Structure of Invariant Subspaces for Left-Invertible Operators on Hilbert Space

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(ABSTRACT)

This dissertation is primarily concerned with studying the invariant subspaces of left-invertible, weighted shifts, with generalizations to left-invertible operators where applicable. The two main problems that are researched can be stated together as When does a weighted shift have the one-dimensional wandering subspace property for all of its closed, invariant subspaces? This can fail either by having a subspace that is not generated by its wandering subspace, or by having a subspace with an index greater than one. For the former we show that every left-invertible, weighted shift is similar to another weighted shift with a residual space, with respect to being generated by the wandering subspace, of dimension n, where n is any finite number. For the latter we derive necessary and sufficient conditions for a pure, left-invertible operator with an index of one to have a closed, invariant subspace with an index greater than one. We use these conditions to show that if a closed, invariant subspace for an operator in a class of weighted shifts has a vector in l^1, then it must have an index equal to one, and to produce closed, invariant subspaces with an index of two for operators in another class of weighted shifts.

Praise

To God, for:

It is the glory of God to conceal a thing: but the honour of kings is to search out a matter.

–Proverbs 25:2 KJV

And especially for:

If any of you lack wisdom, let him ask of God, that giveth to all men liberally, and upbraideth not; and it shall be given him.

–James 1:5 KJV

Dedication

To those that I have loved, especially Diddy and Golnar

Contents

Chapter 1

Introduction

The purpose of this dissertation is to help better understand the structure of the invariant subspaces of weighted shifts. To motivate the importance of studying weighted shifts we provide a brief overview of their history. Then we discuss the two main problems considered in this dissertation and the results in the literature that are related to them. Finally we give an outline of the remainder of the dissertation.

1.1 History of Weighted Shifts

The Hilbert space l^2 is the space of sequences that are square-summable, that is

$$l^2 = \{\{l_i\}_{i=1}^{\infty},\ \ l_i \in \mathbb{C} :\ \sum_{i=1}^{\infty} |l_i|^2 < \infty\}.$$

Its vectors can be represented as $l = (l_1, l_2, l_3, \ldots)$. The shift operator is the operator defined by the mapping

$$S : (l_1, l_2, l_3, \ldots) \to (0, l_1, l_2, l_3, \ldots). \tag{1.1}$$

The shift operator has long been used in examples and counterexamples due to its unique behavior and how easily the rule by which it maps can be understood. One of its first uses was in the theory of operators, which has its roots in extending the analysis of matrices to operators with domains that have an infinite number of dimensions, in integral equations, and in quantum mechanics. One of the first achievements of the theory of operators was the spectral theorem, first for self-adjoint and then for normal operators. The spectral theorem in essence showed that normal operators could be viewed as the continuous, direct sum of scalar operators, each of which acted on an invariant subspace for the normal operator (an invariant subspace $\mathcal{M}$ for an operator T is a subspace that satisfies the relation $T\mathcal{M} \subset \mathcal{M}$).

Moreover, the invariant subspaces were actually eigenspaces where the operator acted as a scalar multiple of the identity. The next step towards an understanding of all operators was studying the structure of invariant subspaces for nonnormal operators. One of the first, interesting examples was the shift, which has since, among other reasons, been studied in the hopes of a fuller understanding of the invariant subspaces of arbitrary operators. It gained importance after the works of von Neumann [46] and Wold [47] showed that any isometry on a Hilbert space is the direct sum of a unitary operator and a generalized shift operator, which shifts sequences in an arbitrary Hilbert space instead of just sequences in $\mathbb{C}$. Related to the study of the invariant subspaces of an arbitrary operator was the *Invariant Subspace Problem*, which asked: *Does every bounded linear operator on a separable, complex Hilbert space have an invariant subspace other than* $\{0\}$ *and* $\mathcal{H}$*?*

The seminal paper of Beurling [9] in a sense completely characterized the closed, invariant subspaces of the shift operator on l^2. It worked in the Hardy space, which is the space of functions that are analytic in the unit disk and such that

