{\alpha m}} y^*_{2^{n\beta}}, \tag{11.11}$$

where

$$R_{\alpha\beta mn} = \{y^*_{1\alpha\beta}a^{mn}\}^{-1}. \tag{11.12}$$

Thus, denoting $v^* = \{N^{\alpha\beta}, Q^{m\alpha}\}$ we have

$$G^*_L(v^*) = \int_S g^*_L(v^*)\sqrt{a}\, dy = \int_S \bar{a}_{\alpha\beta\sigma\tau} N^{\sigma\tau} N^{\alpha\beta}\sqrt{a}\, dy + \frac{1}{2}\int_w R_{\alpha\beta mn} Q^{m\alpha} Q^{n\beta}\sqrt{a}\, dy. \tag{11.13}$$

We now obtain the polar functional related to the external load.

11.3 The Polar Functional Related to $F : U \to \mathbb{R}$

The polar functional related to $F : U \to \mathbb{R}$, for $v^* = \{N^{\alpha\beta}, Q^{m\alpha}\}$ is expressed by

$$F^*(\Lambda^* v^*) = \sup_{\eta\in U}\{\langle \eta, \Lambda^* v^*\rangle_U - F(u)\}. \tag{11.14}$$

That is,

$$F^*(\Lambda^* v^*) = \sup_{\eta\in U}\{\langle \Lambda_{1\alpha\beta}\eta, N^{\alpha\beta}\sqrt{a}\rangle_{L^2(S)} + \langle \Lambda_{2m\alpha}\eta, Q^{m\alpha}\sqrt{a}\rangle_{L^2(S)} - \int_S p^i \eta_i \sqrt{a}\, dy\}. \tag{11.15}$$

Thus

$$F^*(\Lambda^* v^*) = \begin{cases} 0, & if \quad v^* \in C^*, \\ +\infty, & otherwise, \end{cases} \tag{11.16}$$

where

$$v^* \in C^* \Leftrightarrow \begin{cases} -(N^{\alpha\beta} + Q^{\alpha\beta})|_\beta + b^\alpha_\beta Q^{3\beta} = p^\alpha \quad in \quad S, \\ \\ -b_{\alpha\beta}(N^{\alpha\beta} + Q^{\alpha\beta}) - Q^{3\beta}|_\beta = p^3 \quad in \quad S, \\ \\ (N^{\alpha\beta} + Q^{\alpha\beta})\nu_\beta = 0 \quad on \quad \Gamma_1, \\ \\ Q^{3\beta}|_\beta \nu_\beta = 0 \quad on \quad \Gamma_1. \end{cases} \tag{11.17}$$

11.4 The Final Format of First Duality Principle

The duality principle presented in Theorem 9.5.1 is applied to the present case. Thus we obtain

$$\inf_{\eta\in U}\{J(\eta)\} \leq \inf_{v^*\in C^*}\{\frac{1}{2K}\langle v^*, v^*\rangle_{L^2(S,\mathbb{R}^{10})} - G^*_L(v^*) + K\left\|\Lambda\hat{u} - \frac{\partial g^*_L(v^*)}{\partial y^*}\right\|^2_{L^2(S)}.$$

More explicitly

$$\inf_{\eta \in U}\left\{\frac{1}{2}\int_S a_\varepsilon^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\eta) G_{\alpha\beta}(\eta)\sqrt{a}\,dy - \int_S p^i \eta_i \sqrt{a}\,dy\right\}$$

$$\leq$$

$$\inf_{v^* \in C^*}\left\{\frac{1}{2K}\left(\int_S N^{\alpha\beta}N^{\alpha\beta}\sqrt{a}\,dy + \int_S Q^{m\alpha}Q^{m\alpha}\sqrt{a}\,dy\right)\right.$$

$$-\frac{1}{2}\int_S \bar{a}_{\alpha\beta\sigma\tau}N^{\sigma\tau}N^{\alpha\beta}\sqrt{a}\,dy - \frac{1}{2}\int_S R_{\alpha\beta mn}Q^{m\alpha}Q^{n\beta}\sqrt{a}\,dy$$

$$\left. + \sum_{\alpha,\beta=1}^{2}\sum_{m=1}^{3}\left(K\left\|\eta_{\alpha\|\beta} - \bar{a}_{\alpha\beta\sigma\tau}N^{\sigma\tau} - a^{kl}v_{02^{k\alpha}}v_{02^{l\beta}}\right\|_{L^2(S)} + K\|\eta_{m\|\alpha} - v_{02^{m\alpha}}\|_{L^2(S)}\right)\right\}$$

11.5 The Second Duality Principle

Our objective now is to establish a convex primal-dual formulation. First we will write the primal formulation as the difference of two convex functionals and then will obtain the second duality principle, as indicated in the next theorem.

Theorem 11.5.1. *Define $J : U \to \mathbb{R}$ as*

$$J(\eta) = \hat{G}(\Lambda\eta) - F(\Lambda_2\eta) \tag{11.18}$$

where

$$\hat{G}(\Lambda\eta) = G^{**}(\Lambda\eta) - \langle \eta, p\rangle, \tag{11.19}$$

$$G(\Lambda\eta) = \frac{1}{2}\int_S a_\varepsilon^{\alpha\beta\sigma\tau}G_{\sigma\tau}(\eta)G_{\alpha\beta}(\eta)\sqrt{a}\,dy + \frac{K}{2}\int_S \eta_{m\|\alpha}\eta_{m\|\alpha}\sqrt{a}\,dy, \tag{11.20}$$

$$F(\Lambda_2\eta) = \frac{K}{2}\int_S \eta_{m\|\alpha}\eta_{m\|\alpha}\sqrt{a}\,dy, \tag{11.21}$$

$$\langle \eta, p\rangle = \int_S p^i\eta_i\sqrt{a}\,dy, \tag{11.22}$$

and $\Lambda : U \to Y_1^* \times Y_2^* \equiv L^2(S;\mathbb{R}^4) \times L^2(S;\mathbb{R}^6)$ is defined as

$$\Lambda\eta = \{\Lambda_1\eta, \Lambda_2\eta\} \equiv \left\{\left\{\frac{1}{2}(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha})\right\}, \{\eta_{m\|\alpha}\}\right\}. \tag{11.23}$$

Thus, we may write

$$\inf_{\eta\in U}\{J(\eta)\} \leq \inf_{u^*\in U^*}\sup_{v^*\in C^*(u^*)}\{F^*(u^*) - G^*(v^*)\} \tag{11.24}$$

where

$$v^* \equiv \{N, Q\}, \tag{11.25}$$

$$F^*(u^*) = \frac{1}{2K}\int_S u^*_{mi} u^*_{mi} \sqrt{a}\; dy, \tag{11.26}$$

$$G^*(v^*) = G^*_L(v^*) = \int_S g^*_L(v^*)\sqrt{a} dy = \frac{1}{2}\int_S \bar{a}_{\alpha\beta\sigma\tau} N^{\sigma\tau} N^{\alpha\beta}\sqrt{a} dy + \frac{1}{2}\int_S R_{\alpha\beta mn} Q^{m\alpha} Q^{n\beta}\sqrt{a}\; dy \tag{11.27}$$

if $N_K = \{N^{\alpha\beta} a^{mn} + K\delta_{\alpha\beta}\delta^{mn}\}$ *is positive definite in* S*. Furthermore*

$$\{R_{\alpha\beta mn}\} = N_K^{-1} \tag{11.28}$$

and

$$\{\bar{a}_{\alpha\beta\sigma\tau}\} = \{a_\varepsilon^{\alpha\beta\sigma\tau}\}^{-1}. \tag{11.29}$$

Finally,

$$v^* \equiv \{N, Q\} \in C^*(u^*) \Leftrightarrow \begin{cases} -(N^{\alpha\beta} + Q^{\alpha\beta} - u^*_{\alpha\beta})|_\beta + b^\alpha_\beta(Q^{3\beta} - u^*_{3\beta}) = p^\alpha \quad in \quad S, \\ -b_{\alpha\beta}(N^{\alpha\beta} + Q^{\alpha\beta} - u^*_{\alpha\beta}) - (Q^{3\beta} - u^*_{3\beta})|_\beta = p^3 \quad in \quad S, \\ (N^{\alpha\beta} + (Q^{\alpha\beta} - u^*_{\alpha\beta}))\nu_\beta = 0 \quad on \quad \Gamma_1, \\ (Q^{3\beta} - u^*_{3\beta})|_\beta \nu_\beta = 0 \quad on \quad \Gamma_1. \end{cases} \tag{11.30}$$

Proof: Defining $\alpha \equiv \inf_{\eta\in U}\{J(\eta)\}$ we have

$$\hat{G}(\Lambda\eta) - F(\Lambda_2\eta) \geq \alpha, \quad \forall \eta \in U, \tag{11.31}$$

or

$$-F(\Lambda_2\eta) \geq -\hat{G}(\Lambda\eta) + \alpha, \quad \forall \eta \in U, \tag{11.32}$$

so that

$$\sup_{\eta\in U}\{\langle \Lambda_2\eta, u^*\rangle - F(\eta)\} \geq \sup_{\eta\in U}\{\langle \Lambda_2\eta, u^*\rangle - \hat{G}(\Lambda\eta)\} + \alpha. \tag{11.33}$$

However,

$$F^*(u^*) \geq \sup_{\eta\in U}\{\langle \Lambda_2\eta, u^*\rangle - F(\Lambda_2\eta)\}. \tag{11.34}$$

On the other hand, from Theorem 7.2.5

$$(\hat{G}\circ\Lambda)^*(\Lambda_2^* u^*) = \sup_{\eta\in U}\{\langle \Lambda_2\eta, u^*\rangle - \hat{G}(\Lambda\eta)\} = \inf_{v^*\in C^*(u^*)}\{G^*(v^*)\} \tag{11.35}$$

where

$$v^* \equiv \{N, Q\} \in C^*(u^*) \Leftrightarrow \begin{cases} -(N^{\alpha\beta} + Q^{\alpha\beta} - u^*_{\alpha\beta})|_\beta + b^\alpha_\beta(Q^{3\beta} - u^*_{3\beta}) = p^\alpha \quad in \quad S, \\ -b_{\alpha\beta}(N^{\alpha\beta} + Q^{\alpha\beta} - u^*_{\alpha\beta}) - (Q^{3\beta} - u^*_{3\beta})|_\beta = p^3 \quad in \quad S, \\ (N^{\alpha\beta} + (Q^{\alpha\beta} - u^*_{\alpha\beta}))\nu_\beta = 0 \quad on \quad \Gamma_1, \\ (Q^{3\beta} - u^*_{3\beta})|_\beta \nu_\beta = 0 \quad on \quad \Gamma_1. \end{cases} \tag{11.36}$$

We can summarize the last results by

$$\inf_{\eta \in U}\{J(\eta)\} = \alpha \leq F^*(u^*) - (\hat{G} \circ \Lambda)^*(\Lambda_2^* u^*) = F^*(u^*) + \sup_{v^* \in C^*(u^*)}\{-G^*(v^*)\}, \tag{11.37}$$

$\forall u^* \in U^*$.

Finally, we have to prove that $G^*_L(v^*) = G^*(v^*)$ if N_K is positive definite in S. We start by formally calculating $g^*_L(y^*)$, the Legendre transform of $g(y)$, where $g : \mathbb{R}^{10} \rightarrow \mathbb{R}$ is expressed as:

$$g(y) = \frac{1}{2}a_\varepsilon^{\alpha\beta\sigma\tau}(y_{1^{\alpha\beta}} + \frac{1}{2}a^{mn}y_{2^{m\alpha}}y_{2^{n\beta}})(y_{1^{\sigma\tau}} + \frac{1}{2}a^{kl}y_{2^{k\sigma}}y_{2^{l\tau}}) + \frac{K}{2}y_{2^{m\alpha}}y_{2^{m\alpha}}. \tag{11.38}$$

Observe that $g^*_L : R^{10}_L \rightarrow \mathbb{R}$ is given by

$$g^*_L(y^*) = \langle y, y^* \rangle_{\mathbb{R}^{10}} - g(y) \tag{11.39}$$

where $y \in \mathbb{R}^{10}$ is solution of the system

$$y^* = \frac{\partial g(y)}{\partial y}. \tag{11.40}$$

That is,

$$y^*_{1^{\alpha\beta}} = a_\varepsilon^{\alpha\beta\sigma\tau}(y_{1^{\sigma\tau}} + \frac{1}{2}a^{kl}y_{2^{k\sigma}}y_{2^{l\tau}}) \tag{11.41}$$

and,

$$y^*_{2^{m\alpha}} = a_\varepsilon^{\alpha\beta\sigma\tau}(y_{1^{\sigma\tau}} + \frac{1}{2}a^{kl}y_{2^{k\sigma}}y_{2^{l\tau}})a^{mn}y_{2^{n\beta}} + Ky_{2^{m\alpha}} \tag{11.42}$$

or

$$y^*_{2^{m\alpha}} = y^*_{1^{\alpha\beta}}a^{mn}y^*_{2^{n\beta}} + Ky_{2^{m\alpha}}. \tag{11.43}$$

Therefore, after simple algebraic manipulations we would obtain

$$g^*_L(y^*) = \frac{1}{2}\bar{a}_{\alpha\beta\sigma\tau}y^*_{1^{\sigma\tau}}y_{1^{\alpha\beta}} + \frac{1}{2}R_{\alpha\beta mn}y^*_{2^{\alpha m}}y^*_{2^{n\beta}} \tag{11.44}$$

where

$$R_{\alpha\beta mn} = \{y^*_{1^{\alpha\beta}}a^{mn} + K\delta_{\alpha\beta}\delta^{mn}\}^{-1}. \tag{11.45}$$

Thus, through the notation $v^* = \{N^{\alpha\beta}, Q^{m\alpha}\}$ we would obtain

$$G_L^*(v^*) = \int_w g_L^*(v^*)\sqrt{a}\,dy = \frac{1}{2}\int_S \bar{a}_{\alpha\beta\sigma\tau} N^{\sigma\tau}N^{\alpha\beta}\sqrt{a}\,dy + \frac{1}{2}\int_S R_{\alpha\beta mn}Q^{m\alpha}Q^{n\beta}\sqrt{a}\,dy \quad (11.46)$$

where

$$R_{\alpha\beta mn} = \{N^{\alpha\beta}a^{mn} + K\delta_{\alpha\beta}\delta^{mn}\}^{-1}. \quad (11.47)$$

Now we will prove that $g_L^*(v^*) = g^*(v^*)$ if N_K is positive definite. First observe that

$$g^*(v^*) = \sup_{y\in\mathbb{R}^{10}} \{\langle y_1, N\rangle_{\mathbb{R}^4} + \langle y_2, Q\rangle_{\mathbb{R}^6}$$

$$-\frac{1}{2}a_\varepsilon^{\alpha\beta\sigma\tau}[y_{1\alpha\beta} + \frac{1}{2}a^{mn}y_{2m\alpha}y_{2n\beta}][y_{1\sigma\tau} + \frac{1}{2}a^{kl}y_{2k\sigma}y_{2l\tau}] - \frac{K}{2}y_{2m\alpha}y_{2m\alpha}\} =$$

$$\sup_{(\bar{y}_1,y_2)\in\mathbb{R}^4\times\mathbb{R}^6} \{\langle \bar{y}_{1\alpha\beta} - \frac{1}{2}a^{mn}y_{2m\alpha}y_{2m\beta}, N^{\alpha\beta}\rangle_{\mathbb{R}} + \langle y_2, Q\rangle_{\mathbb{R}^6} - \frac{1}{2}a_\varepsilon^{\alpha\beta\sigma\tau}[\bar{y}_{1\alpha\beta}][\bar{y}_{1\sigma\tau}] - \frac{K}{2}y_{2m\alpha}y_{2m\alpha}\}. \quad (11.48)$$

The result follows just observing that

$$\sup_{\bar{y}_1\in\mathbb{R}^4} \{\langle \bar{y}_{1\alpha\beta}, N^{\alpha\beta}\rangle_{\mathbb{R}} - \frac{1}{2}a_\varepsilon^{\alpha\beta\sigma\tau}[\bar{y}_{1\alpha\beta}][\bar{y}_{1\sigma\tau}] = \frac{1}{2}\bar{a}_{\alpha\beta\sigma\tau}N^{\sigma\tau}N^{\alpha\beta} \quad (11.49)$$

and

$$\sup_{y_2\in\mathbb{R}^6} \{\langle -\frac{1}{2}a^{mn}y_{2m\alpha}y_{2m\beta}, N^{\alpha\beta}\rangle_{\mathbb{R}} + \langle y_2, Q\rangle_{\mathbb{R}^6} - \frac{K}{2}y_{2m\alpha}y_{2m\alpha}\} = \frac{1}{2}R_{\alpha\beta mn}Q^{m\alpha}Q^{n\beta} \quad (11.50)$$

if N_K is positive definite. $\square$

11.6 The Convex Primal Dual Formulation

The next theorem is concerned with the convex primal-dual formulation.

Theorem 11.6.1. *The solution of the boundary value problem of shell model described above minimizes the functional $\hat{J}^* : U \times Y^* \to \mathbb{R}$ under the constraint $(\eta, v^*) \in A^*$, where*

$$\hat{J}^*(\eta, v^*) = G^*(v^*) - \langle \Lambda\eta, v^*\rangle + G^{**}(\Lambda\eta) \quad (11.51)$$

where

$$G(\Lambda\eta) = \frac{1}{2}\int_S a_\varepsilon^{\alpha\beta\sigma\tau}G_{\sigma\tau}(\vec{\eta})G_{\alpha\beta}(\vec{\eta})\sqrt{a}\,dy + \frac{K}{2}\int_S \eta_{m\|\alpha}\eta_{m\|\alpha}\sqrt{a}\,dy, \quad (11.52)$$

$$G^*(v^*) = G_L^*(v^*) = \int_S g_L^*(v^*)\sqrt{a}\,dy = \frac{1}{2}\int_S \bar{a}_{\alpha\beta\sigma\tau}N^{\sigma\tau}N^{\alpha\beta}\sqrt{a}\,dy + \frac{1}{2}\int_S R_{\alpha\beta mn}Q^{m\alpha}Q^{n\beta}\sqrt{a}\,dy \quad (11.53)$$

if $N_K = \{N^{\alpha\beta}a^{mn} + K\delta_{\alpha\beta}\delta^{mn}\}$ *is positive definite in* S. *Furthermore*

$$\{R_{\alpha\beta mn}\} = N_K^{-1}, \tag{11.54}$$

$$\{\bar{a}_{\alpha\beta\sigma\tau}\} = \{a_\varepsilon^{\alpha\beta\sigma\tau}\}^{-1} \tag{11.55}$$

and

$$\langle \Lambda\eta, v^*\rangle = \int_S \frac{1}{2}(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha})N^{\alpha\beta}\sqrt{a}\,dy + \int_S \eta_{m\|\alpha}Q^{m\alpha}\sqrt{a}\,dy. \tag{11.56}$$

Finally,

$$v^* \equiv \{N, Q\} \in A^* \Leftrightarrow \begin{cases} -(N^{\alpha\beta} + Q^{\alpha\beta} - K\eta_{\alpha\|\beta})|_\beta + b^\alpha_\beta(Q^{3\beta} - K\eta_{3\|\beta}) = p^\alpha \quad in \quad S, \\ -b_{\alpha\beta}(N^{\alpha\beta} + Q^{\alpha\beta} - K\eta_{\alpha\|\beta}) - (Q^{3\beta} - K\eta_{3\|\beta})|_\beta = p^3 \quad in \quad S, \\ (N^{\alpha\beta} + (Q^{\alpha\beta} - K\eta_{\alpha\|\beta}))\nu_\beta = 0 \quad on \quad \Gamma_1, \\ (Q^{3\beta} - K\eta_{3\|\beta})|_\beta\nu_\beta = 0 \quad on \quad \Gamma_1. \end{cases} \tag{11.57}$$

Proof: For the solution $\eta \in U$ of the boundary value problem, if K is big enough, we can define

$$v^* = \frac{\partial G(\Lambda\eta)}{\partial v}, \tag{11.58}$$

so that (η, v^*) minimizes $\hat{J}^*$ and satisfies A^*. $\square$

11.7 Conclusion

We obtained three different dual variational formulations for the non-linear elastic shell model studied in P.Ciarlet [12] (a membrane shell model).The first duality principle presented is an extension of a theorem found in Toland [35]. The solution behavior as $K \to +\infty$ is of particular interest for a future work.

The second duality principle relaxes the condition of positive definite membrane tensor, and thus the constant K must be chosen so that the matrix N_K is positive definite at the equilibrium point, where $N_K = \{N^{\alpha\beta}a^{mn} + K\delta_{\alpha\beta}\delta^{mn}\}$.

Finally, we also obtain a convex primal dual variational formulation. In fact it was our long term objective to obtain a convex variational formulation for such a class of non-convex problems.

Chapter 12

Duality Applied to Phase Transition Problems

12.1 Introduction

In this chapter, our first objectives are to show existence and develop dual formulations concerning the semi-linear Ginzburg-Landau equation. We start by describing the primal formulation.

By $S \subset \mathbb{R}^3$ we denote an open connected bounded set with a sufficiently regular boundary $\Gamma = \partial S$ (regular enough so that the Sobolev Imbedding Theorem holds). The Ginzburg-Landau equation is given by:

$$\begin{cases} -\nabla^2 u + \alpha(\frac{u^2}{2} - \beta)u - f(x) = 0 \quad a.e \quad in \quad S, \\ u = 0 \quad on \quad \Gamma. \end{cases} \tag{12.1}$$

Here $u : S \to \mathbb{R}$ denotes the primal field, $f(x) \in L^2(S)$, and α, β are real positive constants.

The corresponding variational formulation is given by the functional $J : U \to \mathbb{R}$ where,

$$J(u) = \frac{1}{2}\int_S |\nabla u|^2 dS + \frac{\alpha}{2}\int_S (\frac{u^2}{2} - \beta)^2 dx - \int_S f u dx \tag{12.2}$$

and $U = \{u \in W^{1,2}(S) \mid u = 0 \quad on \quad \Gamma\} = W_0^{1,2}(S)$.

Equations indicated in (12.1) are necessary conditions for the solution of Problem $\mathcal{P}$, where

$$\text{Problem } \mathcal{P}: \text{ to determine } u_0 \in U \text{ such that } J(u_0) = \min_{u \in U}\{J(u)\}.$$

For this problem, our first result is a convex primal-dual formulation. It is not difficult to show that the solution of Ginzburg-Landau equations indicated in (12.1), minimizes the functional $\hat{J} : U \times Y^* \to \mathbb{R}$ where

$$\hat{J}(u,v^*) = \frac{1}{2}\int_S |v_1^*|^2 dx + \frac{3}{4(\alpha/2)^{1/3}}\int_S (v_2^*)^{\frac{4}{3}} dx - \langle \nabla u, v_1^* \rangle_{L^2(S;\mathbb{R}^3)}$$
$$-\langle u, v_2^* \rangle_{L^2(S)} + \frac{1}{2}\int_S |\nabla u|^2 dx + \frac{\alpha}{8}\int_S u^4\, dx$$

under the constraint

$$-div(v_1^*) + v_2^* - \alpha\beta u = f, \quad a.e. \;\; in \;\; S.$$

The second duality principle presented gives us another convex dual variational formulation, through which optimality conditions for the primal problem may be obtained. Such a principle is expressed as

$$\begin{aligned} \inf_{u \in U}\{J(u)\} \;\; \geq & \sup_{(z^*,v_1^*,v_0^*) \in B^*} \left\{ -\frac{1}{2K^2}\int_S |\nabla z^*|^2 dx + \frac{1}{2K}\int_S (z^*)^2 dx \right. \\ & - \left. \frac{1}{2}\int_S \frac{(v_1^*)^2}{v_0^* + K} dx - \frac{1}{2\alpha}\int_S (v_0^*)^2 dx - \beta\int_S v_0^* dx \right\}, \end{aligned} \quad (12.3)$$

where

$$\begin{aligned} B^* = \Big\{(z^*, v_1^*, v_0^*) \in L^2(S;\mathbb{R}^3) \;\; | \;\; & -\frac{1}{K}\nabla^2 z^* + v_1^* - z^* = f, \\ & v_0^* + K > 0, \;\; a.e. \;\; in \;\; S, \;\; z^* = 0 \;\; on \;\; \Gamma \Big\}. \end{aligned} \quad (12.4)$$

Sufficient conditions for optimality concerning the primal formulation are given by $\delta J(u_0) = \theta$ and $v_0^*(u_0) + K > 0$ a.e. in S where $K = 1/K_0$, here K_0 is the constant related to Poincaré inequality, and $v_0^*(u_0) = \alpha(u_0^2/2 - \beta)$.

As earlier mentioned, our second objective in this work is to provide, through the tools of convex analysis, duality principles which are valid even for the vectorial case in the calculus of variations. We obtain a simple result, namely Theorem 12.6.1, through which we establish a duality principle for a phase transition problem. Thus, considering the phases $\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^N$, here $\mathbf{e}^k \in \mathbb{R}^{3\times 3}, \forall k \in \{1,...,N\}$, that a elastic solid may present, we denote the primal functional as $J : U \to \mathbb{R}$, where

$$J(u) = \int_S \min_{k \in \{1,...,N\}}\{g_k(\nabla u)\} dx + \frac{\alpha}{2}\langle u_i, u_i \rangle_{L^2(S)} - \langle u, f \rangle_{L^2(S;\mathbb{R}^3)},$$

$$\nabla u = \left\{ \frac{\partial u_i}{\partial x_j} \right\},$$

$$g_k(\nabla u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \mathbf{e}_{ij}^k \right) C_{ijlm}^k \left(\frac{\partial u_l}{\partial x_m} - \mathbf{e}_{lm}^k \right),$$

and,

$$U = \{u \in W^{1,2}(S;\mathbb{R}^3) \;\; | \;\; u = (0,0,0) \;\; on \;\; \Gamma\}.$$

Here $\{C^k_{ijlm}\}$ are positive definite matrices for each $k \in \{1, ..., N\}$ and $f \in L^2(S; \mathbb{R}^3)$ is a external load. We have obtained a duality principle expressed as

$$\begin{aligned}
\inf_{u \in U}\{J(u)\} &= \sup_{v^* \in A^*} \inf_{\lambda \in B} \sup_{\sigma \in C^*} \left\{ -\int_S (v^*_{1^{ij}} + \sigma_{1^{ij}})(D_{ijlm})(v^*_{1^{lm}} + \sigma_{1^{lm}})dx \right. \\
&- \int_S (\sigma_{1^{ij}} + v^*_{1^{ij}}) D_{ijlm} \lambda_k C^k_{mlop} \mathbf{e}^k_{op} dx - \frac{1}{2\alpha} \langle v^*_{2^i} + \sigma_{2^i}, v^*_{2^i} + \sigma_{2^i} \rangle_{L^2(S)} \\
&+ \left. \frac{1}{2} \int_S (D_{ijop}(\sigma_{op} + v^*_{op} + \eta_{op}) - \mathbf{e}^k_{ij})(\lambda_k C^k_{ijlm})(D_{lmrs}(\sigma_{rs} + v^*_{rs} + \eta_{rs}) - \mathbf{e}^k_{lm}) dx \right\},
\end{aligned}$$

where

$$\{\eta_{ij}\} = \{\lambda_k C^k_{ijlm} \mathbf{e}^k_{lm}\},$$
$$\{D_{ijlm}\} = \{\lambda_k C^k_{ijlm}\}^{-1},$$

$$C^* = \{\sigma \in Y^* \mid \sigma_{1^{ij},j} - \sigma_{2^i} = 0, \quad a.e. \quad in \quad S, \forall i \in \{1,2,3\}\},$$
$$A^* = \{v^* \in Y^* \mid v^*_{1^{ij},j} - v^*_{2^i} + f_i = 0, \quad a.e. \quad in \quad S, \forall i \in \{1,2,3\}\},$$

and

$$B = \left\{ (\lambda_1(x), ..., \lambda_N(x)), \mid \lambda_k(x) \geq 0, \forall k \in \{1, ..., N\} \; and \; \sum_{k=1}^{N} \lambda_k(x) = 1, \quad a.e. \quad in \quad S \right\}.$$

Remark 12.1.1. *The dual formulation is concave in v^* and, as mentioned above the solution of dual problem reflects the average behavior of minimizing sequences for the primal problem, when this latter problem has not solutions in the classical sense.*

12.2 Existence of Solution for the Ginzburg-Landau Equation

Remark 12.2.1. *From the Sobolev Imbedding Theorem (Adams [1]) for*

$$mp < n, \quad n - mp < n, \quad p \leq q \leq p^* = np/(n - mp),$$

we have

$$W^{j+m,p}(\Omega) \to W^{j,q}(\Omega).$$

Therefore, considering $n = 3$, $m = 1$, $j = 0$, $p = 2$, and $q = 4$, we obtain

$$W^{1,2}(\Omega) \subset L^4(\Omega) \subset L^2(\Omega)$$

and thus

$$\|u\|_{L^4(\Omega)} \to +\infty \Rightarrow \|u\|_{W^{1,2}(\Omega)} \to +\infty.$$

Furthermore, from above and the Poincaré Inequality it is clear that for J given by (12.2), we have

$$J(u) \to +\infty \quad as \quad \|u\|_{W^{1,2}(S)} \to +\infty,$$

that is, J is coercive.

