Exploring Payoffs and Beliefs in Game Theory

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Abstract

This dissertation explores the importance of the payoff structure and beliefs for noncooperative games.

Chapter 2 considers instances where the payoffs and the objectives may not be very clear to the players or the analyst. We develop a model for analyzing such games using a vector of *reference utilities* which are included in the definition of the game and assumed to be shared by all players. These are used to obtain the *true utilities* in the game. Conditions for the existence of equilibrium are identified by allowing players to have beliefs about the others. The solution concept is applied to the Traveler's Dilemma and a duopoly.

In Chapter 3 a non-cooperative model of network formation is developed. Agents form links based on the cost of the link and its assessed benefit. Link formation is one-sided, i.e., agents can initiate links with other agents without their consent, provided the agent forming the link makes the appropriate investment. The model builds on the work of Bala and Goyal, but allows for agent heterogeneity by allowing for different failure probabilities. We investigate Nash networks that exhibit connectedness and redundancy and provide an explicit characterization of star networks. Efficiency and Pareto-optimality issues are discussed through examples. We also explore the consequences of three alternative specifications which address criticisms of such communication networks.

Chapter 4 examines noncooperative fuzzy games. Both in fuzzy noncooperative games and in abstract economies, players impose mutual restrictions on their available strategic choices. Here we combine these two theories: A player tries to minimize the restrictions he imposes on others, while respecting the restrictions imposed by others on him, but does not explicitly pursue any other objectives of his own. We establish existence of an equilibrium in this framework.

In Chapter 5 normal form game is modeled using tools from fuzzy set theory. We extend the decision theory framework of Bellman and Zadeh (1970) to a game-theoretic setting. The formulation is preliminary with some results and examples.

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Chapter 1

Introduction

Game theory is the study of mathematical models of conflict and cooperation between intelligent individual decision makers. The term game theory is somewhat of a misnomer since game theory goes well beyond recreational activities like parlor games. It models interactive decision making. A *game* refers to any social situation involving two or more individuals who are called players. The rules governing interaction between the two players are well defined and their joint actions or strategies lead to outcomes in the game which are also a part of the description of the game. Often, though not always game theory assumes that the players in a game are *rational*, i.e., each player pursues her own self interested objective. Game theorists try to understand cooperation and conflict by studying quantitative models and hypothetical examples formulated in the abstract using the terminology mentioned above.

Modern game theory may be said to begin with the work of Zermelo (1913), Borel (1921), von Neumann (1928) and von Neumann and Morgenstern (1944). The next major development was John Nash's modification of the von Neumann and Morgenstern approach. Nash (1950) formally defined an equilibrium of a noncooperative game to be a profile of strategies, one for each player in the game, such that each player's strategy maximizes his expected utility payoff against the given strategies of the other players.[1]

[1]A notable precursor to the work of Nash was Cournot (1838). He constructed a theory of duopolistic

The impact of Nash's reconstruction of game theory spread slowly, raising with it a host of new questions. Among the major contributors to game theory following the work of Nash were Reinhard Selten and John Harsanyi. Selten (1965, 1975) showed that for many games, normal-form (simultaneous move games) Nash equilibrium can sometimes generate too many equilibria, including some that seem implausible and absurd when examined in the extensive form (sequential move games). This led to the entire equilibrium refinements literature. Harsanyi (1967-68) introduced the formal modeling of uncertainty into game theory by introducing Bayesian game models which opened the door for models of incomplete information and their many different applications.

The formulation of Nash equilibrium and the subsequent developments in game theory have had a fundamental and pervasive impact on economics and social sciences (Myerson, 1999). Economic interactions often involve inter-dependent decision making and are inherently game-theoretic. Game theory has provided economics with a rich set of tools to describe strategic interactions and for predicting what will happen in economic contexts entirely on the basis of theory. It has provided economics with a clear and precise language for communicating and insights and intuitions and the ability to transfer those from one context to another. It allows economics to subject particular insights and notions to a rigorous test of logical consistency. By providing a systematic way for abstraction, game theory also allows us trace back from 'observations' to assumptions in order to understand what behavioral hypotheses lie at the heart of particular conclusions.

At the same time economic applications have inspired extensions and refinements in game theory and have transformed game theory from a branch of mathematics with a primarily normative focus into a powerful tool for positive analysis. While much of this dialogue between game theory and economics has been quite fruitful, it has also revealed the limitations of game theory. Most strategic applications raise questions about the principles that govern behavior that are not sufficiently resolved by theory. This is in

competition that includes monopoly and perfect competition as the limiting extermes. Cournot had already used the methodology of Nash equilibrium without its formalism.

addition to questions about the principles that govern behavior that are not convincingly resolved by theory and to questions about preferences and the environment like those encountered in non-strategic applications. Binmore (1990) provides an excellent discussion about the aims and limitations of game theory. He suggests that the limitations of game theory arise due to four facts − the players may lack adequate knowledge of the physical nature of the game, the players may have inadequate behavioral knowledge, the theory itself lacks sufficient explanatory power, and the players may have insufficient computational ability. A large body of accumulating experimental evidence also indicates that the predictions of game theory are often inaccurate, at least in describing the behavior of human subjects in the laboratory. Camerer (1997) provides examples of many instances where players deviate from the standard game-theoretic predictions.[2] Often, though not always these deviations are quite systematic. For instance, in ultimatum games with a pie of $10, the proposer usually keeps $6 and offers $4, while game theory predicts that the respondent should settle for nothing and the proposer should demand the entire pie in equilibrium. (For more examples of such violations I refer to Kagel and Roth (1995)). This has led to new directions in game theory which attempts to bridge the gap between the achievements and aspirations of game theory by incorporating experimental findings and new concepts.

In the next section I discuss some of the new trends in game theory and how these have motivated my work. Section 3 provides a brief overview of my dissertation and discusses the main themes present in my work. Section 4 concludes by providing succinct summary of the main findings and their relationships with the predominant themes.

[2] A list of the violation of game-theoretic predictions in experiments can be found in Table 6, (page 183) of Camerer (1997).

1.1 New Directions in Game Theory

I now provide some examples of new trends in game theoretic modelling that are similar to ideas explored here and have inspired large part of the motivation for my dissertation.[3] The two main themes of this dissertation are alternative ways of modeling the payoff function and the importance of the beliefs of the players.[4]

In general game theory assumes that the players are aware of the payoffs in the game. Relaxing this assumption usually affects strategic behavior in the game. The structure of the payoff function has been explored in many different contexts in game theory. Seltens's (1975) trembling hand perfection is one such example where trembles by the players lead to perturbed games. The equilibrium is then defined in the limit as the mistake probability goes to zero. Fudenberg, Kreps and Levine (1988) investigate robustness of different equilibria in the context of iterated deletion of weakly dominated strategies by considering equilibria in games whose payoffs differ from those of the original game by a small amount (denoted by ϵ, where $\epsilon > 0$). Dekel and Fudenberg (1990) extend this result to extensive form games, where the find that iterated elimination of weakly dominated strategies will not be followed by rational players, where rationality is dependent on the payoff uncertainty in the game. Rational players are defined to be those who will engage in one round of eliminating weakly dominated strategies, after which only strictly dominated strategies can be eliminated.

Behavioral game theory goes in the same direction by modeling cognitive processes and psychological aspects of the players' reasoning process through additional payoffs. These additional payoffs are endogenous to the game since they depend on the chosen strategy profile. Camerer (1997) catalogues the progress in behavioral game theory and its attempt to augment traditional game theory models using experimental findings in

[3] One major new development which we will not discuss here is evolutionary game theory. Evolutionary game theory considers agents with zero intelligence, born with their stratgies, who then are allowed to experiment, learn or imitate their neighbors. See Samuelson (1998) for an introduction to this topic.

[4] Apart from these two main themes the links between the different chapters may seem somewhat tenuous. General conclusion however can still be drawn.

order to bridge the gap between theory and its predictions. Such models incorporate psychological aspects in to the game which gives additional payoffs. Geanakplous, Pearce and Staccehtti (1989) develop a formal model where emotions are used to generate additional payoffs. Rabin (1993) uses reciprocity to model fairness as a form of strategic behavior. In a similar vein Guth (1995) outlines a behavioral theory for ultimatum bargaining games which incorporates a dynamic reasoning process. An eclectic discussion of the implications of psychology for economics can also be found in Rabin (1998). Slonim and Roth (1998) focuses on learning in ultimatum games to explain the interaction between rejection frequencies and financial stakes. McKelvey and Palfrey (1995) explore the use of standard statistical models for quantal choice in a game theoretic setting. Players in this setting choose strategies based on relative expected utility and assume other players do the same. Chen, Friedman and Thisse (1997) provide a bounded rationality interpretation of aforementioned discrete choice model. In an interesting paper Costa-Gomez, Crawford and Broseta (1999) argue that strategic behavior depends on the extent to which it reflects players' analyses of their environment as a game, taking its structure and other players' incentives into account. This notion which they call *strategic sophistication* is the main difference between the behavioral assumptions of traditional noncooperative and cooperative game theory, which take it to be to be unlimited and evolutionary game theory or adaptive learning models which take it to be non-existent or severely limited.

Another idea that has recently gained popularity is the notion of interaction between agents through social and economic networks. The networks literature defines an interaction structure for agents which is not localized and usually generates externalities. These non-rival externalities which could be positive or negative are the distinguishing feature of this topic. The networks literature can be broadly categorized into three main strands: (i) Social networks: which are generally the domain of sociologists and deal with issues like marriage, power and demographics. Astone, Nathanson, Schoen and Kim (1999) for example deals with family demographics and investment in social capital, while Johnson and Gilles (2000) explore social distances and their consequences for network formation from an economist's perspective. (*ii*) Exchange networks: concerns the trading of commodities. See Kranton and Minehart (1998) for trade among a small set on inter-linked buyers and sellers while Ioannides (1990) and Haller (1990) are examples of commodity exchange in general equilibrium settings. (*iii*) Information networks, which is the strand of literature that concerns us, can be viewed as reduced forms of economic phenomena. Bala and Goyal (1999a, b) investigate such networks using Nash equilibrium as the solution concept. Bala and Goyal (1999a) analyzes deterministic networks with and without information decay in the network. Bala and Goyal (1999b) study properties of random networks where each link fails to transmit information with a certain probability which is identical across all links. Jackson and Wolinsky (1996) use another equilibrium concept called *pairwise stability* which requires the acceptance of both types of agents to establish a relationship between agents. The main concern in such models is on the tension between stability and efficiency issues their implications for social and economic networks. Stability derives from the equilibrium concept, while efficiency tries to maximize aggregate gains given costly link formation.

Another element that plays a major role in my dissertation with the possible exception of the second essay is the importance of beliefs in a game. In recent years there has been a renewed interest in the relationship between beliefs and the equilibrium concept of a game. The earlier work in this are is due Harsanyi (1967-68) where beliefs are used to define types of players. Mertens and Zamir (1985) provide a beliefs based foundation to the Bayesian formulation of Harsanyi. More recent work on this topic has been done by Brandenburger and Dekel (1993) and Aumann and Brandenburger (1995). The former investigates common knowledge using beliefs and the later provides epistemic foundations for Nash equilibrium. Let us first consider the interpretation of an equilibrium using a belief hierarchy. Assume that the structure of the game and rationality are mutual knowledge and that the players have a common prior. Then any differences in beliefs have to be attributed to differences in information. Further assume that beliefs are common knowledge. Then following Aumann and Brandenburger's (1995) work on epistemic foundations, any two players beliefs about a third player's strategy must be the same and these common beliefs viewed as mixed strategies must be the same in equilibrium. In this equilibrium in beliefs, a player's mixed strategy represents other players beliefs about his realized pure strategy about which he himself need not be uncertain, and players beliefs determine their optimal strategies and expected payoffs. Hence equilibrium requires that in addition to rationality, players beliefs are coordinated on the same outcome.

Another important development in the beliefs literature in recent years has been the use of non-additive probabilities to model strategic uncertainty in game theory. There are two main strands in this area: newer literature relying on *capacities* and a somewhat older literature using fuzzy set theoretic tools (Zadeh, 1965). Our focus here will be on exploring the relationship between fuzzy sets and game theory.[5] Fuzzy games are based on concepts from fuzzy set theory which differs from the standard set theory by allowing the characteristic function to take values in the interval [0,1]. For example, a glass with some water in it can belong to the set of both "empty glasses" and "full glasses" up to a certain degree. As argued by Bellman and Zadeh (1970) in their model about decision making in a fuzzy environment, there is a crucial distinction between *fuzziness* and *randomness*. First, fuzziness allows us to quantify linguistic and vague verbal descriptions of economic phenomena. Secondly, uncertainty or randomness in the probabilistic sense describes the uncertainty about the membership or non-membership of a particular element in a set. Fuzziness on the other hand describes the degree of this membership or belongingness to a particular set. By assigning a degree to which an element of a set posses the defining characteristics of set, fuzzy sets provide an excellent tool to model imprecisely defined situations. This becomes crucial when problems are vaguely defined in linguistic terms which many real world problems are apt to be. By being able to quantify the a linguistic term like "good outcome" fuzzy sets provide us a

[5]For modelling payoffs using capacities I refer the reader to Gilboa (1987), Schmeidler (1989) and Haller (1999) where the decision theoretic model of capacities has been extended to a game theoretic setting.

way to analyze situations fraught with ambiguity.

The fuzzy formulation of noncooperative games was introduced by Butnariu (1978, 1979) and later refined by Billot (1992). The most interesting feature of this approach is its ability to ensure the existence of results in games while allowing players to have their own subjective interpretations of the game. A very curious feature that is present in the Butnariu-Billot formulation is its ability to define an equilibrium in terms of beliefs without using an explicit objective function for the players.

My dissertation follows these new trends in game theory. Unsettling experimental evidence and the desire to explore some of the new techniques discussed above have been the primary motivation for the next four chapters. The primary motivation for my dissertation has been to investigate how sensitive are the theoretical predictions of a game to the structure of the game. Game theory requires that the players have complete knowledge of the set of players, their strategies and the payoffs in a game. This dissertation tries to explore how subjective perceptions about these components of a game affect the outcomes.

The models developed in here are all static models and in the words of Binmore (1990) belong to the *eductive* or deductive approach which differs from the *evolutive* approach relying experimentation, learning and imitation. In my models there is no repeated interaction or learning of any sort and the players are also assumed to be fully rational. This choice of modeling technique has been deliberate since I believe it is harder to provide convincing explanations of behavior that differs from theoretical predictions without resorting to an ad hoc learning process. This has in part been the primary reason that the structure of the payoff function and beliefs that players have about each other and the game, plays a crucial reason in my dissertation. Another motivating factor behind this work has been my desire to develop models that can describe real life phenomena. Fuzzy set theory is an ideal toolkit for this purpose since it allows players to formulate their own version of the strategic situation and allows for a substantial subjective component in the game and is used extensively in the last two essays. The chapter on formation of networks is closer to standard theory and investigates the consequences of having substantial non-rival benefits and possible free riding behavior in agent interaction.

In the first essay the subjective payoffs of the player are derived from a social norm based objectively given vector of payoffs. In the second essay we study Nash networks in random networks with non-rival benefits. The third essay uses a game form that does not have a payoff function. In the fourth essay payoffs are transformed into a goal function that takes values in the interval [0,1] reflecting a player's aspirations. Different notions of beliefs are explored in this dissertation. The first essay uses beliefs in the epistemic logic sense. In the second essay, beliefs play a relatively minor role but their importance for networks with incomplete information becomes evident even in a simple model. In the last two essays non-additive beliefs play a predominant role. In the next section I present a brief overview of the four different models developed in this dissertation and relate them to the above discussion.

1.2 A Brief Overview

The first essay is an examination of a problem where the outcome of strategic situation is quite sensitive to how the structure of the game is presented and hence to the environment of the game itself. I claim that in many situations the players or the analyst may have their own perception of the game. We model this by introducing a vector of payoffs called reference payoffs which depend on the cultural context and the social beliefs of the players. This vector captures different notions like being fair or the desire to win and it is quite possible that these different objectives are in conflict with each other. It is the logical culmination of Kavka's (1991) notion of *inter-acting sub-agents* in a decision theory problem. He argues that an individual faced with choosing between alternative may end up playing an internal Prisoner's Dilemma since he may have conflicting ethical and moral viewpoints. While the dimensions of this vector of payoffs are the same for all players the precise elements of the vector can be different for each player for any given strategy profile.

We assume that the reference payoffs are common for all players by appealing to social norms. Considerations like keeping their own social capital intact may force agents to pay attention to more than selfish utility maximization. For example, Ball, Eckel, Grossman and Zame (2000) in a recent paper show that experimental subjects care about status, and interaction among agents is influenced by whether they are dealing with a person with high status or low status. The sociology literature provides many examples of how interactions between agents are affected by social capital, making it important to invest in this social resource (see Portes, 1998). Since a convex combination determines the actual payoffs, the weights represent the importance assigned by a player to the different components of the payoff vector. The players then create their own true payoffs by using a convex combination of the reference payoffs where the weights in the convex combination reflect the priorities that players may assign to the different objectives at hand. The model is somewhat similar to the work of Geanakplous, Pearce and Staccehtti (1989) where the payoffs are payoffs are determined endogenously by the strategy profile after taking psychological implications of the chosen strategies into account. After proving the existence of a Nash equilibrium the model is used to analyze a game called the Traveler's Dilemma introduced by Basu (1994). This game is a generalized version of the Prisoner's Dilemma requiring iterated elimination of weakly dominated strategies. Using the notion of reference payoffs it is shown that the predictions of standard game theory can be reversed for the Traveler's Dilemma. The model is also applied to mixed duopolies and the usual subgame perfect equilibrium is investigated using reference payoffs.

The analysis is static and hence the reference payoffs can also be used to introduce dynamic considerations like reputational effects indirectly. This plays a crucial role in allowing us to obtain results quite different from those predicted by standard Nash equilibrium type reasoning for the Traveler's Dilemma. Note that most attempts to explain anomalous behavior in games either use an extensive form game or appeal to repeated play. We prefer instead to exploit the structure of the payoff function. One could argue that the learning models like Mckelvey and Palfrey (1995) and Chen *et al.* (1997) which are based on repeated play describe a process that is captured by our reference payoffs in a static manner. Furthermore, we find that this technique allows us to sustain as equilibria outcomes obtained in laboratory experiments (Capra *et al.* (1999)). The reference payoffs also provide us an alternative way to analyze mixed duopoly game where the reference payoffs are the different possible objectives that a firm might pursue.

Although players have the same set of reference payoffs, but choose the weights of the convex combination individually to derive their own payoff function. Hence we have a game of incomplete information where we model the "type" of a player by using beliefs as the primitive. This requires a hierarchy of beliefs that are *consistent* and *coherent* in the sense of Brandenburger and Dekel (1993) or Aumann and Brandenburger (1995). The existence of equilibrium in such a game crucially depends on the belief system, which allows players to coordinate on the same outcome. Also, the importance of having strategic beliefs and is shown in the example relating to duopolies. Strategic beliefs in a game of reference payoffs can sustain refinements of Nash equilibrium like subgame perfection.

The second essay deals with the formation of networks. A network is an interaction structure where the players are represented by the nodes or vertices of a graph and a link or the relationship between two players is denoted by an edge. Networks are becoming increasingly popular in economics as a way to model interactions among agents. In our model a link between two players leads to a symmetric two-way flow of benefits, while the agent initiating the link incurs the cost of the link. The flow of these non-rival benefits is stochastic in nature. Links between agents can fail to work with a certain probability (that can differ across agents) independently of each other. This chapter builds on the work of Bala and Goyal (1999a, b). We focus on the structural properties of Nash equilibria in the context of heterogenous link success. We analyze the popular star networks and look at examples of some other types of networks as well. We identify conditions under which the empty network, the complete network and redundant links can arise. Finally, three different variations of the original model are also considered in this chapter.

As there is no congestion or negative externality more links increase the gross benefits for each player in this game. This positive externality is the driving force for the main result in networks with homogeneous probabilities for link success (Bala and Goyal, 1999b). They find that with homogeneous imperfect reliability as the size of the player set increases, the externalities and, hence, the payoffs from the network increase dramatically and ensuring link success becomes absolutely vital for the players. Consequently, redundant links between players always arise as a form of insurance. In contrast the model presented here allows for heterogeneous probabilities of link success and the Bala and Goyal (1999b) results no longer hold always. By choosing the probabilities for the different links in an appropriate manner one can always ensure that the network is not even connected (in a graph-theoretic sense), let alone have superfluous links.[6] In one section of the paper we introduce uncertainty about the reliability of indirect links. Instead of introducing the usual Bayesian formulation with a continuum of types, we allow agents to assign the average probability of their own link success to all indirect links. Our assumption about the beliefs of agents regarding indirect links imparts a symmetry to the model where each agent has the same type of beliefs about the others. We finds that uncertainty has serious consequences for star networks. Star networks may fail to be Nash, even if an agent's belief about one indirect link is lower than its actual probability. These incorrect beliefs lead to superfluous connections destroying the star architecture.

The third essay looks at noncooperative fuzzy games. In this chapter we re-interpret the work of Butnariu (1978, 1979) and Billot (1992) and show that the existing model relies on a very strong assumption which is often not satisfied. This restrictive assumption says that a player is only concerned about minimizing the restrictions he imposes on the others. The player does not pursue any other self-interested goals in this framework. We

[6] A sequence of probabilities that decreases fast enough relative to the costs of link formation will lead to empty networks.

augment this model by borrowing from the abstract economies literature. In our model a player tries to minimize the restrictions imposed on him by the others while taking into account the restrictions imposed on him by the others. This essay establishes the existence of an equilibrium and uses an example of an abstract economy to highlight the differences from the work of Butnariu and Billot. The most interesting feature of this chapter is that it uses a game form without specifying a payoff function. Players do not explicitly maximize any objectives of their own. Each player is only concerned about minimizing the constraints he imposes on the actions of the others and respecting the constraints they impose on his behavior. While the possibility of defining an equilibrium without any payoff function is an interesting idea, one might argue that this is also the model's most serious weakness.

The fourth essay is of an exploratory nature where we try to make the leap from decision making in a fuzzy environment to analyzing noncooperative games in such situations. This is an extension of the work of Bellman and Zadeh (1970). The payoffs and strategy sets are both fuzzy sets and their membership function denotes the degree to which each is feasible. The payoff function is translated into a membership function of a fuzzy set that denotes a players goals. Thus given a strategy profile, the membership functions tells us how desirable is that outcome with respect to the player's goals. Such membership functions need not change the payoff ranking of the original game. But it is also possible that they alter the payoff ranking to incorporate the players own perceptions of the game to introduce notions of altruism or fairness as in behavioral game theory Camerer (1997). By reducing both the strategies (or constraints on a player's choices) and payoffs (or a players goals) to a common platform, the game has a symmetry that might be considered intuitively appealing. Some preliminary results and directions for future research have been outlined in this chapter. A simple example is used to illustrate how a fuzzy payoff or goal function can easily sustain collusion in a duopoly.

Beliefs also play a very crucial role in the last two essays. These beliefs are however, in the form of fuzzy membership functions and hence can be described as non-additive probabilities.[7] Each player tries to minimize the restrictions he imposes on others. This is modeled through beliefs that players associate with the feasibility of different strategy profiles. The last essay also uses fuzzy sets to define a game in a fuzzy environment. Both payoffs and strategies are redefined using membership functions. By being able to put both the payoffs and strategies on a common platform, we have an innovative way of analyzing the game. This allows us to compare the tensions between a player's aspirations in terms of payoffs and feasible choices in terms of strategies.

1.3 Conclusion

In this section we discuss the implications of payoffs and beliefs in the context of the models developed here. The lessons learnt from these models should be borne in mind while exploring these issues further. These conclusions are intended to be the most general message that one could distil from each of the models. Detailed conclusions regarding each essay can be found at the end of each chapter.

Games with reference payoffs are similar to what behavioral game theory refers to as "games as social allocations." In such models the players rather than focussing self-interestedly on their own payoff alone, might take other social norms like fairness, altruism, etc. into consideration while choosing strategies. The main message that emerges from the model developed here seems to be the fact that such approaches can explain observed behavior in games like the Traveler's Dilemma or the ultimatum game.

It might be possible to construct payoff matrices where players concerned with fairness will eventually end up playing dominated strategies in games like the Prisoner's Dilemma. Clearly, one must proceed with caution down this slippery slope! Conclusions from any

[7] Here we do not use the standard *capacities* approach based on Choquet integrals. For an exposition of Nash equilibrium in capacities see Haller (1999). Billot (1990) studies fuzzy belief measures called *possibility* and *necessity* measures and concludes that they are normalized capacities.

such model where additional payoffs are introduced by appealing to psychological and behavioral considerations should be generalized with care. The context of the game and its strategic nature should be borne in mind while extending the conclusions or making inferences about other games. Another consequence of the interaction between payoffs and beliefs emerges from the mixed duopoly models. Games with reference payoffs where players use best response strategies and have strategic beliefs, i.e., follow Nash type reasoning in choosing beliefs as well as strategies lead to the subgame perfect equilibrium of the traditional mixed duopoly models. This finding also has interesting implications for behavioral game theory. Players beliefs and additional payoffs stemming from psychological and emotional considerations makes it possible to sustain as equilibria, outcomes that cannot be explained by traditional game theory. Introducing strategic beliefs that lead to the psychic payoffs can be used to sustain refinements of Nash equilibria in such models.

The investigation of Nash networks with heterogeneous agents demonstrates that this general model can encompass results of the deterministic models and models with homogenous link reliability. Keeping this in mind, it would be interesting to endogenize the link success probabilities based on the number of links per agent. Nash networks do not require mutual consent for establishing relationships. This issue needs to be investigated further, in particular to compare results of Nash networks with those relying on other equilibrium concepts like pairwise stability (Jackson and Wolinksy, (1996)). Another issue that is directly related to payoffs that comes up at least twice in our model is the appropriate modelling of the costs of link formation. Future research in this context would be to implement bargaining over the costs and benefits of link formation or to allow cost sharing among agents interested in forming links. Finally, I believe that it is important to pay heed to congestion issues in a network, and explore the consequences of negative externalities.

