Homology of Group Von Neumann Algebras

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(ABSTRACT)

In this paper the following conjecture is studied: the group von Neumann algebra $\mathcal{N}(G)$ is a flat $\mathbb{C}G$-module if and only if the group G is locally virtually cyclic. This paper proves that if G is locally virtually cyclic, then $\mathcal{N}(G)$ is flat as a $\mathbb{C}G$-module. The converse is proved for the class of all elementary amenable groups without infinite locally finite subgroups. Foundational cases for which the conjecture is shown to be true are the groups $G = \mathbb{Z}$, $G = \mathbb{Z} \oplus \mathbb{Z}$, $G = \mathbb{Z} * \mathbb{Z}$, Baumslag-Solitar groups, and some infinitely-presented variations of Baumslag-Solitar groups. Modules other than $\mathcal{N}(G)$, such as ℓ^p-spaces and group C^*-algebras, are considered as well. The primary tool that is used to achieve many of these results is group homology.

Contents

Chapter 1

Preliminary Material

1.1 Introduction and Main Results

This paper is mainly concerned with the connections between discrete groups G and their associated "group von Neumann algebras" $\mathcal{N}(G)$. Since group von Neumann algebras can be explored in either analytic contexts (since they are von Neumann algebras) or algebraic contexts (since they are rings, among other things), there are many possible avenues for studying $\mathcal{N}(G)$. In this paper, we will explore the algebraic side of $\mathcal{N}(G)$ and how it relates to certain group properties. In particular, we will primarily think of $\mathcal{N}(G)$ as a module over the complex group ring $\mathbb{C}G$. My original motivation was a conjectured connection between $\mathcal{N}(G)$ and the amenability of a group, posed by Wolfgang Lück (see [31], page 262):

Conjecture (A)**.** *A group G is amenable if and only if the group von Neumann algebra $\mathcal{N}(G)$ is dimension-flat as a $\mathbb{C}G$-module.*

The "dimension" being referred to is the von Neumann dimension function, which will be described below. And to say that $\mathcal{N}(G)$ is "dimension-flat" means that for every $\mathbb{C}G$-module M, $\dim_{\mathcal{N}(G)} \operatorname{Tor}_1^{\mathbb{C}G}(\mathcal{N}(G), M) = 0$. Lück has proved the "only if" direction of this conjecture (see [31], Theorem 6.37). However, the other half of Conjecture A is still open.

Much of my work has been on another conjecture which is closely related to Conjecture A. The following conjecture was also introduced by Lück in [31]:

Conjecture (B)**.** *A group G is locally virtually cyclic if and only if the group von Neumann algebra $\mathcal{N}(G)$ is flat as a $\mathbb{C}G$-module.*

Since both halves of this conjecture will be featured extensively in this paper, it will be helpful for the purpose of self-reference to split it into two smaller conjectures:

Conjecture 1.1.1. *Let G be a group.*

(A) If G is locally virtually cyclic, then $\mathcal{N}(G)$ is flat as a $\mathbb{C}G$-module.

(B) If $\mathcal{N}(G)$ is flat as a $\mathbb{C}G$-module, then G is locally virtually cyclic.

The most standard definition of a flat module is as follows:

Definition 1.1.2. Let R be a ring, and let M be a right R-module. Then M is flat if the tensor functor $M \otimes_R -$ is exact. Similarly, if M is a left R-module, then M is called flat if $- \otimes_R M$ is exact.

However, there is an equivalent definition which is more useful in the present context (see [40], Theorem 8.9):

Definition 1.1.3. Let R be a ring, and let M be a right R-module. Then M is flat if for every left R-module N, $\operatorname{Tor}_1^R(M, N) = 0$. Similarly, if M is a left R-module, then M is flat if for every right R-module N it is true that $\operatorname{Tor}_1^R(N, M) = 0$.

Note that, as a consequence of Definition 1.1.3, the dimension-flatness of $\mathcal{N}(G)$ is a weaker condition than flatness of $\mathcal{N}(G)$ over the ring $\mathbb{C}G$. And, naturally, the property of a group being amenable is a weaker condition than requiring it to be locally virtually cyclic. One of the results of this paper is a proof of Conjecture 1.1.1A (see Chapter 3). However, just as with Lück's first conjecture, the half in Conjecture 1.1.1B is the more difficult piece. It is still open in its most general form, but the main results of this paper prove it for certain classes of groups (see Chapter 5). In particular, it will be shown that Conjecture 1.1.1B is true for groups G such that G is an elementary amenable group with no infinite locally finite subgroups.

The foundational cases for this result are the rank-two free abelian group $\mathbb{Z} \oplus \mathbb{Z}$, certain Baumslag-Solitar groups $B(1, n)$, and other groups we are denoting $G_{m,n}$, which are related to Baumslag-Solitar groups. The first of these three cases is studied in Chapter 2. The latter two cases are studied in Chapter 4.

Other special cases which fall outside the purview of those theorems have also been considered. In particular, certain groups with infinite locally finite subgroups (such as the Lamplighter group) have been shown to also be consistent with Conjecture 1.1.1B (see Chapter 6).

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1.2 Group Theory Terminology

The term "group" will always refer to a discrete group unless more structure is explicitly mentioned. In Conjecture A, the idea is that one can use certain "L^2-invariants" to determine if a group is amenable. There are many equivalent ways of defining amenability, but the most classical definition is as follows.

Definition 1.2.1. A group G is amenable if there is a measure μ on G such that

1. The measure is a probability measure, meaning $\mu(G) = 1$.
2. The measure is finitely additive, meaning $\mu\left(\bigcup_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mu(X_i)$ if X_i are disjoint subsets of G.
3. The measure is left-invariant, meaning $\mu(gA) = \mu(A)$ for all $A \subset G$ and $g \in G$.

One of the more commonly used of the equivalent definitions of amenability, and the one which Lück uses in his proof of the known half of Conjecture A, is the following: a group is amenable if and only if it satisfies the "Følner Condition" (see Theorem F.6.8 in [4]).

Definition 1.2.2. Let G be a group. Then G satisfies the Følner Condition if for any finite set $S \subset G$ such that $s \in S$ implies $s^{-1} \in S$, and for any $\epsilon > 0$, there exists a finite nonempty subset $A \subset G$ such that for its S-boundary $\partial_S A = \{a \in A \mid \text{ there is } s \in S \text{ with } as \notin A\}$ we have $|\partial_S A| \leq \epsilon \cdot |A|$.

Of particular interest in this paper is a subclass of the class of amenable groups called "elementary amenable groups," and this subclass will be defined and extensively featured in Chapter 5.

With regard to Conjecture 1.1.1, it is critical to understand what it means for a group to be locally virtually cyclic. First, it is necessary to define what it means for a group to be virtually cyclic.

Definition 1.2.3. A group G is called virtually cyclic if either:

1. G is finite, or

2. G has an infinite cyclic subgroup H of finite index.

Thinking in the context of Geometric Group Theory, a group is finite if and only if it has zero ends, and it is known that a group is infinite virtually cyclic if and only if it has exactly two ends [43]. Now these groups can be used to define locally virtually cyclic groups.

Definition 1.2.4. A group is called locally virtually cyclic if every finitely generated subgroup is virtually cyclic.

1.3 Motivation for Group Von Neumann Algebras

Most prototypical examples of von Neumann algebras can be described as follows: a subset X of $\mathcal{B}(H)$ for some complex Hilbert space H, which is closed under all of the algebraic operations on $\mathcal{B}(H)$ (addition, multiplication, scalar multiplication), is closed with respect to the weak operator topology, and is closed under the adjoint operation. For example, the ring of essentially bounded functions on the real line $L^\infty(\mathbb{R})$ is a von Neumann algebra for the Hilbert space of square-integrable functions $L^2(\mathbb{R})$. More generally, von Neumann algebras are defined as follows.

Definition 1.3.1. An involution on a complex Banach algebra A is a map $* : A \to A$ such that $(aS + bT)^* = \overline{a}S^* + \overline{b}T^*$, $(ST)^* = T^*S^*$, and $(T^*)^* = T$ for all $S, T \in A$ and $a, b \in \mathbb{C}$. A von Neumann algebra is a complex Banach algebra (with a unit element I) with an involution that is closed under the weak operator topology.

The concept of von Neumann algebras was first introduced in 1929 by John von Neumann [45]. He and Francis Murray developed the basic theory of von Neumann algebras in the 1930s and 1940s, primarily by classifying the types of factors they can have [46]. Von Neumann was well-known for being extremely influential in a multitude of topics in and around Mathematics, and he found von Neumann algebras to be relevant in several different contexts including Operator Theory, Ergodic Theory, quantum mechanics, and group representations [23].

Indeed, while group representations for finite groups can be completely classified using Character Theory ([11], Chapter 18), von Neumann algebras arise naturally when studying representations for infinite groups. The following is a classical description of the von Neumann algebras in Representation Theory of groups (see chapter 1 of [33]). Define a *group representation* of a group G to be a homomorphism $G \to GL(V)$, where $GL(V)$ is the general linear group on a vector space V. In particular, consider group representations in which V is a complex separable Hilbert space, and denote the Hilbert space of a representation R by $H(R)$. Let L and M be representations of a group G, and let $T : H(L) \to H(M)$ be a bounded linear operator. Then T is called an *intertwining operator* for L and M if $TL_x = M_xT$ for all $x \in G$; let $R(L, M)$ denote the vector space of all such intertwining operators. These spaces can be useful in studying the representations of an infinite group. For example, $R(L, M) = 0$ if and only if no subrepresentation of L is equivalent to a subrepresentation of M. If $L = M$, then $R(L, L)$ is a von Neumann algebra. In particular, let L be the left regular representation, where $H(L) = \ell^2(G)$ and $L_x(\alpha) = x \cdot \alpha$ for all $x \in G$ and $\alpha \in \ell^2(G)$. Then $R(L, L)$ is the group von Neumann algebra $\mathcal{N}(G)$.

Group von Neumann algebras have played a role in both Group Theory and the analytical study of von Neumann algebras. They are vital to the study of general von Neumann algebra theory, since they provide an abundance of interesting examples of von Neumann algebras. For example, group von Neumann algebras were used to create the first example of a von Neumann algebra with uncountably many different separable type II_1 factors [34]. As noted above, von Neumann algebras are relevant to Group Theory by way of group representations. In contrast to the algebraic nature of the featured conjectures of this paper, there is also much active research in which groups are studied by investigating von Neumann algebras in an analytic context. For instance, for certain classes of groups, $\mathcal{N}(G) \cong \mathcal{N}(\Gamma)$ as von Neumann algebras if and only if $G \cong \Gamma$ as groups [22]. And sometimes an algebraic property of a group can be recognized from an analytic property of $\mathcal{N}(G)$; e.g., a group is exact if and only if its group von Neumann algebra is "weakly exact" [36]. Von Neumann algebras other than $\mathcal{N}(G)$ can also be used to study G. For example, every measure-preserving group action on a probability space yields a von Neumann algebra, and information about the orbits of the action can be gleaned from the isomorphism class of this von Neumann algebra [38].

Group von Neumann algebras are also the foundation of so-called "L^2-invariants," which the Tor-groups and dimensions of the main conjectures are examples of. One way that L^2-invariants have been important is by providing new methods for solving seemingly unrelated problems. A few of these are listed in the preface of [31]. An example of such a result is the existence of finitely generated groups which are quasi-isometric but not measurably equivalent. L^2-invariants have also led to several high-profile open conjectures which have inspired much study. Perhaps the most famous one is the Atiyah Conjecture. Consider a ring A with $\mathbb{Z} \subseteq A \subseteq \mathbb{C}$. The Atiyah Conjecture for A and G says that for each finitely presented AG-module M we have:

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G)\otimes_{AG} M) \in \frac{1}{\mathcal{FIN}(G)}\mathbb{Z}.$$

For an overview of groups for which the Atiyah Conjecture is known to be true, see section 10.1 of [31], [28], and [42]. The Atiyah Conjecture is related to another well-known conjecture: the "Zero Divisor Conjecture." One version of the ZDC guesses that if G is torion-free and $0 \neq \alpha, \beta \in \mathbb{C}G$, then $\alpha\beta \neq 0$. This question has also been studied with $\beta \in \ell^2(G)$, $\beta \in \mathcal{N}(G)$, or $\beta \in \ell^p(G)$ for other p-values. Some of the results in this paper will depend on known cases of the ZDC. For an overview of results related to this conjecture see [28], [39], and [29].

All the examples above show that group von Neumann algebras have proven to be relevant and useful in a vast variety of contexts. The particular context of $\mathcal{N}(G)$ featured in this paper, which has not been studied as extensively as many of the contexts above, is $\mathcal{N}(G)$ as a $\mathbb{C}G$-module and a ring.

1.4 Group Von Neumann Algebras

In this section, group von Neumann algebras will be defined explicitly and placed into the context of Conjecture 1.1.1. Let G be a group. First, ℓ^p-spaces can be defined on G as follows:

Definition 1.4.1. Let G be a group and $p \in \mathbb{R}^+$. The $\underline{\ell^p\text{-space}}$ of G, denoted $\ell^p(G)$, is the vector space of p-summable formal sums of complex numbers over G. In other words:

$$\ell^p(G) = \{\sum_{g\in G} a_g \cdot g \mid a_g \in \mathbb{C} \text{ and } \sum_{g\in G} |a_g|^p < \infty\}.$$

For all $p \in \mathbb{N}$, $\ell^p(G)$ is a normed space. In the special case of $p = 2$, $\ell^p(G)$ is a Hilbert space. In particular, $\ell^2(G)$ is the Hilbert space will Hilbert basis G, and the inner product is defined as:

$$\left\langle \sum_{g\in G} a_g \cdot g, \sum_{g\in G} b_g \cdot g \right\rangle = \sum_{g\in G} a_g \overline{b_g}.$$

Note that for all p, the complex group ring $\mathbb{C}G$ is contained within $\ell^p(G)$. Furthermore, there is a natural action of $\mathbb{C}G$ on $\ell^p(G)$, so that $\ell^p(G)$ can viewed as a $\mathbb{C}G$-module. This places the ℓ^p-spaces into an algebraic context. Since $\mathbb{C}G \subseteq \ell^2(G)$, one might wonder if the multiplication which makes $\mathbb{C}G$ a ring can be extended to make $\ell^2(G)$ a ring. In fact, the multiplication on the group ring can be extended in a natural way to $\ell^2(G)$, but the operation is usually not closed, and thus $\ell^2(G)$ is not a ring. Specifically, this multiplication is defined as follows:

$$\ell^2(G) \times \ell^2(G) \to \ell^\infty(G) = \{\sum_{g\in G} a_g g \mid \sup_{g\in G} |a_g| < \infty\},$$

$$\sum_{g\in G} a_g g \sum_{g\in G} b_g g = \sum_{h,g\in G} a_h b_g hg = \sum_{g\in G} (\sum_{x\in G} a_{gx^{-1}} b_x) g.$$

At this point, the most concise way to define the group von Neumann algebra is that $\mathcal{N}(G)$ is the largest subspace of $\ell^2(G)$ which is a ring under this operation of multiplication. More precisely:

Definition 1.4.2. Let G be a group. The group von Neumann algebra of G is defined as:

$$\mathcal{N}(G) = \{\alpha \in \ell^2(G) \mid \alpha\beta \in \ell^2(G) \text{ for all } \beta \in \ell^2(G)\}.$$

For all groups G, $\mathcal{N}(G)$ is an algebra, and $\mathbb{C}G \subseteq \mathcal{N}(G) \subseteq \ell^2(G)$. In the most natural way, we will consider $\mathcal{N}(G)$ to be a $\mathbb{C}G$-module. And since $\mathcal{N}(G)$ is a ring, it is also natural to consider $\ell^2(G)$ as a $\mathcal{N}(G)$-module, which will occasionally be quite useful. As a ring, $\mathcal{N}(G)$ is noncommutative if G is nonabelian. It typically has many zero divisors, as will become evident in the examples of $G = \mathbb{Z}$ and $G = \mathbb{Z} \oplus \mathbb{Z}$ in Chapter 2. As a ring, the group von Neumann algebra also has the property of being semihereditary, which means any finitely generated submodule of a projective $\mathcal{N}(G)$-module is itself projective (see [31], Theorem 6.5 and Theorem 6.7).

The definition above of $\mathcal{N}(G)$ is an algebraic definition. One useful aspect of group von Neumann algebras is that there is also an equivalent analytic way of defining $\mathcal{N}(G)$. Consider all bounded linear operators from the Hilbert space $\ell^2(G)$ to itself, denoted $\mathcal{B}(\ell^2(G))$. In particular, so-called G-equivariant operators will be of interest.

Definition 1.4.3. Let F be a map from $\ell^2(G)$ to $\ell^2(G)$. Then F is called G-equivariant if $F(x \cdot g) = F(x) \cdot g$ for all $g \in G$ and $x \in \ell^2(G)$, with respect to the natural right G-action on $\ell^2(G)$.

The set of all such bounded linear operators constitutes $\mathcal{N}(G)$, putting it into an analytic context.

Definition 1.4.4. Let G be a group. The group von Neumann algebra $\mathcal{N}(G)$ is defined as the algebra of G-equivariant bounded linear operators from $\ell^2(G)$ to $\ell^2(G)$. Symbolically, $\mathcal{N}(G) = \mathcal{B}(\ell^2(G))^G$.

The equivalence of Definition 1.4.2 and Definition 1.4.4 can be achieved with the following correspondence: for every $\alpha \in \mathcal{N}(G)$ (in the algebraic sense), define an operator $F_\alpha : \ell^2(G) \to \ell^2(G)$ by $F_\alpha(x) = \alpha \cdot x$ for all $x \in \ell^2(G)$. Many of the facts about $\mathcal{N}(G)$ which are critical for the main results of this paper are proved using this analytic perspective, as will be noted when such facts are referenced.

1.5 Von Neumann Dimension

In this section and the next, two concepts related to group von Neumann algebras will be discussed: the existence of a trace on $\mathcal{N}(G)$ and an Ore localization of $\mathcal{N}(G)$. The trace will be the foundation for the so-called von Neumann dimension function, which is a centerpiece of Conjecture A. The Ore localization will be critical for showing $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$ for certain groups.

For every group G, there is a complex-valued "trace" which can be defined on $\mathcal{N}(G)$. Within the context of the algebraic definition of $\mathcal{N}(G)$ (i.e., Definition 1.4.2), it is defined as follows:

Definition 1.5.1. Let G be a group, and let $\alpha = \sum_{g \in G} a_g \cdot g \in \mathcal{N}(G)$. Then define $tr_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$, the von Neumann trace of $\mathcal{N}(G)$, to be $tr_{\mathcal{N}(G)}(\alpha) = a_e$, where $e \in G$ is the identity element.

There is also an equivalent definition with respect to the analytic version of group von Neumann algebras (Definition 1.4.4):

Definition 1.5.2. Let G be a group, and consider an operator $\alpha : \ell^2(G) \to \ell^2(G)$ in $\mathcal{N}(G)$. Then define $tr_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$, the von Neumann trace of $\mathcal{N}(G)$, to be $tr_{\mathcal{N}(G)}(\alpha) = \langle \alpha(e), e \rangle$.

This trace can be extended in a natural way to be defined on square matrices with entries in $\mathcal{N}(G)$; if n is a natural number and $A = (a_{ij}) \in M_n(\mathcal{N}(G))$, then define $tr_{\mathcal{N}(G)}(A) = \sum_{i=1}^{n} tr_{\mathcal{N}(G)}(a_{ii})$. Now let $\mathcal{M}$ be the category of right $\mathcal{N}(G)$-modules. We would like to use the above definition of trace to define a dimension function $\dim_{\mathcal{N}(G)} : \mathcal{M} \to [0, \infty]$. Note that the definition outlined below has a straightforward analogue for the category of left $\mathcal{N}(G)$-modules. First, let P be a finitely generated projective $\mathcal{N}(G)$-module. Since P is finitely generated, there is a finitely generated free module $\mathcal{N}(G)^n$ which maps onto P. And since P is projective, P is isomorphic to the image of a $\mathcal{N}(G)$-homomorphism $\mathcal{N}(G)^n \to \mathcal{N}(G)^n$. There exists $A \in M_n(\mathcal{N}(G))$ with $A^2 = A$ such that the map above can realized as left-multiplication by A. Now define the von Neumann dimension of P as $\dim_{\mathcal{N}(G)}(P) = tr_{\mathcal{N}(G)}(A)$. It is not obvious that this definition of the dimension function on finitely-generated projective modules is well-defined, but there is an explanation in Section 6.1 of [31] of why it is indeed well-defined. This dimension function may now be extended to be defined on all of $\mathcal{M}$:

Definition 1.5.3. For a group G and $M \in \mathcal{M}$, define the von Neumann dimension function $\dim_{\mathcal{N}(G)} : \mathcal{M} \to [0, \infty]$ by:

$$\dim_{\mathcal{N}(G)}(M) = \sup\{\dim_{\mathcal{N}(G)}(P) \mid P \subseteq M \text{ finitely generated projective submodule}\}.$$

This is the dimension function featured in Conjecture A. It comes equipped with several useful properties (see [31], Theorem 6.7), such as the following two:

Theorem 1.5.4. *Let G be a group.*

1. *Additivity: If $0 \to A \to B \to C \to 0$ is an exact sequence of $\mathcal{N}(G)$-modules, then $\dim_{\mathcal{N}(G)}(B) = \dim_{\mathcal{N}(G)}(A) + \dim_{\mathcal{N}(G)}(C)$.*

2. *Cofinality: Let $\{M_i \mid i \in I\}$ be a cofinal system of submodules of an $\mathcal{N}(G)$-module M. In other words, $M = \bigcup_{i \in I} M_i$, and for all $i, j \in I$, there exists $k \in I$ such that M_i, $M_j \subset M_k$. Then $\dim_{\mathcal{N}(G)}(M) = \sup\{\dim_{\mathcal{N}(G)}(M_i) \mid i \in I\}$.*

The von Neumann dimension function leads to many "L^2-invariants," such as the dimensions of the Tor groups mentioned earlier, which can be useful in studying groups. The most commonly studied of these invariants are the L^2-Betti numbers.

Definition 1.5.5. Let G be a group. Then the n-th $\underline{L^2\text{-Betti number}}$, denoted $b_n^{(2)}(G)$, is the von Neumann dimension of the n-th group homology with coefficients in $\mathcal{N}(G)$:

$$b_n^{(2)}(G) = \dim_{\mathcal{N}(G)}(H_n(G, \mathcal{N}(G)).$$

1.6 Algebras of Affiliated Operators

There is another algebra, $\mathcal{U}(G)$, which contains the group von Neumann algebra of a group. Specifically, it is the Ore localization of $\mathcal{N}(G)$. In order for an Ore localization to exist for a ring, a certain condition must be satisfied:

Definition 1.6.1. Let R be a ring, and let S be a multiplicatively closed subset of R. The pair (R, S) satisfies the (right) Ore condition if

1. for all $(r, s) \in R \times S$, there is $(r', s') \in R \times S$ such that $rs' = sr'$, and

2. for all $r \in R$ and $s \in S$ with $sr = 0$, there is $t \in S$ with $rt = 0$.

The first condition guarantees that any "left fraction" $s^{-1}r$ in the desired localization can be converted into a "right fraction" $r'(s')^{-1}$. Note that the second condition is automatically met if S has no zero divisors. If a pair (R, S) satisfies the Ore condition, consider the following equivalence relation on $R \times S$: $(r, s) \sim (r', s')$ if there exist $u, u' \in R$ such that $ru = r'u'$, $su = s'u'$, and $su \in S$. Now the Ore localization may be defined.

Definition 1.6.2. If (R, S) satisfies the Ore condition, define the (right) Ore localization to be the following ring RS^{-1}. Elements of RS^{-1} are elements of $(R \times S)/\sim$, with the following operations:

1. Addition: $(r, s) + (r', s') = (rc + r'd, t)$, where $t = sc = s'd \in S$.
2. Multiplication: $(r, s) \cdot (r', s') = (rc, s't)$, where $sc = r't$ for $t \in S$.