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

If a function is represented by its Taylor series $f = \sum a_n z^n$, then this is the space of functions such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, and if f is viewed as $f = (a_1, a_2, a_3, \ldots)$ then the shift operator has its standard form as in equation (1.1). Alternatively, the shift can be viewed as the operator M_z of multiplication by the complex coordinate function z on the space H^2, that is $M_z : f \to zf$, and as this perspective focuses more on the space than the operator, it is sometimes called the Hardy shift. These two views yield unitarily equivalent operators, as any concrete f can be viewed abstractly as $f = (a_1, a_2, a_3, \ldots)$ where $a_n = \hat{f}(n-1)$ are the Taylor coefficients of f, $\hat{f}(n) = \int_0^{2\pi} f(e^{i\theta})e^{-in\theta} \frac{d\theta}{2\pi}$, so that if $U : (a_1, a_2, a_3, \ldots) \to \sum_{n=1}^{\infty} a_n z^{n-1}$, then $S = U^{-1} M_z U$. The viewpoint of Beurling's paper was mostly based on the theory of functions in the Hardy space where it viewed the shift operator as the operation of multiplication by the complex coordinate function z. It showed that for any closed, invariant subspace $\mathcal{M}$ of S, there is a function ϕ satisfying the condition of being a so called inner function such that $\mathcal{M} = \phi\mathcal{H}$. It also showed that any vector could be uniquely factored as $f = \phi F$ where ϕ is an inner function and F is an outer function, that is, it is cyclic for M_z. Lax [32] generalized to shifts that operate on sequences of vectors in $\mathbb{C}^n$ instead of complex numbers and Halmos [24] and Helson and Lowdenslager [28] generalized to shifts that operate on sequences of sequences.

A weighted shift is any operator defined by the mapping $T : (l_1, l_2, l_3, \ldots) \to (0, \alpha_1 l_1, \alpha_2 l_2, \alpha_3 l_3, \ldots)$ where $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence of bounded complex numbers. The shift operator is seen to be the weighted shift whose weights are all ones, and thus is usually called the unweighted shift. An operator T is called left-invertible if there is an operator L such that $LT = I$. Weighted shifts are left-invertible if and only if $\inf_i |\alpha_i| > 0$. Analogous to the Hardy space are the Dirichlet space D of functions that are analytic in the unit disk and such that

$$\|f\|_D = \sqrt{\|f\|_{H^2}^2 + \frac{1}{\pi}\int_0^1 \int_0^{2\pi} |f'(re^{i\theta})|^2 r dr d\theta} < \infty$$

and the Bergman space L_a^2 of functions that are analytic in the unit disk and such that

$$\|f\|_{L_a^2} = \left(\frac{1}{\pi}\int_0^1 \int_0^{2\pi} |f(re^{i\theta}|^2 r dr d\theta\right)^{1/2} < \infty.$$

Both of these spaces can be viewed abstractly as the space of Taylor coefficients $\{a_n\}_{n=1}^\infty$ for which $\sum_{n=1}^\infty w_n|a_n|^2 < \infty$ where $\{w_n\}_{n=1}^\infty$ is a sequence of positive real numbers (which depends on the space). With this perspective multiplication by the complex coordinate is a weighted shift, and is called the Dirichlet shift or Bergman shift, according to the space on which it acts. Beurling's theorem led to research into the closed, invariant subspaces of weighted shifts on other spaces such as the Dirichlet and Bergman spaces, again viewing the shift operator as multiplication by the complex coordinate, and also produced a search for factorization theorems for functions in these spaces.

The papers of Julia [31, 29, 30], Halmos [23] and Sz.-Nagy [44] showed that any contraction can be extended to be the adjoint of an isometry. In this sense isometries are models for all operators, and in view of the result of von Neumann-Wold that any isometry splits into the direct sum of a unitary operator and a generalized unweighted shift, it appeared that understanding well the structure of generalized unweighted shifts would lead to an understanding of the structure of all operators. This was actually the approach of the Sz.-Nagy-Foiaş model theory for a contraction operator [45]. While the approach led to a number of sufficient conditions for an operator to have a nontrivial, invariant subspace, the problem in full generality remains open. It was a surprise to many that for any strict contraction A there are two invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ for the Bergman shift, T, which is only a regular, weighted shift on sequences in $\mathbb{C}$, such that $\mathcal{N} \subset \mathcal{M}$ and $A = P_{\mathcal{M}\cap\mathcal{N}^\perp} T|_{\mathcal{M}\cap\mathcal{N}^\perp}$. This was shown by Apostol, Bercovici, Foiaş, and Pearcy [6] and was a surprise to many, as it showed that the Bergman shift, in a sense has every other operator contained inside of it. This was drastically different from the Hardy shift, whose behavior was known to be much nicer, and made the task of characterizing all of the invariant subspaces of the Bergman shift seem intractable. More generally, for the last result about "containing" any strict contraction, the Bergman shift can be replaced by any weighted shift whose powers converge strongly to zero and has both a spectral radius and norm equal to one. In this way studying the invariant subspaces of weighted shifts will yield information for all operators. This increased the importance of studying weighted shifts, and both the result from Apostol, et alii, and that of Sz.-Nagy showed that the *Invariant Subspace Problem* could be reformulated to be a problem about just the invariant subspaces of a single weighted shift. Also, weighted shifts have been generalized in different directions, including to spaces with an indefinite inner product [8], and to the commuting and noncommuting multivariable cases [36].