The central conclusion that emerges from the re-interpretation and reformulation of fuzzy games a la Butnariu and Billot, seems to be the fact that this approach in its current form is quite limited. While it allows us to establish an equilibrium that is quite subjective, its lack of an explicit objective function which players can maximize seems to be a rather serious drawback. To best evaluate this approach, it is necessary to explore applications where there is no conflict between the objectives of the like players. Simple situations like unanimity games would be the right starting point in this direction.

The last essay is rather preliminary and holds promise for future work. While the existence of equilibrium is easy to establish using standard techniques, and the usefulness of this subjective approach has been demonstrated through a simple example, more general applications are needed to asses the usefulness of this approach. By reducing strategies and payoffs to a common denominator however, it does provide us with a novel way of looking at noncooperative games.

Chapter 2

Games with Reference Payoffs

2.1 Introduction

Game theory normally requires that a precise definition of the game is available. The components of the game, i.e., the set of players, their strategies and the associated payoffs are presumed to be given by a sort of *deus ex machina*. In real situations however, the players in a game may lack a complete description of the strategy space and the payoffs. Harsanyi's (1967/68) formulation provided a way to incorporate this uncertainty into game theoretic situations. His notion of "types" enabled game theory to capture this uncertainty in an elegant fashion. In this paper we focus on the "black box" around the payoff functions by exploring an alternative way of obtaining the payoffs of the game. In an interesting paper Geneakoplous, Pearce and Stacchetti (1989; hereafter referred to as GPS) pursue this question by incorporating emotions in the game. GPS refer to these games as *psychological games* since the realized utility or payoff depends on the emotional reactions to opponents play in the game.[1] Rabin (1993) extends this idea to explain fairness as reciprocal behavior. He provides a method for constructing

[1] It is worth mentioning that Gilboa and Schmeidler's (1988) "information dependent games" is a precursor of the psychological games. While the formulation is somewhat rudimentary from a psychological standpoint, it was developed independently of psychological games and concerns itself with the information structure and associated knowledge axioms in a game.

psychological games from material games (in Rabin's terminology the standard normal form game is the material game since it is devoid of any emotions). In this essay we develop a more general descriptive theory to explain certain types of observed outcomes by imposing a more realistic structure on the payoffs.

The preferences of an economic agent are shaped by their experiences. Cultural context, social norms and institutions influence our way of thinking which may allow different societies to create their own perception of the same situation. The experimental study of bargaining and market behavior in Jerusalem, Ljubljana, Pittsburgh and Tokyo by Roth *et al.* (1991) is an excellent example. Their study supports the hypothesis that the significant differences in bargaining behavior in these locations can be tentatively attributed to cultural differences. Moreover, human ideas are under the constant influence of evolution and education and change over time. Game theoretic models that neglect the environment of the game do not always provide a satisfactory way to deal with these different attributes and the conflicts arising there from. While emotions are one aspect of this, other tensions may exist due to differences in moral principles or inter-temporal considerations leading to differences in perception.

Models of intelligent agents or evolutionary selection is generally used to predict behavior and explain outcomes in games. In many circumstances the structure of payoff functions maybe provide a better explanation for observed behavior and the structure of the payoff function has been used for many different purposes in game theory. Fudenberg, Kreps and Levine (1988) for instance, perturb the payoffs of a given game to check for robustness of different equilibria. Behavioral game theory (Camerer, 1998), also uses this argument and attempts to modify game theory by incorporating psychological explanations of findings that arise repeatedly in experimental situations. The payoffs of the original game are augmented by additional psychic payoffs. Rabin's (1993) explanation of fairness is an ideal example of this type of work. The payoff structure of certain types of games will be explored in this paper to provide alternative explanations. Consider for instance situations where the payoffs of the players are fixed by an external referee. Examples could be economic experiments or any kind of remuneration system. This is also true of some sporting tournaments. In long distance car rallies, for example, a player earns points for covering a leg of the race in the pre-selected optimal time and also for his relative position in the race. A player who chooses to drive fast and improve his ranking might sacrifice points for arriving at the next destination before the optimal time. So, by selecting one strategy i.e., "drive fast" the player earns payoffs from two sources. Bolton's (1991) work on alternating bargaining provides an excellent example of this type of situation. He postulates that players in an alternative bargaining game care about "absolute money" and "relative money," where relative money is the difference between the absolute payoffs. He further provides experimental evidence to support his claim. Clearly the choice of optimal strategy depends on a how a player the relative importance of these two sources of payoffs. A similar issue also arises in games where the final objective may not be very clear. Take for example a public firm. Does it maximize profits or does it maximize welfare? It is obvious that the optimal strategy against such a firm depends on what its ultimate goal is (see for example Fershtman, 1990).

Kavka (1991) talks about a similar situation in individual decision making. He argues that if individuals evaluate a set of outcomes using more than one criterion, they might end up facing an *internal* Prisoner's Dilemma played by interacting subagents whose preference ordering over the different evaluation criteria varies. Kavka explains[2]

> . . . these internal value conflicts encompass such familiar phenomena as conflicts between prudence and conscience, between long- and short term interests, between different moral principles, between the interests of different people, the agent cares about, and between different private interests or desires that the agent has.

Feld and Grofman (1990) use the notion of value conflict to show that transitive preferences may exist in voting games. Some empirical confirmation of a similar hypothesis

[2]While the notion dates back to Hobbes (1651), Levi (1986) also discusses this idea.

can be found in Shaw's (1982) studies of perception and decision problems. Elster (1985) also makes similar arguments in an edited volume called *The Multiple Self* which explores interacting subagents in some detail.[3]

While Kavka modeled the internal struggle an individual with a value conflict may face, this paper examines how she makes strategic decisions. We carry the idea to its logical culmination by introducing different perspectives for a given game, which create smaller component games each having its own strategic structure. We define a vector of payoffs which embody the alternative viewpoints on the game, for example, the payoff associated with being fair and the payoff associated with being nasty. Note that these do not refer to strategy choices but to the players' perceptions of the context of interaction. They may embody social norms and perceptions about the consequences of repeated play. Also these are assumed to be the same for all players and objectively known. We refer to these as the *reference utilities.* The actual utility in the game is based on players' subjective evaluation of these different criteria and is called the *true utility.* True utilities are derived by considering convex combinations of the reference utilities. In our model, agents can resolve their conflicts and establish an individual payoff function which weights the payoffs from different component games. Thus, the emphasis is shifted from requiring sophisticated reasoning on the part of agents by allowing them to have a subjective evaluation of the game and follow simple Nash equilibrium type reasoning in this game. Since our analysis is essentially static we do not derive the reference utilities and assume that they are exogenously given. A more dynamic evolutionary framework is needed to explore this issue.

The primary motivation for this formulation is to examine the Traveler's Dilemma (Basu, 1994). This is a generalized version of the Prisoner's Dilemma with an expanded strategy set allowing for iterated elimination of weakly dominant strategies. Since the game induces backward induction in the normal form explanations relying on learning or adaptive behavior would not resolve the paradox implicit in the one-shot version. The

[3] He refers to the study of such issues as *egonomics*!

model presented here has been kept deliberately static to avoid such criticisms, while using the reference utilities to provide a dynamic flavor.

Our characterization is also consistent with an observation made by Glazer and Rosenthal (1992) in the context of Abreu-Matsushima mechanisms. They claim that players might be willing to behave cooperatively in games involving iterated elimination of weakly dominated strategies, since they do not mind forgoing the small gains from non-cooperative play. Thus, players discount the small payoffs from the "competition" component game in favor of larger payoffs from the "compensation" component game.

The optimal strategy choice for such games requires that players have beliefs about what the other players care about in the game. The Nash equilibrium requires an infinite hierarchy of beliefs of the sort used by GPS with each player's payoff depending only on their own choice of weights. Unlike GPS however, there is no endogeniety in the payoffs based of the choice of strategies. This enables us to modify games which may require extraordinary depths of reasoning to ones where the equilibrium maybe fairly simple to identify. The technique is explicitly applied to a quantity setting duopoly and the Traveler's Dilemma (Basu, 1994) which illustrates the centipede game in the normal form. The former represents a game where the goal of each player may not be clear to the others, although they are cognizant of the various possibilities. The second game's unusual structure makes it an interesting study in itself. We use it to analyze consequences of payoffs determined by an external referee.

The following section presents the basic model and the Nash equilibrium for such games. Section 2.3 provides the illustrative examples. Discussions and conclusions are found in the final section.

2.2 The Model

Assume that there are N players in the game. For each $i \in N$ let S^i denote the nonempty finite set of actions available to player i. For any set X we use $\Delta(X)$ to denote the set of (Borel) probability measures on X. So $\Sigma^i = \Delta(S^i)$ is the set of mixed strategy profiles of player i. Let $\Sigma = \times_{i \in N} \Sigma^i$ and $\Sigma^{-i} = \times_{j \neq i} \Sigma^i$, $i \in N$. Then each strategy profile $\sigma \in \Sigma$ represents canonically a probability distribution over the set of pure strategy profiles $S = \times_{i \in N} S^i$.

We will now introduce the notion of reference payoffs. Consider for example a game like the ultimatum game. Numerous experiments have shown that players in this game usually choose to divide the pie in an equitable manner. It is argued that this occurs due to some notion of fairness or altruism ingrained in the subject's mind. In principle it is conceivable to think of this as a situation where players evaluate the game using different criteria. We can postulate that they get utility from maximizing their own payoff and they also derive utility from being fair. We label the payoffs from these different perspectives on the game as the reference payoffs. For the ultimatum game we can assume that these alternative payoffs are objectively known by the players in the game. Of course, it is also easy to conceptualize this in terms of Bolton's (1991) hypothesis where players care about both absolute and relative money in a bargaining game. In formal terms we will denote for each $i \in N$, the m different reference utilities by $\Pi_1, \ldots, \Pi_m$. They represent the alternative perspective on the game to be played. Since they are dictated by prevailing social and cultural beliefs, we assume that they are objectively given. This makes it possible for them to be shared and known by all the players in the game. To summarize, everyone knows what the alternative goals are for each player and the players have different utilities depending on how they combine these goals. So we have now a game form denoted by $\Gamma = \langle S, \Pi_1, \ldots, \Pi_M \rangle_{i=1}^n$ from which we will construct the normal form of the game.

Before proceeding further, it is worth pointing out that it is possible to interpret the above game form as a game with vector payoffs and analyze it with the techniques developed for such games (see for example Shapley (1959), Zeleny (1978) and Zhou (1991)). The equilibria for these games must satisfy two conditions: (i) the equilibrium vector of payoffs must be Pareto efficient, and (ii) it must be the solution of a vector maximization problem for each $i \in N$, keeping the strategies of the other players fixed and for all $s^i \in S^i$. However, these games were meant to describe situations where the payoffs were material objects like apples and oranges. Besides, as we will show later in the section on examples, their predictive ability is rather weak. Another interpretation would be to think of the game form as being composed of several component games all of which have the same set of players and strategies. The payoffs $\Pi_1, \ldots, \Pi_m$ enable us to define the m different component games which are just played in the agents minds. As we will see later this interpretation also has some advantages.

In order to convert this game into a game form, we introduce some additional notation. The utility of player i from the m$-th$ reference payoff function is given by $\Pi_m^i : \Sigma \to \mathbb{R}$. We assume that players in this game wish to maximize expected utility. While the reference utilities are given, player i's true utility depends on his subjective evaluation of the relative importance of these different perspectives. So, in the ultimatum game described above this depends on how much importance the player assigns to the reference utility from being fair and how much the player cares about the reference payoff from obtaining a larger share of the pie. The true utilities are obtained by convex combinations of the reference payoff functions. Let $C = \{(\lambda_1, \ldots, \lambda_m) : \sum \lambda_j = 1\} \subset [0,1]^m$ denote the set of all possible convex combinations. We assume that the weights for the convex combination are chosen by a dummy player in the game. For $i \in N$, we denote the choice of player i's weights by C^i. We now define a variant of a psychological game in the normal form as follows:

Definition 1 *: $\Gamma_P = \langle S, \Pi_P \rangle_{i=1}^n$ where the set of strategies is the same as before and all other definitions hold except that $\Pi_p^i : \Sigma \times C^i \to \mathbb{R}$.*

Π_p^i is the superposition function that combines together what we labelled as the component games and tells us how much importance each player attaches to each of these games. More formally we can define this as $\Pi_p^i = \{\sum_{j=1}^m \lambda_j^i \Pi_j^i(\sigma) : \sigma \in \Sigma$ and $\lambda^i \in C^i\} \subset \mathbb{R}$ for all $i \in N$. Note that this is different from the GPS or Rabin framework.

In both of these formulations a player's payoffs depends on his beliefs about what other players will play. This allows him to form opinions about what they feel about him. These feelings induce additional psychic payoffs which are then used to create a new game called a psychological game. Hence the payoffs in their version of the psychological game depends endogenously on the strategies being chosen in the game.

In order to find an equilibrium however, each player must have beliefs about how the other players define Γ_p, i.e., how Π_p^j, $j \neq i$, is defined. At this point we could introduce the notion of types as in Harsanyi and treat it as a Bayesian game. Instead, we will follow a formulation which defines a type using beliefs as the primitive. Detailed discussions of this formulation can be found in Mertens and Zamir (1985) and Tan and Werlang (1988). Our model is based on the work of Brandenburger and Dekel (1993). This line of reasoning has the advantage of making the application of common knowledge to a game direct and more appealing. The construction of types follows two stages in this formulation. First, an individual's belief is defined to be *coherent* if the induced hierarchy of beliefs over the types of the other players is not self contradictory. Second, we require common knowledge of coherent of beliefs. A simple inductive definition imposes the requirement that each type knows (in the probabilistic sense of assigning probability one) that the other individuals' types are coherent, that each type knows that the other type knows this, and so on. This serves to close the model of beliefs. Of course it is assumed that each player knows his type.

A *first order belief* is a probability measure over the other players' payoffs in the psychological game defined above. This is player i's belief about what the other players' view of the world is, in terms of their objectives. This is basically player i's beliefs about the dummy player for all the other players. We define it as follows

$B_1^i = \Delta(\times_{j \neq i} C^j) = \Delta(C^{-i})$.

Since C^{-i} is a subset of the Euclidean space it is a separable metric space. Endow B_1^i with the weak topology. This gives B_1^i the hereditary properties and it makes a separable metric space too. However, players need to have higher order beliefs as well, i.e., beliefs about beliefs. So player i's second order belief which is defined over how the other players' payoffs and their first order beliefs.

$B_2^i = \Delta(C^{-i} \times B_1^{-i})$.

We define the set of higher order beliefs for $k \geq 1$ inductively as follows:

$B_{k+1}^i = \Delta(C^{-i} \times B_1^{-i} \times \cdots \times B_k^{-i})$ and $B^i = \times_{k=1}^{\infty} B_k^i$.

Also $B_{k+1}^{-i} = \times_{j \neq i} B_{k+1}$ and $B_{k+1} = \times_{i \in N} B_k$.

This set however is too large and allows for the existence of inconsistent beliefs. So, we impose a consistency requirement which states that the probability of an event evaluated by the k$-th$ order belief and the (k+1)$-th$ order belief must coincide. This is called coherency of beliefs. Note that the marginals of a probability measure P which is defined on a product space $U \times V$ are given by $marg_U(A) = P(A \times V)$ and $marg_V(B) = P(B \times U)$ for any event A in U and B in V.

Definition 2 $b^i = (b_1^i, b_2^i, \cdots) \in \times_{k=1}^{\infty} B_k^i = B^i$ *is a coherent belief if for each* $k \geq 1$ $marg(b_{k+1}^i, B_k^{-i}) = b_k^i$. *Denote by* $\hat{B}^i(0)$ *the set of player* i*'s coherent beliefs.*

Note that the infinite hierarchy of beliefs defined above is a proxy for the description of an agent's psychology or type.[4] We now proceed to the next stage in the construction of types. Since all players in this game are assumed to be rational, it implies that coherency of beliefs is should be common knowledge. So players in this game are not allowed to entertain absurd beliefs and everybody knows that everybody knows this and so on. Basically for player i the support of the marginal (defined over the others' beliefs) of a coherent belief is the set of types of the j$-th$ agent that player i considers possible. This allows us to define *collective coherency.*

For each $j \in N$, and $k \geq 1$, let $Y_k^j = \times_{l=1}^{k} B_l^j$.

We also define in an inductive fashion for $\alpha = 0, 1, \ldots,$ the following sets:

$X_k^j(\alpha) =$ projection of $\hat{B}^i(\alpha)$ into Y_k^j, $j \in N$ and $k \geq 1$,

[4] In our existing framework an actual psychology includes both the infinite heirarchy of beliefs and the way in which a player combines his reference utilities.

$X_k^{-i}(\alpha) = \times_{j \neq i} X_k^j(\alpha)$, which can be thought of as a subset of $\times_{l=1}^{k} B_l^{-i}$, and

$\hat{B}^i(\alpha+1) = \{b^i \in \hat{B}^i(\alpha) \mid$ for every $k \geq 1$, $b_{k+1}^i(B^{-i} \times X_k^{-i}(\alpha)) = 1\}$.

Definition 3 $\bar{B}^i = \cap_{\alpha>0} \hat{B}^i(\alpha)$ *is the set of collectively coherent beliefs of player* $i \in N$.

In equilibrium, all beliefs must conform to the commonly held view of reality. Players must have consistent beliefs and must be playing their best response to the others' strategies given these beliefs. Let $\bar{B} = \times_{i \in N} \bar{B}^i$. Let $c = \mathrm{c}_1, \ldots, \mathrm{c}_n$ be a profile of beliefs, one for each $i \in N$, which is a part of the equilibrium in question. It is imperative that each player's perception of the game must be consistent with their opponents perception of the game as well, or

$\beta(c) = (\beta^1(c), \beta^2(c), \cdots, \beta^n(c)) \in \bar{B}$.

Definition 4 *A psychological Nash equilibrium of this game is a pair* $(b^*, \sigma^*) \in \bar{B} \times \Sigma$ *such that*

(i) $b^* = \beta(c)$ *and*

(ii) for each $i \in N$, *and* $\sigma^i \in \Sigma^i$, $\Pi_P^{i*}(\sigma^{i*}, \sigma^{-i*}) \geq \Pi_P^{i*}(\sigma^i, \sigma^{-i*})$.

THOEREM: *Every normal form psychological game* $\Gamma_P = \langle S, \Pi_p \rangle_{i=1}^n$ *defined as above has a Nash equilibrium.*

Proof: In order for the existence of the Nash equilibrium defined in (ii) we need that the payoff function for each $i \in N$ be quasiconcave in its own strategies. Since we create the payoff functions using convex combinations, this implies that $\Pi_1, \ldots, \Pi_M$ must be concave. By *Theorem 3.1* of Tan and Werlang (1988) we know that b^* defined in (i) will also exist for all agents. The definition of $C \subset [0,1]^m$ ensures that the requirements of this theorem are satisfied. Given that such a profile of types exists and the conditions for existence of a Nash equilibrium are satisfied, every psychological game of this type will have an equilibrium. ■

Before we proceed to the examples it is worth mentioning that the GPS formulation can also be subsumed into our framework by allowing for all different possible emotional payoffs. The psychological game can then be defined using these references based on to which of the fates or furies the player assigns greater importance.

2.3 Examples

In this section we will provide two examples of how the solution concept outlined above extends the existing framework of game theory. The first example concerns the Traveler's Dilemma. This game has a obvious reference points and hence belongs to the first category of games discussed earlier. We analyze this game by creating two different component games for which players have different priorities. The second example explores a quantity duopoly using several different objective functions for the firm. We also discuss scenarios under which these alternatives to profit maximization might be relevant. The purpose of this example will be to explain some empirical facts with regard to the objective function of the firm. Note that in both these examples we will not focus on the belief structures since conditions for identifying the other agents type are always satisfied.

2.3.1 The Traveler's Dilemma

The Traveler's Dilemma (Basu, 1994) is a tale of two tourists who, on arriving at their final destination from the same holiday trip find, much to their chagrin, that their souvenirs have been destroyed in flight. The airline company comes up with a compensation rule for the two travelers. Believing that the value of the damaged goods does not exceed $100, the airline allows the travelers to each choose a number from the interval [2, 100]. If they both choose the same number the compensation amount corresponds to the number chosen. However, if the numbers chosen are different, then the one choosing the lower number (say x) is paid $x + 2$ and the one who chose the higher number is paid $x - 2$ dollars respectively. The payment scheme thus rewards honesty and punishes the dishonest traveler!

The strict Nash equilibrium of this game is to choose (2, 2) which is also the only rationalizable outcome. This is also the unique equilibrium. The reasoning proceeds as follows: The best response to the strategy 100 is to choose 99 since it yields a payoff of 101. Since both agents are rational and will reason in the same manner, they would both pick 99. The best response to 99 is to select 98. Repeated application of the above reasoning will now occur. The game unfolds in this fashion until both players end up choosing 2 each. Since both players do better when they both pick 100, this is not an intuitively appealing outcome.[5] Recent experiments by Capra *et al.* (1997) provide concrete evidence for the dilemma inherent in this game. They find that human subjects do indeed react to the incentives in this game by choosing high numbers when the incentive from winning is not very high and vice versa. Basu himself suggests two possible ways to tackle the problem and provides solution using a modified version of curb sets. He then proceeds to dismiss this solution by asserting that it relies on mathematical technicalities to resolve a paradox of human behavior.

The Traveler's Dilemma has several interesting features. It can be viewed as a generalized version of the Prisoner's Dilemma with a unique Nash equilibrium. The desire to maximize individual well being leads to sub-optimal outcomes in both games. The Traveler's Dilemma differs from the Prisoner's Dilemma in that there is no dominant strategy. This game induces backward induction (iterated elimination of weakly dominated strategies) in a normal form and highlights the conflict between game-theoretic reasoning and intuition. It is essentially a centipede game in normal form. Since this game involves iterated elimination of weakly dominated strategies the equilibrium outcome may not be obvious to the players. The lack of a sequential structure is also problematic since it does not permit an examination of altruism, as in McKelvey and Palfrey (1992), or the presence of an irrational player as in Kreps, Milgrom, Roberts and Wilson (1982) to justify non Nash behavior or regret equilibria as in Droste *et al.* (1999). Thus, the real challenge of Traveler's Dilemma is that it requires us to provide a rational justification

[5]The reason for chosing to call this game a Dilemma was Basu's assertion that rational player will select high numbers.

for non Nash play in a one shot game.

The original game can be viewed as two component games each with their own reference payoffs which capture one of the strategic elements of the original game. Note that the set of strategies in both these games coincides with those in the Traveler's Dilemma. The first game is the *compensation game* $\Gamma_C = \langle S, \Pi_1 \rangle_{i=1}^{2}$, where subjects get paid the lowest number that was chosen. Formally the payoffs in this game are defined as follows:

$$\Pi_C^i(s^i, s^j) = s^i \text{ if } s^i = s^j,$$
$$\Pi_C^i(s^i, s^j) = s^i \text{ if } s^i < s^j, \text{ and}$$
$$\Pi_C^i(s^i, s^j) = s^j \text{ if } s^i > s^j \text{ for } i = 1, 2.$$

This is shown in Table 1, which is a truncated Traveler's Dilemma with players restricted to choosing numbers between [98, 100]. The compensation game has three equilibria which are all located on the main diagonal.

Γ_C

1\2	100	99	98
100	100, 100	99, 99	98, 98
99	99, 99	99, 99	98, 98
98	98, 98	98, 98	98, 98

Table 1

The second game is the *punishment/reward game* $\Gamma_{PR} = \langle S, \Pi_2 \rangle_{i=1}^{2}$ which reflects only the punishment and reward payoffs. These are given by:

$$\Pi_2^i(s^i, s^j) = 0 \text{ if } s^i = s^j,$$
$$\Pi_2^i(s^i, s^j) = 2 \text{ (or } r) \text{ if } s^i < s^j, \text{ and}$$
$$\Pi_2^i(s^i, s^j) = -2 \text{ (or } p) \text{ if } s^i > s^j \text{ for } i = 1, 2.$$

This is a zero sum game (see Table 2) where the only Nash equilibrium is to choose the lowest possible number. Notice also that the payoffs for this game happen to satisfy $\Pi_C(\sigma)+\Pi_{PR}(\sigma) = \Pi_{TD}(\sigma)$ where Π_{TD} is the payoff from the game as defined by Basu. By considering the first game as the base game and the second game as a deviation game, the formal construction used here might be reminiscent of Kahneman and Tversky's (1979) *prospect theory* for games of chance.

Γ_{PR}

1\2	100	99	98
100	$0,0$	$-2,2$	$-2,2$
99	$2,-2$	$0,0$	$-2,2$
98	$2,-2$	$2,-2$	$0,0$

Table 2

The original Traveler's Dilemma game assumes that the players payoffs are linear in the payoffs of the two component games. It assumes that the desire to *do well* as reflected in the compensation game, is equal to the desire to *win* as reflected in the punishment\reward game. We now define a modified Traveler's Dilemma $\Gamma_P = \langle S, \Pi_P \rangle_{i=1}^2$ which reflects the importance players attach to Γ_C and Γ_{PR}. The true payoffs in this psychological game are a convex combination of payoffs in the component games.[6]

$$\Pi_P^i(s^i, s^j) = \alpha_i \Pi_C^i(s^i, s^j) + (1-\alpha_i)\Pi_{PR}^i(s^i, s^j) \text{ for } i = 1, 2.$$

The table below shows the payoffs from the Traveler's Dilemma game outlined above with $\alpha_i = 0.9$ for $i = 1, 2$. This transformed game has symmetric equilibria located on the main diagonal. Note that for the game shown below this will happen for all $\alpha_i \geq \frac{2}{3}, i = 1, 2$ as long as long as the reward parameter does not vary, since this is

[6] The possibility of more general forms of constructing payoffs of the pyschological game remains to be explored.

enough to rule out the weak dominance. Also there are no weakly dominated strategies in this modified game.