In this paper, elements of RS^{-1} will usually be written with inverse notation rs^{-1} rather than ordered-pair notation (r, s). There are two facts about Ore localizations in particular that will be useful (see page 51 of [44]).

Theorem 1.6.3. *Suppose (R, S) satisfies the Ore condition. Then:*

1. *The map $f : R \to RS^{-1}$ defined by $f(r) = (r, 1)$ is a ring homomorphism, and if S has no zero divisors then f is injective.*
2. *The functor $- \otimes_R RS^{-1}$ is exact.*

A key fact about group von Neumann algebras, which is proved using the analytic version of $\mathcal{N}(G)$, is in the next theorem (Theorem 8.22(1) in [31]):

Theorem 1.6.4. *Let G be a group, and let S be the set of non-zero-divisors of $\mathcal{N}(G)$. Then the pair $(\mathcal{N}(G), S)$ satisfies the Ore condition.*

This leads to the definition of $\mathcal{U}(G)$.

Definition 1.6.5. Let G be a group. Define the algebra of affiliated operators, denoted as $\mathcal{U}(G)$, as the Ore localization $\mathcal{N}(G)S^{-1}$, where S is the set of non-zero-divisors in $\mathcal{N}(G)$.

The definition above puts $\mathcal{U}(G)$ into an algebraic context. However, just as $\mathcal{N}(G)$ had, the algebra of affiliated operators has an equivalent definition which is more in the realm of analysis.

Definition 1.6.6. Let G be a group.

1. An unbounded linear operator $f : \ell^2(G) \to \ell^2(G)$ is called affiliated (to $\mathcal{N}(G)$) if f is densely defined, closed, and G-equivariant.
2. The algebra of affiliated operators $\mathcal{U}(G)$ is the algebra of operators $f : \ell^2(G) \to \ell^2(G)$ affiliated to $\mathcal{N}(G)$.

The equivalence of these two definitions for $\mathcal{U}(G)$ is proved in Theorem 8.22 in [31].

Just as the category of $\mathcal{N}(G)$-modules has a dimension function, the category of $\mathcal{U}(G)$-modules has one as well. The foundation for this dimension function is the following connection between $\mathcal{N}(G)$-modules and $\mathcal{U}(G)$-modules (Theorem 8.22(7) in [31]):

Theorem 1.6.7. *Given a finitely generated projective $\mathcal{U}(G)$-module Q, there is a finitely generated projective $\mathcal{N}(G)$-module P such that $Q \cong \mathcal{U}(G) \otimes_{\mathcal{N}(G)} P$ as $\mathcal{U}(G)$-modules. If P_0 and P_1 are two finitely generated projective $\mathcal{N}(G)$-modules, then $P_0 \cong P_1$ if and only if $\mathcal{U}(G) \otimes_{\mathcal{N}(G)} P_0 \cong \mathcal{U}(G) \otimes_{\mathcal{N}(G)} P_1$ as $\mathcal{U}(G)$-modules.*

The fact that every finitely generated projective $\mathcal{U}(G)$-module can be realized as an induced module from an $\mathcal{N}(G)$-module facilitates the extension of the von Neumann dimension function.

Definition 1.6.8. Let Q be a finitely generated projective $\mathcal{U}(G)$-module. Define $\dim_{\mathcal{U}(G)}(Q) := \dim_{\mathcal{N}(G)}(P)$, where P is as described in Theorem 1.6.7. Now let M be a $\mathcal{U}(G)$-module. Define $\dim_{\mathcal{U}(G)}(M) := \sup\{\dim_{\mathcal{U}(G)}(Q) \mid Q \subseteq M \text{ finitely generated projective submodule}\}$.

In particular, this dimension function provides a new way to define L^2-Betti numbers for groups. For a group G, define the p-th L^2-Betti number as $b_p^{(2)}(G) = \dim_{\mathcal{U}(G)}(H_1(G, \mathcal{U}(G)))$. This definition of $b_p^{(2)}(G)$ coincides the first one because of Theorem 1.6.3(2).

1.7 Left Modules vs. Right Modules

It may be mentioned at this point that the flatness of $\mathcal{N}(G)$ as a left $\mathbb{C}G$-module is equivalent to the flatness of $\mathcal{N}(G)$ as a right $\mathbb{C}G$-module. In Lück's original statement of the conjecture, $\mathcal{N}(G)$ is considered as a right module. In some of the calculations of this paper, $\mathcal{N}(G)$ is considered as a left module. Since $\mathcal{N}(G)$ is a von Neumann algebra, it is closed under taking adjoints. Indeed, if $\alpha = \sum_{g\in G} a_g \cdot g \in \mathcal{N}(G)$, then the adjoint of α is $\alpha^* = \sum_{g\in G} \overline{a_g} \cdot g^{-1}$, which is also in $\mathcal{N}(G)$. Adjoints can be used to convert left-actions into right-actions, and vice versa, since $(\alpha\beta)^* = \beta^*\alpha^*$ for all $\alpha, \beta \in \mathcal{N}(G)$. To show the equivalence of left-flatness and right-flatness of $\mathcal{N}(G)$, it will be convenient to use the following equivalent definition of flatness (Theorem 3.53 in [40]).

Theorem 1.7.1. *Let R be a ring. If B is a right R-module such that $0 \to B \otimes_R I \to B \otimes_R R$ is exact for every finitely generated left ideal I of R, then B is flat.*

For a finitely-generated left ideal I of $\mathbb{C}G$ generated by $\{b_1, \ldots, b_n\}$, define a finitely-generated right ideal I^* to be generated by $\{b_1^*, \ldots, b_n^*\}$. And if I is a finitely-generated right ideal, similarly define a finitely-generated left ideal I^*. Consider the following property of these ideals.

Lemma 1.7.2. *Let I be a finitely generated right ideal of $R = \mathbb{C}G$, and let $\alpha = \sum x_i \otimes a_i \in I \otimes_R \mathcal{N}(G)$. Then $\alpha = 0$ if and only if $\alpha^* = \sum a_i^* \otimes x_i^* = 0$ in $\mathcal{N}(G) \otimes_R I^*$.*

Proof. Define abelian group homomorphisms $\varphi : I \otimes_R \mathcal{N}(G) \to \mathcal{N}(G) \otimes_R I^*$ by $\varphi(x \otimes a) = a^* \otimes x^*$ and $\psi : \mathcal{N}(G) \otimes_R I^* \to I \otimes_R \mathcal{N}(G)$ by $\psi(a \otimes x) = x^* \otimes a^*$. The map φ is well-defined since:

$$\varphi(xr \otimes a) = a^* \otimes (xr)^* = a^* \otimes r^*x^* = a^*r^* \otimes x^* = (ra)^* \otimes x^* = \varphi(x \otimes ra).$$

Similarly, the map ψ is well-defined. Now clearly $\varphi\psi = id$ and $\psi\varphi = id$, and so $\alpha = 0$ if and only if $\varphi(\alpha) = 0$. $\square$

With this in mind, it is now routine to show the equivalence of left-flatness and right-flatness for $\mathcal{N}(G)$.

Theorem 1.7.3. *The group von Neumann algebra $\mathcal{N}(G)$ is a flat left $\mathbb{C}G$-module if and only if $\mathcal{N}(G)$ is a flat right $\mathbb{C}G$-module.*

Proof. Let $R = \mathbb{C}G$ and $A = \mathcal{N}(G)$. Suppose A is a flat right R-module. Let I be a finitely generated right ideal of R, and consider the sequence $0 \to I \otimes A \to R \otimes A$; we would like to show this sequence is exact. Suppose $\sum x_i \otimes a_i \in I \otimes A$ is such that $\sum x_i \otimes a_i = 0$ in $R \otimes A$. Then $\sum x_i a_i = 0$, and hence $\sum a_i^* x_i^* = (\sum x_i a_i)^* = 0$. This means $\sum a_i^* \otimes x_i^* = 0$ in $A \otimes R$. Since A is a flat right R-module, it follows that $\sum a_i^* \otimes r_i^* = 0$ in $A \otimes I^*$. By the lemma, $\sum x_i \otimes a_i = 0$ in $I \otimes_R \mathcal{N}(G)$. Therefore the sequence is exact and A is a flat left R-module. Similarly, if A is a flat left R-module, then A is a flat right R-module. $\square$

Similarly, the right module $\ell^p(G)$ is flat over $\mathbb{C}G$ if and only if the left module $\ell^p(G)$ is flat over $\mathbb{C}G$ for any $1 \leq p \in \mathbb{R}$.

Chapter 2

First Calculations

2.1 Introduction

Recall Conjecture 1.1.1, which is the primary topic we are considering:

Conjecture. *A group G is locally virtually cyclic if and only if the group von Neumann algebra $\mathcal{N}(G)$ is flat as a $\mathbb{C}G$-module.*

To prove the (A) direction of this conjecture, the strategy will be to show that certain Tor-groups vanish when G is locally virtually cyclic. When dealing with the other half of the conjecture, it will usually be most convenient to consider its contrapositive: if G is not locally virtually cyclic, then $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$. So we will consider groups which are not locally virtually cyclic and aim to show the non-vanishing of at least one relevant Tor-group.

One critical case for proving the (A) direction will be the case of the infinite cyclic group, $G = \mathbb{Z}$. The most straightforward examples of groups which are not locally virtually cyclic are the free abelian group of rank two, $G = \mathbb{Z} \times \mathbb{Z}$, and the free group on two generators, $G = \mathbb{Z} * \mathbb{Z}$; these groups will be important for our consideration of the (B) direction of the conjecture. This chapter includes several calculations on these groups, which will serve as cornerstones for the main results in subsequent chapters. Many of the Tor calculations in this paper are, more specifically, "group homology" calculations. Thus, this chapter also reviews the relevant facts for computing group homology. And another section of this chapter outlines various relationships between $\mathcal{N}(G)$, $\ell^2(G)$, and $\mathcal{U}(G)$. These relationships will be critical, because some calculations which are impenetrable for the module $\mathcal{N}(G)$ are manageable when the module is $\ell^2(G)$ or $\mathcal{U}(G)$.

2.2 Group Homology

The flatness of an R-module N can be tested by calculating various associated Tor groups to see if they vanish. The Tor groups are calculated as follows. Let R be a ring, let M be a right R-module, and let N be a left R-module. First, take a free (or projective) resolution of M; i.e., construct a long exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

in which each F_i is a free R-module. Next, create a deleted complex (not necessarily exact) by applying the functor $- \otimes_R N$:

$$\cdots \to F_2 \otimes_R N \to F_1 \otimes_R N \to F_0 \otimes_R N \to 0.$$

If f_i is the map from F_i to F_{i-1} in the free resolution and f_i' is the corresponding induced map of the deleted complex, then the n-th Tor group is defined as:

$$\mathrm{Tor}_n^R(M,\, N) = \ker(f_i')/\mathrm{Im}(f_{i+1}').$$

Equivalently, $\mathrm{Tor}_n^R(M,\, N)$ may be defined by taking a free (or projective) resolution of N, applying $M \otimes_R -$ to create a deleted complex, and then taking the homology of this complex ([40], Theorem 7.9). The 0-th Tor group $\mathrm{Tor}_0^R(M,\, N)$ is always isomorphic to $M \otimes_R N$, but this group does not provide information about the flatness of N. Rather, N is flat if and only if $\mathrm{Tor}_n^R(M,\, N) = 0$ for all $n \geq 1$ and all choices of M ([40], Theorem 8.4). Furthermore, by a dimension-shifting argument, it can be seen that N is flat if and only if $\mathrm{Tor}_1^R(M,\, N) = 0$ for all choices of M. This is the characterization of flatness that will be used throughout this paper. Hence, to show N is not flat, it suffices to find one choice of

M for which $\mathrm{Tor}_1^R(M, N) \neq 0$. For $R = \mathbb{C}G$, the most convenient choice of M will often be $M = \mathbb{C}$ with trivial G-action.

Definition 2.2.1. Let N be a left $\mathbb{C}G$-module. Then the n-th group homology of N is $H_n(G, N) = \mathrm{Tor}_n^{\mathbb{C}G}(\mathbb{C}, N)$. Similarly, if M is a right $\mathbb{C}G$-module, then the n-th group homology of M is $H_n(G, M) = \mathrm{Tor}_n^{\mathbb{C}G}(M, \mathbb{C})$.

It should be mentioned at this point that group homology is usually defined in the context of $\mathbb{Z}G$-modules and calculated using resolutions of $\mathbb{Z}$. However, if M is a $\mathbb{C}G$-module, then it is also a $\mathbb{Z}G$-module and the two notions of $H_n(G, M)$ are isomorphic (see p. 4 of [5]). By the previous definition, to calculate $H_1(G, N)$ one can find a free $\mathbb{C}G$-resolution of $\mathbb{C}$, apply the functor $- \otimes_{\mathbb{C}G} N$, and then take the homology of the resulting deleted complex. To calculate H_1, the resolution only needs to be taken out a few steps:

$$\mathbb{C}G^{\alpha_2} \to \mathbb{C}G^{\alpha_1} \to \mathbb{C}G^{\alpha_0} \xrightarrow{\epsilon} \mathbb{C} \to 0.$$

For any presentation of G, α_1 may be chosen to correspond to the number of generators in the presentation, and α_2 can be the number of relations. This fact will be expanded upon in a later section. And we can always choose $\alpha_0 = 1$ with ϵ being the "augmentation map" defined by $\epsilon(\sum a_g \cdot g) = \sum a_g$. The kernel of ϵ is called the augmentation ideal, denoted $\Delta(G)$, and it is the $\mathbb{C}$-module generated by elements $g - 1$ for all $g \in G$ ([7], p. 12). For a right $\mathbb{C}G$-module M, the group of co-invariants of M, denoted M_G, is defined as $M_G = M \otimes_{\mathbb{C}G} \mathbb{C}$. If M is a left module, then $M_G = \mathbb{C} \otimes_{\mathbb{C}G} M$. Roughly speaking, M_G is obtained from M by "dividing out" the G-action. For our purposes, the most useful definition of M_G will be $\frac{M}{\Delta(G)M}$ if M is a left module and $\frac{M}{M\Delta(G)}$ if M is a right module.

Since Tor and group homology calculations will be one of the primary tools we use, there are a few critical facts about these groups that will be used extensively. First, as a consequence of the Hochschild-Serre spectral sequence ([7], Section 7.7), there is a useful relationship of group homology for group extensions.

Theorem 2.2.2. *For any group extension* $1 \to H \to G \to Q \to 1$ *and any* $\mathbb{C}G$*-module* M*,* *there is an exact sequence on homology:*

$$H_2(G,\, M) \to H_2(Q,\, M_H) \to H_1(H,\, M)_Q \to H_1(G,\, M) \to H_1(Q,\, M_H) \to 0.$$

Another important fact is the relationship between group homology and direct limits of groups ([7], p. 121).

Theorem 2.2.3. *Let $\{G_\alpha\}$ be a direct system of groups, and suppose $G = \varinjlim G_\alpha$. Then for any $\mathbb{C}G$-module M and any $n \in \mathbb{N}$, there is an isomorphism $H_1(G,\, M) \cong \varinjlim H_1(G_\alpha,\, M)$.*

Since we will occasionally need to show all Tor groups vanish for a certain choice of G and M, and since the tricks above are in terms of group homology, it will be helpful to sometimes convert Tor calculations to group homology calculations. The following theorem ([7], Proposition 2.2) makes this possible.

Theorem 2.2.4. *Let M and N be $\mathbb{C}G$-modules. If M is $\mathbb{Z}$-torsion-free, then for all $n \in \mathbb{N}$*

$$Tor_n^{\mathbb{C}G}(M,\, N) \cong H_n(G,\, M \otimes N),$$

where G acts diagonally on $M \otimes N$.

Since flatness is the main module-theoretic property being studied in this paper, most calculations will be group homology or other Tor calculations. However, for the groups $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$, some comments about group cohomology will also be made. To understand how to define group cohomology, one must first understand how to define Ext-groups, which is quite analogous to the way Tor-groups are defined. Let R be a ring, and let M and N be left R-modules. Construct a resolution of M comprising of projective R-modules; call this resolution F. Then obtain a deleted complex by applying the functor $\mathrm{Hom}_R(-,\, N)$ to F. Then the n-th Ext-group is the n-th cohomology group of this complex:

$$\mathrm{Ext}_n^R(M,\, N) = H^n\left(\mathrm{Hom}_R(F,\, N)\right).$$

This leads to the following definition of group cohomology, where the $\mathbb{C}G$-module $\mathbb{C}$ has trivial G-action.

Definition 2.2.5. Let G be a group and let N be a left $\mathbb{C}G$-module. Then the n-th group cohomology group of N is:

$$H^n(G,\, N) = \mathrm{Ext}_n^{\mathbb{C}G}(\mathbb{C},\, N).$$

2.3 Fox Derivatives

For finitely-presented groups, one tool for calculating $H_1(G,\, M)$ is the so-called Fox derivative. In particular, these derivatives can be used to partially build a free $\mathbb{C}G$-resolution of $\mathbb{C}$. The theory of Fox derivatives was developed by Ralph Fox in a series of five papers, beginning with [12] in 1953 and culminating with [13] in 1960. They are defined as follows.

Definition 2.3.1. If F is a free group with identity e and generators g_i, then the Fox derivative with respect to g_i is a function $G \to \mathbb{Z}G$, denoted $\frac{\partial}{\partial g_i}$, which obeys the following axioms:

1. $\frac{\partial}{\partial g_i}(g_j) = \delta_{ij}$,
2. $\frac{\partial}{\partial g_i}(e) = 0$, and
3. $\frac{\partial}{\partial g_i}(uv) = \frac{\partial}{\partial g_i}(u) + u\frac{\partial}{\partial g_i}(v)$ for all $u, v \in G$.

Note the similarities between the axioms above and the properties of partial derivatives in Multivariable Calculus; this is why they are called Fox "derivatives." A consequence of the three axioms above is a fourth property, which is of practical importance for explicitly calculating Fox derivatives:

4. $\frac{\partial}{\partial g_i}(u^{-1}) = -u^{-1}\frac{\partial}{\partial g_i}(u)$ for all $u \in G$.

Here is the connection between Fox derivatives and group homology. Let G be a finitely-presented group with generators $g_1, \ldots, g_n$ and defining relations $r_1, \ldots r_m$. Let $\epsilon : \mathbb{C}G \to \mathbb{C}$ be the augmentation map. Let $d_1 : \mathbb{C}G^n \to \mathbb{C}G$ be the map that sends $(x_1, \ldots, x_n)$ to $x_1(g_1 - 1) + \cdots + x_n(g_n - 1)$. And let $d_2 : \mathbb{C}G^m \to \mathbb{C}G^n$ be represented by multiplication on the right by the matrix $\left(\frac{\partial r_i}{\partial x_j}\right)$. Then the following sequence is the start of a free left $\mathbb{C}G$-resolution of $\mathbb{C}$ (see [32], p. 100):

$$\mathbb{C}G^m \xrightarrow{d_2} \mathbb{C}G^n \xrightarrow{d_1} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0.$$

In general, this resolution may need to continue infinitely to the left with non-trivial terms. However, even if that is the case, the resolution segment above is enough to calculate the first homology group. And, in the special case of a one-relator group, the resolution may be extended trivially (see [32], p. 101):

$$0 \to \mathbb{C}G \xrightarrow{d_2} \mathbb{C}G^n \xrightarrow{d_1} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0.$$

There are three groups relevant to this chapter: $\mathbb{Z}$, $\mathbb{Z}\oplus\mathbb{Z}$, and $\mathbb{Z}*\mathbb{Z}$. The infinite cyclic group $G = \mathbb{Z} = \langle g_1\rangle$ is a one-generator, zero-relation group. Hence, $\mathbb{C}$ has a free $\mathbb{C}G$-resolution:

$$0 \to \mathbb{C}G \xrightarrow{d_1} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where $d_1(x) = x(g_1 - 1)$. The free group $G = \mathbb{Z}*\mathbb{Z} = \langle g_1,\, g_2\rangle$ is a two-generator, zero-relation group. Hence, the corresponding resolution is:

$$0 \to \mathbb{C}G^2 \xrightarrow{d_1} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where $d_1(x,\, y) = x(g_1 - 1) + y(g_2 - 1)$. The free abelian group of rank 2, $G = \mathbb{Z} \oplus \mathbb{Z} = \langle g_1,\, g_2 \mid g_1 g_2 g_1^{-1} g_2^{-1}\rangle$, is a two-generator, one-relation group, and the resolution is:

$$0 \to \mathbb{C}G \xrightarrow{d_2} \mathbb{C}G^n \xrightarrow{d_1} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where d_1 is the same as it was for $\mathbb{Z} * \mathbb{Z}$, and $d_2(x) = (x(1 - g_2),\, x(g_1 - 1))$. The first component of the map d_2 was calculated using the following Fox derivative calculation:

$$\frac{\partial}{\partial g_1}(g_1 g_2 g_1^{-1} g_2^{-1}) = 1 + g_1 \frac{\partial}{\partial g_1}(g_2 g_1^{-1} g_2^{-1}) = 1 + g_1 g_2 \frac{\partial}{\partial g_1}(g_1^{-1} g_2^{-1})$$
$$= 1 - g_1 g_2 g_1^{-1} g_2^{-1} \frac{\partial}{\partial g_1}(g_2 g_1) = 1 - \frac{\partial}{\partial g_1}(g_2 g_1) = 1 - g_2 \frac{\partial}{\partial g_1}(g_1) = 1 - g_2.$$

A similar calculation shows that $\frac{\partial}{\partial g_2}(g_1 g_2 g_1^{-1} g_2^{-1}) = g_1 - 1$. Fox derivative calculations will also be used in Chapter 4 to calculate group homology for particular two-generator, one-relation groups called Baumslag-Solitar groups.

2.4 Fourier Transforms

When working with the algebraic definition of $\mathcal{N}(G)$, one problem that arises is the difficulty of producing concrete elements of $\mathcal{N}(G)$. It is not difficult to create elements of $\ell^2(G)$; if $\alpha = \sum_{g \in G} a_g \cdot g$, then $\alpha \in \ell^2(G)$ if and only if $||\alpha||_2 < \infty$. Verifying the finiteness of this norm for an element α is often relatively easy. However, demonstrating the property that $\alpha\beta \in \ell^2(G)$ for all $\beta \in \ell^2(G)$ is generally much harder. Therefore, whenever it is possible, it will be helpful to change the perspective on $\mathcal{N}(G)$ to one in which elements are easier to identify and work with. In particular, when G is abelian, Fourier analysis can be used to make useful identifications.

On every LCA (locally compact abelian) group, there exists a non-negative regular measure m called the *Haar measure* of G (see [18]). It is nonzero, and it is translation invariant: $m(E + x) = m(E)$, for all $x \in G$ and all Borel sets $E \subseteq G$. Haar measure is unique up to positive constant multiple. If G is compact, it is customary to make $m(G) = 1$. And if G is discrete, it is customary to make $m(\{g\}) = 1$ for all $g \in G$. When we speak of $L^2(G)$, we mean with respect to this measure. A complex function γ on a LCA group G is a *character* of G if $|\gamma(x)| = 1\ \forall x \in G$ and if $\gamma(x + y) = \gamma(x)\gamma(y)\ \forall x, y \in G$. The set of all continuous characters of G forms a group Γ, the *dual group* of G, where addition is defined by $(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x)$. Now we can define Fourier transforms:

Definition 2.4.1. If $f \in L^1(G)$, then $\hat{f}$ defined on Γ by

$$\hat{f}(\gamma) = \int_G f(x)\gamma(-x)dx \quad , \gamma \in \Gamma,$$

is the Fourier transform of f. Let $A(\Gamma)$ denote the set of all Fourier transforms.

