The Quantum Automorphism Group and Undirected Trees

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(ABSTRACT)

A classification of all undirected trees with automorphism group isomorphic to $(Z_2)^l$ is given in terms of a vertex partition called a refined star partition. Recently the notion of a quantum automorphism group has been defined by T. Banica and J. Bichon. The quantum automorphism group is similar to the classical automorphism group, but has relaxed commutivity. The classification of all undirected trees with automorphism group isomorphic to $(Z_2)^l$ along with a similar classification of all undirected asymmetric trees is used to give some insight into the structure of the quantum automorphism group for such graphs.

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Chapter 1

Introduction

1.1 Graph Invariants

The problem of determining whether two graphs are isomorphic is an NP-hard problem and is referred to as the graph isomorphism problem. Graph invariants are helpful in determining when two graphs are not isomorphic. One such graph invariant is the automorphism group of a graph. There is no known complete graph invariant which will distinguish between all non-isomorphic graphs. A graph and its non-isomorphic complement are examples of two non-isomorphic graphs which have the same automorphism group.

1.2 Overview of Results

This dissertation was motivated by a new graph invariant, the quantum automorphism group of a graph. The quantum automorphism group is closely related to the Hopf dual of the group algebra of the classical automorphism group although the commutativity conditions are relaxed. The quantum automorphism group defined in [2] is able to distinguish between some non-isomorphic graphs that have the same classical automorphism group. In particular, the complete graph K_n for $n > 3$ and its complement have distinct quantum automorphism groups. The motivation for this dissertation came from considering the basic question about quantum automorphism groups:

When is the quantum automorphism group a new invariant? More precisely, for what graphs is the quantum automorphism group non-isomorphic to the Hopf dual of the group algebra of the classical automorphism group?

Note that the Hopf dual of a group algebra is commutative. Hence if the quantum automorphism group of a given graph is non-commutative, then the quantum automorphism group is indeed a new invariant. On the other hand, if the relations of the quantum automorphism group force it to be commutative then we just get the Hopf dual of the group algebra of the classical automorphism group. Thus we investigate the following version of the above question:

For which graphs is the quantum automorphism group commutative and for which graphs is it non-commutative?

Our investigation focuses on undirected trees. In [5], W. Kocay characterizes all asymmetric trees, trees which have the trivial classical automorphism group, in terms of a specific vertex partition called a star partition. We use this characterization to prove the following theorem:

Theorem 1.2.1 *Let $\mathcal{T}$ be an undirected tree. Then $\mathcal{T}$ is an asymmetric tree if and only if the quantum automorphism group is isomorphic to* $\mathbf{C}$.

One of the main results of this dissertation extends Kocay's result and classifies all trees that have automorphism group congruent to $(Z_2)^l$. We do this by modifying his star partition to what we call a refined star partition. The following theorem gives this classification.

Theorem 1.2.2 *Let $\mathcal{T}$ be an undirected tree. Then $Aut(\mathcal{T}) \cong (Z_2)^l$ if and only if there are exactly 2^l refined star partitions of $\mathcal{G}$.*

To aid the reader, this theorem will be proved in two parts. In Chapter 5, we will prove this theorem for the case when $l = 1$. In Chapter 6, this proof will be modified to work for all l. Both proofs are very similar, but the second proof is more tedious.

We use this classification of trees with automorphism group congruent to $(Z_2)^l$ along with the **Refine** algorithm which is discussed in section 4.3 to prove the following two theorems about the quantum automorphism group of such trees. The distinction between the quantum automorphism group for $l = 1$ and $l > 1$ is due to the larger number of refined star partitions for graphs in which $l > 1$.

Theorem 1.2.3 *If $\mathcal{G}$ is a tree with $Aut(\mathcal{G}) \cong Z_2$, then the quantum automorphism group is commutative.*

Theorem 1.2.4 *If $\mathcal{G}$ is a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$ with $l \geq 2$, then the quantum automorphism group is non-commutative.*

This thesis is organized as follows. Chapter 2 sets basic graph notation and provides definitions for the quantum automorphism group. Chapter 3 uses linear algebra and the adjacency matrix to obtain a new characterization of the quantum automorphism group. Using this new characterization as well as information about paths in graphs, we deduce when certain generators, X_{ij}, of the quantum automorphism group are zero.

Chapter 4 discusses star partitions and the link between asymmetric trees and star partitions from [5]. Using this characterization of asymmetric trees, we prove some results about the generators of the quantum automorphism group. These results will ultimately lead to proving that the quantum automorphism group of an asymmetric tree is isomorphic to $\mathbf{C}$.

In Chapter 5, we will refine the star partition and use this new refined star partition to classify all trees with automorphism group congruent to Z_2. This classification will lead to the proof that all trees, $\mathcal{T}$ with $Aut(\mathcal{T}) \cong Z_2$ have commutative quantum automorphism groups. In Chapter 6, we extend the result from Chapter 5 and classify all trees with automorphism group congruent to $(Z_2)^l$. We use this classification to prove that all trees, $\mathcal{T}$ with $Aut(\mathcal{T}) \cong (Z_2)^l$ for $l > 1$ have non-commutative quantum automorphism groups. In Chapters 4, 5, and 6, we also show why these classifications do not hold for graphs which are not trees and discuss what is known about the quantum automorphism groups of such graphs.

Chapter 2

Background

2.1 Notation

The following notation will be used throughout the dissertation. A finite digraph $\mathcal{G} = (V, E)$ consists of a finite set of vertices V labelled $1, ..., n$, and a set of directed edges $E \subseteq V \times V$ labelled $\gamma_1, ..., \gamma_m$. A directed edge from vertex u to vertex v is denoted by (u, v). If $\gamma = (a, b) \in E$ then $s(\gamma) = a$ and $t(\gamma) = b$. Let $\mathbf{id}(v)$ denote the indegree of vertex v and $\mathbf{od}(v)$ denote the outdegree of vertex v. Set

$$D = (d_{ij})_{1 \leq i,j \leq n}$$

to be the adjacency matrix of $\mathcal{G}$, where $d_{ij} = 1$ if $(i, j) \in E$ and $d_{ij} = 0$ if $(i, j) \notin E$. Throughout this dissertation we will restrict ourselves to simple digraphs which contain no self-loops or multi-edges. We will view undirected graphs as digraphs by viewing an undirected edge connecting u and v as two directed edges (u, v) and (v, u) in E. We will set $deg(i) = \mathbf{id}(i) = \mathbf{od}(i)$ when viewing undirected graphs as digraphs.

2.2 Automorphism Group of Finite Graphs

An automorphism of a graph $\mathcal{G}$ is a bijection $\sigma : V \to V$ such that $\sigma(x)$ is adjacent to $\sigma(y)$ if and only if x is adjacent to y for all $x, y \in V$. Thus an automorphism σ of $\mathcal{G}$ induces an action on the edge set E sending (x, y) to $(\sigma(x), \sigma(y))$. The automorphism group of a graph $\mathcal{G}$, denoted $Aut(\mathcal{G})$, is the set of all automorphisms of $\mathcal{G}$. We can view $Aut(\mathcal{G})$ as a subgroup of the permutation matrices in the following way: $Aut(\mathcal{G})$ is the set of permutation matrices $\sigma = (a_{ij})_{1 \leq i,j \leq n}$ such that $\sigma D = D\sigma$. This gives us the following equations which are satisfied if and only if a permutation matrix σ is an element of $Aut(\mathcal{G})$

$$\sum_{k=1}^{n} d_{ik} a_{kj} = \sum_{l=1}^{n} d_{lj} a_{il} \quad 1 \leq i, j \leq n$$

The action of $Aut(\mathcal{G})$ on $\mathcal{G}$ satisfies the following commutative diagram:

$$\begin{tikzcd} V \times V & V \times V \times Aut(\mathcal{G}) \arrow[l] \\ E \arrow[u] & E \times Aut(\mathcal{G}) \arrow[l] \arrow[u] \end{tikzcd}$$

where the vertical maps represent inclusion and the horizontal maps correspond to the action of elements in $Aut(\mathcal{G})$.

2.3 Hopf algebra and co-actions

Denote the complex numbers as $\mathbf{C}$. Throughout this dissertation all algebras will be over $\mathbf{C}$.

Definition 2.3.1 *([6],Definition 1.1.1) A $\mathbf{C}$-algebra (with unit) is a $\mathbf{C}$-vector space A together with two $\mathbf{C}$-linear maps, multiplication $m : A \otimes A \to A$ and unit $u : \mathbf{C} \to A$, such that the following diagrams are commutative:*

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

In the following diagram the unlabelled maps are given by scalar multiplication.

$$\begin{tikzcd} & A \otimes A \arrow[dd, "m"] & \\ \mathbf{C} \otimes A \arrow[ur, "u \otimes id"] \arrow[dr] & & A \otimes \mathbf{C} \arrow[ul, "id \otimes u"'] \arrow[dl] \\ & A & \end{tikzcd}$$

Definition 2.3.2 *([6], Definition 1.1.3) A* $\mathbf{C}$*-coalgebra (with counit) is a* $\mathbf{C}$*-vector space* D *together with two* $\mathbf{C}$*-linear maps, comultiplication* $\Delta : D \to D \otimes D$ *and counit* $\epsilon : D \to \mathbf{C}$*, such that the following diagrams are commutative.*

$$\begin{tikzcd} D \arrow[r, "\Delta"] \arrow[d, "\Delta"] & D \otimes D \arrow[d, "\Delta \otimes id"] \\ D \otimes D \arrow[r, "id \otimes \Delta"] & D \otimes D \otimes D \end{tikzcd}$$

$$\begin{tikzcd} & D \arrow[dl, "1\otimes"'] \arrow[dr, "\otimes 1"] \arrow[dd, "\Delta"'] & \\ \mathbf{C} \otimes D & & D \otimes \mathbf{C} \\ & D \otimes D \arrow[ul, "\epsilon \otimes id"] \arrow[ur, "id \otimes \epsilon"'] & \end{tikzcd}$$

Definition 2.3.3 *([6],Definition 1.3.1) A* $\mathbf{C}$*-space* B *is a bialgebra if* (B, m, u) *is an algebra,* (B, Δ, ϵ) *is a coalgebra, and either of the following (equivalent) conditions holds:*

1. Δ *and* ϵ *are algebra morphisms*
2. m *and* u *are coalgebra morphisms*

Definition 2.3.4 *An antipode map* S *is an element of* $Hom_{\mathbf{C}}(H, H)$ *such that* $\sum S(u_{(1)})u_{(2)} = \sum u_{(1)}S(u_{(2)}) = \epsilon(u)$ *where* $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$ *in Sweedler notation.*

Definition 2.3.5 *([6], Definition1.5.1) Let* $(H, m, u, \Delta, \epsilon)$ *be a bialgebra. Then* H *is a Hopf algebra if there exists an element* $S \in Hom_{\mathbf{C}}(H, H)$ *which is an antipode map.*

2.4 $\mathcal{A}(\mathcal{G})$, the Hopf dual of $\mathbf{C}[Aut(\mathcal{G})]$

Let $\mathcal{A}(\mathcal{G})$ be the commutative algebra with generators $\{X_{ij}\}_{1\le i,j\le n}$ such that for $1 \le i, j \le n$ the following hold:

$$X_{ij}^2 = X_{ij} \;\; ; \;\; \sum_{l=1}^{n} X_{il} = 1 = \sum_{l=1}^{n} X_{li}$$

$$\sum_{k=1}^{n} d_{ik} X_{kj} = \sum_{l=1}^{n} d_{lj} X_{il} \tag{2.4.1}$$

$\mathcal{A}(\mathcal{G})$ has the following Hopf algebra structure.

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj} \ \ ; \ \ \epsilon(X_{ij}) = \delta_{ij} \ \ ; \ \ S(X_{ij}) = X_{ji}$$

As an algebra, $\mathcal{A}(\mathcal{G})$ is just the coordinate ring for $Aut(\mathcal{G})$ viewed as a subvariety of affine n^2 space. Here we are identifying $M_{n\times n}(\mathbf{C})$ with affine n^2 space over $\mathbf{C}$ and $Aut(\mathcal{G})$ with a subset of the $n \times n$ permutation matrices. Let $I(Aut(\mathcal{G}))$ denote the set $\{f \in \mathbf{C}[x_{11}, \dots x_{nn}] | f(\sigma) = 0$ for all $\sigma \in Aut(\mathcal{G})\}$. The following is a more explicit description of $I(Aut(\mathcal{G}))$.

$$I(Aut(\mathcal{G})) = \langle x_{ij}^2 - x_{ij} \ ; \ \sum_{k=1}^{n} x_{kj} - 1 \ ; \ \sum_{k=1}^{n} x_{jk} - 1 \ ; \ \sum_{k=1}^{n} d_{ik}x_{kj} - \sum_{k=1}^{n} d_{lj}x_{il} \rangle$$

The coordinate ring for $Aut(\mathcal{G})$, $\mathbf{C}[x_{11}, \dots x_{nn}]/I(Aut(\mathcal{G}))$, is isomorphic to $\mathcal{A}(\mathcal{G})$ as an algebra.

We know show that we can identity $\mathcal{A}(\mathcal{G})$ with the dual of $\mathbf{C}[Aut(\mathcal{G})]$. Note that $Aut(\mathcal{G})$ is a finite group. Hence the group algebra $\mathbf{C}[Aut(\mathcal{G})]$ is a finite-dimensional vector space over $\mathbf{C}$. It follows from [6] that the vector space dual of $\mathbf{C}[Aut(\mathcal{G})]$ can be given the structure of a Hopf algebra and is the Hopf dual of $\mathbf{C}[Aut(\mathcal{G})]$. Elements of $\mathcal{A}(\mathcal{G})$ can be interpreted as elements of $Hom_{\mathbf{C}}(\mathbf{C}[Aut(\mathcal{G})], \mathbf{C})$ where

$$X_{ij}(g) = \begin{cases} 1 & : \quad g_{ij} = 1 \\ 0 & : \quad g_{ij} = 0 \end{cases}$$

for $g \in Aut(\mathcal{G})$. Again we view elements of $Aut(\mathcal{G})$ as $n \times n$ permutation matrices. Thus we view g as the $n \times n$ matrix $(g_{ij})_{1\le i,j \le n}$. This leads to a Hopf algebra embedding of $\mathcal{A}(\mathcal{G})$ inside the Hopf dual of $\mathbf{C}[Aut(\mathcal{G})]$. Moreover for each $g \in Aut(\mathcal{G})$, one can find an $X_g \in \mathcal{A}(\mathcal{G})$ such that $X_g(g) = 1$ and $X_g(h) = 0$ for $h \neq g$, $h \in Aut(\mathcal{G})$. Hence we can identify $\mathcal{A}(\mathcal{G})$ with the dual of $\mathbf{C}[Aut(\mathcal{G})]$.

2.5 Quantum Automorphism Group

In [7], S. Wang defined the quantum permutation group. This quantum permutation group, denoted $A_{aut}(X_n)$, is generated as an algebra by $\{a_{ij}\}_{1\le i,j\le n}$. These a_{ij}'s satisfy the following relations:

$$a_{ij}^2 = a_{ij} \ \ ; \ \ \sum_{l=1}^{n} a_{il} = 1 = \sum_{l=1}^{n} a_{li} \qquad \text{for each } 1 \leq i,j \leq n.$$

$A_{aut}(X_n)$ has a Hopf algebra structure given by:

$$\Delta(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj} \ \ ; \ \ \epsilon(a_{ij}) = \delta_{ij} \ \ ; \ \ S(a_{ij}) = a_{ji} \quad 1 \leq i,j \leq n.$$

Note we are not assuming the a_{ij}'s commute with each other.

The notation of quantum permutation groups has been generalized to a quantum version of automorphism groups of graphs. In this context the quantum permutation group, $A_{aut}(X_n)$, corresponds to the quantum automorphism group of the graph with n vertices and no edges.

There are two different definitions of the quantum automorphism group in the literature. In [1], T. Banica and J. Bichon define the quantum automorphism group, $QAut(\mathcal{G})$, as the quotient $A_{aut}(X_n)/(AD = DA)$ where $A = (a_{ij})_{1\leq i,j\leq n}$ and D is the adjacency matrix of the graph $\mathcal{G}$. Bichon defines the quantum automorphism group, $A_{aut}(\mathcal{G})$, in [2], as the universal complex algebra with generators $\{X_{ij}\}_{1\leq i,j,\leq n}$ satisfying the following:

$$X_{ij}^2 = X_{ij} \ \ ; \ \ \sum_{l=1}^{n} X_{il} = 1 = \sum_{l=1}^{n} X_{li} \tag{2.5.1}$$

$$X_{s(\gamma_j)i}X_{t(\gamma_j)k} = X_{t(\gamma_j)k}X_{s(\gamma_j)i} = 0 \quad , \quad \gamma_j \in E, \quad (i,k) \notin E \tag{2.5.2}$$

$$X_{is(\gamma_j)}X_{kt(\gamma_j)} = X_{kt(\gamma_j)}X_{is(\gamma_j)} = 0 \quad , \quad \gamma_j \in E, \quad (i,k) \notin E \tag{2.5.3}$$

$$\sum_{l=1}^{m} X_{s(\gamma_l)s(\gamma_j)}X_{t(\gamma_l)t(\gamma_j)} = 1 \ \ ; \ \ \sum_{l=1}^{m} X_{s(\gamma_j)s(\gamma_l)}X_{t(\gamma_j)t(\gamma_l)} = 1 \ \ , \ \ \gamma_j \in E \tag{2.5.4}$$

$$X_{s(\gamma_j)s(\gamma_l)}X_{t(\gamma_j)t(\gamma_l)} = X_{t(\gamma_j)t(\gamma_l)}X_{s(\gamma_j)s(\gamma_l)} \quad , \quad \gamma_j, \gamma_l \in E \tag{2.5.5}$$

The Hopf structure is given by the following:

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj} \;\; ; \;\; \epsilon(X_{ij}) = \delta_{ij} \;\; ; \;\; S(X_{ij}) = X_{ji}$$

Bichon's quantum automorphism group, $A_{aut}(\mathcal{G})$, can also be viewed in terms of the following universal property. Let $C(V)$ be the *-algebra with generators $\{e_i\}_{1\leq i\leq n}$ and relations $e_i^* = e_i$; $e_ie_j = \delta_{ij}$; $\sum\limits_{i=1}^{n} e_i = 1$. The elements of $C(V)$ can be interpreted as functions on the vertices $1, \ldots n$ where $e_i(k) = \delta_{ik}$ for $1 \leq k \leq n$. Similarly, let $C(E)$ be the *-algebra with generators $\{f_j\}_{1\leq j\leq m}$ and relations $f_j^* = f_j$; $f_jf_l = \delta_{jl}$; $\sum\limits_{j=1}^{m} f_j = 1$. The elements of $C(E)$ can be interpreted as functions on the edges $\gamma_1, \ldots \gamma_m$ where $f_j(\gamma_k) = \delta_{jk}$ for $1 \leq k \leq m$. The action of a Hopf-algebra $\mathcal{A}$ on a finite graph $\mathcal{G}$ consists of maps $\alpha \; : \; C(V) \longrightarrow C(V)\otimes\mathcal{A}$ and $\beta \; : \; C(E) \longrightarrow C(E) \otimes \mathcal{A}$ such that the following commutes:

$$\begin{tikzcd}
C(V)\otimes C(V) \arrow[r, "\alpha\otimes\alpha"] \arrow[d, "(m\circ(s_*\otimes t_*))"'] & C(V)\otimes C(V)\otimes\mathcal{A} \arrow[d, "(m\circ(s_*\otimes t_*))\otimes id_A"] \\
C(E) \arrow[r, "\beta"] & C(E)\otimes\mathcal{A}
\end{tikzcd}$$

Here $(m\circ(s_*\otimes t_*))(e_i\otimes e_j) = f_k$ where $s(\gamma_k) = i$ and $t(\gamma_k) = j$. The *quantum automorphism group* of $\mathcal{G}$ is a compact quantum group $\mathcal{A}$ acting on $\mathcal{G}$ by the following maps:

$\alpha : C(V) \to C(V)\otimes\mathcal{A}$ and $\beta : C(E) \to C(E)\otimes\mathcal{A}$

such that if $\mathcal{B}$ is a compact quantum group acting on $\mathcal{G}$ by

$\alpha' : C(V) \to C(V)\otimes\mathcal{B}$ and $\beta' : C(E) \to C(E)\otimes\mathcal{B}$

there there exists a unique $\phi : \mathcal{A} \to \mathcal{B}$ such that the following diagrams commute:

$$\begin{tikzcd}
C(V) \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & C(V)\otimes\mathcal{A} \arrow[d, "id\otimes\phi"] \\
& C(V)\otimes\mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
C(E) \arrow[r, "\beta"] \arrow[dr, "\beta'"'] & C(E)\otimes\mathcal{A} \arrow[d, "id\otimes\phi"] \\
& C(E)\otimes\mathcal{B}
\end{tikzcd}$$

2.6 Examples

The following are several examples which show the quantum automorphism group can be commutative or non-commutative, finite dimensional or infinite dimensional. These examples also illustrate that graphs can have the same classical automorphism group, but different quantum automorphism groups. Thus the quantum automorphism group is a stronger graph invariant than the classical automorphism group.

- Example 1: Let $\mathcal{G}_1 = (V,\ E)$ be the graph with n vertices and no edges. $A_{aut}(\mathcal{G}_1) = QAut(\mathcal{G}_1)$ which equals the quantum permutation group defined in [7]. As stated in [2] this is non-commutative and infinite dimensional for $n \geq 4$.

Note that graph complements have the same classical automorphism group. We will now look at $\mathcal{G}_2$, the graph complement of $\mathcal{G}_1$. We will see that $A_{aut}(\mathcal{G}_1) \neq A_{aut}(\mathcal{G}_2)$.

- Example 2: Let $\mathcal{G}_2 = (V,\ E)$ be the complete graph on n vertices. Since there are edges between any two vertices we see that $A_{aut}(\mathcal{G}_2)$ is commutative for all n and isomorphism to $A(\mathcal{G})$, the algebra of functions on S_n. While $QAut(\mathcal{G}_2)$ is equal to the quantum permutation group defined in [7].

We see from this example that $A_{aut}(\mathcal{G})$ and $QAut(\mathcal{G})$ can differ.