Γ_P

1\2	100	99	98
100	90,90	88.9,89.3	88,88.4
99	89.3,88.9	89.1,89.1	88,88.4
98	88.4,88	88.4,88	88.2,88.2

Table 3

PROPOSITION 1: *If* $\Gamma_P = \langle S, \Pi_p \rangle_{i=1}^{2}$ *is derived by using the same convex combination of the payoffs from* Γ_C *and* Γ_{PR} *, then for every positive reward there exists a critical level of importance* α^* *that will create multiple equilibria in the psychological game.*

Proof: Let n denote the number chosen by a player and let $p > 0$ and $r > 0$ stand for the punishment and reward parameter respectively. We do not assume that $p = r$. We need to ensure that the iterated elimination of weakly dominated strategies is no longer possible. Observe that if weak dominance is not applicable for any $(n+1)$ and n then it is automatically invalidated for the entire strategy space. Note also from the structure of Traveler's Dilemma, that the payoff from the cells on the diagonal below the leading diagonal is the cause of weak dominance. For instance choosing 99 is better than 100 since it gives 101. In order to ensure that weak dominance does not hold we express this in a general form in the following manner:

$$\alpha n + r(1-\alpha) \leq (n+1)\alpha$$

$$\Rightarrow \alpha^* \geq \frac{r}{1+r}$$

where the lefthand side is the payoff from the cell below the main diagonal and the right hand side is the payoff from the cell on the main diagonal above it. ■

Note that p has no bearing on the above proof except being a part of the payoffs. Also, note that as r increases players need to attach greater importance to Γ_C. This model allows rational players to select high numbers since (2, 2) is no longer the unique Nash equilibrium. In the next proposition we investigate the importance of penalty parameter.[7]

PROPOSITION 2: *If $\alpha^* \leq \alpha_i$, and $\alpha_j < \alpha^*$ and $p > 0$ and $r > 0$ (where α^* is the critical level defined in the previous Proposition) then the Traveler's Dilemma has only one Nash equilibrium where both players choose the lowest available number.*

Proof : Assume all conditions for the previous propositions hold as well. We know that for player j weak dominance does not hold any more. She has an incentive to choose lower numbers. So we know that she prefers 99 over 100. Now consider player i. Knowing this he also prefers 99. If he chooses $(n+1)$ and player j chooses n, then player $i's$ payoff is given by $n\alpha - p(1-\alpha)$. Comparing payoffs we get $n\alpha > n\alpha - p(1-\alpha)$ for all $p > 0$. Hence player i will also choose lower numbers if he knows $j's$ beliefs. So the only equilibrium with consistent beliefs will be to choose the lowest number available. ■

Let us examine what happens when we set p or r equal to zero. For $r = 0$ and $p > 0$, we find that irrespective of the value of α is, there are multiple Nash equilibria. These are located on the main diagonal. This occurs since no player has an incentive to choose a lower number anymore and the game reduces to a coordination game. When $p = 0$ and $r > 0$, we have two cases. If $\alpha_i \geq \alpha^*$ for $i = 1, 2$, then we have multiple Nash equilibria which are all on the main diagonal. However, if α is less than the critical value for one of the players then either all elements on the diagonal above or all elements on the diagonal below the main diagonal are Nash equilibria. If α is below the critical value for player 2, then all elements on the diagonal above the leading diagonal are equilibria and vice versa.

[7]In terms of Fudenberg, Kreps and Levine (1988), where $\alpha \geq \alpha^*$ implies that the games are not arbitrarily "close by". We do not however suggest any comparison with their work, since they check for robustness of different equilibria only in games that close to the original game. Since the Traveler's Dilemma has only one equilibrium the issue robustness is not an issue.

In either case, the lowest number in the strategy set will also be a Nash equilibrium. The reason for this can be found in the proofs of the two propositions above.

Remarks about Capra et al.: Capra *et al.* use a Traveler's Dilemma where players select numbers between [80, 200]. These experiment were run with 9-12 subjects and the penalty/reward parameters used were 5, 10, 20, 25, 50 and 80. Each experiment session consisted of 20 rounds with the punishment/reward parameter being changed after the tenth round. After eliminating treatment effects they estimate a logit equilibrium for the model. This is based on Anderson, Goeree and Holt's (1997) variant of the quantal response model of McKelvey and Palfrey (1995). They conclude that in the logit equilibrium, an increase in the penalty/reward parameter will reduce claims in the sense of first order stochastic dominance. The table below shows the average claims for each of the values mentioned earlier. The trend is very much in line with our results.

Penalty/Reward	Mean payoff
5	82 (3)
10	92 (15)
20	146 (15)
25	116 (15)
50	186 (7)
80	196 (4)

Table 4[8]

The presence of multiple Nash equilibria bodes well for the descriptive power of our model. Experiments (including our own pilot experiments) indicate that players pick numbers that are spread over a substantial range with the majority being in the upper range of the strategy set. In the context of our theory it would be interesting to determine the beliefs that subjects predominantly use in Traveler's Dilemma. Although it does

[8]The numbers in the parenthesis are the standard deviations.

predict that higher numbers can be chosen in the Traveler's Dilemma and they will decrease as the reward parameter changes, the presence of multiple equilibria is not very attractive. Also it seems to be more responsive to the reward than a penalty and not responsive to any asymmetry in rewards and punishments. This might indeed be true in reality. Some of these problems can perhaps be avoided by using more general versions of the true utility function. A more promising direction would be to endogenize α itself and make it a function of the punishment and rewards. This would of course require a more sophisticated modeling of the payoffs and the environment of the game. On the other hand, this model does better than using a game with vector payoffs. It can be easily checked that the vector payoffs model has multiple equilibria that are not responsive to any changes in the punishment/reward parameters.

2.3.2 Firm Performance Games

We will now look at a quantity setting duopoly from the perspective of firm performance and managerial compensation issues. Unlike the Traveller's Dilemma where the reference payoffs were relatively obvious, the reference payoffs in this case are not so apparent since it is difficult to argue against the Darwinian logic of profit maximization. However, there is substantial evidence from the marketing literature that firms are not always profit maximizing entities. In case of a private firm one could argue that the relevant reference payoffs are maximizing sales and profits. Sales maximization might be hard to justify, but empirical evidence clearly suggests that firms do not maximize profits alone. Besides, all popular corporate literature reports market share as the primary yardstick for evaluating firm performance. This in itself might compel firms to take sales maximization into account. In a setting characterized by separation of ownership and control, it is easier to justify sales maximization as an alternative payoff relevant objective (see for instance Fershtman and Judd, 1987). If we assume that the basic decision making entity in a firm are its managers, it is possible to justify a lot more than pure profit maximization. Arguably the managers care about their own compensation package which could include stock options. They may care about profits as well as their reputation as a manager which usually depends on a firm's market share. One could also argue that a the managers care about their relationship with the Board of Directors of a firm.

In case of a public firm, it is possible to argue that the reference payoffs could be maximizing profits and social welfare. The more interesting case is that of partly nationalized firms which is the main type of public firm considered here. A complete analysis of the behavior of a partly nationalized firm must consider the conflict situation between the managers representing the private owners' interest and the managers representing the government's interest. If decisions are made by voting using a majority voting rule, the firms may behave either as profit-maximizing or as welfare-maximizing. Here we will assume that the conflict between the two groups is resolved by a comprise, which captures the payoffs from the two sources.

Analyzing mixed duopolies using a vector of reference payoffs has a very simple implication. It allows us to re-interpret the subgame perfect equilibrium in mixed duopoly game as an equilibrium of a game with reference payoffs where firms follow a Nash type reasoning for both strategies and their beliefs. We show that the weights in the convex combination can be chosen to obtain the subgame perfect equilibrium profits.

In what follows we will set up a basic duopoly model and present the results for various possible forms of the true utility function. While it is possible to show that once a set of beliefs is chosen it is easy to find the equilibrium corresponding to it, we will present results for the cases that imply firms choosing optimal beliefs as well. This amounts to looking for subgame perfect equilibria for the entire game. Assume that all decisions are taken by managers.

We will consider a single period homogeneous product Cournot duopoly. The inverse demand function in this market is given by

$$p = a - bQ,\ Q = \sum_{i=1}^{2} q_{i,}\ \text{ and } a, b > 0.$$

We also assume that both firms have identical constant marginal cost functions given by $C(q_i) = cq_i$, $i = 1, 2$. We can now write the profit function as

$$\Pi_i(q_1, q_2) = (a - bQ - c)q_i, \ i = 1, 2.$$

The sales revenue function is given by

$$S_i(q_1, q_2) = (a - bQ)q_i, \ i = 1, 2.$$

The social welfare function is given by

$$W(q_1, q_2) = (1/2)[a + \{a - b(q_1 + q_2)\}](q_1 + q_2) - c(q_1 + q_2), \ i = 1, 2.$$

Suppose we now consider a mixed duopoly with two private firms which are maximizing a combination of profits and sales. Denote the true utility function of such firms by

$$O_i = \alpha\Pi_i(q_1, q_2) + (1 - \alpha)S_i(q_1, q_2).$$

PROPOSITION 3: *In the mixed duopoly described above, equilibrium price increases as α declines. If each firm uses the Nash equilibrium type reasoning for their quantity choices and beliefs, output exceeds the standard Cournot output and profits are lower. In equilibrium $\alpha^* = (6c - a)/5c$ and $q_i^* = 2(a - c)/5b$.*

Proof : See Fershtman and Judd (1987).

In our model while a Nash equilibrium is possible for any belief, we can show that the subgame perfect equilibrium requires firms to use more than Nash reasoning in their strategies. They need to find beliefs that are also best responses. While this interpretation is not the one used by Fershtman and Judd (1987), they note that this is an equivalent version the managerial incentives game, in which the competing managers make proposals to a capital market. The equilibrium identified above is the subgame-perfect equilibrium that maximizes profits and is the one that the capital market would select. It would be interesting to verify Fershtman and Judd's claim empirically since they show that maximizing this combination of profits and outputs is a dominant strategy for the owners of a firm, provided managerial compensation depends on O_i. This example demonstrates that choosing optimal beliefs and quantities following a Nash type reasoning gives us the subgame perfect equilibrium of the managerial incentives game, where the choice of q_i is made after the choice of α. The same is true of all the subsequent examples which only differ from the example above in terms of what constitutes the reference payoffs.

We will now look at an alternative model based on Fershtman (1990). This considers a state owned enterprise competing with a private firm that maximizes profits. The objective function of the public firm is given by (without loss of generality assume that the firm 1 is the public firm):

$$O_1 = \alpha W(q_1, q_2) + (1-\alpha)\Pi(q_1, q_2).$$

The private firm is assumed to be a standard profit maximizing firm. So its objective function is: $O_2 = \Pi_2(q_1, q_2)$.

PROPOSITION 4: *In a mixed duopoly described above, prices are lower as α increases. In the subgame perfect equilibrium (i.e., one in which firms select the optimal beliefs) the public firm produces the Stackelberg leader output and the private firm produces the output of the follower.*

Proof: See Fershtman (1990).

Basically what happens in both of these models is that when a firm maximizes a convex combination of different objectives, the reaction functions shift outwards, and are to the right of the reaction functions of the standard Cournot model where each firm maximizes profits. In case of sales maximization or the maximization of welfare a weight is assigned to an objective that ignores costs. While empirical testing of these outcomes might explain behavior of firms in the real world, we will now develop a model of mixed duopoly which is particularly appropriate for transition economies. The petroleum industry in Russia is still under substantial state control. However, the firms have relative autonomy in managing their everyday affairs and are also increasingly required to finance their own operations. At the same they are subject to a number of state directives. For example, the state resorts to fixing prices during the cold winter months in order to make oil available for heating purposes to everyone. So, we assume that the objective function of the state firm is given by:

$$O_1 = \alpha W(q_1, q_2) + (1-\alpha)\Pi_1(q_1, q_2).$$

In recent years, the Russian oil industry has been invaded by a number of foreign firms as well, who have set up joint ventures. We will assume that this firm which is called the private firm maximizes

$$O_2 = \beta S_2(q_1, q_2) + (1-\beta)\Pi_2(q_1, q_2).$$

The reasons for this could be attributed either to the problem of monitoring managers or because of a social norm which requires firms to pay attention to sales as well. Detailed information about the Russian oil industry can be found in a weekly called *Petroleum Reports* and a monthly publication called the *Russian Petroleum Investors,* both of which are aimed at the Western investor. For simplicity we will also the unrealistic assumption that both firms have constant marginal costs of production. This model will have different equilibria based on choice of α and β by the two firms. As a benchmark case we will derive the Nash equilibrium for this model. This amounts to finding a subgame perfect equilibrium where firms first select the weights and make their quantity decisions.

PROPOSITION 5: *In the duopolistic setting described above, the equilibrium price* *declines as α or β increase.*

Proof: Solving for $\partial O_1/\partial\alpha = 0$ and $\partial O_2/\partial\beta = 0$ gives us the reaction functions. We use these to obtain

$$q_1 = \frac{(a - c(1-\beta))}{(3-2\alpha)\beta} \text{ and } q_2 = \frac{(a - c(1 - 2\beta - \alpha + \alpha\beta))}{(3-2\alpha)b}$$

Using these two together we obtain the price:

$$p = \frac{(a(1-\alpha) - c(1 - 2\beta - \alpha + \alpha\beta))}{(3-2\alpha)b}$$

It is easy to check that $\partial p/\partial\alpha > 0$ and $\partial p/\partial\beta > 0$. ■

This is also intuitive, as a higher α or β implies a greater weight on the non-profit maximizing component. The next proposition describes the optimal weights.

PROPOSITION 6: *In this setting profits of the state owned enterprise are maximized at $\alpha = 1/2$ and profits of the private firm are maximized at $\beta = (a-c)/3c$. Also the public firm earns lower profits than its rivals.*

Proof: Using the optimal quantities and price from before we get

$$O_1(q_1, q_2) = \frac{(1-\alpha)\{a - c(\beta+1)\}^2}{(3-2\alpha)^2}$$

Solving for α, we get $\alpha = 1/2$. Notice that this is independent of β. Using this for the second firm we get

$$O_2(q_1, q_2) = \frac{\{a - c(1-3\beta)\}\{a - c(1+\beta)\}}{16b}$$

Setting the derivative to zero we get $\beta = (a-c)/3c$. Using all of this we get $O_1^* = (a-c)^2/18b$ and the profits of the second firm are given by $O_2^* = (a-c)^2/12b$. ■

Clearly both firms earn lower profits than in the standard Cournot case. However, the private firm does better than in the Fershtman and Judd model. While the public firm earns lower profits than in the Fershtman model, social welfare is highest in this framework This occurs due to the fact that in our model the public firm produces the regular Stackelberg output while the private firm produces the regular Cournot output. It is also worth noting that when $0 < \alpha < 3/4$ and $\beta = 0$, the profits of the public firm in this setting are higher than regular Cournot profits. Similarly when $\beta < (a - c)/2c$ and $\alpha = 0$, the profits of the private firm are higher than Cournot profits. We believe that this model of mixed duopoly is more realistic than the other models as it might be possible to argue that both the private and public firms are pleased with the final outcome. This might also be the reason why reforms in many industries in the transition economies have been slow. However, this also raises questions about the nature of the reform process in such economies. Notice that throughout this analysis it was assumed that agents have consistent beliefs about each other. In practice that may not be the case. This implies that *ex ante* it is difficult to justify one approach over the other unless the agents can identify the type of the other players.

2.4 Conclusion

In principle games with reference payoffs are just a kind of Bayesian game. While they may provide intuitively appealing solutions for certain types of games, they also suffer from some severe limitations. There are two major problems with the modeling: *(i)* we assume knowledge of reference payoffs by appealing to external factors, and *(ii)* in equilibrium players are required to have consistent beliefs in spite of the static interaction. The first problem can only be addressed to a limited extent. Even if one allows for the evolution of the reference payoffs on the basis on some history, they can only be endogenized partially. Since a base set of reference payoffs is still required to define the game, this *ad hoc* nature of the analysis cannot be eliminated. The second issue can be perhaps explored better in a dynamic setting using the notion of "experimentation" following the work of Mirman *et al.* (1993).

The lack of predictive power is a source of concern for models of psychological games including the type discussed here. This fact is emphasized by modelling the "type" of a player through an infinite hierarchy of beliefs which immediately shows that there is a multiplicity of equilibria. One way to resolve this would be to use the experimentation idea mentioned earlier which can help select between beliefs. Alternatively, the presence of a vector of payoffs endows these static games with a some what dynamic structure. So, it would be interesting to explore the existence of a folk theorem type result in this context.

In terms of Camerer's (1998) taxonomy, games with reference payoffs are similar to what behavioral game theory classifies as "games as social allocations", i.e., games where players are not just self-interested agents but also have social concerns. It is important to keep in mind that of this nature which tamper with the payoff function as a means of explaining behavior require extreme caution. While they may provide us with solutions to games like the Traveler's Dilemma, it is also possible that one could create a Traveler's Dilemma in a game where none existed before by selecting the payoffs in an appropriate manner. Hence results from any such analysis are very sensitive to the context and the environment of the game.

Despite all the above criticisms, this exercise highlights some simple problems. By shifting the emphasis from sophisticated reasoning to alternative payoffs, it suggests a cautious modeling of experiments with complicated structures. For example, it would be interesting to study the *guessing game* (Nagel, 1995) using reference payoffs. Since much of the motivation here stems from the attempt to explain behavior in the Traveler's Dilemma, it is encouraging to note some concurrence with experimental results. The analysis of the duopoly game suggests a wary approach to problems involving regime shifts. Finally, the theory developed here is intended to be descriptive in nature and to that extent provides a reasonable method to explain apparently anomalous behavior.

Chapter 3

Nash Networks with Heterogeneous Agents

3.1 Introduction

Information dissemination affects all aspects of economic activity. The *Internet* provides ample testimony to this fact by creating globalization that has hitherto been unprecedented in human history. Financial crises in one country now have devastating consequences for other economies as the contagion moves across boundaries with relative ease. Fashion and fads emerging in one country are easily communicated across the world with almost no time lag. However, the East Asian financial crises also demonstrated that Asian economies where information networks were relatively primitive remained largely insulated from the crisis, indicating that the structure and technology of information dispersion are very important economic characteristics. Information in most societies can either be obtained in the market-place or through a non-market environment like a social network. For isntance, in developed countries credit agencies provide credit ratings for borrowers, while in many developing countries credit worthiness is assessed through a social network organized along ethnic lines.

In this paper we look at the formation of social networks which serves as a mechanism for information transmission. Such networks have played a vital role in the diffusion of information across society in settings as diverse as referral networks for jobs (Granovetter (1974) and Loury (1977)) and in assessing quality of products ranging from cars to computers (Rogers and Kincaid (1981)). Our agents are endowed with some information which can be accessed by other agents forming links with them. Link formation is costly and the links transmit information randomly − which introduces uncertainty into the social network. In terms of the networks literature ours is a non-cooperative model of network formation that closely follows Bala and Goyal (1999a and 1999b). However, we introduce heterogeneous agents by allowing link failures to be different across agents. This may describe the nature of the transmission technology or the quality of the information. The generalization provides a richer model in terms of answering theoretical as well as practical questions. Models with heterogeneous agents have not been explored much in the networks literature. A notable exception are Johnson and Gilles (1999) who introduce spatial heterogeneity of agents. Their model and ours differ in two respects: the kind of agent heterogeneity and the equilibrium concept. They follow Jackson and Wolinsky (1996) and use pairwise stability as the equilibrium. We analyze Nash networks.

Agents in our model can form links and participate in a network by incurring a cost for each, which may be interpreted in terms of time, money or effort. The cost of establishing a link is incurred only by the agent who initiates it, while the flow of benefits is both ways. The initiating agent has access to the other agent's information with a certain probability. In addition, he has access to the information from all the links of the other agent. Thus each link can generate substantial externalities of a non-rival nature in the network. Since the strength of ties varies across agents and links fail with possibly different probabilities, the flow of benefits differs across agents. This reflects the fact that in reality, communication often embodies a degree of costly uncertainty. We frequently have to ask someone to reiterate what they tell us, explain it again and even seek second opinions.[1]

Bala and Goyal suggest telephone calls as an example of such networks. Another example (especially of the star networks considered here) of this type is a LISTSERVE or an e-mail network. Costs have to be incurred in setting up and joining such electronic networks, but being a part of the network does not automatically ensure access to the information of other agents. Members participation in the electronic network can vary, and mail sent by a member may even get lost as in the celebrated "E-mail Game" (Rubinstein, (1989)).[2]

Foreign immigrants are often members of such networks. When an immigrant lands on the shores of a foreign country he usually has a list of people from the home country to get in touch with. Once contacted, some compatriots are more helpful than others. Often a substantial information exchange takes place in this process, where the new arrival learns about the foreign country, while providing the established immigrants news about the home country and allowing them to indulge in nostalgia. Sometimes, the new immigrant might also bring small gifts and hard to find delicacies from the home country. Some process of this kind is usually a part of the graduate school experience of every foreign student.

Motivated by these examples, we follow Bala and Goyal (1999a, b) to develop a non-cooperative model of network formation. Our focus lies on the structural properties of Nash networks. Agents choose to form links on the basis of costs and stochastic flow of benefits that accrue from links. Unlike Bala and Goyal (1999a, b) we introduce agent

[1]In the intelligence community where the sole commodity is information, a link is established when a "handkerchief is dropped" by one group and is picked up by the potential spy. However, the information provided by the new recruit (if any) may not be useful or might even be of a dubious nature. This leads to a large network of agents in order to gather and substantiate the information.

[2]Like most of the networks literature we preclude the possibility of wasteful information like phone calls which are a nuisance. The same is true for intermediate agents (or indirect links) in a network who function as purveyors of informations between other agents.

heterogeneity by allowing for the probability of success to differ across links.

Besides imparting greater realism to the model, the introduction of heterogeneous agents alters results significantly. We are able to show that social networks can exhibit widely different features with heterogeneous agents. We find that Nash networks may be nested and Pareto-ranked. We find inefficient Nash networks that are Pareto-optimal. Bala and Goyal show that Nash networks must either be connected or empty. With heterogeneous agents, this is true only when the probabilities of success are not very different from each other. The range in which the probabilities must lie depends on the cost of links and the cardinality of the player set. Bala and Goyal (1999b) show that for a given level of costs and probability of link success, asymptotically redundant networks always exist. With agent heterogeneity neither connectedness nor super-connectedness need arise asymptotically. For star networks we find that while in equilibrium probabilities must lie in a certain range exceeding costs, they must satisfy additional conditions. In particular, it never pays in the Bala and Goyal framework to connect to the center of the star indirectly. In our case, this connection might be beneficial and further conditions on probabilities are required to prevent such connections. Moreover, the coordination problem inherent in selecting the central agent with a common probability of link success is no longer very serious now.

Three criticisms of the non-cooperative approach to network formation are also addressed. We extend the model to allow for duplication of links and to analyze Nash networks with incomplete information. We find that sometimes redundant links may be established if the agents beliefs about the probabilities of the indirect links are less than the actual ones even for a single pair of agents. Finally, the implications of mutual consent for such Nash networks are discussed. We present a simple model of incomplete information where agents are unaware of the success probabilities of indirect links.

In the next section we review the most relevant literature. Section 3.3 introduces the basic notation and terminology used throughout the paper. In Section 3.4, we present some general results on Nash networks. We consider three alternative formulations of the model are considered in Section 3.5. Section 3.6 concludes.

3.2 The Networks Literature

There is an earlier literature in industrial organization which considers (positive) consumption and production externalities called network externalities; see Economides (1996) for an excellent survey. This literature points out the similarities between the economic structure of networks and the structure of vertically related industries. The impact of externalities on pricing and market structures is analyzed along with their implications for issues like coordination on technical standards and compatibility.

There is a partly informal, partly empirical literature in sociology on social networks. This literature focuses mainly on gender relationships and family demographics (Astone, *et al.* 1999). There is also a more formal modelling in sociology where social networks are treated as stochastic phenomena; see for instance, Wasserman (1980) and Mayer (1984). Their objective is to identify the steady state of the evolving network using differential equations. This literature does not incorporate any strategic or incentive-based behavior and all interaction between agents is random. The institution of a social network can also be related to the concept of social capital. Social capital which has gained wide currency in popular discourse due to the work of James Coleman (1998, 1990) may be defined as the resources that emerge from one's social ties. Information flows are an important component of social capital. Besides the examples cited above, they have been crucial in explaining the formation of cities (Jacobs (1961)), informal lending behavior (Biggs *et al.* (1996)) and diffusion of innovations (Rogers and Shoemaker, 1971).

Many substantive areas in marketing research also use social networks and dyadic relationships as modelling techniques. Such models are utilized in analyzing, among other things, issues like conflict resolution in family purchasing, coalition formation in buying centres, power and concentration in marketing channel dyads and identification of opinion leaders in word-of-mouth networks (Iacobucci and Hopkins, (1992)). The emphasis in this literature is usually on the network density, identifying the more active members of the networks and the implications of the hierarchical structure of the network.

Recently, the idea of local interactions has made inroads into economically motivated evolutionary game theory, see for example, Ellison (1993), Blume (1995), Young (1998), and Baron *et al.* (2000). Models of neural and other networks developed in neuroscience, computer science, electrical engineering and physics lend themselves for adoption and adaptation in social science, in particular to formulate local interaction between boundedly rational agents; see for instance Berninghaus and Schwalbe (1996) and Haller and Outkin (1999). Droste *et al.* (2000), to be discussed below, is the only paper in this tradition that addresses network formation.

There are three main strands of literature that have recently emerged in the context of networks in economics and game theory of concern to us. They are differentiated by their use of cooperative game theory, the notion of pairwise stability and the noncooperative game formulation, respectively.

One approach in the cooperative framework concentrates on the costs of forming social and economic relationships. Debreu (1969), Haller (1994) and Gilles and Ruys (1990) theorize that costs may be described by a topological structure on the set of individuals called *cost topology*. In Debreu (1969) and Gilles and Ruys (1990) for instance, the space in which agents are located is a topological characteristic space where "neighbors" denotes agents who are similar in characteristics. Haller (1994) studies more general cost topologies. However, neglecting the benefits from network formation prevents these theories from dealing with the hypothesis that greater disparity between agents can lead to more beneficial interactions between them. Another strand of the literature tries to rectify this problem, by assuming a given cost structure and focusing on the benefits from agent interaction. The allocation problem is examined by treating the costs as a set of constraints on coalition formation (see for example, Myerson (1977), Kalai *et al.* (1978) and Gilles *et al.* (1994)). An excellent survey of this literature can be found in van den Nouweland (1993), and Borm, van den Nouweland and Tijs (1994). Aumann and Myerson (1988) have developed a model that incorporates both costs and benefits within the cooperative game theory framework. This line of research has been extended by Slikker and van den Nouweland (1999). They provide a full characterization of the three-person case and show that further assumptions are necessary for a meaningful examination of cases involving more than three players.