Now we establish the existence of a minimizer for $J : U \to \mathbb{R}$. It is a well known procedure (the direct method of calculus of variations). We present it here for the sake of completeness.

Theorem 12.2.2. *For $\alpha, \beta \in \mathbb{R}^+$, $f \in L^2(S)$ there exists at least one $u_0 \in U$ such that*

$$J(u_0) = \min_{u \in U}\{J(u)\}$$

where

$$J(u) = \frac{1}{2}\int_S |\nabla u|^2 dx + \frac{\alpha}{2}\int_S (\frac{u^2}{2} - \beta)^2 dx - \int_S f u dx$$

and $U = \{u \in W^{1,2}(S)| \quad u = 0 \quad on \quad \Gamma\} = W_0^{1,2}(S)$.

Proof: From Remark 12.2.1 we have

$$J(u) \to +\infty \quad as \quad \|u\|_U \to +\infty.$$

Thus as J is strongly continuous, there exists $\alpha_1 \in \mathbb{R}$ such that $\alpha_1 = \inf_{u \in U}\{J(u)\}$, so that, for $\{u_n\}$ minimizing sequence, in the sense that

$$J(u_n) \to \alpha_1 \quad as \quad n \to +\infty \tag{12.5}$$

we have that $\|u_n\|_U$ is bounded, and thus, as $W_0^{1,2}(S)$ is reflexive, there exists $u_0 \in W_0^{1,2}(S)$ and a subsequence $\{u_{nj}\} \subset \{u_n\}$ such that

$$u_{nj} \rightharpoonup u_0, \text{ weakly in } W_0^{1,2}(S). \tag{12.6}$$

From (12.6), by the Rellich-Kondrachov theorem, up to a subsequence, which is also denoted by $\{u_{nj}\}$, we have

$$u_{nj} \to u_0, \text{ strongly in } L^2(S). \tag{12.7}$$

Furthermore, defining $J_1 : U \to \mathbb{R}$ as

$$J_1(u) = \frac{1}{2}\int_S |\nabla u|^2 dx + \frac{\alpha}{8}\int_S u^4 dx - \int_S f u dx$$

we have that $J_1 : U \to \mathbb{R}$ is convex and strongly continuous, therefore weakly lower semi-continuous, so that

$$\liminf_{j \to +\infty}\{J_1(u_{nj})\} \geq J_1(u_0). \tag{12.8}$$

On the other hand, from (12.7)

$$\int_S (u_{nj})^2 dx \to \int_S u_0^2 dx, \quad as \quad j \to +\infty \tag{12.9}$$

and thus, from (12.8) and (12.9) we may write

$$\alpha_1 = \inf_{u \in U}\{J(u)\} = \liminf_{j \to +\infty}\{J(u_{nj})\} \geq J(u_0). \quad \square$$

12.3 Convex Dual Formulations for the Ginzburg-Landau Equation

We start this section by stating the following theorem which was proved in F.Botelho [3].

Theorem 12.3.1. *Let U be a reflexive Banach space, $(G \circ \Lambda) : U \to \bar{\mathbb{R}}$ a convex Gâteaux differentiable functional and $(F \circ \Lambda_1) : U \to \bar{\mathbb{R}}$ convex, coercive and lower semi-continuous (l.s.c.) such that the functional*

$$J(u) = (G \circ \Lambda)(u) - F(\Lambda_1 u) - \langle u, u_0^* \rangle_U$$

is bounded from below , where $\Lambda : U \to Y$ and $\Lambda_1 : U \to Y_1$ are continuous linear operators.

Then we may write

$$\inf_{z^* \in Y_1^*} \sup_{v^* \in B^*(z^*)} \{F^*(z^*) - G^*(v^*)\} \geq \inf_{u \in U}\{J(u)\}$$

where $B^*(z^*) = \{v^* \in Y^*$ such that $\Lambda^* v^* - \Lambda_1^* z^* - u_0^* = 0\}$.

We may apply the last theorem to the variational formulation of Ginzburg-Landau equation. Just defining

$$(G \circ \Lambda)(u) = \frac{1}{2}\int_S |\nabla u|^2 dx + \frac{\alpha}{8}\int_S u^4 dx, \tag{12.10}$$

$$(F \circ \Lambda_1)(u) = \frac{\alpha\beta}{2}\int_S u^2 dx, \tag{12.11}$$

$$\langle u, u_0^* \rangle_U = \langle u, f \rangle_{L^2(S)} \tag{12.12}$$

and,

$$\Lambda u \equiv \{\nabla u, u\} \quad and \quad \Lambda_1 u \equiv u.$$

Thus,

$$G^*(v^*) = \frac{1}{2}\int_S |v_1^*|^2 dx + \int_S \frac{3}{4(\alpha/2)^{1/3}} (v_2^*)^{\frac{4}{3}} dx,$$

$$F^*(z^*) = \frac{1}{2\alpha\beta}\int_S (z^*)^2 dx$$

and

$$B^*(z^*) = \{v^* \in Y^* \mid -div(v_1^*) + v_2^* - z^* = f, \quad a.e. \quad in \quad S\}.$$

Finally,

$$\inf_{u \in U}\{J(u)\} \leq \inf_{z^* \in Y_1^*} \sup_{v^* \in B^*(z^*)} \left\{ \frac{1}{2\alpha\beta}\int_S (z^*)^2 dx - \frac{1}{2}\int_S |v_1^*|^2 dx - \frac{3}{4(\alpha/2)^{1/3}}\int_S (v_2^*)^{\frac{4}{3}} dx \right\}.$$

We are now ready to establish the convex primal dual formulation. First, observe that

$$G^*(v^*) \geq \langle \Lambda u, v^* \rangle_Y - G(\Lambda u), \forall u \in U, \quad v^* \in Y^*,$$

or

$$-\frac{1}{2\alpha\beta}\int_S (z^*)^2 + G^*(v^*) \geq -\frac{1}{2\alpha\beta}\int_S (z^*)^2 dx + \langle \Lambda u, v^* \rangle_Y - G(\Lambda u)$$

subject to

$$-div(v_1^*) + v_2^* - z^* = f, \quad a.e. \quad in \quad S.$$

Therefore we can write

$$\frac{\alpha\beta}{2}\int_S u^2 dx - \langle u, z^* \rangle_{L^2(S)} + \frac{1}{2\alpha\beta}\int_S (z^*)^2 dx + G^*(v^*) \geq \langle \Lambda u, v^* \rangle_Y - G(\Lambda u). \tag{12.13}$$

In particular for $z^* = \alpha\beta u$, from (12.13) we have

$$G^*(v^*) - \langle \Lambda u, v^* \rangle_Y + G(\Lambda u) \geq 0,$$

subject to

$$(v^*, u) \in C^* \equiv \{(v^*, u) \in Y^* \times U \mid -div(v_1^*) + v_2^* - \alpha\beta u = f, \quad a.e. \quad in \quad S\}.$$

Thus, we can state the next theorem.

Theorem 12.3.2. *The solution of Ginzburg-Landau equations indicated in (12.1), minimizes the functional $\hat{J} : U \times Y^* \to \mathbb{R}$ where*

$$\begin{aligned} \hat{J}(u, v^*) = {} & \frac{1}{2}\int_S |v_1^*|^2 dx + \frac{3}{4(\alpha/2)^{1/3}}\int_S (v_2^*)^{\frac{4}{3}} dx - \langle \nabla u, v_1^* \rangle_{L^2(S;\mathbb{R}^3)} \\ & -\langle u, v_2^* \rangle_{L^2(S)} + \frac{1}{2}\int_S |\nabla u|^2 dx + \frac{\alpha}{8}\int_S u^4 dx \end{aligned} \tag{12.14}$$

under the constraint

$$-div(v_1^*) + v_2^* - \alpha\beta u = f, \quad a.e. \quad in \quad S. \tag{12.15}$$

Proof: Just consider a solution u_0 for the boundary value problem related to Ginzburg-Landau equation. Defining $v_0^* = \frac{\partial G(\Lambda u_0)}{\partial v}$, we have that (u_0, v_0^*) minimizes the functional above indicated in (12.14) and satisfies (12.15). $\square$

Remark 12.3.3. *Observe that the primal-dual formulation presented in (12.14) is convex, but one shortcoming of such a variational approach is that any critical point of the original problem works as (global) minimizer. However, now we will indicate a procedure that leads to optimization of our primal variational formulation through the dual one.*

Our next result refers to a convex dual variational formulation, through which we obtain sufficient conditions for optimality.

Theorem 12.3.4. *Consider $J : U \to \mathbb{R}$, where*

$$J(u) = \int_S \frac{1}{2}|\nabla u|^2 dx + \int_S \frac{\alpha}{2}(\frac{u^2}{2} - \beta)^2 dx - \int_S f u dx,$$

and $U = W_0^{1,2}(S)$. For $K = 1/K_0$, where K_0 stands for the constant related to Poincaré inequality, we have the following duality principle

$$\inf_{u \in U}\{J(u)\} \geq \sup_{(z^*, v_1^*, v_0^*) \in B^*} \{-G_L^*(v^*, z^*)\}$$

where

$$G_L^*(v^*, z^*) = \frac{1}{2K^2}\int_S |\nabla z^*|^2 dx - \frac{1}{2K}\int_S (z^*)^2 dx + \frac{1}{2}\int_S \frac{(v_1^*)^2}{v_0^* + K} dx + \frac{1}{2\alpha}\int_S (v_0^*)^2 dx + \beta \int_S v_0^* dx,$$

and

$$\begin{aligned} B^* = \{(z^*, v_1^*, v_0^*) \in L^2(S;\mathbb{R}^3) \quad | \quad & -\frac{1}{K}\nabla^2 z^* + v_1^* - z^* = f, \\ & v_0^* + K > 0, \quad a.e. \ \ in \ \ S, \quad z^* = 0 \quad on \quad \Gamma\}. \end{aligned} \tag{12.16}$$

If in addition there exists $u_0 \in U$ such that $\delta J(u_0) = \theta$ and $\bar{v}_0^* + K = (\alpha/2)u_0^2 - \beta + K > 0$, a.e. in S, then

$$J(u_0) = \min_{u \in U}\{J(u)\} = \max_{(z^*, v_1^*, v_0^*) \in B^*}\{-G_L^*(v^*, z^*)\} = -G_L^*(\bar{v}^*, \bar{z}^*),$$

where

$$\bar{v}_0^* = \frac{\alpha}{2}u_0^2 - \beta,$$

$$\bar{v}_1^* = (\bar{v}_0^* + K)u_0$$

and

$$\bar{z}^* = K u_0.$$

Proof: Observe that we may write

$$J(u) = G(\Lambda u) - F(\Lambda_1 u) - \int_S fudx,$$

where

$$G(\Lambda u) = \int_S \frac{1}{2}|\nabla u|^2 dx + \int_S \frac{\alpha}{2}(\frac{u^2}{2} - \beta + 0)^2 dx + \frac{K}{2}\int_S u^2 dx,$$

$$F(\Lambda_1 u) = \frac{K}{2}\int_S u^2 dx,$$

where

$$\Lambda u = \{\Lambda_0 u, \Lambda_1 u, \Lambda_2 u\},$$

and

$$\Lambda_0 u = 0, \quad \Lambda_1 u = u, \quad \Lambda_2 u = \nabla u.$$

From Theorem 12.3.1 (here this is an auxiliary theorem through which we obtain A^*, indicated below), we have

$$\inf_{u \in U}\{J(u)\} = \inf_{z^* \in Y_1^*} \sup_{v^* \in A^*}\{F^*(z^*) - G^*(v^*)\}.$$

Here

$$F^*(z^*) = \frac{1}{2K}\int_S (z^*)^2 dx,$$

and

$$G^*(v^*) = \frac{1}{2}\int_S |v_2^*|^2 dx + \frac{1}{2}\int_S \frac{(v_1^*)^2}{v_0^* + K}dx + \frac{1}{2\alpha}\int_S (v_0^*)^2 dx + \beta\int_S v_0^* dx,$$

if $v_0^* + K > 0, \quad a.e. \ \ in \ \ S$, and

$$A^* = \{v^* \in Y^* \ \mid \ \Lambda^* v^* - \Lambda_1^* z^* - f = 0\},$$

or

$$A^* = \{(z^*, v^*) \in L^2(S) \times L^2(S; \mathbb{R}^5) \ \mid \ -div(v_2^*) + v_1^* - z^* - f = 0, \quad a.e. \ \ in \ \ S\}.$$

Observe that

$$G^*(v^*) \geq \langle \Lambda u, v^* \rangle_Y - G(\Lambda u), \quad \forall u \in U, \quad v^* \in Y^*,$$

and thus

$$-F^*(z^*) + G^*(v^*) \geq -F^*(z^*) + \langle \Lambda_1 u, z^* \rangle_{L^2(S)} + \langle u, f \rangle_U - G(\Lambda u). \tag{12.17}$$

Hence, making z^* an independent variable through A^*, from (12.17) we may write

$$\sup_{z^* \in L^2(S)}\{-F^*(z^*) \ + \ G^*(v_2^*(v_1^*, z^*), v_1^*, v_0^*)\}$$

$$\geq \sup_{z^* \in L^2(S)}\{-F^*(z^*) + \langle \Lambda_1 u, z^* \rangle_{L^2(S)} + \int_S fu \, dx - G(\Lambda u)\}. \tag{12.18}$$

Thus,

$$\begin{aligned}
\sup_{z^* \in L^2(S)} \left\{ -\frac{1}{2K}\int_S (z^*)^2\,dx \right. &+ \frac{1}{2}\int_S (v_2^*(z^*,v_1^*))^2\,dx + \frac{1}{2}\int_S \frac{(v_1^*)^2}{v_0^*+K}\,dx \\
&+ \left. \frac{1}{2\alpha}\int_S (v_0^*)^2\,dx + \beta\int_S v_0^*\,dx \right\} \\
&\geq F(\Lambda_1 u) + \int_S fu\,dx - G(\Lambda u), \forall u \in U. \qquad (12.19)
\end{aligned}$$

Therefore if $K \leq 1/K_0$ (here K_0 denotes the constant concerning Poincaré Inequality), the supremum in the left side of (12.19) is attained through the relations

$$v_2^* = \frac{\nabla z^*}{K} \quad \textit{and} \quad z^* = 0 \quad \textit{on} \quad \Gamma.$$

Hence the final format of our duality principle is given by

$$\begin{aligned}
\inf_{u \in U}\{J(u)\} \quad \geq \quad & \sup_{(z^*,v_1^*,v_0^*) \in B^*} \left\{ -\frac{1}{2K^2}\int_S |\nabla z^*|^2 dx + \frac{1}{2K}\int_S (z^*)^2 dx \right. \\
- \quad & \left. \frac{1}{2}\int_S \frac{(v_1^*)^2}{v_0^*+K}dx - \frac{1}{2\alpha}\int_S (v_0^*)^2 dx - \beta\int_S v_0^* dx \right\}, \qquad (12.20)
\end{aligned}$$

where

$$\begin{aligned}
B^* = \left\{ (z^*,v_1^*,v_0^*) \in L^2(S;\mathbb{R}^3) \quad | \quad \right. & -\frac{1}{K}\nabla^2 z^* + v_1^* - z^* = f, \\
& \left. v_0^* + K > 0, \text{ a.e. in } S, \quad z^* = 0 \quad on \quad \Gamma \right\}. \qquad (12.21)
\end{aligned}$$

The remaining conclusions follow from an application of Theorem 7.1.27. $\square$

Remark 12.3.5. *The relations*

$$v_2^* = \frac{\nabla z^*}{K} \quad \textit{and} \quad z^* = 0 \quad \textit{on} \quad \Gamma,$$

are sufficient for attainability of supremum indicated in (12.19) but just partially necessary, however we assume them because the expression of dual problem is simplified without violating inequality (12.20) (in fact the difference between the primal and dual functionals even increases under such relations).

In a similar fashion, we have also the following result.

Theorem 12.3.6. *Considering the functionals $(G \circ \Lambda) : U \to \mathbb{R}$ and $(F \circ \Lambda_1) : U \to \mathbb{R}$ defined in (12.10) and (12.11) respectively, we can write*

$$\begin{aligned}
\inf_{u \in U}\{G(\Lambda u) - F(\Lambda_1 u) - \langle u, f\rangle_{L^2(S)}\} \quad \geq \quad & \sup_{(v_1^*,z^*) \in D^*} \left\{ \frac{1}{2\alpha\beta}\int_S (z^*)^2 dx \right. \\
- \quad & \left. \frac{1}{2}\int_S |v_1^*|^2 dx - \frac{3\alpha/2}{4(\alpha\beta)^4}\int_S (z^*)^4 dx \right\} \qquad (12.22)
\end{aligned}$$

where

$$D^* = \left\{(v^*, z^*) \in L^2(S;\mathbb{R}^3) \times L^2(S) \ \mid \ \frac{\alpha}{2}(\frac{z^*}{\alpha\beta})^3 - (div(v_1^*) + z^* + f) = 0, \quad a.e. \ \ in \ \ S\right\}.$$

The equation in the definition of D^* represents the attainment of supremum in z^* for the left side of (12.25), indicated below.

Proof: Again, we have

$$G^*(v^*) \geq \langle \Lambda u, v^* \rangle_Y - G(\Lambda u), \quad \forall u \in U, \quad v^* \in Y^*$$

or

$$\begin{aligned} -\frac{1}{2\alpha\beta}\int_S (z^*)^2 dx \ \ &+ \ \ \frac{1}{2}\int_S |v_1^*|^2 dx + \frac{3}{4(\alpha/2)^{1/3}}\int_S (v_2^*)^{4/3} dx \\ &\geq \ \ -\frac{1}{2\alpha\beta}\int_S (z^*)^2 dx + \langle \Lambda u, v^* \rangle_Y - G(\Lambda u) \end{aligned} \tag{12.23}$$

subject to

$$-div(v_1^*) + v_2^* - z^* = f, \quad a.e. \ \ in \ \ S$$

or

$$v_2^* = div(v_1^*) + z^* + f. \tag{12.24}$$

Thus, replacing equation (12.24) into (12.23) we obtain

$$\begin{aligned} -\frac{1}{2\alpha\beta}\int_S (z^*)^2 dx \ \ &+ \ \ \frac{1}{2}\int_S |v_1^*|^2 dx + \frac{3}{4(\alpha/2)^{1/3}}\int_S (div(v_1^*) + z^* + f)^{4/3} dx \\ &\geq \ \ -\frac{1}{2\alpha\beta}\int_S (z^*)^2 dx + \langle u, z^* \rangle_{L^2(S)} + \langle u, f \rangle_{L^2(S)} - G(\Lambda u) \end{aligned} \tag{12.25}$$

and taking the supremum in z^* in both sides of (12.25) we have

$$-\frac{1}{2\alpha\beta}\int_S (z^*)^2 dx + \frac{1}{2}\int_S |v_1^*|^2 dx + \frac{3\alpha/2}{4(\alpha\beta)^4}\int_S (z^*)^4 dx \geq \frac{\alpha\beta}{2}\int_S (u)^2 dx + \langle u, f \rangle_{L^2(S)} - G(\Lambda u) \tag{12.26}$$

subject to

$$\frac{\alpha}{2}(\frac{z^*}{\alpha\beta})^3 - (div(v_1^*) + z^* + f) = 0 \tag{12.27}$$

where the relation (12.27), as above mentioned, represents the solution of supremum in z^* concerning the left side of inequality (12.25). $\square$

The cubic equation (12.27) has three roots. Each one leads to a different local extremum of primal variational problem.

Numerical results seems to indicated that the global minimum is obtained through the only root that is always real.

12.4 Applications to Phase Transition in Polymers

We consider now a variational problem closely related to the Ginzburg-Landau formulation. See [9] and other references therein for more information how this applies to phase transition in polymers). For an open bounded $S \subset \mathbb{R}^3$ with a sufficient regular boundary denoted by Γ, let us define $J : U \times V \to \mathbb{R}$, as

$$J(u,v) = \frac{\varepsilon}{2}\int_S |\nabla u|^2 dx + \frac{1}{4\varepsilon}\int_S (u^2-1)^2 dx + \frac{\gamma}{2}\int_S |\nabla v|^2 dx,$$

under the constraints,

$$\frac{1}{|S|}\int_S u dx = m, \tag{12.28}$$

$$-\nabla^2 v = u - m, \tag{12.29}$$

and

$$\int_S v dx = 0. \tag{12.30}$$

Here $U = V = W^{1,2}(S)$, ε is a small constant and $-1 < m < 1$. We may rewrite the primal functional, now denoting it by $J : U \times V \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ as

$$J(u,v) = G_1(\Lambda(u,v)) - F(u).$$

Here $\Lambda : U \times V \to Y \equiv L^2(S) \times L^4(S) \times L^2(S;\mathbb{R}^3) \times L^2(S;\mathbb{R}^3)$ is defined as

$$\Lambda(u,v) = \{\Lambda_0 u = 0, \Lambda_1 u = u, \Lambda_2 u = \nabla u, \Lambda_3 v = \nabla v\},$$

also

$$G_1(\Lambda(u,v)) = G(\Lambda(u,v)) + Ind_1(u,v) + Ind_2(u,v) + Ind_3(u,v).$$

Hence

$$G(\Lambda(u,v)) = \frac{\varepsilon}{2}\int_S |\nabla u|^2 dx + \frac{1}{4\varepsilon}\int_S (u^2-1+0)^2 dx + \frac{K}{2}\int_S u^2 dx + \frac{\gamma}{2}\int_S |\nabla v|^2 dx,$$

$$Ind_1(u,v) = \begin{cases} 0, & if \ \ \frac{1}{|S|}\int_S u dx = m, \\ +\infty, & otherwise, \end{cases}$$

$$Ind_2(u,v) = \begin{cases} 0, & if \ \ \nabla^2 v + u - m = 0, \ \ a.e. \ \ in \ \ S, \\ +\infty, & otherwise, \end{cases}$$

$$Ind_3(u,v) = \begin{cases} 0, & if \ \ \int_S v dx = 0, \\ +\infty, & otherwise, \end{cases}$$

and,

$$F(u) = \frac{K}{2}\int_S u^2 dx.$$

Similarly to Theorem 12.3.1, through appropriate Lagrange multipliers for the constraints, we may write

$$\inf_{(u,v)\in U\times V}\{J(u,v)\} \leq \inf_{z^*\in Y_1^*}\sup_{(u^*,v^*,\lambda)\in A^*}\left\{F^*(z^*)-G^*(u^*,v^*)+\lambda_1 m+\int_S \lambda_2 m dx\right\},$$

where

$$G^*(u^*,v^*)=\frac{1}{2\varepsilon}\int_S |u_1^*|^2dx+\frac{1}{2}\int_S\frac{(u_2^*)^2}{2u_0^*+K}dx+\varepsilon\int_S (u_0^*)^2dx+\int_S u_0^*dx+\frac{1}{2\gamma}\int_S |v^*|^2dx,$$

if $2u_0^*+K>0$, *a.e. in* S, and

$$F(z^*)=\frac{1}{2K}\int_S (z^*)^2dx.$$

Also, defining $\hat{Y}^*=L^2(S)\times Y^*\times\mathbb{R}\times L^2(S)\times\mathbb{R}$, we have

$$A^*=A_1^*\cap A_2^*\cap A_3^*,$$

$$A_1^*=\{(z^*,u^*,\lambda)\in\hat{Y}^* \mid div(u_1^*)-u_2^*+z^*-\frac{\lambda_1}{|S|}-\lambda_2=0 \text{ a.e. in } S \text{ and } u_1^*.n=0 \text{ on } \Gamma\},$$

$$A_2^*=\{(z^*,u^*,\lambda)\in\hat{Y}^* \mid \nabla^2\lambda_2+\lambda_3-div(v^*)=0 \text{ a.e. in } S,\ v^*.n+\frac{\partial\lambda_2}{\partial n}=\lambda_2=0, \text{ on } \Gamma\},$$

and

$$A_3^*=\{(z^*,u^*,\lambda)\in\hat{Y}^* \mid 2u_0^*+K>0, \text{ a.e. in } S\}.$$

Similarly to the case of Ginzburg-Landau formulation we can obtain

$$\inf_{(u,v)\in U\times V}\{J(u,v)\}\geq$$

$$\sup_{(z^*,u^*,v^*,\lambda)\in C^*}\{-\frac{\varepsilon}{2K^2}\int_S|\nabla z^*|^2dx+\frac{1}{2K}\int_S (z^*)^2dx-\frac{1}{2}\int_S\frac{(u_2^*)^2}{2u_0^*+K}dx-$$
$$-\varepsilon\int_S (u_0^*)^2dx+\int_S u_0^*dx-\frac{1}{2\gamma}\int_S|v^*|^2dx-\lambda_1 m-\int_S\lambda_2 m dx\},$$

where

$$C^*=C_1^*\cap A_2^*\cap A_3^*,$$

$$C_1^*=\{(z^*,u^*,\lambda)\in\hat{Y}^* \mid \frac{\varepsilon\nabla^2 z^*}{K}-u_2^*+z^*-\frac{\lambda_1}{|S|}-\lambda_2=0 \text{ a.e. in } S \text{ and } \frac{\partial z^*}{\partial n}=0 \text{ on } \Gamma\}.$$

Remark 12.4.1. *It is important to emphasize that by analogy to Section 12.3, we may obtain sufficient conditions for optimality. That is, if there exists a critical point for the dual formulation for which* $2u_0^*+K>0$ *a.e. in* S *and* $K=\varepsilon/K_0$ *(where* K_0 *stands for the constant related to Poincaré inequality), then the corresponding primal point through Legendre transform relations is also a global minimizer and the last inequality is in fact an equality, as has been pointed out in Theorem 12.3.4.*

12.4.1 Another Two Phase Model in Polymers

The following problem has applications in two phase models in Polymers. To minimize the functional $J : U \times V \to \mathbb{R}$ (here consider $S \subset \mathbb{R}^3$ as above), where

$$J(u, v) = |Du|(S) + \frac{\gamma}{2}\int_S |\nabla v|^2 dx,$$

under the constraints

$$\int_S u dx = m,$$

$$-\nabla^2 v = u - m, \tag{12.31}$$

where

$$U = BV(S, \{-1, 1\}), \tag{12.32}$$

and $V = W^{1,2}(S)$ (here BV denotes the space of functions with bounded variation in S and $|Du|(S)$ denotes the total variation of u in S).