Chapter 3

Quantum Automorphism Group

In this chapter, we will prove several lemmas that will ultimately lead to alternate definitions of $A_{aut}(\mathcal{G})$ and $QAut(\mathcal{G})$ in terms of the adjacency matrix. We will also prove some basic results about the generators of the quantum automorphism group, $\{X_{ij}\}_{1\leq i,j\leq n}$, using paths in the underlying graph.

3.1 Alternate Definitions

The following lemma gives equivalent relations for some of the relations of $A_{aut}(\mathcal{G})$ using the adjacency matrix of the graph.

Lemma 3.1.1 *If relation 2.5.1 holds for the generators $\{X_{ij}\}1 \leq i,j \leq n$, then relations 2.5.2, 2.5.3, and 2.5.4 are equivalent to $X = (X_{ij})_{1\leq i,j\leq n}$ commuting with the adjacency matrix D.*

Proof: Suppose that X commutes with the adjacency matrix $D = (d_{ij})$. Then for each $1 \leq i, j \leq n$

$$\sum_{k=1}^{n} d_{ik}X_{kj} = \sum_{l=1}^{n} d_{lj}X_{il}. \tag{3.1.1}$$

Multiplying the above equation on the left by X_{im} for some m such that $(m, j) \notin E$ gives us:

$$\sum_{k=1}^{n} d_{ik}X_{im}X_{kj} = \sum_{l=1}^{n} d_{lj}X_{im}X_{il}.$$

Applying 2.5.1 to the above equation gives us:

$$\sum_{\{\gamma|s(\gamma)=i\}} X_{s(\gamma)m}X_{t(\gamma)j} = d_{mj}X_{im}$$

$$\sum_{\{\gamma|s(\gamma)=i\}} X_{s(\gamma)m}X_{t(\gamma)j} = 0.$$

Multiply the above equation on the left by $X_{s(\gamma)m}$ and on the right by $X_{t(\gamma)j}$ for some $\gamma \in E$. This shows that $X_{s(\gamma)m}X_{t(\gamma)j} = 0$ for $\gamma \in E$ and $(m, j) \notin E$. Similar arguments show that

$$X_{ms(\gamma)}X_{jt(\gamma)} = 0 \text{ for } (m,\ j) \notin E$$

$$X_{jt(\gamma)}X_{ms(\gamma)} = 0 \text{ for } (m,\ j) \notin E$$

$$X_{t(\gamma)j}X_{s(\gamma)m} = 0 \text{ for } (m,\ j) \notin E.$$

This proves that X commuting with D implies 2.5.2 and 2.5.3.

To show that X commuting with D implies Equation 2.5.4, multiply Equation 3.1.1 on the left by X_{im} for some m such that $\gamma = (m,\ j) \in E$. This gives us:

$$\sum_{\{k|(i,\ k)\in E\}} X_{im}X_{kj} = X_{im}.$$

Sum the above equation over i:

$$\sum_{i=1}^{n} \sum_{\{k|(i,\ k)\in E\}} X_{im}X_{kj} = \sum_{i=1}^{n} X_{im}.$$

Simplifying the above using Equations 2.5.1 and 2.5.3 gives:

$$\sum_{\gamma_i\in E} X_{s(\gamma_i)s(\gamma)}X_{t(\gamma_i)t(\gamma)} = 1 \quad \text{for } \gamma \in E.$$

Similarly by multiplying Equation 3.1.1 on the right by X_{kj} for some k such that $\gamma = (i, k) \in E$ and using arguments similar to above one shows that

$$\sum_{\gamma_i\in E} X_{s(\gamma)s(\gamma_i)}X_{t(\gamma)t(\gamma_i)} = 1 \quad \text{for } \gamma \in E.$$

This shows that X commuting with D implies 2.5.4

Now suppose that Equations 2.5.3 and 2.5.4 hold and show that Equation 3.1.1 holds for each $1 \leq i, j \leq n$.

$$\begin{aligned}
\sum_{k=1}^{n} d_{ik} X_{kj} &= \sum_{l=1}^{n} X_{il} \sum_{k=1}^{n} d_{ik} X_{kj} \\
&= \sum_{k,l \in V} d_{ik} X_{il} X_{kj} \\
&= \sum_{k,l \in V} d_{ik} d_{lj} X_{il} X_{kj}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{l=1}^{n} d_{lj} X_{il} &= \sum_{l=1}^{n} d_{lj} X_{il} \sum_{k=1}^{n} X_{kj} \\
&= \sum_{k,l \in V} d_{lj} X_{il} X_{kj} \\
&= \sum_{k,l \in V} d_{ik} d_{lj} X_{il} X_{kj}
\end{aligned}$$

This implies that

$$\sum_{l=1}^{n} d_{lj} X_{il} = \sum_{k=1}^{n} d_{ik} X_{kj}$$

and shows that Equations 2.5.3 and 2.5.4 imply that X commutes with D. This concludes the proof that Equations 2.5.3 and 2.5.4 and X commuting with D are equivalent. □

In the following lemma, we use Lemma 3.1.1 to give an alternate definition of $A_{aut}(\mathcal{G})$.

Lemma 3.1.2 *The quantum automorphism group $A_{aut}(\mathcal{G})$ is equivalent to the universal complex algebra generated by $\{X_{ij}\}_{1 \leq i,j \leq n}$ satisfying the following:*

$$X_{ij}^2 = X_{ij} \;\; ; \;\; \sum_{l=1}^{n} X_{il} = 1 = \sum_{l=1}^{n} X_{li} \tag{3.1.2}$$

$$\sum_{k=1}^{n} d_{ik} X_{kj} = \sum_{l=1}^{n} d_{lj} X_{il} \;\; 1 \leq i, j \leq n \tag{3.1.3}$$

$$X_{s(\gamma_j)s(\gamma_l)}X_{t(\gamma_j)t(\gamma_l)} = X_{t(\gamma_j)t(\gamma_l)}X_{s(\gamma_j)s(\gamma_l)} \quad , \quad \gamma_j, \gamma_l \in E \tag{3.1.4}$$

The Hopf structure is given by the following:

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj} \quad ; \quad \epsilon(X_{ij}) = \delta_{ij} \quad ; \quad S(X_{ij}) = X_{ji}$$

Proof: The proof follows from Lemma 3.1.1. □

The following lemma shows that $A_{aut}(\mathcal{G})$ is a quotient of $QAut(\mathcal{G})$.

Lemma 3.1.3 $A_{aut}(\mathcal{G}) \cong QAut(\mathcal{G})/(X_{s(\gamma_j)s(\gamma_l)}X_{t(\gamma_j)t(\gamma_l)} = X_{t(\gamma_j)t(\gamma_l)}X_{s(\gamma_j)s(\gamma_l)} \ , \ \gamma_j, \gamma_l \in E)$

Proof: The proof follows from Lemma 3.1.2. □

We will use these equivalent definitions of $A_{aut}(\mathcal{G})$ interchangeably throughout the rest of this paper. A consequence of the above lemma is that the relations for Banica's quantum automorphism group, $QAut(\mathcal{G})$, are equivalent to Relations 2.5.1, 2.5.2, 2.5.3, 2.5.4.

3.2 Specifics about Generators

Recall we proved in Lemma 3.1.1 that $XD = DX$ for the matrix X whose entries are the generators of a quantum automorphism group of a graph. The following lemma is easy to verify using this fact and Relation 2.5.1. Since the results do not rely on the additional commuting of $A_{aut}(\mathcal{G})$ in relation 2.5.5, this lemma applies to both $QAut(\mathcal{G})$ and $A_{aut}(\mathcal{G})$.

Lemma 3.2.1 *View $X = (X_{ij})_{1 \le i,j \le n}$ as a matrix whose entries are the generators of a quantum automorphism group of a graph.*

(i) $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$ *is a left eigenvector for the matrix X with eigenvalue 1.*

(ii) $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} D$ *is a left eigenvector for the matrix X with eigenvalue 1.*

(iii) $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} D^l$ *is a left eigenvector for the matrix X with eigenvalue 1 for $l \in Z^+$.*

(iv) $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$ *is a right eigenvector for the matrix X with eigenvalue 1.*

(v) $D\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ *is a right eigenvalue for the matrix X with eigenvalue 1.*

(vi) $D^l\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ *is a right eigenvalue for the matrix X with eigenvalue 1 for* $l \in Z^+$.

Recall, that the (i,j) entry of D^l, $d_{ij}^{(l)}$, is the number of distinct $i-j$ walks of length l. Set $\mathbf{M} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$ and $\mathcal{M} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. Let $A \subset V$. Set $w_{\text{in}}^l(i,\ A)$ to be the number of distinct walks of length l into vertex i from the vertices in A.

$$w_{\text{in}}^l(i,\ A) = \sum_{v\in A} d_{vi}^{(l)}.$$

Note that $w_{\text{in}}^1(i, V) = \mathbf{id}(i)$. Set $w_{\text{out}}^l(i,\ A)$ to be the number of distinct walks of length l from vertex i into the vertices in A.

$$w_{\text{out}}^l(i,\ A) = \sum_{v\in A} d_{iv}^{(l)}$$

Note that $w_{\text{out}}^1(i, V) = \mathbf{od}(i)$.

The previous lemma is useful in proving the following two lemmas regarding walks of a specific length. We will use these lemmas to help determine the quantum automorphism groups of specific classes of graphs in the following chapters.

Lemma 3.2.2 *If* $w_{in}^l(i,\ V) = w_{in}^l(j,\ V)$ *for some* l, *then* $X_{ij} = 0$. *Similarly, if* $w_{out}^l(i,\ \ V) \neq w_{out}^l(j,\ \ V)$ *for some* l, *then* $X_{ij} = 0$.

Proof: By Lemma 3.2.1 iii, we know that $\mathbf{M}D^l$ is an eigenvector for matrix X. By equating elements in the j^{th} column of the vectors $\mathbf{M}D^lX$ and $\mathbf{M}D^l$ we have the following equations:

$$w_{\text{in}}^l(1,\ V)X_{1j} + \cdots + w_{\text{in}}^l(n,\ V)X_{nj} = w_{\text{in}}^l(j,\ V) \text{ for } 1 \leq j \leq n \tag{3.2.1}$$

Suppose that

$$w^l_{\text{in}}(i,\ V) \neq w^l_{\text{in}}(j,\ V) \text{ for some } i, j \text{ such that } 1 \leq i, j, \leq n.$$

Multiplying equation 3.2.1 by X_{ij} yields:

$$(w^l_{\text{in}}(i,\ V) - w^l_{\text{in}}(j,\ V))X_{ij} = 0. \tag{3.2.2}$$

Since $w^l_{\text{in}}(i,\ V) - w^l_{\text{in}}(k,\ V) \neq 0$ this implies that $X_{ij} = 0$. The second assertion is proved in similar manner using $D^l\mathcal{M}$ instead of $\mathbf{M}D^l$. □

Lemma 3.2.3 *If $d^{(l)}_{ii} \neq d^{(l)}_{jj}$ for some l, then $X_{ij} = 0$*

Proof: Recall $DX = XD$. By induction on l, we see that $D^l X = XD^l$ for any $l \in Z^+$. The ij^{th} entry of these matrices gives us the following equations:

$$\sum_{k=1}^{n} d^{(l)}_{ik} X_{kj} = \sum_{k=1}^{n} d^{(l)}_{kj} X_{ik}$$

Multiplying the above equation by X_{ij} gives,

$$d^{(l)}_{ii} X_{ij} = d^{(l)}_{jj} X_{ij}.$$

This implies that $X_{ij} = 0$ if $d^{(l)}_{ii} \neq d^{(l)}_{jj}$. □

Chapter 4

Undirected Asymmetric Trees

In this chapter, we use a classification of asymmetric trees to obtain a result about the quantum automorphism group of asymmetric trees.

4.1 Star Partitions

The following is the definition used by Weichsel in [8] for a star partition of a graph $\mathcal{G}$.

Definition 4.1.1 *A star partition of an undirected graph, $\mathcal{G}$, is a partition, $\mathcal{P}$, of the vertices of $\mathcal{G}$ such that:*

1. *All vertices in $A \in \mathcal{P}$ have the same degree.*
2. *Let $A \in \mathcal{P}$ and $B \in \mathcal{P}$. If $a \in A$ and $b \in B$ are such that $ab \in E$, then for each $a' \in A$ there exists $b' \in B$ such that $a'b' \in E$.*

If $A \in \mathcal{P}$, we call A a cell of the partition $\mathcal{P}$. Note that for $\alpha \in Aut(\mathcal{G})$, the orbits of α are a star partition of $\mathcal{G}$.

The next definition, also due to Weichsel [8], defines a graph $\overline{\mathcal{P}}$ associated to the partition $\mathcal{P}$. The vertices of $\overline{\mathcal{P}}$ are just the cells of $\mathcal{P}$.

Definition 4.1.2 *Let $\mathcal{G}$ be a simple graph and $\mathcal{P}$ a star partition of $\mathcal{G}$. The star partition graph of $\mathcal{G}$ associated to $\mathcal{P}$ is the graph $\overline{\mathcal{P}}$ defined as follows. The vertex set is just the set $\mathcal{P}$. Given $A \in \mathcal{P}$ and $B \in \mathcal{P}$ with $A \neq B$, there is an edge from A to B if and only if there exists $a \in A$ and $b \in B$ such that $ab \in E$.*

Then we may regard $\mathcal{P}$ as a graph with A adjacent to B, if $A \neq B$ and there is $a \in A$ and $b \in B$ such that $ab \in E$. We call $\mathcal{P}$ a star partition graph of $\mathcal{G}$, and denote it by $\overline{\mathcal{P}}$.

As noted in [8], every graph has at least one star partition–the trivial partition consisting of singleton sets. We also know from [8] that the star partition graph of a tree is a tree.

4.2 Distinct Walks

Recall that $w^l_{\text{in}}(i,\ A)$ is the number of distinct walks of length l into vertex i from the vertices in A. Similarly, $w^l_{\text{out}}(i,\ A)$ is the number of distinct walks of length l from vertex i into the vertices in A. The following lemma obtains some information about $QAut(\mathcal{G})$ and $A_{aut}(\mathcal{G})$ using these quantities.

Lemma 4.2.1 *Let $\mathcal{P}$ be any partition V such that if $c \in C \in \mathcal{P}$ and $d \in D \in \mathcal{P}$ with $C \neq D$ then $X_{cd} = 0$. If $a, a' \in A \in \mathcal{P}$, $B \in \mathcal{P}$ and $w^l_{in}(a, B) \neq w^l_{in}(a', B)$ for some $l \in Z^+$, then $X_{aa'} = 0$. Similarly, if $a, a' \in A \in \mathcal{P}$, $B \in \mathcal{P}$ and $w^l_{out}(a, B) \neq w^l_{out}(a', B)$ for some l, then $X_{aa'} = 0$.*

Proof: We prove this lemma by looking at two cases: $A = B$ and $A \neq B$.

Case 1: $A = B$
Without loss of generality, rename the vertices of $\mathcal{G}$ so that the vertices in A are labelled $1, 2, \ldots, m$. Since $X_{ac} = 0$ for $a \in A$ and $c \notin A$ then the column vector with 1's in the first m spots and 0's elsewhere is a right eigenvector with eigenvalue 1 for the matrix X. This can be seen from a simple matrix calculation. By a similar calculation, we see that $D^l \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}^T$ is also a right eigenvector with eigenvalue 1 for the matrix X for any l. Using the same calculations as those in Lemma 3.2.2, but replacing the vector $D^l\mathcal{M}$ with the vector $D^l \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}^T$ yields this result.
Case 2: $A \neq B$
Without loss of generality, rename the vertices of $\mathcal{G}$ so that the vertices in A are labelled $1, 2, \ldots, m$ and the vertices in B are labelled $m+1, \ldots, k$. By the same argument as in case 1, one can see that $D^l \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}^T$ is a right eigenvector with eigenvalue 1 for the matrix X, where the 1's are in the $m+1$ through k entries in the vector. Comparing the a^{th} entry in the vectors
$XD^l \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}^T$ and $D^l \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}^T$ gives us the following equation:

$$w^l_{\text{out}}(1,\ B)X_{a1} + \ldots (w^l_{\text{out}}(n,\ B)X_{an} = w^l_{\text{out}}(a,\ B)$$

Multiplying this equation by $X_{aa'}$ yields the following:

$$(w^l_{\text{out}}(a',\ B) - w^l_{\text{out}}(a,\ B))X_{aa'} = 0$$

Since $w^l_{\text{out}}(a',\ B) - w^l_{\text{out}}(a,\ B) \neq 0$ this implies that $X_{aa'} = 0$.

The second assertion is proved in similar manner using left eigenvectors instead of right eigenvectors. □

4.3 Stable Partition Algorithm

Definition 4.3.1 *Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of a graph $\mathcal{G}$. Partition $\mathcal{P}$ is smaller than $\mathcal{P}'$ if it is a refinement of $\mathcal{P}'$. Similarly, $\mathcal{P}'$ is larger than $\mathcal{P}$ if $\mathcal{P}$ is a refinement of $\mathcal{P}'$.*

A partition $\pi = \{C_1, \ldots, C_p\}$ is called stable if whenever $u, v \in C_i$ then u and v are both joined to the same number of vertices of C_j for all $j = 1, 2 \ldots p$ ([5],Section 2).

Definition 4.3.2 *Let $v_{C_i}(a)$ denote the number of vertices in cell C_i that are adjacent to vertex a.*

In [5], an algorithm was introduced in order to better understand graph isomorphism programs since they are generally based on partition refinement. We rewrite this algorithm and then use it to refine any partition of an undirected graph into a stable partition.

Let π be a partition with some ordering on the cells. Let $[\pi]_j$ denote the j^{th} cell of partition π.

Refine(π)
  Set $\pi(1) := \pi$
  {each cell of π is initially marked uncounted}
  While(there exists a cell of $\pi(i)$ that is uncounted)
    j = first uncounted cell of $\pi(i)$
      $C := [\pi(i)]_j$
      Calculate $v_C(w)$ for each $w \in V$
      Mark C counted
        for each $[\pi(i)]_k \in \pi(i)$
          If $v_C(\cdot)$ is not constant on $[\pi(i)]_k$ then split $[\pi(i)]_k$ into new cells with equal value on $v_C(\cdot)$. These new cells are in $\pi(i+1)$ and are marked uncounted.

If $v_C(\cdot)$ is constant on $[\pi(i)]_k$ then this cell is in $\pi(i+1)$ and has the same counted status as $[\pi(i)]_k$.

end

$i = i + 1$

End

The **Refine** algorithm ends either when all cells are counted and $\pi(i) = \pi(i-1)$ or when the trivial partition is reached. From [5], we know that this algorithm produces the largest stable partition that is smaller than the partition that is input. Additional information regarding implementation of this algorithm can be found in [5].

The next lemma shows that the partition resulting from this algorithm is a star partition. It is a specific type of star partition, called a refined star partition, that we will define in the next chapter.

Lemma 4.3.1 *The partition $\mathcal{P}$ that results from **Refine** is a star partition.*

Proof: Let $A_1, \ldots, A_m$ be the cells of the partition $\mathcal{P}$ resulting from **Refine** and let i be a vertex. We have $deg(i) = v_{A_1}(i) + \ldots + v_{A_m}(i)$. We know from the algorithm that $v_{A_k}(i) = v_{A_k}(j)$ for all $i, j \in A_l$ and $1 \le k \le m$. Therefore $deg(i) = deg(j)$ for $i, j \in A_l$. For $i, j \in A_l \in \mathcal{P}$ and $A_k \in \mathcal{P}$, $v_{A_k}(i) = v_{A_k}(j)$. Hence there is an edge from vertex i to a vertex in A_k if and only if there is an edge from vertex j to a vertex in A_k. Therefore $\mathcal{P}$ is a star partition of the graph. $\square$

4.4 Quantum Automorphism Group of Asymmetric Trees

The following theorems which were proven in [8] will be used to help determine the quantum automorphism groups of asymmetric trees and certain other asymmetric graphs.

Theorem 4.4.1 *([8], Theorem 2.4) An undirected tree is asymmetric if and only if its only star partition is the trivial one.*

The following is a weaker theorem that applies to all graphs.

Theorem 4.4.2 *([8], Theorem 2.2) Let $\mathcal{G}$ be a graph whose only star partition is the trivial partition. Then $\mathcal{G}$ is asymmetric.*

The following theorem computes the quantum automorphism group of all asymmetric trees using the result about star partitions and the algorithm **Refine**.

Theorem 4.4.3 *Let $\mathcal{T}$ be an undirected tree. Then $\mathcal{T}$ is an asymmetric tree if and only if $QAut(\mathcal{T}) \cong \mathbf{C}$.*

Proof: If $QAut(\mathcal{T}) \cong \mathbf{C}$ then $\mathcal{A}(\mathcal{T}) \cong \mathbf{C}$. Hence $\mathcal{T}$ is asymmetric. Now assume $\mathcal{T}$ is an undirected asymmetric tree. From Theorem 4.4.1, we know that $\mathcal{T}$ has only the trivial star partition. Hence if we place the partition $\pi = \{1, 2, \ldots n\}$ through the algorithm **Refine** it should produce the trivial partition. The first time through, the algorithm sorts the vertices according to $v_v(\cdot) = deg(\cdot)$. Note that $w^1_{out}(a, V) = deg(a)$ and $w^1_{out}(a, B) = v_B(a)$ for $a \in V$ and $B \in \mathcal{P}$. We know that if $a \in A$ and $b \notin A$ at the end of the first iteration through the algorithm $deg(a) \neq deg(b)$. Hence by Lemma 4.2.1 $X_{ab} = 0$. The partition $\pi(2)$ satisfies the conditions of Lemma 4.2.1. By Lemma 4.2.1, after the second iteration of the algorithm if $a \in A$ and $b \in B$ with $A \neq B$ then $X_{ab} = 0$. Each successive iteration of the algorithm produces a partition that satisfies Lemma 4.2.1. Since $\mathcal{G}$ has only the trivial star partition, the algorithm must produce the trivial star partition. By successive use of Lemma 4.2.1, we see that if $a, b \in V$ with $a \neq b$ then $X_{ab} = 0$. Therefore $QAut(\mathcal{T}) =< 1 >\cong \mathbf{C}$. □

We use the previous theorem to help compute the quantum automorphism group of graphs that have only the trivial star partition.