Jackson and Wolinsky (1996) introduced the concept of pairwise stability (known from the matching literature) as an equilibrium concept leading to a completely new strand of the literature. Pairwise stability bridges the cooperative and non-cooperative elements in network formation and requires mutual consent of both agents for link formation. A pairwise stable network consists of a set of links such that no two individuals would choose to create a link if none exists between them and no pair would like to sever the link between them either. Thus it is a weak stability concept where no pair of players regrets that the "game is over." This concept is quite desirable as it relies on a cost-benefit analysis of network formation and allows for both link formation and severance of links. Jackson and Wolinsky (1996) characterize all pairwise stable networks that result in their framework and find that the star network and the complete network are the most predominant ones. Dutta and Muttuswami (1997) and Watts (1997) refine the Jackson-Wolinksy framework further by introducing other stability concepts and derive implementation results for these concepts. Jackson and Watts (1998) consider dynamic network formation where the network structure determines payoffs from social or economic activity. Agents make and break links over time based on payoff comparison with the network in place. They focus on the sequence of networks called improving paths that lead to higher payoffs. The evolution of these networks is stochastic due to the presence of a small probability of error. They find that in some cases the evolutionary process selects inefficient networks, even though the efficient ones are statically stable. In a more recent paper, Jackson and Watts (1999) consider coordination games played on a network. The choice of partners in the game is endogenous and players are periodically allowed to add or sever links. Each player uses the same strategy with all her partners and prospective partners on chosen based on their past behavior. They find multiple stochastically stable states, some of which involve strategies that are neither efficient nor risk dominant. Goyal and Vega-Redondo (1999) examine the effect of incentives on endogenous network formation. The formation links in their model is costly and agents must choose the same strategy for all games they play in any network. Agents are allowed to revise their links periodically. When agents are allowed to play only if they have direct links with, the complete network emerges as the equilibrium. The star network is the equilibrium network if agents are allowed to transact even if they are only indirectly linked. They find that for low costs of link formation, agents coordinate on the risk dominant action, while with high costs of link formation the efficient outcome is chosen as the equilibrium.

Johnson and Gilles (1999) introduce a spatial dimension to the Jackson-Wolinsky model. The spatial nature of the costs may be interpreted as geographic, social or individual differences. Their model shows that the complete network and the star are no longer so ubiquitous. For low link costs they find a rich pattern of pairwise stable networks in which locally complete networks play a prominent role. The spatial formulation reduces the number of cases where coordination might be an issue among the agents. They also address efficiency and implementation issues. Droste *et al.* (2000) also analyze a spatial model where agents are assigned fixed locations on a circle. Players create their own interaction neighborhood by forming and severing links with other players based on mutual consent. The cost of link formation is based on the distance between agents. In the second stage of the game agents play a coordination game. Players react to their environment myopically by deciding about both pure strategies in the coordination game and link formation based on a best-reply dynamics. The risk-dominant one is found to be the unique stochastically stable convention in the long run.

The noncooperative model of network formation is developed in two papers by Bala and Goyal (1999a and 1999b) leading to Nash networks. They assume that a player can create a one-sided link with another player by making the appropriate investment. This assumption differs fundamentally from the concept of pairwise stability since mutual consent of both the players is no longer required for link formation. They find that the set of Nash networks can differ from those obtained under pairwise stability. They also investigate the reliability issue in networks by allowing links to fail independently of each other with a certain probability. Both in the deterministic and stochastic model they find that Nash networks are either empty or connected, a fact which does not always hold up with heterogeneous agents. One of the central findings of their papers is that imperfect reliability has very different effects on network formation compared to information decay. With information decay they find that minimally connected networks (notably the star) are Nash for a large range of costs and the decay parameter, independently of the size of society. However, with imperfect reliability, they find that minimally connected networks are increasingly replaced by super-connected networks as the player set increases and link formation is only moderately expensive. This constitutes another result that is not always sustained under agent heterogeneity. They also study the formation of Nash networks using a modified version of the best-response dynamic and identify strict Nash networks.

Our model belongs to the non-cooperative tradition and is a generalization of Bala and Goyal (1999b).[3] The remainder of the paper is devoted to developing the model and analyzing results.

3.3 The Model

Let $N = \{1, \ldots, n\}$ denote the set of agents and let i and j be typical members of this set. For ordered pairs $(i, j) \in N \times N$, the shorthand notation ij is used. The symbol $\subset$ for set inclusion permits equality. We assume throughout that $n \geq 3$. Each agent has some information of value to the other agents and can have access to more information by forming links with the other agents. The formation of links is costly and each link denotes a connection between a pair of agents. Agents form links simultaneously in this model. However, links are not fully reliable and may fail to transmit information with a positive probability than can differ across links.

Each agent's strategy is a vector $g_i = (g_{i1}, \ldots, g_{ii-1}, g_{ii+1}, \ldots, g_{in})$ where $i \in N$ and $g_{ij} \in \{0, 1\}$ for each $j \in N \backslash \{i\}$. The value $g_{ij} = 1$ means that agents i and j have a link initiated by i whereas $g_{ij} = 0$ means that agent i does not initiate the link. This does not preclude the possibility of agent j initiating a link with i. A link between agents i and j potentially allows for **two-way (symmetric) flow of information**. The set of all pure strategies of agent i is denoted by $\mathcal{G}_i$. We focus only on pure strategies in this paper. Given that agent i has the option of forming or not forming a link with each of the remaining $n - 1$ agents, the number of strategies available to agent i is $|\mathcal{G}_i| = 2^{n-1}$. The strategy space of all agents is given by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$. A strategy profile $g = (g_1, \ldots, g_n)$ can be represented as a **directed graph** or **network**. Notice that there is a one-to-one correspondence between the set of all directed networks with n vertices

[3]In both papers, an imperfectly reliable network is modelled as a random graph. Random graphs have been used to model communication opportunities among traders in large markets (Kirman(1983), Kirman *et al.* (1986), Haller (1990) and Ioannides (1990)). For work in sociology refer to Holland and Leinhardt (1977) and Kindermann and Snell (1980).

or nodes and the set of strategies $\mathcal{G}$. The link g_{ij} will be represented pictorially by an edge starting at j with the arrowhead pointing towards i to indicate that agent i has initiated the link. This is shown in the figure below where agents 1 and 2 establish the links with agent 3 and bear the cost of forming the links. Thus the arrowhead always points towards the agent who pays for the link.

Insert **Figure 1** here:

For describing information flows in the network, let for $i \in N$ and $g \in \mathcal{G}$, $\mu_i^d(g_i) = |\{k \in N : g_{ik} = 1\}|$ denote the number of links in g initiated by i which is used in the determination of i's costs. Next we define the closure of g which is instrumental for computing benefits, since we are concerned with the symmetric, two-way flow of benefits.

Definition 5 *The* ***closure*** *of g is a non-directed network denoted by $h = cl(g)$ and defined as $cl(g) = \{ij \in N \times N : i \neq j$ and $g_{ij} = 1$ or $g_{ji} = 1\}$.*

Pictorially the closure of a network is equivalent to replacing each directed edge of g by a non-directed one. In Figure 1, this amounts to assuming away the arrowheads.

Benefits. The benefits from network g are derived from $h = cl(g)$. Each link $h_{ij} = 1$ succeeds with probability $p_{ij} \in (0, 1)$ and fails with probability $(1 - p_{ij})$ where p_{ij} is not necessarily equal to p_{ik} for $j \neq k$. It is assumed, however, that $p_{ij} = p_{ji}$. Furthermore, the success or failure of different links are assumed to be independent events. Thus, h may be regarded as a random network with possibly non-equiprobable probabilities. We define h' as a realization of h (denoted by $h' \subset h$) if for all i, j with $i \neq j$ we have $h'_{ij} \leq h_{ij}$.

At this point the concept of a path (in h') between two agents proves useful.

Definition 6 *For $h' \subset h$, a* ***path*** *of length m from an agent i to a different agent j is a finite sequence $i_0, i_1, \ldots, i_m$ of pairwise distinct agents such that $i_0 = i$, $i_m = j$, and $h_{i_k i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. We say that player i* ***observes*** *player j in the realization h', if there exists a path from i to j in h'.*

Invoking the assumption of independence, the probability of the network h' being realized given h is

$$\lambda(h' \mid h) = \prod_{ij \in h'} p_{ij} \prod_{ij \in h \setminus h'} (1 - p_{ij}).$$

Let $\mu_i(h')$ be the number of players that agent i observes in the realization h', i.e. the number of players to whom i is directly or indirectly linked in h'. Each observed agent in a realization yields a benefit $V > 0$ to agent i. Without loss of generality assume that $V = 1$.[4]

Given the strategy tuple g agent i's expected benefit from the random network h is given by the following benefit function $B_i(h)$:

$$B_i(h) = \sum_{h' \subset h} \lambda(h' \mid h) \mu_i(h')$$

where $h = cl(g)$. The probability that network h' is realized is $\lambda(h' \mid h)$, in which case agent i gets access to the information of $\mu_i(h')$ agents in total. Note that the benefit function is clearly non-decreasing in the number of links for all the agents.

[4]Another formulation could be used to obtain agent heterogeneity. Under this formulation, the value of agent i's information would be given by V_i which differs across agents, while p, the probability of the link success, is identical for all agents $i \in N$. The direct expected benefit from a link g_{ij} would now be given by pV_i which would then differ across links. Alternatively, instead of depending on the recipient, the expected benefit might depend on the source of information and thus be pV_j. In contrast, Johnson and Gilles (2000) assume $p = 1$ and $V = 1$, but differences in costs based on a spatial distribution of agents.

Payoffs. We assume that each link formed by agent i costs $c > 0$. Agent i's expected payoff from the strategy tuple g is

$$\Pi_i(g) = B_i(cl(g)) - \mu_i^d(g_i)c.$$

Given a network $g \in \mathcal{G}$, let g_{-i} denote the network that remains when all of agent i's links have been removed. Clearly $g = g_i \oplus g_{-i}$ where the symbol $\oplus$ indicates that g is formed by the union of links in g_i and g_{-i}.

Definition 7 *A strategy g_i is said to be the **best response** of agent i to g_{-i} if*

$$\Pi_i(g_i \oplus g_{-i}) \geq \Pi_i(g'_i \oplus g_{-i}) \qquad \text{for all } g'_i \in \mathcal{G}_i.$$

Let $BR_i(g_{-i})$ denote the set of agent i's best response to g_{-i}. A network $g = (g_1, \ldots, g_n)$ is said to be a **Nash network** if $g_i \in BR_i(g_{-i})$ for each i, i.e., agents are playing a Nash equilibrium. A strict Nash network is one where agents are playing strict best responses.

Agent i's benefit from the direct link ij to agent j is at most $p_{ij}(n-1)$. Set $p_0 = p_0(c, n) = c \cdot (n-1)^{-1}$. If $p_{ij} < p_0$, it never benefits agent i to initiate a link from i to j, no matter how reliably agent j is linked to other agents and, therefore, $g_{ij} = 0$ in any Nash equilibrium g.

We now introduce some additional definitions which are of a more graph-theoretic nature.

A network g is said to be **connected** if there is a path in $h = cl(g)$, between any two agents i and j. A connected network g is said to be **super-connected**, if the network is still connected after any link is deleted. A connected network g is **minimally connected**, if it is no longer connected after the deletion of any link. A network g is called **complete**, if all links exist in $cl(g)$. A network with no links is called an **empty**

network. The geodesic distance between two agents i and j denoted by $d(i, j; h)$ is the number of links in the shortest path between them in h.

Definition 8 *A set $C \subset N$ is called a **component** of g if there is a path in $cl(g)$ for every pair of agents i and j and there is no strict superset C' of C for which this is true.*

The welfare measure here is defined as the sum of the welfare of all the agents. Formally, let $W : \mathcal{G} \rightarrow \mathcal{R}$ be defined as

$$W(g) = \sum_{i=1}^{n} \Pi_i(g) \qquad \text{for } g \in \mathcal{G}$$

Definition 9 *A network g is **efficient** if $W(g) \geq W(g')$ for all $g' \in \mathcal{G}$.*

An efficient network is one that maximizes the total value of information made available to all agents less the aggregate costs of forming the links. The definition of (strict) Pareto-optimality is the usual one: A network g is **Pareto-optimal**, if there does not exist another network g' such that $\Pi_i(g') \geq \Pi_i(g)$ for all i and $\Pi_i(g') > \Pi_i(g)$ for some. Obviously, every efficient network is Pareto-optimal. However, we will show that not every Pareto-optimal network is efficient. In fact, we will present an example with a Pareto-optimal Nash network which is inefficient while the unique efficient network is not Nash. Hence a violation of the very popular efficiency criterion has to be interpreted with some caution.

We finally introduce the notion of an essential network. A network $g \in \mathcal{G}$ is **essential** if $g_{ij} = 1$ implies $g_{ji} = 0$. Note that if $g \in \mathcal{G}$ is a Nash network or an efficient network, then it must be essential. This follows from the fact that the benefits from a link are given by the closure of the link $h_{ij} = \{\max g_{ij}, g_{ji}\}$ (making the probability of failure independent of whether it is a single link or a double link) and from the fact that the information flow is symmetric and independent of which agent invests in forming the link. If $g_{ij} = 1$, then by the definition of Π_j agent j pays an additional cost c for setting

$g_{ji} = 1$, while neither he nor anyone else gets any benefit from it. Hence if g is not essential it cannot be Nash or efficient.

3.4 Nash Networks

In this section we look at Nash networks. We first discuss efficiency issues by means of examples. This discussion is followed by an analysis of connectedness and redundancy in Nash networks. We also identify conditions under which the complete network and the empty network, respectively, will be Nash. A further subsection covers the popular star networks. Finally, we examine some examples of other network architectures.

3.4.1 Efficiency Issues

Efficiency is a key issue in Jackson and Wolinsky (1996), Bala and Goyal (1999a,b), Johnson and Gilles (2000). When costs are very high or very low, or when links are highly reliable, there is virtually no conflict between Nash networks and efficiency in the Bala and Goyal (1999b) framework. This observation still holds in our context. However, there is a conflict between Nash networks and efficiency for intermediate ranges of costs and link reliability, even with the same probability of link failure for all links. In particular, Nash networks may be under-connected relative to the social optimum as the subsequent example shows.

Let us add two important observations not made before. First, it is possible that Nash networks are nested and Pareto-ranked. Second, at least in our context, the following can coexist: a Nash network which is not efficient, but Pareto-optimal and a unique efficient network which is not Nash and does not weakly Pareto-dominate the Nash network. The first observation is supported by the following example: $c = 1$, $n = 4$ and $p_{ij} = 0.51$ for all ij. In this case, both the empty network and the outward pointing star with center 4, are Nash networks. The "outward pointing star" consisting of the links 14, 24 and 34 contains and strictly Pareto-dominates the empty network. Moreover, the empty network is under-connected. Our second observation is based on the following example. See Figure 2 below.

Insert **Figure 2** here:

Example 1: $c = 1$, $n = 7$. $p_{16} = p_{26} = p_{37} = p_{47} = p = 0.6181$, $p_{56} = a = 0.2$, $p_{67} = b = 0.3$, and corresponding probabilities for the symmetric links. All other links have probabilities $p_{ij} < p_0$. Now g given by $g_{16} = g_{26} = g_{37} = g_{47} = 1$ and $g_{ij} = 0$ otherwise is a Nash network. Indeed, p is barely large enough to make this a Nash network. The critical value for p satisfies $p(1 + p) = 1$ with solution 0.6180.... But g is not efficient. Linking also 5 with 6 and 6 with 7 provides the following added benefits where we use $2p = 1.2362$ and $1 + 2p = 2.2362$:

For 1+2:	$1.2362 \cdot (a + b \cdot 2.2362)$	$=$	1.07656
For 3+4:	$1.2362 \cdot b \cdot (a + 2.2362)$	$=$	0.90349
For 5:	$a \cdot 2.2362 + ab \cdot 2.2362$	$=$	0.58141
For 6:	$a + b \cdot 2.2362$	$=$	0.87086
For 7:	$b \cdot (a + 2.2362)$	$=$	0.73086
Total:			4.16318

Hence the total added benefit exceeds the cost of establishing these two additional links. The network thus created would be efficient. But neither 6 nor 7 benefits enough from the additional link between them to cover the cost of the link. Hence the enlarged efficient network is not Nash. Since the rules of the game stipulate that one of the two agents assumes the entire cost of the new link, the enlarged efficient network cannot weakly Pareto-dominate g. In fact, g is Pareto-optimal while inefficient. Reconciling efficiency and Pareto-optimality would require the possibility of cost sharing and side payments. ■

3.4.2 Connectivity and Super-Connectivity

With homogeneous agents, Nash networks are either connected or empty (Bala and Goyal (1999a)). With heterogeneous agents, this dichotomy does not always hold as the previous example shows. The proposition below identifies conditions under which Nash networks will show this property.

PROPOSITION 1: *If* $p_{ij} \geq \dfrac{1}{1+c/n^2}\, p_{mk}$ *for any* $i \neq j$ *and* $m \neq k$*, then every Nash network is either empty or connected.*

Proof: Consider a Nash network g. Suppose g is neither empty nor connected. Then there exist three agents i, j, and k such that i and j belong to one connected component of $cl(g)$, C_1 and k belongs to a different connected component of $cl(g)$, C_2. Then $g_{ij} = 1$ or $g_{ji} = 1$, whereas $g_{mk} = g_{km} = 0$ for all $m \in C_1$. Without loss of generality assume $g_{ij} = 1$. Then the incremental benefit to i of having the link from i to j is $b_1 \geq c$. Let g' denote the network which one obtains, if in g all direct links with i as a vertex are severed. The incremental expected benefit to i of having the link ij in g' is $b_2 \geq b_1 \geq c$ and can be written as $b_2 = p_{ij}(1+V_j)$ where V_j is the j's expected benefit from all the links j has in addition to ij.

Now consider a link from k to j, given $g' \oplus g_{ij}$. This link is worth $b_3 = p_{kj}(p_{ij}+1+V_j)$

to k. A link from k to j, given g, is worth $b_4 \geq b_3$ to k. We claim that $b_3 > b_2$, i.e.,

$$p_{kj} > \frac{1 + V_j}{1 + V_j + p_{ij}}$$

Since g is Nash and $g_{ij} = 1$, we know $p_{ij} \geq p_0 > c/n$. By assumption, $p_{kj} \geq \dfrac{1}{1 + c/n^2} p_{ij}$. Therefore,

$$p_{kj} > \frac{1}{1 + p_{ij}/n} p_{ij} = p_{ij} \frac{1 + n - 1}{1 + n - 1 + p_{ij}} \geq \frac{1 + V_j}{1 + V_j + p_{ij}}$$

where we use the fact that V_j is bounded above by $n - 1$. This shows the claim that $b_4 \geq b_3 > b_2 \geq b_1 \geq c$. Initiating the link kj is better for k than not initiating it, contradicting that g is Nash. Hence to the contrary, g has to be either empty or connected. ■

This result means that if the probabilities are not too widely divergent, then the empty versus connected dichotomy still holds. If however, the probabilities are widely divergent, then a host of possibilities can arise and a single dichotomous characterization is no longer adequate.

Bala and Goyal (1999b) further show that under imperfect reliability, for homogeneous agents, Nash networks become super-connected as the size of the society increases. This is really easy to see for the star network. With an increase in the number of players, each player's connection to the star becomes more and more important since it remains the lifeline to all the other agents' information. At some point players will insure themselves against failure of this linkage by establishing more connections. This result warrants several qualifications. The first one concerns an obvious trade-off even in the case of homogeneous agents. While it is correct that for any given probability of success $p > 0$, super-connectivity obtains asymptotically, the minimum number of players it takes to get super-connectivity goes to infinity as p goes to zero. Let n^* be any number of agents. If $p < p_0(c, n^*)$, then it takes at least n^*+1 agents to obtain even a connected Nash network.

Secondly, in our model with heterogeneous agents, asymptotic connectivity need no longer obtain, eliminating any scope for super-connectivity. Consider an infinite sequence of agents $i = 1, 2, \ldots, n, \ldots$ and a sequence of probabilities $p_2, p_3, \ldots$ such that $p_{ij} = p_{ji} = p_j$ for $i < j$. Then the sequence $p_k, k \geq 2$, can be constructed in such a way that the empty network is the only Nash network for any agent set $I_n = \{1, \ldots, n\}$, $n \geq 2$.

Thirdly, with heterogeneous agents, asymptotic super-connectivity obtains, if there exists a $q_0 > 0$ such that $p_{ij} \geq q_0$ for all ij. The argument for homogeneous agents can easily be adapted to this case.

Fourthly, the lack of a common positive lower bound for the success probabilities does not necessarily rule out asymptotic super-connectivity, provided the probabilities do not drop too fast. A positive example is given by $c = 1$ and $p_{ij} = p_{ji} = p_j = j^{-1/2}$ for $i < j$. Basically, the argument developed for homogeneous agents can be applied here, too. This follows from the fact that for $1 < m < n$,

$$\sum_{i=m}^{n} p_{1i} > \int_{m}^{n+1} s^{-1/2} ds = [2s^{1/2}]_{m}^{n+1} = 2((n+1)^{1/2} - m^{1/2}).$$

Furthermore, with heterogeneous agents, other possibilities exist. For instance, super-connectivity may be established at some point, but connectivity may break down when further agents are added and reemerge later, etc. Or several connected components can persist with super-connectivity within each component. Thus the Bala and Goyal result is altered significantly in our model.

3.4.3 The Polar Cases

The next proposition identifies conditions under which the complete network and the empty network are Nash. Let $\underline{p} = \min\{p_{ij}\}$ denote the probability of the least reliable link.

PROPOSITION 2: *If* $\underline{p} \in (0,1)$, *then there exists* $c(\underline{p}) > 0$ *such that each complete network* g^c *is (strict) Nash for all* $c \in (0, c(\underline{p}))$. *The empty network is strict Nash for* $c > \max\{p_{ij}\}$.

Proof: Let $g = g_i \oplus g_{-i}$ be a complete network and suppose that agent i has one or more links in his strategy g_i. Let g^o be a network where at least one of agent i's links have been deleted, ceteris paribus. We already know that $B_i(h^o) < B_i(h)$, where $h^o = cl(g^o)$ and $h = cl(g)$. Clearly it can be seen that if $c = 0$ then g_i is a strict best response for agent i. By continuity, there exists $c_i(\underline{p}) > 0$ for which g_i is a strict best response for all $c \in (0, c_i(\underline{p}))$. Now let $c(\underline{p}) = \min_i c_i(\underline{p})$ over all agents i who have one or more links in their strategy g_i. The first part of the claim follows from this.

For the second part, if $c > \max\{p_{ij}\}$ and no other agent forms a link, then it will not be worthwhile for agent i to form a link. Hence the empty network is strict Nash under these conditions. ■

3.4.4 Star Networks

We now identify conditions under which different types of star networks are Nash equilibria. Star networks are characterized by one agent who is at the center of the network and the property that the other players can only access each other through the central agent. There are three possible types of star networks. The inward pointing (center-sponsored) star has one central agent who establishes links to all other agents and incurs the cost of the entire network. An outward pointing (periphery-sponsored) star has a central agent with whom all the other $n-1$ players form links. A mixed star is a combination of the inward and outward pointing stars.

Insert **Figures 3(a)-(c)** here:

While the method of computing Nash networks does not change with the introduction of heterogeneous agents, the process of identifying the different parameter ranges for Nash networks is now significantly complicated. We will next establish two claims about Nash networks in our setting to illustrate the complex nature of the problem with agent heterogeneity. Without loss of generality we will assume that player n is the central agent in all the three types of stars. Define M to be the set of all the agents except n or $M = N\backslash\{n\}$ and let $K_m = M\backslash\{m\}$ be the set M without agent m. Also let $J_k = K_m\backslash\{k\}$ be the set K_m without agent k.

PROPOSITION 3: *Given $c \in (0, 1)$ there exists a threshold probability $\delta \in (c, 1)$, such that for $p_{ij} \in (\delta, 1)$:*
(a) The inward pointing star is Nash when the central agent's worst link yields higher benefits than c.
(b) The outward pointing star is Nash if either every non-central agent ($i \neq n$) has her best link with the central agent or the benefits that accrue to this agent through two indirect links outweigh the benefits through more direct links.
(c) The mixed star is Nash when both the conditions above are satisfied, the first one for the central agent, and the second one for all agents who form links with the central agent.

Proof: The proposition consists of a common statement and condition(s) that are particular to each type of star. The common statement identifies a threshold probability level for each given cost. So, it demarcates the range in which the probabilities can lie, while the individual condition(s) relevant to each star architecture are condition(s) that must be satisfied by the probabilities of link success lying in the range defined by the common condition.