Redefining U as $U = W^{1,1}(S) \cap L^2(S)$, we rewrite the primal formulation, through suitable Lagrange multipliers, now denoting it by $\hat{J}(u, v, \lambda_1, \lambda_2, \lambda_3)$ ($\hat{J} : U \times V \times L^2(S) \times \mathbb{R} \times L^2(S) \to \mathbb{R}$), as

$$\begin{aligned} \hat{J}(u, v, \lambda) &= \int_S |\nabla u| dx + \frac{\gamma}{2}\int_S |\nabla v|^2 dx + \int_S \frac{\lambda_1}{2}(u^2 - 1) dx \\ &+ \lambda_2\left(\int_S u dx - m\right) + \int_S \lambda_3(\nabla^2 v + u - m) dx, \end{aligned} \tag{12.33}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. We may write

$$J(u, v, \lambda) = G(\Lambda(u, v)) + F(u, v, \lambda),$$

where

$$G(\Lambda(u, v)) = \int_S |\nabla u| dx + \frac{\gamma}{2}\int_S |\nabla v|^2 dx,$$

here $\Lambda : U \times V \to Y = L^2(S; \mathbb{R}^3) \times L^2(S; \mathbb{R}^3)$ is defined as

$$\Lambda(u, v) = \{\Lambda_1 u = \nabla u, \Lambda_2 v = \nabla v\}$$

and

$$F(u, v, \lambda) = \lambda_2\left(\int_S u dx - m\right) + \int_S \lambda_3(\nabla^2 v + u - m) dx + \int_S \frac{\lambda_1}{2}(u^2 - 1) dx.$$

Defining

$$\hat{U} = \left\{(u, v) \in U \times V \ \mid \ \int_S u dx = m, \quad \nabla^2 v + u - m = 0, \quad u^2 = 1 \ \ a.e. \ \ in \ \ S\right\},$$

from the Lagrange Multiplier version of Theorem 7.2.5 we have that

$$\inf_{(u,v)\in\hat{U}}\{G(\Lambda(u,v)) + F(u,v,\theta)\} = \sup_{(v^*,\lambda)\in Y^*\times B^*}\{-G^*(v^*) - F^*(-\Lambda^* v^*, \lambda)\},$$

where

$$G^*(v^*) = \sup_{v\in Y}\{\langle v, v^*\rangle_Y - G(v)\} = \frac{1}{2\gamma}\int_S |v_2^*|^2 dx + Ind_0(v_1^*),$$

and

$$Ind_0(v_1^*) = \begin{cases} 0, & if \ \ |v_1^*|_2 \leq 1, \ \ a.e. \ \ in \ \ S, \\ +\infty, & otherwise. \end{cases}$$

We may define

$$Ind_1(v_1^*) = \begin{cases} 0, & if \ \ v_1^*.n = 0, \ \ on \ \ \Gamma, \\ +\infty, & otherwise, \end{cases}$$

and

$$Ind_2(v_2^*) = \begin{cases} 0, & if \ \ div(v_2^*) - \nabla^2\lambda_3 = 0, \ \ a.e. \ \ in \ \ S, \ \ v_2^*.n + \frac{\partial\lambda_3}{\partial n} = \lambda_3 = 0 \ \ on \ \ \Gamma, \\ +\infty, & otherwise, \end{cases}$$

so that

$$F^*(-\Lambda^* v^*, \lambda) = \int_S |div(v_1^*) - \lambda_2 - \lambda_3| dx + Ind_2(v_2^*) + Ind_1(v_1^*) + \lambda_2 m + \int_S \lambda_3 m dx.$$

Therefore, we can summarize the last results by the following duality principle,

$$\begin{aligned} \inf_{(u,v)\in\hat{U}}\{G(\Lambda(u,v)) + F(u,v,\theta)\} &= \sup_{(v^*,\lambda)\in A^*\cap B^*}\left\{-\frac{1}{2\gamma}\int_S |v_2^*|^2 dx\right. \\ &- \left.\int_S |div(v_1^*) - \lambda_2 - \lambda_3| dx - \lambda_2 m - \int_S \lambda_3 m dx\right\}, \end{aligned}$$

where

$$A^* = \{v^* \in Y^* = L^2(S;\mathbb{R}^3) \times L^2(S;\mathbb{R}^3) \ \mid \ |v_1^*(x)|_2 \leq 1, \ \ a.e. \text{ in S and } v_1^*.n = 0, \ \ on \ \ \Gamma\}.$$

and

$$\begin{aligned} B^* = \{(v_2^*, \lambda_3) \in L^2(S;\mathbb{R}^3) \times L^2(S) \ \mid \ & div(v_2^*) - \nabla^2\lambda_3 = 0, \text{ a.e. in S}, \\ & v_2^*.n + \frac{\partial\lambda_3}{\partial n} = \lambda_3 = 0 \ \ on \ \ \Gamma\}. \end{aligned} \tag{12.34}$$

Remark 12.4.2. *It is worth noting that the last dual formulation represents a standard convex non-smooth optimization problem. The non-smoothness is the responsible by a possible micro-structure formation. Furthermore such a formulation seems to be amenable to numerical computation (in a simpler way as compared to the primal approach).* □

12.5 The Multi-Well Problem

This section is dedicated to analysis of the Multi-well problem via duality. We start with the scalar case, for which the results may be obtained through the theory developed in Ekeland and Témam [14] (These authors surely deserves most of the credit on the analysis for the scalar case. In fact, what we do here is to connect parts I and III of mentioned book). In Section 12.5.3 we present an example which is completely solved through the dual formulation.

12.5.1 The Primal Variational Formulation

Consider an open bounded set $S \subset \mathbb{R}^n$ with a regular boundary denoted by Γ. For $i \in \{1, ..., N\}$, also consider the convex differentiable functions $g_i : \mathbb{R}^n \to \mathbb{R}$ and $(g \circ \nabla)$ non-convex defined by

$$g(\nabla u) = \min_{i \in \{1,\ldots,N\}} \{g_i(\nabla u)\}, \tag{12.35}$$

such that

$$\frac{G(\nabla u)}{\|u\|_U} \to +\infty \quad as \quad \|u\|_U \to \infty, \tag{12.36}$$

where

$$G(\nabla u) = \int_S \min_{i \in \{1,\ldots,N\}} \{g_i(\nabla u)\} dx = \int_S g(\nabla u) dx$$

and

$$U = \{u \in W^{1,2}(S) | \quad u = u_0 \quad on \quad \Gamma\}. \tag{12.37}$$

As a preliminary result, we recall Theorem 7.3.6.

Theorem 12.5.1. *Let f be a Carathéodory function from $\Omega \times (\mathbb{R} \times \mathbb{R}^n)$ into $\mathbb{R}$ which satisfies*

$$a_2(x) + c_2|\xi|^\alpha \le f(x, s, \xi) \le a_1(x) + b|s|^\alpha + c_1|\xi|^\alpha$$

where a_1, $a_2 \in L^1(\Omega)$, $1 < \alpha < +\infty$, $b \ge 0$ and $c_1 \ge c_2 > 0$. Let $u_0 \in W^{1,\alpha}(\Omega)$. Under such assumptions, defining $\hat{U} = \{u \ | \ u - u_0 \in W_0^{1,2}(\Omega)\}$, we have

$$\inf_{u \in \hat{U}} \left\{ \int_\Omega f(x, u, \nabla u) dx \right\} = \min_{u \in \hat{U}} \left\{ \int_\Omega f^{**}(x, u; \nabla u) dx \right\}$$

The solutions of relaxed problem are weak cluster points in $W^{1,\alpha}(\Omega)$ of the minimizing sequences of primal problem.

Now we can state the following result.

Theorem 12.5.2. *Let $(G \circ \nabla) : U \to \mathbb{R}$ satisfies (12.35) and (12.36). Also assume the hypothesis of Theorem 12.5.1. Then,*

$$\inf_{u \in U}\{G(\nabla u) - \langle u, f\rangle_{L^2(S)}\} = \inf_{u \in U}\{G^{**}(\nabla u) - \langle u, f\rangle_{L^2(S)}\},$$

and there exists $\hat{u} \in U$ such that

$$\min_{u \in U}\{G^{**}(\nabla u) - \langle u, f\rangle_{L^2(S)}\} = G^{**}(\nabla \hat{u}) - \langle \hat{u}, f\rangle_{L^2(S)}.$$

The proof follows directly from Theorem 12.5.1.

Our next proposition is very important to establish the duality principle. It is a well know result in convex analysis so we do not prove it.

Proposition 12.5.3. *Consider $g : \mathbb{R}^n \to \mathbb{R}$ defined as*

$$g(v) = \min_{i \in \{1,\ldots,N\}}\{g_i(v)\}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ are not necessarily convex functions. Under such assumptions, we have

$$g^*(v^*) = \max_{i \in \{1,\ldots,N\}}\{g_i^*(v^*)\} \tag{12.38}$$

Now we present the duality principle.

Theorem 12.5.4. *For $(G \circ \nabla) : U \to \mathbb{R}$ defined as above, that is,*

$$G(\nabla u) = \int_S \min_{i \in \{1,\ldots,N\}}\{g_i(\nabla u)\}dx,$$

where here $g_i : \mathbb{R}^n \to \mathbb{R}$ is convex, for all $i \in \{1, ..., N\}$ and $F : U \to \mathbb{R}$, defined as

$$F(u) = \langle u, f\rangle_{L^2(S)},$$

we have

$$\min_{u \in U}\{G^{**}(\nabla u) - F(u)\} = \sup_{v^* \in C^*}\{-G^*(v^*) + \langle u_0, v^*.n\rangle_{L^2(\Gamma)}\}.$$

Here

$$G^*(v^*) = \int_S \max_{i \in \{1,\ldots,N\}}\{g_i^*(v^*)\}dx$$

and

$$C^* = \{v^* \in Y^* | \;\; div(v^*) + f(x) = 0, \;\; a.e. \;\; in \;\; S\}.$$

Proof: We have

$$G^*(v^*) = G^{***}(v^*) = \sup_{v \in Y}\{\langle v, v^*\rangle_Y - G^{**}(v)\}$$

that is,

$$\begin{aligned} G^*(v^*) &\geq \langle \nabla u, v^*\rangle_Y - G^{**}(\nabla u) \\ &= \langle u, -div(v^*)\rangle_{L^2(S)} + \langle u_0, v^*.n\rangle_{L^2(\Gamma)} - G^{**}(\nabla u), \quad \forall u \in U, \quad v^* \in Y^*. \end{aligned} \tag{12.39}$$

Thus, for $v^* \in C^*$ we can write:

$$G^*(v^*) \geq \langle u_0, v^*.n\rangle_{L^2(\Gamma)} + \langle u, f\rangle_{L^2(S)} - G^{**}(\nabla u), \quad \forall u \in U,$$

or

$$\inf_{u \in U}\{G^{**}(\nabla u) - \langle u, f\rangle_{L^2(S)}\} \geq \sup_{v^* \in C^*}\{-G^*(v^*) + \langle u_0, v^*.n\rangle_{L^2(\Gamma)}\}. \quad \square$$

The equality in the last line follows from hypothesis (12.36) and Theorem 7.2.5.

Observe that the dual formulation is convex but non-smooth. In the next lines we will see, through the dual formulation, how the micro-structure is formed, particularly when the original primal formulation has no minimizers in the classical sense.

12.5.2 A Scalar Multi-Well Formulation

To start this section, we present duality for the solution of a scalar multi-well problem (in Firoozye and Kohn [16] you may find a similar vectorial formulation). Consider the open bounded set $S \subset \mathbb{R}^3$ with a regular boundary Γ and the function $(W \circ \nabla)$ defined as

$$W(\nabla u) = \min_{i \in \{1,...,N\}}\left\{\frac{1}{2}|\nabla u - a_i|^2\right\}$$

where

$$U = \{u \in W^{1,2}(S) \ \mid \ u = u_0 \ \ on \ \ \Gamma\}$$

a_i are known matrices, for all $i \in \{1, ..., N\}$. The energy of the system is modeled by $J : U \to \mathbb{R}$, where

$$J(u) = \int_S W(\nabla u)dx - \langle u, f\rangle_{L^2(S)}$$

or

$$J(u) = \frac{1}{2}\int_S \min_{i \in \{1,...,N\}}\left\{|\nabla u - a_i|^2\right\}dx - \langle u, f\rangle_{L^2(S)}.$$

From Theorem 12.5.4 we have

$$\inf_{u \in U}\{J(u)\} = \sup_{v^* \in C^*}\left\{-\int_S \max_{i \in \{1,...,N\}}\left\{\frac{1}{2}|v^*|^2 + v^{*T}a_i\right\}dx + \langle u_0, v^*.n\rangle_{L^2(\Gamma)}\right\}$$

or

$$\inf_{u\in U}\{J(u)\} = \sup_{v^*\in C^*}\left\{-\int_S \max_{\lambda\in B}\left\{\frac{1}{2}|v^*|^2 + \sum_{i=1}^{N}\lambda_i v^{*T}a_i\right\}dx + \langle u_0, v^*.n\rangle_{L^2(\Gamma)}\right\}$$

where

$$B = \{(\lambda_1(x), ..., \lambda_N(x)), \;\; | \;\; \lambda_k(x) \geq 0, \forall k \in \{1, ..., N\} \;\; and \;\; \sum_{k=1}^{N}\lambda_k(x) = 1, \;\; in \;\; S\},$$

and

$$C^* = \{v^* \in Y^* \;\; | \;\; div(v^*) + f = 0, \;\; a.e. \;\; in \;\; S\}$$

The solution of the dual problem seems not to be difficult. However, it is important to emphasize that, in general, this kind of problem does not present minimizers in the classical sense. The solution of the dual problem (which is well-posed and convex), reflects the average behavior of minimizing sequences, as weak cluster points of such sequences.

12.5.3 An Example - A Two-dimensional Two-Well Problem

Consider the same hypothesis as above for $S \subset \mathbb{R}^2$ and $J : U \to \mathbb{R}$ defined as:

$$J(u) = \int_S \min\{g_1(\nabla u), g_2(\nabla u)\}dx - \langle u, f\rangle_{L^2(S)},$$

where

$$U = \{u \in W^{1,2}(S) \;\; | \;\; u = u_0 \;\; on \;\; \Gamma\},$$

$$g_1(\nabla u) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - 1\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + 1\right)^2$$

and

$$g_2(\nabla u) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + 1\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} - 1\right)^2.$$

From Theorem 12.5.4 we can write

$$\inf_{u\in U}\{J(u)\} = \sup_{v^*\in A^*}\left\{-\int_S \max\{g_1^*(v^*), g_2^*(v^*)\}dx + \langle u_0, v^*.n\rangle_{L^2(\Gamma)}\right\}$$

where

$$A^* = \{v^* \in Y^* \;\; | \;\; div(v^*) + f = 0, \;\; a.e. \;\; in \;\; S\},$$

$$g_1^*(v^*) = \frac{1}{2}(v_1^*)^2 + v_1^* + \frac{1}{2}(v_2^*)^2 - v_2^*$$

and

$$g_2^*(v^*) = \frac{1}{2}(v_1^*)^2 - v_1^* + \frac{1}{2}(v_2^*)^2 + v_2^*.$$

We solve the dual problem. The corresponding Euler-Lagrange equations are given by:

$$\delta\left\{-\langle u, div(v^*)+f\rangle_U+\int_S\{-g_1^*(v^*)+\lambda(g_1^*(v^*)-g_2^*(v^*))\}dx+\langle u_0, v^*.n\rangle_{L^2(\Gamma)}\right\}=\theta$$

or more explicitly

$$\begin{aligned}\delta\left\{\langle\frac{\partial u}{\partial x}, v_1^*\rangle_Y+\langle\frac{\partial u}{\partial y}, v_2^*\rangle_Y\right. &- \left(\frac{1}{2}\int_S(v_1^*)^2dx+\int_S v_1^*dx+\frac{1}{2}\int_S(v_2^*)^2dx-\int_S v_2^*dx\right)\\ &+ \left.\int_S 2\lambda(v_1^*-v_2^*)dx\right\}=\theta. \qquad (12.40)\end{aligned}$$

Remark 12.5.5. *Suppose that $|f(x,y)|$ is almost everywhere small enough so that the optimum for the dual formulation occurs at a point in which $g_1^*(v^*)=g_2^*(v^*)$ what justify the Lagrange multiplier λ.*

Thus, we obtain the following system

$$\frac{\partial u}{\partial x}-v_1^*-1+2\lambda=0,$$

$$\frac{\partial u}{\partial y}-v_2^*+1-2\lambda=0$$

and

$$v_1^*=v_2^*$$

so that

$$\lambda=\frac{1}{2}+\frac{1}{4}\left(\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}\right),$$

$$v_1^*=v_2^*=\frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)$$

and therefore, from $div(v^*)+f=0$, the variable u must satisfy the equations

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}+f(x,y)=0, \quad a.e. \;\; in \;\; S, \quad u=u_0 \;\; on \;\; \Gamma. \; \square$$

Finally, the duality principle could be summarized in the general case as

$$\inf_{u\in U}\{J(u)\}=\sup_{v^*\in A^*}\left\{-\int_S\max_{\lambda\in[0,1]}\{(1-\lambda)g_1^*(v^*)+\lambda g_2(v^*)\}dx+\langle u_0, v^*.n\rangle_{L^2(\Gamma)}\right\}$$

which is in fact a quadratic constrained optimization problem in λ amenable to numerical computation. Furthermore, the variable $\lambda(x)$ gives the proportion of mixture between the phases.

12.6 Duality Suitable for Vectorial Variational Problems

We just recall the very simple result, namely Theorem earlier labeled as 7.4.7 and respective Remark 7.4.8

Theorem 12.6.1. *Consider $(G \circ \Lambda) : U \to \mathbb{R}$ (not necessarily convex) such that $J : U \to \mathbb{R}$ defined as*

$$J(u) = G(\Lambda u) - \langle u, f \rangle_U, \forall u \in U,$$

is bounded from below (here as usual $\Lambda : U \to Y$ is a continuous linear operator). Under such assumptions, we have

$$\inf_{u \in U}\{J(u)\} = \sup_{v^* \in A^*}\{-(G \circ \Lambda)^*(\Lambda^* v^*)\}$$

where

$$A^* = \{v^* \in Y^* \mid \Lambda^* v^* - f = 0\}.$$

12.6.1 The Multi-Well Formulation Applied to Phase Transitions

Consider an open bounded connected set $S \subset \mathbb{R}^3$ with a regular boundary Γ and the field of displacements $u = (u_1, u_2, u_3)$ of a solid that can present the phases $\{\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^N\}$, here $\mathbf{e}^k \in \mathbb{R}^{3\times 3}, \forall k \in \{1, ..., N\}$. The elastic energy of the system is given by $J : U \to \mathbb{R}$ where

$$J(u) = \int_S \min_{k \in \{1,...,N\}}\{g_k(\nabla u)\}dx - \langle u, f \rangle_{L^2(S;\mathbb{R}^3)},$$

$$\nabla u = \left\{\frac{\partial u_i}{\partial x_j}\right\},$$

$$g_k(\nabla u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \mathbf{e}^k_{ij}\right) C^k_{ijlm}\left(\frac{\partial u_l}{\partial x_m} - \mathbf{e}^k_{lm}\right), \tag{12.41}$$

and,

$$U = \{u \in W^{1,2}(S;\mathbb{R}^3) \mid u = \theta \;\; on \;\; \Gamma\} = W^{1,2}_0(S;\mathbb{R}^3).$$

Here $\{C^k_{ijlm}\}$ are positive definite matrices for each $k \in \{1, ..., N\}$, which guarantees coercivity for the primal formulation and $f \in L^2(S;\mathbb{R}^3)$ is a external load. We apply Theorem 12.6.1 and obtain

$$\inf_{u \in U}\{J(u)\} = \sup_{v^* \in A^*}\{-(G \circ \Lambda)^*(\Lambda^* v^*)\} \tag{12.42}$$

where

$$\Lambda u = \nabla u$$

and

$$(G \circ \Lambda)^*(\Lambda^* v^*) = \sup_{u \in U}\{\langle \nabla u, v^* \rangle_{L^2(S;\mathbb{R}^9)} - G(\nabla u)\}.$$

Here

$$G(\Lambda u) = \int_S \min_{k \in \{1,\ldots,N\}}\{g_k(\nabla u)\}dx$$

and

$$A^* = \{v^* \in Y^* \mid v^*_{ij,j} + f_i = 0, \quad a.e. \ \ in \ \ in \ \ S\}. \tag{12.43}$$

From (12.41), we have

$$(G \circ \Lambda)^*(\Lambda^* v^*) = \sup_{u \in U}\left\{\langle \nabla u, v^* \rangle_{L^2(S;\mathbb{R}^9)} - \int_S \min_{k \in \{1,\ldots,N\}}\{g_k(\nabla u)\}dx\right\} \tag{12.44}$$

or

$$(G \circ \Lambda)^*(\Lambda^* v^*) = \sup_{u \in U}\left\{\langle \nabla u, v^* \rangle_{L^2(S;\mathbb{R}^9)} - \inf_{\lambda \in B}\int_S \sum_{k=1}^N \lambda_k g_k(\nabla u)dx\right\} \tag{12.45}$$

where

$$B = \left\{(\lambda_1(x), \ldots, \lambda_N(x)) \mid \lambda_k(x) \geq 0, \forall k \in \{1, \ldots, N\} \ \ and \ \ \sum_{k=1}^N \lambda_k(x) = 1, \quad a.e. \ \ in \ \ S\right\}.$$

Hence from (12.45) we can write

$$(G \circ \Lambda)^*(\Lambda^* v^*) = \sup_{u \in U}\sup_{\lambda \in B}\left\{\int_S \left\{\nabla u.v^* - \sum_{k=1}^N \lambda_k g_k(\nabla u)\right\} dx\right\}.$$

That is,

$$(G \circ \Lambda)^*(\Lambda^* v^*) = \sup_{\lambda \in B}\sup_{u \in U}\left\{\int_S \left\{\nabla u.v^* - \sum_{k=1}^N \lambda_k g_k(\nabla u)\right\} dx\right\},$$

and thus from Theorem 7.2.5

$$\begin{aligned}(G \circ \Lambda)^*(\Lambda^* v^*) = \sup_{\lambda \in B}\inf_{\sigma \in C^*}\bigg\{&\int_S (v^*_{ij} + \sigma_{ij})(D_{ijlm})(v^*_{lm} + \sigma_{lm})dx \\ &+ \int_S (\sigma_{ij} + v^*_{ij})D_{ijlm}\lambda_k C^k_{mlop}\mathbf{e}^k_{op}dx \\ - \frac{1}{2}\int_S (D_{ijop}(\sigma_{op} + v^*_{op} + \eta_{op}) - \mathbf{e}^k_{ij})(\lambda_k C^k_{ijlm})&(D_{lmrs}(\sigma_{rs} + v^*_{rs} + \eta_{rs}) - \mathbf{e}^k_{lm})dx\bigg\},\end{aligned}$$

where

$$\{\eta_{ij}\} = \{\lambda_k C^k_{ijlm}\mathbf{e}^k_{lm}\}$$

$$\{D_{ijlm}\} = \{\lambda_k C^k_{ijlm}\}^{-1},$$

and

$$C^* = \{\sigma \in Y^* \mid \sigma_{ij,j} = 0, \quad a.e. \quad in \quad S\}. \tag{12.46}$$

So to summarize, the duality principle for the multi-well problem may be written as

$$\begin{aligned}
\inf_{u \in U}\{J(u)\} \quad &= \quad \sup_{v^* \in A^*} \inf_{\lambda \in B} \sup_{\sigma \in C^*} \left\{ -\int_S (v^*_{ij} + \sigma_{ij})(D_{ijlm})(v^*_{lm} + \sigma_{lm})dx \right. \\
&- \quad \int_S (\sigma_{ij} + v^*_{ij}) D_{ijlm} \lambda_k C^k_{mlop} \mathbf{e}^k_{op} dx \\
&+ \quad \left. \frac{1}{2} \int_S (D_{ijop}(\sigma_{op} + v^*_{op} + \eta_{op}) - \mathbf{e}^k_{ij})(\lambda_k C^k_{ijlm})(D_{lmrs}(\sigma_{rs} + v^*_{rs} + \eta_{rs}) - \mathbf{e}^k_{lm})dx \right\}
\end{aligned}$$

where A^* is indicated in (12.43) and C^* in (12.46). Even though we have not performed yet numerical results the dual formulation seems to be simple to compute.

A numerical example including algorithm to compute the solution of dual formulation is planned for future work.

12.6.2 A More Complex Phase Transition Problem

As in the last section, consider an open bounded connected set $S \subset \mathbb{R}^3$ with a regular boundary Γ and the field of displacements $u = (u_1, u_2, u_3)$ of a solid that can present the phases $\{\mathbf{e}^1, \mathbf{e}^2, ..., \mathbf{e}^N\}$, here $\mathbf{e}^k \in \mathbb{R}^{3\times 3}, \forall k \in \{1, ..., N\}$. Now we are concerned with the minimization of $J : U \to \mathbb{R}$ where for $\alpha > 0$,

$$J(u) = \int_S \min_{k \in \{1,...,N\}} \{g_k(\nabla u)\}dx + \frac{\alpha}{2}\langle u_i, u_i\rangle_{L^2(S)} - \langle u, f\rangle_{L^2(S;\mathbb{R}^3)},$$

$$\nabla u = \{\frac{\partial u_i}{\partial x_j}\},$$

$$g_k(\nabla u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \mathbf{e}^k_{ij}\right) C^k_{ijlm} \left(\frac{\partial u_l}{\partial x_m} - \mathbf{e}^k_{lm}\right),$$

and,

$$U = \{u \in W^{1,2}(S;\mathbb{R}^3) \mid u = \theta \quad on \quad \Gamma\}.$$

As above $\{C^k_{ijlm}\}$ are positive definite matrices for each $k \in \{1, ..., N\}$ and $f \in L^2(S;\mathbb{R}^3)$ is a external load.

Similarly to last section, considering again Theorem 12.6.1, we may obtain the following duality principle

$$\begin{aligned}
\inf_{u \in U}\{J(u)\} &= \sup_{v^* \in A^*} \inf_{\lambda \in B} \sup_{\sigma \in C^*} \left\{ -\int_S (v^*_{1ij} + \sigma_{1^{ij}})(D_{ijlm})(v^*_{1lm} + \sigma_{1^{lm}})dx \right. \\
&- \int_S (\sigma_{1^{ij}} + v^*_{1^{ij}}) D_{ijlm} \lambda_k C^k_{lmop} \mathbf{e}^k_{op} dx - \frac{1}{2\alpha}\langle v^*_{2^i} + \sigma_{2^i}, v^*_{2^i} + \sigma_{2^i} \rangle_{L^2(S)} \\
&+ \left. \frac{1}{2}\int_S (D_{ijop}(\sigma_{op} + v^*_{op} + \eta_{op}) - \mathbf{e}^k_{ij})(\lambda_k C^k_{ijlm})(D_{lmrs}(\sigma_{rs} + v^*_{rs} + \eta_{rs}) - \mathbf{e}^k_{lm})dx \right\},
\end{aligned}$$

where

$$\{\eta_{ij}\} = \{D_{ijlm}\lambda_k C^k_{lmop}\mathbf{e}^k_{op}\},$$

$$\{D_{ijlm}\} = \{\lambda_k C^k_{ijlm}\}^{-1},$$

$$C^* = \{\sigma \in Y^* \mid \sigma_{1^{ij},j} - \sigma_{2^i} = 0, \quad a.e. \quad in \quad S, \forall i \in \{1,2,3\}\},$$

and

$$A^* = \{v^* \in Y^* \mid v^*_{1^{ij},j} - v^*_{2^i} + f_i = 0, \quad a.e. \quad in \quad S, \forall i \in \{1,2,3\}\}.$$

Remark 12.6.2. *It is worth noting that dual formulation is concave in v^* and the dual problem has a solution, which is related to the average behavior of minimizing sequences even as the primal formulation has no solution in the classical sense.*

12.7 Another Multi-Well Problem

In this section we consider duality for another class of multi-well problems similar as those found in [7]. The format of our problem is more general, not restricted to two-well formulations.