Lemma 4.4.1 *If the only star partition for graph $\mathcal{G}$ is the trivial partition, then $QAut(\mathcal{G}) \cong \mathbf{C}$*

Proof: Same as for Theorem 4.4.3. □

The graph in Figure 4.1 shows that a graph can be asymmetric, but have a non-trivial star partition. Note that this is a regular graph. Therefore, this asymmetric graph has the non-trivial star partition $\mathcal{P}$ where

$$\mathcal{P} = \{\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\}.$$

In particular, any regular graph $\mathcal{G}$ will have at least two star partitions, the trivial partition and the partition consisting of one cell with all of the vertices of the graph.

Hence we see Theorem 4.4.1 cannot be extended to graphs in general. But in this example it can be shown that $QAut(\mathcal{G}) \cong A_{aut}(\mathcal{G}) \cong \mathbf{C}$. So it is unknown whether Theorem 4.4.3 can be generalized to arbitrary undirected asymmetric graphs.

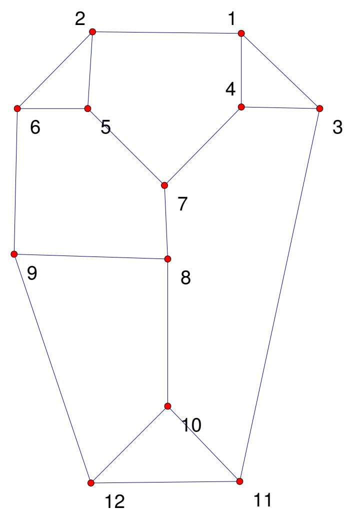

Figure 4.1: Asymmetric Graph

Chapter 5

Z_2 Trees

In this chapter, we obtain a classification, in terms of a specific vertex partition, of all trees $\mathcal{T}$ satisfying $Aut(\mathcal{T}) \cong Z_2$ and use this classification to calculate the quantum automorphism group of such trees. This result is similar to the result classifying asymmetric trees in the previous chapter. We first introduce a refined version of the star partition of a graph.

5.1 Refined Star Partitions

Definition 5.1.1 *Let $\mathcal{G}$ be a graph and $\mathcal{P}$ a partition of its vertices satisfying the following*

1. *If $A \in \mathcal{P}$, then all elements of A have the same degree*
2. *If $a, a' \in A \in \mathcal{P}$, then for each $B \in \mathcal{P}$, $v_B(a) = v_B(a')$*

Call such a partition a refined star partition.

Recall that the partition, $\pi = \{C_1, \ldots C_p\}$, that results from **Refine** has the property that if whenever $u, v \in C_i$ then u and v are both joined to the same number of vertices in cell C_i for all $j = 1, 2, \ldots, p$. Thus we see that the partition resulting from **Refine** is a refined star partition. Every graph has a refined star partition, namely the trivial partition. Let $\mathcal{P}$ be a refined star partition of a graph with cells A and B. Since $v_B(a) = v_B(a')$ for any $a, a' \in A$, we will denoted this quantity by $v_B(A)$. Note that every refined star partition of a graph $\mathcal{G}$ is a star partition. But as the following example shows, not every star partition is a refined star partition. One can check that the following partition, $\mathcal{P}$, is a star partition of the graph in Figure 4.1.

$$\begin{aligned}
&\mathcal{P} : \{A, B, C, D\} \text{ where} \\
&A = \{1, 2\} \\
&B = \{3, 4, 5, 6\} \\
&C = \{7, 9, 11\} \\
&D = \{8, 10, 12\}
\end{aligned}$$

But $\mathcal{P}$ is not a refined star partition of the graph, since $v_D(8) \neq v_D(10)$.

Since a refined star partition is a star partition, if $\mathcal{P}$ is a refined star partition of a tree then from [8] we know that $\overline{\mathcal{P}}$ is a tree.

As with star partitions, one way to construct refined star partitions uses elements of the automorphism group.

Lemma 5.1.1 *If $\sigma \in Aut(\mathcal{G})$, then the orbits of σ are a refined star partition of $\mathcal{G}$.*

Proof: Let $\sigma \in Aut(\mathcal{G})$ and let $\mathcal{P}$ be the partition of the vertices given by the orbits of σ. It is immediate that $\mathcal{P}$ satisfies condition one of a refined star partition. Let $a \in A \in \mathcal{P}$ with $v_B(a) = m$ and $ab_i \in E$ for distinct $b_i \in B$ labelled $1 \leq i \leq m$. Let $a' \in A$, then there exists some $j \in Z^+$ such that $\sigma^j(a) = a'$. Since $\sigma \in Aut(\mathcal{G})$, $\sigma^j(a)\sigma^j(b_i) \in E$ for each $1 \leq i \leq m$ and $\sigma^j(b_i) \neq \sigma^j(b_k)$ for $i \neq k$. Since $\mathcal{P}$ is the partition given by the orbits of σ, then $\sigma^j(a) \in A$ and $\sigma^j(b_i) \in B$ for $1 \leq i \leq m$. Thus $\sigma^j(b_i)$ is adjacent to $\sigma^j(a)$ for $1 \leq i \leq m$ and $v_B(a') = v_B(a) = m$. Hence the orbits of σ are a refined star partition. $\square$

This next lemma compares the distances of two different vertices in a cell to all of the vertices in V.

Lemma 5.1.2 *Let $\mathcal{P}$ be a refined star partition of the tree $\mathcal{G}$ with $a, a' \in A \in \mathcal{P}$ then*

$$\max_{v \in V} dist(a, v) = \max_{v \in V} dist(a', v).$$

Proof: Let $A \in \mathcal{P}$ with $i, j \in A$. Let $i, a_1, \ldots, a_s$ be a path of longest length from i with $a_j \in A_j \in \mathcal{P}$. Note that $deg(a_s) = 1$, since if $deg(a_s) > 1$ then the path from i could be made longer. Since $\mathcal{P}$ is a refined star partition and i is adjacent to $a_1 \in A_1$ then j is adjacent to a vertex in A_1. Hence there exists $b_1 \in A_1$ such that $jb_1 \in E$. Since $\mathcal{P}$ is a refined star partition and a_1 is adjacent to $a_2 \in A_2$ then b_1 is adjacent to a vertex in A_2. Hence there exists $b_2 \in B_2$ such that $b_1b_2 \in E$. Continue in a similar fashion with the rest of the a_r.

We have now found a path of length s from j, namely this path is $j, b_1, \dots b_s$. We cannot continue this path any further, since a_s and b_s are in the same element of the partition and $deg(a_s) = deg(b_s) = 1$. Hence $\max_{v \in V} dist(a, v) = \max_{v \in V} dist(a', v)$. $\Box$

5.2 The center of $\mathcal{G}$

We use the notion of graph centers to give us necessary conditions a refined star partition must satisfy.

Definition 5.2.1 *The center of a graph is the set of vertices such that* $\max_{v \in V} dist(\cdot, v)$ *is as small as possible. Let* $Z(\mathcal{G})$ *denote the center of the graph* $\mathcal{G}$.

Lemma 5.2.1 *Suppose* $x, y \in V$ *with* $x \in Z(\mathcal{G})$ *and* $y \notin Z(\mathcal{G})$. *If* $\mathcal{P}$ *is a refined star partition and* $x \in A \in P$ *then* $y \notin A$.

Proof: Follows from Lemma 5.1.2. $\Box$

From [3], we know that the center of a tree consists of exactly one vertex or two adjacent vertices. In particular, a refined star partition will have the center of a tree either as one cell or split into two cells. The next lemma shows that every vertex in a specific cell of a refined star partition has the same distance from $Z(\mathcal{G})$.

Lemma 5.2.2 *If* $\mathcal{G}$ *is a tree and* $\mathcal{P}$ *is a refined star partition with* $a, a' \in A \in \mathcal{P}$ *then*

$$\min_{c \in Z(\mathcal{G})} dist(a, c) = \min_{c \in Z(\mathcal{G})} dist(a', c).$$

Proof: Suppose $a \in A$ and $a, b_1, \dots, b_m, c$ is the shortest path from a to an element in $Z(\mathcal{G})$ with $b_i \in B_i$. Let $a' \in A$. Since $\mathcal{P}$ is a refined star partition, $a'b'_1 \in E$ for some $b'_1 \in B$, $b'_{i-1}b'_i \in E$ for some $b'_i \in B_i$ for each $1 \le i \le m$, and $b'_m c' \in E$ for some $c' \in Z(\mathcal{G})$. Hence $a', b'_1, \dots, b'_m, c'$ is a path from a' to an element in $Z(\mathcal{G})$. If there was a shorter path from a' to an element in $Z(\mathcal{G})$, then there would be two distinct paths from a' to $c' \in Z(\mathcal{G})$. Therefore $\min_{c \in Z(\mathcal{G})} dist(a, c) = \min_{c \in Z(\mathcal{G})} dist(a', c)$. $\Box$

The following lemma shows that, except for the possibility of a cell consisting of a pair of adjacent vertices from the center, two vertices in the same cell of a refined star partition cannot be adjacent.

Lemma 5.2.3 *If $\mathcal{P}$ is a refined star partition of a tree $\mathcal{G}$ and $A \in \mathcal{P}$ with A containing no elements of $Z(\mathcal{G})$ then $v_A(a) = 0$ for each $a \in A$.*

Proof: Let $a, a' \in A$ with $aa' \in E$. There are two cases to consider: $Z(\mathcal{G}) = \{c\}$ and $Z(\mathcal{G}) = \{c, c'\}$.
Case 1: $Z(\mathcal{G}) = \{c\}$
Suppose the path from a to c is of length d. By Lemma 5.2.2 we know the path from a' to c must be of length d and this path cannot contain the vertex a. Since $aa' \in E$ there is another path from a to c that goes through a'. This contradicts that fact that $\mathcal{G}$ is a tree.
Case 2: $Z(\mathcal{G}) = \{c, c'\}$
Suppose the path from a to c is of length d. Lemma 5.2.2 tells us there is a path of length d from $c \in Z(\mathcal{G})$ to a' or a path of length d from $c' \in Z(\mathcal{G})$ to a'. Since $cc' \in E$, both of these result in two distinct paths from a to c contradicting the fact that $\mathcal{G}$ is a tree.
Thus $aa' \notin E$ and $v_A(a) = 0$ for each $a, a' \in A$. $\square$

Let A be a cell of a refined star partition. Since $\min_{c \in Z(\mathcal{G})} dist(a, c)$ is independent of the choice of $a \in A$, we use this to define the notion of distance from a cell to the center of the graph.

Definition 5.2.2 *Let $\mathcal{P}$ be a partition of the graph $\mathcal{G}$ and $A \in \mathcal{P}$. The distance of A from $Z(\mathcal{G})$, denoted $dist(A, Z(\mathcal{G}))$, is defined to be $\min_{c \in Z(\mathcal{G})} dist(a, c)$ for $a \in A$.*

We use the distance from cells to the center in the following lemma to determine the adjacency of vertices between two cells of a refined star partition.

Lemma 5.2.4 *Let $\mathcal{G}$ be a tree and $\mathcal{P}$ be a refined star partition of $\mathcal{G}$. If $A, B \in \mathcal{P}$ with A adjacent to B in $\overline{\mathcal{P}}$ and $dist(A, Z(\mathcal{G})) > dist(B, Z(\mathcal{G}))$, then $v_B(A) = 1$.*

Proof: Since A adjacent to B in $\overline{\mathcal{P}}$ we know that $v_B(a) \geq 1$. Suppose $v_B(a) > 1$ for some $a \in A$. Let $b, b' \in B$ with ab and ab' both in E and let $c \in Z(\mathcal{G})$. Since $\mathcal{G}$ is a tree there is a path from b to c as well as a path from b' to c. Hence there is a path from a to c through b as well as a path from a to c through b'.

Figure 5.1 shows a subgraph of $\mathcal{G}$ that includes the vertices a, b, b', c. The thin lines represent edges and the thick lines represent paths. One can see that there exist two paths from a to c contradicting the fact that $\mathcal{G}$ is a tree. Hence $v_B(a) = 1$ for each $a \in A$ and thus $v_B(A) = 1$. $\square$

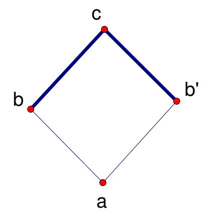

Figure 5.1: Lemma 5.2.4

The following lemma defines a relationship between the number of vertices in cells in terms of the adjacency between cells.

Lemma 5.2.5 *Let $\mathcal{G}$ be a tree and $\mathcal{P}$ be a refined star partition of $\mathcal{G}$. If $A, I_1, \ldots I_{r-1}, C$ is a path in $\overline{\mathcal{P}}$ with $dist(A, Z(\mathcal{G})) > dist(C, Z(\mathcal{G}))$, then*

$$v_A(I_1)v_{I_1}(I_2)\ldots v_{I_{r-1}}(C) = \frac{|A|}{|C|}.$$

Moreover, if A and C are cells of $\mathcal{P}$ with A adjacent to C in $\overline{\mathcal{P}}$ and $dist(A, Z(\mathcal{G})) > dist(C, Z(\mathcal{G}))$, then $\frac{|A|}{|C|} = v_A(C)$.

Proof: First assume that A is adjacent to C. Note that $\sum_{c\in C} v_A(C) = \sum_{a\in A} v_C(A)$. Since $v_A(\cdot)$ is constant on C and $v_C(\cdot)$ is constant on A it follows that $|C|v_A(C) = |A|v_C(A)$. By Lemma 5.2.4, we know that $v_C(A) = 1$. Hence $|C|v_A(C) = |A|$ and $\frac{|A|}{|C|} = v_A(C)$.
Now suppose that $A, I_1, \ldots I_{r-1}, C$ is a path in $\overline{\mathcal{P}}$ with $dist(A, Z(\mathcal{G})) > dist(C, Z(\mathcal{G}))$. Set $A = I_0$ and $C = I_r$. Note that $dist(I_j, Z(\mathcal{G})) > dist(I_{j+1}, Z(\mathcal{G}))$ for $0 \leq j \leq r-1$. Hence by repeated application of the previous paragraph we know that

$$\begin{aligned} v_A(I_1)v_{I_1}(I_2)\ldots v_{I_r}(C) &= \tfrac{|A|}{|I_1|}\tfrac{|I_1|}{|I_2|}\ldots\tfrac{|I_r|}{|C|} \\ &= \tfrac{|A|}{|C|}. \; \square \end{aligned} \tag{5.2.1}$$

The following lemma uses the distance from the center to gives some restrictions on the number of vertices in a cell of a refined star partition.

Lemma 5.2.6 *Let $\mathcal{G}$ be a tree and $\mathcal{P}$ be a refined star partition of $\mathcal{G}$. If A and B are cells of $\mathcal{P}$ with A adjacent to B in $\overline{\mathcal{P}}$ and $dist(A, Z(\mathcal{G})) > dist(B, Z(\mathcal{G}))$, then $|A| \geq |B|$.*

Proof: By Lemma 5.2.5, we know that $v_A(B) = \frac{|A|}{|B|}$. If $|A| < |B|$, then $v_A(B)$ is not an integer. This contradicts the definition. Hence if $dist(A, Z(\mathcal{G})) > dist(B, Z(\mathcal{G}))$, then $|A| \geq |B|$. $\square$

We use the notion of distance from the center to determine possible adjacency between two cells in $\overline{\mathcal{P}}$ in the following two lemmas.

Lemma 5.2.7 *Let $\mathcal{G}$ be a tree with refined star partition $\mathcal{P}$. Let $A, B \in \mathcal{P}$ such that $dist(A, Z(\mathcal{G})) = dist(B, Z(\mathcal{G}))$. Then A is not adjacent to B in $\overline{\mathcal{P}}$.*

Proof: Let $A, B \in \mathcal{P}$ with $dist(A, Z(\mathcal{G})) = dist(B, Z(\mathcal{G})) = \zeta$. Suppose $a \in A$ and $b \in B$ with $ab \in E$. Since $dist(A, Z(\mathcal{G})) = dist(B, Z(\mathcal{G})) = \zeta$, there exists a path from a to an element of $Z(\mathcal{G})$ of length ζ that does not include any elements from B. Similarly there exists a path from b to an element of $Z(\mathcal{G})$ of length ζ that does not include any elements from A.

Case 1: Suppose that both a and b have paths of length ζ to $c \in Z(\mathcal{G})$. Then there are two distinct paths from a to c in $\mathcal{G}$.
Case 2: Suppose there is a path of length ζ from a to $c \in Z(\mathcal{G})$ and a path of length ζ from b to $c' \in Z(\mathcal{G})$. Since $\mathcal{G}$ is a tree and elements of the center are adjacent, we know that $cc' \in E$. Again we have two distinct paths from a to c in $\mathcal{G}$.

Both cases contradict the fact that $\mathcal{G}$ is a tree. Hence A is not adjacent to B in $\overline{\mathcal{P}}$ if $dist(A, Z(\mathcal{G})) = dist(B, Z(\mathcal{G}))$. $\square$

Lemma 5.2.8 *If $A \in \mathcal{P}$ then A is adjacent to only one element, B, in $\overline{\mathcal{P}}$ such that $dist(A, Z(\mathcal{G})) > dist(B, Z(\mathcal{G}))$*

Proof: Suppose that A is adjacent to B and A is adjacent to D in $\overline{P}$ with $dist(A, Z(\mathcal{G})) > dist(B, Z(\mathcal{G}))$, and $dist(A, Z(\mathcal{G})) > dist(D, Z(\mathcal{G}))$. Then there is a path from A to $Z(\mathcal{G})$ through B as well as a path from A to $Z(\mathcal{G})$ through D. Figure 5.2 represents a subgraph of $\overline{\mathcal{P}}$ containing cells $A, B, D, Z(\mathcal{G})$ where the bold lines represent a path between the two cells and the thin lines represent an edge between the two cells.

Note that these two paths are different which contracts the fact that $\overline{\mathcal{P}}$ is a tree. Hence A is adjacent to only one element, B, in $\mathcal{P}$ such that $dist(A, Z(\mathcal{G})) > dist(B, Z(\mathcal{G}))$. $\square$

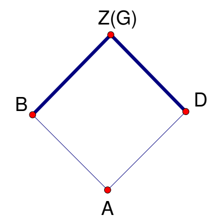

Figure 5.2: Lemma 5.2.8

5.3 Z_2 Graphs

In the next lemma, we build an automorphism of a graph from a special class of refined star partitions. The automorphism that we build will be used in the classification of all trees with automorphism group isomorphic to Z_2.

Lemma 5.3.1 *If $\mathcal{G}$ is a graph with a non-trivial refined star partition $\mathcal{P}$ such that for each $A \in \mathcal{P}$, $|A| \leq 2$, then there exists $\sigma \in Aut(G)$ with $|\sigma| = 2$.*

Proof: Given $D \in \mathcal{P}$ with $|D| = 2$ and $x \in D$. We write x' for the other element in D. We further set $x'' = x$. Define

$$\sigma(x) = \begin{cases} x & : \quad \{x\} \in \mathcal{P} \\ x' & : \quad \{x, x'\} \in \mathcal{P} \end{cases}$$

Note that σ is one to one on $V(\mathcal{G})$. Now we need to show that if $uz \in E$ then $\sigma(u)\sigma(z) \in E$. Suppose $uv \in E$ with $u \in U \in \mathcal{P}$ and $z \in Z \in \mathcal{P}$. There are four cases to consider:

CASE 1: $|U| = |Z| = 2,\ v_U(z) = 2$
Since $\mathcal{P}$ is a refined star partition, we know that $v_U(z') = 2$ and $uz, uz', u'z, u'z' \in E$. Hence $\sigma(u)\sigma(z) = u'z' \in E$

CASE 2: $|U| = |Z| = 2,\ v_U(z) = 1$
Then $v_U(z') = 1$ and $uz, u'z' \in E$. Hence $\sigma(u)\sigma(z) = u'z' \in E$.

CASE 3: $|U| = 2$ and $|Z| = 1$ then $uz, u'z \in E$. Hence $\sigma(u)\sigma(z) = u'z \in E$.

CASE 4: $|U| = |Z| = 1$ then $\sigma(u)\sigma(z) = uz \in E$.
Hence $\sigma \in Aut(\mathcal{G})$. Since $\mathcal{P}$ is a non-trivial partition, there is at least one $A \in \mathcal{P}$ with

$|A| = 2$. Therefore σ is a non-trivial automorphism of the graph. It is easy to check that $\sigma^2 = id$, therefore $|\sigma| = 2$. $\square$

From Lemma 5.3.1 and Lemma 5.1.1, we have sufficient conditions for a graph to have automorphism group isomorphic to Z_2.

Theorem 5.3.1 *Let $\mathcal{G}$ be a graph whose only refined star partitions are the trivial partition and a non-trivial partition $\mathcal{P}$. If all cells of $\mathcal{P}$ contain no more than two vertices, then $Aut(\mathcal{G}) \cong Z_2$.*

The follow lemma illustrates that refined star partitions of trees are determined by what they do on cells containing degree one elements.

Lemma 5.3.2 *Let $\mathcal{P}$ be a refined star partition of the graph $\mathcal{G}$. If for each $A \in \mathcal{P}$ containing vertices of degree 1, $|A| \leq 2$, then $|B| \leq 2$ for all $B \in \mathcal{P}$.*

Proof: Suppose A is adjacent to B in $\overline{\mathcal{P}}$. Since the vertices in A are of degree 1, then $v_B(A) = 1$.

$$\sum_{a\in A} v_B(A) = \sum_{b\in B} v_A(B)$$

$$|A| v_B(A) = |B| v_A(B)$$

$$|A| = |B| v_A(b)$$

Since $|A| = 1$ or $|A| = 2$ and $v_A(B) \geq 1$ this implies that $|B| \leq 2$. Remove all vertices of degree 1 from the tree and call the subtree H. H is a tree with the same hypothesis as $\mathcal{G}$. Continuing in a similar manner one shows that $|B| \leq 2$ for all $B \in \mathcal{P}$. $\square$

The following example shows that if $\mathcal{G}$ is not a tree then the converse of Theorem 5.3.1 does not hold. The graph in Figure 5.3 has automorphism group isomorphic to Z_2, but has more than two refined star partitions.