(a) Consider first the inward pointing star with the central agent making all the connections. Choose the threshold probability $\delta \in (c, 1)$ to satisfy the following inequality

$$\max_{m \in M} \left[(1 - p_{nm}) + \left(1 - p_{nm} \sum_{k \in K_m} p_{nk} \right) \right] < c$$

for all $p_{ij} \in (\delta, 1)$. For the central agent to be maintaining all the links we need that $\sum_{m \in M} p_{nm} - (n-1)c > 0$. In order to ensure this we just need that n does not wish to sever his worst link, i.e., $\min_{m \in M} p_{nm} > c$. Further, we must verify that no agent $m \in M$ wants to add an extra link. Now consider an agent $m \in M$. His payoff from the inward pointing star then is given by $\Pi_m(g^{in}) = p_{nm} + p_{nm} \sum_{k \in K_m} p_{nk}$. Supposing agent m now wishes to form an extra link. He will never wish to form a link with n since the flow of benefits from any link is two ways. So, denote this link as g_{mk} where $k \in K_m$. However, we know that payoffs from $\Pi_m(g^{in} \oplus g_{mk})$ are bounded above by $(n-1) - c$. Taking the difference between this and $\Pi_m(g^{in})$, an extra link will not be added when $\Pi_m(g^{in} \oplus g_{mk}) - \Pi_m(g^{in}) < 0$, so in particular not when:

$$(n-1) - c - \left(p_{nm} + p_{nm} \sum_{k \in K_m} p_{nk} \right) < 0$$

This gives us $[(1 - p_{nm}) + (1 - p_{n1}p_{nm}) + \cdots + (1 - p_{nm-1}p_{nm}) + (1 - p_{nm+1}p_{nm}) + \cdots + (1 - p_{nn-1}p_{nm})] < c$. In order to ensure that no agent $m \in M$ wants to form an extra link we must verify that this inequality holds for all of them, in particular, the agent with the lowest payoffs in the Nash equilibrium does not wish to deviate:

$$(n-1) - c - \min_{m \in M} \left(p_{nm} + p_{nm} \sum_{k \in K_m} p_{nk} \right) < 0$$

which gives us $\max_{m \in M} [(1 - p_{nm}) + (1 - p_{n1}p_{nm}) + \cdots + (1 - p_{nm-1}p_{nm}) + (1 - p_{nm+1}p_{nm}) + \cdots + (1 - p_{nn-1}p_{nm})] < c$, the condition stated above is used determine δ. This completes the claim since no agent wants to break a link or form a link provided $min\ p_{ij} > c$.

(b) Next consider the outward pointing star and once again let agent n be the central agent. Choose the threshold probability $\delta \in (c, 1)$ to satisfy the following inequality

$$\max_{m\in M}\left[(1-p_{nm})+\left(1-p_{nm}\sum_{k\in K_m}p_{nk}\right)\right]<c$$

for all $p_{ij}\in(\delta,1)$. Let us first identify the conditions under which no player wants to deviate. We know that n has no links to sever, and cannot add a link since $g_{mn}=1$ for all $m\in M$ and the flow of benefits is two-way. Now consider an agent $m\neq n$ who might wish to sever the link with n and instead link with some other $k\in K_m$. Player m's payoff from the outward pointing star is $\Pi_m(g^{ot})=p_{mn}+p_{mn}\sum_{k\in K_m}p_{kn}-c$. His payoff from deviating and forming the new link is $\Pi_m(g^{ot}-g_{mn}+g_{mk})=p_{mk}+p_{mk}p_{kn}+p_{mk}p_{kn}\sum_{j\in J_k}p_{jn}-c$. Solving for the case of no deviation we get $\Pi_i(g^{ot})-\Pi_i(g^{ot}-g_{in}+g_{ik})$:

$$(p_{mn}-p_{mk})+p_{kn}(p_{mn}-p_{mk})+(p_{mn}-p_{mk}p_{kn})\sum_{j\in J_k}p_{jn}$$

This is clearly positive when $p_{mn}>p_{mk}$ for all $m\in M$. So, no agent will wish to deviate by breaking the link with n and forming a link with another agent if their best link is with n. However, when the inequality is reversed, we need $p_{mn}>p_{mk}p_{kn}$ or agent k's link with n is so weak that it is not worthwhile for m to form this link. Also, the benefits from the indirect links should outweigh the benefits from the direct ones or we need that $(p_{mn}-p_{mk}p_{kn})\sum_{j\in J_k}p_{jn}>(p_{mn}-p_{mk})+p_{kn}(p_{mn}-p_{mk})$. Next we need to check that no agent wants to add an extra link. This means that no $m\in M$ wants to form a link with any $k\in K_m$. Note that payoffs from this additional link are bounded above by $(n-1)-2c$. Taking the difference between $\Pi_m(g^{ot}+g_{mk})$ and $\Pi_m(g^{ot})$ we get $[(1-p_{mn})+(1-p_{1n}p_{mn})+\cdots+(1-p_{m-1n}p_{mn})+(1-p_{m+1n}p_{mn})+\cdots+(1-p_{n-1n}p_{mn})]<c$. Verifying that this is satisfied for all $m\in M$, gives us $\max_{m\in M}[(1-p_{mn})+(1-p_{1n}p_{mn})+\cdots+(1-p_{m-1n}p_{mn})+(1-p_{m+1n}p_{mn})+\cdots+(1-p_{n-1n}p_{mn})]<c$, which is the same condition used to define δ for the previous case. Under these conditions no agent will form an additional link and the inward pointing star will be Nash.

(c) The mixed star is a combination of the inward and outward pointing stars where n is the central agent, but does not form links to all the other $n-1$ players. Some of these players instead establish links with n. For the mixed star to be Nash we need the conditions in *(a)* to be satisfied for n and the conditions in *(b)* to be satisfied for all $m \in M$ who establish the link with n. We can also calculate a condition under which no $i \in N$ will want to form an additional link using the reasoning established above making the mixed star a Nash. ■

Relative to the Bala and Goyal framework, the introduction of heterogeneous agents alters the situation significantly. While part of the difference involves more complex conditions required for establishing star networks, heterogeneity comes with its own reward. A different probability for the success of each link resolves the coordination problem implicit in the Bala and Goyal framework. With a constant probability of success, once we identify conditions under which different stars will be Nash, the role of the central agent can be assigned to any player. With heterogeneous agents however, there are some natural candidates for the central agent. For instance, for the inward pointing star, the agent whose payoff net of costs from being the central agent is the highest is the obvious choice for this position. Similarly, for an outward pointing star, the agent who has the least benefit net of costs from a single link, is the natural choice for the central agent. There are also some other differences with the Bala and Goyal framework. Notice that the determination of δ involves all probabilities of other links, making it quite complicated. Further, the benefits from deviation are also altered now. Consider the outward pointing star. In the Bala and Goyal framework, no agent will ever deviate by severing a link with the central agent. In our model, links to the central agent will be severed unless the probabilities in the relevant range satisfy some additional conditions.

We next consider the situation where $c > 1$. In this case, $c > p_{ij}$ for all links g_{ij}. Hence the inward pointing star and the mixed star will never be Nash. We identify conditions under which the outward pointing star can be Nash.

Proposition 4: *Given $c \in (1, n-1)$ there exists a threshold probability $\delta < 1$ such that for $p_{ij} \in (\delta, 1)$ the outward pointing star is Nash.*

Proof: Since $c \in (1, n-1)$, we know that $c > p_{ij}$. Again let agent n be the center of the star with whom all the other players establish links. The proof for the outward pointing star proceeds along the lines outlined for the previous claim. Since the cost of linking are high, it is easy to show that any agent $m \in M$ will not wish to form an additional link. Once again player m's payoff from the outward pointing star is $\Pi_m(g^{ot}) = p_{mn} + p_{mn} \sum\limits_{k \in K_m} p_{kn} - c$. From the previous case we already know, that for $c \in (0, 1)$, the outward pointing star can be Nash. Using the continuity of c it can be shown there exist $p_{ij} \in (\delta, 1)$ for which $\Pi_m(g^{ot}) > 0$ and each agent will maintain his link with n. Further, the value of δ can be determined using the same condition as in *Proposition 3,* for which no agent will wish to deviate and form a link with some $k \in K_m$. Hence, the outward pointing star will be Nash. ■

While this result is quite similar to what Bala and Goyal obtain with uniform probabilities, once again it is possible to identify a natural candidate for the position of the central player.

3.4.5 Other Nash Networks

Besides the star networks identified above, several other types of architectures can be Nash as well. This is illustrated through two examples. The first one considers a line network which is quite popular in the networks literature and is shown in the figure below.

Insert **Figure 4** here:

Example 2: Consider the line network with $N = 4$. The payoffs to each player from this line network are as follows:

$$\Pi_1^* = p_{12} + p_{12}p_{23} + p_{12}p_{23}p_{34} - c$$
$$\Pi_2^* = p_{12} + p_{23} + p_{23}p_{34} - c$$
$$\Pi_3^* = p_{12}p_{23} + p_{23} + p_{34} - c$$
$$\Pi_4^* = p_{12}p_{23}p_{34} + p_{34}p_{23} + p_{34} - c$$

To verify that this network is indeed Nash, we have to ensure that no agent wants to use another strategy involving either a different link or an additional link. For any line network, agents closer to the origin of the line will have more strategies and hence will require more conditions to be satisfied. The Nash link should have a higher success probability than a link involving deviation. Moreover, when deviations lead to direct links to players who are relatively closer, the probability of accessing this player through indirect links using the Nash strategy should outweigh the benefits from the direct link. The same must be true for all other players who will be accessed by the link formed through deviation. Stated differently, the benefits of accessing neighbors through the Nash link should dominate the benefits from using other links. Further, the gains from an additional link must yield a lower payoff. The above line network will be a Nash equilibrium for $c = 0.5$, and $p_{12} = 0.8$, $p_{23} = 0.75$ and $p_{34} = 0.85$. In order to make the example interesting, we set some of the other probabilities above 0.5 as well, i.e., $p_{13} = 0.54$, $p_{14} = 0.56$, and $p_{24} = 0.6$. All other probabilities are assumed to be very close to zero. A precise set of conditions with details is provided in the Appendix. Using 3 as the upper bound on payoffs, for $c = 0.5$ and using common probabilities, requires $p \geq 0.912$ for the line network to be Nash. Thus, with heterogeneous agents the line network is Nash under a much wider range of probabilities.

Most of the literature in networks concentrates on a few popular architectures like the star, the wheel and the line. The next example considers a network of six agents, who are arranged in two stars with a single link between the two central agents.

Insert **Figure 5** here:

Example 3: The figure shows an outward pointing *twin star* network with players 3 and 4 being the central agents. The twin stars are connected since player 3 has a link with player 4. In order to show that this is Nash we need to verify that no player wishes to deviate and form another link and no one wants to add an extra link either. The above configuration can be easily supported as Nash, provided each non-central player's link to their central agent has a higher probability than any other link with a non-central player. Also the probability of accessing the other central player indirectly, is higher than the probability of a successful direct link with the other central player. For example player 5's probability of accessing agent 3 indirectly through player 4, should be higher than the probability of accessing player 3 directly. For $c = 0.5$, this requires a set of probabilities like $p_{54} = 0.8$, $p_{56} = p_{51} = p_{52} = 0.55$, and $p_{53} = 0.52$. A similar set of probabilities for the other non-central players, including this set vis-a-vis the set of players they can form links with ensures that the twin star will be Nash. All other probabilities for the non-central players are assumed to be close to zero. For the central player who initiates the link to the other central player (Player 3) we require this link to have a higher probability than a link to any non-central player. Based on this we can assign $p_{34} = 0.75$, and $p_{35} = p_{36} = 0.52$. In order to ensure that the central player with no links does not wish to add a link (Player 4), we need to ensure that all her remaining links have probability of success close to zero. The Appendix provides a set of precise conditions with the details of derivations which make the twin star network Nash. Using 5 as the upper bound on payoffs, for $c = 0.5$ and using common probabilities, requires $p \geq 0.953$ for the twin star network to be Nash. Once again we find that with heterogeneous agents the twin star network is Nash under a much wider range of probabilities

3.5 Alternative Model Specifications

In this section we will consider three alternative specifications of our current model. The first variation introduces more realism in the formation of networks by allowing agents to duplicate existing links. The second specification considers network formation under incomplete information. Here, each agent $i \in N$ is aware of the success probability $p_{ij}, i \neq j$ of her links, but is ignorant of the probabilities of link successes of the other agents. Finally we discuss the implications for Nash networks in a model with consent, where a link is established only when the non-initiating agent consents to the link.

3.5.1 An Alternative Formulation of Network Reliability

The payoff function in the previous section is based on the closure of the network implying that the links $g_{ij} = 1$ and $g_{ji} = 1$ are perfectly correlated. Thus, no agent will ever duplicate a link if it already exists. This gives us the *essential* network property described earlier. A more accurate way of modelling information flows would be to assume that $g_{ij} = 1$ and $g_{ji} = 1$ are independent. Then, the link $h_{ij} = \max\{g_{ij}, g_{ji}\}$ exists with probability $r_{ij} = 1 - (1 - p_{ij})^2$. This provides an incentive for a two-way connection between agents i and j. This never occurs in the previous model since duplicating a link can only increase costs. Note we retain the assumption that $p_{ij} = p_{ji}$. Also, the flow of benefits is symmetric. The consequences of the new formulation will now be explored by reexamining *Proposition 3* in the light of this alternative assumption. The incentives for modifying links by deviating do not change under this formulation. The main impact is on the threshold probability value δ, thereby altering the range of costs and probabilities under which different architectures can be supported as Nash. Note that the payoff function used earlier for determining the payoff from an additional link gets around this issue by assuming that payoffs have an upper bound of $(n-1) - \alpha c$ where α denotes the number of links formed. In order to see how this formulation will affect reliability we need to compute the precise value of the payoffs from additional links instead of using the upper bound. We denote the resulting new threshold value by $\widetilde{\delta}$.

PROPOSITION 5: *When the links $g_{ij} = 1$ and $g_{ji} = 1$ are independent, and $c \in (0,1)$ the inward pointing star, the outward pointing star and the mixed star can be supported as Nash under the threshold probability value $\widetilde{\delta}$ for a given level of the costs of link formation.*

Proof: Let us first consider the inward pointing star. Agent n establishes all the links and hence will not deviate as long as her worst link yields higher benefits then c. The only change will occur in the payoffs from an extra link. Given an inward pointing star, agent n has no extra links to form. Hence consider agent $m \in M$. His payoff from the inward pointing star is given by $\Pi_m(g^{in}) = p_{nm} + p_{nm} \sum_{k \in K_m} p_{nk}$. He can form an additional link with either *(i)* agent n, or *(ii)* with some $k \in K_m$ and we take both possibilities into account. First, assume that this agent forms a link with player n. The probability of obtaining player n's information with this two-way link is given by $r_{nm} = [1-(1-p_{nm})^2]$. Payoffs from the augmented network are given by

$$\Pi_m(g^{in} + g_{mn}) = r_{nm} + r_{nm} \sum_{k \in K_m} p_{nk} - c$$

This link will not to be formed when $\Pi_m(g^{in} + g_{mn}) - \Pi_m(g^{in}) < 0$. The inequality will be used to determine the threshold probability value δ_n^m, pertaining to links with agent n. It can be written as: $p_{nm}(1 - p_{nm}) + p_{nm}(1 - p_{nm}) \sum_{k \in K_m} p_{nk} < c$. Note that there will be one such inequality for each $m \in M$. Now consider case *(ii)*. We need to compute $\Pi_m(g^{in} + g_{mk})$ which is the sum of payoffs from three different terms $-$ payoff from player n, payoff from player k, and the payoff from all others players $j \in J_k$. Payoffs from player n are given by $p_{nm}(1-p_{mk}p_{nk}) + (1-p_{nm})p_{mk}p_{nk} + p_{nm}p_{mk}p_{nk} = r'_{nm}$. Note the first term denotes the payoffs from agent n, when the links between the pairs mk, and nk are not working simultaneously. In any other situation involving these three agents, m can access n irrespective of which of the other two links are working. The second term denotes the fact that if the link g_{nm} has failed, m can access n only if the links between the pairs mk and kn are both working simultaneously. The third term captures the possibility that all three links are working simultaneously. Similarly payoffs from player k are given by $p_{mk}(1-p_{nm}p_{nk})+(1-p_{mk})p_{nm}p_{nk}+p_{nm}p_{mk}p_{nk}=r'_{mk}$. Finally, the payoff from all other players is given by $r'_{nm}\sum_{j\in J_k} p_{nj}$. Adding all this up gives us

$$\Pi_m(g^{in}\oplus g_{mk})=r'_{nm}+r'_{mk}+r'_{nm}\sum_{j\in J_k} p_{nj}$$

Checking that $\Pi_m(g^{in}\oplus g_{mk})-\Pi_m(g^{in})<0$ implies δ_k^m has to satisfy the following condition: $(1-p_{nm})p_{mk}p_{nk}+p_{mk}(1-p_{nm}p_{nk})+(1-p_{nm})p_{mk}p_{nk}\sum_{j\in J_k} p_{nj}<c$. Again there is one such inequality for each $m\in M$. Using these inequalities we obtain $\widetilde{\delta}$, the new threshold probability that satisfies: $\widetilde{\delta}=\max_{m\in M}\max_{n,k}\{\delta_n^m,\delta_k^m\}$ where $k\in K_m$. Hence for $p_{ij}\in(\widetilde{\delta},1)$ we can support the inward pointing star as Nash.

Now consider the outward pointing star. All agents $m\in M$ have a link with the central agent, and the conditions for deviating from the Nash strategy identified in *Proposition 3(b)* remain unchanged. However, we must check that no $m\in M$ wants to form an extra link. Also agent n should not gain by adding a link. We know that for $m\in M$, $\Pi_m(g^{ot})=p_{mn}+p_{mn}\sum_{k\in K_m} p_{kn}-c$. Payoffs from establishing a link with some $k\in K_m$ are given by $\Pi_m(g^{ot}\oplus g_{mk})=r'_{mn}+r'_{mk}+r'_{mn}\sum_{j\in J_k} p_{jn}-2c$. Agent m will not form this link if $\Pi_m(g^{ot}\oplus g_{mk})-\Pi_m(g^{ot})<0$, which simplifies to the condition for δ_k^m obtained for the inward pointing star discussed above. Now consider agent n. His payoffs are given by $\Pi_n(g^{ot})=p_{mn}+\sum_{k\in K_m} p_{kn}$. Suppose he now adds a link to some $m\in M$. Payoffs from this link are given by $\Pi_n(g^{ot}\oplus p_{nm})=r_{mn}+\sum_{k\in K_m} p_{kn}-c$, where $r_{mn}=[1-(1-p_{nm})^2]$.

Taking the difference between this and the Nash payoffs gives us the condition

$$p_{mn}(1 - p_{mn}) < c$$

from when we can obtain the value of δ_m^n. Thus, $\widetilde{\delta}$ must now satisfy the following condition: $\widetilde{\delta} = \max_{n \in N} \max_{m,k} \{\delta_m^n, \delta_k^m\}$ where $k \in K_m$. Hence for $p_{ij} \in (\widetilde{\delta}, 1)$ we can support the outward pointing star as Nash.

Similar arguments can be used to show that the mixed star will also be Nash with a different threshold value for c, the cost of link formation since it is just a combination of the inward and outward pointing stars. ■

For our previous formulation, instead of using the upper bound on the payoffs, a more precise value of δ for the different stars could be based on $\max_{m \in M} \{\delta_k^m\}$. The set of δ_n^m defined for the inward pointing star contains higher values than the set of δ_k^m, since it contains involves fewer indirect links. Hence, it forms the binding constraint and will raise the threshold probability value. A similar situation occurs in case of the outward pointing star, since the set δ_k^m involves networks containing more indirect links than $p_{mn}(1 - p_{mn})$. Also note that for the outward star, the central agent can also duplicate links. So, now we have to consider all agents $n \in N$, while solving for the threshold probability. Consequently, the respective constraints will be more binding in a mixed star, too. Thus, in general $\widetilde{\delta} > \delta$. It is worth reiterating that the value of δ computed in the previous section will be unaffected since it uses the upper bound on the payoffs. Finally, this formulation can lead to super-connected networks of a different sort − one where agents may reinforce existing higher probability links instead of forming new links with other players.

3.5.2 Nash Networks under Incomplete Information

The previous sections have assumed that the agents are fully aware of all link success probabilities. However, this is not always a very realistic assumption. In this subsection we introduce incomplete information in the game. Each agent $i \in N$, has knowledge of the probability of success of all her direct links. However, she is not aware of the probability of success of indirect links, i.e., agent i know the value of p_{ij}, but is unaware of the value of p_{jk}, where $i \neq j, k$. Note the assumption that $p_{ij} = p_{ji}$ is still retained. We will re-examine *Proposition 3* for this specification. In order to solve for equilibria, each agent i must now have some beliefs about the indirect links. We argue that each agent postulates that, on average, every other agent's world is identical to her own. She assigns the average success value of her own direct links to all the indirect links, thus imparting a kind of symmetry to the problem of indirect links. Thus, agent i assigns a value of $p_i = \frac{1}{n-1} \sum\limits_{N \setminus \{i\}} p_{im}$, to all indirect links p_{jk} for $i \neq j, k$. This has some immediate consequences for the payoff function. Consider the inward pointing star and some agent $m \in M$. This agent now believes that her payoff from the inward pointing star is given by $\Pi_m(g^{in}) = p_{nm} + p_{nm} \sum\limits_{k \in K_m} p_m$. Since the cardinality of the set K_m is $n-2$, her payoffs are given by $\Pi_m(g^{in}) = p_{nm} + (n-2)p_{nm}p_m$, which are clearly different from her actual payoffs in this equilibrium. In what follows we will assume that *Proposition 3* holds, i.e., the conditions required by the proposition are satisfied.

PROPOSITION 6*: Given each agent's beliefs about her indirect links:*
(a) The inward pointing star is Nash when the central agent's worst link yields higher benefits than c and for each agent $m \in M$, the inequality $(n-2)(1 - p_{nm}p_m) < (1 - p_{nm} \sum_{k \in K_m} p_{nk})$ holds.
(b) The outward pointing star is Nash if every non-central agent ($i \neq n$) has her best link with the central agent and for each agent $m \in M$, the inequality $(n-2)(1 - p_{nm}p_m) < (1 - p_{nm} \sum_{k \in K_m} p_{nk})$ holds.
(c) The mixed star is Nash when both the conditions above are satisfied, the first one for the central agent, and the second one for all agents who form links with the central agent and for each agent $m \in M$, the inequality $(n-2)(1 - p_{nm}p_m) < (1 - p_{nm} \sum_{k \in K_m} p_{nk})$ holds.

Proof: (a) Consider first the inward pointing star. The first half of *Condition (a)* is the same as before. The central agent n has no strategy to chose from and every non-central agent $m \in M$ cannot break any links. Agent m can only add a link to some $k \in K_m$. Her payoffs from this strategy are bounded above by $(n-1)-c$ and she will not form an additional link if $\Pi_m(g^{in} \oplus g_{mk}) - \Pi_m(g^{in}) < 0$. This gives us the following condition:

$$1 - p_{nm} + (n-2)p_{nm}p_m < c$$

However the actual condition is given by

$$\left[(1-p_{nm}) + \left(1 - p_{nm} \sum_{k \in K_m} p_{nk}\right)\right] < c$$

Hence agent m will not form an additional link if $(n-2)(1-p_{nm}p_m) < (1-p_{nm}\sum_{k\in K_m} p_{nk})$. This completes the proof.

(b) In this instance the central agent n has no role to play. Every agent m receives a perceived payoff given by $\Pi_m(g^{ot}) = p_{mn} + (n-2)p_{mn}p_m - c$. Let us consider the possibility that agent m wants to deviate and form a link with some $k \in K_m$. Her payoffs from this are given by $\Pi_m(g^{ot} + g_{mk} - g_{mn}) = p_{mk} + (n-2)p_{mk}p_m - c$. Hence the condition for no deviation is given by

$$p_{mn} - p_{mk} + (n-2)(p_{mn} - p_{mk})p_m > 0$$

which is only true when $p_{mn} > p_{mk}$. In order to rule out additional links, just as before we require that $(n-2)(1-p_{nm}p_m) < (1-p_{nm}\sum_{k\in K_m} p_{nk})$. This completes the proof for this part.

(c) The mixed star is a combination of the inward and outward pointing stars where n is the central agent, but does not form links to all the other $n-1$ players. Some of these players instead establish links with n. For the mixed star to be Nash we need the conditions in *(a)* to be satisfied for n and the conditions in *(b)* to be satisfied for all $m \in M$ who establish the link with n. Additionally imposing the requirement that $(n-2)(1-p_{nm}p_m) < (1-p_{nm}\sum_{k \in K_m} p_{nk})$ for all agents will ensure that no agent wants to add an extra link. This will make the mixed star a Nash equilibrium as well. ■

This formulation provides us with some interesting insights about the role of the indirect links. First, it is possible that $(n-2)(1-p_{nm}p_m) > c > (1-p_{nm}\sum_{k \in K_m} p_{nk})$, in which case agents will add links, resulting in lower payoffs when the network is realized. Notice that if $p_m = p_{nk}$ for all $k \in K_m$, and $p_m < p_{nk}$ for even one agent, then redundant links can be established. This is an instance when the inward star is Nash under incomplete information, but due to incomplete information about indirect links, agents switch to different strategies involving additional links. Consider the outward pointing star. Under incomplete information, this star will be Nash only when every non-central agent has her best link with the central agent. However, as shown in *Proposition 3(b)*, the outward pointing star can still be Nash if this condition is not satisfied. Under incomplete information such Nash networks will never be formed. Agents will switch to strategies yielding lower payoffs.

3.5.3 Nash Networks with Mutual Consent

In our present setting and in most of the literature, it is assumed that agent i does not need the consent of agent j to initiate a link from i to j. All it takes is that agent i incurs the cost c. This may be construed as a drawback of the non-cooperative formulation. Arguably however, if asked agent j might give her permission anyway, since she would only benefit from an additional link that does not cost her anything.[5] Thus it appears that introducing an implicit consent requirement is inconsequential, a descriptive improvement at best, a notational burden at worst. Yet Nash networks have another more serious weakness. It still seems somewhat preposterous that agent j should divulge all the information from her other links without her consent. Hence we now

[5] This argument is less compelling in the case of one-way information flow.

discuss the implications of a consent game. Formally, such a requirement can be accommodated by replacing each player's strategy set $\mathcal{G}_i$ by $\mathcal{G}_i \times \mathcal{G}_i$ with generic elements $(g_i, a_i) = (g_{i1}, \ldots, g_{ii-1}, g_{ii+1}, \ldots, g_{in}; a_{i1}, \ldots, a_{ii-1}, a_{ii+1}, \ldots, a_{in})$ where the second component, a_i, stands for i's consent decisions. A link from i to j is initiated by mutual consent if and only if $g_{ij} = 1$ and $a_{ji} = 1$. Agents incur only the cost of links that are permitted, denied links being absolutely costless.