Now we describe the primal formulation. For an open bounded set $S \subset \mathbb{R}^3$ with a sufficiently regular boundary denoted by ∂S, consider the functional $J : U \to \mathbb{R}$ where

$$J(u) = \int_S \min_{k \in \{1,\ldots,N\}} \{g_k(\epsilon(u)) + \beta_k\}dx - \langle u, f \rangle_{L^2(S;\mathbb{R}^3)},$$

$$\epsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

$$g_k(\epsilon(u)) = \frac{1}{2}(\epsilon_{ij}(u) - \mathbf{e}^k_{ij})C^k_{ijlm}(\epsilon_{lm}(u) - \mathbf{e}^k_{lm}).$$

As above $\{C^k_{ijlm}\}$ are positive definite matrices for each $k \in \{1, ..., N\}$ and $f \in L^2(S;\mathbb{R}^3)$ is a external load. Here again $\mathbf{e}^k \in \mathbb{R}^{3\times 3}$, $\ for \ \ k \in \{1, ..., N\}$ represent the phases presented by a solid with field of displacements $(u_1, u_2, u_3) \in W^{1,2}(S; \mathbb{R}^3)$ due to a external load $f \in L^2(S; \mathbb{R}^3)$. Also

$$U = \{u \in W^{1,2}(S; \mathbb{R}^3) \ \mid \ u = (0,0,0) \equiv \theta \ \ on \ \ \partial S\} \equiv W_0^{1,2}(S; \mathbb{R}^3).$$

We may write

$$J(u) = G(u) - F(\epsilon(u))$$

where

$$G(u) = J(u) + \frac{K}{2}\int_S (\epsilon_{ij}(u))H_{ijlm}(\epsilon_{lm}(u))dx,$$

$$F(\epsilon(u)) = \frac{K}{2}\int_S (\epsilon_{ij}(u))H_{ijlm}(\epsilon_{lm}(u))dx,$$

and $\{H_{ijlm}\}$ is a positive definite matrix. Observe that $\epsilon : U \to Y$ is given by

$$\epsilon(u) = \{\epsilon_{ij}(u)\},$$

so that from Toland [35], we have

$$\inf_{u \in U}\{G^{**}(u) - F(\epsilon(u))\} = \inf_{z^* \in Y^*}\{F^*(z^*) - G^*(\epsilon^*(z^*))\} \tag{12.47}$$

where

$$F^*(z^*) = \sup_{z \in Y}\{\langle z_{ij}, z^*_{ij}\rangle_{L^2(S)} - F(z)\} = \frac{1}{2K}\int_S z^*_{ij}\hat{H}_{ijlm}z^*_{lm}dx,$$

and

$$\{\hat{H}_{ijlm}\} = \{H_{ijlm}\}^{-1}.$$

Also,

$$G^*(\epsilon^*(z^*)) = \sup_{u \in U}\{\langle \epsilon_{ij}(u), z^*_{ij}\rangle_{L^2(S)} - G(u)\}$$

and thus

$$G^*(\epsilon^*(z^*)) = \inf_{v^* \in A^*}\{\tilde{G}^*(z^*, v^*)\},$$

where

$$A^* = \{v^* \in Y^* \ \mid \ v^*_{ij,j} + f_i = 0, \ \ a.e. \ \ in \ \ S\},$$

$$\begin{aligned} \tilde{G}(v, z^*) \ &= \ -\langle v_{ij}, z^*_{ij}\rangle_{L^2(S)} + \frac{1}{2}\int_S \min_{k \in \{1,\ldots,N\}}\{(v_{ij} - \mathbf{e}^k_{ij})C^k_{ijlm}(v_{lm} - \mathbf{e}^k_{ij}) + \beta_k\}dx \\ &+ \ \frac{K}{2}\int_S v_{ij}H_{ijlm}v_{lm}dx, \end{aligned} \tag{12.48}$$

and

$$\tilde{G}^*(z^*, v^*) = \sup_{v \in Y}\{\langle v_{ij}, v^*_{ij}\rangle_{L^2(S)} - \tilde{G}(v, z)\}.$$

Therefore

$$\tilde{G}^*(v,z^*)=\int_S \max_{k\in\{1,...,N\}}\left\{\frac{1}{2}(v^*_{ij}+z^*_{ij})D^k_{ijlm}(v^*_{lm}+z^*_{lm})+(v^*_{ij}+z^*_{ij})\hat{C}^k_{ijlm}\mathbf{e}^k_{lm}-\beta_k\right\}dx,$$

where

$$\{D^k_{ijlm}\}=\{C^k_{ijlm}+KH_{ijlm}\}^{-1},$$

and

$$\{\hat{C}^k_{ijlm}\}=\{\{C^k_{ijlm}+KH_{ijlm}\}^{-1}C^k_{ijlm}\}.$$

Hence, the duality principle given by (12.47) may be expressed as

$$\begin{aligned}\inf_{u\in U}\{G^{**}(u)-F(\epsilon(u))\} &= \inf_{z^*\in Y^*}\left\{\frac{1}{2K}\int_S z^*_{ij}\hat{H}_{ijlm}z^*_{lm}dx\right.\\ &+ \sup_{v^*\in A^*}\left\{\int_S \min_{k\in\{1,...,N\}}\left\{-\frac{1}{2}(v^*_{ij}+z^*_{ij})D^k_{ijlm}(v^*_{lm}+z^*_{lm})\right.\right.\\ &- \left.\left.\left.(v^*_{ij}+z^*_{ij})\hat{C}^k_{ijlm}\mathbf{e}^k_{lm}+\beta_k\right\}dx\right\}\right\}\end{aligned}$$

Interchanging the infimum and supremum in the right side of last equality, we obtain

$$\begin{aligned}\inf_{u\in U}\{G^{**}(u)-F(\epsilon(u))\} &\geq \sup_{v^*\in A^*}\left\{\inf_{z^*\in Y^*}\left\{\frac{1}{2K}\int_S z^*_{ij}\hat{H}_{ijlm}z^*_{lm}dx\right.\right.\\ &+ \int_S \min_{k\in\{1,...,N\}}\left\{-\frac{1}{2}(v^*_{ij}+z^*_{ij})D^k_{ijlm}(v^*_{lm}+z^*_{lm})\right.\\ &- \left.\left.\left.(v^*_{ij}+z^*_{ij})\hat{C}^k_{ijlm}\mathbf{e}^k_{lm}+\beta_k\right\}dx\right\}\right\}\end{aligned}$$

so that

$$\begin{aligned}\inf_{u\in U}\{G^{**}(u)-F(\epsilon(u))\} &\geq \sup_{v^*\in A^*}\left\{\inf_{t\in B}\left\{\inf_{z^*\in Y^*}\left\{\frac{1}{2K}\int_S z^*_{ij}\hat{H}_{ijlm}z^*_{lm}dx\right.\right.\right.\\ &- \int_S \frac{t_k}{2}(v^*_{ij}+z^*_{ij})D^k_{ijlm}(v^*_{lm}+z^*_{lm})dx\\ &- \left.\left.\left.\int_S (v^*_{ij}+z^*_{ij})t_k\hat{C}^k_{ijlm}\mathbf{e}^k_{lm}dx+\int_S t_k\beta_k dx\right\}\right\}\right\}\end{aligned}$$

where

$$B=\{(t_1,...,t_N)\ \ measurable\ \ |\ \ t_k(x)\in[0,1]\ \ \forall k\in\{1,...,N\},\ \ \sum_{k=1}^N t_k(x)=1,\ \ a.e.\ \ in\ \ S\}.$$

Observe that the infimum in z^* is attained for functions satisfying

$$\frac{1}{K}\hat{H}_{ijlm}z^*_{lm}-\sum_{k=1}^N\{t_kD^k_{ijlm}(v^*_{lm}+z^*_{lm})\}-\sum_{k=1}^N t_k\hat{C}^k_{ijlm}\mathbf{e}^k_{lm}=0. \tag{12.49}$$

The final format of concerned duality principle is given by

$$\begin{aligned}\inf_{u\in U}\{G^{**}(u)-F(\epsilon(u))\} \geq & \sup_{v^*\in A^*}\left\{\inf_{t\in B}\left\{\frac{1}{2K}\int_S z^*_{ij}(v^*,t)\hat{H}_{ijlm}z^*_{lm}(v^*,t)dx\right.\right. \\ - & \int_S \frac{t_k}{2}(v^*_{ij}+z^*_{ij}(v^*,t))D^k_{ijlm}(v^*_{lm}+z^*_{lm}(v^*,t))dx \\ - & \left.\left.\int_S (v^*_{ij}+z^*_{ij}(v^*,t))t_k\hat{C}^k_{ijlm}\mathbf{e}^k_{lm}dx+\int_S t_k\beta_k dx\right\}\right\}\end{aligned}$$

where,

$$A^*=\{v^*\in Y^* \mid v^*_{ij,j}+f_i=0, \quad a.e. \ in \ S\},$$

$$B=\{(t_1,...,t_N) \ measurable \mid t_k(x)\in[0,1] \ \forall k\in\{1,...,N\}, \ \sum_{k=1}^N t_k(x)=1, \quad a.e. \ in \ S\}.$$

Finally, $z^*(v^*,t)$ is obtained through equation (12.49).

Remark 12.7.1. *The final dual formulation is concave in v^* (as the infimum of concave functionals) and, if K is big enough (we may choose H as the identity matrix), so that for a minimizing sequence $\{u_n\}$ we have $G^{**}(u_n)=G(u_n)$ (as $n\to+\infty$), the duality gap is zero (we have not proved it in the present work) and the last inequality is in fact an equality.*

12.8 A Numerical Example

In this section we present numerical results for the one-dimensional example (originally due to Bolza, see P.Pedregal [25]).

Consider $J: U\to\mathbb{R}$ expressed as

$$J(u)=\frac{1}{2}\int_0^1((u_{,x})^2-1)^2dx+\frac{1}{2}\int_0^1(u-f)^2dx$$

or, defining $S=[0,1]$,

$$G(\Lambda u)=\frac{1}{2}\int_0^1((u_{,x})^2-1)^2dx$$

and

$$F(u)=\frac{1}{2}\int_0^1(u-f)^2dx$$

we may write

$$J(u)=G(\Lambda u)+F(u)$$

where, for convenience we define, $\Lambda: U\to Y\equiv L^4(S)\times L^2(S)$ as

$$\Lambda u=\{u_{,x},0\}.$$

Furthermore, we have

$$U = \{u \in W^{1,4}(S) \mid u(0) = 0 \;\; and \;\; u(1) = 0.5\}$$

For $Y = Y^* = L^4(S) \times L^2(S)$, defining

$$G(\Lambda u + p) = \frac{1}{2}\int_S ((u_{,x} + p_1)^2 - 1.0 + p_0)^2 dx$$

for $v_0^* > 0$ we obtain

$$G(\Lambda u) + F(u) \geq \inf_{p \in Y}\{-\langle p_0, v_0^*\rangle_{L^2(S)} - \langle p_1, v_1^*\rangle_{L^2(S)} + G(\Lambda u + p) + F(u)\}$$

or

$$G(\Lambda u) + F(u) \geq \inf_{p \in Y}\{-\langle q_0, v_0^*\rangle_{L^2(S)} - \langle q_1, v_1^*\rangle_{L^2(S)} + G(q) + \langle 0, v_0^*\rangle_{L^2(S)} + \langle u', v_1^*\rangle_{L^2(S)} + F(u)\}.$$

Here $q = \Lambda u + p$ so that

$$G(\Lambda u) + F(u) \geq -G^*(v^*) + \langle 0, v_0^*\rangle_{L^2(S)} + \langle u_{,x}, v_1^*\rangle_{L^2(S)} + F(u).$$

That is

$$G(\Lambda u) + F(u) \geq -G^*(v^*) + \inf_{u \in U}\{\langle 0, v_0^*\rangle_{L^2(S)} + \langle u_{,x}, v_1^*\rangle_{L^2(S)} + F(u)\},$$

or

$$\inf_{u \in U}\{G(\Lambda u) + F(u)\} \geq \sup_{v^* \in A^*}\{-G^*(v^*) - F^*(-\Lambda^* v^*)\}$$

where

$$G^*(v^*) = \frac{1}{2}\int_S \frac{(v_1^*)^2}{v_0^*} dx + \frac{1}{2}\int_S (v_0^*)^2 dx,$$

if $v_0^* > 0, \;\; a.e. \;\; in \;\; S$. Also

$$F^*(-\Lambda^* v^*) = \frac{1}{2}\int_S [(v_1^*)_{,x}]^2 dx + \langle f, (v_1^*)_{,x}\rangle_{L^2(S)} - v_1^*(1)u(1)$$

and

$$A^* = \{v^* \in Y^* \mid v_0^* > 0, \;\; a.e. \;\; in \;\; S\}.$$

Remark 12.8.1. *Through the extremal condition $v_0^* = ((u_{,x})^2 - 1)$ and Weierstrass condition $(u_{,x})^2 - 1.0 \geq 0$ we can see that the dual formulation is convex for $v_0^* > 0$, however it is possible that the primal formulation has no minimizers, and we could expect a microstructure formation through $v_0^* = 0$ (that is, $u_{,x} = \pm 1$, depending on f(x)). To allow $v_0^* = 0$ we will redefine the primal functional as below indicated.*

Define $G_1 : U \to \mathbb{R}$ and $F_1 : U \to \mathbb{R}$ by

$$G_1(u) = G(\Lambda u) + F(u) + \frac{K}{2}\int_S (u_{,x})^2 dx$$

and

$$F_1(u) = \frac{K}{2}\int_S (u_{,x})^2 dx.$$

Also defining $\hat{G}(\Lambda u) = G(\Lambda u) + \frac{K}{2}\int_S (u_{,x})^2 dx$, from Theorem 12.3.1 we can write

$$\inf_{u \in U}\{J(u)\} \leq \inf_{z^* \in Y^*} \sup_{v^* \in B^*(z^*)} \{F_1^*(z^*) - \hat{G}^*(v_0^*, v_2^*) - F^*(v_1^*)\} \tag{12.50}$$

where

$$F_1^*(z^*) = \frac{1}{2K}\int_S (z^*)^2 dx,$$

$$\hat{G}^*(v_0^*, v_2^*) = \frac{1}{2}\int_S \frac{(v_2^*)^2}{v_0^* + K} dx + \frac{1}{2}\int_S (v_0^*)^2 dx,$$

$$F^*(v_1^*) = \frac{1}{2}\int_S (v_1^*)^2 dx + \langle f, v_1^* \rangle_{L^2(S)} - v_2^*(1)u(1)$$

and

$$B^*(z^*) = \{v^* \in Y^* \mid -(v_2^*)_{,x} + v_1^* - z^* = 0 \;\; and \;\; v_0^* \geq 0 \;\; a.e. \;\; in \;\; S\}.$$

We developed an algorithm based on the dual formulation indicated in (12.50). It is relevant to emphasize that such a dual formulation is convex for $v_0^* \geq 0$ (this results follows from the traditional Weierstrass condition, so that there is no duality gap between the primal and dual formulations and the inequality indicated in (12.50) is in fact an equality).

We present numerical results for $f(x) = 0$ (figure 12.1), $f(x) = 0.3 * Sin(\pi * x)$ (figure 12.2) and $f(x) = 0.3 * Cos(\pi * x)$ (figure 12.3). The solutions indicated as optima through the dual formulations (denoted by u_0), are in fact weak cluster points of minimizing sequences for the primal formulations.

12.9 Conclusion

In this chapter we developed dual variational formulations for the Ginzburg-Landau equations. Also we present a study about the Multi-Well problem, introducing duality as an efficient tool to tackle the problem. It is fascinating how the standard results of Convex Analysis can be used to clarify the understanding of mixture of the phases, as illustrated in Section 12.5.3.

In our view, the importance of duality for theoretical and numerical analysis of the Multi-Well and related phase transition problems seems to have been clarified.

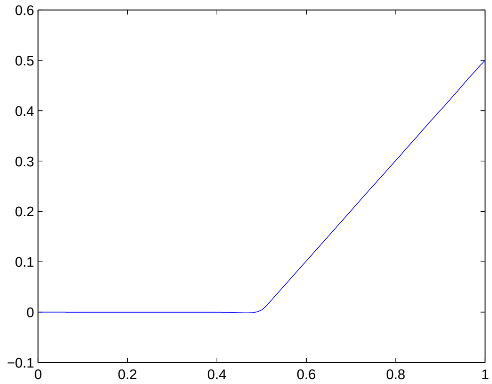

Figure 12.1: Vertical axis: $u_0(x)$-weak limit of minimizing sequences for f(x)=0

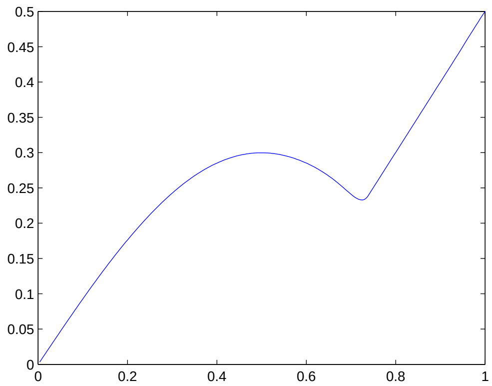

Figure 12.2: Vertical axis: $u_0(x)$-weak limit of minimizing sequences for $f(x) = 0.3 * Sin(\pi * x)$

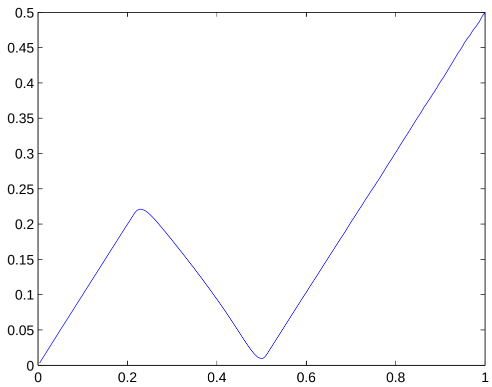

Figure 12.3: Vertical axis: $u_0(x)$-weak limit of minimizing sequences for $f(x) = 0.3 * Cos(\pi * x)$

Chapter 13

Duality Applied to Conductivity in Composites

13.1 Introduction

For the primal formulation we repeat the statements found in reference [15] (U.Fidalgo, P.Pedregal). Consider a material confined into a bounded domain $\Omega \subset \mathbb{R}^N$, $N > 1$. The medium is obtained by mixing two constituents with different electric permitivity and conductivity. Let Q_0 and Q_1 denote the two $N \times N$ symmetric matrices of electric permitivity corresponding to each phase. For each phase, we also denote by L_j, $j = 0, 1$, the anisotropic $N \times N$ symmetric matrix of conductivity. Let $0 \leq t_1 \leq 1$ be the proportion of the constituent 1 into the mixture. Constituent 1 occupies a space in the physical domain Ω which we denote by $E \subset \Omega$. Regarding the set E as our design variable, we introduce the characteristic function $\chi : \Omega \to \{0, 1\}$:

$$\chi(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{otherwise}, \end{cases} \tag{13.1}$$

Thus,

$$\int_E dx = \int_\Omega \chi(x) dx = t_1 \int_\Omega dx = t_1 |\Omega|. \tag{13.2}$$

The matrix of conductivity corresponding to the material as a whole is $L = \chi L_1 + (1 - \chi) L_0$.

Finally, the electrostatic potential, denoted by $u : \Omega \to \mathbb{R}$ is supposed to satisfy the equation

$$div[\chi L_1 \nabla u + (1 - \chi) L_0 \nabla u] = P(x), \quad in \quad \Omega, \tag{13.3}$$

with the boundary conditions

$$u = u_0, \quad on \quad \partial\Omega \tag{13.4}$$

where $P : \Omega \to \mathbb{R}$ is a given source or sink of current (we assume $P \in L^2(\Omega)$).

13.2 The Primal Formulation

Now and on we assume $N = 3$. Consider the problem of minimizing the cost functional,

$$I(\chi, u) = \int_\Omega \frac{\chi}{2}(\nabla u)^T Q_1 \nabla u + \frac{(1-\chi)}{2}(\nabla u)^T Q_0 \nabla u dx \tag{13.5}$$

subject to

$$div[\chi L_1 \nabla u + (1-\chi) L_0 \nabla u] = P(x) \tag{13.6}$$

where $u \in U$, here $U = \{u \in W^{1,2}(\Omega) \ \mid \ u = u_0 \ \ on \ \ \partial\Omega\}$.

We will rewrite this problem as the minimization of $J : U \times Y \to \mathbb{R}$, where $Y = L^2(S; \mathbb{R}^3)$,

$$J(u, f) = \inf_{t \in B} \int_\Omega (\{\frac{t}{2}(\nabla u)^T Q_1 \nabla u + \frac{(1-t)}{2}(\nabla u)^T Q_0 \nabla u) + Ind_1(u, f)\} dx + Ind_2(u, f),$$

$$Ind_1(\nabla u, f) = \begin{cases} 0, & if \ \ (tL_1 + (1-t)L_0)\nabla u - f = 0, \\ +\infty, & otherwise, \end{cases}$$

and

$$Ind_2(u, f) = \begin{cases} 0, & if \ \ div(f) = P \ \ a.e. \ \ in \ \ \Omega, \\ +\infty, & otherwise. \end{cases}$$

Here

$$B = \{t \ \ measurable \ \mid \ t(x) \in \{0, 1\}, \ \ a.e. \ \ in \ \ \Omega, \ \ \int_\Omega t(x) dx = t_1 |\Omega|\}.$$

13.3 The Duality Principle

Observe that we may write

$$J(u, f) = \inf_{t \in B} \{G(\Lambda u, f, t) + F(u, f)\},$$

where $\Lambda : U \to Y$ is given by

$$\Lambda u = \nabla u,$$

$$G(\Lambda u, f, t) = \int_\Omega \{(\frac{t}{2}(\nabla u)^T Q_1 \nabla u + \frac{(1-t)}{2}(\nabla u)^T Q_0 \nabla u) + Ind_1(\Lambda u, f)\} dx,$$

and

$$F(u, f) = Ind_2(u, f).$$

Also, we have that

$$\inf_{(u,f) \in U \times Y} \{J(u, f)\} = \inf_{t \in B} \inf_{(u,f) \in U \times Y} \{G(\Lambda u, f, t) + F(u, f)\}.$$

However, from Theorem 7.2.5 we obtain

$$\inf_{(u,f)\in U\times Y}\{G(u,f,t)+F(u,f)\}=\sup_{(v^*,f^*)\in Y^*\times Y^*}\{-G^*(v^*,f^*,t)-F^*(-\Lambda^*v^*,-f^*)\}.$$

Thus,

$$G^*(v^*,f^*,t)=\sup_{(v,f)\in Y\times Y}\{\langle v,v^*\rangle_{L^2(\Omega;\mathbb{R}^2)}+\langle f,f^*\rangle_{L^2(\Omega;\mathbb{R}^2)}-G(v,f,t)\}$$

or

$$\begin{aligned}G^*(v^*,f^*,t) &= \sup_{(v,f)\in Y\times Y}\{\langle v,v^*\rangle_{L^2(\Omega;\mathbb{R}^2)}+\langle f,f^*\rangle_{L^2(\Omega;\mathbb{R}^2)}\\ &- \int_\Omega\{(\frac{t}{2}(v)^TQ_1v+\frac{(1-t)}{2}(v)^TQ_0v)+Ind_1(v,f)\}dx\}\end{aligned} \tag{13.7}$$

so that

$$\begin{aligned}G^*(v^*,f^*,t) &= \sup_{(v,f)\in Y\times Y}\{\langle v,v^*\rangle_{L^2(\Omega;\mathbb{R}^2)}+\langle (tL_1+(1-t)L_0)v,f^*\rangle_{L^2(\Omega;\mathbb{R}^2)}\\ &- \int_\Omega\{(t\frac{1}{2}(v)^TQ_1v+(1-t)\frac{1}{2}(v)^TQ_0v)\}dx\}\end{aligned} \tag{13.8}$$

or

$$G^*(v^*,f^*,t)=$$

$$=\frac{1}{2}\int_\Omega\{(v^*+(t(L_1)+(1-t)L_0)^Tf^*)(tQ_1+(1-t)Q_0)^{-1}(v^*+(t(L_1)+(1-t)L_0)^Tf^*)\}dx.$$

On the other hand

$$F^*(-\Lambda^*v^*,-f^*)=\sup_{(u,f)\in U\times Y}\{-\langle\nabla u,v^*\rangle_{L^2(\Omega;\mathbb{R}^2)}-\langle f,f^*\rangle_{L^2(\Omega;\mathbb{R}^2)}-F(u,f)\},$$

or

$$F^*(-\Lambda^*v^*,-f^*)=\sup_{(u,f)\in U\times Y}\{-\langle\nabla u,v^*\rangle_{L^2(\Omega;\mathbb{R}^2)}-\langle f,f^*\rangle_{L^2(\Omega;\mathbb{R}^2)}-Ind_2(u,f)\}.$$

That is

$$F^*(-\Lambda^*v^*,-f^*)=\sup_{(u,f)\in U\times Y}\{-\langle\nabla u,v^*\rangle_{L^2(\Omega;\mathbb{R}^2)}-\langle f,f^*\rangle_{L^2(\Omega;\mathbb{R}^2)}-\langle\lambda,div(f)-P\rangle_{L^2(\Omega)},$$

where λ is an appropriate Lagrange Multiplier. Hence we have

$$F^*(-\Lambda^*v^*,-f^*)=\begin{cases}-\langle\lambda,P\rangle_{L^2(\Omega)}+\langle u_0,v^*.n\rangle_{L^2(\partial\Omega)}, & if\ \ (v^*,f^*)\in B^*,\\ +\infty,\ \text{otherwise},\end{cases}$$

where

$$B^*=\{(v^*,f^*)\in Y^*\times Y^*\ \mid\ div(v^*)=0,\quad f^*=\nabla\lambda,\quad a.e.\ \ in\ \ \Omega,\quad \lambda=0\ \ on\ \ \partial\Omega\}.$$

Therefore, we may summarize the last results by the following duality principle,

$$\inf_{(u,f)\in U\times Y}\{J(u,f)\} =$$

$$\inf_{t\in B}\{\sup_{(v^*,\lambda)\in C^*}\{-\frac{1}{2}\int_\Omega\{(v^*+(t(L_1)+(1-t)L_0)^T\nabla\lambda)\tilde{Q}(t)(v^*+(t(L_1)+(1-t)L_0)^T\nabla\lambda)\}dx+$$

$$\langle\lambda, P\rangle_{L^2(\Omega)} - \langle u_0, v^*.n\rangle_{L^2(\partial\Omega)}\}\},$$

where

$$\tilde{Q}(t) = (tQ_1 + (1-t)Q_0)^{-1},$$

$$C^* = \{(v^*,\lambda)\in Y^*\times U \ \mid \ div(v^*) = 0, \ \ a.e. \ \ in \ \ \Omega, \ \ \lambda = 0 \ \ on \ \ \partial\Omega\},$$

and

$$B = \{t \ \ measurable \ \mid \ t(x)\in\{0,1\}, \ \ a.e. \ \ in \ \ \Omega, \ \ \int_\Omega t(x)dx = t_1|\Omega|\}. \ \ \Box$$

13.4 Conclusion

In this chapter we developed duality for a two-phase non-convex variational problem in conductivity. As we may not have minimizers for this kind of problem, a possible solution of the dual variational formulation reflects the average behavior of minimizing sequences, as a weak cluster point of such (minimizing) sequences. Finally, it seems that the solution of dual problem is not difficult to compute.

Chapter 14

Duality Applied to the Optimal Design in Elasticity

14.1 Optimal Design of a Plate

The first objective of the present chapter is the establishment of a dual variational formulation for the optimal design, concerning the minimization of internal work, of a plate of variable thickness. Such a thickness is denoted by $h(x)$ and allowed to assume the values between a minimum h_0 and maximum h_1. The total plate volume, assume fixed, is a design constraint denoted by $\bar{V}$.