These refined star partitions include:

- the trivial partition
- $\{\{1, 2, 3, 4, 5, 6, 7, 8, 9\}\}$
- $\{\{1\}, \{5, 6\}, \{4, 7\}, \{3, 8\}, \{2, 9\}\}$

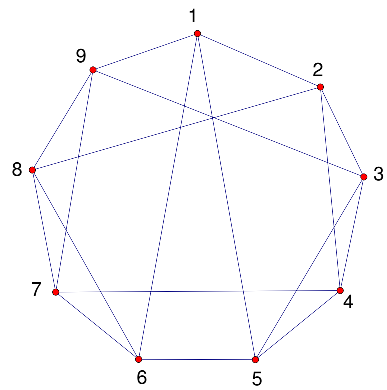

Figure 5.3: Z_2 graph

As we saw in the previous chapter all regular graphs will have at least two refined star partitions, the trivial partition and the partition with all of the vertices in one cell. Thus a regular graph with automorphism group congruent to Z_2 will have at least three refined star partitions: the trivial partition, the partition consisting of one cell with all of the vertices of the graph, and the partition induced by the non-trivial automorphism of the graph.

5.4 Classification of Z_2 trees

In the case of trees, we obtain the converse of Theorem 5.3.1. Before we can prove the converse, we need the following lemma, which inductively builds an automorphism of the whole graph from an automorphism of a subgraph.

Lemma 5.4.1 *Suppose that $\mathcal{G}$ is a tree and $\mathcal{P}$ is a non-trivial refined star partition of $\mathcal{G}$. Let A be a cell of $\mathcal{P}$. Suppose H is a subgraph of $\mathcal{G}$ such that the vertices in H consist of the vertices in cell A along with the vertices in cells of $\mathcal{P}$ whose path to the center does not include cell A. If σ_0 is an automorphism of H, then σ_0 can be extended to an automorphism of the whole graph.*

Proof: We will inductively build an automorphism of the whole graph $\mathcal{G}$ by extending σ_0. Let $|A| = s$ and label the vertices in cell A as $a_0, a_1, \ldots, a_{s-1}$. Let V_H be the set of vertices in subgraph H. Relabel the cells in $\mathcal{P}$ that are not in V_H based on their distance from A, so that cells distance r from A are $\{M[F] \mid F \in \mathbf{N}^r\}$ with $F = (f_1, \ldots, f_r)$. Do this so that if $M[F]$ is adjacent to $M[J]$ distance $r-1$ from A in $\overline{\mathcal{P}}$ then $J = (f_2, \ldots, f_r)$. Label the vertices in $M[F]$ as $\{a_{Fi} | 0 \leq i \leq |M[F]| - 1\}$, so that a_{Fk} is adjacent to $a_{Jj} \in M[J]$ when

$$k = \left(i + j\frac{|M[F]|}{|M[J]|}\right) \bmod |M[F]| \text{ for } 0 \leq i \leq v_F(J) - 1.$$

If $M[F]$ is adjacent to A then label the vertices as $\{a_{Fi} | 0 \leq i \leq |M[F]| - 1\}$, so that a_{Fk} is adjacent to $a_s \in A$ when

$$k = \left(i + s\frac{|M[F]|}{|A|}\right) \bmod |M[F]| \text{ for } 0 \leq i \leq v_F(J) - 1.$$

By Lemma 5.2.8, we know that $M[F]$ is adjacent to only one cell that is distance $r-1$ from A and by Lemma 5.2.4 we know that a_{Fk} is adjacent to only one vertex in this cell, so this labelling is well defined.

We know that σ_0 is an automorphism on the subgraph induced by V_H. Thus the base case of the induction holds.

Now we continue with the induction. Set $V_q = V_{q-1} \cup \{\text{cell that are distance } q \text{ from } A\}$
Define $\sigma_q : V_q \to V_q$ by

$$\sigma_q(x) = \begin{cases} \sigma_{q-1}(x) & : \quad x \in V_{q-1} \\ a_{Fj} & : \quad x = a_{Fk} \end{cases}$$

with

$$k = \left(\alpha + \beta\frac{|M[F]|}{|M[B]|}\right) \bmod |M[F]|$$

and

$$j = \left(\alpha + \zeta\frac{|M[F]|}{|M[B]|}\right) \bmod |M[F]|$$

where $a_{B\beta}a_{Fk} \in E$ with $M[B] \in V_{q-1}$ and $\sigma_{q-1}(a_{B\beta}) = \sigma_q(a_{B\beta}) = a_{B\zeta}$. By Lemma 5.2.8, we know that $M[F]$ is adjacent to only one cell in V_{q-1} and by Lemma 5.2.4 we know that a_{Fk} is adjacent to only one vertex in V_{q-1}, so σ_q is well defined. Now we show that σ_q is an automorphism of V_q. It can be shown that σ_q is one-to-one. It remains to be shown that if $uv \in E$ then $\sigma_q(u)\sigma_q(v) \in E$. We need only verify this when at least one of u, v is not in V_{q-1} since $\sigma_q = \sigma_{q-1}$ on elements of V_{q-1}. We consider these remaining two possibilities:

1. $u \in V_{q-1}$ and $v \notin V_{q-1}$. Let $u \in M[G]$ and $v \in M[B]$ where $M[G]$ and $M[B]$ are

adjacent in $\overline{P}$. Hence $u = a_{Gg}$ and $v = a_{Bb}$, where

$$b = (z + g\frac{|M[B]|}{|M[G]|}) \bmod |M[B]|.$$

Suppose

$$\sigma_q(a_{Gg}) = \sigma_{q-1}(a_{Gg}) = a_{Gw}.$$

Then $\sigma_q(a_{Bb}) = a_{Bm}$, where

$$m = (z + w\frac{|M[B]|}{|M[G]|}) \bmod |M[B]|.$$

We can see by our labelling that $\sigma_q(a_{Gg})\sigma_q(a_{Bb}) = a_{Gw}a_{Bm} \in E$. Hence σ_q is an automorphism of V_q.

2. $u \notin V_{q-1}$ and $v \notin V_{q-1}$. Suppose $u \in M[F]$ and $v \in M[G]$, then $M[F]$ and $M[G]$ are the same distance from A. Therefore $M[F]$ and $M[G]$ are the same distance from $Z(\mathcal{G})$. Hence by Lemma 5.2.7, we know that this is not possible.

Continue in a similar manner until $V_r = V$, then set $\sigma_r = \sigma$. And $\sigma : V \to V$, is an automorphism of graph $\mathcal{G}$. □

The following two remarks allow us to use the above lemma if the hypothesis are slightly varied and will be used in the proof of the Theorem 5.4.1.

Remark 5.4.1 *In the previous lemma, if we let $\sigma'_i = \sigma_i$ for $1 \leq i < j$ and let $\sigma'_j : V_j \to V_j$ be an automorphism of V_j that is different from σ_j replacing the σ_{q-1} with σ'_{q-1} and the σ_q with σ'_q, then we will build an automorphism σ' of the graph that is distinct from σ.*

Remark 5.4.2 *We can extend this lemma to work for the case when A and B are cells of $\mathcal{P}$ with neither A nor B along the path of the other to $Z(\mathcal{G})$. Suppose H is a subgraph of $\mathcal{G}$ such that the vertices in H consist of the vertices in cells A and B along with the vertices in cells of $\mathcal{P}$ whose path to the center does not include cell A or cell B. If σ_0 is an automorphism of H then the methods of the above proof will lead to an inductively built automorphism, σ, of the tree $\mathcal{G}$.*

We will use the above lemma to help inductively build automorphisms and ultimately to help show the converse of Theorem 5.3.1 holds for trees.

Theorem 5.4.1 *If $\mathcal{G}$ is a tree and $Aut(\mathcal{G}) \cong Z_2$, then there are exactly two refined star partitions of $\mathcal{G}$.*

Proof: Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong Z_2$. Let $\gamma \in Aut(\mathcal{G})$ and $|\gamma| = 2$. Since the orbits of an automorphism of $\mathcal{G}$ are a refined star partition then there exist at least 2 refined star partitions of $\mathcal{G}$, the trivial partition and the partition induced by γ. Suppose there exists a third refined star partition of $\mathcal{G}$. Call this partition $\mathcal{P}$. There are four cases to consider.

CASE 1: *Each $A \in \mathcal{P}$ containing degree 1 vertices has $|A| \leq 2$.*

By Lemma 5.3.2, we know that all $B \in \mathcal{P}$ have $|B| \leq 2$. Then the partition $\mathcal{P}$ induces $\sigma \in Aut(\mathcal{G})$ with $|\sigma| = 2$ as seen from Lemma 5.3.1. And the automorphism σ induces the refined star partition $\mathcal{P}$. The automorphisms γ and σ must differ since the refined star partitions they induce differ. Hence there are at least two elements in $Aut(\mathcal{G})$ of order 2, so $Aut(\mathcal{G}) \ncong Z_2$.

CASE 2: *There exists $A \in \mathcal{P}$ containing degree 1 vertices with $|A| > 2$, and $|B| = 1$ for some $B \in \mathcal{P}$ with A adjacent to B in $\overline{\mathcal{P}}$.*

Let $A = \{a_0, a_1, \ldots, a_{m-1}\}$ with $m > 2$ and $B = \{b\}$. Define

$$\sigma(x) = \begin{cases} x & : \ x \notin A \\ a_{(i+1) \bmod m} & : \ x = a_i \end{cases}$$

Now we verify that σ is an automorphism of $\mathcal{G}$. It is easy to see that σ is one-to-one on the vertices of $\mathcal{G}$. All that remains is to show if $uz \in E$ then $\sigma(u)\sigma(z) \in E$. There are 3 possibilities to consider. Let $uz \in E$.

1. $u \in A$ and $z \in A$. From Lemma 5.2.3 we can see this is only possible in a connected tree if $u, z \in Z(\mathcal{G})$. Since $|Z(\mathcal{G})| \leq 2$ and $|A| > 2$, this is not possible.

2. $u \in A$ and $z \notin A$. It follows that $z = b$ and $\sigma(u)\sigma(b) = \sigma(u)b$. Since $\mathcal{P}$ is a refined star partition and $\{b\} \in \mathcal{P}$, then all vertices in A are adjacent to b. Hence $\sigma(u)\sigma(b) = \sigma(u)b \in E$, since $\sigma(u) \in A$.

3. $u \notin A$ and $z \notin A$. It follows that $\sigma(u)\sigma(z) = uz \in E$

By calculating $\sigma^i(a_1)$, we see that $|\sigma| = m > 2$. Hence $Aut(\mathcal{G}) \ncong Z_2$.

Recall the comment listed after Lemma 5.5.3, thus we know a refined star partition will either have one cell consisting of two elements of the center or at least one cell consisting of just one element of the center.

CASE 3: *There exists $A \in \mathcal{P}$ containing degree 1 vertices with $|A| > 2$, and $|B| \geq 2$ for some $B \in \mathcal{P}$ with A adjacent to B in $\overline{\mathcal{P}}$. There exists $c \in Z(\mathcal{G})$, with $C = \{c\} \in \mathcal{P}$.*

We will build an automorphism of $\mathcal{G}$ inductively using Lemma 5.4.1. If this automorphism is the same as σ then we will modify it slightly and show that we have at least two non-trivial automorphisms of $\mathcal{G}$. Label elements in $\mathcal{P}$ based on their distance from C, so that cells distance r from C are $\{M[I] \mid I \in N^r\}$ with $I = (i_1, \ldots, i_r)$. Do this so that if $M[I]$ is adjacent to $M[J]$ distance $r-1$ from C in $\overline{\mathcal{P}}$ then $J = (i_2, \ldots, i_r)$. Label the vertices in $M[I]$ as $\{a_{Ii} | 0 \leq i \leq |M[I]| - 1\}$, so that a_{Ik} is adjacent to $a_{Jj} \in M[J]$ when

$$k = \left(i + j\frac{|M[I]|}{|M[J]|}\right) \bmod |M[I]| \text{ for } 0 \leq i \leq v_I(a_{J0}) - 1.$$

Let $A, M[I_{(r-1)}], M[I_{(r-2)}], \ldots, M[I_1], C$ be the path from A to C in the tree $\overline{\mathcal{P}}$. Set $A = M[I_r]$. By Lemma 5.2.5, we know that

$$v_{I_r}(I_{(r-1)})v_{I_{(r-1)}}(I_{(r-2)}) \ldots v_{I_1}(C) = \frac{|M[I_r]|}{|C|}.$$

Since $|C| = 1$ and $|M[I_r]| > 2$, we know there exists an a such that $v_{I_a}(I_{(a-1)}) > 1$. Choose a such that it is the first to have this property. Hence $v_{I_1}(C) = v_{I_2}(I_1) = \ldots = v_{I_{a-1}}(I_{a-2}) = 1$. By Lemma 5.2.5 and the fact that $|C| = 1$ it can be shown that

$$|C| = |M[I_1]| = |M[I_2]| = \ldots = |M[I_{a-1}]| = 1 \text{ and } |M[I_a]| > 1.$$

Set $\overline{I} = \{v \in V | \text{path from v to c doesn't include vertices in } M[I_a]\}$. Set $V_0 = \overline{I} \cup M[I_a]$ and set $\mathcal{G}_0$ to be the graph induced by V_0.

Define $\sigma_0 : V_0 \to V_0$ by

$$\sigma_0(x) = \begin{cases} x & : \quad x \in \overline{I} \\ a_{I_a j} & : \quad x = a_{I_a k} \end{cases}$$

with $j = (k+1) \bmod |M[I_a]|$. It is easily verified that σ_0 is one to one on $\mathcal{G}_0$. We verify σ_0 is an automorphism of the subgraph $\mathcal{G}_0$ by checking that if $uv \in E$ then $\sigma_0(u)\sigma_0(v) \in E$. Suppose that $uv \in E$. There are 3 possibilities to consider:

1. $u \in \overline{I}$ and $v \in \overline{I}$, then $\sigma_0(u)\sigma_0(v) = uv \in E$

2. $u \in \overline{I}$ and $v \notin \overline{I}$. Then $v \in M[I_a]$ and $u \in M[I_{a-1}]$ with $|M[I_{a-1}]| = 1$. Since $\mathcal{P}$ is a refined star partition, any w in $M[I_a]$ is adjacent to u. Hence $\sigma_0(u)\sigma_0(v) = u\sigma_0(v) \in E$ since $\sigma_0(v) \in M[I_a]$.

3. $u \in M[I_a]$ and $v \in M[I_a]$. By Lemma 5.2.3, there cannot be an edge between u, v, so this case is not possible.

Therefore σ_0 is an automorphism of $\mathcal{G}_0$. Furthermore we see that σ_0 is a non-trivial automorphism of $\mathcal{G}_0$ since $\sigma_0(a_{I_a0}) \neq a_{I_a0}$.

By Lemma 5.4.1 we can inductively extend σ_0 and build an automorphism, σ, of the whole graph $\mathcal{G}$. Since σ_0 is not the identity on $\mathcal{G}_0$ and $\sigma = \sigma_0$ on V_0, we know that σ is a non-trivial automorphism of the graph.

We know by considering $\sigma^i(a_{I_ak}) = \sigma_0^i(a_{I_ak})$ for some $a_{I_ak} \in M[I_a]$, that $|\sigma| \geq |M[I_a]|$. If $|M[I_a]| > 2$, then we have found a non-trivial automorphism of $\mathcal{G}$ of order greater than 2. Hence $Aut(\mathcal{G}) \ncong Z_2$. If $|M[I_a]| = 2$ and $\sigma \neq \gamma$, then we have found another non-trivial automorphism of the graph and $Aut(\mathcal{G}) \ncong Z_2$. If $\sigma = \gamma$, then we know that $M[I_a] = 2$ and we modify σ slightly as follows to build a second non-trivial automorphism of $\mathcal{G}$.

Use the original labelling of the cells of $\mathcal{P}$ based on the distance from $Z(\mathcal{G})$. There must exist a $M[I_{a+w}]$ for some $w > 0$ such that $v_{I_{a+w}}(I_{a+w-1}) > 1$, since

$$v_{I_r}(I_{(r-1)})v_{I_{(r-1)}}(I_{(r-2)}) \ldots v_{I_1}(C) = \frac{|M[I_r]|}{|C|},$$

with $|C| = 1$, $|M[I_r]| > 2$, and $M[I_a] = 2$. Relabel the cells not in V_0 as in Lemma 5.4.1 by labelling according to the distance from $M[I_a]$. Suppose $M[I_{a+w}] = M[W]$ by the new labelling where $W \in N^w$. We follow the construction in Lemma 5.4.1 and define $\sigma_i' = \sigma_i$ on V_i, for $i < w$. Then set $V_w = V_{w-1} \cup \{$cells that are distance w from $M[I_a]\}$ and set $\mathcal{G}_w$ to be the graph induced by V_w. Define $\sigma_w' : V_w \to V_w$ as follows:

$$\sigma'_w(x) = \begin{cases} \sigma'_{w-1}(x) & : \quad x \in V_{w-1} \\ a_{Fj} & : \quad x = a_{Fk}; M[F] \neq M[W] \\ a_{Wt} & : \quad x = a_{Wm} \end{cases}$$

where $a_{Wm}a_{Hg} \in E$ with $\sigma'_w(a_{Hg}) = \sigma'_{w-1}(a_{Hg}) = a_{Hh}$ and $M[H] \in V_{w-1}$ and

$$m = \left(f + g\frac{|M[W]|}{|M[H]|}\right) \bmod |M[W]|$$

and

$$t = \left((f+1) \bmod \frac{|M[W]|}{|M[H]|} + h\frac{|M[W]|}{|M[H]|}\right) \bmod |M[W]|$$

And

$$k = \left(s + b\frac{|M[F]|}{|M[T]|}\right) \bmod |M[F]|$$

and

$$j = \left(s + p\frac{|M[F]|}{|M[T]|}\right) \bmod |M[F]|$$

where $a_{Fk}a_{Tb} \in E$ with $M[T] \in V_{w-1}$ and $\sigma'_w(a_{Tb}) = a_{Tp}$.

It is easily verified that σ'_w is one to one on $\mathcal{G}_w$. We verify σ'_w is an automorphism of the subgraph $\mathcal{G}_w$ by checking that if $uv \in E$ then $\sigma'_w(u)\sigma'_w(v) \in E$. Suppose that $uv \in E$. There are three possibilities to consider:

1. $u \in V_{w-1}$ and $v \in V_{w-1}$, then $\sigma'_w = \sigma'_{w-1}$ which we already know to preserve adjacency.

2. $u \in M[W]$ and $v \in V_{w-1}$. Suppose $u = A_{Wf} \in M[W]$ and $v = a_{Gg} \in M[G]$. Since $M[G] \in V_{w-1}$, we know that $dist(M[G], Z(\mathcal{G})) < dist(M[W], Z(\mathcal{G}))$. Therefore

$$f = (i + g\frac{|M[W]|}{|M[G]|}) \bmod |M[W]| \text{ for some } 0 \leq i \leq \frac{|M[W]|}{|M[G]|} - 1.$$

Suppose $\sigma'_w(a_{Gg}) = \sigma'_{w-1}(a_{Gg}) = a_{Gh}$. Then $\sigma'_w(a_{Ff}) = a_{Fj}$ where

$$j = ((i+1) \bmod \frac{|M[W]|}{|M[G]|} + h\frac{|M[F]|}{|M[G]|}) \bmod |M[W]|.$$

Thus we see by our labelling that $a_{Gh}a_{Fj} \in E$ and hence $\sigma'_w(u)\sigma'_w(v) \in E$.

3. $u \in M[F] \in N^w$ with $F \neq W$, and $v \in V_{w-1}$. Suppose $u = A_{Ff} \in M[F]$ and $v = a_{Gg} \in M[G]$. Since $M[G] \in V_{w-1}$, we know that $dist(M[G], Z(\mathcal{G})) < dist(M[F], Z(\mathcal{G}))$. Therefore

$$f = (i + g\frac{|M[F]|}{|M[G]|}) \bmod |M[F]| \text{ for some } 0 \leq i \leq \frac{|M[F]|}{|M[G]|} - 1.$$

Suppose $\sigma'_w(a_{Gg}) = \sigma'_{w-1}(a_{Gg}) = a_{Gh}$. Then $\sigma'_w(a_{Ff}) = a_{Fj}$ where

$$j = (i + h\frac{|M[F]|}{|M[G]|}) \bmod |M[F]|.$$

Thus we see by our labelling that $a_{Gh}a_{Fj} \in E$ and hence $\sigma'_w(u)\sigma'_w(v) \in E$.

4. $u \notin V_{w-1}$ and $v \notin V_{w-1}$. Then u, v are both distance w from $M[I_a]$ and therefore are the same distance from $Z(\mathcal{G})$. Hence by Lemma 5.2.3, there cannot be an edge between u, v.

Therefore σ'_w is an automorphism of $\mathcal{G}_w$.

We continue with the induction from Lemma 5.4.1, using σ'_w in place of σ_w producing a non-trivial automorphism σ' that is not equal to σ and therefore not equal to γ. Hence $Aut(\mathcal{G}) \ncong Z_2$ since it contains at least two non-trivial automorphisms.