Every graph g that was a Nash network before is still a Nash network. But now there is room for mutual obstruction: $g_{ij} = 0$ is always a best response to $a_{ji} = 0$ and vice versa. Therefore, the empty network is always Nash under a mutual consent requirement. More generally, take any set of potential edges $E \subseteq N \times N$ and replace p_{ij} by $q_{ij} < p_0$ for all $ij \in E$ in the original model. Then any Nash network of the thus defined hypothetical game constitutes a Nash network of the network formation game requiring mutual consent. In particular, for any $N' \subseteq N$, the Nash networks with reduced player set N' form Nash networks (as long as the architecture of the network is preserved) of the network formation game requiring mutual consent with player set N, if one adds the agents in $N \setminus N'$ as isolated nodes. One could modify the mutual consent game by requiring that agents must incur the cost of all links they initiate, irrespective of consent. Since agents are rational and have complete information, in equilibrium, links that will be denied will never be initiated. The Nash networks will be identical under this specification.

All the new equilibria from the mutual consent game will be eliminated, if one imposes 2-player coalition-proofness or introduces conjectural variations of the kind that a player interested in initiating a link presumes the other's consent. A more serious issue is why two agents cannot split the cost in a Pareto-improving way when both would benefit from an additional link. Addressing endogenous cost sharing in a satisfactory way necessitates a radically different approach which is beyond the scope of the present generation of models.[6]

3.6 Concluding Remarks

The model developed here as well as a substantial part of the networks literature is concerned with information flows. Such models may then be interpreted as the reduced form of a certain economic phenomenon where all the costs and benefits have been attributed to information flows. Under perfect reliability, the primary focus lies on the size and efficiency of networks. With imperfect reliability the strength of social ties, or the nature and quality of information can be discussed as well. In our model for instance, one could argue that the information exchange between ij is valuable with probability p_{ij} and is of a dubious nature with the complementary probability. Thus, imperfect reliability raises questions about the quantity-quality trade-off as well as the related efficiency issues.

Agent heterogeneity in the form of imperfect reliability in social networks provides a richer set of results. In conjunction with our adopted solution concept, Nash equilibrium, it accentuates open questions that also arise in the context of pairwise stability though perhaps to a lesser degree. An example is the issue of cost sharing and side payments. Twice in the course of our current investigation we came across this issue; first, in the discussion of efficiency and Pareto-optimality; for a second time in the context of the mutual consent model. Cost sharing and bargaining over the costs of link formation is especially crucial when benefits mainly accrue from indirect links. This indicates an important direction for future research.[7]

[6] The Jackson-Wolinsky "connections model" assumes exogenous cost sharing.

[7] Currarini and Morelli (2000) take a first step in this direction. They introduce a noncooperative game of sequential network formation in which players propose links and demand payoffs. They show that for networks which satisfy size monotonicity, there is no conflict between efficiency and stability.

The earlier literature on Nash networks shows that results under imperfect reliability are quite different from those in a deterministic setting. With the introduction of heterogeneity this clear distinction no longer prevails, and our findings encompass results of both types of models. For example, with perfect reliability and information decay, Nash networks are always minimally connected, irrespective of the size of society (Bala and Goyal (1999a)). In contrast with homogeneous imperfect reliability and no information decay redundant links between agents always arise asymptotically (Bala and Goyal 1999b). In our model with heterogeneous imperfect reliability and no information decay both types of outcomes can be generated through appropriate choice of p_{ij}. For instance, decay models (with perfect reliability) compute benefits by considering only the shortest path between agents; extra indirect links do not contribute to benefits. Given a resulting minimally connected Nash network g of such a model, there exists a parameter specification of our model that gives rise to g as a Nash network. In our framework this requires lowering the p_{ij} to zero or below p_0 for all ij with $g_{ij} = 0$ and $g_{ji} = 0$ and choosing sufficiently high probabilities p_{ij} for all other ij so that all benefits accrue from the direct links only. This can give rise to minimally connected networks. On the other hand, as already shown, choosing the p_{ij}'s appropriately leads to super-connected networks as well.

Finally, to end on a cautionary note it is only appropriate to mention Greif's (1994) tale of two historical societies – the Maghribi traders, with an Islamic culture who shared trading information widely, and the Genoese traders exemplifying the Latin world, who operated individually and did not share information amongst each other, relying more on legal contracts. He argues that the culture and social organization of these two communities ultimately determined their long-run survival. The Genoese kept business secrets from each other, improved their contract law and operated through the market. Consequently they ended up with an efficient society. The Maghribis on the other hand operated through an informal network where behavior of a single pair of agents affected everyone in the network. As opposed to the Genoese traders the Maghribis invested considerable time and effort to gather information about their network. Since one bad link could adversely affect the entire network, the Maghribis often had to engage in superfluous links as well without adequate concern for efficiency. Efficiency became a critical issue once new business opportunities arose in far away lands, where operating through an ethnically based network became very expensive. In the end these organizational differences created by the cultural beliefs of the two societies led to the survival of the more efficient of the two. Thus social networks may be good substitutes for anonymous markets in certain societies, but the market paired with the proper infrastructure may be a more efficient institution. In fact for trade in standardized commodities, a frictionless and informationally efficient anonymous market, if feasible, would be best.

3.7 Appendix

1. *Nash Conditions for a Example 2*: This example illustrates a Line Network ($N = 4$). We now derive conditions under which no player will deviate to another strategy. Consider agent 1 first. He can form a link with either agent 3 or 4. Payoffs from these two strategies are respectively given by

$$\Pi_1' = p_{13} + p_{13}p_{23} + p_{13}p_{34} - c$$
$$\Pi_1'' = p_{14} + p_{14}p_{34}p_{23} + p_{14}p_{34} - c$$

Next consider player 2. By deviating she can only form a link with player 4. Payoff from this strategy is given by

$$\Pi_2' = p_{12} + p_{24} + p_{24}p_{34} - c$$

Player 3 can only form a link with player 1 by deviating from her current strategy. However, this is a sub-optimal choice since it destroys the sole link to player 4 reducing the benefits from this network by one unit. Since player 4 does not have any links to begin with, she cannot switch to another strategy.

For the line network to be Nash, we need to ensure that the difference between Π_i^* (defined earlier) and Π_i' (or Π_i'') is always positive. Checking that $\Pi_1^* - \Pi_1' > 0$, requires that one of the following two conditions must be satisfied at all times:

(*i*) $p_{12} > p_{13}$ and $p_{12}p_{23} > p_{13}$, or

(*ii*) $p_{12} > p_{13}$ and $p_{12}p_{23} < p_{13}$, but the benefits accruing from agents 2 and 3 from using the Nash strategy outweigh the benefits from agent 4.

Checking that $\Pi_1^* - \Pi_1'' > 0$, also gives us two conditions, of which one must be satisfied at all times:

(*iii*) $p_{12} > p_{14}$ and $p_{12}p_{23} > p_{14}p_{34}$, or

(*iv*) if one of the above inequalities in (*iii*) is reversed the payoffs to player 1 from the one that holds must outweigh the payoffs from the one that does not hold.[8]

Checking that $\Pi_2^* - \Pi_2' > 0$, results in the condition that her Nash strategy must have a greater probability of success than the deviation strategy, or

(*v*) $p_{23} > p_{24}$.

Note that when condition (*v*) is satisfied all indirect links also yield higher payoffs. Finally, we know that the payoffs from the above network are bounded above for each player by $3 - \alpha c$, where α is the number of links they form. Using this it is easy to formulate the conditions for δ under which no agent will add an extra link. The exercise is identical to the one done in the propositions. ■

2. *Nash Conditions for a Example 3*: We will now identify conditions under which the twin star configuration will be a Nash equilibrium. We need to check that no player wants to deviate and form a different link. Clearly player 4 cannot do so. Now consider one of the non-central players, say, player 5. We know that his payoff from the above

[8]Note however, that the possibility of $p_{12} < p_{14}$ and $p_{12}p_{23} > p_{14}p_{34}$, along with the latter yielding higher benefits is rather unlikely.

configuration which we want to establish as Nash is

$$\Pi_5^* = p_{54} + p_{54}p_{64} + p_{54}p_{34} + p_{54}p_{34}p_{13} + p_{54}p_{34}p_{23} - c.$$

Player 5 can deviate and form a link with one of the non-central players like 1, 2 and 6 or with player 3. Supposing he forms a link with player 6. Then

$$\Pi_5^{'} = p_{56} + s_{54} + s_{54}p_{34} + s_{54}p_{34}p_{13} + s_{54}p_{34}p_{23} - c,$$

where $s_{54} = p_{56}p_{64}$. We need to ensure that $\Pi_5^* - \Pi_5^{'} > 0$. This is possible when one of the two conditions given below is satisfied.

(i) The Nash connection is better than the connection with agent 6, i.e., $p_{54} > p_{56}$, or

(ii) If $p_{54} < p_{56}$, then we need $p_{54} > p_{56}p_{64}$, or the Nash connection is better than connecting to agent 4 through agent 6, and the sum of the payoffs from agents 1, 2, and 3 exceeds the payoffs from players 4 and 6.

The same calculation can be applied to player 5's links with the other non-central agents 1 and 2. Since we will get two such conditions from each non-central player for player 5, and similar conditions when we check for deviations by all the other non-central players, the second condition is less likely to hold. Now let us consider his connection to agent 3. From this we get

$$\Pi_5^{''} = p_{53} + p_{53}p_{13} + p_{53}p_{23} + p_{53}p_{34} + p_{53}p_{34}p_{64} - c.$$

The difference between the Nash payoffs and $\Pi_5^{''}$ is positive when the one of the two conditions below holds:

(iii) $p_{54} > p_{53}$, $p_{54}p_{34} > p_{53}$, and $p_{54} > p_{53}p_{34}$, or

(iv) $p_{54} < p_{53}$, $p_{54}p_{34} > p_{53}$, and $p_{54} < p_{53}p_{34}$, and the benefits from players 1 and 2 outweigh benefits from all other players as well.

Note that when the first two inequalities in (*iii*) are satisfied, the third automatically holds. Clearly identifying networks where condition (*iv*) holds will be more difficult, than those arising under condition (*iii*). Similar arguments can be made for all the other non-central players 1, 2 and 6, each giving us its own set of inequalities, that will sustain the twin stars network as a Nash equilibrium. It is easy to verify that the probabilities required by these conditions are consistent with each other.

Now consider player 3, who is a central player. He will not form any links with player 1 or 2. So, by breaking a link he can only add one to player 5 or 6. His payoffs from deviating and establishing a link to player 5 results in

$$\Pi_3^{'} = p_{35} + p_{13} + p_{23} + p_{35}p_{54} + p_{35}p_{54}p_{64} - c.$$

His payoffs from the Nash configuration are given by

$$\Pi_3^{*} = p_{34} + p_{13} + p_{23} + p_{34}p_{54} + p_{34}p_{64} - c.$$

The difference $\Pi_3^{*} - \Pi_3^{'} > 0$, when one of the following two inequalities is satisfied

(*v*) $p_{34} > p_{35}$, or

(*vi*) If $p_{34} < p_{35}$ then we need that $p_{34} > p_{35}p_{54}$ and the benefits from agent 6 must outweigh the combined benefits from players 4 and 5.

A similar argument applies to his forming a link with player 6. Hence as argued earlier, condition (*vi*) is less likely to be satisfied. Player 4, who is the other central agent can only form links with players 1 or 2, and a similar set of inequalities can also be deduced for her. Finally, it is easy to rule out additional link formation by any agent using an inequality for δ similar to the ones used in the proposition. ■

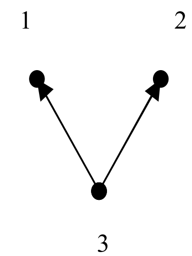

Figure 3-1: A Simple Network (n = 3)

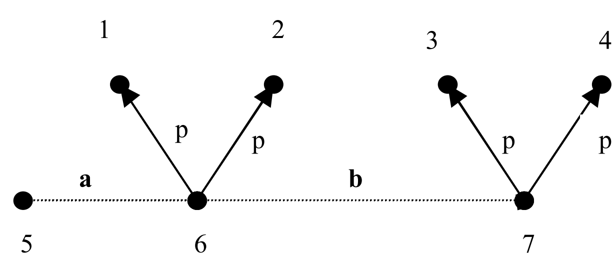

Figure 3-2: Nash, Pareto and Inefficient Network

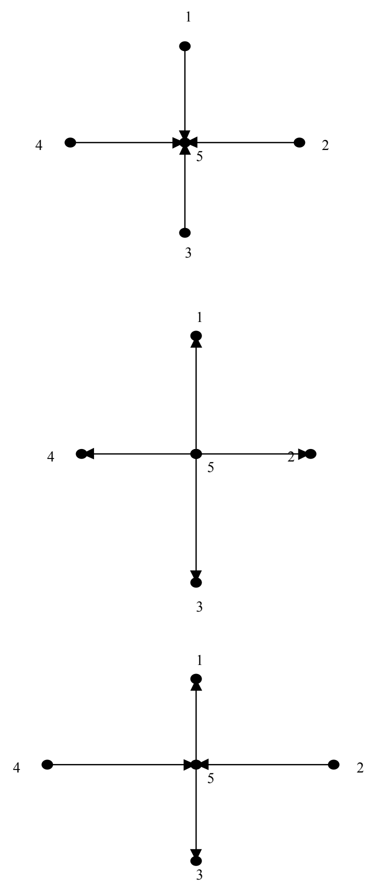

Figure 3-3: Inward Star, Outward Star and Mixed Star

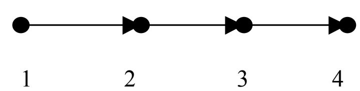

Figure 3-4: A Line Network (n = 4)

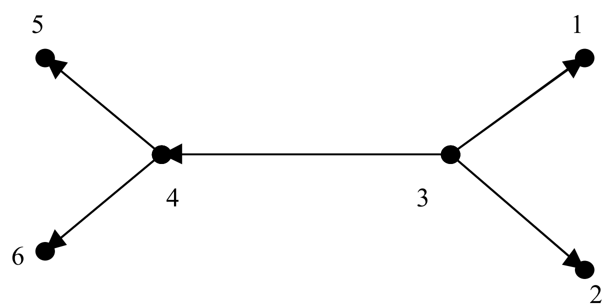

Figure 3-5: The Twin Stars Network

Chapter 4

Revisiting Fuzzy Game Theory

4.1 Introduction

This paper provides a new formulation of noncooperative fuzzy games. Such games were originally developed by Butnariu (1978, 1979) and later revised by Billot (1992). A fuzzy game is modeled using the notion of a fuzzy set introduced in a seminal paper by Zadeh (1965). A fuzzy set differs from a classical set (from now on referred to as a *crisp set*) in that the characteristic function can take any value in the interval [0,1]. In the Butnariu-Billot formulation each player's beliefs about the actions of the other players are modeled as fuzzy sets. These beliefs need to satisfy an axiom that constrains the actions available to others. The equilibrium relies on a restrictive assumption involving only those situations where players' beliefs are commonly known. Since these beliefs are now perfect information, they do not constrain the actions of the other players with any degree of uncertainty. Without this assumption the above model need not have an equilibrium, though possible solutions to the game may still exist.

The existing verbal and formal descriptions of fuzzy noncooperative games by Butnariu (1978, 1979) and Billot (1992) seem somewhat enigmatic and perhaps "fuzzy" to the unfamiliar reader. In this paper we first recast the Butnariu-Billot model using standard game-theoretic and crisp set terminology. We then relax the Butnariu-Billot axiom by requiring that player i's beliefs should allow the other players to choose any mixed strategy from their available choices. Further, in equilibrium instead of requiring full information about each others' beliefs, we only impose mutual consistency of beliefs. Despite these two modifications, the equilibrium concept is still quite weak. We demonstrate this through the example of an abstract economy. In an abstract economy each player maximizes his own objectives subject to the constraints imposed on his actions by the others. We develop a new model of fuzzy noncooperative games by marrying the two types of possible restrictions on the actions of the players, one derived from the Butnariu-Billot formulation and one from the model of abstract economies. In our model, each player tries to minimize the restrictions he imposes on others while respecting the constraints imposed on his own actions by the others, but does not explicitly pursue any objectives of his own. This allows us to ensure the existence of an equilibrium in the reformulated fuzzy game.

The remaining sections of the paper are organized as follows. In Section 4.2 mathematical tools required in the rest of the paper are presented. These cover concepts from fuzzy set theory as well as abstract economies. Section 4.3 contains the crisp version of the Butnariu-Billot model as well as our reformulation. Section 4.4 concludes and the Appendix provides a brief summary of Butnariu's formulation.

4.2 Mathematical Preliminaries

In this section we set forth the basic mathematical definitions that will be used in later sections of the paper. We will first introduce the notion of fuzzy sets. This is followed by the introduction of relevant material on abstract economies that are necessary for our re-formulation.

4.2.1 Relevant Concepts From Fuzzy Set Theory

The earliest formulation of the concepts of fuzzy sets is due to Zadeh (1965) who tried to generalize the idea of a classical set by extending the range of its characteristic function. Informally, a fuzzy set is a class of objects for which there is no sharp boundary between those objects that belong to the class and those that do not. Here we provide some definitions that are pertinent to our work.

Let X denote the universe of discourse. We distinguish between "crisp" or traditional and fuzzy subsets of X.

Definition 10 *The* ***characteristic function*** Ψ_A *of a* ***crisp set*** A *maps the elements of* X *to the elements of the set* $\{0, 1\}$*, i.e.,* $\Psi_A : X \rightarrow \{0, 1\}$*. For each* $x \in X$*,*

$$\begin{aligned} \Psi_A(x) &= 1 \text{ if } x \in A \\ &= 0 \text{ otherwise} \end{aligned}$$

To go from here to a fuzzy set we need to expand the set {0,1} to the set [0,1] with 0 and 1 representing the lowest and highest grades of membership respectively.

Definition 11 *The* ***membership function*** μ_A *of a* ***fuzzy set*** A *maps the elements of* X *to the elements of the set* $[0, 1]$*, i.e.,* $\mu_A : X \rightarrow [0, 1]$*. For* $x \in X$*,* $\mu_A(x)$ *is called the degree or grade of membership.*

Membership functions have also been used as belief functions and can be viewed as non-additive probabilities. For a discussion of these issues see Klir and Yuan (1995) and Billot (1991). The **fuzzy set** A itself is defined as the graph of μ_A:

$$A = \{(x, y) \in X \times [0, 1] : y = \mu_A(x)\}.$$

The only purpose of this definition is to have something at hand that is literally a set. All the properties of fuzzy sets are defined in terms of their membership functions.

For example, the fuzzy set A is called **normal** when $\sup_x \mu_A(x) = 1$. To emphasize that, indeed, all the properties of fuzzy sets are actually attributes of their membership functions, suppose that X is a nonempty subset of a Euclidean space. Then A is called **convex**, if μ_A is quasi-concave. This does not mean, however, that the graph of μ_A is convex. Take in particular a crisp set A. Then A as a subset of X is convex if and only if its characteristic function Ψ_A is quasi-concave. The latter does not imply, however, that the graph of Ψ_A is convex. We highlight two further important definitions, again in terms of membership functions.

Definition 12 *The fuzzy set B is a subset of the fuzzy set A if and only if*

$$\mu_B(x) \leq \mu_A(x)$$

for all $x \in X$.

For an axiomatic discussion of the standard set operations like union, intersection, etc. in the context of fuzzy sets see Bellman and Giertz (1973). The upper contour sets of a fuzzy set are called α-cuts and introduced next.

Definition 13 *Let $\alpha \in [0,1]$. The crisp set A_α of elements of X that belong to the fuzzy set A at least to the degree α is called the α-**cut** of the fuzzy set A:*

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$$

Moreover, we define the strict α-cut A^*_α of A as the crisp set

$$A^*_\alpha = \{x \in X : \mu_A(x) > \alpha\}.$$

*In particular, $A_0 = X$ and $A^*_1 = \emptyset$. A^*_0 is called the **support** of A or μ_A.*

Detailed expositions of all aspects of fuzzy set theory and their numerous applications can be found in the textbooks by Zimmermann (1990) and Klir and Yuan (1995).

4.2.2 Relevant Concepts From Abstract Economies

In this section we introduce some basic elements of abstract economies. The standard model of **strategic games** assumes in fact that each player is free to choose whatever action he pleases from his strategy set, regardless of the actions of others. The objectives of the player are represented by a utility or payoff function defined on the set of joint strategies or action profiles which add a cardinal flavor to the model. We shall turn to abstract economies which allow for (a) dependence of a player's feasible actions on the choices made by other players and (b) an ordinal concept of (not necessarily transitive and complete) preferences. This setting lends itself quite naturally to fuzzification.

Preferences and Constraints in Abstract Economies

For Nash equilibria in pure strategies of strategic or normal form games only the ordinal preferences of players matter. But one resorts frequently to Nash equilibria in mixed strategies and then the cardinal aspects of the payoff functions become essential. The situation is quite different in the context of **abstract economies**, also known as "generalized games" or "pseudo-games". An abstract economy assumes the form

$$\Gamma = (I; (S_i)_{i \in I}; (P_i)_{i \in I}; (F_i)_{i \in I})$$

where

1. I is a non-empty set of players;
2. S_i is a non-empty strategy set (strategy space), representing the strategies s_i for player $i \in I$;
3. $P_i : S \Longrightarrow S$ is a strict preference relation on $S \equiv \times_{j \in I} S_j$ for each player $i \in I$;
4. $F_i : S \Longrightarrow S_i$ is the constraint relation for each player $i \in I$.

F_i tells which strategies are actually feasible for player i, given the strategy choices of the other players. For technical convenience, we have written F_i as a function of the strategies of all the players including player i. In most applications, F_i is independent of i's choice. For instance, i cannot take a chair taken by somebody else. In an economic context, a fictitious player known as the auctioneer may set prices and thus determine the budget sets of other players. The jointly feasible strategies are the fixed points of the relation $F = \times_{j \in I} F_j : S \Longrightarrow S$. In principle, $F_i(s)$ can be empty for some $i \in I$ and $s \in S$. However, if this happens too often and F does not have a fixed point, then the theory becomes vacuous. If at the other extreme, $F_i(s) = S_i$ for all i and s, then Γ is an ordinal game. Following Border (1985), let us define, for each $i \in I$, the **good reply relation** $U_i : S \Longrightarrow S_i$ by $U_i(s) \equiv \{s'_i \in S_i : (s'_i, s_{-i}) \in P_i(s)\}$ for $s = (s_i, s_{-i}) \in S$. An **equilibrium** of the abstract economy Γ is a strategy profile $s \in S$ which is jointly feasible (a fixed point of F, i.e. $s \in F(s)$), and does not permit a feasible good reply, i.e. $U_i(s) \cap F_i(s) = \emptyset$ for all $i \in I$. The following existence result which is also stated and demonstrated in Border (1985) is of particular interest to us, since it does not require transitivity or completeness of preferences.

THEOREM 1 (SHAFER AND SONNENSCHEIN (1975)) *Suppose that for each i,*

(i) *S_i is a nonempty, compact and convex subset of a Euclidean space;*

(ii) *F_i is continuous and has nonempty, compact and convex values;*

(iii) *U_i has open graph in $S \times S_i$;*

(iv) *s_i does not belong to the convex hull of $U_i(s)$ for all $s \in S$.*

Then the abstract economy Γ has an equilibrium.

Fuzzification of Preferences and Constraints

Binary relations from a set Y to a set Z are easily fuzzified. Namely, a binary relation R from Y to Z can be identified with its graph, $Gr_R = \{(y,z) \in Y \times Z : z \in R(y)\}$, a subset of $X = Y \times Z$. In that sense, the binary relations from Y to Z are the crisp subsets of X. Accordingly, the fuzzy binary relations from Y to Z are the fuzzy subsets of X. Fuzzy preferences and choice based on such preferences have been explored among others by Basu (1984), Barret *et al.* (1990) Sengupta (1999), and Pattanaik and Sengupta (2000). Basu (1984) fuzzifies revealed preference theory, where fuzzy preferences lead to exact choices. It is shown that the a choice rule $C(.)$ which can be rationalized by the fuzzy preferences exists. The paper also provides comparisons with the traditional theory. Barret *et al.* (1990) argue that while people in general have vague preferences they make exact choices. They investigate plausible rationality properties for two different types of choice rules. The first is called a binary choice rule under which the choice from any set is basically derived from choices made from two-element. Non-binary choice rules where this assumption is relaxed are also explored in this paper. Sengupta (1999) considers agents with fuzzy preferences making exact choices. He provides an axiomatic characterization of a class of binary choice rules called $\alpha-$rules. According to this rule, for any $\alpha \in [0,1]$, an alternative x is chosen over y when in any pairwise comparison involving the two, x is preferred over y by at least a degree of α. Pattanaik and Sengupta also consider a situation where an agent with fuzzy preferences makes exact choices. They confine attention only to feasible sets containing no more than two alternatives and provide an axiomatic characterization of two broad classes of decision rules called ratio rules and difference rules. For a given fuzzy preference relation R, an alternative x is chosen over y according to the ratio rule if there exists $\alpha_{\{x,y\}} \in [0,1]$ such that $R(x,y) \geq \alpha_{\{x,y\}} R(y,x)$. An alternative ratio rule can be defined by using the strict inequality. A difference rule on the other hand requires that the difference between $R(x,y)$ and $R(y,x)$ be bounded by $\varepsilon_{\{x,y\}} \in [0,1]$. The appeal of this theory clearly lies in the fact that it allows agents to be somewhat fuzzy in their ranking of alternatives, thus embodying different degrees of rationality and yet making exact choices.

Returning to abstract economies Γ, we can replace both the preference relations P_i or U_i, respectively, and the constraint relations F_i by fuzzy versions. The above existence theorem readily applies to the various crisp relatives of these relations:

- If we merely require that a relation holds with a nonzero degree, we can work with the support of the relation.
- If we require that the relation holds with at least a given degree α, we can work with the corresponding upper contour set (α-cut) of the relation.

Notice that a higher α for the F_i makes joint feasibility harder whereas a higher α for the F_i and U_i furthers the absence of feasible good replies. The first effect may eliminate some equilibria. The second effect may create new equilibria.