1.2 Two Problems and Current Results

The two main problems of this dissertation find their origins in ongoing research on spaces of functions including the Hardy and Bergman spaces. A wandering subspace $\mathcal{E}$ for an operator T is a subspace such that $T^i\mathcal{E} \perp \mathcal{E}\ \forall i \geq 1$. We say that a space is generated by the subspace X for the operator T if the linear span of elements of the form $T^i x : \ x \in X \ i \geq 0$ is dense in the space. We will simply say that a space is generated by the subspace X if there is no confusion as to the operator T. Halmos showed in [24] that if a closed, invariant subspace $\mathcal{M}$ is generated by a wandering subspace $\mathcal{E}$, then it must be that $\mathcal{E} = \mathcal{M} \cap (T\mathcal{M})^{\perp}$. The index of a closed, invariant subspace $\mathcal{M}$ is defined to be the dimension of $\mathcal{M} \cap (T\mathcal{M})^{\perp}$. After Beurling [9] showed that every closed, invariant subspace of the unweighted shift has an index of one and the subspace is generated by its wandering vector (that is $\mathcal{M} \cap (T\mathcal{M})^{\perp}$ always has a dimension of one and this subspace generates $\mathcal{M}$), it was natural to ask whether this property also holds for other weighted shifts. The first, main problem is: *Is every closed, invariant subspace of an arbitrary, left-invertible, weighted shift generated by its wandering subspace?* This was an open question but will be shown here to have a negative answer. It is known that in the Dirichlet and Bergman spaces every closed, invariant subspace is generated by its wandering subspace [38, 4].

For general operators there are several results that give examples of operators with closed, invariant subspaces that are generated by their wandering subspaces. These results can be split into two categories: (i) those that are expressed in terms of a model from the theory of functions, where the operator is represented as M_z on a space of analytic functions, and (ii) those that are expressed directly in terms of a model on l^2, where the operator is represented as a weighted shift. This dissertation will focus on the model using weighted shifts. To be closer to completeness we list some of the results from both categories; as it is often difficult to translate a result given in the context of a model from the theory of functions to an equivalent form in a model for a weighted shift, we leave many of the results given below in their formulation from the theory of functions, and do not define the terminology. Note that an operator T is *pure* if $\bigcap_{i=0}^{\infty} T^i\mathcal{H} = \{0\}$. Some closed, invariant subspaces that are generated by their wandering subspaces are:

1. Multiplier invariant subspaces when the subspace has a Bergman-type reproducing kernel [33].

2. All invariant subspaces if T is a pure, left-invertible contraction such that its left-inverse has a spectral radius of one and there is a family of functions satisfying other properties; this class of operators includes weighted shifts with decreasing weights such that the recursive sequence $\beta_1 = 4\alpha_2 + \alpha_3 - \alpha_1$, $\beta_{n+1} = 4\alpha_{n+2} + \alpha_{n+3} - \alpha n + 1 - \frac{4\alpha_{n+2}^2}{\beta_n}$ is always greater than zero [43].

