Universal Localizations and Group Cohomology

Mark S. Grinshpon

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Peter Linnell, Chair
William Floyd
Peter Haskell
Martin Klaus
Charles Parry

August 10, 2006
Blacksburg, Virginia

Keywords: Universal Localization, Division Closure, Rational Closure, Group Cohomology.

Universal Localizations and Group Cohomology

Mark S. Grinshpon

(ABSTRACT)

Two results are obtained in this work. First, we prove that for a commutative ring embedded in a larger ring, which is not necessarily commutative, its division and rational closures coincide. Second, for an infinite discrete group G, we investigate group cohomology and homology with coefficients in $\ell^p(G)$. We prove that if G is of type FP_n, then all its homology and cohomology groups up to n are either zero or infinite dimensional. This generalizes one of the results obtained by Bekka and Valette.

Acknowledgments

First of all, I wish to thank my parents — for everything.

I would like to thank Prof. Bill Greenberg for bringing me to Virginia Tech and Prof. Dan Farkas for welcoming me here.

I owe a lot more than just an acknoledgement to my advisor Prof. Peter Linnell — not only for all the mathematics that I learned from him, not only for all the mathematical help that I received from him, but also for his patience, support and encouragement during my work.

I wish to thank the many faculty members of Mathematics Department at Virginia Tech for making my being here both edcational and enjoyable.

And last but not least — I wish to thank my wife Irina and the children for bearing with me during my work on the thesis.

Contents

Chapter 1

Introduction

In some very general sense, the research leading to this dissertation has been concentrating on "L^p-theory" of groups. There is a wide range of various questions to consider in this area. The following two chapters of this dissertation provide an account of two different results that have been obtained in this study.

The chapter on universal localizations presents a result which does not seem to be related to the study of groups, but motivation for this did come from the main line of research. For a group G, there are several algebras that one is interested in studying, among them the group algebra $\mathbb{C}G$, the group von Neumann algebra $\mathcal{N}(G)$, the algebra of affiliated operators $\mathcal{U}(G)$ [14]. One of of our goals was to understand how $\mathbb{C}G$ "sits" inside $\mathcal{U}(G)$, or more precisely, to find and investigate the properties of its division closure — the minimal division subring of $\mathcal{U}(G)$ that contains $\mathbb{C}G$. This can be described along the lines of Paul Cohn's theory of localization. This leads to studying matrix rings, localizations and division and rational closures of a ring in an overring. In particular, during this work a question arose to describe the division and rational closures of a commutative subring in a larger ring, which is not assumed to be commutative. Chapter 2 is devoted to this question, its main results being the following.

Theorem 1.1. *Let S be a ring, $R \subseteq S$ a subring, and assume that R is commutative. If a matrix $A \in \mathrm{M}_n(R)$ is invertible in $\mathrm{M}_n(S)$, then $\det A$ is invertible in S.*

Corollary 1.2. *Let S be a ring, $R \subseteq S$ a subring, and assume that R is commutative. Then $\mathcal{D}(R, S) = \mathcal{R}(R, S) = RT^{-1}$, a commutative subring of S, where $T = \{t \in R \mid t^{-1} \in S\}$ and $RT^{-1} = \{rt^{-1} \mid r \in R, t \in T\}$.*

A different subject is presented in Chapter 3. There we study cohomology of a discrete group G with coefficients in $\ell^p(G)$. The main result of this chapter was motivated by and generalizes one of the results of Bekka and Valette [2], who in turn were motivated by Guichardet's work [7]. Using a geometric description of the first cohomology group, they proved (among other things) that for a discrete finitely generated group $H^1(G, \ell^2(G))$ is either zero or infinite dimensional. Using a different approach, we show that under a finiteness condition — G be of type FP_n, which guarantees a free resolution with modules of finite rank up to degree n, — all cohomology groups $H^k(G, \ell^p(G))$ with any $1 < p < \infty$ and any integer $1 \leq k \leq n$ are either zero or infinite dimensional. Our approach was based on investigating G-invariant subspaces of the module $\ell^p(G)^m$. Once we prove the main theorem on such invariant subspaces, the desired result on cohomology follows almost immediately. More precisely, our results are:

Theorem 1.3. *Let m be a non-negative integer. Let G be an infinite discrete group, and let $Y \subseteq X$ be closed G-invariant subspaces of $\ell^p(G)^m$. Then either $Y = X$ or Y has infinite codimension in X.*

Corollary 1.4. *Let G be an infinite discrete group.*

(1) *If G is of type FP_∞ over $\mathbb{C}$, then each ℓ^p-cohomology group $H^n(G, \ell^p(G))$ is either zero or infinite dimensional.*

(2) *If G is of type FP_n over $\mathbb{C}$, then each ℓ^p-cohomology group $H^k(G, \ell^p(G))$, $0 \leq k \leq n$, is either zero or infinite dimensional.*

Both chapters are arranged in the same manner: their first halves present some background theory, and then in the second halves we prove our main results.

The result presented in Chapter 2 has been accepted for publication in “Communications in Algebra” [6].

Chapter 2

Universal Localization

2.1 Localizations and Embedding into Division Rings

2.1.1 Field of Quotients and Ore Localization

The classical way of constructing the ring of quotients of a commutative ring, such as $\mathbb{Q}$ from $\mathbb{Z}$, can be described as follows (see e.g. [12]). Let R be a commutative ring with identity and $T = \{t \in R \,|\, tr \neq 0 \text{ for all } r \in R \setminus 0\}$ the set of all non-zerodivisors. Then we can form the ring RT^{-1}, which consists of elements, or more precisely, of equivalence classes of elements, of the form rt^{-1}, also written as r/t, with $r \in R$ and $t \in T$. Two fractions r_1/t_1 and r_2/t_2 are equivalent if and only if $r_1t_2 = t_1r_2$, and in this case we write $r_1/t_1 = r_2/t_2$ in RT^{-1}. The original ring R can be identified with a subring of RT^{-1} via $r \mapsto r/1$ for $r \in R$. Thus R is embedded into a larger ring RT^{-1} which has the property that every element is either invertible or a zero divisor. In case R is an integral domain, $T = R \setminus 0$ and RT^{-1} is a field.

More generally, let R be an arbitrary ring with 1, and consider any **multiplicatively closed** subset $T \subseteq R$, i.e. $1 \in T$ and $s, t \in T$ implies $st \in T$. Then we would like to form in a similar fashion a ring of quotients of R with respect to T (with "denominators" in T). But as we turn to noncommutative rings, the situation becomes more complicated. If we choose to work, for example, with right fractions rt^{-1}, then we run into the problem of multiplying them: in a product like $r_1t_1^{-1}r_2t_2^{-1}$ we need somehow move t_1^{-1} to the right over r_2. It is well known that one needs the Ore condition to form such a ring.

Definition 2.1. [17, Chapter II, §1] *Let R be a ring and let T be a multiplicatively closed subset of R. A* ***right ring of fractions*** *of R with respect to T is a ring RT^{-1} together with a ring homomorphism $\varphi\colon R \to RT^{-1}$ satisfying:*

(1) *$\varphi(t)$ is invertible for every $t \in T$;*

(2) *every element in RT^{-1} has the form $\varphi(r)\varphi(t)^{-1}$ with $t \in T$;*

(3) *$\varphi(r) = 0$ if and only if $rt = 0$ for some $t \in T$.*

*RT^{-1} is also called a **right Ore localization** of R with respect to T.*

One often considers the case, as we did above for commutative rings, where $T = \mathrm{NZD}(R)$, the set of all non-zerodivisors of R. In this case, if the ring RT^{-1} exists, it is called the **classical right ring of quotients** (or sometimes the **total right ring of fractions**) of R. Since we are not going to discuss left-sided versions in this work, hereafter we will be omitting the word "right" from most of our terminology.

It follows directly from condition (3) of the definition that the canonical homomorphism $\varphi\colon R \to RT^{-1}$ is a monomorphism if and only if T is contained in the set of non-zerodivisors. In this case, R is usually identified with its image under φ in RT^{-1}, whose elements thus may be written as rt^{-1} with $r \in R$ and $t \in T$.

Proposition 2.2. [17, Chapter II, Proposition 1.4] *Let T be a multiplicatively closed subset of a ring R. RT^{-1} exists if and only if R and T satisfy the following two **right Ore conditions**:*

(1) *if $r \in R$ and $t \in T$ there exist $r' \in R$ and $t' \in T$ such that $rt' = tr'$; and*

(2) *if $tr = 0$ with $t \in T$, then $rt' = 0$ for some $t' \in T$.*

When RT^{-1} exists, its elements are the equivalence classes of pairs $(r, t) \in R \times T$, where (r_1, t_1) and (r_2, t_2) are equivalent if and only if there exist $u, v \in R$ such that $r_1 u = r_2 v$ and $t_1 u = t_2 v \in T$.