Consider a plate which the middle surface is denoted by $S \subset \mathbb{R}^2$, where S is an open bounded connected set with a sufficiently regular boundary denoted by Γ. The plate thickness is assumed to be the design variable and, as mentioned above, is denoted by $h(x)$, where $x = (x_1, x_2) \in S \subset \mathbb{R}^2$ and $h_0 \leq h(x) \leq h_1$. The field of normal displacements to S, due to a external load $P \in L^2(S)$, is denoted by $w : S \to \mathbb{R}$.

The optimization problem is the minimization of $J : U \to \mathbb{R}$, where

$$J(w) = \inf_{t \in C} \int_S \left\{ \frac{H_{\alpha\beta\lambda\mu}(t)}{2} w_{,\alpha\beta} w_{,\lambda\mu} \right\} dx, \tag{14.1}$$

subject to

$$(H_{\alpha\beta\lambda\mu}(t) w_{,\lambda\mu})_{,\alpha\beta} = P, \quad in \quad S \tag{14.2}$$

and

$$\int_S (th_1 + (1-t)h_0) dS = t_1 h_1 |S| = \bar{V}, \tag{14.3}$$

where

$$C = \{t \;\; measurable \;\; t(x) \in [0,1], \;\; a.e. \;\; in \;\; S\},$$

$0 < t_1 < 1$ and $|S|$ denotes the Lebesgue measure of S and

$$U = W_0^{2,2}(S) = \{w \in W^{2,2}(S) \;\mid\; w = \frac{\partial w}{\partial \mathbf{n}} = 0 \;\; on \;\; \Gamma\}. \tag{14.4}$$

Finally,

$$H_{\alpha\beta\lambda\mu}(t) = (th_1 + (1-t)h_0)^3 \mathcal{A}_{\alpha\beta\lambda\mu} \tag{14.5}$$

where $h(x) = t(x)h_1 + (1 - t(x))h_0$ represents the plate thickness and $\{\mathcal{A}_{\alpha\beta\lambda\mu}\}$ is a positive definite matrix related to Hooke's Law. Observe that $0 \leq t(x) \leq 1, \;\; a.e. \;\; in \;\; S.$

14.1.1 The First Duality Principle

Now we rewrite the primal formulation, so that we express $J : U \times Y \to \bar{\mathbb{R}}$, as

$$J(w,f) = \inf_{t \in B}\left\{\int_S \left\{\frac{H_{\alpha\beta\lambda\mu}(t)}{2} w_{,\alpha\beta}w_{,\lambda\mu} + Ind_1(\Lambda w, f)\right\} dx\right\} + Ind_2(w,f), \tag{14.6}$$

where,

$$Ind_1(\Lambda w, f) = \begin{cases} 0, & if \;\; f_{\alpha\beta} = H_{\alpha\beta\lambda\mu}(t)w_{,\lambda\mu}, \\ +\infty, & otherwise, \end{cases} \tag{14.7}$$

$$Ind_2(w, f) = \begin{cases} 0, & if \;\; f_{\alpha\beta,\alpha\beta} = P, \;\; a.e \;\; in \;\; S, \\ +\infty, & otherwise, \end{cases} \tag{14.8}$$

$\Lambda : U \to Y$ is given by

$$\Lambda w = \{w_{,\alpha\beta}\},$$

and

$$B = \{t \text{ measurable} \mid t(x) \in [0,1] \; a.e. \; in \; S, \; \int_S (th_1 + (1-t)h_0)dS = t_1 h_1 |S| = \bar{V}\}, \tag{14.9}$$

and also $Y = L^2(S;\mathbb{R}^4)$. Observe that we may write

$$\inf_{(w,f) \in U \times Y}\{J(w,f)\} = \inf_{t \in B}\{\inf_{(w,f) \in U \times Y}\{G(\Lambda w, f, t) + F(w,f)\}\}, \tag{14.10}$$

where

$$G(\Lambda w, f, t) = \int_S \frac{H_{\alpha\beta\lambda\mu}(t)}{2} w_{,\alpha\beta}w_{,\lambda\mu} + Ind_1(\Lambda w, f)\}dx,$$

and

$$F(w,f) = Ind_2(w,f).$$

From Theorem 7.2.5, we may write

$$\inf_{(w,f) \in U \times Y}\{G(\Lambda w, f, t) + F(w,f)\} = \sup_{(v^*,f^*) \in Y^* \times Y^*}\{-G^*(v^*, f^*, t) - F^*(-\Lambda^* v^*, -f^*)\}, \tag{14.11}$$

where

$$G^*(v^*, f^*, t) = \sup_{(v,f)\in Y\times Y}\{\langle v_{\alpha\beta}, v^*_{\alpha\beta}\rangle_{L^2(S)} + \langle f_{\alpha\beta}, f^*_{\alpha\beta}\rangle_{L^2(S)} - G(v,f,t)\},$$

or

$$\begin{aligned} G^*(v^*, f^*, t) &= \sup_{(v,f)\in Y\times Y}\{\langle v_{\alpha\beta}, v^*_{\alpha\beta}\rangle_{L^2(S)} + \langle f_{\alpha\beta}, f^*_{\alpha\beta}\rangle_{L^2(S)} \\ &- \int_S\{\frac{H_{\alpha\beta\lambda\mu}(t)}{2}v_{\alpha\beta}v_{\lambda\mu} + Ind_1(v,f)\}dx\}. \end{aligned} \tag{14.12}$$

Thus,

$$G^*(v^*, f^*, t) = \sup_{(v,f)\in Y\times Y}\left\{\langle v_{\alpha\beta}, v^*_{\alpha\beta}\rangle_{L^2(S)} + \langle H_{\alpha\beta\lambda\mu}(t)v_{\lambda\mu}, f^*_{\alpha\beta}\rangle_{L^2(S)} - \int_S \frac{H_{\alpha\beta\lambda\mu}(t)}{2}v_{\alpha\beta}v_{\lambda\mu}dx\right\},$$

and we may write

$$G^*(v^*, f^*, t) = \frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}(t)f^*_{\alpha\beta}f^*_{\lambda\mu}dS + \langle v^*_{\alpha\beta}, f^*_{\alpha\beta}\rangle_{L^2(S)} + \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}(t)v^*_{\alpha\beta}v^*_{\lambda\mu}dS\}.$$

where

$$\{\bar{H}_{\alpha\beta\lambda\mu}(t)\} = \{H_{\alpha\beta\lambda\mu}(t)\}^{-1}.$$

On the other hand

$$F^*(-\Lambda^*v^*, -f^*) = \sup_{(w,f)\in U\times Y}\{-\langle w_{,\alpha\beta}, v^*_{\alpha\beta}\rangle_{L^2(S)} - \langle f_{\alpha\beta}, f^*_{\alpha\beta}\rangle_{L^2(S)} - F(w,f)\},$$

or

$$F^*(-\Lambda^*v^*, -f^*) = \sup_{(w,f)\in U\times Y}\{-\langle w_{,\alpha\beta}, v^*_{\alpha\beta}\rangle_{L^2(S)} - \langle f_{\alpha\beta}, f^*_{\alpha\beta}\rangle_{L^2(S)} - Ind_2(w,f)\}.$$

That is,

$$F^*(-\Lambda^*v^*, -f^*) = \sup_{(w,f)\in U\times Y}\{-\langle w_{,\alpha\beta}, v^*_{\alpha\beta}\rangle_{L^2(S)} - \langle f_{\alpha\beta}, f^*_{\alpha\beta}\rangle_{L^2(S)} + \langle \hat{w}, f_{\alpha\beta,\alpha\beta} - P\rangle_{L^2(S)}\},$$

where $\hat{w}$ is an appropriate Lagrange Multiplier. Thus

$$F^*(-\Lambda^*v^*, -f^*) = \begin{cases} -\langle \hat{w}, P\rangle_{L^2(S)}, & if \quad (v^*, f^*) \in B^*, \\ +\infty, & otherwise, \end{cases} \tag{14.13}$$

where

$$B^* = \{(v^*, f^*) \in Y^* \times Y^* \mid f^*_{\alpha\beta} = \hat{w}_{\alpha\beta}, \quad v^*_{\alpha\beta,\alpha\beta} = 0, \quad a.e. \ \ in \ \ S\}.$$

Hence, the duality principle indicated in (14.11), may be expressed as

$$\inf_{(w,f)\in U\times Y}\{G(\Lambda w, f, t) + F(w,f)\} =$$

$$\sup_{(v^*,\hat{w})\in B^*}\{-\frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}(t)\hat{w}_{\alpha\beta}\hat{w}_{\lambda\mu}dS - \langle v^*_{\alpha\beta},\hat{w}_{\alpha\beta}\rangle_{L^2(S)} - \frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}(t)v^*_{\alpha\beta}v^*_{\lambda\mu}dS + \langle \hat{w},P\rangle_{L^2(S)}\},$$

We may evaluate the last supremum and obtain $v^* = \theta$. Therefore,

$$\inf_{(w,f) in U\times Y}\{G(\Lambda w,f,t)+F(w,f)\} = \sup_{\hat{w}\in U}\left\{-\frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}(t)\hat{w}_{\alpha\beta}\hat{w}_{\lambda\mu}dS + \langle \hat{w},P\rangle_{L^2(S)}\right\},$$

However, from Theorem 7.2.5, we may conclude that

$$\sup_{\hat{w}\in U}\left\{-\frac{1}{2}\int_S H_{\alpha\beta\lambda\mu}(t)\hat{w}_{\alpha\beta}\hat{w}_{\lambda\mu}dS + \langle \hat{w},P\rangle_{L^2(S)}\right\} = \inf_{\{M_{\alpha\beta}\}\in D^*}\left\{\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}(t)M_{\alpha\beta}M_{\lambda\mu}dS\right\},$$

where

$$D^* = \{\{M_{\alpha\beta}\}\in Y^* \mid M_{\alpha\beta,\alpha\beta}+P=0, \quad a.e. \ \ in \ \ S\}.$$

And thus, the final format of the duality principle would be

$$\inf_{(w,f)\in U\times Y}\{J(w,f)\} = \inf_{(t,\{M_{\alpha\beta}\})\in B\times D^*}\left\{\frac{1}{2}\int_S \bar{H}_{\alpha\beta\lambda\mu}(t)M_{\alpha\beta}M_{\lambda\mu}dS\right\}. \quad \square$$

14.2 Optimal Design in Three-Dimensional Elasticity

In this section we develop duality for a two phase problem in elasticity. Consider $V\subset\mathbb{R}^3$, and open connected bounded set with a sufficiently regular boundary denoted by ∂V. Here V stands for the volume of a elastic solid under the action of a load $P\in L^2(V,\mathbb{R}^3)$. The field of displacements is denoted by $u=(u_1,u_2,u_3)\in U$ where

$$U=\{u\in W^{1,2}(V;\mathbb{R}^3) \mid u=(0,0,0) \ \ on \ \ \partial V\} = W^{1,2}_0(V;\mathbb{R}^3). \tag{14.14}$$

The strain tensor, denoted by $e=\{e_{ij}\}$, is defined as

$$e_{ij}(u)=\frac{1}{2}(u_{i,j}+u_{j,i}). \tag{14.15}$$

The solid V is composed by mixing two constituents, namely 1 and 0, with elasticity matrices related to Hooke's Law denoted by H^1_{ijkl} and H^0_{ijkl}, respectively. The part occupied by constituent 1 is denoted by E and represented by the characteristic function $\chi: V\to\{0,1\}$ where

$$\chi(x)=\begin{cases}1, & if \ \ x\in E,\\ 0, & otherwise,\end{cases}$$

Now we define the optimization problem of minimizing $J(u,\chi)$ where

$$J(u,\chi)=\frac{1}{2}\int_V(\chi H^1_{ijkl}e_{ij}e_{kl}+(1-\chi)H^0_{ijkl}e_{ij}e_{kl})dV, \tag{14.16}$$

subject to

$$(\chi H^1_{ijkl} e_{kl} + (1-\chi) H^0_{ijkl} e_{kl})_{,j} + P_i = 0, \quad in \quad V, \tag{14.17}$$

$u \in U$ and

$$\int_V \mathcal{X} dV \leq t_1 |V|, \tag{14.18}$$

where $0 < t_1 < 1$ and $|V|$ denotes the Lebesgue measure of V.

We rewrite the primal formulation, now denoting it by $J : U \times Y \to \mathbb{R}$ as

$$J(u, f) = \inf_{t \in B} \int_V \left\{ \frac{H_{ijkl}(t)}{2} e_{ij} e_{kl} + Ind_1(\{e_{ij}(u)\}, f) \right\} dV + Ind_2(u, f), \tag{14.19}$$

where

$$H_{ijkl}(t) = t H^1_{ijkl} + (1-t) H^0_{ijkl},$$

$$Ind_1(\{e_{ij}(u)\}, f) = \begin{cases} 0, & if \quad f_{ij} = t H^1_{ijkl} e_{kl} + (1-t) H^0_{ijkl} e_{kl}, \\ +\infty, & otherwise, \end{cases}$$

$$Ind_2(u, f) = \begin{cases} 0, & if \quad f_{ij,j} + P_i = 0, \quad a.e \quad in \quad S, \\ +\infty, & otherwise, \end{cases}$$

$$B = \{ t \ \ measurable \ \mid \ t(x) \in \{0, 1\}, \ \ a.e. \ \ in \ \ V, \ \ \int_V t(x) dx \leq t_1 |V| \},$$

and also $Y = L^2(V; \mathbb{R}^9)$.

By analogy to last section, we may obtain

$$\inf_{(u,f) \in U \times Y} \{ J(u, f) \} = \inf_{(t,\sigma) \in B \times B^*} \left\{ \frac{1}{2} \int_V \bar{H}_{ijkl}(t) \sigma_{ij} \sigma_{kl} dx \right\},$$

where

$$\{\bar{H}_{ijkl}(t)\} = \{H_{ijkl}(t)\}^{-1},$$

$$B^* = \{ \sigma \in Y^* \ \mid \ \sigma_{ij,j} + P_i = 0, \quad a.e. \quad in \quad V \},$$

and

$$B = \{ t \ \ measurable \ \mid \ t(x) \in \{0, 1\}, \ \ a.e. \ \ in \ \ V, \ \ \int_V t(x) dx \leq t_1 |V| \}.$$

14.3 A Numerical Example

Consider a plate which the middle surface is represented by $S = [0, 1] \times [0, 1]$, over which is applied the distributed vertical external load $P = 2000$. The plate is supposed to be a composite of materials 1 and 2, with stiffness coefficients $E_1 = 20000$, $E_2 = 5000$, respectively

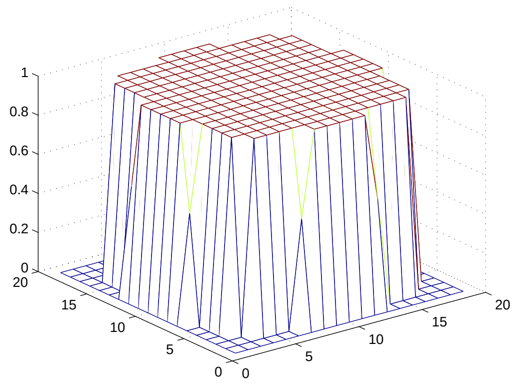

Figure 14.1: Vertical axis: solution $t(x, y)$ for the dual problem with $\int_S t dS \leq 0.68$

(unities related to international system). We present numerical results for the problem of obtaining the optimal mixture between these two constituents , in order to minimize $J : U \to \mathbb{R}$, the internal work produced by the displacement field $w : S \to \mathbb{R}$, where

$$J(w) = \inf_{t \in B} \left\{ \frac{1}{2} \int_S (tE_1 + (1-t)E_2)(\nabla^2 w)^2 dS \right\}, \tag{14.20}$$

subject to

$$\nabla^2((tE_1 + (1-t)E_2)\nabla^2 w) = P \tag{14.21}$$

and

$$\int_S t dS \leq 0.68, \tag{14.22}$$

where

$$U = \{w \in W^{2,2}(S) \ \mid \ w = 0 \ \ on \ \ \partial S\}. \tag{14.23}$$

We compute the optimal composite through the dual problem, as above, given by,

$$\inf_{t \in B} \left\{ \sup_{\hat{w} \in U} \left\{ -\frac{1}{2} \int_S (tE_1 + (1-t)E_2)(\nabla^2 \hat{w})^2 dS + \langle \hat{w}, P \rangle_{L^2(S)} \right\} \right\}, \tag{14.24}$$

where

$$B = \{t \ \ measurable \ \mid \ t(x) \in [0,1] \ \ a.e. \ \ in \ \ S, \ \int_S t dS \leq 0.68\}. \tag{14.25}$$

See figure 14.1 for the results for $t(x, y)$, which expresses the proportion of constituent of stiffness E_1. The field of displacements, denoted by $\hat{w}_0$ is indicated in figure 14.2.

For $\int_S t dS \leq 0.60$, for the same problem see figure 14.3.

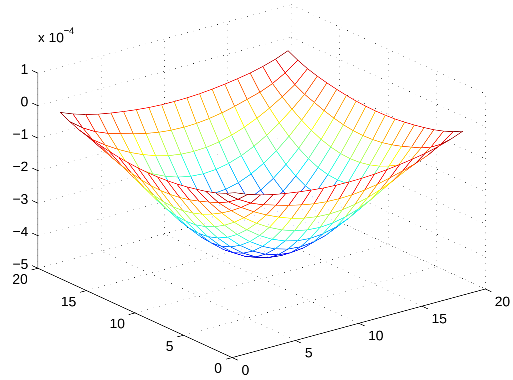

Figure 14.2: Vertical axis: Field of displacements $\hat{w}_0(x, y)$ (in m) for the dual problem, with $\int_S t dS \leq 0.68$

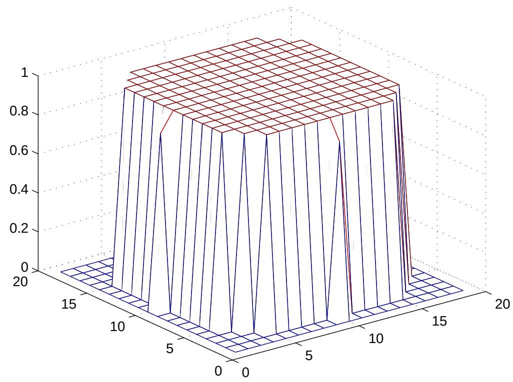

Figure 14.3: Vertical axis: solution $t(x, y)$ for the dual problem with $\int_S t dS \leq 0.60$

14.4 Conclusion

In this chapter we developed dual variational formulations for the optimal design of the variable thickness of a plate and for a two-phase problem in elasticity. The infima in t indicated in the dual formulations represent the structure search for stiffness in the optimization process, which implies the minimization of the internal work. In some cases, the primal problem may not have solutions, so that the solution of dual problem is a weak cluster point of minimizing sequences for the primal formulation. Finally, about the numerical results, we may see a clear preference of material with greater stiffness to concentrate in the central region of the plate (see figures 14.1 and 14.3), where we have the greatest deformations and moments (see figure 14.2).

Chapter 15

Duality Applied to Micro-Magnetism

15.1 Introduction

In this chapter we develop dual variational formulations for models in micro-magnetism. For the primal formulation we refer to P.Pedregal and B.Yan [26] for details.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a finite Lebesgue measure and a regular boundary denoted by $\partial\Omega$. Consider the model of micro-magnetism in which the magnetization $m : \Omega \to \mathbb{R}^3$, is given by the minimization of the functional

$$J(m, f) = \frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx + \int_\Omega \varphi(m(x))dx - \int_\Omega H(x).m dx + \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2 dz, \tag{15.1}$$

$$m \in W^{1,2}(\Omega; \mathbb{R}^3) \equiv Y_1, \quad |m(x)| = 1, \quad a.e. \ \ in \ \ \Omega \tag{15.2}$$

and $f \in L^2(\mathbb{R}^3; \mathbb{R}^3) \equiv Y_2$ is the unique field determined by the simplified Maxwell's equations

$$Curl(f) = 0, \quad div(-f + m\chi_\Omega) = 0, \text{ a.e. in } \mathbb{R}^3. \tag{15.3}$$

Here $H \in L^2(\Omega; \mathbb{R}^3)$ is a known external field and χ_Ω is a function defined by

$$\chi_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases} \tag{15.4}$$

The term

$$\frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx$$

is called the exchange energy. Finally, $\varphi(m)$ represents the anisotropic contribution and is given by a multi-well functional whose minima establish the prefered directions of magnetization.

15.2 The Primal formulations and the Duality Principles

15.2.1 Summary of Results for the Hard Uniaxial Case

We examine first the case of uniaxial material with no exchange energy. That is, $\alpha = 0$ and $\varphi(x) = \beta(1 - |m.e|)$.

Observe that

$$\varphi(m) = \min\{\beta(1 + m.e), \beta(1 - m.e)\}$$

where $\beta > 0$ and $e \in \mathbb{R}^3$ is a unit vector. Thus we can express the the functional $J : V \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$, (here $V \equiv Y_1 \times Y_2$), as

$$J(m, f) = G(m, f) + F(m)$$

where

$$G(m, f) = \int_\Omega \min\{g_1(m), g_2(m)\}dx + \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2 dz + Ind_0(m) + Ind_1(f) + Ind_2(m, f),$$

and

$$F(m) = -\int_\Omega H(x) m dx.$$

Here,

$$g_1(m) = \beta(1 + m.e),$$

$$g_2(m) = \beta(1 - m.e),$$

$$Ind_0(m) = \begin{cases} 0, & if \;\; |m(x)| = 1 \;\; a.e. \;\; in \;\; \Omega, \\ +\infty, & otherwise, \end{cases}$$

$$Ind_1(m, f) = \begin{cases} 0, & if \;\; div(-f + m\chi_\Omega) = 0 \;\; a.e. \;\; in \;\; \mathbb{R}^3, \\ +\infty, & otherwise, \end{cases}$$

and

$$Ind_2(f) = \begin{cases} 0, & if \;\; Curl(f) = 0, \;\; a.e. \;\; in \;\; \mathbb{R}^3, \\ +\infty, & otherwise. \end{cases}$$

The dual functional for such a variational formulation can be expressed by the following duality principle:

$$\begin{aligned} \inf_{(m,f) \in Y_1 \times Y_2} \{J(m, f)\} = & \sup_{(\lambda_1, \lambda_2) \in \hat{Y}^*} \left\{ \inf_{t \in B} \left\{ -\int_\Omega \left(\sum_{k=1}^3 \left(\frac{\partial \lambda_2}{\partial x_i} + H_i + \beta(1 - 2t)e_i\right)^2\right)^{1/2} dx \right.\right. \\ & \left.\left. -\frac{1}{2}\int_{\mathbb{R}^3} |Curl^* \lambda_1 + \nabla \lambda_2|^2 \, dx \right\} \right\} + \int_\Omega \beta \, dx \qquad (15.5) \end{aligned}$$

where

$$B = \{t \ \ measurable \ \mid \ t(x) \in [0,1], \ \ a.e. \ \ in \ \ \Omega\}.$$

and

$$\hat{Y}^* = \{(\lambda_1, \lambda_2) \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3) \ \mid \ \lambda_2 = 0 \ \ on \ \ \partial\Omega\}.$$

15.2.2 The Results for the Full Semi-linear Case

Now we present the duality principle for the full semi-linear case, that is, for $\alpha > 0$. First we define $G : Y_1 \times Y_2 \to \bar{\mathbb{R}}$ and $F : Y_1 \times Y_2 \to \bar{\mathbb{R}}$ as

$$\begin{aligned} G(m,f) &= \frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx + \int_\Omega \min\{g_1(m), g_2(m)\} dx + \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2 dz \\ &\quad + Ind_0(m) + Ind_1(m) + Ind_2(m), \end{aligned} \tag{15.6}$$

and

$$F(m,f) = -\int_\Omega H.m dx.$$

Also,

$$g_1(m) = \beta(1 + m.e),$$

$$g_2(m) = \beta(1 - m.e),$$

$$Ind_0(m) = \begin{cases} 0, & if \ \ |m(x)| = 1 \ \ a.e. \ \ in \ \ \Omega, \\ +\infty, & otherwise, \end{cases}$$

$$Ind_1(m,f) = \begin{cases} 0, & if \ \ div(-f + m\chi_\Omega) = 0 \ \ a.e. \ \ in \ \ \mathbb{R}^3, \\ +\infty, & otherwise, \end{cases}$$

and

$$Ind_2(f) = \begin{cases} 0, & if \ \ Curl(f) = 0, \ \ a.e. \ \ in \ \ \mathbb{R}^3, \\ +\infty, & otherwise. \end{cases}$$

For $J(m,f) = G(m,f) + F(m,f)$, the dual variational formulation is given by the following duality principle

$$\begin{aligned} \inf_{(m,f)\in Y_1\times Y_2}\{J(m,f)\} &= \sup_{(\lambda_1,\lambda_2,y^*)\in \hat{Y}^*\times Y_0^*} \left\{ \inf_{t\in B} \left\{ -\frac{1}{2\alpha}\int_\Omega |y^*|^2 dx \right.\right. \\ &\quad \left. - \int_\Omega \Big(\sum_{i=1}^3 (div(y_i^*) + H_i + (1-2t)\beta e_i + \frac{\partial \lambda_2}{\partial x_i})^2\Big)^{1/2} \, dx \right\} \\ &\quad \left. - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \lambda_2 + Curl^* \lambda_1|^2 dz \right\} + \int_\Omega \beta dx, \end{aligned} \tag{15.7}$$

where

$$B = \{t \ \ measurable \ \mid \ t(x) \in [0,1], \ \ a.e. \ \ in \ \ \Omega\},$$

$$Y_0^* = \{y^* \in W^{1,2}(\Omega; \mathbb{R}^{3\times 3}) \ \mid \ y_i^*.\mathbf{n} = 0 \ \ on \ \ \partial\Omega, \forall i \in \{1,2,3\}\},$$

and

$$\hat{Y^*} = \{(\lambda_1, \lambda_2) \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3) \ \mid \ \lambda_2 = 0 \ \ on \ \ \partial\Omega\}. \ \square$$

Remark 15.2.1. *It is important to emphasize that in both cases the dual formulations are concave. Thus the dual problems always have solutions, even when for the hard uniaxial case the minimizer in the primal problem is not attained. In this latter case the solution of dual problem reflects the average behavior of minimizing sequences, as a weak limit of such sequences.*

15.3 A Preliminary Result

Now we recall a simple but very useful result, through which we establish our duality principles.