3. All invariant subspaces if T is pure and there exists an infinite matrix $A = (a_{k,l})$ that satisfies

$$(i)\ \sum_{l\geq 0} |a_{k,l}|\|T^l L^l\| < \infty\ \ \forall k \geq 0,$$

$$(ii)\ \lim_{k\to\infty} \sum_{l\geq 0} a_{k,l} = 1\ \ \forall l \geq 0,$$

$$(iii)\ \lim_{k\to\infty} a_{k,l} = 0\ \ \forall l \geq 0,$$

and

$$(iv)\ \sup_{k\geq 0} \|\sum_{l\geq 0} a_{k,l} T^l L^l\| < \infty.$$

A sufficient condition for the existence of such an A is that $\alpha_1 \leq 2\alpha_2$ and the sequence $\{\frac{1}{\alpha_i}\}_{i=1}^{\infty}$ is concave or $\{\alpha_i\}_{i=1}^{\infty}$ comes from a logarithmically subharmonic weight function (weighted shifts are always pure) [42].

4. All invariant subspaces if T is a pure, expansive operator such that

$$\|T^k x\|^2 \leq c_k(\|Tx\|^2 - \|x\|^2) + c\|x\|^2$$

for all $x \in \mathcal{H}$, $k \geq 2$, with $\sum_{k=2}^{\infty} \frac{1}{c_k} = \infty$ [35]. This condition will be slightly generalized in the second chapter.

The closest result to showing that there are weighted shifts with closed, invariant subspaces that are not generated by their wandering subspaces is by Hedenmalm and Zhu [27] who show that there are zero-set subspaces that do not have the optimal factorization property. This does not prove that there are subspaces that are not generated by their wandering subspaces, but suggests it. In this dissertation we will show that there are weighted shifts that have closed, invariant subspaces that are not generated by their wandering subspaces.

The second, main problem is: *When does a weighted shift have only closed, invariant subspaces of index equal to one?* It has been known for a while that there are weighted shifts, including the Bergman shift, that have closed, invariant subspaces with arbitrary index, finite or infinite, according to the results in [6]. As this paper showed the existence of such subspaces but did not explicitly construct them, there have been papers that have produced such subspaces [25, 26, 11], as well as work in l^p and other spaces [2, 10]. There are a few results for when the index is always one [3] based on the division property, as well as criteria for when the index is or is not equal to one [5], based on the results in [6]. Also, [34] has some results on when a subnormal operator has only invariant subspaces with an index of one or has invariant subspaces with an index of any size. Note that a subnormal operator is one that can be extended to be a normal operator, and for weighted shifts this necessitates that the weights be nondecreasing. In this dissertation we derive equivalent conditions for a pure, left-invertible operator whose adjoint has a kernel of dimension equal to one to have only closed, invariant subspaces with an index equal to one. These conditions are derived by techniques formulated in the context of the models for weighted shifts in l^2. Then we demonstrate the usefulness of the conditions by proving that a class of weighted shifts cannot have any vectors contained in l^1 in a closed, invariant subspace with an index greater than one, and by constructing closed, invariant subspaces with an index of two for another class of weighted shifts. The results for the second class strictly include those of [6], in regards to their results on when a left-invertible, weighted shift has closed, invariant subspaces with an index of two.

It has been the opinion of the author while reading many different papers that problems that are innately based on the theory of operators are being solved using methods from the theory of spaces of functions. As it appears that the primary goal is the solution of the problems and moreover, most authors refer to solutions based on the theory of Hilbert spaces and operators when available (as these are usually shorter and more elegant), the author suggests that more research be devoted to solving these problems using the methods of the theory of operators. Of course for those problems that are innately based on the theory of the underlying spaces of functions the techniques based on those spaces should be employed.

1.3 Remaining Chapters

The remaining chapters are organized as follows: The second chapter contains the essential information regarding left-invertible operators and wandering subspaces. It introduces the *Wandering Subspace Property* and gives different kinds of conditions for an operator to possess the *Wandering Subspace Property*. The main results are very slight generalizations of material from [35]. It concludes with basic results on isometries, with the emphasis on shifts, leading into the introduction of weighted shifts in the third chapter. The main content of the third chapter is the treatment of the first, main problem – that of when every closed, invariant subspace is generated by its wandering subspace. A new proof using only the theory of operators acting on l^2 gives a part of the Carathéodory Interpolation theorem, a result which is usually proven by using the setting of the theory of functions in H^2. It solves in a certain sense when a weighted shift that is similar to the unweighted shift does not have the *Wandering Subspace Property.* We also show that given any weighted shift, there is another weighted shift, similar to the first, that has a closed, invariant subspace with a residual space having any preassigned, finite dimension of n. As such it is the first concrete example of a weighted shift that has a closed, invariant subspace that is not generated by its wandering subspace, and hence is also the first proof that such weighted shifts exist. The fourth chapter is concerned with the treatment of the second, main problem – that of when a weighted shift has only closed, invariant subspaces with indices equal to one. It includes many equivalent conditions for when this happens that hopefully will help to illuminate the answer to this question in more generality. It concludes by demonstrating that a certain class of weighted shifts cannot have any vectors contained in l^1 in a closed, invariant subspace with an index greater than one, and constructs closed, invariant subspaces with an index of two for another class of weighted shifts. The fifth and final chapter is mostly a list of some of the topics that the author would like to see researched in the future.