Now we can give Γ the weak topology induced by $A(\Gamma)$. With this topology, if G is discrete, then Γ is compact. Conversely, if G is compact, then Γ is discrete. Here we reach the important consequence of Fourier transforms for our purposes ([41], p. 26):

Theorem 2.4.2. *The Fourier transform, restricted to $(L^1 \cap L^2)(G)$, is an isometry (with* *respect to L^2-norms) onto a dense linear subspace of $L^2(\Gamma)$. Hence it may be extended, in a unique manner, to an isometry of $L^2(G)$ onto $L^2(\Gamma)$.*

For instance, if $G = \mathbb{Z}$ we can convert our calculations into a more workable context by using its dual group instead:

Theorem 2.4.3. *If $G = \mathbb{Z}$, then $\Gamma = S^1$.*

Proof. If $G = \mathbb{Z}$ and $\gamma \in \Gamma$, then $\gamma(1) = e^{i\alpha}$ for some $\alpha \in \mathbb{R}$. Because γ is a character, this implies $\gamma(n) = e^{in\alpha}$. The correspondence $\gamma \mapsto e^{i\alpha}$ is an isomorphism $\Gamma \to S^1$. □

This means we can convert from the context of square-summable sequences of complex numbers to the context of square-integrable functions on the circle.

Theorem 2.4.4. *If $G = \mathbb{Z}$, then the isometry $\ell^2(G) \to L^2(S^1)$ is given by*

$$(a_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} a_n e^{-inx}$$

where S^1 is identified with the quotient space of the interval $[-\pi, \pi]$.

Proof. Let $f = (a_n)_{n\in\mathbb{Z}} \in \ell^2(G)$ and $\gamma = e^{i\alpha} \in S^1$. Recall the formula for Fourier transform:

$$\hat{f}(\gamma) = \int_G f(x)\gamma(-x)dx.$$

In our setting, that translates to:

$$\hat{f}(\gamma) = \sum_{x\in\mathbb{Z}} f(x)\gamma(-x) = \sum_{x\in\mathbb{Z}} a_x e^{-inx}.$$

□

Notice that under this identification, $\mathbb{C}G$ corresponds to finite sums $\sum_{finite} a_n e^{-inx}$. And $\mathcal{N}(G) \cong \{f \in L^2(S^1) \mid fg \in L^2(S^1),\ \forall g \in L^2(S^1)\} = L^\infty(S^1)$. These identifications will be useful later for precisely the reasons stated above; it is often easier to demonstrate bounded functions on the circle than to demonstrate elements in the algebraic definition of $\mathcal{N}(G)$.

As described above, identifying $\mathcal{N}(\mathbb{Z})$ with $L^\infty(S^1)$ will be useful for doing calculations for the group $G = \mathbb{Z}$. And while much of the results for $G = \mathbb{Z}$ can be managed without this help, these kinds of identifications will be crucial for the group $G = \mathbb{Z} \oplus \mathbb{Z}$. For $G = \mathbb{Z} \oplus \mathbb{Z}$, the results are analogous to the results above.

Theorem 2.4.5. *For $G = \mathbb{Z} \oplus \mathbb{Z}$, $\Gamma = T^2$ the two-dimensional torus.*

Proof. Let $\gamma \in \Gamma$. Then $\gamma(1,0) = e^{i\alpha}$ and $\gamma(0,1) = e^{i\beta}$ for some $\alpha, \beta \in \mathbb{R}$. Since γ is a character, $\gamma(n,m) = \gamma((n,0) + (0.m)) = \gamma(n,0)\gamma(0,m) = e^{in\alpha}e^{im\beta}$. The map $\gamma \mapsto (e^{i\alpha}, e^{i\beta})$ is an isomorphism $\Gamma \to T^2$. $\square$

Since the dual group is the torus, this means the context of sequences of complex numbers indexed by $\mathbb{Z} \oplus \mathbb{Z}$ can be traded for the context of functions defined on the torus.

Theorem 2.4.6. *If $G = \mathbb{Z} \times \mathbb{Z}$, then the isometry $\ell^2(G) \to L^2(T^2)$ is given by*

$$(a_{n,m})_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} \mapsto \sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} a_{n,m}e^{-in\alpha}e^{-im\beta}$$

where T^2 is identified with the quotient space of $[-\pi, \pi] \times [-\pi, \pi]$.

Proof. Let $f = (a_{n,m})_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} \in \ell^2(G)$ and $(e^{i\alpha}, e^{i\beta}) \in T^2$. Recall the formula for Fourier transform above. In this setting, that translates to:

$$\hat{f}(\gamma) = \sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} f(n,m)\gamma(-n,-m) = \sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} a_{n,m}e^{-in\alpha}e^{-im\beta}.$$

$\square$

Under these identifications, $\mathbb{C}G$ corresponds to $\sum_{finite} a_{n,m}e^{-in\alpha}e^{-im\beta}$ and $\mathcal{N}(G) \cong L^\infty(T^2)$. In a later section, the calculation of $H_1(G, \mathcal{N}(G)) \neq 0$ for $G = \mathbb{Z} \oplus \mathbb{Z}$ will rely on using bounded functions on the torus to construct elements of $\mathcal{N}(G)$.

Notice the abundance of zero divisors in the rings $\mathcal{N}(\mathbb{Z})$ and $\mathcal{N}(\mathbb{Z} \oplus \mathbb{Z})$. For instance, if $f \in L^\infty(S^1)$ and $A = \{x \in S^1 \mid f(x) = 0\}$, then f is a zero divisor if and only if $m(A) > 0$. In the following sections it will often be necessary to demonstrate that certain elements of $\mathcal{N}(G)$ are not zero divisors.

2.5 The Case $G = \mathbb{Z}$

Let G be the infinite cyclic group $\mathbb{Z}$. The goal of this section is to show that $H_1(G, M)$ and $H^1(G, M)$ are trivial for $M = \mathcal{N}(G)$ and $M = \ell^p(G)$ for all $1 \leq p \in \mathbb{R}$. First consider when the module is the group von Neumann algebra.

Theorem 2.5.1. *If $G = \mathbb{Z}$ and $M = \mathcal{N}(G)$, then $H_1(G, M) = 0$.*

Proof. Take the standard free $\mathbb{C}G$-resolution of $\mathbb{C}$:

$$0 \to \mathbb{C}G \xrightarrow{f} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where $G = \langle t \rangle$, ϵ is the augmentation map, and f is multiplication by $t - 1$ on the right. Now apply $- \otimes_{\mathbb{C}G} \mathcal{N}(G)$ to create the complex:

$$0 \to \mathcal{N}(G) \xrightarrow{f_*} \mathcal{N}(G) \to 0,$$

where f_* is still multiplication by $t-1$. To show $H_1(G, M) = 0$ it suffices to show the kernel of f_* is trivial. One way to see this is to use Fourier transforms to convert the sequence into the context of functions on the circle S^1, seen as a quotient space of $[-\pi, \pi]$:

$$0 \to L^\infty(S^1) \xrightarrow{f_*} L^\infty(S^1) \to 0,$$

where f_* is multiplication on the left by the function $e^{-ix} - 1$. Since $e^{-ix} - 1$ vanishes only on a set of measure 0, if it is multiplied against any other function with support of full measure, the result will also be a function that vanishes only on a set of measure 0. In other words, the kernel of f_* is trivial. $\square$

The key fact of the previous proof is that $t - 1$ is not a zero-divisor in $\mathcal{N}(G)$, and this was realized by using the Fourier transform $\mathcal{N}(G) \cong L^\infty(S^1)$. However, this result could also be realized without the aid of Fourier transforms by considering that the Zero Divisor Conjecture is known to be true for $G = \mathbb{Z}$. In fact, it is known that if G is a torsion-free elementary amenable group, $0 \neq \alpha \in \mathbb{C}G$, and $0 \neq \beta \in \ell^2(G)$, then $\alpha\beta \neq 0$ ([27], Theorem 2). Using this fact, and the same method of proof as above, we can see that $H_1(G, M) = 0$ if $M = \ell^2(G)$. Furthermore, it is known that if $G = \mathbb{Z}$, $0 \neq \alpha \in \mathbb{C}G$, and $0 \neq \beta \in \ell^p(G)$ for $1 \leq p \in \mathbb{R}$, then $\alpha\beta \neq 0$ [39]. Hence, we get the following theorem:

Theorem 2.5.2. *If $G = \mathbb{Z}$ and $M = \mathcal{N}(G)$ or $M = \ell^p(G)$ for any $1 \leq p \in \mathbb{R}$, then $H_1(G, M) = 0$.*

We can also get a similar result for group cohomology.

Theorem 2.5.3. *If $G = \mathbb{Z}$ and $M = \mathcal{N}(G)$ or $M = \ell^p(G)$ for any $1 \leq p \in \mathbb{R}$, then $H^1(G, M) = 0$.*

Proof. Once again, take the standard free $\mathbb{C}G$-resolution of $\mathbb{C}$:

$$0 \to \mathbb{C}G \xrightarrow{f} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where $G = \langle t \rangle$, ϵ is the augmentation map, and f is multiplication by $t - 1$ on the right. Now apply the functor $\mathrm{Hom}_{\mathbb{C}G}(-, M)$ to obtain the complex:

$$0 \to M \xrightarrow{f^*} M \to 0,$$

where f^* is multiplication by $t - 1$. Then $H^1(G, M) = \ker(f^*) = 0$ by the Zero Divisor Conjecture. $\square$

2.6 The Case $G = \mathbb{Z} \oplus \mathbb{Z}$

The main goal of this section is to prove $\mathcal{N}(G)$ is not a flat $\mathbb{C}G$-module for $G = \mathbb{Z} \oplus \mathbb{Z}$. This will be accomplished by showing $H_1(G, \mathcal{N}(G)) \neq 0$, and the group homology calculation will be greatly aided by using Fourier transforms to identify $\mathcal{N}(G)$ with $L^\infty(T^2)$.

Theorem 2.6.1. *If $G = \mathbb{Z} \oplus \mathbb{Z}$, then $H_1(G, \mathcal{N}(G)) \neq 0$.*

Proof. Identify T^2 with the square $[-\pi, \pi] \times [-\pi, \pi]$ with opposite sides glued together. Recall that by using Fourier transforms we can identify $\mathcal{N}(G)$ with $L^\infty(T^2)$. And under this identification, $\mathbb{C}G$ is identified with functions of the form $\sum_{finite} a_{n,m} e^{-inx} e^{-imy}$. Recall that to calculate $H_1(G, \mathcal{N}(G))$, we first need a free $\mathbb{C}G$-resolution of $\mathbb{C}$, such as the following:

$$0 \to \mathbb{C}G \xrightarrow{\gamma} \mathbb{C}G^2 \xrightarrow{\beta} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where $\gamma : x = \sum a_{n,m} \cdot t^n s^m \mapsto (-(s-1)x, (t-1)x)$,
and $\beta : (x, y) \mapsto (t-1)x + (s-1)y$, for $G = \langle t, s \mid ts = st \rangle$.
Next we need to apply the functor $\mathcal{N}(G) \otimes -$ to obtain a deleted complex:

$$0 \to L^\infty(T^2) \otimes_{\mathbb{C}G} \mathbb{C}G \xrightarrow{\gamma'} L^\infty(T^2) \otimes_{\mathbb{C}G} \mathbb{C}G^2 \xrightarrow{\beta'} L^\infty(T^2) \otimes_{\mathbb{C}G} \mathbb{C}G \to 0.$$

This can be simplified to

$$0 \to L^\infty(T^2) \xrightarrow{\gamma'} L^\infty(T^2)^2 \xrightarrow{\beta'} L^\infty(T^2) \to 0,$$

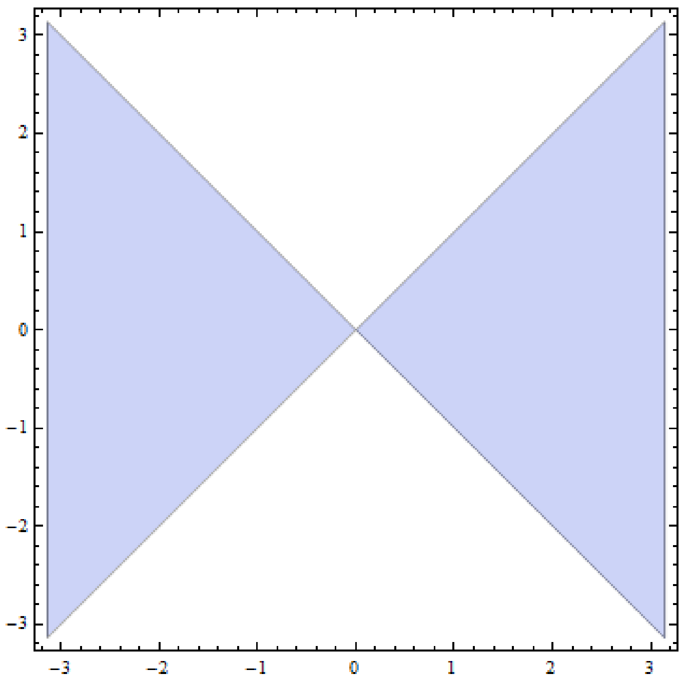
where $\gamma' : f \mapsto (f \cdot (1 - e^{-iy}), f \cdot (e^{-ix} - 1))$,
and $\beta' : (g, h) \mapsto g \cdot (e^{-ix} - 1) + h \cdot (e^{-iy} - 1)$.
Now we are ready to calculate the first group homology: $H_1(G, \mathcal{N}(G)) \cong \ker(\beta')/\mathrm{Im}(\gamma')$.
Suppose $(g, h) \in \ker(\beta')$. Then $g \cdot (e^{-ix} - 1) + h \cdot (e^{-iy} - 1) = 0$. Define

$$f = \frac{g}{1 - e^{-iy}} = \frac{h}{e^{-ix} - 1}.$$

The group homology vanishes if every such f must be in $L^\infty(T^2)$. We can show that the group homology is nontrivial if there exists $(g, h) \in \ker(\beta')$ such that $f \notin L^\infty(T^2)$. In other words, we need to create the following situation:

$$h \in L^\infty(T^2), \quad f = \frac{h}{e^{-ix} - 1} \notin L^\infty(T^2), \quad g = \frac{h \cdot (1 - e^{-iy})}{e^{-ix} - 1} \in L^\infty(T^2).$$

Define $A = \{(x, y) \in T^2 \ : \ \cos y > \cos x\}$, and let $h = \chi_A$. The region A is pictured below:



Clearly, $h \in L^\infty(T^2)$. Since A contains open balls around points with arbitrarily small x-values, it follows that $f \notin L^\infty(T^2)$.

Claim: $g \in L^\infty(T^2)$.

This can be proved by showing the stronger claim: $|g| < 1$ for all $(x, y) \in T^2$. Suppose to the contrary that $|g(x, y)| > 1$ for some (x, y). Then clearly, (x, y) must be in A. And:

$$\left|\frac{1-e^{-iy}}{e^{-ix}-1}\right| > 1 \implies |e^{-iy}-1| > |e^{-ix}-1| \implies (\cos y - 1)^2 + (\sin y)^2 > (\cos x - 1)^2 + (\sin x)^2$$

$$\implies 2 - 2\cos y > 2 - 2\cos x \implies \cos y < \cos x \implies (x, y) \notin A$$

Thus, we have produced a contradiction. This proves the claim and finishes the proof. $\square$

Corollary 2.6.2. *If $G = \mathbb{Z} \oplus \mathbb{Z}$, then $\mathcal{N}(G)$ is not a flat $\mathbb{C}G$-module.*

The preceding corollary is the main attraction of this section. However, it is also interesting to note that the above calculations can be slightly altered to show that the first group cohomology is nontrivial as well.

Theorem 2.6.3. *For $G = \mathbb{Z} \oplus \mathbb{Z}$, we have $H^1(G, \mathcal{N}(G)) = Ext^1_{\mathbb{C}G}(\mathbb{C}, \mathcal{N}(G)) \neq 0$.*

Proof. We want to calculate $H^1(G, \mathcal{N}(G)) = \operatorname{Ext}^1_{\mathbb{C}G}(\mathbb{C}, \mathcal{N}(G))$. We can reuse some of the work from the last proof. Take the same resolution of $\mathbb{C}$ as above (since G is abelian we don't have to worry about the difference between left-actions and right-actions). Now apply the functor $\operatorname{Hom}_{\mathbb{C}G}(-, \mathcal{N}(G))$. This yields the complex:

$$0 \to \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G, L^\infty(T^2)) \xrightarrow{\beta^*} \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G^2, L^\infty(T^2)) \xrightarrow{\gamma^*} \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G, L^\infty(T^2)) \to 0,$$

which becomes:

$$0 \to L^\infty(T^2) \xrightarrow{\beta^*} L^\infty(T^2)^2 \xrightarrow{\gamma^*} L^\infty(T^2) \to 0,$$

where $\beta^* : f \mapsto ((e^{-ix} - 1)f, (e^{-iy} - 1)f)$, and $\gamma^* : (g, h) \mapsto (1 - e^{-iy})g + (e^{-ix} - 1)h$. We would like to find an element of $\ker(\gamma^*)$ which is not in $\operatorname{Im}(\beta^*)$. Suppose $(G, H) \in \ker(\gamma^*)$. Then $(1 - e^{-iy})G + (e^{-ix} - 1)H = 0$. And suppose $(G, H) \in \operatorname{Im}(\beta^*)$. Then there exists $F \in L^\infty(T^2)$ such that $G = (e^{-ix} - 1)F$ and $H = (e^{-iy} - 1)F$. So it suffices to construct the following scenario:

$$G \in L^\infty(T^2), \qquad F = \frac{G}{e^{-ix} - 1} \notin L^\infty(T^2), \qquad H = \frac{(1 - e^{-iy})G}{-(e^{-ix} - 1)} \in L^\infty(T^2).$$

Using the f, g, h as defined in the previous proof, it is clear that if $G = h \in L^\infty(T^2)$, then $F = f \notin L^\infty(T^2)$ and $H = -g \in L^\infty(T^2)$. □

It may be an interesting question to consider for which groups $H^1(G, \mathcal{N}(G))$ vanishes. We have already seen that this cohomology group is trivial for $G = \mathbb{Z}$, and it is nontrivial for $G = \mathbb{Z} \oplus \mathbb{Z}$.

Finally, we would like to show that the modules $\ell^p(G)$ are also not flat over $\mathbb{C}G$. In this case, we do not have Fourier transforms at our disposal. However, since elements of $\ell^p(G)$ are easier to identify than elements of $\mathcal{N}(G)$, the transforms are not needed for this proof.

Theorem 2.6.4. *Let $G = \mathbb{Z} \oplus \mathbb{Z}$ and $1 \le p \in \mathbb{R}$. Then $H_1(G, \ell^p(G)) \neq 0$.*

Proof. We can use the same free $\mathbb{C}G$-resolution as before:

$$0 \to \mathbb{C}G \xrightarrow{f} \mathbb{C}G^2 \xrightarrow{k} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where $f : x = \sum a_{n,m} \cdot t^n s^m \mapsto (-(s-1)x, (t-1)x)$,
and $k : (x, y) \mapsto (t-1)x + (s-1)y$, for $G = \langle t, s \mid ts = st \rangle$. Now apply the functor $\ell^p(G) \otimes_{\mathbb{C}G} -$ to obtain the following deleted complex:

$$0 \to \ell^p(G) \xrightarrow{f_*} \ell^p(G)^2 \xrightarrow{k_*} \ell^p(G) \to 0,$$

where $f_* : x \mapsto (-(s-1)x, (t-1)x)$ and $k_* : (x, y) \mapsto (t-1)x + (s-1)y$. It suffices to find $(\alpha,\, \beta) \in \ker(k_*)$ such that $(\alpha,\, \beta) \notin \mathrm{Im}(f_*)$. So it suffices to find $\alpha, \beta \in \ell^p(G)$ such that $(t-1)\alpha = (s-1)\beta$ but there does not exist $\gamma \in \ell^p(G)$ such that $\alpha = (s-1)\gamma$. Use the following notation:

$$\alpha = \sum_{g \in G} a_g \cdot g, \quad \beta = \sum_{g \in G} b_g \cdot g, \quad \gamma = \sum_{g \in G} c_g \cdot g.$$

Assume $\alpha(t-1) = \beta(s-1)$. Then $\left(\sum a_g \cdot g\right)(t-1) = \left(\sum b_g \cdot g\right)(s-1)$, which implies:

$$\left(\sum a_g \cdot gt\right) - \left(\sum a_g \cdot g\right) = \left(\sum b_g \cdot gs\right) - \left(\sum b_g \cdot g\right),$$

$$\sum (a_{gt^{-1}} - a_g) \cdot g = \sum (b_{gs^{-1}} - b_g) \cdot g.$$

Hence, for all $g \in G$, $a_{gt^{-1}} - a_g = b_{gs^{-1}} - b_g$. Solve this equation for b_g: $b_g = (a_g - a_{gt^{-1}}) + b_{gs^{-1}}$. Substituting this equation into itself repeatedly yields the following relationship between the β coefficients and the α coefficients:

$$b_g = \sum_{k=0}^{\infty} \left(a_{gs^{-k}} - a_{gs^{-k}t^{-1}}\right). \tag{2.1}$$

Now assume that $\alpha = \gamma(s-1)$. Then:

$$\sum a_g \cdot g = \left(\sum c_g \cdot g\right)(s-1) = \left(\sum c_g \cdot gs\right) - \left(\sum c_g \cdot g\right) = \sum (c_{gs^{-1}} - c_g) \cdot g.$$

Therefore, $a_g = c_{gs^{-1}} - c_g$ for all $g \in G$, and $c_g = -a_g + c_{gs^{-1}}$. As before, this leads to a relationship between the γ coefficients and the α coefficients:

$$c_g = -\sum_{k=0}^{\infty} a_{gs^{-k}}. \tag{2.2}$$

Let $r \in \mathbb{R}$ be such that $\{(m+n)^{-r}\} \in \ell^p(\mathbb{N} \times \mathbb{N})$ and $\{(m+n)^{-r+1}\} \notin \ell^p(\mathbb{N} \times \mathbb{N})$. For $g = s^m t^n$, denote a_g with $a_{m,n}$. First define $a_{0,n} = 0$ for all $n \in \mathbb{Z}$. Next define the "first quadrant” of α coefficients. For $m > 0$ and $n \geq 0$, define $a_{m,n} = (m+n)^{-r}$. Next define the “fourth quadrant” of α coefficients. For $m \geq 0$ and $n < 0$, define $a_{m,n} = a_{m,-n}$. Finally, define the second and third quadrants of α coefficients. For $m < 0$ and $n \in \mathbb{Z}$, define $a_{m,n} = -a_{-m,n}$. By the choice of r, it follows that $\alpha \in \ell^p(G)$. For $m < 0$ and $n > 0$:

$$c_{m,n} = -\sum_{k=m}^{-\infty} a_{k,n} = \sum_{k=-m}^{\infty} a_{k,n} = \sum_{k=-m}^{\infty} \frac{1}{(k+n)^r} = \Theta((-m+n)^{-r+1}).$$

By the choice of r, it follows that $\gamma \notin \ell^p(G)$. Now consider β. Note that for $m, n \geq 1$, $(m+n)^{-r} - (m+n-1)^{-r} = \Theta((m+n)^{-r-1})$, with respect to the variable n. Hence, for $m < 0$ and $n > 0$:

$$b_{m,n} = \sum_{k=m}^{-\infty}(a_{k,n} - a_{k,n-1}) = -\sum_{k=-m}^{\infty}(a_{k,n} - a_{k,n-1})$$

$$= -\sum_{k=-m}^{\infty}(k+n)^{-r} - (k+n-1)^{-r} = -\sum_{k=-m}^{\infty}\Theta((k+n)^{-r-1}) = \Theta((-m+n)^{-r}).$$

For the other three quadrants, the bounds of b_g are similar. Hence, by the choice of r, it follows that $\beta \in \ell^p(G)$. □

2.7 Group C^*-algebras

Another class of operator algebras, similar to von Neumann algebras, is the class of C^*-algebras. An example of a prototypical C^*-algebra is a subset X of $\mathcal{B}(H)$ for some complex Hilbert space H, which is closed under all of the algebraic operations on $\mathcal{B}(H)$ (addition, multiplication, scalar multiplication), is closed with respect to the norm topology, and is closed under the adjoint operation. More generally, a C^*-algebra is defined as follows.