Note that if T contains no zero-divisors, the second condition is automatically satisfied. In the special case of $T = \mathrm{NZD}(R)$, one simply says that the ring R itself satisfies the right Ore condition whenever (1) holds. If R has no zero-divisors, then the right Ore condition states that $rR \cap sR \neq 0$ for all nonzero $r, s \in R$.

Example 2.3. If R is a commutative ring, then both Ore conditions are automatically satisfied, so there exist localizations of R with respect to any multiplicative subset. A particular case is the ring of fractions described above.

Example 2.4. [17, Chapter II, §2] Let R be a commutative domain with field of fractions K, and let G be a finite group. Then the group ring RG has KG as its classical rings of quotients.

Example 2.5. Let Γ be the free group on two generators, $\Gamma = \langle x, y \rangle$. Then the group ring $\mathbb{C}\Gamma$ is a domain which does not satisfy the Ore condition and therefore does not have a classical ring of quotients. In fact, $(x-1)\mathbb{C}\Gamma \cap (y-1)\mathbb{C}\Gamma = 0$. For a proof, see [16, Example 13.5] or [12, Proposition 2.2].

Ore localization RT^{-1}, when it exists, has many nice ring-theoretic properties. For example:

- For any T, RT^{-1} is a flat left R-module, i.e. the functor $- \otimes_R RT^{-1}$ is exact [17, Proposition II.3.5];
- For $T = \mathrm{NZD}(R)$, every element of RT^{-1} is either invertible or a zero divisor.

But it also has a very important universal property which will be discussed below.

2.1.2 Ore Localization and Inverting Homomorphisms

Let R be a ring and $X \in R$ any subset. A homomorphism $f\colon R \to S$ is called X**-inverting** if $f(x)$ is invertible in S for any $x \in X$.

Definition 2.6. [16, Definition 13.1] *An X-inverting homomorphism $i\colon R \to R_X$ is called **universal X-inverting** if it has the following universal property: for any X-inverting homomorphism $f\colon R \to S$ there exists a unique homomorphism $g\colon R_X \to S$ such that $gi = f$, i.e. the following diagram commutes:*

$$\begin{tikzcd} & R_X \arrow[d, "g"] \\ R \arrow[ur, "i"] \arrow[r, "f"'] & S. \end{tikzcd}$$

It is a standard argument that R_X, when it exists, is unique up to isomorphism. Without loss of generality, we may always assume that X is multiplicatively closed. Indeed, let $i\colon R \to R_X$ be a universal X-inverting homomorphism and let $\bar{X}$ be the set of all elements in R which become invertible in R_X. Then obviously $X \subseteq \bar{X}$, and it is easy to verify that R_X and $R_{\bar{X}}$ are naturally isomorphic.

Proposition 2.7. [17, Chapter II, Proposition 1.1] *Let T be a multiplicatively closed subset of a ring R. When RT^{-1} exists, it satisfies the universal property from Definition 2.6, i.e. $\varphi\colon R \to RT^{-1}$ is a universal T-inverting homomorphism.*

Note that the universal problem in Definition 2.6 always has a solution: R_X is the ring generated by R and new generators x' for each $x \in X$ and relations being those in R and $xx' = x'x = 1$ for all $x \in X$. However, this construction gives us no idea about the structure of the ring R_X or what its elements look like and is practically useless. Propositions 2.2 and 2.7 provide necessary and sufficient conditions — the Ore conditions — when the solution is given by Ore localization.

2.1.3 Universal Localization

Instead of inverting a subset of elements of the ring, one may want to invert a set of matrices over it. Let R be a ring and let Σ be a set of square matrices over R. A homomorphism $f\colon R \to S$ is called **Σ-inverting** if $f(A)$ becomes invertible as a matrix over S for any $A \in \Sigma$.

Definition 2.8. [16, Definition 13.9] *A Σ-inverting homomorphism $i\colon R \to R_\Sigma$ is called* ***universal*** ***Σ-inverting*** *if it has the following universal property: for any Σ-inverting homomorphism $f\colon R \to S$ there exists a unique homomorphism $g\colon R_\Sigma \to S$ such that $gi = f$, i.e. the following diagram commutes:*

$$\begin{tikzcd} & R_\Sigma \arrow[d, "g"] \\ R \arrow[ur, "i"] \arrow[r, "f"'] & S. \end{tikzcd}$$

Similarly to X-inverting homomorphisms of the previous subsection, one can show that the universal Σ-inverting ring homomorphism always exists and is unique up to isomorphism. To show existence, for each $n \times n$ matrix A in Σ introduce n^2 symbols a'_{ij} and let $A' = \left(a'_{ij}\right)$. Then R_Σ is the ring generated by R and $\left\{a'_{ij} \mid A \in \Sigma\right\}$ with relations being those in R and $AA' = A'A = I$ for all $A \in \Sigma$. But again, this presentation is not convenient for practical purposes. For example, it can be very difficult to determine whether $i\colon R \to R_\Sigma$ is even injective.

A set of matrices Σ is **upper multiplicative** if $I \in \Sigma$ and whenever $A, B \in \Sigma$ then $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \Sigma$ for any matrix C of appropriate size; **lower multiplicative** sets are defined similarly with C in the lower left corner. Σ is **multiplicative** if it is upper (or lower) multiplicative and the result of permuting rows and columns of any matrix in Σ still lies in Σ (the other multiplicativity follows from this condition).

Proposition 2.9. [5, Chapter 7, Proposition 1.1] *Given any homomorphism $f\colon R \to S$, let Σ be the set of all matrices over R whose image under f is invertible over S. Then Σ is multiplicative.*

Given an arbitrary set Σ of matrices over R, let $\varphi\colon R \to R_\Sigma$ be the universal Σ-inverting homomorphism and let $\bar{\Sigma}$ be the set of all matrices over R which become invertible over R_Σ. Then by the proposition $\bar{\Sigma}$ is multiplicative. Obviously $\Sigma \subseteq \bar{\Sigma}$, and it is easy to verify that R_Σ and $R_{\bar{\Sigma}}$ are naturally isomorphic.

2.1.4 Division Closure and Rational Closure

For a ring S and a subring $R \subseteq S$, let $T = T(R \subseteq S)$ be the set of all elements in R which are invertible in S and let $\Sigma = \Sigma(R \subseteq S)$ be the set of all matrices over R which are invertible over S. Considering the universal localizations R_T and R_Σ, the question arises whether these rings can be embedded in S. Unfortunately, this is not always possible. This motivates defining intermediate rings with similar inverting properties.

Definition 2.10. *Let $R \subseteq S$ be rings. The **division closure** $\mathcal{D}(R, S)$ of R in S is the smallest subring D of S containing R with the property: if $d \in D$, $d^{-1} \in S$, then $d^{-1} \in D$. A subring $R \subseteq S$ is called **division closed** in S if $\mathcal{D}(R, S) = R$.*

In other words, R is division closed in S if and only if for every element $r \in R$ invertible in S its inverse r^{-1} already lies in R.

Note that the division closure is well defined, because S is division closed in itself and the intersection of any family of intermediate division closed rings is again division closed. It is obvious from the definition that division closure is an idempotent operation: $\mathcal{D}(\mathcal{D}(R, S), S) = \mathcal{D}(R, S)$.

Definition 2.11. *Let $R \subseteq S$ be rings. The **rational closure** $\mathcal{R}(R, S)$ of R in S is the smallest subring D of S containing R with the property: if A is a matrix over D invertible over S, then A^{-1} has all entries in D. A subring $R \subseteq S$ is called **rationally closed** in S if $\mathcal{R}(R, S) = R$.*

Another approach to the rational closure is given by Paul Cohn in [5, Chapter 7]. Let Σ be a set of square matrices over a ring R and let $f\colon R \to S$ be a Σ-inverting homomorphism (see Subsection 2.1.3). The **Σ-rational closure** of R in S is the set $\mathcal{R}_\Sigma(S)$ of all $t \in S$ such that t is an entry of the inverse of $f(M)$ for some matrix $M \in \Sigma$. Defined this way, the set $\mathcal{R}_\Sigma(S)$ is not necessarily a ring. But if Σ is upper multiplicative, then $\mathcal{R}_\Sigma(S)$ is a subring of S containing $\operatorname{im} f$ [5, Chapter 7, Theorem 1.2]. Now if we start with a homomorphism $f\colon R \to S$, then the set of all matrices over R whose image under f is invertible over S is upper multiplicative by Proposition 2.9, and thus we obtain the Σ-rational closure of R in S which is a subring of S and which we will temporarily denote $\mathcal{R}_S(R)$.