Chapter 2

Left-Invertible Operators and Wandering Subspaces

2.1 Left-Invertible Operators

Let $\mathcal{H}$ be a (always assumed separable and having an infinite number of dimensions) Hilbert space over the Complex Field. Let $e_i = (0, 0, ..., 0, 1, 0, 0, ...)$, where the 1 is in the i^{th} spot; a vector $x \in \mathcal{H}$ can be written in either of the two equivalent forms: $x = (x_1, x_2, x_3, ...)$ or $x = \sum_{i=1}^{\infty} x_i e_i$, where $x_i \in \mathbb{C}\ \ \forall i \geq 1$.

Let $T : \mathcal{H} \to \mathcal{H}$ be an (always assumed bounded and linear) operator.

Definition 2.1.1. *For any operator T from $\mathcal{H}$ to $\mathcal{H}$, the Kernel of T, Ker T, is the set $\{x \in \mathcal{H} :\ Tx = 0\}$. The Image of T, Im T, is the set $\{y \in \mathcal{H} :\ \exists x :\ Tx = y\}$. For any subspace X, the dimension of X, dim X, is the smallest integer n such that there exists a set $\{x_i\}_{i=1}^{n}$ such that X is the closure of the span of the x_i's.*

Definition 2.1.2. *T is left-invertible if there exists an operator L such that $LT = I$, and T is right-invertible if there exists an operator R such that $TR = I$.*

Theorem 2.1.3. *[18, Section A.2.2][7, Section 4.5] The following are equivalent:*
(i) T is left-invertible
(ii) T^* is right-invertible
(iii) T is one-to-one and $Im(T)$ is closed
(iv) $\exists c > 0$ such that $\|Tx\| \geq c\|x\|\ \ \forall x \in \mathcal{H}$

So, if T is left-invertible then $c^2\|x\|^2 \leq \|Tx\|^2 = (Tx, Tx) = (T^*Tx, x)\ \ \forall x \in \mathcal{H}$, which shows that T^*T is coercive, and by Corollary A.50 of [18], T^*T is invertible. Therefore, $L = (T^*T)^{-1}T^*$ is well defined, and $LT = (T^*T)^{-1}T^*T = I$, so L is a left-inverses for T. Since $\mathcal{H}$ has an infinite number of dimensions, if T is not right-invertible then the space of left-inverses for T has an infinite number of dimensions;

the important fact about L is that it is the only left-inverse of T with the property $\mathrm{Ker}(L) = \mathrm{Ker}(T^*)$.

Proposition 2.1.4. *[7, Proposition 4.5.4] Let T be a left-invertible operator, $L = (T^*T)^{-1}T^*$, and $P_{T\mathcal{H}}$ the orthogonal projection onto the (closed) range of T, then L_A is a left-inverse of T if and only if there is an operator A such that $L_A = L + A(I - P_{T\mathcal{H}})$*

Note: Since $\mathrm{Ker}(T^*) = (\mathrm{Im}T)^{\perp} = \mathrm{Im}(I - P_{T\mathcal{H}})$, A only affects L_A on $\mathrm{Ker}(T^*)$, so $L_{A_1} = L_{A_2}$ if $A_1|_{\mathrm{Ker}(T^*)} = A_2|_{\mathrm{Ker}(T^*)}$, that is, if A_1 and A_2 agree on $\mathrm{Ker}(T^*)$.