Definition 2.7.1. A $\underline{C^*\text{-algebra}}$ is a complex Banach algebra (with a unit element I) with an involution that is closed under the norm topology.

Since the norm topology is stronger than the weak topology, it follows that any von Neumann algebra is also a C^*-algebra. For a group G there are a couple of relevant C^*-algebras (which are in general not von Neumann algebras) associated to G, and the questions being asked about $\mathcal{N}(G)$ in this paper can also be asked about these other algebras.

First, let's define what is called a "reduced group C^*-algebra," denoted $C_r^*(G)$, for a group G. Consider the left regular representation $\lambda : G \to \mathcal{B}(\ell^2(G))$, where for any $x \in G$, $\lambda(x) : \ell^2(G) \to \ell^2(G)$ maps $\sum_{g\in G} a_g \cdot g$ to $\sum_{g \in G} a_{x^{-1}g} \cdot g$. This induces a representation of $\ell^1(G)$, $\lambda : \ell^1(G) \to \mathcal{B}(\ell^2(G))$ ([9], p. 183). For any $A = \sum_{g\in G} a_g \cdot g \in \ell^1(G)$, define $\lambda(A)$ to be the map $\left(\sum_{g\in G} a_g \cdot \lambda(g)\right) : \ell^2(G) \to \ell^2(G)$. Then the reduced group C^*-algebra is the norm-closure of all such operators on $\ell^2(G)$; $C_r^*(G) = \overline{\lambda(\ell^1(G))}$. Another algebra associated to G is one called the "group C^*-algebra," denoted $C^*(G)$. Define it to be the enveloping C^*-algebra of $\ell^1(G)$. Much like $\mathcal{N}(G)$, these group C^*-algebras were originally studied in the context of Representation Theory of infinite groups (see [24], section 10.3). In general, $C_r^*(G)$ is a quotient of $C^*(G)$. However, if G is amenable, then $C^*(G) \cong C_r^*(G)$ ([9], Theorem VII.2.5). Just as with $\mathcal{N}(G)$, it can be easier to work with $C_r^*(G)$ when G is abelian. This is because one may work within the context of the dual group, as the next theorem describes ([9], Proposition VII.1.1).

Theorem 2.7.2. *If G is an abelian group, then $C^*(G) \cong C_r^*(G) \cong C_0(\Gamma)$, where Γ is the dual group of G and $C_0(\Gamma)$ is the space of all continuous functions on Γ vanishing at infinity.*

This fact allows us to mimic the group homology and cohomology calculations done earlier for $G = \mathbb{Z}$ and $G = \mathbb{Z} \oplus \mathbb{Z}$. For $G = \mathbb{Z}$, we can use the dual group, and the proof works out analogously to how it did in section 2.4.

Theorem 2.7.3. *If $G = \mathbb{Z}$ and $M = C^*(G) = C_r^*(G)$, then $H_1(G, M) = 0$ and $H^1(G, M) = 0$.*

Proof. Let $G = \langle t \rangle \cong \mathbb{Z}$. Take the standard free $\mathbb{C}G$-resolution of $\mathbb{C}$:

$$0 \to \mathbb{C}G \xrightarrow{f} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where ϵ is the augmentation map and f is multiplication by $t-1$ on the right. Now apply $- \otimes_{\mathbb{C}G} M$ to create the complex:

$$0 \to M \xrightarrow{f_*} M \to 0,$$

where f_* is still multiplication by $t-1$. To show $H_1(G,\, M)=0$ it suffices to show the kernel of f_* is trivial. Convert everything into the context of the dual group $\Gamma \cong S^1$:

$$0 \to C_0(S^1) \xrightarrow{f_*} C_0(S^1) \to 0,$$

where f_* is multiplication on the left by the function $e^{-ix}-1$. Since $e^{-ix}-1$ vanishes only on a set of measure 0, if it is multiplied against any other function with support of full measure, the result will also be a function that vanishes only on a set of measure 0. In other words, the kernel of f_* is trivial. Hence, $H_1(G,\, M)=0$. Similarly, $H^1(G,\, M)=0$. $\square$

For $G=\mathbb{Z}\oplus\mathbb{Z}$, we can again use the dual group, except we must use a continuous analogue of the characteristic function which was used in section 2.5.

Theorem 2.7.4. *If $G=\mathbb{Z}\oplus\mathbb{Z}$ and $M=C^*(G)=C^*_r(G)$, then $H_1(G,\, M)\neq 0$ and $H^1(G,\, M)\neq 0$.*

Proof. Let $G=\langle s,t \mid ts=st\rangle \cong \mathbb{Z}\oplus\mathbb{Z}$. Build a free $\mathbb{C}G$-resolution of $\mathbb{C}$:

$$0 \to \mathbb{C}G \xrightarrow{\gamma} \mathbb{C}G^2 \xrightarrow{\beta} \mathbb{C}G \xrightarrow{\epsilon} \mathbb{C} \to 0,$$

where $\gamma : x=\sum a_{n,m}\cdot t^n s^m \mapsto (-(s-1)x, (t-1)x)$,
and $\beta : (x,y) \mapsto (t-1)x+(s-1)y$, for $G=\langle t,s \mid ts=st\rangle$.
Next apply the functor $\mathcal{N}(G)\otimes -$ to obtain a deleted complex:

$$0 \to C_0(T^2) \xrightarrow{\gamma'} C_0(T^2)^2 \xrightarrow{\beta'} C_0(T^2) \to 0,$$

where $\gamma' : f \mapsto (f\cdot(1-e^{-iy}), f\cdot(e^{-ix}-1))$,
and $\beta' : (g,h) \mapsto g\cdot(e^{-ix}-1)+h\cdot(e^{-iy}-1)$.
Now we are ready to calculate the first group homology: $H_1(G,\, M)\cong \ker(\beta')/\mathrm{Im}(\gamma')$. Suppose $(g,h)\in\ker(\beta')$. Then $g\cdot(e^{-ix}-1)+h\cdot(e^{-iy}-1)=0$. Define

$$f=\frac{g}{1-e^{-iy}}=\frac{h}{e^{-ix}-1}.$$

The group homology vanishes if every such f must be in $C_0(T^2)$. We can show that the group homology is nontrivial if there exists $(g,h)\in\ker(\beta')$ such that $f\notin C_0(T^2)$. In other words, we need to create the following situation:

$$h \in C_0(T^2), \qquad f = \frac{h}{e^{-ix}-1} \notin C_0(T^2), \qquad g = \frac{h \cdot (1-e^{-iy})}{e^{-ix}-1} \in C_0(T^2).$$

Just as in section 2.5, define $A = \{(x, y) \in T^2 \ : \ \cos y > \cos x\}$. This time let $h(x, y)$ be a continuous function with support contained in A such that $h(x, 0) = \sqrt{|x|}$. I claim that $g \in C_0(T^2)$. First, we want to show that $|g| < d$ for all $(x, y) \in T^2$, where $d = ||h||_\infty$. Suppose to the contrary that $|g(x, y)| > d$ for some (x, y). Then clearly, (x, y) must be in A. And:

$$\left|\frac{1-e^{-iy}}{e^{-ix}-1}\right| > 1 \implies |e^{-iy}-1| > |e^{-ix}-1| \implies (\cos y-1)^2+(\sin y)^2 > (\cos x-1)^2+(\sin x)^2$$

$$\implies 2-2\cos y > 2-2\cos x \implies \cos y < \cos x \implies (x, y) \notin A.$$

Thus, we have produced a contradiction. Hence g is bounded. Since g is a bounded quotient of continuous functions, g must be measurably equivalent to a continuous function. To show the group homology is nontrivial, it now suffices to show $f \notin C_0(T^2)$. This is true since:

$$\lim_{x \to 0^+} |f(x, 0)| = \lim_{x \to 0^+} \left|\frac{h(x, 0)}{e^{-ix}-1}\right| = \lim_{x \to 0^+} \left|\frac{\sqrt{x}}{e^{-ix}-1}\right| = \lim_{x \to 0^+} \left|\frac{1}{2\sqrt{x}e^{-ix}}\right| = \infty.$$

Hence, f cannot be extended to be a continuous function, and $H_1(G,\, M) \neq 0$.
Similarly, $H^1(G,\, M) \neq 0$. □

It may be an interesting question to consider for which groups G the $\mathbb{C}G$-module $C^*(G)$ is flat.

2.8 Connections: $\mathcal{N}(G)$, $\mathcal{U}(G)$, and $\ell^2(G)$

The results of this section will be critical for taking the calculations for special cases such as $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ and drawing conclusions about wider classes of groups. Perhaps the most important such result is the following connection between groups and subgroups ([31], Theorem 6.29(1)).

Theorem 2.8.1. *If $H \leq G$ and $\mathcal{N}(G)$ is flat over $\mathbb{C}G$, then $\mathcal{N}(H)$ is flat over $\mathbb{C}H$.*

The contrapositive of the previous theorem will be utilized extensively in this paper:

Corollary 2.8.2. *If $H \leq G$ and $\mathcal{N}(H)$ is not flat over $\mathbb{C}H$, then $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$.*

For example, if we combine this fact with (2.6.2), then we get the next result.

Corollary 2.8.3. *If G contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, then $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$.*

There is an analogous relationship between groups and subgroups with respect to the $\mathbb{C}G$-module $\ell^p(G)$.

Theorem 2.8.4. *Let $1 \leq p \in \mathbb{R}$. If $H \leq G$ and $\ell^p(G)$ is flat over $\mathbb{C}G$, then $\ell^p(H)$ is flat over $\mathbb{C}H$.*

Proof. The first relevant fact is that $\ell^p(H)$ is a summand of $\ell^p(G)$ as $\mathbb{C}H$-modules. Indeed, let X be a transversal for H in G, and assume $1 \in X$. Then $\ell^p(G) = \ell^p(H) \oplus \left(\coprod_{1 \neq x \in X} x\ell^p(H) \right)$. Rewrite this as $\ell^p(G) = \ell^p(H) \oplus Y$. Now let $\mathbf{E}$ be an exact sequence of $\mathbb{C}H$-modules $0 \to A \to B \to C \to 0$. Since $\mathbb{C}G$ is flat over $\mathbb{C}H$, $\mathbf{E} \otimes_{\mathbb{C}H} \mathbb{C}G$ is an exact sequence of $\mathbb{C}G$-modules. Since $\ell^p(G)$ is flat over $\mathbb{C}G$, $\mathbf{E} \otimes_{\mathbb{C}H} \ell^p(G)$ is an exact sequence of $\ell^p(G)$-modules. But $\mathbf{E} \otimes_{\mathbb{C}H} \ell^p(G) \cong (\mathbf{E} \otimes_{\mathbb{C}H} \ell^p(H)) \oplus (\mathbf{E} \otimes_{\mathbb{C}H} Y)$ is exact implies that $\mathbf{E} \otimes_{\mathbb{C}H} \ell^p(H)$ is exact. Therefore $\ell^p(H)$ is flat over $\mathbb{C}H$. $\square$

Hence, we have the following corollary, which is a useful tool for classifying for which groups G $\ell^2(G)$ is not flat over $\mathbb{C}G$.

Corollary 2.8.5. *Let $1 \leq p \in \mathbb{R}$. If $H \leq G$ and $\ell^p(H)$ is not flat over $\mathbb{C}H$, then $\ell^p(G)$ is not flat over $\mathbb{C}G$.*

So, loosely speaking, non-flatness with respect to a subgroup implies non-flatness with respect to the group. In general, the converse is not true (e.g., $G = \mathbb{Z} \oplus \mathbb{Z}$ and $H = \mathbb{Z}$). However, the converse is true if H is a subgroup of finite index. This fact is based on the following identity.

Lemma 2.8.6. *Suppose $H \leq G$ and $M(G) = \mathcal{N}(G)$ or $M(G) = \ell^p(G)$ for some $1 \leq p \in \mathbb{R}$. If $[G : H] < \infty$, then $M(G) \cong M(H) \otimes_{\mathbb{C}H} \mathbb{C}G$ as right $\mathbb{C}G$-modules.*

Proof. Let $[g_1, g_2, \ldots, g_n]$ be a transversal of G with respect to H. For any $g \in G$, let $h_g \in H$ be such that $g = h_g g_i$ for some $i \in [1, 2, \ldots, n]$. Define X_i to be the set of all $g \in G$ which are in the orbit of g_i. Define a $\mathbb{C}G$-map $\varphi : M(G) \to M(H) \otimes_{\mathbb{C}H} \mathbb{C}G$ as follows:

$$\sum_{g \in G} a_g \cdot g \mapsto \sum_{i=1}^{n} \left(\left(\sum_{g \in X_i} a_g \cdot h_g \right) \otimes g_i \right).$$

And define another $\mathbb{C}G$-map $\psi : M(H) \otimes_{\mathbb{C}H} \mathbb{C}G \to M(G)$ by $\alpha \otimes \beta \mapsto \alpha\beta$. Then $\psi\varphi = id$:

$$\sum_{g \in G} a_g \cdot g \overset{\varphi}{\mapsto} \sum_{i=1}^{n} \left(\left(\sum_{g \in X_i} a_g \cdot h_g \right) \otimes g_i \right) \overset{\psi}{\mapsto} \sum_{i=1}^{n} \left(\sum_{g \in X_i} a_g \cdot h_g g_i \right) = \sum_{g \in G} a_g \cdot g.$$

Now let $\alpha = \sum_{h \in H} a_g \cdot h \in M(H)$ and $g \in G$. Suppose $g = h_g g_i$. By linearity, to show $\varphi\psi = id$ it suffices to show $\varphi\psi(\alpha \otimes g) = \alpha \otimes g$. This is true because of the following calculation:

$$\psi(\alpha \otimes g) = \alpha g = \sum_{h \in H} a_h \cdot hg = \sum_{h \in H} a_h \cdot h h_g g_i \overset{\varphi}{\mapsto} \left(\sum_{h \in H} a_h \cdot h h_g \right) \otimes g_i = \left(\sum_{h \in H} a_h \cdot h \right) \otimes h_g g_i = \alpha \otimes g.$$

Hence, φ is an isomorphism, and the result follows. □

Using the previous lemma, we can prove a relationship between groups and subgroups of finite index.

Proposition 2.8.7. *Suppose $H \leq G$ and $M(G) = \mathcal{N}(G)$ or $M(G) = \ell^p(G)$ for some $1 \leq p \in \mathbb{R}$. If $[G : H] < \infty$, and $M(H)$ is a flat $\mathbb{C}H$-module, then $M(G)$ is a flat $\mathbb{C}G$-module.*

Proof. Because of (2.8.6), the functor $M(G) \otimes_{\mathbb{C}G} -$ is the composition of the functors $M(H) \otimes_{\mathbb{C}H} -$ and $\mathbb{C}G \otimes_{\mathbb{C}G} -$, which are both exact by hypothesis. □

In general, there aren't any analogous results for groups with respect to quotient groups. However, if the quotient group is obtained by factoring out a finite normal subgroup, then there are some nice results.

Theorem 2.8.8. *Let H be a finite normal subgroup of G, and let $Q = G/H$. Let $M(G) = \ell^p(G)$ for some $1 \leq p \in \mathbb{R}$ or $M(G) = \mathcal{N}(G)$. If $M(G)$ is flat over $\mathbb{C}G$, then $M(Q)$ is flat over $\mathbb{C}Q$.*

Proof. The first relevant claim is that $M(Q)$ is a summand of $M(G)$ as $\mathbb{C}G$-modules. Consider the following central idempotent element of $M(G)$: $e = \dfrac{1}{|H|}\displaystyle\sum_{h\in H} h$. Since e is a central idempotent, it follows that $M(G) \cong M(G)e \oplus M(G)(1-e)$. So to prove the first claim, it suffices to show $M(Q) \cong M(G)e$. Let $X = \{x_i\}$ be a transversal in G with respect to H. And define $\varphi : M(Q) \to M(G)e$ by $\sum a_i \cdot \overline{x_i} \mapsto \sum a_i \cdot x_i e$. It is clear that φ is an injective homomorphism. It remains to show φ is surjective. Pick an arbitrary element $\alpha \in M(G)e$. Then α can be written as $\sum c_{ij} \cdot h_i x_j e$, where $H = \{h_i\}_{i=1}^N$. Note that $h_i x_j e = h_k x_j e = x_j e$ for all $i, k \in \{1, 2, \ldots, N\}$. Hence $\alpha = \displaystyle\sum_j \left(\sum_{i=1}^N c_{ij}\right) x_j e$ and we see that a pre-image for α is $\displaystyle\sum_j \left(\sum_{i=1}^N c_{ij}\right) \overline{x_j}$. This proves that $M(Q)$ is a summand of $M(G)$. Write $M(G) \cong M(Q) \oplus P$. Now let $\mathbf{E} : 0 \to A \to B \to C \to 0$ be an exact sequence of $\mathbb{C}Q$-modules. By assumption, the sequence $\mathbf{E} \otimes_{\mathbb{C}G} M(G)$ must be exact. But since $M(G) \cong M(Q) \oplus P$, we can rewrite this short exact sequence as $(\mathbf{E} \otimes_{\mathbb{C}G} M(Q)) \oplus (\mathbf{E} \otimes_{\mathbb{C}G} P)$. It follows that $\mathbf{E} \otimes_{\mathbb{C}G} M(Q)$ must be exact. And since the action of H is trivial on all modules in this sequence, it follows that $\mathbf{E} \otimes_{\mathbb{C}Q} M(Q)$ is exact. $\square$

There is also a relationship between homology groups, which requires a lemma.

Lemma 2.8.9. *Let H be a finite normal subgroup of G, and let $Q = G/H$. Then there is a $\mathbb{C}Q$-module isomorphism $\ell^p(G)_H \cong \ell^p(Q)$.*

Proof. Since H is finite, there is a natural surjective $\mathbb{C}G$-homomorphism $\varphi : \ell^p(G) \to \ell^p(Q)$. And $\Delta(H)\ell^p(G) \subseteq \ker\varphi$, and so this induces a map $\varphi : \frac{\ell^p(G)}{\Delta(H)\ell^p(G)} \to \ell^p(Q)$. Claim: φ is injective. Let $R = \{g_i\}$ be a set of coset representatives in G with respect to H. Suppose $x = \sum a(g) \cdot g \in \ell^p(G)$ and $[x] \mapsto 0$. Then $\sum a(g) \cdot \bar{g} = 0$ and hence $\sum_{h \in H} a(hg_i) = 0$ for every $g_i \in R$. Now we can use this fact to show that $[x] = 0$:

$$[x] = \left[\sum_H \sum_R a(hg_i) \cdot hg_i \right] = \sum_H \left[h \sum_R a(hg_i) g_i \right] = \sum_H \left[\sum_R a(hg_i) g_i \right] = \left[\sum_R \left(\sum_H a(hg_i) \right) g_i \right] = 0.$$

Therefore φ is a $\mathbb{C}G$-isomorphism. Since the H-action is trivial on both the domain and codomain of φ, it follows that φ is a $\mathbb{C}Q$-isomorphism. $\square$

Using the previous lemma, we get the following relationship between group homology over G and group homology over Q.

Theorem 2.8.10. *Let H be a finite normal subgroup of G, and let $Q = G/H$. Then $H_1(G, \ell^p(G)) \cong H_1(Q, \ell^p(Q))$.*

Proof. The short exact sequence of groups $1 \to H \to G \to Q \to 1$ induces the following exact sequence on homology:

$$H_2(G, \ell^p(G)) \to H_2(Q, \ell^p(G)_H) \to H_1(H, \ell^p(G))_Q \to H_1(G, \ell^p(G)) \to H_1(Q, \ell^p(G)_H) \to 0.$$

Since H is finite, it follows that $H_1(H, \ell^p(G)) = 0$, and hence $H_1(G, \ell^p(G)) \cong H_1(Q, \ell^p(G)_H)$. The result now follows from the previous lemma. $\square$

Next, we would like to show that there is a relationship between whether $\mathcal{N}(G)$ is flat over $\mathbb{C}G$ and whether $\ell^2(G)$ is flat over $\mathbb{C}G$. The first necessary lemma for this states, in effect, that $\mathcal{N}(G)$ is always a semi-hereditary ring ([31], Theorem 6.7(1)).

Lemma 2.8.11. *Any finitely generated submodule of a projective $\mathcal{N}(G)$-module is projective.*

And since $\mathcal{N}(G)$ is semi-hereditary, submodules of free modules must be flat.

Lemma 2.8.12. *Any submodule of a free $\mathcal{N}(G)$-module is flat.*

Proof. Let M be a submodule of a free $\mathcal{N}(G)$-module. Then M can be expressed as a direct limit of its finitely generated submodules, which each must be projective by the previous lemma. Since M is a direct limit of projective modules, it must be a flat module. $\square$

The final lemma necessary to show the relationship between $\ell^2(G)$ and $\mathcal{N}(G)$ is:

Lemma 2.8.13. *The $\mathcal{N}(G)$-module $\mathcal{U}(G)$ can be written as a direct limit (and a union) of free modules F_α for which every $F_\alpha \cong \mathcal{N}(G)$.*

Proof. Let $X = \{\alpha \in \mathcal{N}(G) \mid \alpha \text{ is a non-zero-divisor}\}$. For $\alpha \in X$, define $F_\alpha = \alpha^{-1}\mathcal{N}(G)$, which is a submodule of $\mathcal{U}(G)$ and a free $\mathcal{N}(G)$-module. Then $\mathcal{U}(G) = \bigcup_{\alpha \in X} F_\alpha$. To show this union is also a direct limit, it suffices to show that $\{F_\alpha\}$ is a direct system. In other words, for every pair of non-zero-divisors α, β, we want to show there is a cofinal F_γ such that $F_\alpha \subseteq F_\gamma$ and $F_\beta \subseteq F_\gamma$. Since $\mathcal{N}(G)$ satisfies the Ore condition, there exists δ, $y \in \mathcal{N}(G)$ such that δ is a non-zero-divisor and $\delta\beta = y\alpha$. Hence, for any $\alpha^{-1}x \in \alpha^{-1}\mathcal{N}(G)$ we have:

$$\alpha^{-1}x = \beta^{-1}\beta\alpha^{-1}x = (\beta^{-1}\delta^{-1})yx \in (\delta\beta)^{-1}\mathcal{N}(G)$$

And for any $\beta^{-1}x \in \beta^{-1}\mathcal{N}(G)$ we have:

$$\beta^{-1}x = \beta^{-1}\delta^{-1}\delta x \in (\delta\beta)^{-1}\mathcal{N}(G)$$

Therefore, defining $\gamma = \delta\beta$ provides the cofinal F_γ that we required. $\square$

Now we are ready to prove the important fact about $\ell^2(G)$ as a module over $\mathcal{N}(G)$.

Theorem 2.8.14. *Let G be a group. Then $\ell^2(G)$ is a flat $\mathcal{N}(G)$-module.*

Proof. By the preceding lemma, express $\mathcal{U}(G)$ as $\bigcup F_\alpha$, which is a direct limit of free modules of rank one. Since $\ell^2(G)$ can be imbedded in $\mathcal{U}(G)$, it can be thought of as a submodule. Hence:

$$\ell^2(G) = \mathcal{U}(G) \cap \ell^2(G) = \left(\bigcup F_i\right) \cap \ell^2(G) = \bigcup \left(F_i \cap \ell^2(G)\right)$$

Note that $F_\alpha \cap \ell^2(G)$ is a flat module by (2.8.12). Hence $\ell^2(G)$ is a direct limit of flat modules and the result follows. $\square$

In particular, this leads to the next corollary, which will be relied upon heavily in Chapter 4.