In particular, for embedded rings $R \subseteq S$, $\mathcal{R}_S(R)$ is the set of all entries of the inverses of matrices over R which are invertible over S. By [12, Proposition 3.3], $\mathcal{R}_S(\mathcal{R}_S(R)) = \mathcal{R}_S(R)$, i.e. $\mathcal{R}_S(R)$ is rationally closed in the sense of Definition 2.11. Since it is obviously contained in $\mathcal{R}(R, S)$, the two rings coincide. This can be summarized as follows:

Proposition 2.12. *Let S be a ring, $R \subseteq S$ a subring. The following three definitions of $\mathcal{R}(R, S)$ are equivalent:*

- *$\mathcal{R}(R, S)$ is the smallest subring D of S containing R with the property: if A is a matrix over D invertible over S, then A^{-1} has all entries in D;*
- *$\mathcal{R}(R, S)$ is the smallest subring D of S containing R with the property: if A is a matrix over R invertible over S, then A^{-1} has all entries in D;*

- $\mathcal{R}(R,S) = \{t \in S : t$ *appears in* A^{-1} *for some matrix* A *over* $R\}$.

More generally, considering any ring homomorphisms $f\colon R \to S$, not necessarily embeddings, we can define the division closure $\mathcal{D}^f(R,S)$ as the smallest subring of S containing $\operatorname{im} f$ and closed under taking inverses of elements invertible in S, i.e. $\mathcal{D}^f(R,S) = \mathcal{D}(\operatorname{im} f, S)$; similarly, the rational closure $\mathcal{R}^f(R,S) = \mathcal{R}(\operatorname{im} f, S)$. (For the rational closure, this is precisely Cohn's original definition.)

2.2 Invertibility of Matrices over Subrings

2.2.1 Main Result

Let S be a ring and $R \subseteq S$ a subring. Define $T = \{t \in R \,|\, t^{-1} \in S\}$ and $RT^{-1} = \{rt^{-1} \,|\, r \in R, t \in T\}$. It is easy to see that in general $RT^{-1} \subseteq \mathcal{D}(R,S) \subseteq \mathcal{R}(R,S)$. Note that RT^{-1} is a ring if and only R and T satisfy the Ore condition, and in this case $RT^{-1} \cong \mathcal{D}(R,S)$ by [16, Proposition 13.17].[1] However, it is the relation between $\mathcal{D}(R,S)$ and $\mathcal{R}(R,S)$ that interests us, and when they coincide with each other and with RT^{-1} (see below), the latter serves as a nice description of what the closures actually are.

If S is commutative, then

$$\mathcal{D}(R,S) = \mathcal{R}(R,S) = RT^{-1}, \tag{2.1}$$

due to the standard formula for matrix inverses, viz.

$$A^{-1} = (\det A)^{-1} \cdot \operatorname{adj} A = \operatorname{adj} A \cdot (\det A)^{-1}. \tag{2.2}$$

Indeed, if formula (2.2) holds for all invertible matrices (as is the case for commutative rings), then $\det A \in T$ and every entry of A^{-1} is in RT^{-1}; therefore $\mathcal{R}(R,S) \subseteq RT^{-1}$, proving (2.1).

Is (2.1) still true if we assume only that R is commutative? In view of the above argument, this question can be rephrased as follows: for $A \in \mathrm{M}_n(R)$, an $n \times n$ matrix over R, invertible in $\mathrm{M}_n(S)$, is $\det A$ invertible in S? The answer is yes, i.e. the following results hold.

Theorem 2.13. *Let S be a ring, $R \subseteq S$ a subring, and assume that R is commutative. If a matrix $A \in \mathrm{M}_n(R)$ is invertible in $\mathrm{M}_n(S)$, then $\det A$ is invertible in S.*

Corollary 2.14. *Let S be a ring, $R \subseteq S$ a subring, and assume that R is commutative. Then $\mathcal{D}(R,S) = \mathcal{R}(R,S) = RT^{-1}$, a commutative subring of S, where $T = \{t \in R \,|\, t^{-1} \in S\}$ and $RT^{-1} = \{rt^{-1} \,|\, r \in R, t \in T\}$.*

[1] In the literature, RT^{-1} often stands for the universal T-inverting ring or would not be used at all unless the Ore condition is satisfied. However, using this notation for the set defined above is convenient for our purposes.

An application of the last corollary to an arbitrary homomorphism $f\colon R \to S$ yields the following:

Corollary 2.15. *Let $f\colon R \to S$ be a ring homomorphism, and assume that* $\operatorname{im} f$ *is commutative. Then $\mathcal{D}^f(R,S) = \mathcal{R}^f(R,S) = (\operatorname{im} f)T^{-1}$, a commutative subring of S, where $T = \{t \in \operatorname{im} f \mid t^{-1} \in S\}$.*

The rest of this section presents a proof of these results.

2.2.2 Preliminary Considerations

A is invertible means that there exists some $B \in \mathrm{M}_n(S)$ such that $AB = BA = I$. In the commutative case, this would imply $\det(A)\det(B) = 1$. But in our setting entries of B lie in an a priori non-commutative ring S, so there is no well-defined determinant of B. However, by mimicking a straightforward proof of the Cauchy-Binet formula (see e.g. [1]), of which this property of determinants is a special case, it is possible to prove that $\det(A)$ is invertible, with the inverse given by a "$\det(B)$" — a specific expansion of the $n \times n$ determinant in which all products are taken in an arbitrary but fixed order.

Notation used in the proof: $\sigma = \{i_1, \ldots, i_n\} \in S_n$ means the permutation in S_n acting via $\sigma(t) = i_t$ for $1 \le t \le n$.

2.2.3 Example: Case 2×2

We have that:

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For convenience, we will use d_{ij} to refer to the entries of the identity matrix. Let us compute the determinant of this identity matrix written as the product of A and B. Of course, the result will be 1. But since the entries of B are possibly non-commuting, we need to adopt a certain way of multiplying and expanding expressions involving b_{ij}. Each product occurring in the expansion of the determinant will be multiplied from left to right and in some sense "from inside out".

Let us start off with the product along the main diagonal $d_{11}d_{22}$. Take

$$1 = d_{11} = a_{11}b_{11} + a_{12}b_{21}.$$

Multiply it by: a_{21} on the left and b_{12} on the right, a_{22} on the left and b_{22} on the right. Get:

$$a_{21}b_{12} = a_{21}1b_{12} = a_{21}(a_{11}b_{11} + a_{12}b_{21})b_{12} = a_{21}a_{11}b_{11}b_{12} + a_{21}a_{12}b_{21}b_{12};$$
$$a_{22}b_{22} = a_{22}1b_{22} = a_{22}(a_{11}b_{11} + a_{12}b_{21})b_{22} = a_{22}a_{11}b_{11}b_{22} + a_{22}a_{12}b_{21}b_{22}.$$

Summing these up, we obtain $1 = d_{11}d_{22}$ as:

$$\begin{aligned}
1 &= a_{21}b_{12} + a_{22}b_{22} = a_{21}1b_{12} + a_{22}1b_{22} \\
&= a_{21}a_{11}b_{11}b_{12} + a_{21}a_{12}b_{21}b_{12} + a_{22}a_{11}b_{11}b_{22} + a_{22}a_{12}b_{21}b_{22} \\
&= \sum_{i,j=1}^{2} a_{2j}a_{1i}b_{i1}b_{j2} = \sum_{i,j=1}^{2} a_{1i}a_{2j}b_{i1}b_{j2},
\end{aligned}$$

since the entries of A commute with each other.

Next, let us evaluate the product along the other diagonal in a similar fashion. Take

$$0 = d_{21} = a_{21}b_{11} + a_{22}b_{21}.$$

Multiply it by: a_{11} on the left and b_{12} on the right, a_{12} on the left and b_{22} on the right. Get:

$$\begin{aligned}
0 = a_{11}0b_{12} = a_{11}(a_{21}b_{11} + a_{22}b_{21})b_{12} = a_{11}a_{21}b_{11}b_{12} + a_{11}a_{22}b_{21}b_{12}; \\
0 = a_{12}0b_{22} = a_{12}(a_{21}b_{11} + a_{22}b_{21})b_{22} = a_{12}a_{21}b_{11}b_{22} + a_{11}a_{22}b_{21}b_{22}.
\end{aligned}$$

Summing these up, we obtain $0 = d_{21}d_{12}$ as:

$$\begin{aligned}
0 &= a_{11}0b_{12} + a_{12}0b_{22} \\
&= a_{11}a_{21}b_{11}b_{12} + a_{11}a_{22}b_{21}b_{12} + a_{12}a_{21}b_{11}b_{22} + a_{11}a_{22}b_{21}b_{22} \\
&= \sum_{i,j=1}^{2} a_{1j}a_{2i}b_{i1}b_{j2}.
\end{aligned}$$

Now, the determinant of the identity matrix is $1 = d_{11}d_{22} - d_{21}d_{12}$ written as:

$$\begin{aligned}
1 &= \sum_{i,j=1}^{2} a_{1i}a_{2j}b_{i1}b_{j2} - \sum_{i,j=1}^{2} a_{1j}a_{2i}b_{i1}b_{j2} \\
&= \sum_{i,j=1}^{2} (a_{1i}a_{2j} - a_{1j}a_{2i})b_{i1}b_{j2} = \sum_{i,j=1}^{2} \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix} b_{i1}b_{j2}.
\end{aligned}$$

Note that $\begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix}$ equals zero if $i = j$, so the corresponding terms vanish. And when i and j are distinct, this is $\det(A)$ up to the sign. Thus:

$$\begin{aligned}
1 &= \sum_{i,j=1}^{2} \begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix} b_{i1}b_{j2} = \sum_{\sigma=\{i,j\}\in S_2} \operatorname{sgn}(\sigma)\det(A)b_{i1}b_{j2} \\
&= \det(A) \cdot \sum_{\sigma=\{i,j\}\in S_2} \operatorname{sgn}(\sigma)b_{i1}b_{j2}.
\end{aligned}$$

So $\det(A)$ is invertible from the right. Similarly from the other side.