CASE 4: *There exists $A \in \mathcal{P}$ containing degree 1 vertices with $|A| > 2$, and $|B| \geq 2$ for some $B \in \mathcal{P}$ with A adjacent to B in $\overline{\mathcal{P}}$. There exists $c_0, c_1 \in Z(\mathcal{G})$, with $Z(\mathcal{G}) = \{c_0, c_1\} \in \mathcal{P}$.* Label cells in $\mathcal{P}$ based on their distance from $Z(\mathcal{G})$, so that elements distance r from $Z(\mathcal{G})$ are

$\{M[I] \mid I \in N^r\}$ with $I = (i_1, \ldots, i_r)$. Do this so that if $M[I]$ is adjacent to $M[J]$ distance $r-1$ from $Z(\mathcal{G})$ in $\overline{\mathcal{P}}$ then $J = (i_2, \ldots, i_r)$. Label the vertices in $M[I]$ as $\{a_{Ii} | 0 \leq i \leq |M[I]| - 1\}$, so that a_{Ik} is adjacent to $a_{Jj} \in M[J]$ when

$$k = \left(i + j\frac{|M[I]|}{|M[J]|}\right) \bmod |M[I]| \text{ for } 0 \leq i \leq v_I(J) - 1$$

Define $\sigma : V \to V$ such that

$$\sigma(x) = \begin{cases} c_{(i+1) \bmod 2} & : \quad x = c_i \\ a_{Il} & : \quad x = a_{Ij} \end{cases}$$

where

$$l = \left(j + \frac{|M[I]|}{|Z(\mathcal{G})|} \right) \bmod |M[I]|.$$

It can be easily verified that σ is one-to-one on V. We verify that σ is an automorphism of $\mathcal{G}$ by checking that if $uv \in E$ then $\sigma(u)\sigma(v) \in E$. Let $a_{Ii}a_{Jj} \in E$ with $M[I]$ distance r from $Z(\mathcal{G})$ and $M[J]$ distance $r-1$ from $Z(\mathcal{G})$. Then by our labelling

$$i = \left(k + j\frac{|M[I]|}{|M[J]|} \right) \bmod |M[I]| \text{ for some } 0 \leq k \leq v_I(J) - 1.$$

Then $\sigma(a_{Ii})\sigma(a_{Jj}) = a_{Ig}a_{Jh}$, where

$$\begin{aligned} g &= \left(i + \frac{|M[I]|}{|Z(\mathcal{G})|} \right) \bmod |M[I]| && (5.4.1) \\ &= \left(\left(k + j\frac{|M[I]|}{|M[J]|} \right) + \frac{|M[I]|}{|Z(\mathcal{G})|} \right) \bmod |M[I]| && (5.4.2) \\ &= \left(k + \left(j + \frac{|M[J]|}{|Z(\mathcal{G})|} \right) \frac{|M[I]|}{|M[J]|} \right) \bmod |M[I]| && (5.4.3) \end{aligned}$$

$$h = \left(j + \frac{|M[J]|}{|Z(\mathcal{G})|} \right) \bmod |M[J]|$$

Hence one can see by our labelling that $\sigma(a_{Ii})\sigma(a_{Jj}) \in E$. Since $\sigma(c_0) \neq c_0$, we know that σ is a non-trivial automorphism of $\mathcal{G}$. If $\sigma \neq \gamma$, then $Aut(\mathcal{G}) \ncong Z_2$ since it contains at least two non-trivial automorphisms.

If $\sigma = \gamma$, then define σ' inductively as follows:

Let $A, M[I_{(r-1)}], M[I_{(r-2)}], \ldots, M[I_1], Z(\mathcal{G})$ be the path from A to $Z(\mathcal{G})$ in $\overline{\mathcal{P}}$. Set $A = M[I_{(r)}]$. By Lemma 5.2.5, we know that

$$v_{I_r}(I_{(r-1)})v_{I_{(r-1)}}(I_{(r-2)}) \ldots v_{I_1}(Z(\mathcal{G})) = \frac{|M[I_r]|}{|Z(\mathcal{G})|}.$$

Since $|Z(\mathcal{G})| = 2$ and $|M[I_r]| > 2$, we know there exists an a such that $v_{I_a}(I_{(a-1)}) > 1$. Choose a such that it is the first to have this property. Hence

$$v_{I_{(a-1)}}(I_{((a-2))}) = \ldots = v_{I_1}(Z(\mathcal{G})) = 1.$$

By Lemma 5.2.5 and the fact that $|Z(\mathcal{G})| = 2$, it can be shown that

$$|Z(\mathcal{G})| = |M[I_1]| = \ldots = |M[I_{a-1}]| = 2$$

and $|M[I_a]| > 2$. Set
$\overline{I} = \{v \in V|\text{path from v to c doesn't include vertices in } M[I_a]\}$. Set $V_0 = \overline{I} \cup M[I_a]$.

Define $\sigma_0' : V_0 \to V_0$ by

$$\sigma_0'(x) = \begin{cases} \sigma(x) & : \quad x \in \overline{I} \\ a_{I_a j} & : \quad x = a_{I_a k} \end{cases}$$

where

$$k = \left(\alpha + \beta \frac{|M[I_a]|}{|M[I_{a-1}]|}\right) \mod |M[I_a]|$$

and

$$j = \left((\alpha + 1) \mod \frac{|M[I_a]|}{|M[I_{a-1}]|} + \zeta \frac{|M[I_a]|}{|M[I_{a-1}]|}\right) \mod |M[I_a]|$$

and

$$\sigma(a_{I_{a-1}}\beta) = \sigma_0'(a_{I_{a-1}}\beta) = a_{I_{a-1}\zeta}.$$

It is easy to show that σ_0' is one-to-one on V_0. We verify σ_0' is an automorphism on V_0 by checking that if $uv \in E$ then $\sigma(u)\sigma(v) \in E$. Since $\sigma_0' = \sigma$ for $x \in \overline{I}$, we need only consider when $uv \in E$ and at least one is not in $\overline{I}$. There are 2 possibilities to consider:

1. $u \notin I_a$ and $v \in I_a$. Then $u \in I_{a-1}$ and $u = a_{I_{(a-1)}k}$ for some $0 \leq k \leq |I_{a-1}| - 1$. By our

labelling $v = a_{I_a q}$ where

$$q = \left(i + k\frac{|M[I_a]|}{|M[I_{(a-1)}]|}\right) \bmod |M[I_a]| \text{ where } 0 \leq i \leq v_{I_a}(I_{(a-1)}) - 1.$$

Then $\sigma_0'(u) = \sigma_0'(a_{I_{(a-1)}k}) = \sigma(a_{I_{(a-1)}k}) = a_{I_{(a-1)}w}$ where

$$w = \left(k + \frac{|M[I_{(a-1)}]|}{|Z(\mathcal{G})|}\right) \bmod |M[I_{(a-1)}]|.$$

And $\sigma_0'(v) = \sigma_0'(a_{I_a q}) = a_{I_a s}$ where

$$s = \left((i+1) \bmod \frac{|M[I_a]|}{|M[I_{(a-1)}]|} + w\frac{|M[I_a]|}{|M[I_{(a-1)}]|}\right) \bmod |M[I_a]|.$$

By our labelling we can see that $a_{I_{(a-1)}w}$ is adjacent to $a_{I_a s}$. Hence $\sigma_0'(u)\sigma_0'(v) \in E$.

2. $u \in M[I_a]$ and $v \in M[I_a]$. By Lemma 5.2.3, we know that this is not possible.

Hence σ_0' is an automorphism of V_0
By Lemma 5.4.1 we can inductive extend σ_0' and build an automorphism, σ', of the whole graph $\mathcal{G}$.

And $\sigma' : V \to V$, is an automorphism of graph $\mathcal{G}$. Since $\sigma'(c_0) \neq c_0$, we know that σ' is a non-trivial automorphism of $\mathcal{G}$. Consider vertex $a_{I_a 0}$. $\sigma(a_{I_a 0}) = a_{I_a k}$ where $k = \frac{|M[I_a]|}{|Z(\mathcal{G})|} \bmod |M[I_a]| = \frac{|M[I_a]|}{2} \bmod |M[I_a]|$. $\sigma'(a_{I_a 0}) = a_{I_a l}$ where

$$l = \left(1 + \frac{|M[I_a]|}{|M[I_{a-1}]|}\right) \bmod |M[I_a]| = \left(1 + \frac{|M[I_a]|}{2}\right) \bmod |M[I_a]|.$$

Since $|M[I_a]| > 2$, these reduce to $k = \frac{|M[I_a]|}{2}$ and $l = 1 + \frac{|M[I_a]|}{2}$. We see that $k \neq l$, hence $\sigma(a_{I_a 0}) \neq \sigma'(a_{I_a 0})$. Therefore we have found two non-trivial automorphisms, σ and σ', of the graph $\mathcal{G}$. Hence $Aut(\mathcal{G}) \ncong Z_2$. □

5.5 Structure of Z_2 trees

This section gives some additional requirements the non-trivial automorphism of Z_2 trees must satisfy. The next two lemmas specify characteristics of the non-trivial partition when it contains a cell consisting of exactly one center vertex. The first lemma characterizes the number of vertices in cells containing degree one vertices.

Lemma 5.5.1 *Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong Z_2$. If $\mathcal{P}$ is a non-trivial refined star partition of $\mathcal{G}$ with $C = \{c\} \in \mathcal{P}$ where $c \in Z(\mathcal{G})$, then there is exactly one $A \in \mathcal{P}$ containing degree 1 elements with $|A| = 2$.*

Proof: We know from Theorem 5.4.1 that $\mathcal{G}$ has exactly two refined star partitions. Let $\mathcal{P}$ be the non-trivial refined star partition of $\mathcal{G}$. Since $\mathcal{P}$ is induced by the non-trivial automorphism of $\mathcal{G}$ we know that $|A| \leq 2$ for every $A \in \mathcal{P}$. Suppose there exist $A, B \in \mathcal{P}$ such that both contain degree 1 elements and have $|A| = |B| = 2$. Define

$$\overline{A} = \{W \in \mathcal{P} |\ |W| = 2 \text{ and is part of the path from } A \text{ to } C \text{ in } \overline{P}\}.$$

Given $D \in \mathcal{P}$ with $|D| = 2$ and $x \in D$. We write x' for the other element in D. We further set $x'' = x$. We can define an automorphism $\sigma_A : V \to V$ as follows:

$$\sigma_A(x) = \begin{cases} x & : \quad x \notin \overline{A} \\ x' & : \quad x \in \overline{A} \end{cases}$$

Now we show that σ_A is an automorphism of $\mathcal{G}$. It is easy to see that σ_A is one-to-one. It remains to be shown that if $uv \in E$ then $\sigma_A(u)\sigma_A(v) \in E$. There are three possibilities to consider:

1. $u \notin \overline{A}$ and $v \notin \overline{A}$. Then $\sigma_A(u)\sigma_A(v) = uv \in E$.

2. $u \in \overline{A}$ and $v \in \overline{A}$. Since $\mathcal{P}$ is a refined star partition, if $uv \in E$ then $u'v' \in E$. Thus we know that $\sigma_A(u)\sigma_A(v) = u'v' \in E$.

3. $u \notin \overline{A}$ and $v \in \overline{A}$. Then $\{u\} \in \mathcal{P}$ with $uv \in E$ and $uv' \in E$. Hence $\sigma_A(u)\sigma_A(v) = uv' \in E$.

Since $\sigma_A(a) \neq a$ for $a \in A$ we see that σ_A is a non-trivial automorphism of $\mathcal{G}$.

Similarly, we can define

$$\overline{B} = \{W \in \mathcal{P} |\ |W| = 2 \text{ and is part of the path from } B \text{ to } C \text{ in } \overline{P}\}.$$

and σ_B, so that

$$\sigma_B(x) = \begin{cases} x & : \quad x \notin \overline{B} \\ x' & : \quad x \in \overline{B} \end{cases}$$

By similar arguments to those above it can be shown that σ_B is a non-trivial automorphism of $\mathcal{G}$. Since B is not along the path from A to C in $\mathcal{P}$, $\sigma_A(b) = b$ while $\sigma_B(b) = b'$ for $b \in B$. Hence $\sigma_A \neq \sigma_B$ and there exist two non-trivial automorphisms of $\mathcal{G}$. This contradicts the fact that $Aut(\mathcal{G}) \cong Z_2$. Hence if $\{c\} \in \mathcal{P}$ then there is only one cell A in $\mathcal{P}$ containing vertices of degree 1 such that $|A| = 2$. $\square$

From the proof of the previous lemma we recall that $C = \{c\} \in \mathcal{P}$ where $c \in \mathcal{G}$ and

$$\overline{A} = \{W \in \mathcal{P} || W| = 2 \text{ and is part of the path from } A \text{ to } C \text{ in } \mathcal{P}\}.$$

The following lemma shows that all cells containing two vertices are contained in $\overline{A}$.

Lemma 5.5.2 *Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong Z_2$. Let $\mathcal{P}$ be a non-trivial refined star partition of $\mathcal{G}$ with $C = \{c\} \in \mathcal{P}$ where $c \in Z(\mathcal{G})$. Then all $W \in \mathcal{P}$ with $|W| = 2$ are in $\overline{A}$.*

Proof: Suppose there exists a cell $W \in \mathcal{P}$ with $|W| = 2$ and $W \notin \overline{A}$. From Lemma 5.5.1, we know that if $W \in \mathcal{P}$ with $|W| = 2$, then W does not contain vertices of degree 1. Hence W is along the path from some cell $H = \{h\}$ to C in $\overline{P}$, where $deg(h) = 1$. Choose W to be the cell of $\mathcal{P}$ that satisfies these properties that is adjacent to $H \in \mathcal{P}$ with $dist(H, C) > dist(W, C)$. Suppose $W = \{w, w'\}$. Then $hw \in E$ and $hw' \in E$. Hence there are two paths from h to c one through w and the other through w'. This contracts the fact that $\mathcal{G}$ is a tree. Hence if $\mathcal{P}$ is a non-trivial refined star partition of $\mathcal{G}$ with $C = \{c\} \in \mathcal{P}$ where $c \in Z(\mathcal{G})$, then all $W \in \mathcal{P}$ such that $|W| = 2$ are in $\overline{A}$. $\square$

The following definition applies to the non-trivial refined star partitions of a graph that contain the cell $Z(\mathcal{G}) = \{c, c'\}$.

Definition 5.5.1 *Let $\mathcal{G}$ be a tree with $Z(\mathcal{G}) = \{c, c'\}$. We call $a \in V$ a c-point if $dist(a, c) < dist(a, c')$. Similarly we call $a \in V$ a c'-point if $dist(a, c') < dist(a, c)$.*

The following lemma shows that if the non-trivial refined star partition of a Z_2 tree contains a cell $Z(\mathcal{G}) = \{c, c'\}$ then each cell of this partition contains exactly two vertices. It furthermore shows that each cell of this partition contains exactly one c-point and one c'-point.

Lemma 5.5.3 *Let $\mathcal{G}$ be a tree. If $\mathcal{P}$ is a non-trivial refined star partition of $\mathcal{G}$ with $Z(\mathcal{G}) = \{c, c'\} \in \mathcal{P}$, then all $A \in \mathcal{P}$ have $|A| = 2$. Moreover, if $A \in \mathcal{P}$ then A contains one c-point and one c'-point.*

Proof: Since $\mathcal{P}$ is induced by the non-trivial automorphism of $\mathcal{G}$, we know if $A \in \mathcal{P}$ then $|A| \leq 2$. Suppose there exists an $A \in \mathcal{P}$ such that $|A| = 1$. Choose A such that $dist(A, Z(\mathcal{G}))$ is minimum. Hence A is adjacent to some B in $\overline{\mathcal{P}}$ where $|B| = 2$ and $dist(A, Z(\mathcal{G})) > dist(B, Z(\mathcal{G}))$. Let $B = \{b, b'\}$. Thus $v_B(A) = 2$ since both ab and ab' are edges of the graph. We showed in the proof of Lemma 5.2.4 that this is not possible. Hence if $\mathcal{P}$ is a non-trivial refined star partition of $\mathcal{G}$ with $Z(\mathcal{G}) = \{c, c'\} \in \mathcal{P}$, then all $W \in \mathcal{P}$ have $|W| = 2$.
Let A be an arbitrary cell of $\mathcal{P}$ not equal to $Z(\mathcal{G})$. Let $A, B_0, B_1, \ldots B_s, Z(\mathcal{G})$ be the path in $\overline{\mathcal{P}}$ connecting A and $Z(\mathcal{G})$. Let $B_s = \{b_s, b'_s\}$. From Lemma 5.2.4 and the fact that B_s is adjacent to $Z(\mathcal{G})$ in $\overline{\mathcal{P}}$, we know that b_s is adjacent to exactly one element of $Z(\mathcal{G})$. Without loss of generality assume $b_s c \in E$. Since $\mathcal{P}$ is a refined star partition with $b_s c \in E$ then $b'_s c' \in E$. Hence B_s contains a c-point, b_s, and a c'-point, b'_s. Similarly from Lemma 5.2.4 and the fact that B_{s-1} is adjacent to B_s in $\overline{\mathcal{P}}$, we know that b_{s-1} is adjacent to exactly one element of B_s. Without loss of generality assume $b_{s-1}b_s \in E$. Then $b'_{s-1}b'_s \in E$. Hence B_{s-1} contains a c-point, b_{s-1}, and a c'-point, b'_{s-1}. By continuing in a similar fashion one shows that A contains exactly one c-point and one c'-point. $\Box$

Let $\mathcal{G}$ be a tree and $\mathcal{P}$ be the refined star partition satisfying the conditions of the previous lemma. If $D \in \mathcal{P}$, then by the previous lemma we know that $|D| = 2$ where D contains a c-point and a c'-point. Throughout the rest of this dissertation we will denote the c-point of D as d and the c'-point of D as d'.

5.6 Quantum Automorphism Group of Z_2 trees

In this section we will prove that $QAut(\mathcal{G})$ and $A_{aut}(\mathcal{G})$ are commutative for all trees that have automorphism group isomorphic to Z_2. We first prove some relations involving the X_{ij}'s. The first of these relations proves that $X_{ij} = 0$ if the non-trivial automorphism of the graph does not map i to j.

Lemma 5.6.1 *If $\mathcal{G}$ is a tree with $Aut(\mathcal{G}) \cong Z_2 =< \sigma >$ and $\sigma(a) \neq b$, then $X_{ab} = 0$.*

Proof: Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong Z_2$. Let $\sigma \in Aut(\mathcal{G})$ be the only non-trivial automorphism of $\mathcal{G}$. By Theorem 5.4.1, we know that $\mathcal{G}$ has exactly one non-trivial refined star partition $\mathcal{P}$, the partition induced by σ. Thus if $A \in \mathcal{P}$, then $|A| \leq 2$ and if $a, b \in V$ with $\sigma(a) \neq b$ then a and b are in separate cells of the partition. Hence if we place the partition $\pi = \{1, 2, \ldots n\}$ through the algorithm **Refine** from Section 4.3 it should produce the non-trivial partition $\mathcal{P}$. This is due to the fact that the algorithm will produce the largest refined star partition that is smaller than the input partition. The first time through, the algorithm sorts the vertices according to $v_v(\cdot) = deg(\cdot)$. Note that $w^1_{out}(a, V) = deg(a)$ and $w^1_{out}(a, B) = v_B(a)$ for $a \in V$ and $B \in \mathcal{P}$. We know that if $a \in A$ and $b \notin A$ at the end of the first iteration through the algorithm $deg(a) \neq deg(b)$. Hence by Lemma 4.2.1 $X_{ab} = 0$. The partition $\pi(2)$ satisfies the conditions of Lemma 4.2.1. Hence after the second iteration of the algorithm, if $a \in A$ and $b \in B$ with $A \neq B$ then $X_{ab} = 0$ by Lemma 4.2.1. Each successive iteration of the algorithm produces a partition that satisfies Lemma 4.2.1. Since $\mathcal{G}$ has $\mathcal{P}$ as its only non-trivial refined star partition, the algorithm must produce the partition $\mathcal{P}$. By successive use of Lemma 4.2.1, we see that if $a, b \in V$ with $\sigma(a) \neq b$ then $X_{ab} = 0$. Thus we see if a and b are in separate cells of the partition then $X_{ab} = 0$. □

Lemma 5.6.2 *Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong Z_2 =< \sigma >$ and let $\mathcal{P}$ be the non-trivial refined star partition of the graph. If $A = \{a\} \in \mathcal{P}$, then $X_{aa} = 1$.*

Proof: This is a result of Lemma 5.6.1. □

The next two lemmas give some conditions for when $X_{ii} = X_{jj}$.

Lemma 5.6.3 *If $a \in V$ and $X_{ab} = X_{a'b} = 0$ for every $b \in V$ such that $b \neq a$ and $b \neq a'$, then $X_{aa} = X_{a'a'}$ and $X_{aa'} = X_{a'a}$.*

Proof: Let $a \in V$ with $X_{ab} = X_{a'b} = 0$ for every $b \in V$ such that $b \neq a$ and $b \neq a'$. From Equations 2.5.1, one sees that $X_{aa} + X_{aa'} = 1$. Multiplying both sides of this equation on the right by $X_{a'a'}$ yields:

$$X_{aa}X_{a'a'} + X_{a'a}X_{a'a'} = X_{a'a'}$$

The above equation simplifies to $X_{aa}X_{a'a'} = X_{a'a'}$ after using relation 2.5.1. Similarly, $X_{aa'} + X_{a'a'} = 1$. Multiplying on the left by X_{aa} yields $X_{aa}X_{a'a'} = X_{aa}$ after simplification. Hence $X_{aa} = X_{a'a'}$. Since $X_{aa'} = 1 - X_{aa}$ and $X_{a'a} = 1 - X_{a'a'}$, we know that $X_{aa'} = X_{a'a}$. □

The previous two lemmas show us that if $\{a, a'\} \in \mathcal{P}$ for a refined star partition $\mathcal{P}$ of tree $\mathcal{G}$ that has automorphism group Z_2, then $X_{aa} = X_{a'a'}$.

Lemma 5.6.4 *Suppose that $\mathcal{G}$ is a tree such that $Aut(\mathcal{G}) \cong Z_2 =< \sigma >$ and that $\mathcal{P}$ is the non-trivial refined star partition of graph $\mathcal{G}$ that is induced by σ. If $A_0, A_1, \ldots A_m$ is a path in $\overline{\mathcal{P}}$ with $|A_0| = \ldots = |A_m| = 2$ with $A_s = \{a_s, a'_s\}$, then $X_{a_0a_0} = X_{a'_0a'_0} = \ldots X_{a_ma_m} = X_{a'_ma'_m} =$ and $X_{a_0a'_0} = X_{a'_0a_0} = \ldots X_{a_ma'_m} = X_{a'_ma_m}$.*

Proof: Let $\mathcal{G}$ be a tree such that $Aut(\mathcal{G}) \cong Z_2 =< \sigma >$ and that $\mathcal{P}$ is the non-trivial refined star partition of graph $\mathcal{G}$ that is induced by σ. Since $\mathcal{P}$ is induced by σ we know by Lemma 5.6.1 that if $\sigma(b) \neq a$ then $b \notin A$ and $X_{ab} = X_{ba} = 0$. By Lemma 5.6.3, we know that if $a, a' \in A$ then $X_{aa} = X_{a'a'}$ and $X_{aa'} = X_{a'a}$. Let $A_0, A_1, \ldots A_m$ be a path in $\overline{P}$ with $|A_0| = \ldots = |A_m| = 2$. Suppose that $A_s = \{a_s, a'_s\}$ for $0 \leq s \leq m$. Without loss of generality, suppose that $dist(A_i, \mathcal{G}) > dist(A_{i+1}, \mathcal{G})$ for $0 \leq i \leq m-1$. From relation 2.5.1, we know that

$$X_{a_s a_s} + X_{a_s a'_s} = 1 \text{ for } 0 \leq s \leq m.$$

By Lemma 5.6.3, we know that $X_{a_s a_s} = X_{a'_s a'_s}$ and $X_{a'_s a_s} = X_{a_s a'_s}$ for $0 \leq s \leq m$. Lemma 5.2.4 tells us that $v_{A_s}(A_{s+1}) = 1$. Thus without loss of generality we can relabel the vertices of A_{s+1} so that the only edges between cells A_s and A_{s+1} are $a_s a_{s+1}$ and $a'_s a'_{s+1}$ for $0 \leq s \leq m-1$.