Corollary 2.8.15. *If $\mathcal{N}(G)$ is flat over $\mathbb{C}G$, then $\ell^2(G)$ is flat over $\mathbb{C}G$.*

The previous Corollary stated that to show $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$, it suffices to show $\ell^2(G)$ is not flat over $\mathbb{C}G$. As it turns out, $\mathcal{N}(G)$ has the same relationship with $\mathcal{U}(G)$, on account of $\mathcal{U}(G)$ being the Ore localization of $\mathcal{N}(G)$ ([44], Proposition II.1.4).

Theorem 2.8.16. *The algebra of affiliated operators $\mathcal{U}(G)$ is flat over $\mathcal{N}(G)$.*

Corollary 2.8.17. *If $\mathcal{U}(G)$ is not flat over $\mathbb{C}G$, then $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$.*

2.9 The Case $G = \mathbb{Z} * \mathbb{Z}$

Next we want to consider the free group on two generators $G = \mathbb{Z} * \mathbb{Z}$. Before calculating the group homology for this group, we will need the following lemma ([31], Lemma 6.36).

Lemma 2.9.1. *Let G be a group, and let H be a subgroup of G. Then $\mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C}[G/H]$ is trivial if and only if H is non-amenable.*

With this lemma in mind, it is relatively straightforward to show the first group homology is nontrivial. In fact, one can show the von Neumann dimension of the homology group (i.e., the first L^2-Betti number) is nontrivial.

Theorem 2.9.2. *If $G = \mathbb{Z} * \mathbb{Z}$ and $M = \mathcal{N}(G)$, then $\dim_{\mathcal{N}(G)}(H_1(G, M)) \neq 0$.*

Proof. Recall the free $\mathbb{C}G$ resolution of $\mathbb{C}$: $0 \to \mathbb{C}G^2 \to \mathbb{C}G \to \mathbb{C} \to 0$. Now apply the functor $\mathcal{N}(G) \otimes_{\mathbb{C}G} -$, which is always right exact: $\mathcal{N}(G)^2 \to \mathcal{N}(G) \to \mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C} \to 0$. By 2.9.1, $\mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C} = 0$ since $\mathbb{Z} * \mathbb{Z}$ is not amenable. Thus we have the exact sequence: $\mathcal{N}(G)^2 \to \mathcal{N}(G) \to 0$. Now $H_1(G, M)$ is just the kernel of this map, which has von Neumann dimension one by additivity (see 1.5.4). $\square$

This implies that for $G = \mathbb{Z}*\mathbb{Z}$, the group von Neumann algebra is neither flat nor dimension-flat over $\mathbb{C}G$. By 2.8.2, for any group G with $\mathbb{Z}*\mathbb{Z}$ as a subgroup, $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$. We would like to go one step further and say that for such groups $\mathcal{N}(G)$ is not dimension-flat. Before that can be proved, we need two lemmas.

Lemma 2.9.3. *For any subgroup $H \leq G$ and any $\mathbb{C}H$-module M we have:*

$$\mathcal{N}(G) \otimes_{\mathcal{N}(H)} Tor_1^{\mathbb{C}H}(\mathcal{N}(H),\, M) \cong Tor_1^{\mathbb{C}G}(\mathcal{N}(G),\, \mathbb{C}G \otimes_{\mathbb{C}H} M).$$

Proof. Since $\mathcal{N}(G)$ is flat over $\mathcal{N}(H)$,

$$\mathcal{N}(G) \otimes_{\mathcal{N}(H)} \operatorname{Tor}_1^{\mathbb{C}H}(\mathcal{N}(H),\, M) \cong \operatorname{Tor}_1^{\mathbb{C}H}(\mathcal{N}(G),\, M).$$

This last Tor-group can be calculated by taking a projective $\mathbb{C}H$-resolution $\mathbf{P}$ of M, applying the functor $\mathcal{N}(G) \otimes_{\mathbb{C}H} -$, and taking homology. But since $\mathcal{N}(G) \cong \mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C}G$, it is equivalent to apply the functor $\mathcal{N}(G) \otimes_{\mathbb{C}G} \mathbb{C}G \otimes_{\mathbb{C}H} -$ to $\mathbf{P}$. And this is equivalent to applying the functor $\mathcal{N}(G) \otimes_{\mathbb{C}G} -$ to $\mathbb{C}G \otimes_{\mathbb{C}H} \mathbf{P}$, which is a projective $\mathbb{C}G$-resolution of $\mathbb{C}G \otimes_{\mathbb{C}H} M$ since $\mathbb{C}G$ is a free $\mathbb{C}H$-module (see [7], exercise I.3.1). Hence, the homology of the resulting complex is isomorphic to $\operatorname{Tor}_1^{\mathbb{C}G}(\mathcal{N}(G),\, \mathbb{C}G \otimes_{\mathbb{C}H} M)$. □

The next lemma, which can be found in [31] (Theorem 6.29(2)), provides a strong relationship between the dimensions of a module and its induced module.

Lemma 2.9.4. *Let H be a subgroup of G, and let M be any $\mathcal{N}(H)$-module. Then* $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathcal{N}(H)} M) = \dim_{\mathcal{N}(H)}(M)$.

Now we can prove the lack of dimension-flatness for any group containing a non-abelian free subgroup.

Proposition 2.9.5. *Let G be a group with a subgroup isomorphic to $\mathbb{Z} * \mathbb{Z}$. Then $\mathcal{N}(G)$ is not dimension-flat over $\mathbb{C}G$.*

Proof. Let $H \cong \mathbb{Z} * \mathbb{Z} \leq G$. Then, by 2.9.2, $\dim_{\mathcal{N}(H)}\left(\operatorname{Tor}_1^{\mathbb{C}H}(\mathcal{N}(H), \mathbb{C})\right) = 1$. By 2.9.3, $\mathcal{N}(G) \otimes_{\mathcal{N}(H)} \operatorname{Tor}_1^{\mathbb{C}H}(\mathcal{N}(H), \mathbb{C}) \cong \operatorname{Tor}_1^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C})$. By 2.9.4, it follows that:

$$\dim_{\mathcal{N}(G)}\left(\operatorname{Tor}_1^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C})\right) = \dim_{\mathcal{N}(G)}\left(\mathcal{N}(G) \otimes_{\mathcal{N}(H)} \operatorname{Tor}_1^{\mathbb{C}H}(\mathcal{N}(H), \mathbb{C})\right)$$

$$= \dim_{\mathcal{N}(H)} \left(\mathrm{Tor}_1^{\mathbb{C}H}(\mathcal{N}(H), \mathbb{C}) \right) = 1.$$

$\square$

This result, due to Lück, proves that Conjecture A is true for any group which contains $\mathbb{Z} * \mathbb{Z}$ as a subgroup, since all such groups are non-amenable ([37], Example 0.6). However, Conjecture A is still open in general, since there exist non-amenable groups which do not have any nonabelian free subgroups [35].

Chapter 3

Proving Half of Conjecture B

3.1 Introduction

In this chapter, the first half of Conjecture 1.1.1 will be proved. In other words, we will prove that if a group G is locally virtually cyclic, then $\mathcal{N}(G)$ is a flat $\mathbb{C}G$-module. First, special cases of locally virtually cyclic groups will be considered: finite groups and infinite locally finite groups. Then the focus will be shifted to locally virtually cyclic groups with elements of infinite order. The most foundational case, of course, will be the infinite cyclic group $G = \mathbb{Z}$. This case will serve as a stepping stone to the case when G is virtually cyclic. Finally, the jump can be made from virtually cyclic groups to locally virtually cyclic groups, thereby proving half of Conjecture 1.1.1. The final step to proving the result for locally virtually cyclic groups will require demonstrating that certain Tor-groups vanish, and this will be accomplished by first converting the Tor calculations into group homology calculations. Then the proof can be completed by using previous results and a few important facts about group homology with respect to subgroups and direct limits.

3.2 Locally Finite Groups

In this section, we will show that Conjecture 1.1.1 is consistent for the class of finite groups and infinite locally finite groups. First, suppose that G is a finite group. Then $\mathbb{C}G = \mathcal{N}(G) = \ell^2(G)$. This implies that $\mathcal{N}(G)$ is a free $\mathbb{C}G$-module, and hence it is flat. When G is locally finite, then $\mathcal{N}(G)$ may not be a free module. However, we can show that it must be a flat module just by looking at the ring $\mathbb{C}G$. In particular, it is known that every $\mathbb{C}G$-module is flat. This can be realized as the corollary of some classical results in Ring Theory. First, recall what a semisimple module is.

Definition 3.2.1. An R-module M is simple if it has no nontrivial, proper submodules. We call M semisimple if it is a direct sum of simple modules. A ring R is called semisimple if it is semisimple as a module over itself.

The following theorem, called Maschke's Theorem ([1], p. 157), describes a sufficient condition for a group ring to be semisimple.

Theorem 3.2.2. *Let G be a finite group, and let k be a field whose characteristic does not divide the order of G. Then the group ring kG is a semisimple ring.*

And a version of the Artin-Wedderburn Theorem ([1], p. 152) gives us a powerful fact about semisimple rings.

Theorem 3.2.3. *A ring R is semisimple if and only if every R-module is projective.*

We will use the previous two theorems to show that $\mathbb{C}G$ is a so-called von Neumann regular ring.

Definition 3.2.4. A ring R is von Neumann regular if for every $a \in R$, there exists $x \in R$ such that $a = axa$.

The definition above is the most classical way to define von Neumann regular rings. However, it is the following characterization that will be most relevant for our purpose (see Theorem 4.2.9 in [47]).

Theorem 3.2.5. *A ring R is von Neumann regular if and only if every R-module is flat.*

These results provide the foundation for showing that every $\mathbb{C}G$-module (and in particular, $\mathcal{N}(G)$) is flat for locally finite groups.

Theorem 3.2.6. *If G is a locally finite group, then $\mathbb{C}G$ is a von Neumann regular ring.*

Proof. To show $\mathbb{C}G$ is von Neumann regular, it suffices to show that for all $x \in \mathbb{C}G$ there exists $a \in \mathbb{C}G$ such that $axa = a$. Let $a = a_1g_1 + \cdots + a_ng_n$ be an arbitrary element of $\mathbb{C}G$. Let H be the finite subgroup generated by $g_1, \ldots, g_n$. Then $a \in \mathbb{C}H$ so it suffices to show $\mathbb{C}H$ is von Neumann regular. And this follows from Maschke's Theorem and the Artin-Wedderburn Theorem, since projective modules are always flat modules. $\square$

This result is nothing new. However, it is interesting to note that I originally set out to prove that $\mathcal{N}(G)$ is flat if and only if G is virtually cyclic. Infinite locally finite groups provided the first example of groups for which $\mathcal{N}(G)$ is flat but G is not virtually cyclic, and so the conjecture had to be revised to include all locally virtually cyclic groups.

3.3 Locally Virtually Cyclic Groups

Now let us reconsider the case $G = \mathbb{Z}$. It was previously shown in 2.5.1 that $H_1(G, \mathcal{N}(G)) = 0$. Now we would like to go one step further and prove that $\mathcal{N}(G)$ is flat over $\mathbb{C}G$. First, a basic fact about modules over Principal Ideal Domains is relevant.

Lemma 3.3.1. *Let R be a PID, and let M be an R-module. If M is R-torsion-free, then M is a flat R-module.*

Proof. Consider M as the direct limit of its finitely generated submodules, $M \cong \varinjlim M_i$. Since M is R-torsion-free, it must be the case that every M_i is R-torsion-free. By the standard structure theorem for finitely generated modules over a PID ([11], Theorem 12.1.5), it follows that each M_i is a free R-module. Since M is a direct limit of free modules, it must be a flat module. $\square$

Theorem 3.3.2. *Let $G = \mathbb{Z}$, and let $M = \mathcal{N}(G)$ or $M = \ell^p(G)$ for any $1 \leq p \in \mathbb{R}$. Then M is a flat $\mathbb{C}G$-module.*

Proof. For $G = \mathbb{Z}$, the ring $\mathbb{C}G$ is a PID (see Corollary 6.4 in [21]). By the previous lemma, it suffices to show M is R-torsion-free. But this is true by the Zero Divisor Conjecture [39]. $\square$

As a result of 2.8.7, we have the following corollary.

Corollary 3.3.3. *Suppose G is virtually cyclic, and let $M = \mathcal{N}(G)$ or $M = \ell^p(G)$ for any $1 \leq p \in \mathbb{R}$. Then M is a flat $\mathbb{C}G$-module.*

The final necessary fact before proving half of Conjecture B is the following version of the Zero Divisor Conjecture.

Lemma 3.3.4. *Suppose G has a subgroup H such that $H \cong \mathbb{Z}$, and let $1 \leq p \in \mathbb{R}$. Then $0 \neq \alpha \in \mathbb{C}H$ and $0 \neq \beta \in \ell^p(G)$ implies that $\alpha\beta \neq 0$.*

Proof. Let $0 \neq \alpha \in \mathbb{C}H$ and $0 \neq \beta \in \ell^p(G)$. Let T be a transversal for H in G. Then $\beta = \displaystyle\sum_{t \in T} \alpha_t t$ for some $\alpha_t \in \ell^p(H)$. Suppose, to the contrary, that $\alpha\beta = 0$. Then $\alpha\beta = \alpha \sum_{t\in T} \alpha_t t = \sum_{t \in T} (\alpha\alpha_t)\, t = 0$ if and only if $\alpha\alpha_t = 0$ for all $t \in T$ if and only if $\alpha_t = 0$ for all $t \in T$ (by the Zero Divisor Conjecture being true for $\mathbb{Z}$; see [39]) if and only if $\beta = 0$; a contradiction. Therefore, $\alpha\beta \neq 0$. $\square$

Now we are ready to prove half of Conjecture B.

Theorem 3.3.5. *Suppose G is locally virtually cyclic, and let $M = \mathcal{N}(G)$ or $M = \ell^p(G)$ for any $1 \leq p \in \mathbb{R}$. Then M is a flat $\mathbb{C}G$-module.*

Proof. Let B be any $\mathbb{C}G$-module. It suffices to show $\mathrm{Tor}_1^{\mathbb{C}G}(M, B) = 0$. Since G is locally virtually cyclic, we can express G as a direct limit of virtually cyclic groups: $G = \varinjlim G_i$. Now utilize 2.2.4 and 2.2.3:

$$\mathrm{Tor}_1^{\mathbb{C}G}(M, B) \cong H_1(G, M \otimes B) \cong \varinjlim H_1(G_i, M \otimes B).$$

So it suffices to show $H_1(G_i,\, M \otimes B) = 0$. If G_i is finite, then this is true because of 3.2.6. Suppose each G_i is infinite. Since G_i is infinite virtually cyclic, there must be a normal infinite cyclic subgroup of finite index (see 7.1.6). In other words, we can write $1 \to K_i \to G_i \to Q_i \to 1$, where $K_i \cong \mathbb{Z}$ and Q_i is finite. If $N = M \otimes B$, then there is the following exact sequence (2.2.2):

$$H_2(Q_i,\, N_{K_i}) \to H_1(K_i,\, N)_{Q_i} \to H_1(G_i,\, N) \to H_1(Q_i,\, N_{K_i}).$$

Since Q_i is finite we know that $H_2(Q_i, N_{K_i}) \cong H_1(Q_i,\, N_{K_i}) \cong 0$, hence $H_1(G_i,\, N) \cong H_1(K_i,\, N)_{Q_i}$. Therefore it suffices to show that $H_1(K_i,\, N) = 0$. Rewrite this as $\mathrm{Tor}_1^{\mathbb{C}K_i}(M,\, B)$. This must be trivial since $\mathbb{C}K_i$ is a PID and since M does not have any $\mathbb{C}K_i$-torsion. This completes the proof.

$\square$

This proves Conjecture 1.1.1(A). The rest of this paper will be concerned with Conjecture 1.1.1(B).

Chapter 4

Baumslag-Solitar Groups

4.1 Introduction

One critical case for Conjecture 1.1.1(B), $G = \mathbb{Z} \oplus \mathbb{Z}$, was considered in Chapter 2. Two more critical cases will be considered in this chapter. The first case consists of a class of groups called "Baumslag-Solitar groups." These groups were introduced by Gilbert Baumslag and Donald Solitar in 1962 to provide finitely-presented examples of Hopfian groups [3]; i.e., some of these groups have epimorphisms onto themselves which are not isomorphisms. These groups are defined with the following presentation. For natural numbers m and n, define $B(m, n) = \langle a, b \mid ab^m a^{-1} = b^n \rangle$. Since these groups are one-relator groups, we will use Fox derivatives to do group homology calculations on them. If $m \neq 1$ and $n \neq 1$, then it can be shown that $B(m, n)$ has a copy of $\mathbb{Z} * \mathbb{Z}$ as a subgroup [8]. We shall look only at the amenable Baumslag-Solitar groups, $B(1, n)$. These groups are of cohomological dimension 2 ([5], Theorem 7). In fact, they are the only finitely-generated elementary amenable groups with this property ([25], Theorem 3). Another way of expressing $B(1, n)$ is as a semi-direct product.

Definition 4.1.1. Let G be a group with a normal subgroup H and a subgroup K. If the natural embedding $K \to G$ composed with the natural projection $G \to G/H$ yields an isomorphism $K \to G/H$, then $G = H \rtimes K$ is the semidirect product of H and K.

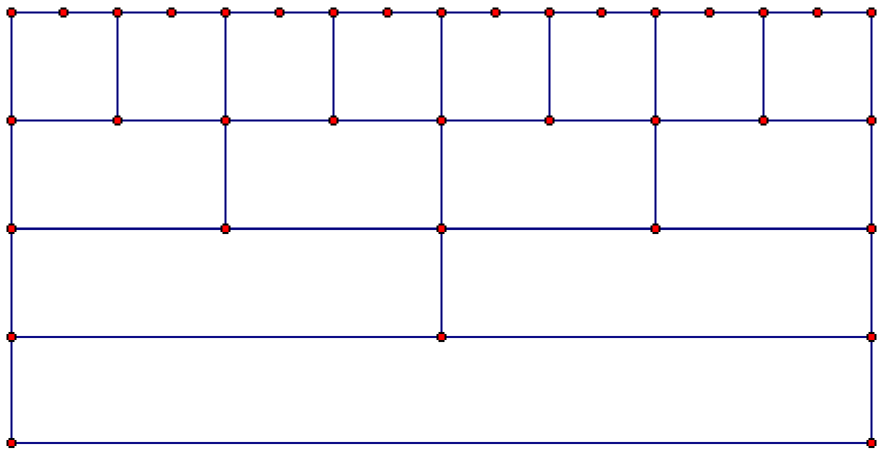
It can be shown that $B(1,n) \cong \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$ (see [14]). In other words, $B(1,n)$ has a normal subgroup H isomorphic to $\mathbb{Z}[\frac{1}{n}]$ with quotient $G/H \cong \mathbb{Z}$, and the generator of the quotient acts on H by multiplication by n.

The other groups considered in this chapter can be similarly expressed as semi-direct products. For distinct natural numbers m, n greater than 1, define $G_{m,n} = \mathbb{Z}[\frac{1}{mn}] \rtimes \mathbb{Z}$, where the generator of the quotient acts on $\mathbb{Z}[\frac{1}{mn}]$ by multiplication by $\frac{m}{n}$. These groups were first introduced within the context of classifying all groups of cohomological dimension 2. For $G = G_{2,3}$, Bieri posed the question (p. 112 in [5]): $cd(G) = 2$ or $cd(G) = 3$? Gildenhuys provided the answer: $cd(G) = 3$ (Theorem 4 in [14]). These groups are finitely-generated but not finitely-presented [2]. Hence, we will not be able to use Fox derivatives for the group homology calculations on these groups.

The goal of this chapter is to prove Conjecture 1.1.1B for the groups $B(1,n)$ and $G_{m,n}$.

4.2 Cayley Graphs for $B(1, n)$

For the groups considered in this chapter, the goal is to show $H_1(G, \ell^2(G)) \neq 0$. The method for showing this will be constructing an $\alpha \in \ell^2(G)$ which satisfies certain properties. Of course, defining an element of $\ell^2(G)$ involves assigning complex numbers to group elements. This process becomes easier to visualize when elements of G are associated with vertices of a Cayley graph. In the notation used below, all the groups considered are two-generator groups with generating set $\Sigma = \{s, t\}$. For such a group we will consider the corresponding Cayley graph $\Gamma = \Gamma(\Sigma)$, where multiplication by s on the right will correspond to leftward edges in Γ, and multiplication by t^{-1} on the right will correspond to upward edges in Γ. These infinite graphs will be somewhat complicated, so we will limit the support of α to just a particular planar subgraph Γ' of Γ, which is defined as follows. First, pick any $g \in G = V(\Gamma)$. Let $V(\Gamma') = \{gt^{-p}s^{-q} \mid p, q \in \mathbb{Z}^{\geq 0}\}$. The edges of Γ' will be inherited from Γ if both ends are vertices in Γ'. For example, if $G = B(1, 2)$, then a piece of Γ' would look like the following:



Not all vertices in Γ' will be used in the support of α. Call a vertex v in Γ' "eligible" if it has both an upward and a downward edge, or if it is on the bottom row with an upward edge. Then $supp(\alpha)$ will be contained in the set of all these vertices, which will be denoted by $EV(\Gamma')$.

4.3 The Case $G = B(1, n)$

Lemma 4.3.1. *Let $G = \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z} = \langle s, t \mid tst^{-1} = s^n \rangle$ and let M denote $\ell^p(G)$ as a right $\mathbb{C}G$-module. Suppose M satisfies the following two conditions:*

(C1) There exist $\alpha, \beta \in M$ such that $\alpha(t-1) = \beta(s-1)$.

(C2) There does not exist $\gamma \in M$ such that $\alpha = \gamma(s-1)$.

Then $H_1(G, M) \neq 0$.