2.2.4 The General Proof

We have that:

$$AB = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)_{i,j=1}^{n} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

As in the sample 2×2 case, let us compute the determinant of this identity matrix written as the product of A and B. We will multiply each term in the expansion of the determinant from left to right, i.e.

$$1 = \sum_{\sigma=\{i_1,\dots,i_n\} \in S_n} \operatorname{sgn}(\sigma) d_{i_1 1} \cdots d_{i_n n}.$$

Start with

$$d_{i_1 1} = \sum_{k_1=1}^{n} a_{i_1 k_1} b_{k_1 1},$$

which is either 0 or 1. In either case,

$$d_{i_2 2} = \sum_{k_2=1}^{n} a_{i_2 k_2} b_{k_2 2} \Longrightarrow d_{i_1 1} d_{i_2 2} = \sum_{k_2=1}^{n} a_{i_2 k_2} d_{i_1 1} b_{k_2 2} = \sum_{k_1,k_2=1}^{n} a_{i_2 k_2} a_{i_1 k_1} b_{k_1 1} b_{k_2 2}.$$

Proceeding in this fashion, we get

$$d_{i_1 1} d_{i_2 2} \cdots d_{i_n n} = \sum_{k_1,k_2,\dots,k_n=1}^{n} a_{i_n k_n} \cdots a_{i_2 k_2} a_{i_1 k_1} b_{k_1 1} b_{k_2 2} \cdots b_{k_n n}.$$

Now, the determinant of the identity matrix can be written as:

$$\begin{aligned} 1 &= \sum_{\sigma=\{i_1,\dots,i_n\} \in S_n} \operatorname{sgn}(\sigma) d_{i_1 1} \cdots d_{i_n n} \\ &= \sum_{\sigma=\{i_1,\dots,i_n\} \in S_n} \operatorname{sgn}(\sigma) \sum_{k_1,k_2,\dots,k_n=1}^{n} a_{i_n k_n} \cdots a_{i_2 k_2} a_{i_1 k_1} b_{k_1 1} b_{k_2 2} \cdots b_{k_n n} \\ &= \sum_{k_1,k_2,\dots,k_n=1}^{n} \left(\sum_{\sigma=\{i_1,\dots,i_n\} \in S_n} \operatorname{sgn}(\sigma) a_{i_n k_n} \cdots a_{i_2 k_2} a_{i_1 k_1} \right) b_{k_1 1} b_{k_2 2} \cdots b_{k_n n} \\ &= \sum_{k_1,k_2,\dots,k_n=1}^{n} \left(\sum_{\sigma=\{i_1,\dots,i_n\} \in S_n} \operatorname{sgn}(\sigma) a_{i_1 k_1} a_{i_2 k_2} \cdots a_{i_n k_n} \right) b_{k_1 1} b_{k_2 2} \cdots b_{k_n n}, \end{aligned}$$

since the entries of A commute with each other.

Note that the expression in parentheses is precisely the determinant of the matrix whose columns, say from left to right, are the columns $k_1, k_2, \ldots, k_n$ of the matrix A. If not all $k_1, k_2, \ldots, k_n$ are distinct, such a determinant is zero. And when they are distinct, this is $\det(A)$ up to the sign. So we can continue:

$$\begin{aligned}
1 &= \cdots \\
&= \sum_{\tau=\{k_1,k_2,\ldots,k_n\}\in S_n} \left(\sum_{\sigma=\{i_1,\ldots,i_n\}\in S_n} \operatorname{sgn}(\sigma) a_{i_1k_1} a_{i_2k_2} \cdots a_{i_nk_n} \right) b_{k_11} b_{k_22} \cdots b_{k_nn} \\
&= \sum_{\tau=\{k_1,k_2,\ldots,k_n\}\in S_n} \operatorname{sgn}(\tau) \det(A) b_{k_11} b_{k_22} \cdots b_{k_nn} \\
&= \det(A) \cdot \sum_{\tau=\{k_1,k_2,\ldots,k_n\}\in S_n} \operatorname{sgn}(\tau) b_{k_11} b_{k_22} \cdots b_{k_nn}.
\end{aligned}$$

So $\det(A)$ is invertible from the right. Similarly from the other side. This finishes the proof of the Main Theorem 2.13.

2.2.5 Some Consequences

For brevity, set $s = (\det A)^{-1}$. Note that while all entries of A lie in the commutative ring R, and of course so does $\det(A)$, s does not have to be in R.

Recall that $A \cdot \operatorname{adj} A = \operatorname{adj} A \cdot A = (\det A)I$, where $\operatorname{adj} A$ is the adjoint matrix of A. Multiplying this from one or the other side by B and then by s, we get:

$$\begin{array}{lclcl}
A \cdot \operatorname{adj} A = (\det A)I & \Longrightarrow & \operatorname{adj} A = B(\det A) & \Longrightarrow & (\operatorname{adj} A)s = B; \\
\operatorname{adj} A \cdot A = (\det A)I & \Longrightarrow & \operatorname{adj} A = (\det A)B & \Longrightarrow & s(\operatorname{adj} A) = B;
\end{array}$$

which is the standard formula for the inverse matrix. This shows that the entries of B lie in RT^{-1}, where $T = \{t \in R \mid t^{-1} \in S\}$, and it is easy to see that RT^{-1} is a commutative subring of S.

Chapter 3

Group Cohomology

3.1 Background Overview

3.1.1 L^p-Spaces of a Group

In this subsection we give a brief overview of a few basic facts about $L^p(G)$ spaces. A more detailed account can be found in many excellent books on harmonic analysis, such as e.g. the one by Loomis [13], which will be our primary source here.

Let G be an arbitrary group and k a commutative ring. The **group ring** kG (sometimes written as $k[G]$) has as its underlying abelian group the free k-module with the basis G, i.e. its elements are formal sums $a = \sum_{g\in G} a_g g$ with all but finitely many a_g being 0. One can also think of them as functions from G to k, $g \mapsto a_g$, which are zero almost everywhere. Multiplication in kG is the one in G extended by linearity: for $a = \sum_{g\in G} a_g g$ and $b = \sum_{g\in G} b_g g$, their product is $ab = \left(\sum_{g\in G} a_g g\right) \cdot \left(\sum_{g\in G} b_g g\right) = \sum_{g,h\in G} a_g b_h gh$. In this work we will be mostly concerned with the complex group ring $\mathbb{C}G$.

When G is a topological group (even if it is a discrete topology), more general spaces can and should be formed to study the group and its representations. The key point here is that on any locally compact group there exists a unique (up to a multiplicative constant) left invariant measure (or integral) called the **Haar measure** (see [13, Chapter VI] or [9, Chapter Four]). Thus we have integration on locally compact groups, which allows us to use the machinery of harmonic analysis.

So, let G be a locally compact group, $1 \leq p < \infty$. Then $L^p(G)$ is defined as the space of all complex functions $f\colon G \to \mathbb{C}$ on G which are p-integrable with respect to the Haar measure on G, i.e. such that $\|f\|_p = \left(\int |f(x)|^p \, dx\right)^{1/p} < \infty$. It is a standard fact that $\|f\|_p$

is indeed a norm, and $L^p(G)$ endorsed with this norm is a Banach space[1], which is reflexive for $1 < p < \infty$, because the dual of $L^p(G)$ is $L^q(G)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Two very special cases are those of $p = 1$ and $p = 2$.

For $p = 2$, $L^2(G)$ is a Hilbert space with the scalar product $(f, g) = \int f(x)\overline{g(x)}dx$.

For $p = 1$, $L^1(G)$ is a Banach algebra with multiplication given by **convolution**:

$$[f * g](x) = \int f(xy)g(y^{-1})dy = \int f(y)g(y^{-1}x)dy,$$

the integrals being equal due to left invariance of the Haar measure [13, Theorem 31B].