Theorem 15.3.1. *Consider $(G \circ \Lambda) : V \to \mathbb{R}$ (not necessarily convex) such that $J : V \to \mathbb{R}$ defined as*

$$J(m) = G(\Lambda m) - \langle m, f \rangle_V, \forall m \in V,$$

is bounded from below (here as usual $\Lambda : U \to Y$ is a continuous linear operator). Under such assumptions, we have

$$\inf_{m \in V}\{J(m)\} = \sup_{y^* \in A^*}\{-(G \circ \Lambda)^*(\Lambda^* y^*)\}$$

where

$$A^* = \{y^* \in Y^* \ \mid \ \Lambda^* y^* - f = 0\}.$$

Remark 15.3.2. *What seems to be relevant is that, when computing $(G \circ \Lambda)^*(\Lambda^* y^*)$, we obtain a duality which is perfect concerning the convex envelop of the primal formulation.*

15.4 The Duality Principle for the Hard Case

We recall the primal formulation for the hard uniaxial case, expressed by $J(m, f)$ where

$$J(m, f) = G(m, f) + F(m),$$

$$G(m, f) = \int_\Omega \min\{g_1(m), g_2(m)\}dx + \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2 dz + Ind_0(m) + Ind_1(f) + Ind_2(m, f),$$

and

$$F(m,f) = -\int_\Omega H(x).m dx.$$

Also,

$$g_1(m) = \beta(1 + m.e),$$

$$g_2(m) = \beta(1 - m.e),$$

$$Ind_0(m) = \begin{cases} 0, & if \quad |m(x)| = 1 \quad a.e. \quad in \quad \Omega, \\ +\infty, & otherwise, \end{cases}$$

$$Ind_1(m,f) = \begin{cases} 0, & if \quad div(-f + m\chi_\Omega) = 0 \quad a.e. \quad in \quad \mathbb{R}^3, \\ +\infty, & otherwise, \end{cases}$$

and

$$Ind_2(f) = \begin{cases} 0, & if \quad Curl(f) = 0, \quad a.e. \quad in \quad \mathbb{R}^3, \\ +\infty, & otherwise. \end{cases}$$

From Theorem 15.3.1, we may write

$$\inf_{(m,f)\in Y_1\times Y_2}\{J(m,f)\} = \sup_{(m^*,f^*)\in Y_1^*\times Y_2^*}\{-G^*(m^*,f^*) - F^*(-m^*,-f^*))\}. \tag{15.8}$$

We now calculate the dual functionals. First we have that

$$G^*(m^*,f^*) = \sup_{(m,f)\in Y_1\times Y_2}\{\langle m,m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f,f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - G(m,f)\},$$

or

$$\begin{aligned} G^*(m^*,f^*) \quad = \quad & \sup_{(m,f)\in Y_1\times Y_2}\left\{\langle m,m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f,f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - \int_\Omega \min\{g_1(m),g_2(m)\}dx \right. \\ & \left. - \int_{\mathbb{R}^3}|f(z)|^2dz - Ind_0(m) - Ind_1(f) - Ind_2(m,f)\right\}. \end{aligned} \tag{15.9}$$

That is,

$$\begin{aligned} G^*(m^*,f^*) \quad = \quad & \sup_{(m,f)\in Y_1\times Y_2}\{\langle m,m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f,f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\ & - \inf_{t\in B}\int_\Omega (tg_1(m) + (1-t)tg_2(m))dx - \int_{\mathbb{R}^3}|f(z)|^2dz \\ & -Ind_0(m) - Ind_1(f) - Ind_2(m,f)\}, \end{aligned} \tag{15.10}$$

where

$$B = \{t \quad measurable \quad | \quad t(x)\in[0,1], \quad a.e. \quad in \quad \Omega\}.$$

Thus,

$$\begin{aligned}G^*(m^*,f^*) &= \sup_{(m,f,t)\in Y_1\times Y_2\times B}\{\langle m,m^*\rangle_{L^2(\Omega;\mathbb{R}^3)}+\langle f,f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\\ &\quad -\int_\Omega (tg_1(m)+(1-t)g_2(m))dx-\int_{\mathbb{R}^3}|f(z)|^2dz\\ &\quad -Ind_0(m)-Ind_1(f)-Ind_2(m,f)\},\end{aligned} \tag{15.11}$$

or,

$$\begin{aligned}G^*(m^*,f^*) &= \sup_{t\in B}\{\sup_{(m,f)\in Y_1\times Y_2}\{\langle m,m^*\rangle_{L^2(\Omega;\mathbb{R}^3)}+\langle f,f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\\ &\quad -\int_\Omega (tg_1(m)+(1-t)g_2(m))dx-\int_{\mathbb{R}^3}|f(z)|^2dz\\ &\quad -Ind_0(m)-Ind_1(f)-Ind_2(m,f)\}\}.\end{aligned} \tag{15.12}$$

Hence

$$\begin{aligned}G^*(m^*,f^*) &= \sup_{t\in B}\{\sup_{(m,f)\in Y_1\times Y_2}\{\langle m,m^*\rangle_{L^2(\Omega;\mathbb{R}^3)}+\langle f,f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\\ &\quad -\int_\Omega (tg_1(m)+(1-t)g_2(m))dx-\int_{\mathbb{R}^3}|f(z)|^2dz\\ &\quad -\int_\Omega \frac{\lambda}{2}(\sum_{i=1}^3 m_i^2-1)dx-\langle Curl(f),\lambda_1\rangle_{L^2(\mathbb{R}^3,\mathbb{R}^3)}\\ &\quad -\langle div(-f+m\chi_\Omega),\lambda_2\rangle_{L^2(\mathbb{R}^3)}\}\},\end{aligned}$$

where λ, λ_1 and λ_2 are appropriate Lagrange Multipliers concerning the respective constraints.

Therefore,

$$\begin{aligned}G^*(m^*,f^*) &= \sup_{t\in B}\{\sup_{(m,f)\in Y_1\times Y_2}\{\langle m,m^*\rangle_{L^2(\Omega;\mathbb{R}^3)}+\langle f,f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\\ &\quad -\int_\Omega \beta(t(1+m.e)+(1-t)(1-m.e))dx-\int_{\mathbb{R}^3}|f(z)|^2dz\\ &\quad -\int_\Omega \frac{\lambda}{2}(\sum_{i=1}^3 m_i^2-1)dx-\langle Curl(f),\lambda_1\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\\ &\quad -\langle div(-f+m\chi_\Omega),\lambda_2\rangle_{L^2(\mathbb{R}^3)}\}\}.\end{aligned}$$

The last indicated supremum is attained for functions satisfying the equations

$$m_i^*+\beta(1-2t)e_i-\lambda m_i+\frac{\partial\lambda_2}{\partial x_i}=0$$

or

$$m_i = \frac{m_i^* + \beta(1-2t)e_i + \frac{\partial \lambda_2}{\partial x_i}}{\lambda} = 0$$

and thus from the constraint

$$\sum_{i=1}^{3} m_i^2 - 1 = 0$$

we obtain

$$\lambda = (\sum_{i=1}^{3}(m_i^* + \beta(1-2t)e_i + \frac{\partial \lambda_2}{\partial x_i})^2)^{1/2}.$$

Also, the supremum in f is achieved for functions satisfying

$$f^* - f - Curl^*\lambda_1 - \nabla\lambda_2 = 0.$$

Observe that we need the condition $\lambda_2 = 0 \ \ on \ \ \partial\Omega$ to have a finite supremum, so that

$$\begin{aligned} G^*(m^*, f^*) &= \sup_{t \in B} \inf_{(\lambda_1,\lambda_2) \in \hat{Y}} \left\{ \int_\Omega (\sum_{i=1}^{3}(m_i^* + \beta(1-2t)e_i + \frac{\partial \lambda_2}{\partial x_i})^2)^{1/2} dx \right. \\ &\left. -\frac{1}{2}\int_{\mathbb{R}^3} |f^* - Curl^*\lambda_1 - \nabla\lambda_2|^2 dx \right\} - \int_\Omega \beta dx. \qquad (15.13) \end{aligned}$$

if $\lambda_2 = 0 \ \ on \ \ \partial\Omega$, $+\infty$ otherwise.

Furthermore

$$F^*(-m^*, -f^*) = \sup_{(m,f) \in Y_1 \times Y_2} \{\langle m, -m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, -f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - F(m,f)\},$$

$$F^*(-m^*, -f^*) = \sup_{(m,f) \in Y_1 \times Y_2} \{\langle m, -m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, -f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - \int_\Omega H(x) m dx\},$$

so that

$$F^*(-m^*, -f^*) = \begin{cases} 0, & if \ \ (m^*, f^*) \in B^*, \\ +\infty, & otherwise, \end{cases} \qquad (15.14)$$

where

$$B^*\{(m^*, f^*) \in Y_1^* \times Y_2^* \ \ | \ \ m^* = H \ \ a.e. \ \ in \ \ \Omega, \ \ \ f^* = \theta, \ \ a.e. \ \ in \ \ \mathbb{R}^3\}.$$

Therefore we may summarize the duality principle indicated in (15.8) as

$$\begin{aligned} \inf_{(m,f) \in Y_1 \times Y_2}\{J(m,f)\} &= \inf_{t \in B} \sup_{(\lambda_1,\lambda_2) \in \hat{Y}} \left\{ -\int_\Omega (\sum_{i=1}^{3}(H_i + \beta(1-2t)e_i + \frac{\partial \lambda_2}{\partial x_i})^2)^{1/2} dx \right. \\ &\left. -\frac{1}{2}\int_{\mathbb{R}^3} |Curl^*\lambda_1 + \nabla\lambda_2|^2 \, dx + \int_\Omega \beta dx \right\}, \qquad (15.15) \end{aligned}$$

where

$$B = \{t \ \ measurable \ \ | \ \ t(x) \in [0,1], \ \ a.e. \ \ in \ \ \Omega\},$$

and

$$\hat{Y}^* = \{(\lambda_1, \lambda_2) \in W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3) \ \ | \ \ \lambda_2 = 0 \ \ on \ \ \partial\Omega\}. \ \ \square$$

15.5 The Full Semi-linear Case

Now we present a study concerning duality for the full semi-linear case, that is, for $\alpha > 0$. First we recall the definition of $G : Y_1 \times Y_2 \to \bar{\mathbb{R}}$ and $F : Y_1 \times Y_2 \to \bar{\mathbb{R}}$, that is,

$$G(m, f) = \frac{\alpha}{2} \int_\Omega |\nabla m|^2 dx + \int_\Omega \min\{g_1(m), g_2(m)\} dx + \frac{1}{2} \int_{\mathbb{R}^3} |f(z)|^2 dz$$

$$+ Ind_0(m) + Ind_1(m) + Ind_2(m),$$

and

$$F(m, f) = -\int_\Omega H.m dx.$$

Also,

$$g_1(m) = \beta(1 + m.e),$$

$$g_2(m) = \beta(1 - m.e),$$

$$Ind_0(m) = \begin{cases} 0, & if \ \ |m(x)| = 1 \ \ a.e. \ \ in \ \ \Omega, \\ +\infty, & otherwise, \end{cases}$$

$$Ind_1(m, f) = \begin{cases} 0, & if \ \ div(-f + m\chi_\Omega) = 0 \ \ a.e. \ \ in \ \ \mathbb{R}^3, \\ +\infty, & otherwise, \end{cases}$$

and

$$Ind_2(f) = \begin{cases} 0, & if \ \ Curl(f) = 0, \ \ a.e. \ \ in \ \ \mathbb{R}^3, \\ +\infty, & otherwise. \end{cases}$$

From Theorem 15.3.1, we have

$$\inf_{(m,f) \in Y_1 \times Y_2} \{G(m, f) + F(m, f)\} = \sup_{(m^*, f^*) \in Y_1^* \times Y_2^*} \{-G^*(m^*, f^*) - F^*(-m^*, -f^*)\}, \quad (15.16)$$

where

$$G^*(m^*, f^*) = \sup_{(m,f) \in Y_1 \times Y_2} \{\langle m, m^* \rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, f^* \rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - G(m, f)\},$$

and

$$F^*(-m^*, -f^*) = \sup_{(m,f) \in Y_1 \times Y_2} \{\langle m, -m^* \rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, -f^* \rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - F(m, f)\}.$$

Thus, from above definitions we may write

$$G^*(m^*, f^*) = \sup_{(m,f)\in Y_1\times Y_2} \left\{ \langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - \frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx \right.$$

$$\left. - \int_\Omega \min\{g_1(m), g_2(m)\}dx - \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2 dz - Ind_0(m) - Ind_1(m) - Ind_2(m) \right\}. \quad (15.17)$$

Or

$$\begin{aligned} G^*(m^*, f^*) &= \sup_{(m,f)\in Y_1\times Y_2} \{\langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - \frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx \\ &\quad - \inf_{t\in B}\int_\Omega (tg_1(m) + (1-t)g_2(m))dx - \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2\, dz \\ &\quad - Ind_0(m) - Ind_1(m) - Ind_2(m)\}, \end{aligned} \quad (15.18)$$

where

$$B = \{t \ \ measurable \ \mid \ t(x) \in [0,1], \ \ a.e. \ \ in \ \ \Omega\}.$$

Hence

$$\begin{aligned} G^*(m^*, f^*) &= \sup_{(m,f,t)\in Y_1\times Y_2\times B} \{\langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - \frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx \\ &\quad - \int_\Omega (tg_1(m) + (1-t)g_2(m))dx - \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2\, dz \\ &\quad - Ind_0(m) - Ind_1(m) - Ind_2(m)\}, \end{aligned} \quad (15.19)$$

or

$$\begin{aligned} G^*(m^*, f^*) &= \sup_{t\in B}\sup_{(m,f)\in Y_1\times Y_2} \{\langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - \frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx \\ &\quad - \int_\Omega (tg_1(m) + (1-t)g_2(m))dx - \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2\, dz \\ &\quad - Ind_0(m) - Ind_1(m) - Ind_2(m)\}. \end{aligned} \quad (15.20)$$

Thus, as the second supremum is a convex optimization problem, there exist $(\lambda, \lambda_1, \lambda_2) \in L^2(S) \times L^2(S;\mathbb{R}^3) \times L^2(S)$, such that

$$\begin{aligned} G^*(m^*, f^*) &= \sup_{t\in B}\sup_{(m,f)\in Y_1\times Y_2} \{\langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, f^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - \frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx \\ &\quad - \int_\Omega (tg_1(m) + (1-t)g_2(m))dx - \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2 dz \\ &\quad - \int_\Omega \frac{\lambda}{2}(\sum_{i=1}^3 m_i^2 - 1)dx - \langle Curl(f), \lambda_1\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - \langle div(-f + m\chi_\Omega), \lambda_2\rangle_{L^2(\mathbb{R}^3)}\}. \end{aligned}$$

Thus we may write,

$$G^*(m^*, f^*) = \sup_{t\in B}\{\sup_{m\in Y_1}\{-\hat{G}(\Lambda m) - \hat{F}(m,t)\} + F_1^*(f^*)\},$$

where

$$F_1^*(f^*) = \frac{1}{2}\int_{\mathbb{R}^3} |f^* + Curl^*\lambda_1 + \nabla\lambda_2|^2 dx$$
$$\hat{G}(\Lambda m) = \frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx,$$
$$\Lambda m = \nabla m,$$

and

$$\begin{aligned}\hat{F}(m,t) &= -\langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \int_\Omega tg_1(m) + (1-t)g_2(m)dx \\ &\quad + \int_\Omega \frac{\lambda}{2}(\sum_{i=1}^3 m_i^2 - 1)dx + \langle div(m\chi_\Omega), \lambda_2\rangle_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{15.21}$$

Therefore, from Theorem 7.2.5,

$$\sup_{(m,f)\in Y_1\times Y_2}\{-\hat{G}(\Lambda m) - \hat{F}(m,t)\} = \inf_{y^*\in Y_0^*}\{\hat{G}^*(y^*) - \hat{F}^*(-\Lambda^* y^*, t)\},$$

where

$$\hat{G}^*(y^*) = \sup_{y\in L^2(\Omega,\mathbb{R}^{3\times 3})}\{\langle y, y^*\rangle_{L^2(\Omega;\mathbb{R}^{3\times 3})} - \frac{\alpha}{2}\int_\Omega |y|^2 dx\} = \frac{1}{2\alpha}\int_\Omega |y^*|^2 dx$$

and

$$\hat{F}^*(-\Lambda^* y^*, t) = \sup_{m\in Y_1}\{-\langle \nabla m_i, y_i^*\rangle_{L^2(\Omega;\mathbb{R}^3)} - F(m,t)\}.$$

Thus we may write

$$\begin{aligned}\hat{F}^*(-\Lambda^* y^*, t) &= \sup_{m\in Y_1}\{-\langle \nabla m_i, y_i^*\rangle_{L^2(\Omega,\mathbb{R}^3)} + \langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} \\ &\quad - \int_\Omega (t(1+m.e) + (1-t)(1-m.e))\beta dx \\ &\quad - \int_\Omega \frac{\lambda}{2}(\sum_{i=1}^3 m_i^2 - 1)dx - \langle div(m\chi_\Omega), \lambda_2\rangle_{L^2(\mathbb{R}^3)}\}.\end{aligned} \tag{15.22}$$

The last supremum is attained for functions satisfying $y_i^*.\mathbf{n} + \lambda_2\mathbf{n}_i = 0$ on $\partial\Omega$, for all $i \in \{1,2,3\}$, where $\mathbf{n}$ denotes the outer normal to $\partial\Omega$ (such a condition is necessary to guarantee a finite supremum). Furthermore

$$div(y_i^*) + m_i^* + (1-2t)\beta e_i - \lambda m_i + \frac{\partial\lambda_2}{\partial x_i} = 0,$$

or

$$m_i = \frac{div(y_i^*) + m_i^* + (1-2t)\beta e_i + \frac{\partial \lambda_2}{\partial x_i}}{\lambda}.$$

From the constraint $\sum_{i=1}^3 m_i^2 - 1 = 0$, we obtain

$$\lambda = \left(\sum_{i=1}^3 \left(div(y_i^*) + m_i^* + (1-2t)\beta e_i + \frac{\partial \lambda_2}{\partial x_i} \right)^2 \right)^{1/2},$$

so that

$$\hat{F}^*(-\Lambda^* y^*, t) = \int_\Omega \left(\sum_{i=1}^3 \left(div(y_i^*) + m_i^* + (1-2t)\beta e_i + \frac{\partial \lambda_2}{\partial x_i} \right)^2 \right)^{1/2} dx - \int_\Omega \beta dx.$$

Therefore, summarizing the last results, we may write

$$\begin{aligned} G^*(m^*, f^*) &= \sup_{t \in B} \inf_{y^* \in Y_0^*} \{ \frac{1}{2\alpha} \int_\Omega |y^*|^2 dx + \int_\Omega (\sum_{i=1}^3 (div(y_i^*) + m_i^* + (1-2t)\beta e_i + \frac{\partial \lambda_2}{\partial x_i})^2)^{1/2} dx \} \\ &+ \frac{1}{2} \int_{\mathbb{R}^3} | - f^* + Curl^* \lambda_1 + \nabla \lambda_2 |^2 dz - \int_\Omega \beta dx, \end{aligned} \tag{15.23}$$

where, $Y_0^* = \{ y^* \in W^{1,2}(\Omega; \mathbb{R}^{3\times3}) \ | \ y_i^*.\mathbf{n} + \lambda_2 \mathbf{n}_i = 0 \ \ on \ \ \partial\Omega, \forall i \in \{1,2,3\} \}$. On the other hand

$$F^*(-m^*, -f^*) = \sup_{(m,f) \in Y_1 \times Y_2} \{ \langle m, -m^* \rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, -f^* \rangle_{L^2(\Omega;\mathbb{R}^3)} + \int_\Omega H.m dx \}$$

so that

$$F^*(-m^*, -f^*) = \begin{cases} 0, & if \ \ (m^*, f^*) \in B^*, \\ +\infty, & otherwise, \end{cases}$$

where

$$B^* = \{ (m^*, f^*) \in Y_1^* \times Y_2^* \ | \ m^* = H, \ \ a.e. \ \ in \ \ \Omega, \ \ f^* = \theta \ \ a.e. \ \ in \ \ \mathbb{R}^3 \}.$$

Therefore, we could summarize the duality principle indicated in (15.16) as

$$\begin{aligned} \inf_{(m,f) \in Y_1 \times Y_2} \{ J(m,f) \} &= \inf_{t \in B} \left\{ \sup_{(\lambda_1, \lambda_2) \in \hat{Y}^*} \sup_{y^* \in Y_0^*} \left\{ -\frac{1}{2\alpha} \int_\Omega |y^*|^2 dx \right. \right. \\ & \left. - \int_\Omega (\sum_{i=1}^3 (div(y_i^*) + H_i + (1-2t)\beta e_i + \frac{\partial \lambda_2}{\partial x_i})^2)^{1/2} \, dx \right\} \\ & \left. - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \lambda_2 + Curl^* \lambda_1|^2 \, dz \right\} + \int_\Omega \beta \, dx, \end{aligned} \tag{15.24}$$

where

$$B = \{t \;\; measurable \;\mid\; t(x) \in [0,1], \;\; a.e. \;\; in \;\; \Omega\},$$

$$Y_0^* = \{y^* \in W^{1,2}(\Omega; \mathbb{R}^{3\times 3}) \;\mid\; y_i^*.\mathbf{n} + \lambda_2 \mathbf{n}_i = 0 \;\; on \;\; \partial\Omega, \forall i \in \{1,2,3\}\},$$

and

$$\hat{Y}^* = W^{1,2}(\mathbb{R}^3; \mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3). \;\; \square$$

15.6 Final Results, Convex Dual Formulations

Consider again the functional given by $J : Y_1 \times Y_2 \to \mathbb{R}$, where

$$\begin{aligned} J(m,f) &= \int_\Omega \min\{g_1(m), g_2(m)\}dx + Ind_0(m) - \int_\Omega H(x).mdx \\ &+ \frac{1}{2}\int_\Omega |f(x)|^2 dx + Ind_1(f) + Ind_2(m,f). \end{aligned} \tag{15.25}$$

Considering the expression of $Ind_0(m)$ given in the last section, may write $J(m,f) = G(m,f) + F(m,f)$ where

$$G(m,f) = \int_\Omega \min\{g_1(m), g_2(m)\}dx + Ind_0(m) - \int_\Omega H(x).mdx + \frac{K}{2}\langle m_i, m_i\rangle_{L^2(\Omega)} - \frac{K}{2}$$

and

$$F(m,f) = \frac{1}{2}\int_\Omega |f(x)|^2 dx + \frac{K}{2}\langle m_i, m_i\rangle_{L^2(\Omega)} - \frac{K}{2} + Ind_1(f) + Ind_2(m,f) + Ind_0(m).$$

It is known that

$$\inf_{(m,f)\in Y_1\times Y_2}\{J(m,f)\} \geq \sup_{z^*\in \hat{Y}^*}\{-G^*(z^*) - F^*(-z^*)\}, \tag{15.26}$$

where

$$G^*(z^*) = \sup_{(m,f)\in Y_1\times Y_2}\{\langle f, z_1^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \langle m, z_2^*\rangle_{L^2(\Omega;\mathbb{R}^3)} - G(m,f)\},$$

and

$$F^*(-z^*) = \sup_{(m,f)\in Y_1\times Y_2}\{-\langle f, z_1^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - \langle m, z_2^*\rangle_{L^2(\Omega;\mathbb{R}^3)} - F(m,f)\}.$$

Thus,

$$\begin{aligned} G^*(z^*) &= \sup_{(m,f)\in Y_1\times Y_2}\{\langle f, z_1^*\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \langle m, z_2^*\rangle_{L^2(\Omega;\mathbb{R}^3)} \\ &- \int_\Omega \min\{g_1(m), g_2(m)\}dx - Ind_0(m) \\ &+ \int_\Omega H(x).mdx - \frac{K}{2}\langle m_i, m_i\rangle_{L^2(\Omega)} + \frac{K}{2}\}, \end{aligned} \tag{15.27}$$

so that

$$G^*(z^*) = \begin{cases} \sup_{t \in B}\{\int_\Omega (\sum_{i=1}^3 (z_{2i}^* + H_i + \beta(1-2t)e_i)^2)^{1/2} dx\} - \beta \int_\Omega dx, & if \quad z^* \in B^*, \\ +\infty, \quad otherwise, \end{cases} \tag{15.28}$$

where

$$B = \{t \;\; measurable \;\; | \;\; t(x) \in [0,1], \;\; a.e. \;\; in \;\; \Omega\},$$

and

$$B^* = \{z^* \in L^2(\mathbb{R}^3;\mathbb{R}^3) \times L^2(\mathbb{R}^3) \;\; | \;\; z_1^* = \theta, \;\; a.e. \;\; in \;\; \mathbb{R}^3\}.$$

Also,

$$\begin{aligned} F^*(-z^*) \;\; = \;\; & \sup_{(m,f) \in Y_1 \times Y_2} \{\langle f, -z_1^* \rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \langle m, -z_2^* \rangle_{L^2(\Omega;\mathbb{R}^3)} - \frac{1}{2}\int_\Omega |f(x)|^2 dx \\ & -\frac{K}{2}\langle m_i, m_i \rangle_{L^2(\Omega)} + \frac{K}{2} - Ind_1(m) - Ind_2(m,f) - Ind_0(m). \end{aligned} \tag{15.29}$$

The calculation of $F^*(-z^*)$ is a standard quadratic optimization problem. Therefore the last supremum indicated is attained through appropriate Lagrange multipliers λ_0, λ_1, λ_2, that is,

$$\begin{aligned} F^*(-z^*) \;\; = \;\; & \sup_{(m,f) \in Y_1 \times Y_2} \{\langle f, -z_1^* \rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \langle m, -z_2^* \rangle_{L^2(\Omega;\mathbb{R}^3)} - \frac{1}{2}\int_\Omega |f(x)|^2 dx \\ & -\frac{K}{2}\langle m_i, m_i \rangle_{L^2(\Omega)} + \frac{K}{2} - \int_{\mathbb{R}^3} \lambda_1 . Curl(f) dx \\ & - \int_\Omega \lambda_2 (div(-f + m\chi_\Omega)) dx - \int_\Omega \frac{\lambda_0}{2}(\sum_{i=1}^3 m_i^2 - 1) dx. \end{aligned} \tag{15.30}$$

Evaluating such a supremum, we obtain

$$F^*(-z^*) = \inf_{(\lambda_1,\lambda_2) \in \hat{Y}^*} \{\int_\Omega (\sum_{i=1}^3 (z_{2i}^* - \frac{\partial \lambda_2}{\partial x_i})^2)^{1/2} dx + \frac{1}{2}\int_{\mathbb{R}^3} |z_1^* + Curl^* \lambda_1 + \nabla \lambda_2|^2 dx\}$$

where

$$\hat{Y}^* = \{(\lambda_1, \lambda_2) \in W^{1,2}(\mathbb{R}^3;\mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3) \;\; | \;\; \lambda_2 = 0 \;\; on \;\; \partial\Omega\}.$$

Observe that if K is big enough so that for a minimizing sequence (m_n, f_n) we have $G(m_n, f_n) = G^{**}(m_n, f_n)$ for all sufficiently big n, then there is no duality gap between the primal and dual formulations. In this case we may replace G by G^{**} and the duality is perfect.