Multiply $X_{a_0 a_0} + X_{a_0 a'_0} = 1$, on the left by $X_{a_1 a_1}$ and then simplify.

$$\begin{aligned} X_{a_1 a_1}(X_{a_0 a_0} + X_{a_0 a'_0}) &= X_{a_1 a_1} \\ X_{a_1 a_1} X_{a_0 a_0} &= X_{a_1 a_1} \end{aligned} \tag{5.6.1}$$

Similarly, we multiply the equation $X_{a_1 a_1} + X_{a_1 a l_1} = 1$ on the right by $X_{a_0 a_0}$ and then simplify we get the following:

$$\begin{aligned} (X_{a_1 a_1} + X_{a_1 a l_1}) X_{a_0 a_0} &= X_{a_0 a_0} \\ X_{a_1 a_1} X_{a_0 a_0} &= X_{a_0 a_0} \end{aligned} \tag{5.6.2}$$

Hence we see that $X_{a_1 a_1} = X_{a'_1 a'_1} = X_{a_0 a_0} = X_{a'_0 a'_0}$. Similarly, one can show that $X_{a_0 a'_0} = X_{a'_0 a_0} = X_{a_1 a'_1} = X_{a'_1 a_1}$.

Continuing in a similar fashion, one shows that

$$X_{a_0 a_0} = X_{a'_0 a'_0} = X_{a_1 a_1} = X_{a'_1 a'_1} = \ldots = X_{a_m a_m} = X_{a'_m a'_m}$$

and

$$X_{a_0 a'_0} = X_{a_0 a'_0} = X_{a_1 a'_1} = X_{a_1 a'_1} = \ldots = X_{a_m a'_m} = X_{a_m a'_m}. \Box$$

The following theorem shows that $QAut(\mathcal{G}) \cong A_{aut}(\mathcal{G})$ and is commutative for trees that have automorphism group congruent to Z_2.

Theorem 5.6.1 *If $\mathcal{G}$ is a tree with $Aut(\mathcal{G}) \cong Z_2$, then both $QAut(\mathcal{G})$ and $A_{aut}(\mathcal{G})$ are commutative. Therefore, for such a tree $\mathcal{G}$, $QAut(\mathcal{G}) \cong A_{aut}(\mathcal{G}) \cong \mathcal{A}(\mathcal{G})$.*

Proof: Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong Z_2$. Let $\sigma \in Aut(\mathcal{G})$ be the only non-trivial automorphism of $\mathcal{G}$. By Theorem 5.4.1, we know that $\mathcal{G}$ has exactly one non-trivial refined star partition $\mathcal{P}$, the partition induced by σ. By Lemma 5.6.1, we know that $X_{ab} = 0$ if $\sigma(a) \neq b$.

There are two cases to consider:
Case 1: $Z(\mathcal{G}) = \{c, c'\} \in \mathcal{P}$.
By Lemma 5.5.3, we know that all $A \in \mathcal{P}$ have $|A| = 2$. Since $\mathcal{P}$ is induced by σ we know that if $\sigma(b) \neq a$ then $b \notin A$ and $X_{ab} = X_{ba} = 0$. By Lemma 5.6.3, we know that if $a, a' \in A$ then $X_{aa} = X_{a'a'}$ and $X_{aa'} = X_{a'a}$. Let $A \in \mathcal{P}$, $A \neq Z(\mathcal{G})$, be an arbitrary cell of $\mathcal{P}$. Let $A, B_s, \ldots, B_0, Z(\mathcal{G})$ be a path in $\overline{P}$. From Lemma 6.4.6, we know that $X_{aa} = X_{a'a'} = X_{b_sb_s} = X_{b'_sb'_s} = \ldots X_{cc} = X_{c'c'}$ and $X_{aa'} = X_{aa'} = X_{b_sb'_s} = X_{b'_sb_s} = \ldots X_{cc'} = X_{cc'}$.

Hence for arbitrary $A \in \mathcal{P}$, $X_{cc} = X_{aa} = X_{a'a'}$ and $X_{cc'} = X_{aa'} = X_{a'a}$. Therefore $QAut(\mathcal{G}) =< X_{cc}, X_{cc'} >$ where $X_{cc}X_{cc'} = 0 = X_{cc'}X_{cc}$. From this we see that $QAut(\mathcal{G})$ is commutative. Since $A_{aut}(\mathcal{G})$ is a quotient of $QAut(\mathcal{G})$, $A_{aut}(\mathcal{G})$ is commutative.

Case 2: $C = \{c\} \in \mathcal{P}$ where $c \in Z(\mathcal{G})$
By Lemma 5.5.1, we know that there is exactly one cell of $\mathcal{P}$ containing degree 1 vertices that contains two vertices. Call this cell A. By Lemma 5.5.2, we know that all cells containing two vertices are in $\overline{A}$. Consider a cell containing degree 1 vertices that is not equal to A. Call this cell B. We know that $|B| = 1$. Let $B, J_1, \ldots J_s, C$ be the path from B to C in $\overline{\mathcal{P}}$. By repeated use of Lemma 5.2.4, we see that $|B| = |J_1| = \ldots = |J_s| = |C| = 1$. By Lemma 5.6.2, we know that $X_{ff} = 1$ if $\{f\} \in \mathcal{P}$. By Lemma 5.6.1 we know if a and b are in different cells of $\mathcal{P}$ then $X_{ab} = 0$. There are two possibilities to consider:

- Let $A, I_1, \ldots I_m, J_1, \ldots J_s, C$ be the path in $\overline{P}$ from A to C where $|J_1| = 1$ and $|I_m| = 2$ then by Lemma 5.2.4 it can be shown that $|I_1| = \ldots = |I_m| = 2$ and $|J_1| = \ldots = |J_s| = |C| = 1$. Thus the only cells in $\mathcal{P}$ of order two are $A, I_1, \ldots I_m$ and this is a path in $\overline{\mathcal{P}}$.

- Let $A, I_1, \ldots I_m, C$ be the path in $\overline{P}$ from A to C where $|I_m| = 2$ then by Lemma 5.2.4 it can be shown that $|I_1| = \ldots = |I_m| = 2$. thus the only cells in $\mathcal{P}$ of order two are $A, I_1, \ldots I_m$ where this is a path in $\overline{\mathcal{P}}$.

Thus we know that $|A| = |I_1| = \ldots = |I_m| = 2$ where $A, I_1, \ldots I_m$ is a path in $\overline{\mathcal{P}}$ and all other cells of $\mathcal{P}$ have only one element. Set $A = I_0$ and $I_s = \{i_s, i'_s\}$. By Lemma 5.6.3, we know that $X_{i_0 i_0} = X_{i'_0 i'_0} = \ldots = X_{i_m i_m} = X_{i'_m i'_m}$ and $X_{i_0 i'_0} = X_{i'_0 i_0} = \ldots X_{i_m i'_m} = X_{i'_m i_m}$. Thus $QAut(\mathcal{G}) =< X_{i_0 i_0}, X_{i_0 i'_0} >$. Since $X_{i_0 i_0} X_{i_0 i'_0} = X_{i_0 i'_0} X_{i_0 i_0} = 0$, $QAut(\mathcal{G})$ is commutative and thus the quotient $A_{aut}(\mathcal{G})$ is also commutative. The second assertion follows immediately from the fact that the relations for $QAut(\mathcal{G}), A_{aut}(\mathcal{G})$, and $\mathcal{A}(\mathcal{G})$ differ only in their commuting relations. □

5.7 Quantum Automorphism Group of Z_2 graphs

We recall the example of the graph in Figure 5.4 with automorphism group isomorphic to Z_2, but having at least three refined star partitions.

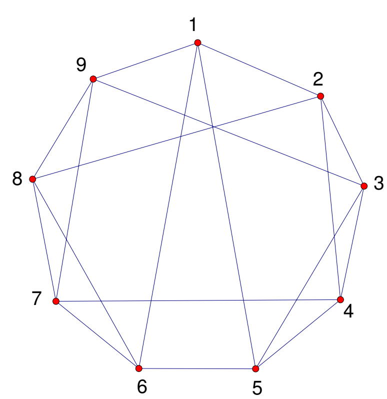

Figure 5.4: Z_2 graph

In this example it can be shown that $QAut(\mathcal{G})$ and $A_{aut}(\mathcal{G})$ are commutative. So it is unknown whether Theorem 5.6.1 can be generalized to show that arbitrary undirected graphs with automorphism group isomorphic to Z_2 have commutative quantum automorphism groups.

Chapter 6

$(Z_2)^l$ Trees

In this chapter, we extend the result from the last chapter and obtain a classification, in terms of refined star partitions, of all trees satisfying $Aut(\mathcal{T}) \cong (Z_2)^l$. We will use this classification to prove that the quantum automorphism group of trees with automorphism group congruent to $(Z_2)^l$ with $l > 1$ is non-commutative.

6.1 $(Z_2)^l$ Graphs

We first show that any graph having exactly 2^l refined star partitions in which each cell of these partitions contains no more than two vertices has automorphism group $(Z_2)^l$.

Theorem 6.1.1 *Let $\mathcal{G}$ be a graph which has exactly 2^l refined star partitions. If each of these partitions consist of cells containing no more than two vertices, then $Aut(\mathcal{G}) \cong (Z_2)^l$.*

Proof: Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{(2^l)}$ denoted the refined star partitions of graph $\mathcal{G}$. Since all cells of partition $\mathcal{P}_i$ contain less than or equal to two vertices, we know from Lemma 5.3.1 that each $\mathcal{P}_i$ induces an automorphism of the graph of order 2 for $2 \leq i \leq 2^l$. We can also see that the trivial partition induced the trivial automorphism. Denote by σ_i the automorphism induced by the partition $\mathcal{P}_i$. Thus we know there are at least 2^l distinct automorphisms of $\mathcal{G}$. From 5.1.1, we know that all automorphisms of $\mathcal{G}$ induce refined star partitions. Hence there are exactly 2^l distinct automorphisms of the graph each or order less than or equal to 2. Thus we know that the group formed by these automorphisms must be isomorphic to $(Z_2)^l$.□

6.2 Classification of $(Z_2)^l$ trees

The following proof is very similar to the proof of Theorem 5.4.1. It is included separately to aid the reader's understanding since it is more tedious than the proof of Theorem 5.4.1.

Theorem 6.2.1 *If $\mathcal{G}$ is a tree and $Aut(\mathcal{G}) \cong (Z_2)^l$ then there are exactly 2^l refined star partitions of $\mathcal{G}$.*

Proof: Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$. Let $\gamma_i \in Aut(\mathcal{G})$ for $1 \leq i < 2^l$ and $|\gamma_i| = 2$ be the non-trivial automorphisms of $\mathcal{G}$. Since we know from Lemma 5.1.1 that the orbits of an automorphism of $\mathcal{G}$ are a refined star partition, we know there exist at least $2^l - 1$ non-trivial refined star partitions of $\mathcal{G}$. Suppose there exists another non-trivial refined star partition of $\mathcal{G}$. Call this partition $\mathcal{P}$.

CASE 1: *Each $A \in \mathcal{P}$ containing degree 1 vertices has $|A| \leq 2$.*

By Lemma 5.3.2, we know that all $B \in \mathcal{P}$ have $|B| \leq 2$. From Lemma 5.3.1, we know that the partition $\mathcal{P}$ induces a $\sigma \in Aut(\mathcal{G})$ with $|\sigma| = 2$. And the automorphism σ induces the refined star partition $\mathcal{P}$. The automorphisms γ_i and σ for each $1 \leq i \leq 2^l - 1$ must differ since the refined star partitions they induce differ. Thus there are at least 2^l elements in $Aut(\mathcal{G})$ of order 2, so $Aut(\mathcal{G}) \ncong (Z_2)^l$.

CASE 2: *There exists $A \in \mathcal{P}$ containing degree 1 vertices with $|A| > 2$, and $|B| = 1$ for some B in $\mathcal{P}$ with A adjacent to B in $\overline{\mathcal{P}}$.*

Let $A = \{a_0, a_1, \ldots, a_{m-1}\}$ with $m > 2$ and $B = \{b\}$. Define

$$\sigma(x) = \begin{cases} x & : \quad x \notin A \\ a_{(i+1) \bmod m} & : \quad x = a_i \end{cases}$$

Now we show that σ is an automorphism of $\mathcal{G}$. It is easy to show that σ is one-to-one on the vertices of $\mathcal{G}$. All that remains is to show if $uz \in E$ then $\sigma(u)\sigma(z) \in E$. There are three possibilities to consider. Let $uz \in E$.

1. $u \in A$ and $z \in A$. From Lemma 5.2.3 we see this is only possible in a connected tree if $u, z \in Z(\mathcal{G})$. Since $|Z(\mathcal{G})| \leq 2$ and $|A| > 2$, this is not possible.

2. $u \in A$ and $z \notin A$. It follows that $z = b$ and $\sigma(u)\sigma(b) = \sigma(u)b$. Since $\mathcal{P}$ is a refined star partition and $\{b\} \in \mathcal{P}$, then all vertices in A are adjacent to b. Hence $\sigma(u)\sigma(b) = \sigma(u)b \in E$, since $\sigma(u) \in A$.

3. $u \notin A$ and $z \notin A$. It follows that $\sigma(u)\sigma(z) = uz \in E$

Thus we have shown that σ is an automorphism of $\mathcal{G}$. By calculating $\sigma^i(a_1)$, we see that $|\sigma| = m > 2$. Hence $Aut(\mathcal{G}) \ncong (Z_2)^l$ since it contains an element, σ, of order greater than 2.

Recall the comment after Lemma 5.5.3, thus a refined star partition will either have one cell consisting of two elements of the center or at least one cell consisting of just one element of the center.

CASE 3: *There exists $A \in \mathcal{P}$ containing degree 1 vertices with $|A| > 2$, and $|B| \geq 2$ for some $B \in \mathcal{P}$ with A adjacent to B in $\overline{\mathcal{P}}$. There exists $c \in Z(\mathcal{G})$, with $C = \{c\} \in \mathcal{P}$.*

We will build an automorphism of $\mathcal{G}$ inductively using Lemma 5.4.1. If this automorphism is the same as one of the γ_i then we will modify it slightly and produce another automorphism distinct from the γ_i. Label elements in $\mathcal{P}$ based on their distance from C, so that cells distance r from C are $\{M[I] \ |I \in N^r\}$ with $I = (i_1, \ldots, i_r)$. Do this so that if $M[I]$ is adjacent to $M[J]$ distance $r-1$ from C in $\overline{\mathcal{P}}$ then $J = (i_2, \ldots, i_r)$. Label the vertices in $M[I]$ as $\{a_{Ii} | 0 \leq i \leq |M[I]| - 1\}$, so that a_{Ik} is adjacent to $a_{Jj} \in M[J]$ when

$$k = \left(i + j\frac{|M[I]|}{|M[J]|}\right) \mod |M[I]| \text{ for } 0 \leq i \leq v_I(a_{J0}) - 1.$$

Let $A, M[I_{(r-1)}], M[I_{(r-2)}], \ldots, M[I_1], C$ be the path from A to C in $\overline{\mathcal{P}}$. Set $A = M[I_r]$. By Lemma 5.2.5, we know that

$$v_{I_r}(I_{(r-1)})v_{I_{(r-1)}}(I_{(r-2)}) \ldots v_{I_1}(C) = \frac{|M[I_r]|}{|C|}.$$

Since $|C| = 1$ and $|M[I_r]| > 2$, we know there exists an a such that $v_{I_a}(I_{(a-1)}) > 1$. Choose a such that it is the first to have this property. Hence

$$v_{I_1}(C) = v_{I_2}(I_1) = \ldots = v_{I_{a-1}}(I_{a-2}) = 1.$$

By 5.2.5 and the fact that $|C| = 1$ it can be shown that $|C| = |M[I_1]| = |M[I_2]| = \ldots = |M[I_{a-1}]| = 1$ and $|M[I_a]| > 1$. Set $\overline{I} = \{v \in V|\text{path from v to c doesn't include vertices in } M[I_a]\}$. Set $V_0 = \overline{I} \cup M[I_a]$.

Define $\sigma_0 : V_0 \to V_0$ by

$$\sigma_0(x) = \begin{cases} x & : & x \in \overline{I} \\ a_{I_a j} & : & x = a_{I_a k} \end{cases}$$

with $j = (k+1) \bmod |M[I_a]|$.

It is easily verified that σ_0 is one to one on V_0. We verify σ_0 is an automorphism of the subgraph V_0 by checking that if $uv \in E$ then $\sigma_0(u)\sigma_0(v) \in E$. Suppose that $uv \in E$. There are 3 possibilities to consider:

1. $u \in \overline{I}$ and $v \in \overline{I}$, then $\sigma_0(u)\sigma_0(v) = uv \in E$
2. $u \in \overline{I}$ and $v \notin \overline{I}$. Then $v \in M[I_a]$ and $u \in M[I_{a-1}]$ with $|M[I_{a-1}]| = 1$. Since $\mathcal{P}$ is a refined star partition, any w in $M[I_a]$ is adjacent to u. Hence $\sigma_0(u)\sigma_0(v) = u\sigma_0(v) \in E$ since $\sigma_0(v) \in M[I_a]$.
3. $u \in M[I_a]$ and $v \in M[I_a]$. By Lemma 5.2.3, there cannot be an edge between u, v, so this case is not possible.

Therefore σ_0 is an automorphism of V_0. Furthermore we see that σ_0 is a non-trivial automorphism of $\mathcal{G}_0$ since $\sigma_0(a_{I_a 0}) \neq a_{I_a 0}$.

Using Lemma 5.4.1, we can extend σ_0 to form σ, an automorphism of the whole graph. We know by considering $\sigma^i(a_{I_a k})$ for some $a_{I_a k} \in M[I_a]$, that $|\sigma| \geq |M[I_a]|$. Hence if $|M[I_a]| > 2$, then we have found a non-trivial automorphism of $\mathcal{G}$ of order larger than 2. Hence if $|M[I_a]| > 2$, then $Aut(\mathcal{G}) \ncong Z_2$. If $|M[I_a]| = 2$ and $\sigma \neq \gamma_i$ for $1 \leq i \leq 2^l - 1$, then we have found another non-trivial automorphism of the graph and $Aut(\mathcal{G}) \ncong (Z_2)^l$. If $|M[I_a]| = 2$ and $\sigma = \gamma_i$ for some $1 \leq i < 2^l$, then we modify σ slightly as follows:

Use the original labelling of the cells of $\mathcal{P}$. There must exist a $M[I_{a+w}]$ for some $b > 0$ such that $v_{I_{a+w}}(I_{a+w-1}) > 1$, since

$$v_{I_r}(I_{(r-1)})v_{I_{(r-1)}}(I_{(r-2)}) \ldots v_{I_1}(C) = \frac{|M[I_r]|}{|C|},$$

$|C| = 1$, and $|M[I_r]| > 2$. Relabel the cells not in V_0 as before by labelling them according to the distance from $M[I_a]$. Suppose $M[I_{a+w}] = M[W]$ by the new labelling where $W \in N^w$. Let $M[H]$ be the cell adjacent to $M[W]$ with $dist(M[W], Z(\mathcal{G})) > dist(M[H], Z(\mathcal{G}))$. Define $\sigma'_i = \sigma_i$, for $i < w$. Then define $\sigma'_w : V_w \to V_w$ as follows:

$$\sigma'_w(x) = \begin{cases} \sigma'_{w-1}(x) & : \quad x \in V_{w-1} \\ a_{Fj} & : \quad x = a_{Fk}; M[F] \neq M[W] \\ a_{Wt} & : \quad x = a_{Wm} \text{ with } 0 \leq m < v_W(M[H]) \\ a_{Ws} & : \quad x = a_{Wr} \text{ with } r \geq v_W(M[H]) \end{cases}$$

where $a_{Wm}a_{Hg} \in E$ with $\sigma'_w(a_{Hg}) = \sigma'_{w-1}(a_{Hg}) = a_{Hh}$ and $M[H] \in V_{w-1}$ and

$$m = \left(f + g\frac{|M[W]|}{|M[H]|}\right) \bmod |M[W]| \text{ for } 0 \leq f \leq \frac{|M[W]|}{|M[H]|} - 1$$

and

$$t = \left((f+1) \bmod v_H(M[W]) + h\frac{|M[W]|}{|M[H]|}\right) \bmod |M[W]|$$

And $a_{Wr}a_{Ha} \in E$ with $\sigma'_w(a_{Ha}) = \sigma'_{w-1}(a_{Ha}) = a_{Hb}$ and $M[H] \in V_{w-1}$ and

$$r = \left(f + a\frac{|M[W]|}{|M[H]|}\right) \bmod |M[W]| \text{ for } 0 \leq f \leq \frac{|M[W]|}{|M[H]|} - 1$$

and

$$s = \left(f + b\frac{|M[W]|}{|M[H]|}\right) \bmod |M[W]|$$

And

$$k = \left(s + b\frac{|M[F]|}{|M[T]|}\right) \bmod |M[F]|$$

and

$$j = \left(s + p\frac{|M[F]|}{|M[T]|}\right) \bmod |M[F]|$$

where $a_{Fk}a_{Tb} \in E$ with $M[T] \in V_{w-1}$ and $\sigma'_w(a_{Tb}) = a_{Tp}$.