Proof. In order to calculate $H_1(G, M)$, we begin with the following free $\mathbb{C}G$-resolution of $\mathbb{C}$:

$$0 \to \mathbb{C}G \xrightarrow{\phi} \mathbb{C}G^2 \xrightarrow{\psi} \mathbb{C}G \to \mathbb{C} \to 0$$

where $\phi(a) = a(t - 1 - s - s^2 - \cdots - s^{n-1}, 1 - s^n)$ and $\psi(a, b) = a(s - 1) + b(t - 1)$. Those maps are calculated using Fox derivatives from the two-generator, one-relation presentation of the group. Now tensor the deleted resolution with M to get the following complex:

$$0 \to M \xrightarrow{\phi_*} M^2 \xrightarrow{\psi_*} M \to 0.$$

Note that $H_1(G, M) = \ker(\psi_*)/\mathrm{Im}(\phi_*)$. Suppose $(x, y) \in \ker(\psi_*)$. Then $x(s-1) = -y(t-1)$; note that this matches with condition (C1). To show $(x, y) \notin \mathrm{Im}(\phi_*)$, it would suffice to show that $y \neq z(s-1)$ for any $z \in M$, since:

$$(x, y) \in \mathrm{Im}(\phi_*) \;\Rightarrow\; \exists z \in M \text{ such that } (x, y) = z(t - 1 - s - s^2 - \cdots - s^{n-1}, 1 - s^n)$$

$$\Rightarrow\; y = z(1 - s^n) \;\Rightarrow\; y = \gamma(s-1) \text{ for some } \gamma \in M.$$

Thus, by contrapositive, $y \neq \gamma(s-1)$ for any $\gamma \in M$ implies $(x, y) \notin \mathrm{Im}(\phi_*)$. And this matches with condition (C2). □

Theorem 4.3.2. *If $G = \mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z} = \langle s, t \mid tst^{-1} = s^2 \rangle$ and $M = \ell^2(G)$, then $H_1(G, M) \neq 0$.*

Proof. It suffices to show M satisfies conditions (C1) and (C2). Use the following notation:

$$\alpha = \sum_{g \in G} a_g \cdot g, \quad \beta = \sum_{g \in G} b_g \cdot g, \quad \gamma = \sum_{g \in G} c_g \cdot g.$$

Assume $\alpha(t-1) = \beta(s-1)$. Then $\left(\sum a_g \cdot g\right)(t-1) = \left(\sum b_g \cdot g\right)(s-1)$, which implies:

$$\left(\sum a_g \cdot gt\right) - \left(\sum a_g \cdot g\right) = \left(\sum b_g \cdot gs\right) - \left(\sum b_g \cdot g\right),$$

$$\sum (a_{gt^{-1}} - a_g) \cdot g = \sum (b_{gs^{-1}} - b_g) \cdot g.$$

Hence, for all $g \in G$, $a_{gt^{-1}} - a_g = b_{gs^{-1}} - b_g$. Solve this equation for b_g: $b_g = (a_g - a_{gt^{-1}}) + b_{gs^{-1}}$. Substituting this equation into itself repeatedly yields the following relationship between the β coefficients and the α coefficients:

$$b_g = \sum_{k=0}^{\infty} \left(a_{gs^{-k}} - a_{gs^{-k}t^{-1}}\right). \tag{4.1}$$

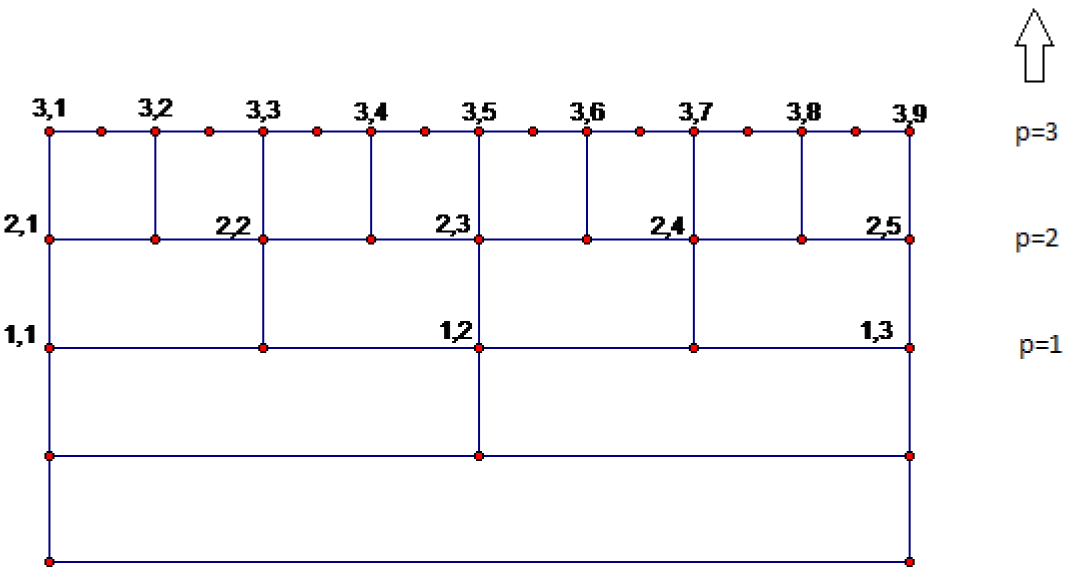
Now assume that $\alpha = \gamma(s-1)$. Then:

$$\sum a_g \cdot g = \left(\sum c_g \cdot g\right)(s-1) = \left(\sum c_g \cdot gs\right) - \left(\sum c_g \cdot g\right) = \sum (c_{gs^{-1}} - c_g) \cdot g.$$

Therefore, $a_g = c_{gs^{-1}} - c_g$ for all $g \in G$, and $c_g = -a_g + c_{gs^{-1}}$. As before, this leads to a relationship between the γ coefficients and the α coefficients:

$$c_g = -\sum_{k=0}^{\infty} a_{gs^{-k}}. \tag{4.2}$$

It now suffices to demonstrate square-summable α-coefficients for which the corresponding β-coefficients in 4.1 are square-summable while the corresponding γ-coefficients in 4.2 are not square-summable. For the remainder of the proof, we will identify elements in the group with vertices in the Cayley graph Γ with respect to the generating set $\Sigma = \{s, t\}$. In particular, the support of α will be contained in the following subset of $EV(\Gamma')$: $S = \{g_{pq} \mid p \in \mathbb{N},\, 1 \leq q \leq 2^p + 1\}$, as labeled below where g_{pq} is denoted by p, q:



For $g = g_{pq}$, denote a_g as a_{pq}. Then $\alpha = \sum_{\substack{p \in \mathbb{N} \\ 1 \leq q \leq 2^p+1}} a_{pq} \cdot g_{pq}$. Define the α-coefficients as follows:

$$a_{pq} = \begin{cases} 2^{-p}, & 1 \leq q \leq 2^{p-1} \\ 0, & q = 2^{p-1} + 1 \\ -2^{-p}, & 2^{p-1} + 2 \leq q \leq 2^p + 1 \end{cases}$$

For every $p \in \mathbb{N}$, there are 2^p α-coefficients of $\pm\frac{1}{2^p}$. Hence:

$$\sum_{p,q} |a_{pq}|^2 = \sum_{p=1}^{\infty} 2^p \left(\frac{1}{2^p}\right)^2 = \sum_{p=1}^{\infty} \frac{1}{2^p} < \infty.$$

Therefore, $\alpha \in \ell^2(G)$. Now consider β. If $g = g_{12}t$ or $g = g_{13}t$, then $b_g = \frac{1}{2}$. If $p \geq 1$ and $1 \leq q \leq 2^{p-1}$, then for $g = g_{pq}s^{-1}$, we have that $b_g = \frac{-1}{2^{p+1}}$. If $p \geq 1$ and $2^{p-1}+2 \leq q \leq 2^p+1$, then for $g = g_{pq}$, we have that $b_g = \frac{1}{2^{p+1}}$. All other β-coefficients are zero. Hence:

$$\sum_{g \in G} |b_g|^2 = 2\left(\frac{1}{2}\right)^2 + \sum_{p=1}^{\infty} 2^p \left(\frac{1}{2^{p+1}}\right)^2 < \infty.$$

Therefore, $\beta \in \ell^2(G)$. Finally, consider γ. For all $p \in \mathbb{N}$, if $q = 2^{p-1}+1$ and $g = g_{pq}$, then $c_g = \frac{-1}{2}$. This implies that $\sum |c_g|^2 = \infty$, and so $\gamma \notin \ell^2(G)$. $\square$

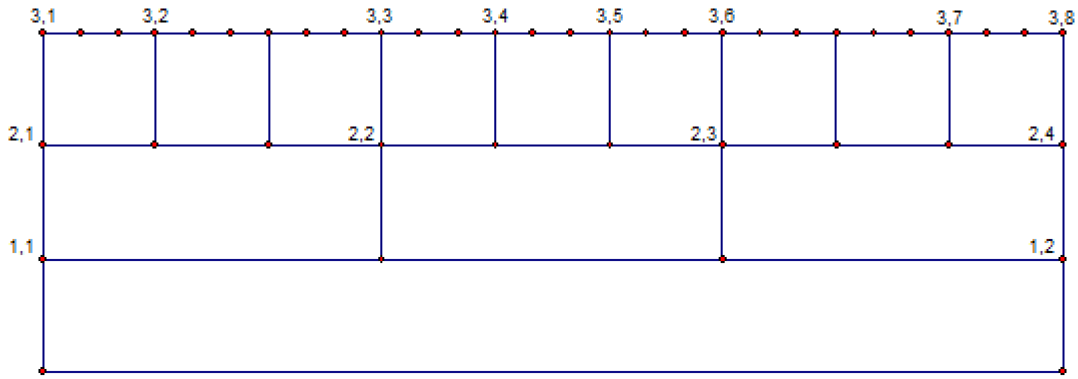
The previous theorem shows the desired result for the group $B(1,n)$ when $n = 2$, and its proof can be generalized to work for when $n \geq 3$.

Theorem 4.3.3. *Let $n \geq 3$ be a natural number. If $G = \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z} = \langle s, t \mid tst^{-1} = s^n \rangle$ and $M = \ell^2(G)$, then $H_1(G, M) \neq 0$.*

Proof. Once again, it suffices to find α, β, γ that satisfy conditions (C1) and (C2). Using the same notation of $\alpha = \sum a_g \cdot g$, $\beta = \sum b_g \cdot g$, and $\gamma = \sum c_g \cdot g$, we see that the coefficients must satisfy the same relationships as before. In particular, see 4.1 and 4.2. This proof will be very similar to the previous one. Indeed, the multi-set of α-coefficients will be identical to the one above, and the support of α will again be contained in $EV(\Gamma')$. The support of α will be equal to a set of vertices $S = \{g_{pq} \mid p \in \mathbb{N}, 1 \leq q \leq 2^p\}$, defined as follows. Pick any $g_{11} \in G$. Define $g_{12} = g_{11}s^{-n}$. Now continue with an inductive definition:

$$g_{p,q} = \begin{cases} \left(g_{p-1,\frac{q+1}{2}}\right)t^{-1}, & \text{if } 1 \leq q \leq 2^{p-1} \text{ is odd} \\ \left(g_{p-1,\frac{q}{2}}\right)t^{-1}s^{-n}, & \text{if } 1 \leq q \leq 2^{p-1} \text{ is even} \\ \left(g_{p-1,\frac{q}{2}}\right)t^{-1}, & \text{if } 2^{p-1} < q \leq 2^p \text{ is even} \\ \left(g_{p-1,\frac{q+1}{2}}\right)t^{-1}s^{-n}, & \text{if } 2^{p-1} < q \leq 2^p \text{ is odd.} \end{cases}$$

In the case $n = 3$, this produces a labeling of $EV(\Gamma')$ pictured below:



Now define the α-coefficients:

$$a_{pq} = \begin{cases} 2^{-p}, & 1 \leq q \leq 2^{p-1} \\ -2^{-p}, & 2^{p-1}+1 \leq q \leq 2^p. \end{cases}$$

Then $\alpha \in \ell^2(G)$:

$$\sum_{p,q} |a_{pq}|^2 = \sum_{p=1}^{\infty} 2^p \left(\frac{1}{2^p}\right)^2 = \sum_{p=1}^{\infty} \frac{1}{2^p} < \infty.$$

And $\beta \in \ell^2(G)$:

$$\sum_{g \in G} |b_g|^2 = \sum_{p=1}^{\infty} 2^{p-1} \left(\frac{1}{2^p}\right)^2 < \infty.$$

However, $\gamma \notin \ell^2(G)$; if $q = 2^{p-1}$ and $g = g_{pq}s^{-1}$, then $c_g = \frac{-1}{2}$.

□

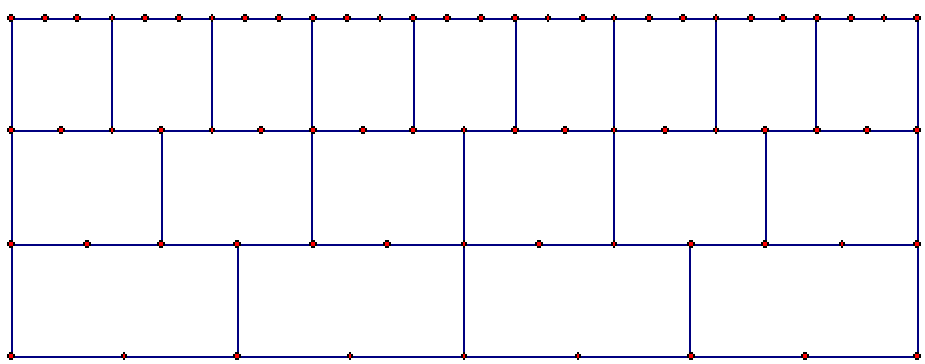
Corollary 4.3.4. *Let $n \geq 2$ be a natural number. If $G = \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z} = \langle s, t \mid tst^{-1} = s^n \rangle$, then $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$.*

The theorems of this section have applied to the module $M = \ell^p(G)$ for $p = 2$. However, they can be easily modified to work for any other p-values such that $1 < p \in \mathbb{R}$.

Theorem 4.3.5. *Let $n \geq 2$ be a natural number, and let $p > 1$ be a real number. If $G = \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z} = \langle s, t \mid tst^{-1} = s^n \rangle$ and $M = \ell^p(G)$, then $H_1(G, M) \neq 0$.*

4.4 Cayley Graphs for $G_{m,n}$

For $G = G_{m,n}$, the method for showing $H_1(G, \ell^2(G)) \neq 0$ will be similar to the methods above; an α in $\ell^2(G)$ will be constructed to satisfy certain properties. Once again, the support of α will be contained in $\Gamma' \leq \Gamma$. In contrast to the Cayley graphs of the last section, in this section's Cayley graphs, leftward edges will represent multiplication by s on the left, and upward edges will represent multiplication by t on the left. The Cayley graphs of this section will be very similar to the ones of the previous section. However, the eligible vertices in Γ' will be more sparsely distributed, meaning the definition of α will be slightly more complicated. If $m = 3$ and $n = 2$, then a representative piece of Γ' looks like the following:



There are some key facts regarding the distribution of the eligible vertices. Define a "horizontal distance" function on $V(\Gamma')$; if $v_1, v_2 \in V(\Gamma')$ then $d_h(v_1, v_2) = k$ if $s^{\pm k}v_1 = v_2$, and define $d_h(v_1, v_2) = \infty$ otherwise. Within rows of Γ' other than the first row, the eligible vertices are spaced distance mn apart. In the most natural way, extend $d(\cdot, \cdot)$ to be a metric on each row of Γ' (i.e., $d(w_1, w_2)$ is defined even for non-vertices w_1, w_2 on the same row of Γ'). For a point w in Γ', define $d(w)$ to be the horizontal distance from w to the left-most column in Γ'. For a point w in Γ', define $F(w)$ to be the point in Γ' on the next row up with $d(F(w)) = \frac{m}{n}d(w)$. In the way Γ' is drawn above, for each $w \in \Gamma'$, the point $F(w)$ is the point in Γ' directly above w. For every $r \in \mathbb{N}$, define a function $T_r : EV(\Gamma') \to EV(\Gamma')$ as follows: $T_r(v)$ is the closest eligible vertex to $F^r(v)$. And if there is a choice for $T_r(v)$ (i.e.,

$F^r(v)$ is midway between two eligible vertices), choose the closest eligible vertex on the right rather than left. For any $w \in \Gamma'$, denote the closest eligible vertex on the right by $R(w)$.

In order to bound norms of elements in $\ell^2(G)$ later on, it will be helpful to consider some bounds on distances in Γ' presently. Note that for all w, w' on the same row of Γ', $d(F(w), F(w')) = \frac{m}{n} d(w, w')$. And for all $v \in EV(\Gamma')$ and $k \in \mathbb{N}$, $d(F^k(v), T_k(v)) \leq mn$. It follows that $d(T_k(v), tT_{k-1}(v)) \leq \frac{m}{n}(mn) + mn$. Suppose that $\frac{m}{n} > 1$, and let p be the smallest natural number such that $\left(\frac{m}{n}\right)^p > 2$. If $v, v' \in EV(\Gamma')$ are such that $d(v, v') = mn$ and v is to the left of v', then $d(T_p(v), T_p(v')) > mn$, which means there is another eligible vertex between $T_p(v)$ and $T_p(v')$. In this case, by the Triangle Inequality, the maximum distance between any two of $T_p(v)$, $tT_{p-1}(v)$, and $R(T_p(v))$ is $\frac{m}{n}(mn) + 2mn$. This constant will be a useful bound, so give it a name; $b = b(m, n) = \frac{m}{n}(mn) + 2mn$. The notation of T_r, R, p, and b as defined here will be utilized below.

4.5 The Case $G = G_{m,n}$

For some natural numbers $m, n \geq 2$, let $G = G_{m,n} = \mathbb{Z}[\frac{1}{mn}] \rtimes \mathbb{Z}$, where the action on $\mathbb{Z}[\frac{1}{mn}]$ is multiplication by $\frac{m}{n}$. Let $M = \ell^2(G)$, considered as a left $\mathbb{C}G$-module. We would like to show that $H_1(G,\, M) \neq 0$. For notation, let $H = \mathbb{Z}[\frac{1}{mn}]$, which is a normal subgroup of G. And let $Q = G/H \cong \mathbb{Z} = \langle t \rangle$. Let $h_i = \left(\frac{1}{mn}\right)^i$, and define $H_i = \langle h_i \rangle \leq H$. Then $H_i \cong \mathbb{Z}$ and $H = \bigcup H_i$.

Lemma 4.5.1. *To prove that $H_1(G,\, M) \neq 0$, it suffices to show $H_1(Q,\, M_H) \neq 0$.*

Proof. There is the short exact sequence of groups $1 \to H \to G \to Q \to 1$, which implies there is an exact sequence on group homology:

$$H_1(H,\, M)_Q \to H_1(G,\, M) \to H_1(Q,\, M_H) \to 0.$$

And by (2.2.3):

$$H_1(H,\, M) = H_1\left(\varinjlim H_i,\, M\right) \cong \varinjlim H_1(H_i,\, M) = 0,$$

since $H_i \cong \mathbb{Z}$ and M has no $\mathbb{C}H_i$-zero-divisors ([27], Theorem 2). Hence $H_1(G,\, M) \cong H_1(Q,\, M_H)$. $\square$

Lemma 4.5.2. *To prove that $H_1(G,\, M) \neq 0$, it suffices to show there exists $\alpha \in M \backslash \Delta(H)M$ such that $(t-1)\alpha \in \Delta(H)M$.*

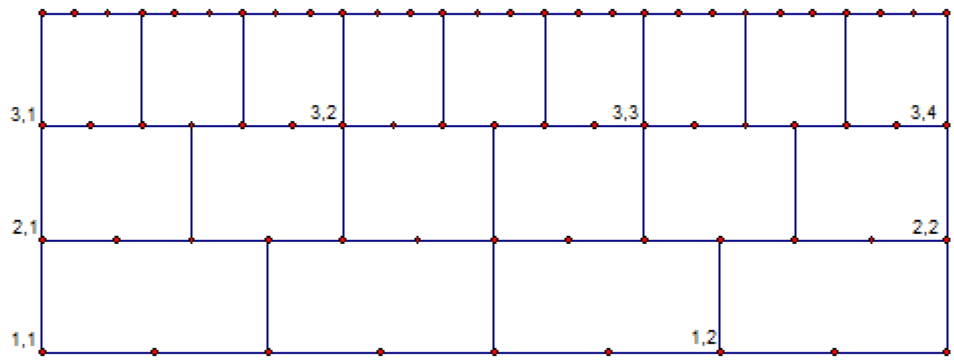
Proof. By the previous Lemma, it suffices to show $H_1(Q,\, M_H) \neq 0$. Since $Q \cong \mathbb{Z}$, it suffices to show M_H has a nontrivial $\mathbb{C}Q$-zero-divisor. Since $M_H \cong \frac{M}{\Delta(H)M}$, and $t-1$ is nonzero in $\mathbb{C}Q$, the existence of the α described above is enough to show $H_1(Q,\, M_H) \neq 0$. $\square$

Theorem 4.5.3. *Suppose $m, n \geq 2$ are distinct natural numbers. If $G = G_{m,n}$ and $M = \ell^2(G)$, then $H_1(G,\, M) \neq 0$.*

Proof. Suppose $m > n$. By the previous Lemma, it suffices to find an $\alpha \in \ell^2(G)$ such that $\alpha \notin \Delta(H)\ell^2(G)$ and $(t-1)\alpha = (s-1)\beta$, where $s = h_1$. Define $\alpha = \sum_{g \in G} a_g \cdot g$ as follows. The support of α will be contained in $EV(\Gamma')$. Before defining $supp(\alpha)$ explicitly, let's take a moment to describe the way $supp(\alpha)$ will be labeled and enumerated. In particular, we will construct $supp(\alpha) = \{g_{i,j} \mid i \in \mathbb{N},\, 1 \leq j \leq 2^{k+1} \text{ if } kp+1 \leq i \leq (k+1)p\} \subset EV(\Gamma')$.
For example, if $m = 3$ and $n = 2$, then $p = 2$. So for $i = 1$ or $i = 2$, there are $2^1 = 2$ corresponding g_{ij} terms; namely, g_{i1} and g_{i2}. For $i = 3$ or $i = 4$, there are $2^2 = 4$ corresponding g_{ij} terms, as j ranges from one to four. And this pattern continues for all i.
Now we are ready to define what each $g_{ij} \in supp(\alpha)$ is. Pick the bottom-left vertex of Γ' to be g_{11}, and pick $g_{12} = s^{-mn}(g_{11})$. Now define the rest of the vertices inductively. First, consider the case when the number of $supp(\alpha)$ vertices on row i is the same as the number in the previous row. If $kp+1 < i \leq (k+1)p$ and $v = g_{kp+1,j}$, then define $g_{ij} = T_{i-kp-1}(v)$. Next, consider the case when there are twice as many $supp(\alpha)$ vertices on row i as there are on the previous row. If $i = kp+1$, then define:

$$g_{ij} = \begin{cases} T_1(g_{i-1,k}), & \text{if } j = 2k-1 \text{ is odd} \\ RT_1(g_{i-1,k}), & \text{if } j = 2k \text{ is even.} \end{cases}$$

For example, if $m = 3$ and $n = 2$, then a piece of Γ' with the first three rows of g_{ij} labeled as i, j looks like the following:



Now each group element $g = g_{ij}$ in $supp(\alpha)$ needs to be assigned a nonzero complex number $a_g = a_{ij}$. If row i has 2^k vertices in $supp(\alpha)$, then define:

$$a_{ij} = \begin{cases} 2^{-k}, & 1 \leq j \leq 2^{k-1} \\ -2^{-k}, & 2^{k-1}+1 \leq j \leq 2^k. \end{cases}$$

It follows that α is in $\ell^2(G)$:

$$\sum_{i,j} |a_{ij}|^2 = \sum_{i=1}^{\infty} p \cdot 2^i \cdot \left(\frac{1}{2^i}\right)^2 < \infty.$$

Now suppose that $(t-1)\alpha = (s-1)\beta$. For precisely the same reasoning as in the $G = B(1,n)$ case, if $\beta = \sum_{g \in G} b_g \cdot g$, then:

$$b_g = \sum_{k=0}^{\infty} \left(a_{s^{-k}g} - a_{t^{-1}s^{-k}g}\right).$$

Using the definition of α, we would like to show that β is in $\ell^2(G)$. First, consider the β-coefficients from the rows of Γ' for which the number of vertices in $supp(\alpha)$ is the same as the row beneath it; let X be the set of all such vertices. Let row i be such a row. For all relevant j-values, consider all vertices between g_{ij} and $tg_{i-1,j}$, including the right endpoint but not the left endpoint; there are at most b of them. And for each of these vertices v, the largest that $|b_v|$ can be is $2^{-(k-1)}$, for k such that $kp+1 < i \leq (k+1)p$. All other b_g on this row are zero. Therefore:

$$\sum_{g \in X} |b_g|^2 \leq p \sum_{k=1}^{\infty} b \cdot 2^k \cdot \left(\frac{1}{2^{k-1}}\right)^2 < \infty.$$

Next, consider the β-coefficients for rows of Γ' for which there are twice as many α-terms as the row beneath it; let Y be the set of all such vertices. Suppose row i is such a row. For all relevant odd j-values, consider all vertices between g_{ij}, $g_{i,j+1}$ and $tg_{i-1,\frac{j+1}{2}}$, including the right endpoint but not the left endpoint; there are at most b of them. For each of these vertices v, the largest that $|b_v|$ can be is $2^{-(i-1)}$. All other b_g on this row are zero. Hence:

$$\sum_{g \in Y} |b_g|^2 \leq \sum_{k=1}^{\infty} b \cdot 2^k \cdot \left(\frac{1}{2^{k-1}}\right)^2 < \infty.$$

All other β-coefficients are zero.