Definition 3.1. *A **representation** T of a group G on a complex vector space X is a strongly continuous homomorphism $s \mapsto T_s$ of G onto a group of bounded linear transformations on X. That is, $T_{st} = T_sT_t$ for all $s, t \in G$, and $T_s(x)$ is a continuous function of s for every $x \in X$. When X is a normed space, T is **bounded** if there is a uniform bound on the norms $\|T_s\|$, $s \in G$.*

For a function f on G and $s \in G$, define the **left translate** f_s via $f_s(x) = f(sx)$. There is an obvious representation of G on $L^p(G)$ given by

$$T_sf = f_{s^{-1}}, \text{ i.e. } [T_sf](t) = f(s^{-1}t). \tag{3.1}$$

This representation is bounded because all transformations T_s are isometries by the left invariance of Haar measure. But any bounded representation of a locally compact group G on a reflexive Banach space, on $L^p(G)$ with $1 < p < \infty$ in particular, is equivalent to a bounded representation of $L^1(G)$ on the same space [13, Theorems 32B and 32C]. For $L^p(G)$, this left action of G corresponds precisely to the left multiplication by (convolution with) functions from $L^1(G)$. This allows us to view $L^p(G)$, $1 < p < \infty$, as an $L^1(G)$-module. Moreover, the following inequality holds [13, 32D]:

$$\|u * v\|_p \leq \|u\|_1 \|v\|_p \quad \text{for any } u \in L^1(G), v \in L^p(G). \tag{3.2}$$

The algebra $L^1(G)$ is unital if and only if G is discrete [13, Theorem 31D]. But in any case it has an approximate identity.

Definition 3.2. [3, Chapter I, §11, Definitions 1 and 8]

- *Let A be a normed algebra. A **left approximate identity** for A is a net $\{e_\lambda\}_{\lambda \in \Lambda}$ in A such that $e_\lambda x \to x$ for any $x \in A$. It is **bounded** if there exists a positive constant C such that $\|e_\lambda\| \leq C$ for all $\lambda \in \Lambda$.*

[1] For $p = \infty$, $L^\infty(G)$ is the Banach space of all essentially bounded functions with $\|f\|_\infty = \operatorname{ess\,sup} |f(x)| < \infty$, but it will not be considered in this work.

- *Let X be a left A-module. A **bounded left approximate identity in** A **for** X is a bounded net $\{e_\lambda\}_{\lambda\in\Lambda}$ in A such that $e_\lambda x \to x$ for any $x \in X$.*

Right and two-sided bounded approximate identities are defined similarly.

The neighborhoods of the identity V form a directed system under inclusion. For each V, let u_V be a non-negative function vanishing off V such that $\int u_V = 1$. Then by [13, 31E], the net $\{u_V\}$ is a bounded approximate identity[2] in $L^1(G)$ for any $L^p(G)$, $1 \le p < \infty$ (in particular, it is a b.a.i. of the algebra $L^1(G)$ itself).

Later we will be interested in left G-invariant subspaces of $L^p(G)$, i.e subspaces invariant under left translates. Not surprisingly, they are precisely the $L^1(G)$-submodules of $L^p(G)$ — but only due to the presence of an approximate identity. To show this, we can start with the algebra $L^1(G)$ itself.

Theorem 3.3. [13, Theorem 31F] *A closed subset of $L^1(G)$ is a left ideal if and only if it is a left invariant subspace.*

The proof of this theorem given in [13] can be used virtually verbatim to yield the analogous result for invariant subspaces of $L^p(G)$. In the proof we will need the following. Let X be a normed space and X^* its dual. For a subset $M \subseteq X$, define $M^\perp = \{y \in X^* \mid (x, y) = 0 \; \forall x \in M\}$; similarly, for a subset $N \subseteq X^*$, define $N^\perp = \{x \in X \mid (x, y) = 0 \; \forall y \in N\}$. A corollary from the Hahn-Banach Theorem states that if M is a closed subspace of a normed space X, then $\left(M^\perp\right)^\perp = M$.

Theorem 3.4. *A closed subset of $L^p(G)$, $1 < p < \infty$, is a left $L^1(G)$-submodule if and only if it is a left G-invariant subspace.*

Proof. Let I be a closed left submodule of $L^p(G)$. Let u run through an approximate identity of $L^1(G)$ for $L^p(G)$, e.g. the net $\{u_V\}$ defined above. Then for any $x \in G$, $u_x * f = (u * f)_x \to f_x$, because $u * f \to f$. Since $u_x * f \in I$ and I is closed, $f_x \in I$.

Conversely, let I be a closed left invariant subspace of $L^p(G)$. Then $I = \left(I^\perp\right)^\perp$, i.e. for any $f \in L^p(G)$, $f \in I$ if and only if $(f, g) = 0$ for all $g \in I^\perp \subseteq L^q(G)$. But if $h \in L^1(G)$, $f \in I$, and $g \in I^\perp$, then $(h * f, g) = \iint h(y) f(y^{-1}x)\overline{g(x)} dy dx = \int h(y) \int \left[f(y^{-1}x)\overline{g(x)} dx \right] dy = \int h(y) \cdot (f_{y^{-1}}, g) \, dy = 0$, since $f_{y^{-1}} \in I$ for any $y \in G$ by the invariance of I, and thus all $(f_{y^{-1}}, g) = 0$. This proves that $h * f \in I$, i.e. I is a left $L^1(G)$-submodule. $\square$

With minor adjustments, all notions and results given here on the left side have right-sided version as well.

[2] two-sided

3.1.2 ℓ^p-Spaces of Discrete Groups

When G is a discrete group, $\ell^p(G)$ has become standard notation for these spaces. In this case, $\ell^p(G)$, $1 \le p < \infty$, can be described as the space of all formal sums $a = \sum_{g\in G} a_g g$ with complex coefficients such that $\|a\|_p = \left(\sum_{g\in G} |a_g|^p\right)^{1/p} < \infty$. The convolution, whenever it makes sense, is simply induced by multiplication in the group.

The representation (3.1) for discrete G is the left regular action of G on $\ell^p(G)$ by multiplication (convolution): for $s \in G$ and $a = \sum_{g\in G} a_g g \in \ell^p(G)$, it is given by $s*a = s*\sum_{g\in G} a_g g = \sum_{g\in G} a_g sg = \sum_{g\in G} a_{s^{-1}g} g$.

Two special features of the discrete case will be important for us. First is that, as mentioned above, the algebra $\ell^1(G)$ is unital. Second, the group ring $\mathbb{C}G$ is a dense subset of each $\ell^p(G)$ in the corresponding p-norm[3]. This simplifies much of the theory. For example, Theorems 3.3 and 3.4 become almost trivial.

For our purposes, this means that studying the left regular action of G on $\ell^p(G)$ and its closed G-invariant subspaces is equivalent to studying $\ell^1(G)$-submodules of $\ell^p(G)$, where the module structure is given by the left multiplication by (convolution with) elements of $\ell^1(G)$.

Later we will be also considering finite direct sums $\ell^p(G)^n$. While different norms can be defined on a finite direct sum of normed spaces, they all are equivalent [15, §1.8]. The most natural and consistent choice for $\ell^p(G)^n$, $1 \le p < \infty$, is using the p-norm:

$$\|(u_1, \ldots, u_n)\|_p = \left(\sum_{k=1}^{n} \|u_k\|_p^p\right)^{1/p} \quad \text{for any } (u_1, \ldots, u_n) \in \ell^p(G)^n.$$

The inequality (3.2) still holds for $u \in \ell^1(G)$, $v \in \ell^p(G)^n$ or $u \in \ell^1(G)^n$, $v \in \ell^p(G)$ with convolution defined componentwise. For example, if $u \in \ell^1(G)$ and $v = (v_1, \ldots, v_n) \in \ell^p(G)^n$, then

$$\begin{aligned}\|u * v\|_p^p = \|u * (v_1, \ldots, v_n)\|_p^p = \sum_{k=1}^{n} \|u * v_k\|_p^p &\le \sum_{k=1}^{n} \left(\|u\|_1 \|v_k\|_p\right)^p \\ &= \sum_{k=1}^{n} \|u\|_1^p \|v_k\|_p^p = \|u\|_1^p \sum_{k=1}^{n} \|v_k\|_p^p = \|u\|_1^p \|v\|_p^p.\end{aligned}$$

3.1.3 Cohomology of Groups

In this subsection we will briefly overview the algebraic definition of group cohomology with coefficients. Again, there are many excellent sources on the subject, such as [10, Chapter VI]

[3]Of course, we are not thinking of $\mathbb{C}G$ as a ring except within $\ell^1(G)$.

or [4] to name a few, but we will not be needing virtually anything of the theory beyond the mere definition of cohomology.