4.3 Fuzzy Games: A Reformulation

In this section we develop a stripped down and crisp version of the standard Butnariu-Billot model. The essential idea is first — and we think better — presented in crisp terms. Fuzzy elements will be introduced later. We begin with the discussion of a certain axiom, labelled Axiom 1, which in our context constitutes the counterpart of Axiom A of the Butnariu-Billot model reported in the Appendix. We put forward an argument that demonstrates the frequent invalidity of Axiom 1. We next discuss the merits of a new and weaker Axiom 2 which is still very restrictive, but not to the extent of being a priori invalid. Finally, we develop a new model of fuzzy noncooperative games which does not rely on any of these axioms and can be cast within the framework of abstract economies.

4.3.1 Preliminary Formulation

Consider an underlying game form $GF = (I; (S_i)_{i \in I})$. A game form is a strategic form without a specification of the payoff functions. For simplicity we assume a finite game form, in particular, $I = \{1, \ldots, n\}$. For each player $i \in I$, let $Y_i = \Delta(S_i)$ denote the set of mixed strategies. Let $Y = \times_{j \in I} Y_j$ and let $Y_{-i} = \times_{j \neq i} Y_j$. Each player i has in addition individual perceptions of which mixed strategy profiles $y \in Y$ are feasible. The perceptions depend on player i's reasoning process as well as her notion of how the other players would reason in the game. These perceptions are represented by a subset π_i of Y. In player i's view if she chooses $y_i \in Y_i$, then only elements y_{-i} in $\pi_i(y_i)$, the y_i-section of π_i are feasible for the other players. Formally,

$$\pi_i(y_i) = \{y_{-i} \in Y_{-i} : (y_i, y_{-i}) \in \pi_i\}.$$

Finally, player i has preferences over subsets of Y_{-i} induced by set inclusion:

$$A_{-i} \precsim_i B_{-i} \Leftrightarrow A_{-i} \subseteq B_{-i} \quad \text{for} \quad A_{-i},\, B_{-i} \subseteq Y_{-i}.$$

We can now define an equilibrium.

Definition 14 *An* ***equilibrium*** *is a profile $y^* = (y_1^*, \ldots, y_n^*) \in Y$, such that the following two conditions hold.*

(a) *Mutual consistency: for all i, $y_{-i}^* \in \pi_i(y_i^*)$.*

(b) *Preference maximization: for all i, there is no $y_i \in Y_i$ such that $\pi_i(y_i) \succ_i \pi_i(y_i^*)$.*

The mutual consistency requirement is a condition on the player's perceptions which requires that in equilibrium, each player's beliefs about the others include the equilibrium strategy profile. Condition (b) means that a player wishes that her own choice restricts the choices available to the others as little as possible.

Now consider the following axiom which is the crisp version of Axiom A suggested in the literature; see Appendix.

Axiom 1: *For each $i \in I$ and $A_{-i} \subseteq Y_{-i}$ with $A_{-i} \neq \emptyset$, there exists $y_i \in Y_i$ such that $A_{-i} = \pi_i(y_i)$.*

Note that this axiom is violated, unless all Y_{-i} are singletons. For suppose Y_{-i} is not a singleton. Then Y_{-i} has the cardinality c of the set of the real numbers. Hence $P(Y_{-i})$, the power set of Y_{-i} has cardinality $2^c > c$. So has $P(Y_{-i})\backslash\emptyset$. On the other hand, Y_i has a cardinality of at most c. Therefore the mapping $y_i \longmapsto \pi_i(y_i)$ cannot have an image that contains $P(Y_{-i})\backslash\emptyset$.[1] Fortunately Axiom 1 can be replaced by a weaker one.

Axiom 2: *For each $i \in I$, there exists $y_i \in Y_i$ such that $Y_{-i} = \pi_i(y_i)$.*

Using this less demanding axiom, we can state the following result.

PROPOSITION 1 *Suppose Axiom 2 holds. Then an equilibrium exists.*

Proof: By Axiom 2, we can choose for each $i \in I$, a $y_i^* \in Y_i$ such that $Y_{-i} = \pi_i(y_i^*)$. Let us choose such a y_i^*. Then $y_{-i}^* \in Y_{-i} = \pi_i(y_i^*)$ for all i. Hence Condition (a) is satisfied. Moreover, for all $i \in I$ and $y_i \in Y_i$, $\pi_i(y_i) \subseteq \pi_i(y_i^*)$. Hence (b) holds as well. □

The appeal of the equilibrium is Condition (b) which lets a player maximize based only on his subjective perception of the others and is not affected by their actual play, akin to the solvability concept of von Neumann and Morgenstern for two-person zero-sum games and of Moulin for dominance solvable games.

The existence result, however, still hinges on the very restrictive Axiom 2. This can be easily demonstrated through a simple example.

[1]Observe that the same reasoning applies to Axiom A in the Appendix and any fuzzy version of Axiom 1.

Example 1 Consider the case where $I = \{1, 2\}$ and $|S_1| = |S_2| = 2$. Then we can set $Y_i = [0, 1]$ for each $i \in I$ where $y_i \in Y_i$ stands for the probability that i's "first action" is played. Let π_1 be given by $\pi_1(y_1) = \{y_2 : 1 - y_1/2 \leq y_2 \leq 1\}$ and π_2 by $\pi_2(y_2) = \{y_1 : 0 \leq y_1 \leq 1 - y_2/2\}$. Then Axiom 2 is violated. Further $y^* = (0, 1)$ is the only point in Y that satisfies condition (a) and $y^* = (1, 0)$ is the only point in Y that satisfies condition (b). Thus no equilibrium exists. The example is depicted in Figure 1.

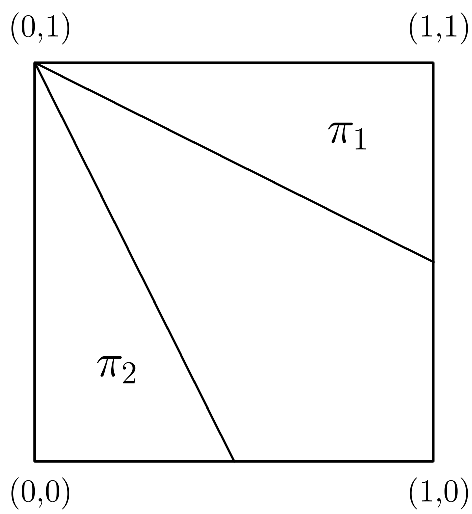

FIGURE 1

Hence, we face the dilemma of either making the strong assumption of Axiom 2 which renders existence almost trivial, or possibly lacking existence of an equilibrium.

4.3.2 The Reformulation

We suggest a way out by moving away from the time-honored von Neumann-Morgenstern approach and closer to the contemporary theory of strategic games. We propose that a player should be aware of the constraints that the choices of others impose on his own play in addition to the perceived restrictions that his play imposes on the choices of others. To illustrate the idea, consider again the parameters used in the previous example: $I = \{1, 2\}$, $|S_1| = |S_2| = 2$ and $Y_1 = Y_2 = [0, 1]$. We assume that all constraints are interval constraints, i.e., $\pi_i(y_i) \subseteq Y_{-i}$ is a non-empty interval $[a_i(y_i),\ b_i(y_i)]$. We also assume that the two functions $a_i : Y_i \to Y_{-i}$ and $b_i : Y_i \to Y_{-i}$ determining π_i are continuous. Then π_i is connected and path-connected. Let us assume, moreover, that $\pi_i \subseteq Y$ is convex. This is equivalent to a_i being convex and b_i being concave and implies that $l_i = b_i - a_i$ is concave.

Now consider the abstract economy $\Gamma = (I;\ Y_1, Y_2;\ P_1, P_2;\ \pi_1, \pi_2)$ with player set $I = \{1, 2\}$, strategy sets Y_1 and Y_2, and constraint relations π_1 and π_2 together with preference relations P_1 and P_2 defined as follows. P_i is the strict preference obtained when $l_i : Y_i \to \mathbb{R}$ is interpreted as a payoff function. Then conditions (i)-(iv) of the Shafer-Sonnenschein theorem are met. Hence we have the following proposition.

PROPOSITION 2 Γ *has an equilibrium.*

Notice that $l_i(y_i)$ is the length of the interval $\pi_i(y_i)$. Hence if we replace the P_i by the preference relation $\succ_i$ induced by set inclusion we have $y_i' \succ_i y_i$ implies $y_i'\ P_i\ y_i$. This gives us the following corollary.

COROLLARY 1 *The abstract economy* $\Gamma^0 = (I;\ Y_1, Y_2; \succ_1, \succ_2;\ \pi_1, \pi_2)$ *has an equilibrium.*

Proof: Let (y_1^*, y_2^*) be an equilibrium of Γ. We claim that (y_1^*, y_2^*) is an equilibrium of the abstract economy Γ^0. (y_1^*, y_2^*) is socially feasible. It remains to show that none of the players has a feasible good reply. Suppose player i has one in Γ^0. Then player i also has one in Γ which yields a contradiction, since (y_1^*, y_2^*) was assumed to be an equilibrium of Γ. $\square$

Comparing feasible sets by their size (Lebesgue measure) is not merely a technical trick, but seems to us an appealing alternative to comparison via set inclusion. The proposition and its corollary ensure that either type of preferences can be accommodated. Incidentally, after the reformulation, the abtract economies Γ and Γ^0 associated with Example 1 have both $(0, 1)$ as the only equilibrium.

This new model of fuzzy games can be generalized to more than two players, more than two actions per player and fuzzy π_i. How the latter can be achieved has been outlined in the previous section.

4.4 Conclusion

The modified version of noncooperative fuzzy games developed here is a merger of two different ideas. By allowing players to minimize the restrictions they impose on others we allow for a larger choice set and in a certain sense a wider set of rational behaviors. Moreover, since we do not require perfect information about beliefs, the equilibrium permits a richer set of possibilities. In the earlier version of fuzzy games, the rationality issue and the models that players have of each others behavior was a moot point owing to the full information requirement. By incorporating the notion of respecting the constraints imposed by others from the abstract economies literature, we restrict the players to choosing feasible actions. This feature enables us to show the existence of equilibrium without requiring complete knowledge of beliefs. Note that like in the earlier literature on fuzzy games, our model still does not involve maximizing an explicit objective.

Finally, it is also worth pointing out that the adopted approach is somewhat akin to Bellman and Zadeh's (1972) work on decision theory in a fuzzy environment. The decision maker's choices are constrained by two types of beliefs in their framework. The first is a goal function which allows an agent to rank the inexact outcomes associated with the different choices, and the second is a constraint function which allows the agent to rank the set of feasible alternatives. The optimal action is determined by the intersection of these two sets. A major difference with our approach is the fact that besides the game-theoretic setting, we have no explicit objectives. Further, in our model the two types of constraints are well defined and quite distinct, while in Bellman and Zadeh's own words in their model "… the goals and/or the constraints constitute classes of alternatives whose boundaries are not sharply defined."

Appendix: Fuzzy Games à la Butnariu and Billot

In this section we provide a brief review of the existing work on fuzzy strategic or noncooperative games. They were first developed by Butnariu (1978, 1979) and later refined by Billot (1992). Billot's main contribution in this area has been to provide a better interpretation of Butnariu's work making it more accessible to the reader. However, as will be evident the notation used in this formulation is still quite cumbersome and many standard concepts have been labelled differently. Our main objective here is to interpret their work using standard terminology and indicate the weaknesses of the formulation.

Butnariu defines a n-person noncooperative fuzzy game in normal form as $\Gamma = (S_i, Y_i, \Pi_i)_{i=1}^n$, where the set of players is denoted by $I = \{1, \ldots, n\}$, such that for any player $i \in I$ the following four conditions are satisfied:

1. Each player's set of pure strategies is given by S_i,

2. We define an element of Y_i as $w^i = (w_1^i, w_2^i, \ldots, w_m^i)$, where w_m^i denotes the weight assigned to player i's to her m-th pure strategy. Each $w^i \in Y_i$ is called a **strategic arrangement** of player $i \in I$. The n-dimensional vector $w = (w^1, \ldots, w^n) \in Y = \times_{i \in I} Y_i$ is called a **strategic choice** in Γ.

3. $\Pi_i \in 2^Y$ and for all $w \in Y$, $\pi_i(w)$ is the possibility level assigned by player i to the strategic choice w. This possibility level is essentially a membership function and denotes the membership value of each mixed strategy profile as assessed by player i.

4. Let $Y_{-i} = \times_{j \neq i} Y_j$ and let $W_i = 2^{Y_{-i}} \times Y_i$. Then $s_i = (A_f^i, w^i) \in W_i$ is player i's **strategic conception** in Γ.

Also the following axiom is assumed to hold:

Axiom A: *If* $A_f^i \in 2^{Y_{-i}}$ *and* $A_f^i \neq \emptyset$*, then* $\pi_i(A_f^i) \neq \emptyset$*, i.e., there exists* $s_i \in W_i$ *such that* $\pi_i(A_f^i)(w^i) \neq \emptyset$*.*

The second condition is just an alternative way of defining mixed strategies, where the players are assumed to know the weights of the mixed strategies. Of course this leads to a certain amount of redundancy, since one could just assume the players know the weights on the mixed strategies, which would imply automatic knowledge of the pure strategies. Alternatively we could assume that they just know the pure strategies, with the set of mixed strategies being all possible probability distributions over the pure strategies. Note that $\Pi_i \in 2^Y$, implying that the beliefs about mixed strategy profiles are actually crisp sets. The strategic conception itself consists of player i's beliefs about the other players and his own mixed strategy. Hence the definition of mixed strategies using probability weights in this formulation has an advantage in the sense that the two components of the strategic conception now lie in the interval [0, 1]. The axiom states that for a nonempty set of beliefs about the strategies of the other players, player i can choose a mixed strategy in response in the game which will constitute a strategic conception. In other words, there exists a w^i such that A_f^i is the w^i- section of π_i. Also let $W = \times_{i \in I} W_i$.

Definition 15 *Let* Γ *be an* n*-person noncooperative game satisfying the 4 conditions and the axiom stated above. A* ***play*** *is a vector* $s = (s_1, \ldots, s_n) \in W$*.*

Definition 16 *Let* s_i^* *and* $\widetilde{s}_i$ *denote two strategic conceptions of player* i*. We say that* s_i^* *is a* ***better strategic conception*** *than* $\widetilde{s}_i$*, or* $s_i^* \succ_i \widetilde{s}_i$ *for player* i *if and only if* $\pi_i(A_f^{i*})(w^{i*}) > \pi_i(\widetilde{A}_f^i)(\widetilde{w}^i)$*.*

In other words, we say that $s_i^* \succ_i \widetilde{s}_i$ for player i if and only if $\widetilde{A}_f^i \subset A_f^{i*}$.

Definition 17 *Let* s^* *and* $\widetilde{s}$ *denote two different plays of the game. We say that* s^* *is* ***socially preferred*** *to* $\widetilde{s}$*, or* $s^* \succ \widetilde{s}$ *if and only if for all* $i \in I$*,* $\pi_i(A_f^{i*})(w^{i*}) > \pi_i(\widetilde{A}_f^i)(\widetilde{w}^i)$*.*

Hence, $s^* \succ \widetilde{s}$ if and only if $s_i^* \succ_i \widetilde{s}_i$ for all $i \in I$.

Definition 18 *A* ***possible solution*** *of the game* Γ *is a play* s^**, such that for any other play* $\widetilde{s}$*, where for all* $i \in I$*,* $s_i^* = (A_f^{i*}, w^{i*})$*, the play* $\widetilde{s}$ *cannot be socially preferable to* s_i^* *if for all* $i \in I$*, and for all* $\widetilde{w}^i \in Y_i$*, we have* $\pi_i(A_f^{i*})(w^{i*}) \geq \pi_i(A_f^{i*})(\widetilde{w}^i)$*.*

The possible solution requires that the w^i- section of π_i corresponding to the equilibrium belief A_f^{i*} is greater for w^{i*} than for $\widetilde{w}^i$. Intuitively, the possible solution can be interpreted as two conditions. The first condition says s^* is a possible solution if s_i^* is feasible, that is A_f^{i*} is the w^i- section of π_i. This implies that Axiom A is satisfied. The second condition requires that there is no $\widetilde{s}$ such that *(a)* $\widetilde{s}_i$ is feasible for all $i \in I$, and *(b)* $\widetilde{s} \succ s^*$.

In order to define an equilibrium Butnariu allows for communication among the players. This communication allows players to reveal their beliefs to each other, while allowing them complete freedom in their choice of strategy. Given that Π_i for all $i \in I$ is already a part of the definition of the game, this can only mean that players reveal their specific π_i to each other. Based on this we only consider what Butnariu calls plays with perfect information which is defined as follows:

Definition 19 *A play* $s^* = (A_f^{i*}, w^{i*})$ *is called a* ***play with prefect information*** *when it is of the form*

$$\begin{aligned} A_f^{i*}(w^{1*}, \ldots, w^{i-1*}, w^{i+1*}, \ldots, w^{n*}) &= 1, \text{ for } w^j = w^{j*} \text{ where } j \neq i \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

We can alternatively replace this with the requirement that players have mutually consistent beliefs, or $w^{-i*} \in A_f^{i*}$ for all $i \in I$. It should also be immediatly obvious that such a play makes the game and the equilibrium concept which only allows for perfect infirmation, quite uninteresting. Using this we can now define an equilibrium of the fuzzy game Γ.

Definition 20 *An* ***equilibrium point*** *of the game* Γ *is a possible solution* s^**, where* $s_i^* = (A_f^{i*}, w^{i*})$ *which satisfies the mutual consistency condition on beliefs for all players* $i \in I$*.*

Two existence proofs are also provided in this literature. The first theorem proves the existence of possible solutions and the second one proves the existence of equilibrium points in Γ. However, in view of our earlier comments about the nature of the equilibrium, details of these proofs are omitted. The interested reader may refer to Butnariu (1979) and Billot (1992). A smorgasbord of fuzzy fixed point theorems can be found in Butnariu (1982).

Chapter 5

From Decision Theory to Game Theory in a Fuzzy Environment

5.1 Introduction

The behavior of players in a game depends on the structure of the game being the decisions they face and the information they have when making decisions, how their decisions determine the outcome and the preferences they have over the outcomes. The structure also incorporates the possibility of repetition, the implemantation of any correlating devices and of alternative forms of communication. Any imprecision regarding the structure of the game has consequences for the outcome. Yet, in the real-world decision making often takes place in an environment in which the goals, the constraints and the outcomes faced by the players are not known in a precise manner. Ambiguities can exist if the components of the game are specified with some vagueness or when the players have their own subjective perception of the game. Psychological games analyzed by Geanakoplous, Pearce and Staccehtti (1989) and the model of fairness developed by Rabin (1993) are two examples of where the players have their own interpretation of the game. The psychological game is defined on an underlying material game (the standard game that one normally assumes the agents are playing). Chen, Freidman and Thisse (1997) have a model of boundedly rational behavior where the players have a latent subconscious utility function and are not precisely aware of the actual utility associated with each outcome.

In this paper we develop a descriptive theory to analyze games with such characteristics. We assume that the components of the game may not be well defined and, hence, involve a considerable amount of subjective perception on the part of the players. The model builds on the work of Bellman and Zadeh (1970) who analyze decision-making in a fuzzy environment, and extends it to a game-theoretic setting. The tools underlying this approach are derived from fuzzy set theory. A fuzzy set differs from a classical set (referred to as a *crisp set* hereafter) in that the characteristic function can take any value in the interval [0,1]. In this manner it replaces the binary (Aristotelian) logic framework of set theory and incorporates "fuzziness" by appealing to multi-valued logic. For instance, a person who is 6 feet tall can have a high membership value (in the characteristic function sense) in the set of "tall people" and a low membership value in the set of "short people". It is an ideal tool for modelling subjective perceptions of problems in a quantitative manner. Consider our example regarding tall people. The notions of "tallness" and "shortness" are themselves context related. Among basketball players a person who is 6 feet tall may be considered "short"; among dwarfs this person would be considered a gaint and have a very high membership value in the set of tall people as defined by dwarfs. Providing general tools to model such issues is one of the main advantages of the fuzzy set theory. Dual membership instances of this type cannot arise in the context of crisp sets. The roots of Zadeh's work on fuzzy sets can be traced back to work on multi-valued logic by the philosopher Max Black (1937). The underlying motive behind much of fuzzy set theory is that by introducing imprecision of this sort in a formal manner into crisp set theory, we can analyze realistic versions of problems of information processing and decision making.

The first part of the paper is devoted to a survey of the existing literature on non-cooperative fuzzy games, including applications. The model developed here is a radical departure from the earlier work. It contains some of results and an application to duopoly.

The paper is exploratory in nature and outlines possibilities for future research.

In the conventional approach to decision-making a decision process is represented by (a) a set of alternatives, (b) a set of constraints restricting choices between the different alternatives,[1] and (c) a performance function which associates with each alternative the gain (or loss) resulting from the choice of that alternative. When we view a decision process from the broader perspective of decision-making in a fuzzy environment, Bellman and Zadeh (1970) argue that a different and perhaps more natural conceptual framework suggests itself. They argue that it is not always appropriate to equate imprecision with randomness and provide a distinction between randomness and fuzziness.[2] Randomness deals with uncertainty concerning membership or non-membership of an object in a non-fuzzy set. Fuzziness on the other hand is concerned with grades of membership in a set, which may take intermediate values between 0 and 1. A fuzzy goal of an agent is a statement like "my payoff should be approximately 50" and a fuzzy constraint may be expressed as "the outcome should lie in the medium range". The most important feature of this framework is its symmetry with respect to goals and constraints − a symmetry which erases the differences between them and makes it possible to relate in a particularly simple way, the concept of decision making to those of the goals and constraints of a decision process.

Our model is similar in spirit to the above approach and models the standard game as a set of constraints and goals which can then be solved like a decision-making problem. The fuzzy extension of the standard two player game in our framework will have fuzzy payoffs which represent the goals of the players. We will define a fuzzy extension of the strategies of both the players effectively limiting the choices of both players. The equilibrium concept will be identical to the Nash equilibrium, except that it will now be defined

[1]Note that the sets of alternatives could be a restricted set. However these are restrictions that are a part of the definition of the problem, while constraints on the choice set arise from the players perception of the decision making situation.

[2]There are several alternative ways to solve fuzzy decison problems. A particularly interesting approach is that of Li and Yen (1996) which relies on linguistic variables. Semantics are used to create a descriptor frame which redefines the decision making problem.

on the fuzzy extension of the game. This is unlike the formulation of Butnariu (1978, 1979) and Billot (1992) where the payoffs functions are completely absent since they are subsumed into abstract beliefs. The solution in their formulation imposes very high information requirements on the definition of a game and it is also rather cumbersome to translate their model into standard game theoretic terms. Our formulation is easier to interpret and is closer to the standard model of noncooperative games. We provide an alternative way to look at noncooperative games that is more appropriate in situations where there might be a highly subjective component to the game. Some results are derived, the most important of which concerns identifying the conditions under which an equilibrium exists. It is also worth mentioning that given the descriptive nature of the formulation, there is a trade-off in terms of its predictive abilities.

The next section describes some of the basic concepts of fuzzy set theory. Section 5.3 provides a review of the existing work on noncooperative fuzzy games. Section 5.4 presents the model along with a few results. The final section has some concluding remarks.

5.2 A Brief Introduction to Fuzzy Sets

The seminal formulation of the concepts of fuzzy sets is due to Zadeh (1965) who tried to generalize the idea of a classical set by extending the range of its characteristic function. Informally, a fuzzy set is a class of objects for which there is no sharp boundary between those objects that belong to the class and those that do not. Here we provide some definitions that are pertinent to our work.

Let X denote a universe of discourse. We distinguish between crisp or traditional and fuzzy subsets of X.

Definition 21 *The* ***characteristic function*** Ψ_A *of a* ***crisp set*** A *maps the elements* *of X to the elements of the set $\{0,1\}$, i.e., $\Psi_A : X \to \{0,1\}$. For each $x \in X$,*

$$\Psi_A(x) = \begin{cases} 1 \ \textit{if} \ x \in A \\ 0 \ \textit{otherwise} \end{cases}$$

To go from here to a fuzzy set we need to expand the set {0,1} to the set [0,1] with 0 and 1 representing the lowest and highest grades of membership respectively.

Definition 22 *The **membership function** μ_A of a **fuzzy set** A maps the elements of X to the elements of the set $[0,1]$, i.e., $\mu_A : X \to [0,1]$. For $x \in X$, $\mu_A(x)$ is called the degree or grade of membership.*

Membership functions have also been used as belief functions and can be viewed as non-additive probabilities. For a discussion of these issues see Klir and Yuan (1995) and Billot (1991). The **fuzzy set** A itself is defined as the graph of μ_A:

$$A = \{(x,y) \in X \times [0,1] : y = \mu_A(x)\}.$$

The only purpose of this definition is to have something at hand that is literally a set. All the properties of fuzzy sets are defined in terms of their membership functions. For example, the fuzzy set A is called **normal** when $\sup_x \mu_A(x) = 1$. To emphasize that, indeed, all the properties of fuzzy sets are actually attributes of their membership functions, suppose that X is a nonempty subset of a Euclidean space. Then A is called **convex**, if μ_A is quasi-concave. This does not mean, however, that the graph of μ_A is convex. Take in particular a crisp set A. Then A as a subset of X is convex if and only if its characteristic function Ψ_A is quasi-concave. The latter does not imply, however, that the graph of Ψ_A is convex. We highlight further some other important definitions, again in terms of membership functions.

Definition 23 *The fuzzy set B is a **subset** of the fuzzy set A if and only if*

$$\mu_B(x) \leq \mu_A(x)$$

for all $x \in X$.

Definition 24 *The* ***complement*** *of a fuzzy set* A *is fuzzy set* CA *with the membership function*

$$\mu_{CA}(x) = 1 - \mu_A(x),\ \ x \in X$$

Elements of X for which $\mu_A(x) = \mu_{CA}(x)$ are sometimes referred to using the misleading term "equilibrium points".

We now define the basic set theoretic notions of union and intersection. Let A and B be two fuzzy sets.