It now only remains to show that $\alpha \notin \Delta(H)\ell^2(G)$. Suppose $\alpha \in \Delta(H)\ell^2(G)$. Then there exists $q \in \mathbb{N}$ such that $\alpha \in \Delta(H_q)\ell^2(G)$. This implies that there exists $\gamma \in \ell^2(G)$ such that $\alpha = (h_q - 1)\gamma$. Define $\sigma = h_q$. If $\gamma = \sum_{g \in G} c_g \cdot g$, then:

$$c_g = -\sum_{k=0}^{\infty} a_{\sigma^{-k} g}.$$

If $g = g_{ij}$, then define $c_{ij} = c_g$. Since there exists $N \in \mathbb{N}$ such that $\sigma^N = s$, it follows that $c_{ij} = -\sum_{k=0}^{\infty} a_{s^{-k} g}$. Therefore, if $i = kp + 1$ and $j = 2^{k-1} + 1$ for any natural number k, then $c_{ij} = \frac{1}{2}$. Hence, γ is not in $\ell^2(G)$; a contradiction. Thus $\alpha \notin \Delta(H)\ell^2(G)$.

If $m < n$, then build the subgraph Γ' downward in the Cayley graph instead of upward. More precisely, replace every "t" with "t^{-1}" in the definition of Γ'. This will lead to analogous definitions of T and p. After these alterations, the rest of the argument is the same as in the case $m > n$.

$\square$

Corollary 4.5.4. *Suppose $m, n \geq 2$ are distinct natural numbers. If $G = G_{m,n}$, then $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$.*

The theorem above applies to the module $M = \ell^p(G)$ with $p = 2$. However, the proof can be easily modified to work for any p-values such that $1 < p \in \mathbb{R}$.

Theorem 4.5.5. *Suppose $m, n \geq 2$ are distinct natural numbers. If $G = G_{m,n}$ and $p > 1$, then $\ell^2(G)$ is not flat over $\mathbb{C}G$.*

Chapter 5

Elementary Amenable Groups

5.1 Introduction

Conjecture 1.1.1(A) was proved in Chapter 3. Conjecture 1.1.1(B) is still open in its most general form, but this chapter will prove it for certain classes of groups. In particular, we will show that the conjecture is true for the class of torsion-free elementary amenable groups. Once that result has been proved, we can then show the conjecture to be true for the class of elementary amenable groups which do not have infinite locally finite subgroups. The reason this conjecture is more approachable for elementary amenable groups than it is in general is the existence of an inductive definition of the class of elementary amenable groups. So the main theorems of this chapter will be proved by induction. The results in this chapter rely heavily upon three special cases which have already been considered: $G = \mathbb{Z} \oplus \mathbb{Z}$, $G = B(1, n)$, and $G = G_{m,n}$. The fact that Conjecture 1.1.1(B) is true for these special cases is foundational to the proofs for the larger classes of groups.

5.2 Elementary Amenable Groups

All of the groups in this chapter are elementary amenable groups. This class of groups is defined as follows.

Definition 5.2.1. The class of elementary amenable groups $\mathcal{EA}$ is the smallest subclass of the class of all groups which satisfies the following conditions:

1. $\mathcal{EA}$ contains all finite groups and all abelian groups,
2. if $G \in \mathcal{EA}$ and $H \cong G$, then $H \in \mathcal{EA}$,
3. $\mathcal{EA}$ is closed under the operations of taking subgroups, forming quotients, and forming extensions, and
4. $\mathcal{EA}$ is closed under directed unions.

The class $\mathcal{EA}$ is contained in the class of all amenable groups ([37], Propositions 0.15 and 0.16). However, these classes are not the same. For instance, Grigorchuk has constructed a finitely presented group which is amenable but not elementary amenable [15].

The most important description of the class $\mathcal{EA}$ for our purposes is the inductive definition given in Section 3 of [26]. First, we need to establish some notation. For two classes of groups $\mathcal{X}$ and $\mathcal{Y}$, say that a group $G \in (L\mathcal{X})\mathcal{Y}$ if G has a normal subgroup H such that H is locally in $\mathcal{X}$ and $G/H \in \mathcal{Y}$. Let $\mathcal{B}$ denote the class of all finitely generated abelian-by-finite groups. For each ordinal α, define $\mathcal{X}_\alpha$ inductively as follows:
$\mathcal{X}_0 = \{1\}$,
$\mathcal{X}_1 = \mathcal{B}$,
$\mathcal{X}_\alpha = (L\mathcal{X}_{\alpha-1})\mathcal{B}$ if α is a successor ordinal, and
$\mathcal{X}_\alpha = \bigcup\limits_{\beta<\alpha} \mathcal{X}_\beta$, if α is a limit ordinal.
With this notation, the class of elementary amenable groups can be built by combining each of these classes; $\mathcal{EA} = \bigcup\limits_{\alpha \geq 0} \mathcal{X}_\alpha$.

The next theorem about the structure of elementary amenable groups should give an indication about why the groups $B(1,n)$ and $G_{m,n}$ in particular are so important ([14], Theorem 5).

Theorem 5.2.2. *Let G be an elementary amenable group of cohomological dimension ≤ 2.*

1. *Suppose that G is finitely generated. Then G possesses a presentation of the form*

$$\langle x, y \mid yxy^{-1} = x^n \rangle.$$

2. *Suppose that G is countable but not finitely generated. Then G is a non-cyclic subgroup of the additive group $\mathbb{Q}$.*

5.3 Hirsch Length

Every elementary amenable group has a "Hirsch length" associated to it. This concept was originally developed for polycyclic groups, which are a particular kind of solvable group.

Definition 5.3.1. A group G is said to be polycyclic if it has a subnormal series

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G,$$

such that G_{i+1}/G_i is cyclic for each $i = 0, 1, \ldots, n-1$. A subnormal series of this form is called a polycyclic series. The Hirsch length of a polycyclic group G, denoted $h(G)$, is the number of infinite factors in a polycyclic series of G.

The class of all polycyclic groups is a subclass of the class of all solvable groups, which in turn is a subclass of $\mathcal{EA}$. Using the inductive definition of $\mathcal{EA}$ above, the notion of Hirsch length may be extended to make sense for all elementary amenable groups. If $G \in \mathcal{X}_1$, then there exists an normal abelian subgroup A such that $[G : A] < \infty$. Let $h(G) = \text{rank}(A)$. Now suppose $h(G)$ is defined for all groups in $\mathcal{X}_\alpha$. If $G \in L\mathcal{X}_\alpha$, then define:

$$h(G) = \sup\{h(G) \mid F \leq G \text{ and } F \in \mathcal{X}_\alpha\}.$$

If $G \in \mathcal{X}_{\alpha+1}$, then there exists a normal subgroup $K \triangleleft G$ with $K \in L\mathcal{X}_\alpha$ and $G/K \in \mathcal{X}_1$. Define $h(G) = h(K) + h(G/K)$.

There are a few facts about Hirsch length that will be utilized, such as the properties in the next theorem ([19], Theorem 1):

Theorem 5.3.2. *Let G be an elementary amenable group. Then:*

1. *Hirsch length $h(G)$ is well-defined.*
2. *If $H \leq G$, then $h(H) \leq h(G)$.*
3. *Furthermore, $h(G) = lub\{h(F) \mid F$ is a finitely generated subgroup of $G\}$.*
4. *If H is a normal subgroup of G, then $h(G) = h(H) + h(G/H)$.*

The only elementary amenable groups that are torsion are the locally finite ones, and these are the only groups in $\mathcal{EA}$ with trivial Hirsch length. More generally ([19], p. 164):

Theorem 5.3.3. *If G is an elementary amenable group and H is a locally finite normal subgroup, then $h(G/H) = h(G)$.*

As a result of Lemma 2 in [19], there is the following bound on $h(G)$.

Theorem 5.3.4. *Hirsch length is bounded above by the rational cohomological dimension.*

Groups of small Hirsch length have the following classification ([19], Theorem 2):

Theorem 5.3.5. *Let G be elementary amenable and let T be its maximal locally finite normal subgroup. Then:*

1. *If $h(G) < \infty$, then $G \in LX_{h(G)+1}$.*
2. *If $h(G) < 3$, then G/T is solvable of derived length at most 5.*
3. *If $h(G) = 1$ or 2 and G is finitely generated, then G/T is virtually torsion-free.*

5.4 Torsion-Free Elementary Amenable Groups

In this section we use the inductive definition of $\mathcal{EA}$ to prove Conjecture 1.1.1 for all torsion-free elementary amenable groups.

Lemma 5.4.1. *Let G be an additive subgroup of $\mathbb{Q}$, and let $\varphi \in Aut(G)$. Then $\varphi(x) = rx$ for some $r \in \mathbb{Q}$.*

Proof. Let $X_n = \{k \in \mathbb{Z}^{\geq 0} \mid \frac{k}{n} \in G\}$, and let $p_n = min(X_n)$. Then $X_n = \{kp_n \mid k \in \mathbb{Z}^{\geq 0}\}$. Suppose $\varphi(\frac{p_n}{n}) = r_n(\frac{p_n}{n})$. Then since φ is additive, $\varphi(x) = r_n x$ for all $x \in G$ such that $x = \frac{kp_n}{n}$. Now note that every nonempty X_n intersects nontrivially with X_1. Hence $r_n = r_1$ for all n, and so we may choose $r = r_1$. □

Lemma 5.4.2. *Let G be a group with a normal subgroup H such that H is an additive subgroup of the rational numbers, and $G/H \cong \mathbb{Z}$. Then G has a subgroup isomorphic to $G_{p,q} = \mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}$.*

Proof. Let $x \in G \setminus H$, and define $f \in Aut(H)$ by $f(h) = xhx^{-1}$. Since H is isomorphic to a subgroup of $\mathbb{Q}$ that includes 1, we can assume that $1 \in H$. By the Lemma above, $f(y) = ry$ for some $r \in \mathbb{Q}$. Suppose r can be written as the reduced fraction $\frac{p}{q}$. Then $f^i(1) = \frac{p^i}{q^i} \in H$ for all $i \in \mathbb{Z}$. Hence $a\frac{p^i}{q^i} + b\frac{q^i}{p^i} = \frac{ap^{2i}+bq^{2i}}{p^iq^i} \in H$, for all $a, b, i \in \mathbb{Z}$. Since $gcd(p, q) = 1$, we can choose a, b such that $\frac{1}{p^iq^i} \in H$ for all $i \in \mathbb{Z}$. This implies that $\mathbb{Z}[\frac{1}{pq}] \leq H$; call this subgroup K. Consider the following subgroup of G: $A = \langle K, x\rangle$. Then K is normal in A with quotient isomorphic to $\mathbb{Z}$, and the conjugation of x on K is equivalent to the homomorphism f above. That is, $A \cong G_{p,q}$. □

The next theorem is the main theorem of this section. For any group G, let $\mathcal{M}(G)$ denote either $\mathcal{N}(G)$ or $\ell^p(G)$ for any $1 < p \in \mathbb{R}$.

Theorem 5.4.3. *Suppose G is a torsion-free elementary amenable group. If $\mathcal{M}(G)$ is flat over $\mathbb{C}G$, then G is locally cyclic.*

Proof. First note that since every virtually cyclic group is either finite, finite-by-(infinite cyclic), or finite-by-(infinite dihedral) and since we are only considering torsion-free groups, we can use the terms "virtually cyclic" and "cyclic" interchangeably (see 7.1.7).

Recall the inductive definition of the class of elementary amenable groups. Let $X_0 = \{1\}$ and let X_1 be the class of finitely generated virtually abelian groups. If X_α has been defined for some ordinal α let $X_{\alpha+1} = (LX_\alpha)X_1$. If X_α has been defined for all ordinals α less than some limit ordinal β let $X_\beta = \bigcup X_\alpha$. Then $\mathcal{EA} = \bigcup X_\alpha$. The result will be proved by induction on α.

Base Case: Consider $G \in X_1$; suppose G is finitely generated virtually abelian. Then there exists $H \leq G$ such that H is abelian and $[G : H] < \infty$. Note that H cannot contain $\mathbb{Z} \times \mathbb{Z}$ as a subgroup by the assumption that $\mathcal{M}(G)$ is flat over $\mathbb{C}G$. Hence $H \cong \mathbb{Z}$ (note: a subgroup of finite index of a finitely generated group is necessarily finitely generated). Hence G is virtually cyclic.

Induction Hypothesis: Suppose the result is true for all groups in X_α.

Induction Step: We wish to show the result is true for all groups in $X_{\alpha+1}$. If $G \in X_{\alpha+1}$, then there exists a normal subgroup $H \in LX_\alpha$ such that $G/H \in X_1$. Since $\mathcal{M}(G)$ is flat over $\mathbb{C}G$, it must be the case that $\mathcal{M}(H)$ is flat over $\mathbb{C}H$ (see 2.8.1). By the induction hypothesis, H is locally cyclic.

Case 1: Suppose G/H is finite. Since H is locally virtually cyclic, it follows that G is locally virtually cyclic (see 7.1.3).

Case 2: Suppose $\mathbb{Z} \leq G/H$. Suppose $x \in G$ is such that $\overline{x}$ generates the copy of $\mathbb{Z}$ in G/H, and consider the subgroup $\tilde{G} = \langle H, x \rangle$. Then H is normal in $\tilde{G}$ and $\tilde{G}/H \cong \mathbb{Z}$. It follows that the Hirsch length is $h(\tilde{G}) = h(H) + h(\tilde{G}/H) = 2$ (see 5.3.2). By a theorem of Hillman, it follows that $\tilde{G}$ is solvable (see 5.3.5). The lowest nontrivial member of the derived series for $\tilde{G}$ is a torsion-free abelian normal subgroup K of $\tilde{G}$. Since $\mathcal{M}(G)$ is flat over $\mathbb{C}G$, this K cannot have $\mathbb{Z} \times \mathbb{Z}$ as a subgroup. Hence, $K \cong \mathbb{Z}$ or K is a subgroup of $\mathbb{Q}$ which is not finitely generated. Suppose $K \cong \mathbb{Z}$. Now pick $g \in G \setminus K$ and consider $G' = \langle K, g \rangle$. This is a finitely generated elementary amenable group of cohomological dimension 2. By a theorem of Kropholler, Linnell, and Lück, it follows that $G' \cong \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$ for some $n \in \mathbb{N}$ (see 5.2.2). Now suppose K is isomorphic to an additive subgroup of the rational numbers $\mathbb{Q}$. In this case still, K is normal in G' and $G'/K \cong \mathbb{Z}$. And by the previous lemma, $G' \cong G_{m,n}$ for some natural numbers m and n. The result now follows. $\square$

5.5 Elementary Amenable Groups With Torsion

While the conjecture is still open for the full class of $\mathcal{EA}$, we can do a little better than the previous section. Again, for any group G, let $\mathcal{M}(G)$ denote either $\mathcal{N}(G)$ or $\ell^p(G)$ for any $1 < p \in \mathbb{R}$.

Theorem 5.5.1. *Let G be an elementary amenable group with no infinite locally finite normal subgroups. If $\mathcal{M}(G)$ is flat over $\mathbb{C}G$, then G is locally virtually cyclic.*

Proof. Just as in the torsion-free case, this will be a proof by induction.
<u>Base Case:</u> Consider $G \in X_1$; suppose G is finitely generated virtually abelian. Then there exists $H \leq G$ such that H is abelian and $[G : H] < \infty$. Note that H cannot contain $\mathbb{Z} \times \mathbb{Z}$ as a subgroup by the assumption that $\mathcal{M}(G)$ is flat over $\mathbb{C}G$. Hence H is virtually $\mathbb{Z}$ or finite (note: a subgroup of finite index of a finitely generated group is necessarily finitely generated). Therefore, G is virtually cyclic.
<u>Induction Hypothesis:</u> Suppose the result is true for all groups in X_α.
<u>Induction Step:</u> We wish to show the result is true for all groups in $X_{\alpha+1}$. If $G \in X_{\alpha+1}$, then there exists normal subgroup $H \in LX_\alpha$ such that $G/H \in X_1$. Since $\mathcal{M}(G)$ is flat over $\mathbb{C}G$, it must be the case that $\mathcal{M}(H)$ is flat over $\mathbb{C}H$. By the induction hypothesis, H is locally virtually cyclic.
<u>Case 1:</u> Suppose G/H is finite. Since H is locally virtually cyclic, it follows from a theorem in the appendix (7.1.3) that G is locally virtually cyclic.
<u>Case 2:</u> Suppose $\mathbb{Z}^n \leq G/H$ for the largest possible $n \in \mathbb{N}$. Suppose $x_1, x_2, \ldots, x_n \in G$ are such that $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}$ generates the copy of $\mathbb{Z}^n$ in G/H, and consider the subgroup $G_1 = \langle H, x_1 \rangle$. Then H is normal in G_1 and $G_1/H \cong \mathbb{Z}$. Since H is locally virtually cyclic, it follows that $h(H) = 1$ and $h(G_1) = h(H) + h(G_1/H) = 2$. Let B_1 be an arbitrary finitely generated subgroup of G_1. Then $h(B_1) \leq h(G_1) = 2$. By a theorem of Hillman (5.3.5), there is a locally finite normal subgroup T_1 of B_1 such that B_1/T_1 is virtually torsion-free. By

assumption, T_1 must be finite. Let K_1 be atorsion-free subgroup of B_1/T_1 of finite index. Since $\mathcal{M}(G)$ is flat over $\mathbb{C}G$, it follows that $\mathcal{M}(B_1)$ is flat over $\mathbb{C}B_1$ (see 2.8.1), and hence $\mathcal{M}(B_1/T_1)$ is flat over $\mathbb{C}[B_1/T_1]$ (see 2.8.8), and so $\mathcal{M}(K_1)$ is flat over $\mathbb{C}K_1$ (see 2.8.1). By the torsion-free case, K_1 must be locally virtually cyclic. Then both B_1/T_1 and B_1 are locally virtually cyclic (see 7.1.3 and 7.1.9). Since B_1 is finitely generated, it follows that B_1 is virtually cyclic. Since B_1 was an arbitrary finitely generated subgroup of G_1, it follows that G_1 is locally virtually cyclic. Consider $G_2 = \langle G_1, x_2 \rangle$. Then G_1 is normal in G_2 and $G_2/G_1 \cong \mathbb{Z}$. By repeating the steps above, for any finitely generated subgroup B_2 of G_2, there is a finite normal subgroup T_2 of B_2 and a torsion-free subgroup K_2 of B_2/T_2 of finite index. And, for the same reasoning as above, it must be the case that G_2 is locally virtually cyclic. Then consider $G_3 = \langle G_2, x_3 \rangle$. We can continue this process until we have used up all of $x_1, x_2, \ldots, x_n$. Since G_n is locally virtually cyclic, it must be the case that G is locally virtually cyclic.

$\square$

Chapter 6

Special Cases and Future Steps

6.1 Introduction

Not all elementary amenable groups were covered in Chapter 5. In particular, groups with infinite locally finite subgroups were not allowed. Conjecture 1.1.1 is still open for most of these groups. However, a few special cases will be covered in this chapter. Consider a group G with infinite locally finite normal subgroup H and quotient $Q = G/H$. First, we will consider the case when $Q = \mathbb{Z}$, such as the Lamplighter Group. In general, the proof used for such groups cannot be extended to the case when the quotient is more complicated, such as $Q = \mathbb{Z} \oplus \mathbb{Z}$. However, if H is assumed to be abelian, then we can show the result to be true for more complicated quotients.

6.2 The Lamplighter Group

Linnell, Schick, and Lück proved that Conjecture 1.1.1 is true for the "Lamplighter Group" [30]. That argument will be discussed in this section, and it will be generalized for a larger class of groups. First, let's define what the Lamplighter Group is. It is defined using a wreath product, which is a semidirect product.

Definition 6.2.1. Let A, H be groups, and let Ω be a set such that H acts on Ω. Let K be the direct sum $\oplus_{\omega\in\Omega}A_\omega$. Then H acts on K: $h\cdot(a_\omega) = (a_{h^{-1}\omega})$. Then the restricted wreath product, denoted $A \wr H$, is the semidirect product $K \rtimes H$. The wreath product is regular if $\Omega = H$.

The Lamplighter Group is the regular restricted wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$. So it has a normal subgroup $N \cong \oplus_{\mathbb{Z}}\mathbb{Z}_2$ such that $G/N \cong \mathbb{Z}$. We will consider all finitely generated groups (including the Lamplighter Group) with an infinite locally finite normal subgroup H such that H is countably generated and $G/H \cong \mathbb{Z}$.

The proof for this class of groups will require some basic knowledge about a group's cohomological and homological dimensions.

Definition 6.2.2. The R-module C has flat dimension n if there exists a flat resolution of length n: $0 \to Q_n \to Q_{n-1} \to \cdots \to Q_0 \to C \to 0$. And C has projective dimension n if there exists a projective resolution of length n: $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to C \to 0$. The cohomological dimension of G over R is defined to be the projective dimension of R as an RG-module, denoted $cd_R G$. The homological dimension of the group G over the ring R is defined to be the flat dimension of R as an RG-module

In particular, the following facts about $cd_R G$ will be necessary (see p. 70, p. 53, Proposition 4.12 and Theorem 4.7 in [5]).

Lemma 6.2.3. *Let G be a group.*

1. *$cd_R G = 0$ if and only if G is a finite group with no R-torsion (i.e., $|G|$ is a unit in R).*

2. *If $G = \varinjlim G_\alpha$ for a direct system of groups $\{G_\alpha \mid \alpha \in I\}$ such that I is countable, then $cd_R G \leq \sup\{cd_R G_\alpha\} + 1$.*

3. *If H is a normal subgroup, then $cd_R G \leq cd_R H + cd_R(G/H)$.*

4. *Suppose $P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to \mathbb{C} \to 0$ is part of a projective resolution of $\mathbb{C}$. Then $cd_{\mathbb{C}} G = n$ if and only if $K = \ker(P_{n-1} \to P_{n-2})$ is projective.*

And the following analogous facts about $hd_R G$ will also be used (see p. 70, p. 52, Proposition 4.12 and Theorem 4.7 in [5]).

Lemma 6.2.4. *Let G be a group.*

1. $hd_R G = 0$ *if and only if G is a locally finite group with no R-torsion.*
2. *If $G = \varinjlim G_\alpha$ for a direct system of groups $\{G_\alpha \mid \alpha \in I\}$, then $hd_R G \leq \sup\{hd_R G_\alpha\}$.*
3. *If H is a normal subgroup, then $hd_R G \leq hd_R H + hd_R(G/H)$.*
4. $hd_{\mathbb{C}} G = n$ *if and only if $Tor_{n+1}^{\mathbb{C}G}(B, \mathbb{C}) = 0$ for all $\mathbb{C}G$-modules B.*

The structure of "almost finitely presented groups" and HNN-extensions will also play a role.

Definition 6.2.5. Let k be a commutative ring with unit. A group G is called almost finitely presented over k if there is a presentation $G = F/R$, such that F is a finitely generated free group and the tensor product $R/[R,R] \otimes_{\mathbb{Z}} k$ is finitely generated as a kG-module.

Definition 6.2.6. Let G be a group with presentation $G = \langle S|R\rangle$, let $H, K \leq G$, and suppose $\alpha : H \to K$ is an isomorphism. Let t denote a new symbol. Then the HNN-extension is defined to be $G*_\alpha = \langle S, t \mid R, tht^{-1} = \alpha(h), \forall h \in H\rangle$, and t is called the stable letter.

The following two facts regarding HNN-extensions will be necessary (see Britton's Lemma on p. 181 of [32], and p. 259 in [6]).

Lemma 6.2.7.