Since every left-invertible operator has a right-invertible adjoint, it will at times be helpful to have a theorem for right-invertible operators similar to Theorem 2.1.3:

Theorem 2.1.5. *[18, Section A.2.2][7, Section 4.6] The following are equivalent:*
(i) T is right-invertible
(ii) T^* is left-invertible
(iii) T is onto
(iv) $\exists c > 0$ such that $\forall y \in \mathcal{H},\ \exists x_y$ such that $Tx_y = y$ and $c\|x_y\| \leq \|y\|$

Note that any operator T with a closed range has a Moore-Penrose pseudoinverse $T^{\#}$ defined by the relations $T^{\#}T = P_{(\mathrm{Ker}T)^{\perp}}$ and $T^{\#}|_{(\mathrm{Ran}T)^{\perp}} = 0$. If T is left-invertible then the Moore-Penrose pseudoinverse is equal to $(T^*T)^{-1}T^*$, that is, it is the unique left-inverse that has the same kernel as that of T^*. In general if one starts with an operator T that has a closed range, calculates the adjoint of its Moore-Penrose pseudoinverse $(T^{\#})^*$, and then takes the adjoint of this new operator's Moore-Penrose pseudoinverse, one will once again have the original operator T. In the case where T is left-invertible, this can be seen from the following: Since $LT = I$, by taking adjoints $T^*L^* = I$, so $L^* = T(T^*T)^{-1}$ is left-invertible. This implies that $L^* = (L_T)^*$ has its own $(L_{L^*})^*$ (by definition the adjoint of the unique left-inverse of L^* which has a kernel equal to the kernel of L^*), which can be calculated with the same equation for L^*, or $(L_{L^*})^* = L^*(L^{**}L^*)^{-1} = T(T^*T)^{-1}((T^*T)^{-1}T^*T(T^*T)^{-1})^{-1} = T(T^*T)^{-1}((T^*T)^{-1})^{-1} = T$. It is because of this that some authors say that there is a duality between the operators T and L^*, and we will see in the next section that this duality also affects the structure of an operator's wandering subspaces.

Theorem 2.1.6. *[15, Theorem 8.18] Let T be a nonzero operator. Then the following are equivalent:*
(i) T has a closed range (that is, a closed image)
(ii) There exists a $c > 0$ such that $\|Tx\| \geq c\|x\|\ \ \forall x \in (Ker\ T)^{\perp}$
(iii) There exists a $c > 0$ such that $\|Tx\| \geq c\|P_{(Ker\ T)^{\perp}}x\|\ \ \forall x \in \mathcal{H}$
(iv) $\inf\{\|Tx\| :\ x \in (Ker\ T)^{\perp},\ \|x\| = 1\} > 0$.

Definition 2.1.7. *Let A and B be subspaces. $A \ominus B$ is defined as $A \cap (B)^\perp$.*

Notice that if $B \subset A$ and B is regarded as a subspace of A, then $B^\perp = A \ominus B$. Using this definition $\text{Ker}(T^*) = \mathcal{H} \ominus T\mathcal{H}$.

Proposition 2.1.8. *Let T be a left-invertible operator on $\mathcal{H}$. Then dim$(\mathcal{H} \ominus T\mathcal{H}) =$ dim$(T^i\mathcal{H} \ominus T^{i+1}\mathcal{H})$ $\forall i \geq 0$.*

Proof Let $\{f_i\}_{i=1}^n$ be an orthornormal basis of $\mathcal{H} \ominus T\mathcal{H}$ where $n = \dim(\mathcal{H} \ominus T\mathcal{H})$. Since T is left-invertible, $T\mathcal{H}$ is closed. By (iv) of Theorem 2.1.3, there is a $c > 0$ such that $\|Tx\| \geq c\|x\|$ $\forall x \in \mathcal{H}$, and hence this c works for T restricted to any of its closed, invariant subspaces, so T restricted to any of its closed, invariant subspaces is also left-invertible. Since $T|_{T\mathcal{H}}$ is left-invertible, $T^2\mathcal{H}$ is closed. If $P_{T\mathcal{H}\ominus T^2\mathcal{H}}Tx = 0$ for some $x \in \text{span}\{f_1, f_2, f_3, \ldots\}$, then $Tx \in T^2\mathcal{H}$. So $Tx = T^2y$ for some $y \in \mathcal{H}$, and applying L to both sides of this equality yields $x = Ty \in T\mathcal{H}$, but $x \in \mathcal{H} \ominus T\mathcal{H}$, so that it must be that $x = 0$. Therefore $P_{T\mathcal{H}\ominus T^2\mathcal{H}}T\text{span}\{f_1, f_2, f_3, \ldots\}$ is a subspace of dimension at least n contained in $T\mathcal{H} \ominus T^2\mathcal{H}$, and hence $\dim(T\mathcal{H} \ominus T^2\mathcal{H}) \geq \dim(\mathcal{H} \ominus T\mathcal{H})$.