Since σ'_w just permutes vertices within a cell, it can be easily verified that σ'_w is one to one on $\mathcal{G}_w$. We verify σ'_w is an automorphism of the subgraph $\mathcal{G}_w$ by checking that if $uv \in E$ then $\sigma'_w(u)\sigma'_w(v) \in E$. Suppose that $uv \in E$. There are four possibilities to consider:

1. $u \in V_{w-1}$ and $v \in V_{w-1}$. For vertices in V_{w-1} $\sigma'_w = \sigma'_{w-1}$ thus we already know that

$\sigma'_w(u)\sigma'_w(v) \in E$.

2. $u \in M[W]$, and $v \in V_{w-1}$. Suppose $v = a_{Hh} \in M[H]$ where $\sigma'_w(a_{Hh}) = \sigma'_{w-1}(a_{Hh}) = a_{Hi}$. Let $u = a_{Ww}$ then by our labelling

$$w = \left(b + h\frac{|M[W]|}{|M[H]|}\right) \bmod |M[W]|$$

with $0 \le b < v_H(M[W])$. If $\sigma'_w(a_{Ww}) = a_{Wy}$ then

$$y = \begin{cases} \left(b + i\frac{|M[W]|}{|M[H]|}\right) \bmod |M[W]| & : \quad 0 \le w < v_H(M[W]) \\ \left((b+1) \bmod v_H(M[W]) + i\frac{|M[W]|}{|M[H]|}\right) \bmod |M[W]| & : \quad w \ge v_H(M[W]) \end{cases}$$

Thus one can see by our labelling of the vertices that $a_{Hi}a_{Wy} \in E$. Hence $\sigma'_w(a_{Hh})\sigma'_w(a_{Ww}) \in E$

3. $u \in M[F] \in N^w$ with $F \neq W$, and $v \in V_{w-1}$. Let $v = a_{Dd} \in M[D]$ with $\sigma'_w(a_{Dd}) = a_{De}$. Let $u = a_{Ff}$ then by our labelling

$$f = \left(b + d\frac{|M[F]|}{|M[D]|}\right) \bmod |M[F]|.$$

If $\sigma'_w(a_{Ff}) = a_{Ff}$ then

$$g = \left(b + e\frac{|M[F]|}{|M[D]|}\right) \bmod |M[F]|.$$

Thus we see by our labelling that $a_{De}a_{Fg} \in E$. Hence $\sigma'_w(a_{Dd})\sigma'_w(a_{Ff}) \in E$.

4. $u \notin V_{w-1}$ and $v \notin V_{w-1}$. Then u, v are both distance w from $M[I_a]$ and therefore are the same distance from $Z(\mathcal{G})$. Hence by Lemma 5.2.3, there cannot be an edge between u, v.

Therefore σ'_w is an automorphism of $\mathcal{G}_w$.

We continue with the induction from Lemma 5.4.1, using σ'_w in place of σ_w producing a non-trivial automorphism σ'. The following properties of σ' are easy to check:

$$\sigma'(a_{W0}) \neq a_{W0} \tag{6.2.1}$$

$$(\sigma')^2(a_{W0}) \neq a_{W0}. \tag{6.2.2}$$

Thus we see that $|\sigma'| > 2$. Hence $Aut(\mathcal{G}) \ncong (Z_2)^l$ since it contains an automorphism of order greater than two.

CASE 4: *There exist $A \in \mathcal{P}$ containing degree one vertices with $|A| > 2$, and $|B| \geq 2$ for some $B \in \mathcal{P}$ adjacent to B in $\overline{\mathcal{P}}$. There exists $c_0, c_1 \in V$, centers of the tree, with $Z(\mathcal{G}) = \{c_0, c_1\} \in \mathcal{P}$.*

Label elements in $\mathcal{P}$ based on their distance from $Z(\mathcal{G})$, so that elements distance r from $Z(\mathcal{G})$ are $\{M[I] \ |I \in N^r\}$ with $I = (i_1, \dots, i_r)$. Do this so that if $M[I]$ is adjacent to $M[J]$ distance $r-1$ from $Z(\mathcal{G})$ in $\overline{\mathcal{P}}$ then $J = (i_2, \dots, i_r)$. Label the vertices in $M[I]$ as $\{a_{Ii} | 0 \leq i \leq |M[I]| - 1\}$, so that a_{Ik} is adjacent to $a_{Jj} \in M[J]$ when

$$k = \left(i + j\frac{|M[I]|}{|M[J]|}\right) \bmod |M[I]| \text{ for } 0 \leq i \leq v_I(J) - 1.$$

Define $\sigma : V \to V$ such that

$$\sigma(x) = \begin{cases} c_{(i+1) \bmod 2} & : \quad x = c_i \\ a_{Il} & : \quad x = a_{Ij} \end{cases}$$

where

$$l = \left(j + \frac{|M[I]|}{|Z(\mathcal{G})|}\right) \bmod |M[I]|.$$

We verify that σ is an automorphism of $\mathcal{G}$ by checking that it is one-to-one on V and that if $uv \in E$ then $\sigma(u)\sigma(v) \in E$. It can be easily verified that σ is one-to-one on V. Let $a_{Ii}a_{Jj} \in E$ with $M[I]$ distance r from $Z(\mathcal{G})$ and $M[J]$ distance r-1 from $Z(\mathcal{G})$. Then by our labelling

$$i = \left(k + j\frac{|M[I]|}{|M[J]|}\right) \bmod |M[I]| \text{ for some } 0 \leq k \leq v_I(J) - 1.$$

Then $\sigma(a_{Ii})\sigma(a_{Jj}) = a_{Ig}a_{Jh}$, where

$$g = \left(i + \frac{|M[I]|}{|Z(\mathcal{G})|}\right) \bmod |M[I]| \tag{6.2.3}$$

$$= \left(\left(k + j\frac{|M[I]|}{|M[J]|}\right) + \frac{|M[I]|}{|Z(\mathcal{G})|}\right) \bmod |M[I]| \tag{6.2.4}$$

$$= \left(k + \left(j + \frac{|M[J]|}{|Z(\mathcal{G})|}\right)\frac{|M[I]|}{|M[J]|}\right) \bmod |M[I]| \tag{6.2.5}$$

$$h = \left(j + \frac{|M[J]|}{|Z(\mathcal{G})|}\right) \bmod |M[J]| \tag{6.2.6}$$

Hence $\sigma(a_{Ii})\sigma(a_{Jj}) \in E$ and σ is a non-trivial automorphism of $\mathcal{G}$. If $\sigma \neq \gamma_1$ for $1 \leq i < 2^l$, then $Aut(\mathcal{G}) \ncong (Z_2)^l$ since it contains 2^l automorphisms of order 2.
If $\sigma = \gamma_i$ for some i such that $1 \leq i < 2^l$, then define σ' as follows:

Let $A, M[I_{(r-1)}], M[I_{(r-2)}], \ldots, M[I_1], Z(\mathcal{G})$ be the path from A to $Z(\mathcal{G})$ in $\overline{\mathcal{P}}$. Set $A = M[I_{(r)}]$. By Lemma 5.2.5, we know that

$$v_{I_r}(I_{(r-1)})v_{I_{(r-1)}}(I_{(r-2)})\ldots v_{I_1}(Z(\mathcal{G})) = \frac{|M[I_r]|}{|Z(\mathcal{G})|}.$$

Since $|Z(\mathcal{G})| = 2$ and $|M[I_r]| > 2$, we know there exists an a such that $v_{I_a}(I_{(a-1)}) > 1$. Choose a such that it is the first to have this property. Set
$\overline{I} = \{v \in V|\text{path from v to c doesn't include vertices in } M[I_a]\}$. Set $V_0 = \overline{I} \cup M[I_a]$.

Define $\sigma_0' : V_0 \to V_0$ by

$$\sigma_0'(x) = \begin{cases} \sigma(x) & : \quad x \in \overline{I} \\ a_{I_a j} & : \quad x = a_{I_a k} \text{ where } 0 \leq k < v_{I_{a-1}}(I_a) \\ a_{I_a s} & : \quad x = a_{I_a t} \text{ where } v_{I_{a-1}}(I_a) \leq t \leq |M[I_a]| - 1 \end{cases}$$

where

$$k = \left(\alpha + \beta\frac{|M[I_a]|}{|M[I_{a-1}]|}\right) \bmod |M[I_a]| \text{ for } 0 \leq k < v_{I_{a-1}}(I_a) \text{ and some } 0 \leq \alpha < v_{I_{a-1}}(I_a)$$

and

$$j = \left((\alpha+1) \bmod \frac{|M[I_a]|}{|M[I_{a-1}]|} + \zeta\frac{|M[I_a]|}{|M[I_{a-1}]|}\right) \bmod |M[I_a]|$$

and

$$\sigma(a_{I_{a-1}}\beta) = \sigma_0'(a_{I_{a-1}}\beta) = a_{I_{a-1}}\zeta.$$

Also

$$t = \left(\delta + \phi\frac{|M[I_a]|}{|M[I_{a-1}]|}\right) \bmod |M[I_a]| \text{ for some } 0 \leq t < v_{I_{a-1}}(I_a) \text{ and some } 0 \leq \delta < v_{I_{a-1}}(I_a)$$

and

$$s = \left(\delta + \xi\frac{|M[I_a]|}{|M[I_{a-1}]|}\right) \bmod |M[I_a]|$$

where

$$\sigma(a_{I_{a-1}\phi}) = \sigma_0'(a_{I_{a-1}\phi}) = a_{I_{a-1}\xi}.$$

The following properties of σ' are easy to check:

$$\sigma'(a_{I_a 0}) \neq a_{I_a 0} \tag{6.2.7}$$

$$(\sigma')^2(a_{I_a 0}) \neq a_{I_a 0}. \tag{6.2.8}$$

Thus we see that $|\sigma'| > 2$. Hence $Aut(\mathcal{G}) \ncong (Z_2)^l$ since it contains an automorphism of order greater than two. $\square$

The following example shows that if $\mathcal{G}$ is not a tree then the converse of Theorem 6.1.1 does not hold. The graph in Figure 6.1 has automorphism group isomorphic to $(Z_2)^2$, but has more than four refined star partitions.

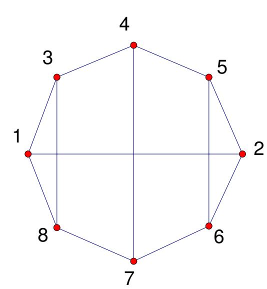

Figure 6.1: $(Z_2)^2$ graph

These refined star partitions include:

- the trivial partition
- $\{\{1, 2, 3, 4, 5, 6, 7, 8\}\}$

- $\{\{1,2\},\{3,5\},\{4\},\{6,8\},\{7\}\}$
- $\{\{1\},\{2\},\{3,8\},\{4,7\},\{5,6\}\}$
- $\{\{1,2\},\{3,6\},\{4,7\},\{5,8\}\}$

A regular graph with automorphism group congruent to $(Z_2)^l$ will have at least 2^l+1 refined star partitions: the trivial partition, the partition consisting of one cell containing all of the vertices of the graph, and 2^l-1 partitions induced by the non-trivial automorphisms of the graph.

6.3 Structure of $(Z_2)^l$ trees

The lemmas in this section will give some additional characteristics about the 2^l-1 non-trivial refined star partitions of a tree with automorphism group $(Z_2)^l$. The first lemma shows that a cell containing two vertices from the center is not in any of the 2^l-1 refined star partitions of the graph.

Lemma 6.3.1 *If $\mathcal{G}$ is a tree with automorphism group congruent to $(Z_2)^l$ for $l>1$, then there does not exist a refined star partition of $\mathcal{G}$, $\mathcal{P}$, such that $\{c_0,c_1\}\in\mathcal{P}$ where $Z(\mathcal{G})=\{c_0,c_1\}$.*

Proof: Suppose that $\mathcal{G}$ has a refined star partition, $\mathcal{P}$, induced by the automorphism σ such that $\{c_0,c_1\}\in\mathcal{P}$ for $c_0,c_1\in Z(\mathcal{G})$. From Lemma 5.5.3, we know that all cells $A\in\mathcal{P}$ have $|A|=2$. Since $Aut(\mathcal{G})\cong(Z_2)^l$ for $l>1$, we know there exists another non-trivial refined star partition, $\mathcal{P}'$ induced by the automorphism σ'. There are two cases to consider.
Case 1: All cells $A\in\mathcal{P}'$ have $|A|=2$. Then by Lemma 5.2.1 and the fact that trees have at most two center vertices we see that $Z(\mathcal{G})=\{c_0,c_1\}\in\mathcal{P}'$. Since $\mathcal{P}$ and $\mathcal{P}'$ are distinct partitions there exists a vertex, a, such that $a\in A\in\mathcal{P}$ and $a\in A'\in\mathcal{P}'$, but $A\neq A'$. Choose a such that $dist(a,Z(\mathcal{G}))$ is as small as possible. Suppose that $A=\{a,b\}$, $A'=\{a,d\}$, and $B=\{d,e\}\in\mathcal{P}$. Since $\mathcal{P}$ is induced by σ and $\mathcal{P}'$ is induced by σ', we know that $\sigma'\circ\sigma$ is an automorphism of order two. By a simple calculation of $\sigma'\circ\sigma(b)$ and the fact that $\sigma'\circ\sigma$ has order two, one discovers that $\sigma'(e)=b$. Thus $B'=\{b,e\}\in\mathcal{P}'$. Define $\overline{AB}$ to be the cells in $\mathcal{P}$ whose paths to the center of $\mathcal{G}$ do not go through cells A or B. Define $\overline{A'B'}$ to be the cells in $\mathcal{P}'$ whose paths to the center of $\mathcal{G}$ do not go through cells A' or B'. Since $A\cup B$ and $A'\cup B'$ contain the same vertices from $\mathcal{G}$, we know that $\overline{AB}$ and $\overline{A'B'}$ contain

the same vertices of $\mathcal{G}$. Since vertex a is minimal with respect to its distance from $Z(\mathcal{G})$ and $\sigma(b) = a$ and $\sigma'(b) = e$, vertices a and e must be adjacent to the same vertex v such that $dist(v, Z(\mathcal{G})) < dist(a, Z(\mathcal{G}))$. Similarly, vertices b and d are adjacent to the same vertex w such that $dist(w, \mathcal{G}) < dist(b, \mathcal{G})$. Thus we see that $\sigma(v) = w$, $\sigma(w) = v$, $\sigma'(v) = w$, and $\sigma'(w) = v$. Set $R = \{$ cells in $\overline{AB} \cup A \cup B\}$. We will define an automorphism on the vertices in R and then extend it using Lemma 5.4.1 to form an automorphism of the whole graph $\mathcal{G}$. Define τ_0 as follows:

$$\tau_0(x) = \begin{cases} \sigma(x) & : \quad x \in \overline{AB} \\ b & : \quad x = a \\ e & : \quad x = b \\ d & : \quad x = e \\ a & : \quad x = d \end{cases}$$

It is easy to see that τ is one-to-one on the vertices in R since σ is one-to-one on the vertices in $\overline{AB}$. Using Lemma 5.4.1 and Remark 5.4.2, we extend τ_0 to form τ an automorphism of the graph $\mathcal{G}$. We see by calculating $\tau^i(a) = \tau_0^i(a)$ that $|\tau| > 2$. Thus $Aut(\mathcal{G}) \ncong (Z_2)^l$ since it contains an automorphism, τ, of order larger than two and we see this case is not possible.
Case 2: There exists a cell $A \in \mathcal{P}'$ such that $|A| = 1$. Thus by Lemma 5.5.3 $\{c_0, c_1\} \notin \mathcal{P}'$ and $\sigma'(c_0) = c_0$. Since $Aut(\mathcal{G})$ is a group, we know that $\sigma \circ \sigma' \in Aut(\mathcal{G})$ and it induces, $\overline{\mathcal{P}}$, a refined star partition of the graph. Since $\sigma \circ \sigma'(c_0) = c_1$, we know that $\overline{\mathcal{P}}$ contains the cell $\{c_0, c_1\}$ and thus we know that all $A \in \overline{\mathcal{P}}$ have $|A| = 2$. Thus we have two refined star partition $\mathcal{P}$ and $\overline{\mathcal{P}}$ both with the property that all cells contain exactly two vertices. Hence case 1 of this lemma applies and we see that there will be an element in $Aut(\mathcal{G})$ of order larger than two. Thus this case is not possible.
We have just shown that if $\mathcal{G}$ is a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$ for $l > 1$, then there does not exist a refined star partition of $\mathcal{G}$, $\mathcal{P}$, such that $\{c_0, c_1\} \in \mathcal{P}$ where $c_0, c_1 \in Z(\mathcal{G})$.□

It follows from the previous lemma that every refined star partition of the tree $\mathcal{G}$ with $Aut(\mathcal{G}) \cong (Z_2)^l$ will contain a cell, $\{c\}$, for a vertex $c \in Z(\mathcal{G})$. The next lemma shows the possible relationship between cells $A \in \mathcal{P}$ and $B \in \mathcal{P}'$ where $\mathcal{P}$ and $\mathcal{P}'$ are both refined star partitions.

Lemma 6.3.2 *Let $\mathcal{G}$ be a tree with automorphism group congruent to $(Z_2)^l$ for $l > 1$. If $\mathcal{P}$ and $\mathcal{P}'$ are non-trivial refined star partitions of $\mathcal{G}$ with $A \in \mathcal{P}$ and $A' \in \mathcal{P}'$ then either $A \subseteq A'$, $A' \subseteq A$, or $A \cap A' = \emptyset$.*

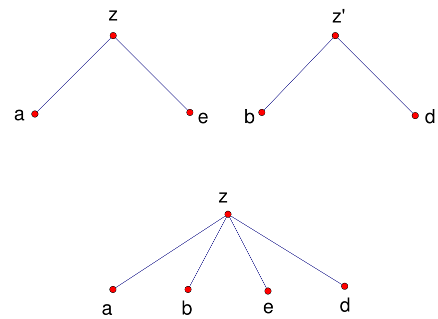

Figure 6.2: Lemma 6.3.2

Proof: Let $\mathcal{P}$ be induced by the automorphism σ and $\mathcal{P}'$ be induced by the automorphism σ'. From Lemma 6.3.1 we know that $C = \{c\} \in \mathcal{P}$ and $C' = \{c\} \in \mathcal{P}'$ for some $c \in Z(\mathcal{G})$. Note if $Z(\mathcal{G}) = \{c, c'\}$ then $\{c\}$ and $\{c'\}$ are both cells in $\mathcal{P}$ and $\mathcal{P}'$. Suppose that there exist cells $A \in \mathcal{P}$ and $A' \in \mathcal{P}'$ with $A \not\subseteq A'$, $A' \not\subseteq A$, and $A \cap A' \neq \emptyset$. Thus we see that $|A| = |A'| = 2$ with $A = \{a, b\} \in \mathcal{P}$ and $A' = \{a, d\} \in \mathcal{P}'$ for $b, d \in V$ such that $b \neq d$. Choose cells $A \in \mathcal{P}$ and $B \in \mathcal{P}'$ such that $dist(A, C) = dist(A', C')$ is as small as possible. Let $Z \in \mathcal{P}$ be the cell such that Z is adjacent to A in $\overline{\mathcal{P}}$ with $dist(Z, Z(\mathcal{G})) < dist(A, Z(\mathcal{G}))$. Since we know that $\sigma' \circ \sigma(b) = d$, some simple calculations along with that fact that $|\sigma' \circ \sigma| = 2$ show that $\{d, e\} \in \mathcal{P}$ and $\{b, e\} \in \mathcal{P}'$ for some $e \in V$. Since A is the cell closest to $Z(\mathcal{G})$ in $\overline{\mathcal{P}}$ with this property, we see that $v_Z(A) \geq 2$. Thus either $|Z| = 2$ or $|Z| = 1$. The subgraph of $\mathcal{G}$ containing the vertices a, b, d, e and the vertices that are in Z must be isomorphic to one of the subgraph pictured in 6.2

We will build a non-trivial automorphism on a subgraph of $\mathcal{G}$ and then extend it inductively using Lemma 5.4.1. The consider the two cases mentioned above:

Case 1: $|Z| = 2$. Set $z = z_0$ and $z' = z_1$, so that $Z = \{z_0, z_1\}$.

Suppose H is a subgraph of $\mathcal{G}$ such that the vertices in H consist of the vertices in cell Z along with the vertices in cells of $\mathcal{P}$ whose path to the center does not include cell Z. We define an automorphism, σ_0, of H as follows:

$$\sigma_0(x) = \begin{cases} x & : \quad x \notin Z \\ z_{(a+1) \bmod 2} & : \quad x = z_a \in Z \end{cases}$$

It is easy to see that σ_0 is one-to-one and to check that σ_0 preserves adjacency. Thus σ_0 is an automorphism of subgraph H. Let σ_i be the automorphisms from Lemma 5.4.1 used to inductively build an automorphism of the whole graph from σ_0. Suppose $A = M[A]$ with $a = a_{A0}, e = a_{A1}$ and $B = M[B]$ with $b = a_{B0}, d = a_{B1}$ and $A, B \in \mathbf{N}$ by the labelling from Lemma 5.4.1. We will modify σ_1 and call the new automorphism σ_1' and then continue with the induction in the lemma using Remark 5.4.1 to produce an automorphism of the whole graph σ'.

$$\sigma_1'(x) = \begin{cases} \sigma_1(x) & : \quad a \in V_0 \\ a_{Bi} & : \quad x = a_{Ai} \in M[A] \\ a_{A(i+1) \bmod 2} & : \quad x = a_{Bi} \in M[B] \end{cases}$$

Again we see that it is easy to check that σ_1' is one-to-on and that it preserves adjacency on V_1. Thus we can use Lemma 5.4.1 and Remark 5.4.1 to extend σ_1' to σ', an automorphism of the whole graph. Since $\sigma'(a_{A0}) = \sigma_1'(a_{A0}) = a_{B0}$ and $\sigma'(a_{B0}) = \sigma_1'(a_{B0}) = a_{A1}$ we see that $|\sigma'| > 2$. Thus this case in not possible since $Aut(\mathcal{G}) \cong (Z_2)^l$ forcing all automorphisms of the graph have order less than or equal to two.