1. *In the notation above for HNN-extensions, if $H \neq G$ and $K \neq G$, then the HNN-extension $G*_\alpha$ contains a subgroup isomorphic to $\mathbb{Z} * \mathbb{Z}$.*
2. *Let G be a group containing a normal subgroup N with infinite cyclic quotient G/N, and let $t \in G$ be an element with $gp(t, N) = G$. If G is almost finitely presented over some ring k, then G is an HNN-group with stable letter t such that both base group and associated subgroups are finitely generated and contained in N.*

An interesting feature of this section, in contrast to the previous sections, is that a result for $\mathcal{N}(G)$ will be reached by way of a result for $\mathcal{U}(G)$. In cases such as this, it can be useful to have an abundance of invertible elements around. For a group G, let $\mathcal{D}(G)$ denote the division closure of $\mathbb{C}G$ in $\mathcal{U}(G)$. If k is a field and $g \in G$ has infinite order, then $1-g$ is a non-zero-divisor in kG, and in the case $k = \mathbb{C}$ we also have that $1-g$ is invertible in $\mathcal{D}(G)$ (see [30], p. 2). In particular, the resulting lemma will be utilized.

Lemma 6.2.8. *If G contains an element of infinite order, then $\mathcal{U}(G) \otimes_{\mathbb{C}G} \mathbb{C} = 0$.*

Proof. If $g \in G$ has infinite order, then we know that $1-g$ is invertible in $\mathcal{U}(G)$. This means for an element of the form $x \otimes y$ of the tensor product, $1-g$ can be factored out of x and brought across the tensor, at which point it acts on y, making it 0. $\square$

The final necessary lemma can be found on p. 3 of [30].

Lemma 6.2.9. *Let R be a subring of the ring S and let P be a projective R-module. If $P \otimes_R S$ is finitely generated as an S-module, then P is finitely generated.*

Now we are ready to prove the main result of this section.

Theorem 6.2.10. *Suppose G is a finitely generated group with an infinite locally finite normal subgroup H such that H is countably generated and $G/H \cong \mathbb{Z}$. Then $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$.*

Proof. First note that since G contains an element of infinite order that $\mathcal{U}(G) \otimes_{\mathbb{C}G} \mathbb{C} = 0$ (see 6.2.8). Since G is finitely generated, we have the following two exact sequences of $\mathbb{C}G$-modules:

$$0 \to I \to \mathbb{C}G \to \mathbb{C} \to 0$$

$$0 \to P \to \mathbb{C}G^d \to I \to 0$$

where I represents the augmentation ideal, and d is the minimum number of group elements needed to generate G. Now suppose that $\mathrm{Tor}_1^{\mathbb{C}G}(\mathcal{U}(G), \mathbb{C}) = 0$ (this assumption will lead to a contradiction). In this case, applying the functor $\mathcal{U}(G) \otimes_{\mathbb{C}G} -$ to the first piece of the resolution yields the exact sequence:

$$0 \to \mathcal{U}(G) \otimes_{\mathbb{C}G} I \to \mathcal{U}(G) \otimes_{\mathbb{C}G} \mathbb{C}G \to \mathcal{U}(G) \otimes_{\mathbb{C}G} \mathbb{C} = 0.$$

It follows that $\mathcal{U}(G) \otimes_{\mathbb{C}G} I \cong \mathcal{U}(G) \otimes_{\mathbb{C}G} \mathbb{C}G \cong \mathcal{U}(G)$. Note that $hd_{\mathbb{C}} H = 0$ by 6.2.4(1). Since $hd_{\mathbb{C}}(\mathbb{Z}) = 1$, it follows that $hd_{\mathbb{C}}(G) = 1$ (see 6.2.4(3)). By 6.2.4(4), $0 = \operatorname{Tor}_2^{\mathbb{C}G}(\mathcal{U}(G), \mathbb{C}) = \operatorname{Tor}_1^{\mathbb{C}G}(\mathcal{U}(G), I)$, and thus we can apply $\mathcal{U}(G) \otimes_{\mathbb{C}G} -$ to the second piece of the resolution to obtain the exact sequence:

$$0 \to \mathcal{U}(G) \otimes_{\mathbb{C}G} P \to \mathcal{U}(G) \otimes_{\mathbb{C}G} \mathbb{C}G^d \to \mathcal{U}(G) \otimes_{\mathbb{C}G} I \to 0$$

which can be rewritten as:

$$0 \to \mathcal{U}(G) \otimes_{\mathbb{C}G} P \to \mathcal{U}(G)^d \to \mathcal{U}(G) \to 0.$$

This implies that $\mathcal{U}(G) \otimes_{\mathbb{C}G} P$ is a finitely generated projective $\mathcal{U}(G)$-module. But since $cd_{\mathbb{C}} H \leq 1$ (see 6.2.3(1) and 6.2.3(2)), we know that $cd_{\mathbb{C}}(G) \leq 2$ (see 6.2.3(3)). Thus P must be a projective $\mathbb{C}G$-module (see 6.2.3(4)). Thus P must be a finitely generated projective $\mathbb{C}G$-module (see 6.2.9). However, if $G \cong F/R$ where F is a free group on d generators, then we know that $P \cong R/[R,R] \otimes \mathbb{C}$ [20]. By definition, G is almost finitely presented over $\mathbb{C}$. Thus G is an HNN-extension (see 6.2.7(1)) and must therefore have a subgroup isomorphic to $\mathbb{Z} * \mathbb{Z}$ (see 6.2.7(2)). This is a contradiction. Hence $\operatorname{Tor}_1^{\mathbb{C}G}(\mathcal{U}(G), \mathbb{C}) \neq 0$, and so $\mathcal{U}(G)$ is not flat over $\mathbb{C}G$. It now follows that $\mathcal{N}(G)$ is not flat over $\mathbb{C}G$ (see 2.8.17). $\square$

6.3 Other Groups

Say that a group G has property (P1) if it is finitely generated and there exists an infinite locally finite normal subgroup H, which is countably generated, such that $G/H \cong \mathbb{Z}$. And say that a group G has property (P2) if there exists a finite normal subgroup H such that $G/H \cong \mathbb{Z} \times \mathbb{Z}$. Say that a group has property (P3) if there exists a finite normal subgroup H such that $G/H \cong \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$.

Theorem 6.3.1. *Let G be a group, and let H be an abelian infinite locally finite normal subgroup, which is countably generated, such that $G/H \cong \mathbb{Z} \times \mathbb{Z}$. Then G either has a subgroup with property (P1) or has a subgroup with property (P2). Hence, Conjecture 1.1.1 holds for G.*

Proof. Let $x, y \in G$ be such that $G/H = \langle \overline{x}, \overline{y} \rangle$. Let $a = [x, y]$. Since $\overline{a} = 1$ in G/H it follows that $a \in H$. Let $X = \{a^{x^i y^j} \mid i, j \in \mathbb{Z}\}$, $B = \langle X \rangle$, and $C = \langle B, x, y \rangle$. Assume that G does not have a subgroup with property (P1). Then it suffices to prove the following three claims:

1. B is finite,
2. B is normal in C, and
3. $C/B \cong \mathbb{Z} \times \mathbb{Z}$.

Claim 1: Define $X_1 = \{a^{x^i} \mid i \in \mathbb{Z}\}$. If we define $K = \langle a, x \rangle$ and assume X_1 is infinite then $H \cap K$ will be infinite. And hence $H \cap K$ is infinite locally finite and normal in K such that $K/(H \cap K) \cong \langle x \rangle \cong \mathbb{Z}$. In other words, K satisfies property (P1). Therefore X_1 is finite. For every $b \in X_1$, define $X_2(b) = \{b^{y^i} \mid i \in \mathbb{Z}\}$. Similar to the previous argument, if any $X_2(b)$ is infinite then $\langle b, y \rangle$ will have property (P1). Therefore $X_2(b)$ is finite for every $b \in X_1$. It follows that $X = \bigcup_{b \in X_1} X_2(b)$ is finite. Since H is locally finite, B must be finite.
Claim 2: Note that if $h \in H$ and $b \in B$ then $b^h = b$, since H is abelian. Hence for all $i, j \in \mathbb{Z}$ there exists $h \in H$ such that $\left(a^{x^i y^j}\right)^x = a^{h x^{i+1} y^j} = a^{x^{i+1} y^j} \in B$. It follows that B is normal in C.
Claim 3: This follows from the fact that $a \in B$ (and nothing else is in B that can't be built with conjugates of a). $\square$

Theorem 6.3.2. *Let G be a group, and let H be an abelian infinite locally finite normal subgroup, which is countably generated, such that $G/H \cong \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$. Then G either has a subgroup with property (P1) or has a subgroup with property (P3). Hence, Conjecture 1.1.1 holds for G.*

Proof. Let $x, y \in G$ generate G/H such that $\overline{yxy^{-1}x^{-n}} = 1$ in G/H. Define $a = yxy^{-1}x^{-n} \in H$, and let $A = \langle a \rangle$. Let $X = \{a^{y^i x^j y^k} \mid i, j, k \in \mathbb{Z}\}$. Define $B = \langle X \rangle$ and $C = \langle B, x, y \rangle$. Assume G has no subgroups with property (P1). Then it suffices to prove the following three claims:

1. B is finite,
2. B is normal in C, and
3. $C/B \cong \mathbb{Z}[\frac{1}{n}] \rtimes \mathbb{Z}$.

Claim 1: Let $X_1 = \{a^{y^i} \mid i \in \mathbb{Z}\}$. If X_1 is infinite, then the subgroup $\langle a, y \rangle$ has property (P1). Hence X_1 must be finite. For each $b \in X_1$, define $X_2(b) = \{b^{x^i} \mid i \in \mathbb{Z}\}$. If any $X_2(b)$ is infinite then $\langle b, x \rangle$ has property (P1). Hence $X_2(b)$ must be finite for every $b \in X_1$. If $c \in X_2(b)$ for some $b \in X_1$, then define $X_3(c) = \{c^{y^i} \mid i \in \mathbb{Z}\}$. Then each such $X_3(c)$ must also be finite. It follows that $X = \displaystyle\bigcup_{b \in X_1} \left(\bigcup_{c \in X_2(b)} X_3(c) \right)$ is finite. Since H is locally finite, B must be finite.

Claim 2: The key fact here is that each element of G/H can be represented by an element of the form $y^i x^j y^k$ [8]. Note also that $b^h = b$ for every $b \in B$ and $h \in H$, since H is abelian. In particular, for every $i, j, k \in \mathbb{Z}$ there exist $p, q, r \in \mathbb{Z}$ and $h \in H$ such that $\left(a^{y^i x^j y^j}\right)^x = a^{hy^p x^q y^r} = a^{y^p x^q y^r} \in B$. Therefore B is normal in C.

Claim 3: This follows from the fact that $a \in B$ (and nothing else is in B that can't be built with conjugates of a).

□

6.4 Future Steps

So far, Conjecture (B) has been proved for the class of elementary amenable groups without infinite locally finite subgroups. It is not yet clear how to extend this result to the class of all elementary amenable groups. In order to make that generalization, it will probably be necessary to prove a version of 2.8.8 in which H is allowed to be infinite locally finite. It is even less clear how to prove this result for the class of amenable groups (including groups which are not elementary amenable), however that may be an interesting question for future study. As for the class of non-amenable groups, the result is known for groups that have nonabelian free subgroups, but the general case is still open. Since such groups are far from being in the conjectured class of acceptable groups (locally virtually cyclic groups), proving the conjecture for the class of non-amenable groups could very well be doable.

Other variations of Conjecture (B) could also be studied. For instance, in this paper, many of the results for the module $\mathcal{N}(G)$ were also shown to be true for the modules $\ell^p(G)$. Perhaps the $\mathbb{C}G$-module $C^*(G)$, the group C^*-algebra of G, could be similarly explored. Some basic results were obtained in section 2.7 concerning the groups $G = \mathbb{Z}$ and $G = \mathbb{Z} \oplus \mathbb{Z}$, but much of the work in this paper has not yet been shown to hold for this module. Another variation of the work in this paper would be to study group cohomology rather than group homology. Once again, there are some elementary results in sections 2.5 and 2.6, but perhaps much more could be said. One could also consider module-theoretic properties of $\mathcal{N}(G)$ other than flatness. Is $\mathcal{N}(G)$ a free module only when G is finite? Is $\mathcal{N}(G)$ a projective module only when G is finite? For which groups, if any, is $\mathcal{N}(G)$ an injective module? What about $\mathcal{U}(G)$?

Conjecture (A) is still open as well. Proving it in its full generality would be a big accomplishment, but perhaps there are weaker versions of it that could be proved more easily. Or it might be worthwhile to attempt to calculate some L^2-invariants for groups which are non-amenable to see if they vanish or not. However, since this conjecture is known for groups with nonabelian free subgroups, that limits the choices of groups that would be interesting in this context. This conjecture could also be studied with regard to Thompson's group. Since it is unknown whether Thompson's group is amenable or not, calculating any of the dimensions in Conjecture (A) would be of interest. It is already known that the L^2-Betti numbers of Thompson's group vanish (Corollary 6.8 in [10]).

Chapter 7

Appendix

7.1 Locally Virtually Cyclic Groups

There are several facts about virtually cyclic groups that are utilized in the proofs by induction in Sections 5.4 and 5.5. Here are those facts, with lemmas interspersed as necessary.

Lemma 7.1.1. *If G is a finitely generated group and H is a subgroup of finite index, then H is finitely generated.*

Proof. This is a consequence of "Schreier's formula." See Proposition 12.1 in [32]. □

Lemma 7.1.2. *If H and K are subgroups of a group G, then $[H : H \cap K] \leq [G : K]$.*

Proof. See Proposition 4.8 in [21]. □

Theorem 7.1.3. *Suppose H is a locally virtually cyclic subgroup of G of finite index. Then G is locally virtually cyclic.*

Proof. Let $G_1 \leq G$ be a finitely generated subgroup. We wish to show that G_1 is virtually cyclic. Let $H_1 = H \cap G_1 \leq G_1$. Then $H_1 \leq H$ implies H_1 is locally virtually cyclic. And $[G_1 : H_1] = [G_1 : H \cap G_1] \leq [G : H] < \infty$. Since G_1 is finitely generated and $[G_1 : H_1] < \infty$, it follows that H_1 is finitely generated. Therefore H_1 is virtually cyclic. Since H_1 is virtually cyclic and $[G_1 : H_1] < \infty$, it follows that G_1 is virtually cyclic. $\square$

Theorem 7.1.4. *If a group is finite-by-cyclic, then it is virtually cyclic.*

Proof. The finite-by-finite case is obvious, so suppose the quotient is infinite cyclic. Let $H = \langle h_1, h_2, \ldots, h_n \rangle$ be a finite normal subgroup such that $Q = G/H \cong \mathbb{Z}$. Suppose $x \in G$ is such that $\overline{x}$ generates Q. Consider $K = \langle x \rangle$. Since K is cyclic, it suffices to show that $[G : K] < \infty$. Note that G has the following left cosets with respect to H:

$$H, xH, x^2H, x^3H \ldots$$

Take one element from each of these cosets as follows:

$$h_1, xh_1, x^2h_1, x^3h_1, \ldots$$

Note that these elements form the right coset Kh_1. Similarly, form the right cosets $Kh_2, Kh_3, \ldots, Kh_n$. Since the left cosets above with respect to H exhaust G, so must the right cosets with respect to K. Clearly these right cosets are disjoint since the H cosets are disjoint and since:

$$x^i h_j = x^i h_m \;\Rightarrow\; h_j = h_m \;\Rightarrow\; j = m$$

$\square$

Lemma 7.1.5. *If H is a subgroup of G, then the "normal core" of H satisfies the following inequality:* $[G : Core(H)] \leq [G : H]!$.

Proof. See Exercise 20 on page 48 in [17]. $\square$

Theorem 7.1.6. *If a group G is infinite virtually cyclic, then there exists a normal infinite cyclic subgroup of finite index.*

Proof. Since G is infinite virtually cyclic, there exists an infinite cyclic subgroup H of finite index. Let $K = Core(H)$. Since G is infinite and $[G : K] \leq [G : H]! < \infty$, it follows that K must be nontrivial. Since H is infinite cyclic and $K \leq H$, K must also be infinite cyclic. The result now follows. $\square$

The following theorem can be found in Lemma 11.4 of [16].

Theorem 7.1.7. *Groups of the following three types are virtually cyclic. Moreover, every virtually cyclic group is exactly one of these types:*

1. *finite*
2. *finite-by-(infinite cyclic)*
3. *finite-by-(infinite dihedral)*

Theorem 7.1.8. *Suppose H is a finite normal subgroup of G such that G/H is virtually cyclic. Then G is virtually cyclic.*

Proof. If G/H is finite, then G is finite and hence virtually cyclic. Suppose $Q = G/H$ is infinite. By Theorem 7.1.7, there exists a finite normal subgroup A of Q such that Q/A is either infinite cyclic or infinite dihedral. By the Correspondence Theorem, there exists a normal subgroup K of G, which contains H, such that $A \cong K/H$. Since A and H are finite, we know that K is finite also. By the Third Isomorphism Theorem, $G/K \cong Q/A$, which implies G/K is either infinite cyclic or infinite dihedral. Therefore, G is either finite-by-(infinite cyclic) or finite-by-(infinite dihedral). By 7.1.7, G must be virtually cyclic. □

Corollary 7.1.9. *Suppose H is a finite normal subgroup of G such that G/H is locally virtually cyclic. Then G is locally virtually cyclic.*

Proof. Let K be an arbitrary finitely generated subgroup of G; it suffices to show K is virtually cyclic. Consider $\tilde{K} = \langle K, H \rangle$, which is still finitely generated and $H \triangleleft \tilde{K}$. Thus $\tilde{K}/H$ is a finitely generated subgroup of G/H, which implies $\tilde{K}/H$ is virtually cyclic. By the previous theorem, it follows that $\tilde{K}$ is virtually cyclic, which implies that K is virtually cyclic. □

7.2 Left Modules vs. Right Modules

This paper has primarily sought out connections between the structure of groups and the module-theoretic structure of $\mathcal{N}(G)$. In particular, the module-theoretic property that has been featured is flatness. There are other module properties one could also explore, such as freeness, projectivity, or injectivity. No conjectures have been made yet connecting properties of G to these potential properties of $\mathcal{N}(G)$ (although, a reasonable guess is that $\mathcal{N}(G)$ is free only if G is finite). However, if one wanted to study such questions, one practical matter is this: is it possible for $\mathcal{N}(G)$ to have one of those properties as a left-module while not having it as a right-module? This section provides that question with an answer of "no."

First, consider the question of when $\mathcal{N}(G)$ is a free module. Recall that an R-module M is called free if M is isomorphic to a direct sum of copies of R.

Theorem 7.2.1. *If G is a group, then $\mathcal{N}(G)$ is free as a left $\mathbb{C}G$-module if and only if $\mathcal{N}(G)$ is free as a right $\mathbb{C}G$-module.*

Proof. Suppose $\mathcal{N}(G)$ is free as a left $\mathbb{C}G$-module. Then there exists an isomorphism $f : \mathbb{C}G^\alpha \to \mathcal{N}(G)$. Construct a map of right $\mathbb{C}G$-modules $f' : \mathbb{C}G^\alpha \to \mathcal{N}(G)$ by $f'(x) = f(x^*)^*$. Then for any $x \in \mathbb{C}G^\alpha$ and any $r \in \mathbb{C}G$ we have:

$$f'(x) \cdot r = f(x^*)^* r = (r^* f(x^*))^* = (f(r^* x^*))^* = f'(xr).$$

And for any $x, y \in \mathbb{C}G^*$:

$$f'(x + y) = f(x^* + y^*)^* = f(x^*)^* + f(y^*)^* = f'(x) + f'(y).$$

Hence, f' is a right $\mathbb{C}G$-homomorphism. Suppose $f'(x) = 0$. Then $f(x^*)^* = 0$, which implies $f(x^*) = 0$. Since f is injective, this means $x^* = 0$, and hence $x = 0$. Therefore f' is injective. And since f is surjective, it clearly follows that f' is surjective too. Therefore, $\mathcal{N}(G)$ is free as a right $\mathbb{C}G$-module. Similarly, the converse is also true. $\square$

Now consider the property of projectivity. An R-module M is called projective if and only if M is a summand of a free module. Another equivalent way to define projective modules is with the existence of a "projective basis," as described in the next lemma ([40], Theorem 3.15).

Lemma 7.2.2. *A left module A is projective if and only if there exist elements $\{a_k \mid k \in K\} \subset A$ and R-maps $\{f_k : A \to R \mid k \in K\}$ such that*

1. *if $x \in A$, then almost all $f_k(x) = 0$, and*
2. *if $x \in A$, then $x = \sum_{k \in K} f_k(x) a_k$.*

There is an analogous characterization of projective right modules.

Theorem 7.2.3. *If G is a group, then $\mathcal{N}(G)$ is projective as a left $\mathbb{C}G$-module if and only if $\mathcal{N}(G)$ is projective as a right $\mathbb{C}G$-module.*

Proof. Suppose $\mathcal{N}(G)$ is projective as a left $\mathbb{C}G$-module. Then there exists a projective basis as described in the previous lemma. For all $k \in K$ define $f_k'(x) = f_k(x^*)^*$. These are right $\mathbb{C}G$-homomorphisms. We'd like to show, for any x, that $f_k'(x) = 0$ for almost all k. If $f_k'(x) = 0$, then $f_k(x^*)^* = 0$, which means $f_k(x^*) = 0$. By property (1) in the lemma, this must be true for almost all $k \in K$. Finally:

$$x^* = \left(\sum f_k(x) a_k\right)^* = \sum a_k^* \left(f_k x\right)^* = \sum a_k^* f_k'(x^*).$$

Hence, property (2) of the lemma is satisfied for $\mathcal{N}(G)$ as a right module with respect to the maps $\{f_k'\}$ and the elements $\{a_k^*\}$. Therefore $\mathcal{N}(G)$ is projective as a right $\mathbb{C}G$-module. Similarly, the converse is also true. □

Finally, we will consider the property of injectivity. The most useful characterization of injective modules will be the "Baer Criterion," which can be found in Theorem 3.20 in [40] and is listed below.

Lemma 7.2.4. *A left R-module E is injective if and only if every map $f : I \to E$, where I is a left ideal of R, can be extended to R. There is an analogous definition for right modules.*

Theorem 7.2.5. *If G is a group, then $\mathcal{N}(G)$ is injective as a left $\mathbb{C}G$-module if and only if $\mathcal{N}(G)$ is injective as a right $\mathbb{C}G$-module.*

Proof. Suppose $\mathcal{N}(G)$ is injective as a left $\mathbb{C}G$-module. Let I be a right ideal of $\mathbb{C}G$ generated by $\{a_i\}$, and let $f : I \to \mathcal{N}(G)$ be a right $\mathbb{C}G$-map. We wish to show that f can be extended to all of $\mathbb{C}G$. Consider the right ideal I' generated by $\{a_i^*\}$ and the left $\mathbb{C}G$-map $f' : I' \to \mathcal{N}(G)$ defined by $f'(x) = f(x^*)^*$. By the previous lemma, f' has an extension $f'_e : \mathbb{C}G \to \mathcal{N}(G)$. Now define a right $\mathbb{C}G$-map $f_e : \mathbb{C}G \to \mathcal{N}(G)$ by $f_e(x) = f'_e(x^*)^*$. Then I claim that f_e is an extension of f. Let $a_1r_1 + \cdots + a_nr_n$ be an arbitrary element of I. Then:

$$f_e(a_1r_1 + \cdots + a_nr_n) = f'_e(r_1^*a_1^* + \cdots + r_n^*a_n^*)^* = f'(r_1^*a_1^* + \cdots + r_n^*a_n^*)^* = f(a_1r_1 + \cdots + a_nr_n).$$

Therefore, $\mathcal{N}(G)$ is injective as a right $\mathbb{C}G$-module. Similarly, the converse is also true. □

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