To prove the reverse inequality, suppose that $g \in T\mathcal{H} \ominus T^2\mathcal{H}$ with $g \neq 0$. Then $Lg \notin T\mathcal{H}$, because this would mean that there is some $x \in \mathcal{H}$ such that $Lg = Tx$, or $TLg = g = T^2x$ where $TLg = g$ because $g \in T\mathcal{H}$. This implies that $g \in T^2\mathcal{H}$, but $g \in T\mathcal{H} \ominus T^2\mathcal{H}$ is a contradiction to $g \neq 0$. Therefore $Lg = f + Ty$ for some $f \in \text{span}\{f_1, f_2, f_3, \ldots\}$, and hence $TLg = g = Tf + T^2y$ and $P_{T\mathcal{H}\ominus T^2\mathcal{H}}g = g = P_{\mathcal{H}\ominus T^2\mathcal{H}}Tf$, so that $g \in P_{T\mathcal{H}\ominus T^2\mathcal{H}}T\text{span}\{f_1, f_2, f_3, \ldots\}$. Since $g \in T\mathcal{H} \ominus T^2\mathcal{H}$ was arbitrary, it follows that $\dim(T\mathcal{H} \ominus T^2\mathcal{H}) \geq \dim(\mathcal{H} \ominus T\mathcal{H})$. Combining this with the previous paragraph yields $\dim(T\mathcal{H} \ominus T^2\mathcal{H}) = \dim(\mathcal{H} \ominus T\mathcal{H})$. Using the fact that $T|_{T^i\mathcal{H}}$ is left-invertible for every $i \geq 0$, by induction we obtain $\dim(\mathcal{H} \ominus T\mathcal{H}) = \dim(T^i\mathcal{H} \ominus T^{i+1}\mathcal{H})$ $\forall i \geq 0$.

Corollary 2.1.9. *Let T be a left-invertible operator such that dim$(\mathcal{H} \ominus T\mathcal{H}) = 1$, then there is a sequence of orthogonal wandering vectors $\{e_i\}_{i=1}^\infty$ such that $\mathcal{H} =$ (span$\{e_1\} \oplus$ span$\{e_2\} \oplus$ span$\{e_3\} \oplus \ldots) \oplus \bigcap_{i=0}^\infty T^i\mathcal{H}$.*

Proof By Proposition 2.1.8 $\dim(T^i\mathcal{H} \ominus T^{i+1}\mathcal{H}) = 1$ $\forall i \geq 0$. Let e_i be a nonzero vector in $T^{i-1}\mathcal{H} \ominus T^i\mathcal{H}$ for every $i \geq 1$. Then since

$$\mathcal{H} = \big((\mathcal{H} \ominus T\mathcal{H}) \oplus \big(T\mathcal{H} \ominus T^2\mathcal{H}\big) \oplus \big(T^2\mathcal{H} \ominus T^3\mathcal{H}\big) \oplus \ldots\big) \oplus \bigcap_{i=1}^{\infty} T^i\mathcal{H},$$

the relation $\mathcal{H} = (\text{span}\{e_1\} \oplus \text{span}\{e_2\} \oplus \text{span}\{e_3\} \oplus \ldots) \oplus \bigcap_{i=0}^\infty T^i\mathcal{H}$ follows. Since $T^je_i \in T^i\mathcal{H}$ $\forall j \geq 1$, $e_i \perp T^je_i$ $\forall i, j \geq 1$, so that e_i is a wandering vector for every i.

2.2 Wandering Subspaces

Definition 2.2.1. *A closed subspace $\mathcal{E}$ is a wandering subspace for the operator T if $\mathcal{E} \perp T^i\mathcal{E}$ $\forall i \geq 1$. A vector ε is a wandering vector for the operator T if $\varepsilon \perp T^i\varepsilon$ $\forall i \geq 1$. As Halmos showed that if $\mathcal{H}$ is generated by a wandering subspace $\mathcal{E}$, then it must be that $\mathcal{E} = \mathcal{H} \ominus T\mathcal{H}$ ([24]), we will call $\mathcal{H} \ominus T\mathcal{H}$* THE *wandering subspace for T.*