Let G be a group and k a commutative ring. The group ring kG (defined above, p. 13) has the following universal property:

Proposition 3.5. *Let R be a k-algebra. Given any group homomorphism $f\colon G \to U$ from G to the group $U \subseteq R$ of units of R, there exists a unique k-algebra homomorphism $g\colon kG \to R$ such that $f'i = f$, where $i\colon G \to kG$ is the obvious inclusion, i.e. the following diagram commutes:*

$$\begin{tikzcd} & kG \arrow[d, "f'"] \\ G \arrow[ur, "i"] \arrow[r, "f"] & R. \end{tikzcd}$$

Proof. Define $f'\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g f(g)$, which obviously is the only homomorphism satisfying $f'i = f$. $\square$

In particular, the trivial map from G to k, $g \mapsto 1$ for all $g \in G$, gives rise to the homomorphism $\varepsilon\colon kG \to k$ acting via $\varepsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$. This map is called the **augmentation** of kG. Its kernel I is called the **augmentation ideal** of kG. The augmentation ideal is a free k-module with the basis $\{g-1 \mid g \in G, g \neq 1\}$ [4, Chapter I, §2]. The augmentation map turns k into the left kG-module on which G acts trivially. Another way to describe it is by saying that as kG-modules, $k \cong kG/I$.

Now let us specify the ring k to be the the complex numbers $\mathbb{C}$.

Definition 3.6. *Let G be a group. Let $\mathbb{C}$ be the $\mathbb{C}G$-module as described above.*

- *Let N be a right $\mathbb{C}G$-module. Define the n-th **homology group of** G **with coefficients in** N as $H_n(G,N) = \mathrm{Tor}_n^{\mathbb{C}G}(N,\mathbb{C})$.*
- *Let M be a left $\mathbb{C}G$-module. Define the n-th **cohomology group of** G **with coefficients in** M as $H^n(G,M) = \mathrm{Ext}^n_{\mathbb{C}G}(\mathbb{C},M)$.*

Let us give a more detailed description of homology and cohomology groups by expanding the definitions of the Ext and Tor functors.

Definition 3.7. [4, Chapter I, §1] *Let R be a ring and M be a left R-module. A **resolution** $\mathbf{F} \to M$ of M is an exact sequence of R-modules*

$$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0. \tag{3.3}$$

*Here $\mathbf{F}$ is the chain complex $\cdots \to F_2 \to F_1 \to F_0 \to 0$, which is exact everywhere except its zeroth term. It is a **free resolution** if each F_n is free. It is a **projective resolution** if each F_n is projective.*

Free resolutions exist for any module M. To construct a free resolution, first choose a surjection $\varepsilon\colon F_0 \to M$ with F_0 free: let $\{m_i\}_{i\in I}$ be a set of generators for M, then F_0 can be taken as the free module with basis I. Then in the same manner choose a surjection $\partial_1\colon F_1 \to \ker\varepsilon$ with F_1 free, and so on. Note that the initial segment $F_1 \to F_0 \to M \to 0$ is precisely a **presentation** of M by generators and relations.

To define $\operatorname{Ext}^n_R(M,N)$ for R-modules M and N, one takes a projective (which in particular can be free) resolution (3.3) of M and applies the contravariant functor $\operatorname{Hom}_R(-,N)$ to obtain

$$0 \longrightarrow \operatorname{Hom}_R(M,N) \longrightarrow \operatorname{Hom}_R(F_0,N) \longrightarrow \operatorname{Hom}_R(F_1,N) \longrightarrow \cdots.$$

Since $\operatorname{Hom}_R(-,N)$ is only left exact but not exact, the resulting cochain complex may have nontrivial homology beyond its initial term.

Definition 3.8. $\operatorname{Ext}^n_R(M,N) = H^n(\operatorname{Hom}(\mathbf{F},N))$, *the n-th homology group of*

$$0 \longrightarrow \operatorname{Hom}_R(F_0,N) \longrightarrow \operatorname{Hom}_R(F_1,N) \longrightarrow \operatorname{Hom}_R(F_2,N) \longrightarrow \cdots.$$

It is a standard fact in homological algebra that this definition is independent of the choice of a projective resolution of M.

Similarly, to define $\operatorname{Tor}^R_n(M,N)$ for a right R-module M and a left R-module N, one also takes a projective (which in particular can be free) resolution (3.3) of M and applies the covariant functor $-\otimes_R N$ to obtain

$$\cdots \longrightarrow F_1 \otimes_R N \longrightarrow F_0 \otimes_R N \longrightarrow M \otimes_R N \longrightarrow 0.$$

Since $-\otimes_R N$ is only right exact but not exact, the resulting chain complex may have nontrivial homology beyond its last term.

Definition 3.9. $\operatorname{Tor}^R_n(M,N) = H_n(\mathbf{F}\otimes_R N)$, *the n-th homology group of*

$$\cdots \longrightarrow F_2 \otimes_R N \longrightarrow F_1 \otimes_R N \longrightarrow F_0 \otimes_R N \longrightarrow 0.$$

It is a standard fact in homological algebra that these definitions are independent of the choice of a projective resolution of M.

Now that both group cohomology and homology with coefficients in a module are defined, it is a good time to state that the main object of study in this chapter is the group ℓ^p-cohomology $H^n(G,\ell^p(G))$.

3.1.4 Group 1-Cohomology

One can give a direct description of the first cohomology group [2, §2]. Let G be a group and let M be a G-module, that is a complex vector space M with a group homomorphism $\pi\colon G \to$

Aut M. A **1-cocycle** with values in M is a map $b\colon G \to M$ such that $b(gh) = b(g) + \pi(g)b(h)$ for any $g, h \in G$; a **1-coboundary** is a 1-cocycle of the form $b(g) = \pi(g)v - v$ for some fixed $v \in M$ and all $g \in G$. Denote by $Z^1(G, M)$ and $B^1(G, M)$ the spaces of 1-cocycles and 1-coboundaries respectively. Then the quotient $Z^1(G, M)/B^1(G, M)$ is the first cohomology space with coefficients in M, denoted by $H^1(G, M)$ or sometimes $H^1(G, \pi)$.

This definition, while seemingly different from above, actually comes from the so called **bar resolution** of $\mathbb{C}$ by free $\mathbb{C}G$-modules [4, p. 59].

3.2 Dimensions of ℓ^p-Cohomology Groups

3.2.1 Motivation

Motivation for this work came from the cited above paper [2], where Bekka and Valette in particular proved that if $p = 2$ and G is a finitely generated infinite group, then $H^1(G, \ell^p(G))$ is either zero or infinite dimensional. One can ask[4] whether this assertion remains true for arbitrary p, $1 < p < \infty$. One can ask this question for any H^n as well.

A direct approach to the problem would be to study the cohomology groups as the quotients in the cochain complex. So, let G be a group and let

$$\cdots \longrightarrow \mathbb{C}G^{e_2} \longrightarrow \mathbb{C}G^{e_1} \longrightarrow \mathbb{C}G^{e_0} \longrightarrow \mathbb{C} \longrightarrow 0 \tag{3.4}$$

be a resolution with free $\mathbb{C}G$-modules. We may take e_1 to be finite if and only if G is finitely generated. Then one gets a complex

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_0}, \ell^p(G)) \xrightarrow{d^0} \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_1}, \ell^p(G)) \xrightarrow{d^1} \cdots \tag{3.5}$$

and $H^n(G, \ell^p(G)) = \ker d^n / \operatorname{im} d^{n-1}$.

If G is finitely generated infinite amenable, then a result due to Guichardet [7] tells us that $\operatorname{im} d^0$ is not closed in $\ker d^1$, from which one deduces that $H^1(G, \ell^p(G))$ is always infinite dimensional for G amenable. If G is finitely generated nonamenable, then Guichardet also tells us that $\operatorname{im} d^0$ is closed in $\ker d^1$. However, these Guichardet results are special to the case $n = 1$; there are no corresponding results for larger n.

Thus in the nonamenable case, we have two closed G-invariant subspaces $Y \subseteq X$ of the space $\operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_1}, \ell^p(G)) \cong \ell^p(G)^{e_1}$, where e_1 is a positive integer, and we want to show that if $Y \neq X$, then X/Y is infinite dimensional (of course, X is $\ker d^1$ and Y is $\operatorname{im} d^0$).

So we are led to studying pairs of nested G-invariant subspaces of $L^p(G)^e$. Although the motivation for this comes from Guichardet's results for $n = 1$, this approach is more universal and provides a proof for arbitrary n.

[4] as Mike Puls did

3.2.2 Crucial Cases

One of the most important cases to consider is that of the infinite cyclic group. It serves not only as an example, but as a basic case to which the general problem often can be reduced.

Proposition 3.10. *Let $G = \mathbb{Z} = \langle g \rangle$, the infinite cyclic group, $1 < p < \infty$. Let $A \subseteq B$ be closed G-invariant subspaces, i.e. $\ell^1(G)$-submodules, of $\ell^p(G)$, and assume that B/A is finite-dimensional. Then $A = B$.*

Notice that already in the statement of the proposition we are using equivalence of G-invariance and $\ell^1(G)$-submodules established earlier.