We may summarize the last results by the following duality principle

$$\begin{aligned}\inf_{(m,f)\in Y_1\times Y_2}\{J(m,f)\} &= \sup_{(z_2^*,\lambda_1,\lambda_2)\in L^2(\Omega)\times\hat{Y}^*}\{\inf_{t\in B}\{-\int_\Omega(\sum_{i=1}^3(z_{2i}^*+H_i+\beta(1-2t)e_i)^2)^{1/2}dx \\ &\quad -\int_\Omega(\sum_{i=1}^3(z_{2i}^*-\frac{\partial\lambda_2}{\partial x_i})^2)^{1/2}dx-\frac{1}{2}\int_{\mathbb{R}^3}|Curl^*\lambda_1+\nabla\lambda_2|^2dx\}\} \\ &\quad +\int_\Omega\beta dx, \qquad (15.31)\end{aligned}$$

where

$$B=\{t \ \ measurable \ \mid \ t(x)\in[0,1], \ \ a.e. \ \ in \ \ \Omega\}.$$

From (15.15) we have

$$\begin{aligned}\inf_{(m,f)\in Y_1\times Y_2}\{J(m,f)\} &\geq \sup_{(\lambda_1,\lambda_2)\in\hat{Y}^*}\left\{\inf_{t\in B}\left\{-\int_\Omega(\sum_{i=1}^3(\frac{\partial\lambda_2}{\partial x_i}+H_i+\beta(1-2t)e_i)^2)^{1/2}\,dx\right.\right. \\ &\quad \left.\left.-\frac{1}{2}\int_{\mathbb{R}^3}|Curl^*\lambda_1+\nabla\lambda_2|^2\,dx\right\}+\int_\Omega\beta\,dx. \qquad (15.32)\end{aligned}$$

Hence, from (15.31) and (15.32) we finally obtain

$$\begin{aligned}\inf_{(m,f)\in Y_1\times Y_2}\{J(m,f)\} &= \sup_{(\lambda_1,\lambda_2)\in\hat{Y}^*}\left\{\inf_{t\in B}\left\{-\int_\Omega(\sum_{i=1}^3(\frac{\partial\lambda_2}{\partial x_i}+H_i+\beta(1-2t)e_i)^2)^{1/2}\,dx\right.\right. \\ &\quad \left.\left.-\frac{1}{2}\int_{\mathbb{R}^3}|Curl^*\lambda_1+\nabla\lambda_2|^2\,dx\right\}\right\}+\int_\Omega\beta\,dx \qquad (15.33)\end{aligned}$$

where

$$B=\{t \ \ measurable \ \mid \ t(x)\in[0,1] \ \ a.e. \ \ in \ \ \Omega\}.$$

and

$$\hat{Y}^*=\{(\lambda_1,\lambda_2)\in W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)\times W^{1,2}(\mathbb{R}^3) \ \mid \ \lambda_2=0 \ \ on \ \ \partial\Omega\}.$$

Similar results may be obtained for the semi-linear case. The final format of the concerned duality principle is given by

$$\begin{aligned}\inf_{(m,f)\in Y_1\times Y_2}\{J(m,f)\} &= \sup_{(\lambda_1,\lambda_2,y^*)\in\hat{Y}^*\times Y_0^*}\left\{\inf_{t\in B}\left\{-\frac{1}{2\alpha}\int_\Omega|y^*|^2\,dx\right.\right. \\ &\quad -\int_\Omega(\sum_{i=1}^3(div(y_i^*)+H_i+(1-2t)\beta e_i+\frac{\partial\lambda_2}{\partial x_i})^2)^{1/2}\,dx\Bigg\} \\ &\quad \left.-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\lambda_2+Curl^*\lambda_1|^2\,dz\right\}+\int_\Omega\beta\,dx, \qquad (15.34)\end{aligned}$$

where

$$B = \{t \;\; measurable \;\;|\;\; t(x) \in [0,1], \;\; a.e. \;\; in \;\; \Omega\},$$

$$Y_0^* = \{y^* \in W^{1,2}(\Omega;\mathbb{R}^{3\times 3}) \;\;|\;\; y_i^*.\mathbf{n} = 0 \;\; on \;\; \partial\Omega, \forall i \in \{1,2,3\}\},$$

and

$$\hat{Y}^* = \{(\lambda_1,\lambda_2) \in W^{1,2}(\mathbb{R}^3;\mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3) \;\;|\;\; \lambda_2 = 0 \;\; on \;\; \partial\Omega\}. \;\; \Box$$

15.7 The Cubic Case in Micro-magnetism

In this section we present the result for the cubic case. For the primal formulation we refer to references [22, 26] for details.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a finite Lebesgue measure and a regular boundary denoted by $\partial\Omega$ and, consider the model of micro-magnetism in which the magnetization $m : \Omega \to \mathbb{R}^3$, is given by the minimization of the functional $J(m,f)$, where

$$J(m,f) = \frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx + \int_\Omega \varphi(m(x))dx - \int_\Omega H.mdx + \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2 dz,$$

$$m \in W^{1,2}(\Omega;\mathbb{R}^3) \equiv Y_1, \;\; |m(x)| = 1, \;\; a.e. \;\; in \;\; \Omega$$

and $f \in L^2(\mathbb{R}^3;\mathbb{R}^3) \equiv Y_2$ is the unique field determined by the simplified Maxwell's equations

$$Curl(f) = 0, \;\; div(-f + m\chi_\Omega) = 0, \;\; in \;\; \mathbb{R}^3.$$

Here the function $\varphi(m)$, for the cubic anisotropy is given by

$$\varphi(m) = K_0 + K_1 \sum_{i\neq j} m_i^2 m_j^2 + K_2 m_1^2 m_2^2 m_3^2,$$

where $K_1, K_2 > 0$. Also $H \in L^2(\Omega;\mathbb{R}^3)$ is a known external field and χ_Ω is a function defined as

$$\chi_\Omega(x) = \begin{cases} 1, & if \;\; x \in \Omega, \\ 0, & otherwise. \end{cases}$$

The term

$$\frac{\alpha}{2}\int_\Omega |\nabla m|^2 dx$$

stands for the exchange energy. Finally, $\varphi(m)$ represents the anisotropic contribution and is given by a multi-well functional whose minima establish the preferred directions of magnetization.

Remark 15.7.1. *It is worth noting that $\varphi(m)$ has six points of minimum, namely $r_1 = (1,0,0)$, $r_2 = (0,1,0)$ and $r_3 = (0,0,1)$, $r_4 = (-1,0,0)$, $r_5 = (0,-1,0)$ and $r_6 = (0,0,-1)$ which define the preferred directions of magnetization.*

15.7.1 The Primal Formulation

For $\alpha = 0$, we define the primal formulation as $J : Y_1 \times Y_2 \to \bar{\mathbb{R}} = \mathbb{R} \cup +\infty$, where

$$J(m, f) = G(m, f) + F(m, f).$$

Here $G : Y_1 \times Y_2 \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is defined as

$$G(m, f) = \int_\Omega \varphi(m)dx + \frac{K}{2}\int_\Omega (\sum_{k=1}^{3} m_k^2 - 1)dx \tag{15.35}$$

and we shall rewrite φ as the approximation

$$\varphi(m) = \min\{g_1(m), g_2(m), g_3(m), g_4(m), g_5(m), g_6(m)\},$$

where

$$g_k(m) = \varphi(r_k) + \sum_{i=1}^{3} \frac{\partial \varphi(r_k)}{\partial mi}(m_i - r_{k^i}) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial^2 \varphi(r_k)}{\partial m_i \partial m_j}(m_i - r_{k^i})(m_j - r_{k^j}).$$

As above mentioned, $r_1 = (1, 0, 0)$, $r_2 = (0, 1, 0)$, $r_3 = (0, 0, 1)$, $r_4 = (-1, 0, 0)$, $r_5 = (0, -1, 0)$ and $r_6 = (0, 0, -1)$ are the points through which the possible microstructure is formed for the cubic case, what justify such expansions. Also, we define $F : V \to \bar{\mathbb{R}}$, as

$$F(m, f) = \frac{1}{2}\int_{\mathbb{R}^3} |f(z)|^2 dz - \int_\Omega H.mdx + Ind_1(m) + Ind_2(m, f) + Ind_3(f), \tag{15.36}$$

where

$$Ind_1(m) = \begin{cases} 0, & if \ \ |m(x)| = 1 \ \ a.e. \ \ in \ \ \Omega, \\ +\infty, & otherwise, \end{cases}$$

$$Ind_2(m, f) = \begin{cases} 0, & if \ \ div(-f + m\chi_\Omega) = 0 \ \ a.e. \ \ in \ \ \mathbb{R}^3, \\ +\infty, & otherwise, \end{cases}$$

and

$$Ind_3(f) = \begin{cases} 0, & if \ \ Curl(f) = 0, \ \ a.e. \ \ in \ \ \mathbb{R}^3, \\ +\infty, & otherwise. \end{cases}$$

15.7.2 The Duality Principles

From Theorem 7.2.5, we have

$$\inf_{(m,f)\in Y_1\times Y_2}\{G^{**}(m, f) + F(m, f)\} = \sup_{(m^*,f^*)\in Y_1^*\times Y_2^*}\{-G^*(m^*, f^*) - F^*(-m^*, -f^*)\}, \tag{15.37}$$

where

$$G^*(m^*, f^*) = \sup_{(m,f)\in Y_1\times Y_2}\{\langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle f, f^*\rangle_{L^2(\Omega;\mathbb{R}^3)} - G(m,f)\},$$

so that, defining $\hat{g}_k(m) = g_k(m) + \frac{K}{2}(\sum_{k=1}^3 m_k^2)$, we have

$$G^*(m, f^*) = \begin{cases} \int_\Omega \max_{k\in\{1,\ldots,6\}}\{\hat{g}_k^*(m)\}dx + \frac{K}{2}|\Omega|, & if \quad (m^*, f^*) \in B^*, \\ +\infty, \quad otherwise, \end{cases}$$

where

$$B^* = \{(m^*, f^*) \in Y_1^* \times Y_2^* \;\mid\; f^* = (0,0,0) \equiv \theta, \quad a.e. \;\; in \;\; \mathbb{R}^3\}.$$

Also,

$$F^*(-m^*, f^*) = \sup_{(m,f)\in Y_1\times Y_2}\{-\langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} - \langle f, f^*\rangle_{L^2(\Omega;\mathbb{R}^3)} - F(m,f)\},$$

or

$$\begin{aligned} F^*(-m^*, -f^*) \;&=\; \sup_{(m,f)\in Y_1\times Y_2}\{-\langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} - \langle f, f^*\rangle_{L^2(\Omega;\mathbb{R}^3)} - \frac{1}{2}\int_{\mathbb{R}^3}|f(z)|^2dz \\ &+ \int_\Omega H.mdx - Ind_1(m) - Ind_2(m,f) - Ind_3(f)\}, \end{aligned} \tag{15.38}$$

that is

$$\begin{aligned} F^*(-m^*, -f^*) \;&=\; \sup_{(m,f)\in Y_1\times Y_2}\{-\langle m, m^*\rangle_{L^2(\Omega;\mathbb{R}^3)} - \langle f, f^*\rangle_{L^2(\Omega;\mathbb{R}^3)} - \frac{1}{2}\int_{\mathbb{R}^3}|f(z)|^2dz \\ &+ \int_\Omega H.mdx - \int_\Omega \frac{\lambda}{2}(\sum_{k=1}^3 m_k^2 - 1)dx \\ &- \langle Curl(f), \lambda_1\rangle_{L^2(\mathbb{R}^3;\mathbb{R}^3)} - \langle (div(-f + m\chi_\Omega), \lambda_2\rangle_{L^2(\mathbb{R}^3)}\}, \end{aligned} \tag{15.39}$$

so that

$$F^*(-m^*, -f^*) = \int_\Omega (\sum_{i=1}^3 (\frac{\partial \lambda_2}{\partial x_i} + m_i^* + H_i)^2)^{1/2}dx - \frac{1}{2}\int_{\mathbb{R}^3}|f^* + Curl^*\lambda_1 + \nabla\lambda_2|^2dx\}.$$

Hence the duality principle indicated in (15.37) may be expressed as

$$\begin{aligned} \inf_{(m,f)\in Y_1\times Y_2}\{G^{**}(m,f) + F(m,f)\} = \sup_{(m^*,\lambda_1,\lambda_2)\in\hat{Y}^*}\Bigg\{-\int_\Omega \max_{k\in\{1,\ldots,6\}}\{\hat{g}_k^*(m^*)\}\,dx \\ -\int_\Omega(\sum_{i=1}^3(\frac{\partial\lambda_2}{\partial x_i} + m_i^* + H_i)^2)^{1/2}\,dx - \frac{1}{2}\int_{\mathbb{R}^3}|Curl^*\lambda_1 + \nabla\lambda_2|^2dz\Bigg\} - \frac{K}{2}|\Omega|, \end{aligned}$$

where

$$\hat{Y^*} = \{(m^*, \lambda_1, \lambda_2) \in H^{-1}(\Omega;\mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3;\mathbb{R}^3) \times W^{1,2}(\mathbb{R}^3)) \;\mid\; \lambda_2 = 0 \;\; on \;\; \partial\Omega\}. \;\; \square$$

Remark 15.7.2. *If K is big enough so that for a minimizing sequence $\{(m_n, f_n)\}$ (of the original primal approach) we have $G^{**}(m_n, f_n) = G(m_n, f_n)$ for any n sufficiently big, then we may replace G^{**} by G in the last duality principle, with no duality gap between the primal and dual formulations. Also, observe that the minimizing sequences for the primal problem does not depend on K.*

By analogy, we may obtain the results for the semi-linear cubic case (for $\alpha > 0$), namely

$$\inf_{(m,f)\in Y_1\times Y_2}\{G^{**}(m,f)+F(m,f)\} = \sup_{(m^*,\lambda_1,\lambda_2,y^*)\in \hat{Y}^*\times Y_0^*}\left\{-\int_\Omega \max_{k\in\{1,\ldots,6\}}\{\hat{g}_k^*(m^*)\}dx\right.$$
$$-\frac{1}{2\alpha}\int_\Omega |y^*|^2dx - \int_\Omega (\sum_{i=1}^3 (div(y_i^*) + \frac{\partial\lambda_2}{\partial x_i} + m_i^* + H_i)^2)^{1/2}\,dx$$
$$\left. -\frac{1}{2}\int_{\mathbb{R}^3}|Curl^*\lambda_1 + \nabla\lambda_2|^2\,dx\right\} - \frac{K}{2}|\Omega|,$$

where

$$Y_0^* = \{y^* \in W^{1,2}(\Omega;\mathbb{R}^{3\times 3}) \mid y_i^*.n = 0 \ \ on \ \ \partial\Omega\},$$

and

$$\hat{Y^*} = \{(m^*,\lambda_1,\lambda_2) \in H^{-1}(\Omega;\mathbb{R}^3)\times W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)\times W^{1,2}(\mathbb{R}^3)) \mid \lambda_2 = 0 \ \ on \ \ \partial\Omega\}. \ \ \square$$

15.8 Conclusion

In this chapter we develop duality principles for models in ferromagnetism met in references [22, 26], for example . The last dual variational formulations here presented are convex (in fact concave) either for the hard and full (semi-linear) uniaxial cases or for the cubic cases. The results are obtained through standard tools of convex analysis. It is important to emphasize that in some situations (specially the hard cases), the minima may not be attained through the primal approaches, so that the minimizers of the dual formulations reflect the average behavior of minimizing sequences for the primal problems, as weak cluster points of such sequences.

Chapter 16

Duality Applied to Fluid Mechanics

16.1 Introduction and Primal Formulation

In this chapter we develop dual variational formulations for the incompressible two-dimensional steady Navier-Stokes system. We establish as a primal formulation the sum of L^2 norm of each of equations, and obtain the dual formulation through the Legendre Transform concept. Now we present the primal formulation.

Consider $S \subset \mathbb{R}^2$ an open, bounded and connected set, whose the internal boundary is denoted by Γ_0 and, the external boundary is denoted by Γ_1. Denoting by $u : S \to \mathbb{R}$ the field of velocity in direction x of the Cartesian system (x, y), by $v : S \to \mathbb{R}$, the velocity field in the direction y, by $p : S \to \mathbb{R}$, the pressure field, so that $P = p/\rho$, where ρ is the constant fluid density and ν is the viscosity coefficient, the Navier-Stokes system is expressed by

$$\nu\nabla^2 u - u\partial_x u - v\partial_y u - \partial_x P = 0, \quad a.e. \ \ in \ \ S, \tag{16.1}$$

$$\nu\nabla^2 v - u\partial_x v - v\partial_y v - \partial_y P = 0, \quad a.e. \ \ in \ \ S, \tag{16.2}$$

$$\partial_x u + \partial_y v = 0, \quad a.e. \ \ in \ \ S, \tag{16.3}$$

$$u = v = 0, \quad on \ \Gamma_0 \tag{16.4}$$

and

$$u = u_\infty, \quad v = 0, \quad P = P_\infty \ \ on \ \ \Gamma_1. \tag{16.5}$$

The primal variational formulation, denoted by $J : U \to \mathbb{R}$, is expressed as:

$$J(\mathbf{u}) = \frac{1}{2}(\|L_1(\mathbf{u})\|^2_{L^2(S)} + \|L_2(\mathbf{u})\|^2_{L^2(S)} + \|L_3(\mathbf{u})\|^2_{L^2(S)}) \tag{16.6}$$

where $\mathbf{u} = (u, v, P) \in U$, and

$$U = \{\mathbf{u} \in H^2(S)\times H^2(S)\times H^1(S) \mid u = v = 0, \ \Gamma_0, \ and, \ u = u_\infty, \ v = 0, \ P = P_\infty \ on \ \Gamma_1\}. \tag{16.7}$$

Also

$$L_1(\mathbf{u}) = \nu\nabla^2 u - u\partial_x u - v\partial_y u - \partial_x P, \tag{16.8}$$

$$L_2(\mathbf{u}) = \nu\nabla^2 v - u\partial_x v - v\partial_y v - \partial_y P \tag{16.9}$$

and

$$L_3(\mathbf{u}) = \partial_x u + \partial_y v. \tag{16.10}$$

Clearly we can write

$$J(\mathbf{u}) = \int_S g(\Lambda\mathbf{u})dS \tag{16.11}$$

where $\Lambda\mathbf{u} = \{\Lambda_i\mathbf{u}\}$, for $i \in \{1, ..., 14\}$ or more explicitly

$$\Lambda_1\mathbf{u} = \nu\nabla^2 u, \quad \Lambda_2\mathbf{u} = u, \quad \Lambda_3\mathbf{u} = -\partial_x u,$$

$$\Lambda_4\mathbf{u} = v, \quad \Lambda_5\mathbf{u} = -\partial_y u, \quad \Lambda_6\mathbf{u} = -\partial_x P,$$

$$\Lambda_7\mathbf{u} = \nu\nabla^2 v, \quad \Lambda_8\mathbf{u} = u, \quad \Lambda_9\mathbf{u} = -\partial_x v,$$

$$\Lambda_{10}\mathbf{u} = v, \quad \Lambda_{11}\mathbf{u} = -\partial_y v, \quad \Lambda_{12}\mathbf{u} = -\partial_y P,$$

$$\Lambda_{13}\mathbf{u} = \partial_x u, \quad \Lambda_{14}\mathbf{u} = \partial_y v. \tag{16.12}$$

Here

$$g(y) = g_1(y) + g_2(y) + g_3(y) \tag{16.13}$$

where

$$g_1(y) = \frac{1}{2}(y_1 + y_2 y_3 + y_4 y_5 + y_6)^2, \tag{16.14}$$

$$g_2(y) = \frac{1}{2}(y_7 + y_8 y_9 + y_{10} y_{11} + y_{12})^2, \tag{16.15}$$

$$g_3(y) = \frac{1}{2}(y_{13} + y_{14})^2. \tag{16.16}$$

16.2 The Legendre Transform

Applying the definition of Legendre transform to $g(y) = g_1(y) + g_2(y) + g_3(y)$ we obtain $g_L^*(y^*) = g_{1L}^*(y^*) + g_{2L}^*(y^*) + g_{3L}^*(y^*)$, where

$$g_{1L}^*(y^*) = \frac{y_2^* y_3^*}{y_1^*} + \frac{y_4^* y_5^*}{y_1^*} + \frac{(y_1^*)^2}{2}, \tag{16.17}$$

$$g_{2L}^*(y^*) = \frac{y_8^* y_9^*}{y_7^*} + \frac{y_{10}^* y_{11}^*}{y_7^*} + \frac{(y_7^*)^2}{2} \tag{16.18}$$

and

$$g_{3L}^* = \frac{1}{2}(y_{13}^*)^2. \tag{16.19}$$

Remark 16.2.1. *Observe that any solution system*

$$\delta(-G_L(v^*) + \langle u, \Lambda^* v^* \rangle_U) = \theta \tag{16.20}$$

yields a solution of the Navier-Stokes system (the Euler-Lagrange equations of such a system is equivalent to the Navier-Stokes system).

Here, $\Lambda^* v^* = \theta$ denotes:

$$\nu \nabla^2 v_1^* + v_2^* + \partial_x v_3^* + \partial_y v_5^* - \partial_x v_{13}^* = 0, \quad a.e. \quad in \quad S, \tag{16.21}$$

$$\nu \nabla^2 v_7^* + v_{10}^* + \partial_x v_9^* + \partial_y v_{11}^* - \partial_y v_{13}^* = 0, \quad a.e. \quad in \quad S, \tag{16.22}$$

and

$$\partial_x v_1^* + \partial_y v_7^* = 0, \quad a.e. \quad in \quad S. \tag{16.23}$$

and

$$G_L^*(v^*) = \int_S g_{1L}^*(v^*) dS + \int_S g_{2L}^*(v^*) dS + \int_S g_{3L}^*(v^*) dS \tag{16.24}$$

or, more explicitly:

$$\begin{aligned} G_L^*(v^*) \;\; = \;\; & \int_S \frac{v_2^* v_3^*}{v_1^*} dS + \int_S \frac{v_4^* v_5^*}{v_1^*} dS + \frac{1}{2} \int_S (v_1^*)^2 dS \\ & + \int_S \frac{v_8^* v_9^*}{v_7^*} dS + \int_S \frac{v_{10}^* v_{11}^*}{v_7^*} dS + \frac{1}{2} \int_S (v_7^*)^2 dS. \end{aligned} \tag{16.25}$$

16.3 The Dual Variational Formulation

Firstly we define $(\hat{G} \circ \Lambda) : U \to \mathbb{R}$ as

$$\begin{aligned} \hat{G}(\Lambda \mathbf{u}) \;\; = \;\; & \int_S g_1(\Lambda \mathbf{u}) dS + \frac{K}{2} \int_S u^2 dS + \frac{K}{2} \int_S (\partial_x u)^2 dS \frac{K}{2} \int_S v^2 dS + \frac{K}{2} \int_S (\partial_y u)^2 dS \\ & + \int_S g_2(\Lambda \mathbf{u}) dS + \frac{K}{2} \int_S u^2 dS + \frac{K}{2} \int_S (\partial_x v)^2 dS \frac{K}{2} \int_S v^2 dS \\ & + \frac{K}{2} \int_S (\partial_y v)^2 dS + \int_S g_3(\Lambda \mathbf{u}) dS, \end{aligned} \tag{16.26}$$

so that clearly we can write

$$\hat{G}(\Lambda \mathbf{u}) = \int_S \hat{g}_1(\Lambda \mathbf{u}) dS + \int_S \hat{g}_2(\Lambda \mathbf{u}) dS + \int_S \hat{g}_3(\Lambda \mathbf{u}) dS \tag{16.27}$$

where

$$\hat{g}_1(y) = g_1(y) + \frac{K}{2} y_2^2 + \frac{K}{2} y_3^2 + \frac{K}{2} y_4^2 + \frac{K}{2} y_5^2, \tag{16.28}$$

$$\hat{g}_2(y) = g_2(y) + \frac{K}{2}y_8^2 + \frac{K}{2}y_9^2 + \frac{K}{2}y_{10}^2 + \frac{K}{2}y_{11}^2, \tag{16.29}$$

$$\hat{g}_3(y) = g_3(y) \tag{16.30}$$

and hence, through the definition of Legendre transform we have

$$\hat{G}_L^*(v^*) = \int_S \hat{g}_{1L}^*(v^*)dS + \int_S \hat{g}_{2L}^*(v^*)dS + \int_S \hat{g}_{3L}^*(v^*)dS \tag{16.31}$$

where

$$\hat{g}_{1L}^*(y^*) = \frac{1}{2\Delta_1}(K(y_2^*)^2 - 2y_1^*y_2^*y_3^* + K(y_3^*)^2) + \frac{1}{2\Delta_1}(K(y_4^*)^2 - 2y_1^*y_4^*y_5^* + K(y_5^*)^2) + \frac{(y_1^*)^2}{2}, \tag{16.32}$$

$$\hat{g}_{2L}^*(y^*) = \frac{1}{2\Delta_7}(K(y_8^*)^2 - 2y_7^*y_8^*y_9^* + K(y_9^*)^2) + \frac{1}{2\Delta_7}(K(y_{10}^*)^2 - 2y_7^*y_{10}^*y_{11}^* + K(y_{11}^*)^2) + \frac{(y_7^*)^2}{2}, \tag{16.33}$$

where $\Delta_1 = K^2 - (y_1^*)^2$, $\Delta_7 = K^2 - (y_7^*)^2$ and

$$\hat{g}_{3L}^*(y^*) = \frac{(y_{13}^*)^2}{2}. \tag{16.34}$$

The dual variational formulation is indicated in the next theorem.

Theorem 16.3.1. *For $J : U \to \mathbb{R}$ defined as*

$$J(\boldsymbol{u}) = \hat{G}(\Lambda\boldsymbol{u}) - F(\check{\Lambda}\boldsymbol{u}), \tag{16.35}$$

where $\hat{G}(\Lambda\boldsymbol{u})$ is indicated in (16.26) and $F(\check{\Lambda}\boldsymbol{u})$ is defined as

$$\begin{aligned} F(\check{\Lambda}\boldsymbol{u}) \;\; = \;\; & \frac{K}{2}\int_S u^2 dS + \frac{K}{2}\int_S (\partial_x u)^2 dS \frac{K}{2}\int_S v^2 dS + \frac{K}{2}\int_S (\partial_y u)^2 dS \\ & + \frac{K}{2}\int_S u^2 dS + \frac{K}{2}\int_S (\partial_x v)^2 dS \frac{K}{2}\int_S v^2 dS + \frac{K}{2}\int_S (\partial_y v)^2 dS, \end{aligned} \tag{16.36}$$

where

$$\check{\Lambda}\boldsymbol{u} = \{u, \partial_x u, \partial_y u, v, v, \partial_x v, \partial_y v, u\}, \tag{16.37}$$

we may write

$$\inf_{\boldsymbol{u}\in U}\{J(\boldsymbol{u})\} \leq \inf_{u^*\in E^*} \sup_{v^*\in C^*(u^*)} \{F^*(u^*) - \hat{G}^*(v^*)\} \tag{16.38}$$

where $v^* \in C^*(u^*)$ if and only if $v^* \in Y^*$ and

$$\nu\nabla^2 v_1^* + v_2^* + \partial_x v_3^* + \partial_y v_5^* - \partial_x v_{13}^* - u_1^* + \partial_x u_2^* - u_4^* + \partial_y u_3^* = 0, \quad a.e. \;\; in \;\; S, \tag{16.39}$$

$$\nu\nabla^2 v_7^* + v_{10}^* + \partial_x v_9^* + \partial_y v_{11}^* - \partial_y v_{13}^* - u_8^* + \partial_x u_6^* - u_5^* + \partial_y u_7^* = 0, \quad a.e. \quad in \quad S, \tag{16.40}$$

and

$$\partial_x v_1^* + \partial_y v_7^* = 0, \quad a.e. \quad in \quad S. \tag{16.41}$$

Also,

$$\begin{aligned} F^*(u^*) &= \frac{1}{2K}\int_S (u_1^*)^2 dS + \frac{1}{2K}\int_S (u_2^*)^2 dS + \frac{1}{2K}\int_S (u_3^*)^2 dS + \frac{1}{2K}\int_S (u_4^*)^2 dS \\ &+ \frac{1}{2K}\int_S (u_5^*)^2 dS + \frac{1}{2K}\int_S (u_6^*)^2 dS + \frac{1}{2K}\int_S (u_7^*)^2 dS + \frac{1}{2K}\int_S (u_8^*)^2 dS, \end{aligned} \tag{16.42}$$

and

$$\hat{G}^*(v^*) = \hat{G}_L^*(v^*), \quad if \quad v^* \in B^*, \tag{16.43}$$

where

$$B^* = \{v^* \in Y^* \;\mid\; \Delta_1 \geq 0 \;\; and \;\; \Delta_7 \geq 0\}. \tag{16.44}$$

Finally,

$$\inf_{v^* \in C^*(u^*)}\{\hat{G}^*(v^*)\} \equiv \bar{G}^*(u^*) \tag{16.45}$$

and

$$E^* = \{u^* \in U^* \;\mid\; (\hat{G}\circ\Lambda)^*(\check{\Lambda}^* u^*) = \bar{G}^*(u^*)\}. \tag{16.46}$$

Proof: We have that $\inf_{\mathbf{u}\in U}\{J(\mathbf{u})\} = 0$ so that

$$-F(\check{\Lambda}\mathbf{u}) \geq -\hat{G}(\Lambda\mathbf{u}), \quad \forall \mathbf{u} \in U \tag{16.47}$$

and hence we may write

$$F^*(u^*) \geq \sup_{\mathbf{u}\in U}\{\langle \check{\Lambda}\mathbf{u}, u^*\rangle - F(\check{\Lambda}\mathbf{u})\} \geq \sup_{\mathbf{u}\in U}\{\langle \check{\Lambda}\mathbf{u}, u^*\rangle - \hat{G}(\Lambda\mathbf{u})\} \tag{16.48}$$

However, from the Theorem 7.2.5

$$(\hat{G}\circ\Lambda)^*(\check{\Lambda}^* u^*) = \sup_{\mathbf{u}\in U}\{\langle \check{\Lambda}\mathbf{u}, u^*\rangle - \hat{G}(\Lambda\mathbf{u})\} \leq \inf_{v^*\in C^*(u^*)}\{\hat{G}^*(v^*)\} \equiv \bar{G}^*(u^*) \tag{16.49}$$

and thus, recalling that

$$E^* = \{u^* \in U^* \;\mid\; (\hat{G}\circ\Lambda)^*(\check{\Lambda}^* u^*) = \bar{G}^*(u^*)\} \tag{16.50}$$

we have

$$F^*(u^*) + \sup_{v^*\in C^*(u^*)}\{\hat{G}^*(v^*)\} \geq 0 = \inf_{\mathbf{u}\in U}\{J(\mathbf{u})\}, \tag{16.51}$$

$\forall u^* \in E^*$.