Case 2: $|Z| = 1$. Set $Z = \{z\}$

Suppose H is a subgraph of $\mathcal{G}$ such that the vertices in H consist of the vertices in cells $Z, A,$ and B along with the vertices in cells of $\mathcal{P}$ whose path to the center does not include cells A or B. Suppose $A = M[A]$ with $a = a_{A0}, e = a_{A1}$ and $B = M[B]$ with $b = a_{B0}, d = a_{B1}$ by the labelling from Lemma 5.4.1. We define an automorphism, σ_0, of H as follows:

$$\sigma_0(x) = \begin{cases} x & : \quad x \in H, x \notin A, x \notin B \\ a_{Ai} & : \quad x = a_{Bi} \\ a_{B(i+1) \bmod 2} & : \quad x = a_{Ai} \end{cases}$$

We see that it is easy to check that σ_0 is one-to-one and that it preserves adjacency on V_0. Thus we can use Lemma 5.4.1 and Remark 5.4.2 to extend σ_0 to σ an automorphism of the whole graph. Since $\sigma(a_{A0}) = \sigma_0(a_{A0}) = a_{B0}$ and $\sigma(a_{B0}) = \sigma_0(a_{B0}) = a_{A1}$ we see that $|\sigma| > 2$. Thus this case in not possible since $Aut(\mathcal{G}) \cong (Z_2)^l$, forcing all automorphisms of the graph to have order less than or equal to two.

Thus we have shown that if $\mathcal{P}$ and $\mathcal{P}'$ are non-trivial refined star partitions of $\mathcal{G}$ with $A \in \mathcal{P}$

and $A' \in \mathcal{P}'$ then either $A \subseteq A'$, $A' \subseteq A$, or $A \cap A' = \emptyset$.$\square$

Suppose that σ and σ' are both automorphisms of $\mathcal{G}$ that respectively induce $\mathcal{P}$ and $\mathcal{P}'$. By Lemma 6.3.2, we see that if $\{a, a'\} \in \mathcal{P}$ then either $\{a, a'\} \in \mathcal{P}'$ or $\{a\} \in \mathcal{P}'$ and $\{a'\} \in \mathcal{P}'$. In particular, if $\sigma(a) = a'$ for some $a, a' \in V$ then either $\sigma'(a) = a$ or $\sigma'(a) = a'$.

6.4 Quantum Automorphism Group of $(Z_2)^l$ trees

In this section, we will prove some additional information about the non-trivial refined star partitions as well as the generators of the quantum automorphism groups for trees with automorphism group $(Z_2)^l$. These lemmas will be used to prove that the quantum automorphism group of a $(Z_2)^l$ tree for $l \geq 2$ is non-commutative. We first prove a result about the smallest partitions that will result in the knowledge that each non-trivial cell of each partition will be in exactly one smallest partition of the graph. Recall that if $\mathcal{P}$ and $\mathcal{P}'$ are both refined star partitions of a graph then $\mathcal{P}$ is a smaller refined star partition if $\mathcal{P}$ is a refinement of $\mathcal{P}'$. Similarly, $\mathcal{P}'$ is larger than $\mathcal{P}$ if $\mathcal{P}$ is a refinement of $\mathcal{P}'$.

Lemma 6.4.1 *Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$ with $l \geq 2$. Let $\mathcal{P}$ be a non-trivial refined star partition of $\mathcal{G}$ with $C = \{c\} \in \mathcal{P}$ and $c \in Z(\mathcal{G})$. Suppose there exist cells A and B in $\mathcal{P}$ containing degree one elements with $|A| = |B| = 2$ and the paths from A to C and from B to C do not share any cells D such that $|D| = 2$. Then there is a smaller partition, $\mathcal{P}'$, of the graph. Moreover, the partition constructed in this lemma is the smallest refined star partition containing A.*

Proof: From Theorem 6.2.1, we know that all refined star partitions of $\mathcal{G}$ are induced by automorphisms of $\mathcal{G}$. Let $A, A_1, \ldots, A_{s-1}, C$ be the path in $\overline{\mathcal{P}}$ from A to C. Set $A_0 = A$ and $A_s = C$. We know from Lemma 5.2.5 that $v_{A_0}(A_1) \ldots v_{A_{s-1}}(A_2) = \frac{|A_0|}{|C|}$. Since $|A_0| = 2$ and $|C| = 1$ there must exist an A_i such that $v_{A_i}(A_{i+1}) = 2$. Thus from Lemma 5.2.5, we see that $|A_0| = |A_1| = \ldots = |A_i| = 2$ and $|A_{i+1}| = \ldots = |A_s| = 1$. Set $\overline{A}$ to contain the cells in $\mathcal{P}$ whose path to C does not include $A_0, A_1, \ldots, A_i$. Note that $B \in \overline{A}$. Label the vertices in A_i as $a_{A_i 0}, a_{A_i 1}$. We will use Lemma 5.4.1 to inductively build an automorphism of the graph with τ_0 as the initial automorphism. Set $V_0 = \{\overline{A} \cup A_i\}$ We define $\tau_0 : V_0 \to V_0$ as follows:

$$\tau_0(x) = \begin{cases} x & : \quad x \in \overline{A} \\ a_{A_i (j+1) \bmod 2} & : \quad x = a_{A_i j} \end{cases}$$

It is easy to see that τ_0 is one-to-one on V_0 and that it preserves adjacency. Use Lemma 5.4.1 to inductively build the automorphism τ on the whole graph $\mathcal{G}$. From the hypothesis, we know that the path from B to C does not involve $A_0, A_1, \ldots, A_i$. Thus $B = \{b_0, b_1\}$ is in $\overline{A}$ and τ fixes the vertices in B. Hence we see that the refined star partition, $\mathcal{P}'$, induced by τ will have $\{b_0\} \in \mathcal{P}'$ as well as $\{b_1\} \in \mathcal{P}'$. One can see that $\mathcal{P}'$ is a smaller partition of $\mathcal{G}$ since it is a refinement of $\mathcal{P}$.

We argue that the partition, $\mathcal{P}'$, is the smallest refined star partition of the graph that contains A_0. Suppose there is a smaller refined star partition, $\mathcal{P}''$, such that $A_0 = \{a_0, a_1\} \in \mathcal{P}''$. Then by the properties of a refined star partition $A_1, A_2, \ldots, A_i$ all must be in the refined star partition. Thus $\mathcal{P}'$ is the smallest refined star partition containing A_0. □

The previous lemma shows that for every cell A in some refined star partition of $\mathcal{G}$ such that $|A| = 2$ there will be a smallest non-trivial refined star partitions of the graph containing cell A. Since every cell A such that $A = \{a, a'\}$ is in exactly one smallest refined star partition and for every refined star partition, $\mathcal{P}$, of $\mathcal{G}$ either $\{a, a'\} \in \mathcal{P}$ or $\{a\} \in \mathcal{P}$ and $\{a'\} \in \mathcal{P}$, we know there must be l such smallest non-trivial refined star partitions. Denote these smallest non-trivial refined star partitions as $\mathcal{P}_1, \ldots \mathcal{P}_l$ which are generated by $\sigma_1, \ldots \sigma_l$, respectively. Thus if $A \in \mathcal{P}_i$ with $|A| = 2$ for $1 \leq i \leq l$, then $A \notin \mathcal{P}_j$ for $j \neq i$.

The following lemma defines the largest refined star partition of a graph in terms of the partitions $\mathcal{P}_1, \ldots, \mathcal{P}_l$ that resulted from Lemma 6.4.1.

Lemma 6.4.2 *Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$ with $l \geq 2$. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots \mathcal{P}_l$ be the smallest non-trivial refined star partition of $\mathcal{G}$ and σ_i the automorphism that induced $\mathcal{P}_i$. The partition, $\mathcal{P}$, induced by $\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_l$ is the largest refined star partition of $\mathcal{G}$.*

Proof: Set $\mathcal{P}$ to be the refined star partition induced by $\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_l$. Let $A \in \mathcal{P}_i$ for $1 \leq i \leq l$ such that $A = \{a, a'\}$. Then $A \notin \mathcal{P}_j$ for $j \neq i$. Thus $\sigma_i(a) = a'$ and $\sigma_j(a) = a$ and $\sigma_1 \circ \ldots \circ \sigma_l(a) = a'$. Hence we see that $A \in \mathcal{P}$. If B is a cell of some refined star partitions of $\mathcal{G}$ with $|B| = 2$, then B is in one of the smallest refined star partition of $\mathcal{G}$. Thus $B \in \mathcal{P}$. We see that any refined star partition, $\mathcal{P}'$, is a refinement of $\mathcal{P}$. Hence $\mathcal{P}$ is the largest refined star partition of $\mathcal{G}$. □

The following lemma shows that $X_{ab} = 0$ if there is no automorphism that sends vertex a to vertex b.

Lemma 6.4.3 *Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong (Z_2)^l = < \sigma_1, \ldots, \sigma_l >$ and $\sigma_i(a) \neq b$ for every $1 \leq i \leq l$, then $X_{ab} = 0$.*

Proof: Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$. By Theorem 6.2.1 we know that $\mathcal{G}$ has exactly $2^l - 1$ non-trivial refined star partitions, the partitions induced by the non-trivial automorphisms of $\mathcal{G}$. Thus if $\mathcal{P}$ is a refined star partition then all cells $A \in \mathcal{P}$ have $|A| \leq 2$. If $a, b \in V$ with $\gamma(a) \neq b$ for every $\gamma \in Aut(\mathcal{G})$ then $\sigma_i(a) \neq b$ for each $1 \leq i \leq l$. Thus a and b are in separate cells of every refined star partition of $\mathcal{G}$. From [8], we know if we place the partition $\pi = \{1, 2, \ldots n\}$ through the algorithm **Refine** it will produce $\mathcal{P}$ the largest non-trivial partition of $\mathcal{G}$. We saw from Lemma 6.4.2 that $\mathcal{P}$ will be the partition induced by $\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_l$. The first time through, the algorithm sorts the vertices according to $v_v(\cdot) = deg(\cdot)$. Note that $w^1_{out}(a, V) = deg(a)$ and $w^1_{out}(a, B) = v_B(a)$ for $a \in V$ and $B \in \mathcal{P}$. We know that if $a \in A$ and $b \notin A$ at the end of the first iteration through the algorithm $deg(a) \neq deg(b)$. Hence by Lemma 4.2.1 $X_{ab} = 0$. The partition $\pi(2)$ satisfies the conditions of Lemma 4.2.1. Hence after the second iteration of the algorithm, if $a \in A$ and $b \in B$ with $A \neq B$ then $X_{ab} = 0$ by Lemma 4.2.1. Each successive iteration of the algorithm produces a partition that satisfies Lemma 4.2.1. Since $\mathcal{G}$ has $\mathcal{P}$ as its largest non-trivial refined star partition, the algorithm must produce the partition $\mathcal{P}$. By successive use of Lemma 4.2.1, we see if a and b are in separate cells of the partition, $\mathcal{P}$, then $X_{ab} = 0$. $\square$

Let $\mathcal{G}$ is a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$ for $l \geq 2$ and let $A \in \mathcal{P}$ for some refined star partition, $\mathcal{P}$, with $A = \{a, a'\}$. From Lemma 6.3.2, we see that for any refined star partition, $\mathcal{P}'$, of $\mathcal{G}$ either $\{a, a'\} \in \mathcal{P}'$ or $\{a\} \in \mathcal{P},$ and $\{a'\} \in \mathcal{P}'$. Thus we see from the previous lemma and Lemma 5.6.3 that $X_{aa} = X_{a'a'}$ if there exists a refined star partition such that $\{a, a'\}$ is a cell in the partition.

The next two lemmas give some conditions for when $X_{ii} \neq X_{jj}$.

Lemma 6.4.4 *Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$ with $l \geq 2$. Let $\mathcal{P}$ be a non-trivial refined star partition of $\mathcal{G}$ with $C = \{c\} \in \mathcal{P}$ and $c \in Z(\mathcal{G})$. Suppose there exist cells $A = \{a_0, a_1\}$ and $B = \{b_0, b_1\}$ in $\mathcal{P}$ such that the paths from A to C and from B to C do not share any cells D such that $|D| = 2$. Then $X_{a_0a_0} \neq X_{b_0b_0}$.*

Proof: From Lemma 6.4.1, we know there exists a smallest refined star partition with A in the partition and B not in the partition. Also there exists a smallest refined star partition with B in the partition and A not in the partition. Thus since the spectrum of $QAut(\mathcal{G})$ is $Aut(\mathcal{G})$ we know that $X_{a_0a_0} \neq X_{b_0b_0}$. $\square$

Lemma 6.4.5 *Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$ with $l \geq 2$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_l$ be the smallest refined star partitions resulting from Lemma 6.4.1. If $A = \{a, a'\}$ and $B = \{b, b'\}$ are respectively non-trivial cells in partitions $\mathcal{P}_i$ and $\mathcal{P}_j$ with $i \neq j$, then $X_{aa} \neq X_{bb}$ $X_{aa} \neq X_{b'b'}$.*

Proof: This is a result of Lemma 6.4.4. □

The next lemma shows that $X_{aa} = X_{bb}$ for vertices $a \in A$ and $b \in B$ such that the paths from cells A and B share a common cell D where $|D| = 2$.

Lemma 6.4.6 *Let $\mathcal{G}$ be a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$ for $l \geq 2$. Let $\mathcal{P}$ be a smallest refined star partition containing cells $A = \{a, a'\}$ and $C = \{c\}$ where $c \in Z(\mathcal{G})$. Let $B = \{b, b'\}$ be a cell in $\mathcal{P}$ such that the paths from A to C and from B to C share at least one cell D such that $|D| = 2$. Then $X_{aa} = X_{bb}$.*

Proof: Let $A, A_1, \ldots, A_{s-1}, C$ be the path in $\overline{\mathcal{P}}$ from A to C. Set $A_0 = A$ and $A_s = C$. We know from Lemma 5.2.5 that $v_{A_0}(A_1) \ldots v_{A_{s-1}}(A_s) = \frac{|A_0|}{|A_s|}$. Since $|A_0| = 2$ and $|A_s| = 1$ there must exist an A_i such that $v_{A_i}(A_{i+1}) = 2$. Thus from Lemma 5.2.5, we see that $|A_0| = |A_1| = \ldots = |A_i| = 2$ and $|A_{i+1}| = \ldots = |A_s| = 1$. If the path from B to C and the path from A to C share at least one cell, D, such that $|D| = 2$ then the path from B to C must contain A_i. Set $A_j = \{a_j, a'_j\}$ for $0 \leq j \leq i$. Thus we see that $X_{a_j a_j} = X_{a'_j a'_j}$.

Due to Lemma 5.2.4, we can relabel the vertices of A_{s+1} so that the only edges between cells A_s and A_{s+1} are $a_s a_{s+1} \in E$ and $a'_s a'_{s+1} \in E$ for $0 \leq s \leq i$.

Multiply $X_{a_i a_i} + X_{a_i a'_i} = 1$, on the left by $X_{a_{i-1} a_{i-1}}$ and then simplify.

$$\begin{aligned} X_{a_{i-1}a_{i-1}}(X_{a_i a_i} + X_{a_i a'_i}) &= X_{a_{i-1}a_{i-1}} \\ X_{a_{i-1}a_{i-1}} X_{a_i a_i} &= X_{a_{i-1}a_{i-1}} \end{aligned} \tag{6.4.1}$$

Similarly, we multiply the equation $X_{a_{i-1}a_{i-1}} + X_{a_{i-1}a'_{i-1}} = 1$ on the right by $X_{a_i a_i}$ and after simplification we get the following:

$$\begin{aligned} (X_{a_{i-1}a_{i-1}} + X_{a_{i-1}a'_{i-1}}) X_{a_i a_i} &= X_{a_i a_i} \\ X_{a_{i-1}a_{i-1}} X_{a_i a_i} &= X_{a_i a_i} \end{aligned} \tag{6.4.2}$$

Hence we see that $X_{a_{i-1}a_{i-1}} = X_{a'_{i-1}a'_{i-1}} = X_{a_i a_i} = X_{a'_i a'_i}$. Using the identities $X_{a_i a_i} + X_{a_i a'_i} = 1$ and $X_{a_{i-1}a_{i-1}} + X_{a_{i-1}a'_{i-1}} = 1$, one can show that $X_{a_i a'_i} = X_{a'_i a_i} = X_{a_{i-1}a'_{i-1}} = X_{a'_{i-1}a_{i-1}}$. Continuing in a similar fashion, we will eventually show that $X_{a_0 a_0} = X_{a_i a_i}$. Since the path from $B = \{b, b'\}$ to C includes A_i we can repeat the above argument for this path and show that $X_{bb} = X_{b'b'} = X_{a_ia_i} = X_{a'_ia'_i}$. Thus any cell,$D = \{d, d'\}$, of $\mathcal{P}$ whose path to C involves A_i will have $X_{dd} = X_{d'd'} = X_{a_ia_i} = X_{a'_ia'_i}$.$\square$

The above lemma shows us that if $A = \{a, a'\}$ and $B = \{b, b'\}$ are cells in the smallest refined star partition $\mathcal{P}_i$, then $X_{aa} = X_{a'a'} = X_{bb} = X_{b'b'}$.

We prove that $QAut(\mathcal{G})$ and $A_{aut}(\mathcal{G})$ are both non-commutative for trees with $Aut(\mathcal{G}) \cong (Z_2)^l$ for $l \geq 2$ in the following theorem.

Theorem 6.4.1 *If $\mathcal{G}$ is a tree with $Aut(\mathcal{G}) \cong (Z_2)^l$ with $l \geq 2$, then $QAut(\mathcal{G})$ is non-commutative. Moreover, $A_{aut}(\mathcal{G})$ is isomorphic to $QAut(\mathcal{G})$.*

Proof: Let $\mathcal{G}$ be a tree such that $Aut(\mathcal{G}) \cong (Z_2)^l$ for $l \geq 2$. We know from Lemma 6.2.1 that the only refined star partitions of the graph are those induced by the 2^l automorphisms of the graph. Recall from Lemma 6.4.2 that the largest partition of $\mathcal{G}$ is the one induced by $\sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_l$ where $\sigma_1, \sigma_2, \ldots \sigma_l$ are the automorphisms defined in the proof of Lemma 6.4.1. Set $\mathcal{P}$ to be this largest partition of $\mathcal{G}$. If D is a cell in $\mathcal{P}_i$, the refined star partition induced by σ_i, with $|D| = 2$ then D is a cell in $\mathcal{P}$. Let $A \in \mathcal{P}_1$ with $A = \{a_0, a_1\}$. Let $B \in \mathcal{P}_2$ with $B = \{b_0, b_1\}$.

Since these partitions are induced by the smallest partitions σ_1 and σ_2 respectively, we know from Lemma 6.4.1 that $A \notin \mathcal{P}_2$ and $B \notin \mathcal{P}_1$. By the construction of these smallest refined star partitions, we know that the path from A to C and the path from B to C do not share any cells that contain two elements. Thus we know from Lemma 6.4.4 that $X_{a_0a_0} \neq X_{b_0b_0}$. Let A_{ij} with $1 \leq j \leq a_j$ be all the cells of $\mathcal{P}_i$ containing two elements labelled so that the path from A_{ij} to C involves A_{i1}. Suppose $A_{ij} = \{a_{ij}, a'_{ij}\}$. Then by Lemma 6.4.6, we know that $X_{a_{i1}a_{i1}} = X_{a'_{i1}a'_{i1}} = \ldots = X_{a_{ia_j}a_{ia_j}} = X_{a'_{ia_j}a'_{ia_j}}$ for $1 \leq i \leq l$. Thus we know $QAut(\mathcal{G}) =< X_{a_{11}a_{11}}, X_{a_{11}a'_{11}}, \ldots, X_{a_{l1}a_{l1}}, X_{a_{l1}a'_{l1}} >$. We show $QAut(\mathcal{G})$ is non-commutative by finding an algebra homomorphism $\pi : QAut(\mathcal{G}) \to M_2(\mathbf{C})$. Define π as follows:

$$\pi(X_{a_{i1}a_{i1}}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ for } 1 \leq i < l$$

$$\pi(X_{a_{i1}a'_{i1}}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } 1 \leq i < l$$

$$\pi(X_{a_{l1}a_{l1}}) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\pi(X_{a_{l1}a'_{l1}}) = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

The calculation to check that π is an algebra homomorphism are easy to check since the only relations to check are:

$$X_{i1} + X'_{i1} = 1 \ \ 1 \leq i \leq l$$

$$X_{i1}X'_{i1} = 0 \ \ 1 \leq i \leq l.$$

Since $\pi(X_{a_{i1}a_{i1}}X_{a_{l1}a_{l1}}) \neq \pi(X_{a_{l1}a_{l1}}X_{a_{i1}a_{i1}})$, we see that $QAut(\mathcal{G})$ is non-commutative for trees $\mathcal{G}$ such that $Aut(\mathcal{G}) \cong (Z_2)^l$ with $l \geq 2$.

Since there are no edge relations between $X_{11}, X'_{11}, \ldots, X_{l1}, X'_{l1}$, we see that $QAut(\mathcal{G}) = A_{aut}(\mathcal{G})$. Thus $A_{aut}(\mathcal{G})$ is non-commutative and equal to $QAut(\mathcal{G})$ for trees with automorphism group congruent to $(Z_2)^l$ for $l \geq 2$. □

6.5 Quantum Automorphism Group of $(Z_2)^l$ graphs

We recall the example of the graph in Figure 6.3 with automorphism group isomorphic to $(Z_2)^2$, but having at least five refined star partitions.

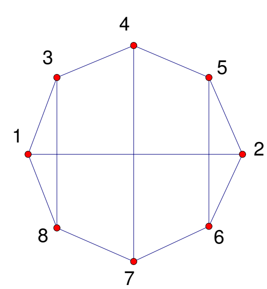

Figure 6.3: $(Z_2)^2$ graph

In this example it can be shown that $QAut(\mathcal{G})$ is non-commutative while $A_{aut}(\mathcal{G})$ is commutative. Thus we have found an example showing that Theorem 6.4.1 does not extent to graphs that are not trees but have $Aut(\mathcal{G}) \cong (Z_2)^l$ for $l \geq 2$.

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