Proof. The action of g on the finite-dimensional space B/A has a minimal polynomial, i.e. there exists a polynomial $F(x) \in \mathbb{C}[x]$ such that $F(g) = 0$ on B/A, and therefore $F(g)b \in A$ for all $b \in B$. Factor $F(x)$ and notice that if $|\omega| \neq 1$, then $(g - \omega)$ is invertible in $\ell^1(G)$. Thus, since A is $\ell^1(G)$-invariant, we may assume that $F(g)$ consists of factors $(g - \omega)$ with $|\omega| = 1$ only. If we prove that $F(g)B$ is dense in B, that will imply that $A = B$.

Let us start with $\omega = 1$. Set $x_n = \sum_{k=1}^{n} \frac{1}{n} g^k \in \mathbb{C}G$ and observe that these elements tend to zero in the p-norm. Indeed,

$$\|x_n\|_p = \left(n \cdot \frac{1}{n^p}\right)^{1/p} = \left(\frac{1}{n^{p-1}}\right)^{1/p} = \frac{1}{n^{\frac{p-1}{p}}} \to 0$$

as $n \to \infty$.

Now, pick arbitrary $b \in B$ and $\varepsilon > 0$. Since $\mathbb{C}G$ is dense in $\ell^p(G)$, there exists a $c \in \mathbb{C}G$ such that $\|b - c\|_p < \varepsilon/2$. From the observation above, we can choose an x_n such that $\|x_n\|_p < \dfrac{\varepsilon}{2\|c\|_1}$. Then:

$$\begin{aligned} \|x_n b\|_p = \|x_n(b - c) + x_n c\|_p &\le \|x_n(b - c)\|_p + \|x_n c\|_p \\ &\le \|x_n\|_1 \|b - c\|_p + \|c\|_1 \|x_n\|_p < 1 \cdot \frac{\varepsilon}{2} + \|c\|_1 \cdot \frac{\varepsilon}{2\|c\|_1} = \varepsilon. \end{aligned}$$

Thus

$$\|b - (1 - x_n)b\|_p = \|x_n b\|_p < \varepsilon,$$

where $1 - x_n \in (g-1)\ell^1(G)$ (the sum of its coefficients is zero) and $(1 - x_n)b \in (g-1)\ell^1(G) * B = (g-1)B$.

It works similarly for any $(g - \omega)$ with $|\omega| = 1$. Let $x_n(\omega) = \sum_{k=1}^{n} \frac{1}{n} \omega^{-k} g^k$, and observe that $1 - x_n(\omega) = (1 - \omega^{-1}g) \sum_{k=0}^{n-1} \frac{n-k}{n} \omega^{-k} g^k = -(g - \omega)\omega^{-1} \sum_{k=0}^{n-1} \frac{n-k}{n} \omega^{-k} g^k \in (g - \omega)\ell^1(G)$. The rest of the proof is the same.

The result for the $F(g)$, i.e. that its range is dense, is true because it is a product of operators $(g-\omega)$ with dense ranges. $\square$

For the sake of completeness, we shall provide here a proof of the last statement above.

Lemma 3.11. *Assume we have operators $T_k : B \to B$ on a normed space B, $1 \le k \le n$, such that the range $T_k(B)$ is dense in B for each T_k. Then the range of $T_1 \cdots T_n$ is also dense.*

Proof. First we can prove the claim for $n = 2$. For any $b \in B$ there exists $b_1 \in B$ such that $\|b - T_1 b_1\| < \frac{\varepsilon}{2}$. Then there exists $b_2 \in B$ such that $\|b_1 - T_2 b_2\| < \frac{\varepsilon}{2\,\|T_1\|}$. Then:

$$\begin{aligned}\|b - T_1T_2b_2\| &\le \|b - T_1b_1\| + \|T_1b_1 - T_1T_2b_2\| \\ &\le \|b - T_1b_1\| + \|T_1\|\,\|b_1 - T_2b_2\| < \frac{\varepsilon}{2} + \|T_1\| \cdot \frac{\varepsilon}{2\,\|T_1\|} = \varepsilon.\end{aligned}$$

Now induction on n finishes the proof. $\square$

The following two corollaries do not technically follow from Proposition 3.10, but rather can be proved using the same argument with minor adjustments.

Corollary 3.12. *Let G be a discrete group containing an infinite cyclic subgroup $H = \langle g \rangle \cong \mathbb{Z}$, $1 < p < \infty$. Let $A \subseteq B$ be closed G-invariant subspaces of $\ell^p(G)$, and assume that B/A is finite-dimensional. Then $A = B$.*

Proof. We can still regard $\ell^p(G)$ as an $\ell^1(H)$-module and $A \subseteq B$ as $\ell^1(H)$-submodules since they are H-invariant. Then the same proof works with obvious changes. $\square$

Corollary 3.13. *Let G be a discrete group containing an infinite cyclic subgroup $H = \langle g \rangle \cong \mathbb{Z}$, $1 < p < \infty$, m an integer. Let $A \subseteq B$ be closed G-invariant subspaces of $\ell^p(G)^m$, and assume that B/A is finite-dimensional. Then $A = B$.*

Proof. The same. $\square$

The core idea of Proposition 3.10 can be applied in some other situations as well. Basically, what we need is an infinite sequence of elements along with "eigenvalues" on B/A. In the above proposition the group G was fixed, but ω could be arbitrary. The other extreme would be to fix a special ω, but make it work for any group. And indeed, for the proof of our main theorem we will need such a result.

Proposition 3.14. *Let G be an infinite discrete group, $1 < p < \infty$. Let $A \subseteq B$ be closed G-invariant subspaces of $\ell^p(G)$ with B/A finite-dimensional. Assume that G acts identically on B/A. Then $A = B$.*

Proof. The condition that G acts identically on B/A means that $(g-1)b \in A$ for all $g \in G$ and $b \in B$, or equivalently that $w(G)B \subseteq A$, where $w(G) = \langle g-1 : g \in G \rangle$ is the augmentation ideal of G in $\ell^1(G)$.

Choose an infinite sequence $\{g_1, g_2, \ldots\}$ of distinct elements in G, and let $x_n = \sum_{k=1}^{n} \frac{1}{n} g_k \in \mathbb{C}G$. Exactly as in the proof of Proposition 3.10, for any $b \in B$ and $\varepsilon > 0$ there exists an index n such that $\|b - (1-x_n)b\|_p < \varepsilon$, where $1 - x_n \in w(G)$, because $1 - x_n = \sum_{k=1}^{n} \frac{1}{n}(1-g_k)$. So $w(G)B$ is dense in B, which along with $w(G)B \subseteq A$ implies $A = B$. □

Again, we can establish two more results by means of virtually the same proof with obvious modifications.

Corollary 3.15. *Let G be a discrete group containing an infinite subgroup H, $1 < p < \infty$. Let $A \subseteq B$ be closed G-invariant subspaces of $\ell^p(G)$ with B/A is finite-dimensional. Assume that H acts identically on B/A. Then $A = B$.*

Proof. This time, the condition is that $(g-1)b \in A$ for all $g \in H$ and $b \in B$, or equivalently that $w(H)B \subseteq A$. Once we show that $w(H)B$ is dense in A, the corollary will be proved. To that end, choose an infinite sequence $\{g_1, g_2, \ldots\}$ of distinct elements in H, and the rest of the proof goes without changes. □

Corollary 3.16. *Let G be a discrete group containing an infinite subgroup H, $1 < p < \infty$, m an integer. Let $A \subseteq B$ be closed G-invariant subspaces of $\ell^p(G)^m$ with B/A is finite-dimensional. Assume that H acts identically on B/A. Then $A = B$.*

Proof. The same. □

3.2.3 Main Theorem

The desired result about group cohomology will be an immediate consequence of the following main theorem.

Theorem 3.17. *Let m be a non-negative integer. Let G be an infinite discrete group, and let $Y \subseteq X$ be closed G-invariant subspaces of $\ell^p(G)^m$. Then either $Y = X$ or Y has infinite codimension in X.*

Proof. If Y has infinite codimension in X, then we are done. What we need to prove is that if Y has finite codimension in X, then $Y = X$. We shall consider several cases for the group G.

<u>Case 1:</u> G has an element of infinite order. So G has an infinite cyclic subgroup H. The result in this case has been proven in Corollary 3.13.

Case 2: G contains a finitely generated infinite subgroup K. In view of Case 1, we may also assume that K is a torsion group. Let H be the kernel of the action of K on X/Y. Since K/H is a finitely generated torsion linear group, it is finite [18, Corollary 4.9]. So H has finite index in K, and in particular H is infinite. Now Corollary 3.16 finishes the proof in this case.

Case 3: G is locally finite. Since every infinite locally finite group contains an infinite abelian subgroup (see [8] or [11]), we find an infinite abelian subgroup H of G, which is locally finite itself. If X/Y is a nonzero finite dimensional G-invariant space, then it is also an H-invarian subspace and thus a $\mathbb{C}H$-module, and it splits into a direct sum of one-dimensional $\mathbb{C}H$-modules by the following argument.