Definition 25 *The membership function* $\mu_F(x)$ *of the* ***intersection*** $F = A \cap B$ *is defined pointwise by*

$$\mu_F(x) = \min\{\mu_A(x), \mu_B(x)\},\ \ x \in X$$

Similarly for the ***union*** *operation, the membership function of* $D = A \cup B$*, we have*

$$\mu_D(x) = \max\{\mu_A(x), \mu_B(x)\},\ \ x \in X$$

also defined pointwise.

These definitions union and intersection in the context of fuzzy sets are due to Zadeh (1965). While alternative formulations of the union and intersection property and other exist, Bellman and Giertz (1973) prove that this is the most consistent way of defining these operations. They also provide an axiomatic discussion of other standard set-theoretic operations in the context of fuzzy sets. The upper contour sets of a fuzzy set are called α-cuts and introduced next.

Definition 26 *Let* $\alpha \in [0, 1]$. *The crisp set* A_α *of elements of* X *that belong to the fuzzy set* A *at least to the degree* α *is called the* α-***cut*** *of the fuzzy set* A*:*

$$A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$$

Moreover, we define the ***strict*** α***-cut*** A^*_α *of* A *as the crisp set*

$$A^*_\alpha = \{x \in X : \mu_A(x) > \alpha\}.$$

In particular, $A_0 = X$ *and* $A^*_1 = \emptyset$*.* A^*_0 *is called the* ***support*** *of* A *or* μ_A*.*

We will define the notion of a fuzzified function and the extension principle for a fuzzy function. We say that a crisp function $f : X \to Y$ is fuzzified when it is extended to act on fuzzy sets defined on X and Y. The fuzzified function for which the same symbol f is usually has the form $f : \mathcal{F}(X) \to \mathcal{F}(Y)$, where $\mathcal{F}(X)$ denotes the fuzzy power set of X (the set of all fuzzy subsets of X). The principle for fuzzifying crisp functions (or crisp relations) is called the extension principle.

Definition 27 ***Extension Principle****: Any given function* $f : X \to Y$ *induces two functions*

$$f : \mathcal{F}(X) \to \mathcal{F}(Y)$$
$$f^{-1} : \mathcal{F}(Y) \to \mathcal{F}(X)$$

defined as $[f(A)](y) = \sup_{x:\, y=f(x)} A(x)$ *for all* $A \in \mathcal{F}(X)$*, and* $[f^{-1}(B)](x) = B(f(x))$ *for* $B \in \mathcal{F}(Y)$ *respectively.*

Detailed expositions of all aspects of fuzzy set theory and their numerous applications can be found in the textbooks by Zimmermann (1990) and Klir and Yuan (1995).

5.3 Review of the Existing Literature

In this section we review alternative approaches to modeling noncooperative fuzzy games. The most prominent work in this area is the formulation of noncooperative fuzzy games due to Butnariu (1978, 1979) and Billot (1992). In the Butnariu-Billot formulation players have the usual strategies and beliefs about what strategies the other players will choose in the game. These beliefs are described by fuzzy sets over the strategy space of the other players. Players in such a fuzzy game choose strategies that maximize the membership value of their belief about the other players and in this tries to minimize the restrictions he imposes on others, without pursuing an explicit objective function. However, the equilibrium concept requires very restrictive assumptions, making the formulation quite uninteresting. A detailed description of this model is given in Haller and Sarangi (2000), and the interested reader may refer to this paper for a reformulation of the Butnariu-Billot model as well.

In this section we will discuss other approaches to noncooperative fuzzy games, as well as some applications. The two other approaches in the literature only provide techniques to analyze zero-sum games. Campos (1989) uses linear programming to model matrix games, and Billot (1992) uses lexicographic fuzzy preferences to identify equilibria in a normal form game. We also discuss two application of fuzzy sets to *Industrial Organization*. The first by Greenhut, Greenhut and Mansur (1995) is an application to modelling quantity setting oligopoly, and the second application due to Goodhue (1998) analyzes collusion through a fuzzy trigger strategy.

5.3.1 The Linear Programming Approach

Campos (1989) introduces a number of different types of linear programming (LP) models to solve zero-sum fuzzy normal form games. In this formulation, each player's strategy set is a crisp set, but players have imprecise knowledge about the payoffs. A zero sum two person fuzzy game is represented by $G = (S_1, S_2, \widetilde{A})$, where S_1 and S_2 denote the pure strategy sets of the two players. We assume that player 1 is the row player and use i for his strategies and player 2 is the column player and hence his strategies will be referred to by j. We assume that player 1 has m strategies and player 2 has n strategies. $\widetilde{A} = (\tilde{a}_{ij})$ is an $m \times n$ matrix of fuzzy numbers, i.e., numbers that lie in the [0,1] interval.

The fuzzy numbers are defined by their membership functions as follows:

$$\mu_{ij} : \mathbb{R} \rightarrow [0,1], \qquad i \in S_1,\ j \in S_2$$

This membership function captures the information that player 1 has about his payoffs and also the information about player 2's payoffs associated with the $i-$th strategy and the $j-$th strategy choices by the two players respectively. Campos (1989) argues that payoffs need to be represented by fuzzy numbers since in many real world situations players may not be aware of their exact payoffs. In standard game-theoretic terms the above operation using the membership function just normalizes the payoffs of each player to the interval [0,1]. However, since the players have imprecise knowledge of their own payoffs, Campos (1989) allows for "soft constraints", i.e., each player is willing to permit some flexibility in satisfying the constraints. Hence we can write down player 1's problem as[3]

$$\begin{aligned}
&Max\ v \\
&\text{s.t. } \sum_i \tilde{a}_{ij} s_i \geq v,\ j \in S_2 \\
&s_i \geq 0,\ i \in S_1,\ \sum_i s_i = 1
\end{aligned}$$

where $\geq$ represents the fuzzy constraint, v represents the security level for player 1 and $s_i \in S_1$. Notice that the problem now involves double fuzziness since the payoff functions are represented membership functions and the constraint is also fuzzy. The LP problem in the above form is intractable and needs to be modified further. For this we define $u_i = s_i/v$, and thus $v = \sum s_i / \sum u_i = 1/\sum u_i$. We can now restate the LP in terms of its dual:

[3]Player 2's problem is a standard minimization problem and for the sake of brevity will not be shown here. The interested reader may refer to Friedman (1990).

$$
\begin{aligned}
&Min \sum u_i \\
&\text{s.t. } \sum_i \widetilde{a}_{ij} u_i \geq 1,\ j \in S_2 \\
&u_i \geq 0,\ i \in S_1
\end{aligned}
$$

The resolution of a fuzzy constraint of the type shown above relies on a technique introduced by Adamo (1980). The fuzzy constraint is now substituted by a convex constraint given by

$$\sum_i \widetilde{a}_{ij} u_i \succeq 1 - \widetilde{p}_j(1-\alpha),\ j \in S_2$$

where $\widetilde{p}_j$ is a fuzzy number that expresses the maximum violation that player 1 will permit in the accomplishment of his constraint, and $\succeq$ is the relation which the decision maker chooses for ranking the fuzzy numbers.[4] Fuzzy set theory provides for numerous ways of ranking fuzzy numbers. Campos (1989) considers five different ways of ranking fuzzy numbers, and for each case rewrites the constraints using fuzzy triangular numbers. Two of these are based on the work of Yager (1981) and involves the use of a ranking function or index that maps the fuzzy numbers onto $\mathbb{R}$. A third approach involves the use of $\alpha-$cuts and is based on the work of Adamo (1980). The last two approaches rank fuzzy numbers using *possibility theory*. This stems from the work of Dubois and Prade (1983). Finally, the five different parametric LP models obtained through this transformation process are solved using conventional LP techniques to identify their fuzzy solutions. This exercise is performed on different numerical examples.

5.3.2 A Fuzzy Game with Lexicographic Preferences

Billot (1992) develops an alternative model of fuzzy games using fuzzy lexicographic preorderings. This is quite a rudimentary formulation and is applicable only to zero-sum

[4] The only requirement for this relation is that it must preserve the ranking of the fuzzy numbers when they are multiplied by a positive scalar.

games. Unlike the model originally developed by Butnariu, this is an ordinal game, and differs from the standard game theoretic formulation only by allowing for fuzzy lexicographic preferences. He then introduces an axiom called the *axiom of Local non-Discrimination* according which a player is assumed to be indifferent between two very close options. Note, however, that this indifference is not unique and that is intensity can vary between 0 and 1, the degree of this intensity being expressed by a membership function. It is further shown that under the axiom, a fuzzy lexicographic preorder can be represented by a continuous utility function defined on a connected referential set $X \subseteq \mathbb{R}$.

A normal form game is defined as $G = (\Sigma_i, P_i)_{i=1}^n$ where Σ_i is the strategy space which is assumed to be a real convex set. An individual strategy is denoted by $\sigma_i \in \Sigma_i$ and the strategies of all the other players by $\sigma_{-i} \in \Sigma_{-i} = \times_{j \neq i} \Sigma_i$. P_i denotes the payoff of function, $P_i : \times_{i=1}^n \Sigma_i \rightarrow \mathbb{R}$ and is assumed to be continuous. Next he introduces a tranformation function that orders the strategies lexicographically based on the payoffs they yield. Recall that under the axiom, a fuzzy lexicographic preorder can be represented by a continuous utility function. Since this utility function can now be defined on the set of strategies, Billot calls this a **strategic utility function**. He then proves an existence result for two-person zero same games under fairly simple conditions. It is shown that if the axiom is satisfied, strategy space is compact and convex, and the payoff function is continuous, then an equilibrium will exist. Further it is shown that for inessential games where the payoffs and the strategies satisfy the conditions listed above, the equilibrium set derived using fuzzy lexicographic preferences contains the usual set of Nash equilibria.

5.3.3 A Fuzzy Approach to Oligopolistic Competition

Greenhut, Greenhut and Mansur (1995) apply fuzzy set theory to model oligopolistic competition. Their objective is to characterize the problem of a real world oligopolistic market from the perspective of the decision maker of a firm. A firm i may be ranked as a strong or a weak rival by firm j depending on the degree of its inclusion in the oligopoly. For example in the soft drink industry Coke and Pepsi are the dominant firms, but smaller rivals also exist and each of the two leading firms may be interested in taking the actions of these smaller rivals into account. The degree of inclusion of these small firms in the oligopoly then quantifies the importance that ought to be given to the actions of the smaller rivals. It is argued that quantification of real world setting in this manner will be of great help to these decision makers.

Greenhut *et al.* (1995) claim that an oligopoly can be described as *competition* among a *few firms* producing *similar products*. They use three different fuzzy sets to model the vague (italicized) linguistic terms in above the definition of an oligopoly.[5] Each fuzzy descriptor captures the degree to which a particular firm belongs to the oligopolistic market when compared with a representative firm F whose membership in the oligopoly is of degree one. The first category is **similar products** and is used to model the notion that firms do not produce exactly identical products. The membership function expresses how a particular firm's product compares to the product of the representative firm. The fuzzy set S^* contains the membership value of each firm in the industry vis-a-vis product similarity. The next aspect of oligopolies that is modeled in the paper is the **degree of inter-dependence between firms**. This is denoted by the fuzzy set I^*, which is the fuzzy set of firms whose membership grades represent the degree of perceived inter-dependence between a firm and the firm F, quantifying the degree of strategic rivalry between firms. The third category mentioned in their formulation is the notion of a **few firms**. The fuzzy set F^* denotes the fuzzy membership of firms in the industry where a degree of membership is assigned to the discrete numbers belonging to $\mathbb{N}$. The authors regard the number of firms in the industry to be inexact by appealing to the possibility of free entry and exit, and the fact that geographical boundaries between competing firms are not well defined. The oligopoly itself is denoted by the fuzzy set O^*

[5] The presence of these imprecise linguistic terms is cited as the main reason for using fuzzy techniques instead of relying on probabilistic methods.

which is a combination of S^*, I^* and F^*. It now expresses the fact that an oligopoly is competition among a few interdependent firms producing similar products. The degree of membership of any particular firm in O^* is obtained by applying Zadeh's *extension principle*. The authors illustrate their point by means of a numerical example. Using numerical examples they also show how fuzzy set theory can be used to compute a fuzzy *Herfindahl Index*.

Although the approach suggested by Greenhut *et al.* (1995) is interesting, it does not provide satisfactory answers to basic oligopoly questions. The membership grades used in their examples are completely subjective and arguably arbitrary. The authors argue that though they use subjective membership functions, accepting the possibility of a fuzzy model will allow us to develop more realistic oligopoly models in conjunction with econometric techniques which amy be used to obtain membership functions. Further, while it may be hard to quibble with the idea of using a fuzzy set to model product homogeneity, the last two fuzzy categories used in defining an oligopoly, namely the notion of inter-dependent firms and a few firms is clearly debatable. In a certain sense the paper also fails to deliver, since it does not suggest how to solve a quantity setting or a price setting game between firms after computing the degree of inclusion of each firm in the oligopoly. This clearly remains an open research question. Their paper concludes on a more philosophical note claiming that fuzzy modeling opens up a host of possibilities despite its subjective elements.

5.3.4 Fuzzy Trigger Strategies

Goodhue (1998) applies fuzzy set theory to model collusive behavior. She examines the Green and Porter (1984) model by assuming that firms can use fuzzy trigger strategies. Prices are expressed as fuzzy sets. There are a finite number (I) of fuzzy sets denoted by P_i describing the level of prices in linguistic terms.. For example, "low prices" denotes one such set. The degree of membership of a price in any particular set captures the extent to which it posses the properties associated with that set. Uncertainty that firms face regarding the realization of demand is also modeled as a fuzzy set and one example of such a set is the set that expresses the fact that "demand is low". There are J such sets each denoted by D_j. The chance of cheating in this model is defined on these two sets, which is made possible through the application of the *Extension Principle*. She finds that the fuzzy trigger pricing game reverses the standard cyclical price war prediction. Collusion sustaining price wars are most likely to occur during times of high demand. The fuzzy model also predicts that markets with relatively volatile prices are more likely to undergo collusion-sustaining price wars.

5.4 The Model

The model developed here uses the Bellman and Zadeh (1970) approach to fuzzy decision making. Let $G = (N, S, \Pi)$ be the triple that defines a standard normal form game where $N = \{1, 2\}$ is the set of players in the game. For each player i we denote the set of strategies by S_i and a particular strategy chosen from this set by s_i. A particular strategy profile is denoted by $(s_1, s_2) = s \in S$ where $S = S_1 \times S_2$. Each player's payoff function is denoted by $\Pi_i : S \to \mathbb{R}$. Since the decision making environment is fuzzy, this is not the game which is actually played. The players create their own fuzzy version of the game. This is similar to the idea of the *subconscious* utility function explored by Chen, Friedman and Thisse (1998) where the players only have a vague notion of their actual utility function. Bacharach's (1993) *variable frame theory* is also similar in the sense that different games are associated with different variable universes and lead to a different focal point in each associated game.

We will now define a fuzzy version of this game. For each player i the *constraint set* is given by

$$\mu_i : S_i \to [0, 1]$$

This suggests that each player does not consider all his strategies as equally feasible. They vary in their degree of feasibility and only some of them might be considered completely feasible, i.e., have a membership value of one. This acts as a constraint on his choice of strategies and can stem from his beliefs about the other player. It might capture for instance player 1's belief about the other player's type or about his rationality. It can also be used to eliminate dominated strategies. We will call this a *perception* constraint, which in this simple case is assumed to be entirely static and non-adaptable. This is clearly an expression of the player's rationality. For example, this can be used to define a curb set (Basu and Weibull, 1994), restricting choices only to a particular set of strategies. Iterated elimination of strictly or weakly dominated strategies can be captured by assigning successively lower values to dominated strategies. Other refinement criteria could also be captured in a similar way. Note also that μ_i must be non-empty, or the player does not think that there are any feasible strategies to choose from.[6]

We also define for each player a non-empty *goal function*

$$\gamma_i : S \to [0, 1]$$

This represents each player's *aspiration* level. This function which is defined over the action space could also be defined over the payoff space by considering a mapping from the action space to the payoff space which is then mapped on to the unit interval.[7] This fuzzy membership function could be used to capture some alternatives to utility maximization or like altruistic behavior or fairness. It could also be used to the model fairness of the type suggested by Rabin (1993), since the goal function reorders he payoff function. In Rabin's formulation players get more or less utility in addition to that feasible from the payoff function depending on whether they feel their opponent is being nice or mean to them. The usual normal form game is now replaced by a modified game in a fuzzy environment which we will call a "fuzzy game." This fuzzy game may be formally expressed by the tripe $G^f = (N, \mu, \gamma)$.

[6] This has some interesting implications. If μ_i is indeed empty, the player does not believe that any strategies are feasible options, and hence is unwilling to participate in the game. In order to avooid issues of this sort we will assume that the constraint set is a *normal* fuzzy set.

[7] Thus γ_i may be construed as the composition of two mappings: $\Pi_i : S \to \mathbb{R}$ and $\Gamma_i : \mathbb{R} \to [0, 1]$.

Thus the two membership functions defined above are quite general and can embody a whole range of possibilities. The membership function can be used to explore sophistication in the players reasoning, while the goal function can capture elements of psychological games and formulations of the sort suggested by behavioral game theory.

Using the two notions developed above we can now determine the player's decision set, which, in the words of Bellman and Zadeh, is the confluence of goals and constraints, defined by $\delta_i : S \to [0,1]$ with

$$\delta_i(s) = \min\{\mu_i(s_i), \gamma_i(s)\}.$$

As can be easily deduced δ_i is basically the intersection of the set of goals and the constraints facing a player. This can be interpreted as follows: Suppose player 1 knew the particular strategy choice $\widehat{s}_2 \in S_2$ of player 2, then $\delta_1(\cdot, \widehat{s}_2)$ represents player 1's response using the Bellman and Zadeh approach. This of course means that player 1 must formulate γ_i accordingly, i.e., player 1 must follow the above rule when computing his goal function. Thus the decision set expresses the degree of compatibility between a players perceptions and his goals. It is obvious that this approach does not have any advantages in simple games, by which we mean games whose structure is so transparent that the components would remain unchanged even in a fuzzy environment. Since it imposes a symmetry between the goals and the constraints, it would be useful in games that involve a large number of strategies and requiring sophistication in the reasoning process, or in games with multiple equilibria. It is perhaps more like a heuristic way of looking at a game, when computing the equilibrium might be difficult. By putting the strategies and payoffs on a common platform, one might argue that it makes it easier to solve such a game.

Definition 28 *A strategy tuple* (s_1^*, s_2^*) *is a Nash equilibrium in* G^f *if for all* $i \in N$, *we have*

$$\delta_i(s^*) \geq \delta_i(s_i^{'}, s_{-i}^*) \text{ for all } s_i^{'} \in S_i$$

Under certain conditions on the membership functions, it will be possible to argue that such an equilibrium will always exist for the fuzzy extension of the game defined above. We will first assume the following.

Assumption: *For all $i \in N$, we assume that S_i is compact and convex and the payoffs are continuous.*

PROPOSITION 1: *For a game $G^f = (N, \mu, \gamma)$, if δ_i is non-empty, continuous, and strictly quasi-concave in a player's own strategies, then G^f has at least one Nash equilibrium.*

Proof. For each player $i \in N$, define the best response function for i as $r_i : S \to S_i$ as follows

$$\text{for all } s \in S,\ r_i(s) = \arg\max \delta_i(t_i, s_{-i})$$

From the above conditions it is obvious that such an $r_i(s)$ must exist and is unique. We also define the best response function $r : S \to S$ if for all $s \in S$, $r(s) = (r_1(s), r_2(s))$. Since S_i is compact and convex for all i, it follows that S is compact and convex. Now through contradiction we will show that r is continuous by showing that $r_i(s)$ is continuous for all i.

Suppose not. Then there exists $s \in S$, and a sequence $\{s^n\}$ in S such that $s^n \to s$, but $r_i(s^n)$ does not converge to $r_i(s)$. This and the compactness of S implies that there is a subsequence which converges to $t_i \neq r_i(s)$. Without loss of generality, suppose that $\{s^n\}$ itself converges to t_i. Since $\delta_i(s^n/r_i(s^n)) \geq \delta_i(s^n/r_i(s))$ for all n, it follows from the continuity of δ_i that $\delta_i(t_i, s_{-i}) \geq \delta_i(r_i(s), s_{-i})$.[8] This is a contradiction since $r_i(s)$ is the unique best response of player i to s.

Since r is continuous and S is compact and convex, we know by Brouwer's fixed point theorem that there exists $s^* \in S$, such that $r(s^*) = s^*$. Thus, for all $i \in N$, $\delta_i(s^*) \geq \delta_i(s_i, s^*_{-i})$ for all $s_i \in S_i$. So s^* is a Nash equilibrium. ■

[8] Note that we abuse notation slightly to denote $(r_i(s), s_{-i})$ by $(s/r_i(s))$.

The fuzzy set theoretic formulation allows us to compare the tension between the player's aspirations and constraints by assigning numerical values to strategies and payoffs in the interval [0,1]. We believe that is the most appealing feature of the version of fuzzy games developed here.

We now investigate an issue that arises quite naturally in this context. Assume that a player has a given goal function. We will now identify conditions on his strategies that will enable him to ensure a certain level of payoff γ_0. For this purpose we assume that the players adopt a cautious approach and follow a maximin type of reasoning. For each player define the following number

$$\begin{aligned} C_i &= \max_{s_i \in S_i} \min_{s_j \in S_j} \delta_i(s) \\ &= \max_{s_i \in S_i} \min\{\mu_i(s_i), \min_{s_j \in S_j} \gamma_i(s)\} \end{aligned}$$

This number defines the maximum payoff a cautious player can ensure for herself. Note also that a low C_i implies that there is a big gap between a players aspirations and her feasible choices. Next we also define the α-cut of the set S_i as $S_i^{\gamma_0} = \{s_i \in S_i : \mu_i(s_i) \geq \gamma_0\}$. Also let $S_i(\gamma_0) = \{s_i \in S_i : \min_{s_j \in S_j} \gamma_i(s) \geq \gamma_0\} \subseteq S_i$.

PROPOSITION 2: *If* $S_i(\gamma_0) \neq \emptyset$ *and* $S_i(\gamma_0) \cap S_i^{\gamma_0} \neq \emptyset$, $C_i \geq \gamma_0$.

Proof. The proof is really simple. If $S_i(\gamma_0) = \emptyset$, then it is possible that $C_i < \gamma_0$ and hence the player cannot always guarantee the desired payoff. If $S_i(\gamma_0) \neq \emptyset$ and if $\mu_i(s_i) \geq \gamma_0$ for at least one $s_i \in S_i(\gamma_0)$, then from the definition of C_i it is easy to check that $C_i \geq \gamma_0$ will always be true. ■

The proposition illustrates for a given goal function what restrictions on the constraint set will ensure a pre-specified payoff like γ_0. This situation however, need not be an equilibrium. Given that players follow a maximin strategy to define C_i, in zero-sum games *Proposition 2* also defines an equilibrium, provided δ_i satisfies the existence conditions.

Pre-play communication has some interesting implications for this model. Suppose the players can communicate before playing the game. This would clearly affect their perception set or the set of feasible choices that each player has. With pre-play communication player i will have a better notion of the strategies that player j will choose. Denote this by $S'_j \subset S_j$. This affects the minimum payoff a player can ensure for herself, i.e.,

$$C'_i = \max_{s_i \in S_i} \min\{\mu_i(s_i), \min_{s_j \in S'_j} \gamma_i(s)\}$$

Provided S'_j is a strict subset of the set of original strategies, $C'_i \geq C_i$. Hence, exchange of information between the two players has potentially interesting possibilities in this context.

5.4.1 A Duopoly Example

In what follows we set up a basic duopoly model and discuss the implication of making it a fuzzy game. We consider a single period homogenous product Cournot duopoly. The inverse demand function in this market is given by the standard linear formulation

$$p = a - bQ,\ Q = q_1 + q_2 \text{ with } a, b > 0.$$

We also assume that both firms have identical constant marginal cost functions given by $C(q_i) = cq_i$, $i = 1, 2$. We can now write the profit function as

$$\Pi_i(q_1, q_2) = (a - bQ - c)q_i,\ i = 1, 2.$$

In the fuzzy version of this game the constraint set is assumed to be a crisp set. Thus, each firm considers all its strategies equally feasible, i.e., $\mu_i(q_i) = 1$ for all q_i and for $i = 1, 2$. In order to make things simple we assume that the strategy set is compact and defined by $q_i \in [0, a/b]$. The goal function however is fuzzy and each firm believes that the collusive outcome is the best possible outcome. Hence the membership function is single peaked such that $\gamma_i(\frac{Q}{4}, \frac{Q}{4}) = 1$ for $i = 1, 2$. An example of a membership function with this property is

$$\gamma_i(q_1, q_2) = \exp(-(q_1 - \frac{Q}{4}) - (q_2 - \frac{Q}{4}))$$

This is a concave function which assumes a value of 1 at $(\frac{Q}{4}, \frac{Q}{4})$. Using $\mu_i(q_i)$ and $\gamma_i(q_1, q_2)$ for $i = 1, 2$ we can define δ_i as the minimum of these two functions for each player. Since the constraint set is a crisp set, the confluence of the goals and constraints will just be the goal function defined above which reaches its maximum at $(\frac{Q}{4}, \frac{Q}{4})$. Since the two firms are symmetric in all respects, $(\frac{Q}{4}, \frac{Q}{4})$ is indeed an equilibrium. Note that it satisfies all the conditions for the existence of the equilibrium given in *Proposition 1.* Hence we see that the collusive outcome can easily be supported as an equilibrium in the fuzzy game. It is also obvious that different types of membership functions can be used to support other situations like the Cournot-Nash outcome as an equilibrium of the fuzzy sets. This illustrates the importance of the beliefs that firms have about each other and the role played by their own goals in strategic interaction. Of course since the membership functions is quite subjective one can argue that this is also a weakness of the approach.

5.5 Further Research

This work is still preliminary and a host of issues need to be addressed before one can accurately asses the usefulness of this approach. While it seems to have a realistic flavor, the first task would be to sophisticated applications and compare the results of this approach with those obtained under the standard game-theoretic formulation. It also seems that most of the results can be generalized to the many players case. Other interesting issues would be to link the constraints faced by a player or his perception set to his level of rationality. This would allow us to investigate equilibrium selection and refinements from a different perspective.

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