Finally, we have to show that $\hat{G}^*(v^*) = \hat{G}_L^*(v^*)$ on B^*. In fact, it is sufficient to show that:

$$\hat{g}_L^*(y^*) \geq \;\langle y, y^*\rangle_{\mathbb{R}^{14}} - \hat{g}(y), \forall y \in \mathbb{R}^{14}, \quad y^* \in B^*, \tag{16.52}$$

or

$$\begin{aligned}
\langle y, y^*\rangle_{\mathbb{R}^{14}} &- \frac{1}{2}(y_1+y_2y_3+y_4y_5+y_6)^2-\frac{K}{2}y_2^2-\frac{K}{2}y_3^2-\frac{K}{2}y_4^2-\frac{K}{2}y_5^2 \\
&- \frac{1}{2}(y_7+y_8y_9+y_{10}y_{11}+y_{12})^2-\frac{K}{2}y_8^2-\frac{K}{2}y_9^2+\frac{K}{2}y_{10}^2+\frac{K}{2}y_{11}^2 \\
&- \frac{1}{2}(y_{13}+y_{14})^2 \\
&\leq \hat{g}_L^*(y^*). \qquad (16.53)
\end{aligned}$$

Through the transformations:

$$\bar{y}_1 = y_1+y_2y_3+y_4y_5+y_6 \qquad (16.54)$$

and

$$\bar{y}_7 = y_7+y_8y_9+y_{10}y_{11}+y_{12} \qquad (16.55)$$

this is equivalent to

$$\begin{aligned}
y_1^*\bar{y}_1 &- y_1^*y_2y_3-y_1^*y_4y_5-y_1^*y_6+y_7^*\bar{y}_7-y_7^*y_8y_9-y_7^*y_{10}y_{11}-y_7^*y_{12} \\
&+ y_2^*y_2+y_3^*y_3+y_4^*y_4+y_5^*y_5+y_6^*y_6+y_8^*y_8 \\
&+ y_9^*y_9+y_{10}^*y_{10}+y_{11}^*y_{11}+y_{12}^*y_{12}+y_{13}^*y_{13}+y_{14}^*y_{14} \\
&- \frac{1}{2}\bar{y}_1^2-\frac{1}{2}\bar{y}_7^2-\frac{1}{2}(y_{13}+y_{14})^2-\frac{K}{2}y_2^2-\frac{K}{2}y_3^2-\frac{K}{2}y_4^2-\frac{K}{2}y_5^2 \\
&- \frac{K}{2}y_8^2-\frac{K}{2}y_9^2-\frac{K}{2}y_{10}^2-\frac{K}{2}y_{11}^2-g_L^*(y^*) \\
&\leq 0. \qquad (16.56)
\end{aligned}$$

On the other hand, if $\Delta_1 \geq 0$ it is easy to see that

$$\sup_{(y_2,y_3)\in\mathbb{R}^2}\{-y_1^*y_2y_3+y_2^*y_2+y_3^*y_3-\frac{K}{2}y_2^2-\frac{K}{2}y_3^2\} = \frac{1}{2\Delta_1}(K(y_2^*)^2-2y_1^*y_2^*y_3^*+K(y_3^*)^2) \qquad (16.57)$$

and also,

$$\sup_{(y_4,y_5)\in\mathbb{R}^2}\{-y_1^*y_4y_5+y_4^*y_4+y_5^*y_5-\frac{K}{2}y_4^2-\frac{K}{2}y_5^2\} = \frac{1}{2\Delta_1}(K(y_4^*)^2-2y_1^*y_4^*y_5^*+K(y_5^*)^2). \qquad (16.58)$$

If $\Delta_7 \geq 0$ we have

$$\sup_{(y_8,y_9)\in\mathbb{R}^2}\{-y_7^*y_8y_9 + y_8^*y_8 + y_9^*y_9 - \frac{K}{2}y_8^2 - \frac{K}{2}y_9^2\} = \frac{1}{2\Delta_7}(K(y_8^*)^2 - 2y_7^*y_8^*y_9^* + K(y_9^*)^2) \quad (16.59)$$

and

$$\sup_{(y_{10},y_{11})\in\mathbb{R}^2}\{-y_7^*y_{10}y_{11} + y_{10}^*y_{10} + y_{11}^*y_{11} - \frac{K}{2}y_{10}^2 - \frac{K}{2}y_{11}^2\} = \frac{1}{2\Delta_7}(K(y_{10}^*)^2 - 2y_7^*y_{10}^*y_{11}^* + K(y_{11}^*)^2) \quad (16.60)$$

so that considering that $y_1^* = y_6^*$, $y_7^* = y_{12}^*$, $y_{13}^* = y_{14}^*$ we obtain

$$\begin{aligned}
\sup_{\hat{y}\in\mathbb{R}^6}\{y_1^*\bar{y}_1 &- y_1^*y_6 + y_7^*\bar{y}_7 + y_6^*y_6 - y_7^*y_{12} + y_{12}^*y_{12} + y_{13}^*y_{13} + y_{14}^*y_{14} \\
&- \frac{1}{2}(\bar{y}_1)^2 - \frac{1}{2}(\bar{y}_7)^2 - \frac{1}{2}(y_{13} + y_{14})^2\} \\
&= \frac{1}{2}(y_1^*)^2 + \frac{1}{2}(y_7^*)^2 + \frac{1}{2}(y_{13}^*)^2 \qquad (16.61)
\end{aligned}$$

where $\hat{y} = (\bar{y}_1, \bar{y}_7, y_6, y_{12}, y_{13}, y_{14})$ so that considering the expression of $g_L^*(y^*)$ in (16.31), we can conclude that (16.56) holds. $\square$

16.4 Conclusion

In this chapter we obtain a dual variational formulations for the two-dimensional incompressible Navier-Stokes system via Legendre Transform. The extension of results to $\mathbb{R}^3$, compressible and time dependent cases is not difficult, but postponed for a future work.

Chapter 17

Duality Applied to a Beam Model

17.1 Introduction and Statement of Primal Formulation

In this chapter we present an existence result and duality theory concerning the non-linear beam model proposed by Gao in [19].

The boundary value form of Gao's beam model is represented by the equation

$$EIw_{,xxxx} - a(w_{,x})^2 w_{,xx} + \lambda w_{,xx} = f, \quad in \quad [0,l] \tag{17.1}$$

subject to the conditions:

$$w(0) = w(l) = w_{,x}(0) = w_{,x}(l) = 0, \tag{17.2}$$

where $w : [0,l] \to \mathbb{R}$ denotes the field of vertical displacements.

The corresponding primal variational formulation for such a model, is expressed by the functional $J : \mathcal{U} \to \mathbb{R}$, where:

$$J(w) = \int_0^l \frac{1}{2}(EI(w_{,xx})^2 + \frac{a}{6}(w_{,x})^4 - \lambda(w_{,x})^2)dx - \int_0^l fwdx \tag{17.3}$$

Here E denotes the Young Modulus related to a specific material, $I = \frac{bh^3}{12}$ for a beam with rectangular cross section (rectangle basis b and height h), a is a constant related to the cross section area. Furthermore, l denotes the beam length (in fact the beam is represented by the set $[0,l] = \{x \in \mathbb{R} \mid 0 \leq x \leq l\}$, λ denotes an axial compressive load applied to $x = l$ and finally, $f(x)$ denotes the distributed vertical load.

Also, we define:

$$\mathcal{U} = \{w \in W^{2,2}([0,l]) \cap W^{1,4}([0,l]) \mid w(0) = w(l) = 0 = w_{,x}(0) = w_{,x}(l)\}. \tag{17.4}$$

Remark 17.1.1. *The boundary conditions refer to a clamped beam at x=0 and x=l.*

We consider two different problems.

$$\text{Problem } \mathcal{P}_1: \text{ To determine } w_0 \in \mathcal{U}, \text{ such that } J(w_0) = \inf_{w \in \mathcal{U}} \{J(w)\} \tag{17.5}$$

$$\text{Problem } \mathcal{P}_2: \text{ To determine } w_0 \in \mathcal{U}^+, \text{ such that } J(w_0) = \inf_{w \in \mathcal{U}^+} \{J(w)\} \tag{17.6}$$

where $\mathcal{U}^+ = \{w \in \mathcal{U}, such \;\; that \;\; w(x) \geq 0, \forall x \in [0,l]\}$

Equation (17.1) stands for the necessary conditions for Problem $\mathcal{P}_1$. The Necessary conditions for Problem $\mathcal{P}_2$ may be similarly obtained, however we postpone their presentation for the next sections.

Remark 17.1.2. *From the Sobolev Imbedding Theorem (Adams [1], page 85) we have the following result (case A for $mp > n$):*

$$W^{j+m,p}(\Omega) \rightarrow W^{j,q}(\Omega), \tag{17.7}$$

for $p \leq q < \infty$. For the present case we have $m = n = 1$, $p = 2$, $j = 1$ and $q = 4$, which means:

$$W^{2,2}([0,l]) \subset W^{1,4}([0,l]) \tag{17.8}$$

so that

$$W^{2,2}([0,l]) \cap W^{1,4}([0,l]) = W^{2,2}([0,l]). \tag{17.9}$$

17.2 Existence and Regularity Results for Problem $\mathcal{P}_1$

In this section we show the existence of a minimizer for the unconstrained problem $\mathcal{P}_1$. More specifically, we establish the following result:

Theorem 17.2.1. *Given $b, h, a, l, E, \lambda \in \mathbb{R}^+$ and $f \in L^2([0,l])$ there exists at least one $w_0 \in \mathcal{U}$ such that*

$$J(w_0) = \inf_{w \in \mathcal{U}} \{J(w)\}, \tag{17.10}$$

where

$$\mathcal{U} = \{w \in W^{2,2}([0,l]) \;\;|\;\; w(0) = w(l) = 0 = w_{,x}(0) = w_{,x}(l)\} \tag{17.11}$$

and

$$J(w) = \int_0^l \frac{1}{2}(EI(w_{,xx})^2 + \frac{a}{6}(w_{,x})^4 - \lambda(w_{,x})^2)dx - \int_0^l fwdx, \forall \;\; w \in \mathcal{U}. \tag{17.12}$$

Proof: From Poincaré Inequality it is clear that J is coercive, that is:

$$\lim_{\|w\|_{\mathcal{U}}\to+\infty} J(w) = +\infty \tag{17.13}$$

where

$$\|w\|_{\mathcal{U}} = \|w\|_{W^{2,2}([0,l])}, \forall w \in \mathcal{U}. \tag{17.14}$$

Therefore since J is strongly continuous, there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = \inf_{w\in\mathcal{U}}\{J(w)\}. \tag{17.15}$$

Thus, if $\{w_n\}_{n\in\mathbb{N}}$ is a minimizing sequence (in the sense that $\lim_{n\to+\infty} J(w_n) = \alpha$), then $\{\|w_n\|_{W_0^{2,2}([0,l])}\}$ and $\{\|w_n\|_{W^{1,4}([0,l])}\}$ are bounded sequences in reflexive Banach spaces (see Remark 17.1.2).

Hence, there exists $w_0 \in W_0^{2,2}([0,L])$ and a subsequence $\{w_{nj}\} \subset \{w_n\}$ such that

$$w_{nj} \to w_0 \ \ as \ \ j \to +\infty, \ \ weakly \ \ in \ \ W_0^{2,2}([0,l]). \tag{17.16}$$

From the Rellich Kondrachov theorem, up to a subsequence, which we also denote by $\{w_{nj}\}$ we have

$$w_{nj,x} \to w_{0,x} \ \ as \ \ j \to +\infty, \ \ strongly \ \ in \ \ L^2([0,l]), \tag{17.17}$$

Furthermore we have

$$J(w) = J_1(w) - \frac{\lambda}{2}\int_0^l (w_{,x})^2 \, dx \tag{17.18}$$

where

$$J_1(w) = \int_0^l \frac{1}{2}(EI(w_{,xx})^2 + \frac{a}{6}(w_{,x})^4) \, dx - \int_0^l fw \, dx. \tag{17.19}$$

As J_1 is convex and continuous, it is also weakly lower semi-continuous, so that

$$\liminf_{k\to+\infty} J_1(w_{nj_k}) \geq J_1(w_0) \tag{17.20}$$

From this and equation (17.17), as $\{w_{nj_k}\}$ is also a minimizing sequence, we can conclude that:

$$\alpha = \inf_{w\in\mathcal{U}}\{J(w)\} = \liminf_{k\to+\infty} J(w_{nj_k}) \geq J(w_0) \tag{17.21}$$

which implies

$$J(w_0) = \alpha = \inf_{w\in\mathcal{U}}\{J(w)\}. \ \square \tag{17.22}$$

Remark 17.2.2. *We recall that from the Rellich-Kondrachov Theorem, Part III (see reference [1], page 168), for $mp > n$, we have the following compact imbedding*

$$W^{j+m,p}(\Omega) \to C^j(\bar{\Omega}_0). \tag{17.23}$$

In our case consider $n = m = 1$, $p = 2$ and $j = 1$, that is, as $w_0 \in W^{2,2}((0,l))$ (here $\Omega = \Omega_0 = (0,1)$) we can conclude that $w_0 \in C^1([0,l])$, which means that w_0 has continuous derivative in $[0,l]$ (no corners). In fact such a regularity result refers to the space $W^{2,2}([0,l])$ as a whole, not only to solution w_0. To obtain deeper results concerning regularity, we would need to evaluate the effect of necessary conditions on the solution w_0.

17.3 A Convex Dual Formulation for the Beam Model

Now, similarly to above, consider the primal variational formulation expressed by $J : U \to \mathbb{R}$, where

$$J(w) = \int_0^l \frac{EI}{2}(w_{,xx})^2 dx + \int_0^l \frac{\alpha}{2}(\frac{w_{,x}^2}{2} - \beta)^2 dx - \int_0^l fw dx, \tag{17.24}$$

where $U = W_0^{2,2}([0,l])$ and $\alpha,\ \beta$ are positive real constants. We may also write

$$J(w) = G(\Lambda w) - F(\Lambda_1 w), \tag{17.25}$$

where

$$G(\Lambda w) = \int_0^l \frac{EI}{2}(w_{,xx})^2 dx + \int_0^l \frac{\alpha}{2}(\frac{w_{,x}^2}{2} - \beta)^2 dx + \frac{K}{2}\int_0^l w_{,x}^2 dx - \int_0^l fw dx, \tag{17.26}$$

$$F(\Lambda_1 w) = \frac{K}{2}\int_0^l w_{,x}^2 dx, \tag{17.27}$$

where

$$\Lambda w = \{\Lambda_1 w, \Lambda_2 w, w\}, \tag{17.28}$$

and

$$\Lambda_1 w = w_{,x}, \quad \Lambda_2 w = w_{,xx}. \tag{17.29}$$

From Theorem 9.6.1, we have

$$\inf_{w\in U}\{J(w)\} = \inf_{z^*\in Y^*}\sup_{v^*\in A^*}\{F^*(z^*) - G^*(v^*)\}, \tag{17.30}$$

where

$$F^*(z^*) = \frac{1}{2K}\int_0^l (z^*)^2 dx, \tag{17.31}$$

and

$$G^*(v^*) = \frac{1}{2EI}\int_0^l (v_2^*)^2 dx + \frac{1}{2}\int_0^l \frac{(v_1^*)^2}{v_0^*+K}dx + \frac{1}{2\alpha}\int_0^l (v_0^*)^2 dx + \beta\int_0^l v_0^* dx, \qquad (17.32)$$

and

$$A^* = \{v^* \in Y^* \mid \Lambda^* v^* - \Lambda_1^* z^* - f = 0\}, \qquad (17.33)$$

or

$$A^* = \{(v^*, z^*) \in L^2([0,l],\mathbb{R}^4) \mid v^*_{2,xx} - v_{1,x} + z^*_{,x} = f, \quad in \quad [0,l]\}. \qquad (17.34)$$

Observe that

$$G^*(v^*) \geq \langle \Lambda w, v^* \rangle_Y - G(\Lambda w), \quad \forall w \in U, \quad v^* \in Y^*. \qquad (17.35)$$

Thus

$$-F^*(z^*) + G^*(v^*) \geq -F^*(z^*) + \langle \Lambda_1 w, z^* \rangle + \langle w, f \rangle_U - G(\Lambda w). \qquad (17.36)$$

We can make z^* an independent variable though A^*, that is, for $v_2^*(z, v_1^*)$ given by

$$v_2^*(z^*, v_1^*) = (v_2^*)'(0)x + v_2^*(0) + \int_0^x v_1^*(t)\,dt - \int_0^x z^*(t)\,dt + \int_0^x \int_0^{t_1} f\,dt\,dt_1. \qquad (17.37)$$

From (17.36), we may write

$$\begin{aligned}\sup_{z^* \in L^2([0,l])} \{-F^*(z^*) \quad &+ \quad G^*(v_2^*(v_1^*, z^*), v_1^*, v_0^*)\} \\ &\geq \sup_{z^* \in L^2([0,1])} \{-F^*(z^*) + \langle \Lambda_1 w, z^* \rangle + \langle w, f \rangle_U - G(\Lambda w)\}, \qquad (17.38)\end{aligned}$$

so that we may infer that

$$\begin{aligned}\sup_{(v_1^*, v_0^*) \in L^2([0,l];\mathbb{R}^2)} \inf_{z^* \in L^2([0,1])} \left\{ \frac{1}{2K}\int_0^1 (z^*)^2 dx \quad \right. &- \quad \frac{1}{2EI}\int_0^1 (v_2^*(z^*, v_1^*))^2 dx \\ &\left. -\frac{1}{2}\int_0^1 \frac{(v_1^*)^2}{v_0^*+K}dx - \frac{1}{2\alpha}\int_0^1 (v_0^*)^2 dx - \beta\int_0^1 v_0^* dx \right\} \\ &\leq \inf_{w \in U}\{J(w)\}. \qquad (17.39)\end{aligned}$$

Observe that the infimum for the dual formulation indicated in (17.39) is attained, for $K < EI/K_0$ (here K_0 denotes the constant concerning Poincaré Inequality), through the relation

$$v_2^* = \frac{EIz_{,x}}{K}, \quad z^*(0) = z^*(l) = 0 \qquad (17.40)$$

so that the final format of our duality principle is given by

$$\inf_{u \in U}\{J(w)\} = \sup_{(z^*,v_1^*,v_0^*)\in B^*}\left\{-\frac{EI}{2K^2}\int_0^1 (z^*_{,x})^2\,dx + \frac{1}{2K}\int_0^1 (z^*)^2\,dx \right.$$
$$\left. -\frac{1}{2}\int_0^1 \frac{(v_1^*)^2}{v_0^*+K}\,dx - \frac{1}{2\alpha}\int_0^1 (v_0^*)^2 dx - \beta\int_0^1 v_0^*\,dx\right\}. \quad (17.41)$$

Defining $Y_0^* = W_0^{1,2}[0,l] \times L^2([0,l],\mathbb{R}^2)$ we have

$$B^* = \{(z^*,v_1^*,v_0^*) \in Y_0^* \mid \frac{EI}{K}z^*_{,xxx} - v^*_{1,x} + z^*_{,x} = f, \;\; and \;\; v_0^* + K > 0, \;\; a.e. \;\; in \;\; [0,l]\}. \quad (17.42)$$

Remark 17.3.1. *It is important to emphasize that the equality indicated in (17.41) holds only if there exists a critical point for the dual formulation such that $v_0^*+K>0$ a.e. in $[0,l]$ and $K < EI/K_0$, where, as above mentioned, K_0 is the constant concerning the Poincaré inequality. In such a case, the dual formulation is convex.*

17.4 A Necessary Condition for Problem $\mathcal{P}_2$

We recall Problem $\mathcal{P}_2$

$$To \;\; determine \;\; w_0 \in \mathcal{U}^+, \;\; such \;\; that \;\; J(w_0) = \inf_{w \in \mathcal{U}^+}\{J(w)\} \quad (17.43)$$

where

$$\mathcal{U}^+ = \{w \in \mathcal{U}, such \;\; that \;\; w(x) \geq 0, \forall x \in [0,l]\}.$$

Given $w \in \mathcal{U}^+$, let us extend it to $z \in \mathcal{U}$ by:

$$w = |z|, \quad (17.44)$$

also, define $J_1 : \mathcal{U} \to \mathbb{R}$, by:

$$J_1(z) = J(|z|) \quad (17.45)$$

Hence we can establish *Problem* $\mathcal{P}_3$ (which is equivalent to *problem* $\mathcal{P}_2$):

$$To \;\; determine \;\; z_0 \in \mathcal{U}, \;\; such \;\; that \;\; J_1(z_0) = \inf_{z \in \mathcal{U}}\{J(|z|)\}. \quad (17.46)$$

The Euler-Lagrange Equation for *Problem* $\mathcal{P}_3$ is expressed through the boundary value problem (which we call the Modular Necessary Condition):

$$EIz_{,xxxx} - a(z_{,x})^2 z_{,xx} + \lambda z_{,xx} - f\frac{z}{|z|} = 0, \forall x \in (0,l) \;\; such \;\; that \;\; z(x) \neq 0, \quad (17.47)$$

$$z(0) = z(l) = z_{,x}(0) = z_{,x}(l) = 0 \quad (17.48)$$

Remark 17.4.1. *Observe that in equation (17.47) the term* $f\frac{z}{|z|}$ *is not defined if* $z(x) = 0$. *We can solve this problem and obtain consistent numerical results, if we regularize the function* $|z| = \sqrt{z^2}$ *through the relation* $|z| \simeq \sqrt{z^2 + \varepsilon}$, *for* $\varepsilon > 0$ *such that* $\mathcal{O}(\varepsilon) \simeq 0$.

17.5 A Similar Two-dimensional Model

Finally, we analyze a two-dimensional model which is similar to the presented beam model. Consider $S \subset \mathbb{R}^2$ open, bounded, connected and with a regular boundary denoted by ∂S, and, $J : U \to \mathbb{R}$ defined as:

$$J(u) = \frac{\varepsilon}{2}\int_S (\nabla^2 u)^2 dS + \frac{1}{2}\int_S (|\nabla u|^2 - 1)^2 dS - \langle u, f\rangle \tag{17.49}$$

where $U = \{u \in W^{1,2}(S) \mid u = u_0 \text{ on } \partial S\}$.

Of great interest in the literature is the system behavior as $\varepsilon \to 0$. Anyway, the problem for which we obtain numerical results is a little different, and is defined by the functional, also denoted by $J : U \to \mathbb{R}$, as indicated below,

$$J(u) = \frac{1}{2}\int_S (|\nabla u|^2 - 1)^2 dS + \frac{1}{2}\int_S u^2 dS - \langle u, f\rangle \tag{17.50}$$

Theorem 17.5.1. *Consider* $J : U \to \mathbb{R}$ *defined as above. Thus we can write*

$$\inf_{u\in U}\{J(u)\} \leq$$

$$\inf_{z^*\in Y^*}\left\{\int_S \frac{1}{2K}|z^*|^2 dS + \sup_{v^*\in A^*}\left\{-\frac{1}{2}\int_S \frac{|v_1^*|^2}{v_0^* + K} dS - \frac{1}{2}\int_S (v_0^*)^2 dS - \frac{1}{2}\int_S (v_2^*)^2 dS\right\}\right\} \tag{17.51}$$

where $A^* = \{v^* \in Y^* \mid div(z^*) + div(v_1^*) - v_2^* + f = 0, \text{ a.e. in } S\}$.

For a proof, we use again Theorem 9.6.1. The idea is to redefine $J : U \to$, as

$$J(u) = G(\Lambda u) - F(\Lambda_1 u) \tag{17.52}$$

where

$$G(\Lambda u) = \frac{1}{2}\int_S (|\nabla u|^2 - 1)^2 dS + \frac{1}{2}\int_S u^2 dS + \frac{K}{2}\int_S |\nabla u|^2 dS \tag{17.53}$$

and

$$F(\Lambda_1 u) = \frac{K}{2}\int_S |\nabla u|^2 dS - \langle u, f\rangle_{L^2(S)}. \tag{17.54}$$

Perhaps, it is not easy to see that the dual formulation is convex for $v_0^* \geq 0$, *a.e. in* S.

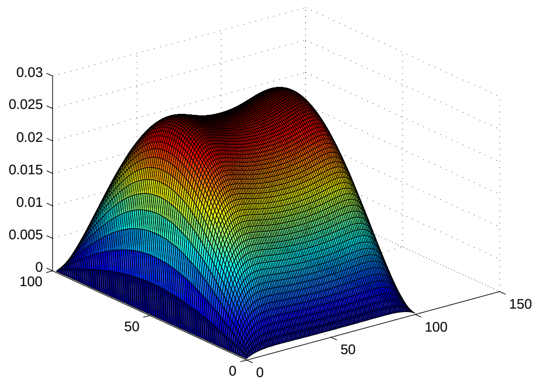

Figure 17.1: Vertical axis: solution $u_0(x, y)$ for the dual problem

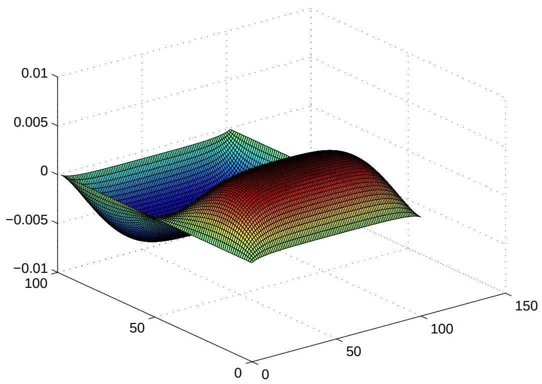

Figure 17.2: Vertical axis: solution $u_0(x, y)$ for the dual problem

However, through the extremal conditions we have $v_0^* = (|\nabla u|^2 - 1)$ and it is clear that for $v_0^* < 0$ the Weierstrass necessary condition is not satisfied, that for such points we have $g_5(x) \neq g_5^{**}(x)$, where $g_5(x) = \frac{1}{2}(x^2-1)^2$. Therefore, the inequality indicated in (17.51) is in fact an equality, and there is no duality gap, if we proceed the maximization indicated in (17.51) restricted to $v_0^* \geq 0$.

We developed numerical results for two examples. We define $S = [0, 1] \times [0, 1]$ and for the example $f(x, y) = 0.3 * sin(\pi x)$, which the graph is indicated in figure 17.1.

For the second example, $f(x, y) = 0.3 * cos(\pi x)$, and the respective graph is indicated in figure 17.2. The computation was done first evaluating the supremum in v^* in (17.51). Then for v^* fixed, the value of z^* is up-dated through the calculation of the infimum (in z^*). Then we calculate the supremum in v^* again and go on until convergence is achieved.

17.6 Conclusion

In this chapter we present existence theory following the direct method of calculus of variations and a duality principle for the non-convex variational formulation concerning Gao's beam model. Also, we introduce the concept of modular necessary condition. Finally, we present duality, in fact a convex dual variational formulation, for a bi-dimensional phase transitional problem closed related to Gao's beam model. The numerical results were obtained through the solution of the dual problem. It is important to emphasize that the solution of the dual problem is not a minimizer for the primal one, but it is a cluster point of a weakly convergent minimizing sequence.

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