Let H be an infinite abelian locally finite group. Then a simple finite dimensional $\mathbb{C}H$-module must have dimension 1. Indeed, such a module must be isomorphic to $\mathbb{C}H/I$, where I is a maximal ideal of $\mathbb{C}H$. Then $\mathbb{C}H/I$ is a field which is algebraic over $\mathbb{C}$, and it follows that $\mathbb{C}H/I = \mathbb{C}$.

So it suffices to obtain a contradiction in the case X/Y is one-dimensional. This can be done in the same fashion as in the previous cases using the "elements and their eigenvalues" idea. H has infinitely many elements, say $g_1, g_2, \ldots$. Since H acts on one-dimensional X/Y, there exist $u_1, u_2, \ldots$ in $\mathbb{C}$ with $|u_i| = 1$ such that $(g_i - u_i)X \subseteq Y$, or equivalently $(u_i^{-1}g_i - 1)X \subseteq Y$. Now by the same argument as in the proof of Proposition 3.14, the elements $(u_i^{-1}g_i - 1)x$ for $x \in X$ and $i = 1, 2, \ldots$ are dense in X, and so we have a contradiction. $\square$

3.2.4 Dimensions of Cohomology Groups

We can restate the Main Theorem 3.17 in a form more suitable for our purposes.

Corollary 3.18. *Let m and n be non-negative integers. Let G be an infinite discrete group, let X be a closed G-invariant subspace of $\ell^p(G)^m$, and let $f\colon \ell^p(G)^n \to X$ be a bounded linear G-map. Then either f is onto or $\operatorname{im} f$ has infinite codimension in X.*

Proof. Assume f is not onto. If $\operatorname{im} f$ is not closed in X, then it has infinite codimension. If $\operatorname{im} f$ is closed in X, then we have exactly the Main Theorem with $Y = \operatorname{im} f$. $\square$

This corollary applied to the coboundary map $d_k\colon \operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_k}, \ell^p(G)) \to \ker d_{k+1}$ (recall that $\operatorname{Hom}_{\mathbb{C}G}(\mathbb{C}G^{e_k}, \ell^p(G)) \cong \ell^p(G)^{e_k}$) in the cochain complex (3.5) proves the desired result on group cohomology, provided the dimensions e_k, e_{k+1} are finite. In particular, the aforementioned result of Bekka and Valette for $H^1(G, \ell^2(G))$ of finitely generated groups is a special case of ours at dimension 1 applied to the resolution (3.4) with $e_0 = 1$ for the augmentation map and e_1 being finite because G is finitely generated.

Before finally stating the main result of this work, let us define classes of groups which it applies to.

Definition 3.19. *Let R be a ring and M a left R-module. A resolution or partial resolution $\mathbf{P} \to M$ is of **finite type** if each module in the resolution is finitely generated. M is of **type** FP_n ($n \geq 0$) if there exists a partial projective resolution $P_n \to \cdots \to P_0 \to M$ of finite type. M is of **type** FP_∞ if there exists a projective resolution $\cdots \to P_n \to \cdots \to P_0 \to M$ of finite type.*

*Let G be a group. We say that G is of **type** FP_n ($0 \leq n \leq \infty$) if $\mathbb{Z}$ as a $\mathbb{Z}G$-module is of type FP_n. We say that G is of **type** FP_n **over** $\mathbb{C}$ ($0 \leq n \leq \infty$) if $\mathbb{C}$ as a $\mathbb{C}G$-module is of type FP_n.*

Clearly if G is of type FP_n (over $\mathbb{Z}$), then it is of type FP_n over $\mathbb{C}$.

It is shown in [4, Chapter VIII, §4] that a module admits a (partial) projective resolution of finite type if and only if it admits a (partial) free resolution of finite type. Therefore this classification is well suited for our purposes.

Corollary 3.20. *Let G be an infinite discrete group.*

1. *If G is of type FP_∞ over $\mathbb{C}$, then each ℓ^p-cohomology group $H^n(G, \ell^p(G))$ is either zero or infinite dimensional.*
2. *If G is of type FP_n over $\mathbb{C}$, then each ℓ^p-cohomology group $H^k(G, \ell^p(G))$, $0 \leq k \leq n$, is either zero or infinite dimensional.*

Proof. Immediate from the last corollary. □

3.2.5 Dimensions of Homology Groups

Even though this work was motivated by and therefore was concentrated on study of cohomology, the method we employed is not specifically "cohomological". It can be equally well applied to any complex with a finiteness condition, e.g. to the one defining group homology with coefficients if the group belongs to one of the FP_n types.

Recall that to define group homology with coefficients in $\ell^p(G)$, we take a resolution (3.4) and tensor the chain complex with this module, obtaining

$$\cdots \xrightarrow{d_1} \mathbb{C}G^{e_1} \otimes_{\mathbb{C}G} \ell^p(G) \xrightarrow{d_0} \mathbb{C}G^{e_0} \otimes_{\mathbb{C}G} \ell^p(G) \longrightarrow 0, \tag{3.6}$$

and then the homology is defined as $H_n(G, \ell^p(G)) = \ker d_{n-1} / \operatorname{im} d_n$.

So again, we have two closed G-invariant subspaces $Y \subseteq X$, where $Y = \operatorname{im} d_n$ and $X = \ker d_{n-1}$ of the space $\mathbb{C}G^{e_n} \otimes_{\mathbb{C}G} \ell^p(G) \cong \ell^p(G)^{e_n}$, where e_n is a positive integer if we assume that G is at least of type FP_n.

Now an application of Corollary 3.18 to the map $d_n \colon \mathbb{C}G^{e_n} \otimes_{\mathbb{C}G} \ell^p(G) \to \ker d_{n-1}$ yields a result on group homology, analogous to the one on cohomology obtained above.

Corollary 3.21. *Let G be an infinite discrete group.*

(1) *If G is of type FP_∞ over $\mathbb{C}$, then each ℓ^p-homology group $H_n(G, \ell^p(G))$ is either zero or infinite dimensional.*

(2) *If G is of type FP_n over $\mathbb{C}$, then each ℓ^p-homology group $H_k(G, \ell^p(G))$, $0 \leq k \leq n$, is either zero or infinite dimensional.*

Bibliography

[1] *Cauchy-Binet formula*, `http://planetmath.org/encyclopedia/CauchyBinetFormula.html`

[2] B. Bekka, A. Valette, *Group Cohomology, Harmonic Functions and the First L^2-Betti Numbers*, Potential Analysis, **6** (1997), pp. 313-326.

[3] F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete **(80)**, Springer-Verlag, New York-Heidelberg-Berlin, 1973.

[4] Kenneth S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics **(87)**, Springer-Verlag, New York-Heidelberg-Berlin, 1982.

[5] P. M. Cohn, *Free rings and their relations*, Academic Press, 1985.

[6] Mark Grinshpon, *Invertibility of matrices over subrings*, to appear in *Communications in Algebra*.

[7] A. Guichardet, *Cohomologie des groupes topologiques et des algèbres de Lie*, Cedic-F.Nathan, 1980.

[8] P. Hall, C. R. Kutilaka, *A property of locally finite groups*, J. London Math. Soc., **39** (1964), pp. 235-239.

[9] E. Hewitt, K. A. Ross, *Abstract Harmonic Analysis I*, Die Grundlehrender der Mathematischen Wissenschaften **(115)**, Springer-Verlag, New York-Heidelberg-Berlin, 1979.

[10] P. J. Hilton, U. Stammbach, *A Course in Homological Algebra*, Graduate Texts in Mathematics **(4)**, Springer-Verlag, New York-Heidelberg-Berlin, 1996.

[11] M. I. Kargapolov, *On a problem of O.Yu. Šmidt* (Russian), Sibirsk. Mat. Ž., **4** (1963), pp. 232-235.

[12] Peter Linnell, *Noncommutative localization in group rings*, in *Noncommutative Localization in Algebra and Topology*, London Mathematical Society Lecture Note Series **(330)**, pp. 40-59, Cambridge University Press, 2006.

[13] Lynn H. Loomis, *An Introduction to Abstract Harmonic Analysis*, Van Nostrand, Toronto-New York-London, 1953.

[14] Wolfgang Lück, *L_2-Invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete **(44)**, Springer-Verlag, New York-Heidelberg-Berlin, 2002.

[15] Robert E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics **(183)**, Springer-Verlag, New York-Heidelberg-Berlin, 1998.

[16] Holger Reich, *Group von Neumann Algebras and Related Algebras*, PhD Thesis, Georg-August-Universität, Göttingen, 1998.

[17] Bo Stenström, *Rings of Quotients*, Die Grundlehrender der Mathematischen Wissenschaften **(217)**, Springer-Verlag, New York-Heidelberg-Berlin, 1975.

[18] B. Wehrfritz, *Infinite Linear Groups*, Queen Mary College Mathematical Notes, London, 1969.