Definition 2.2.2. *For a set $X \subset \mathcal{H}$ and an operator T, the following notation will be used interchangeably: $[X]_T = \bigvee_{i=0}^{\infty} T^iX = \overline{span\{X, TX, T^2X, T^3X, ...\}}$, for the closure of finite linear combinations of elements in T^iX, $i \in \mathbb{N}$, which is also the smallest closed, invariant subspace of T containing X.*

Note: $x \in \bigvee_{i=0}^{\infty} T^iX$ means that for every $\epsilon > 0$ $\exists\{c_i, x_i\}_{i=0}^{n}$ $c_i \in \mathbb{C}$ $x_i \in X$ where n is finite such that $\|x - \sum_{i=0}^{n} c_iT^ix_i\| \leq \epsilon$; however, for a smaller ϵ_1 it may not be true that $c_{i_\epsilon}T^{i_\epsilon}x_{i_\epsilon} = c_{i_{\epsilon_1}}T^{i_{\epsilon_1}}x_{i_{\epsilon_1}}$, so there may not exist one sequence $\{c_i, x_i\}_{i=0}^{\infty}$ such that $x = \lim_{n\to\infty}\sum_{i=0}^{n} c_iT^ix_i = \sum_{i=0}^{\infty} c_iT^ix_i$ and it may not be true that $\bigvee_{i=0}^{\infty} T^iX = \sum_{i=0}^{\infty} T^iX = \{y : \exists\{c_i\}_{i=0}^{\infty}, \{x_i\}_{i=0}^{\infty}\ x_i \in X\ \forall i \geq 0 : y = \lim_{n\to\infty}\sum_{i=0}^{n} c_iT^ix_i\}$.

We will be interested in when the whole space is generated by a wandering subspace for the operator T, that is, when $\mathcal{H} = [\mathcal{E}]_T$ where $\mathcal{E} \subset \mathcal{H}$ satisfies $\mathcal{E} \perp T^i\mathcal{E}$ $\forall i \geq 1$. Notice that (see [24]) if $y \in T\mathcal{H}$, then $y = Tx$ for some $x \in \mathcal{H}$, and if $x \in [\mathcal{E}]_T$ then $y \in T[\mathcal{E}]_T \subset [T\mathcal{E}]_T \perp \mathcal{E}$ where the inclusion follows because T is continuous. Hence if $\mathcal{H} = [\mathcal{E}]_T$, then $\mathcal{E}$ is perpendicular to the range of T, so $\mathcal{E} \subset (\mathrm{Im}T)^{\perp} = \mathrm{Ker}(T^*)$. Likewise, if $\mathcal{E} \neq \mathrm{Ker}(T^*)$, then since both $\mathcal{E}$ and $\mathrm{Ker}(T^*)$ are closed subspaces, there would be a vector $x \in \mathrm{Ker}(T^*)$ such that $x \perp \mathcal{E}$, but also $x \perp T^i\mathcal{E}$ $\forall i \geq 1$ since $T^i\mathcal{E} \subset T\mathcal{H}$ and $x \in \mathrm{Ker}(T^*) = (\mathrm{Im}T)^{\perp}$, so $x \perp [\mathcal{E}]_T$ and $[\mathcal{E}]_T \neq \mathcal{H}$. This shows that the only wandering subspace that could possibly generate the whole space is $\mathcal{E} = \mathrm{Ker}(T^*)$ (but there could be infinitely many wandering subspaces), and an operator with a dense range cannot have a wandering subspace that generates the whole space (since $\mathrm{Ker}(T^*) = \{0\}$).

Definition 2.2.3. *An operator T is called pure if $\bigcap_{i=0}^{\infty} T^i\mathcal{H} = \{0\}$.*

Theorem 2.2.4. *[42, Proposition 2.7] Let T be a left-invertible operator, $\mathcal{E} = \mathcal{H} \ominus T\mathcal{H}$ and $L^* = T(T^*T)^{-1}$ as defined above, then $\mathcal{H} = \bigvee_{i=0}^{\infty} T^i\mathcal{E} \oplus \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H} = \bigvee_{j=0}^{\infty} L^{*j}\mathcal{E} \oplus \bigcap_{i=0}^{\infty} T^i\mathcal{H}$*

Proof Since $L^* = T(T^*T)^{-1}$ and $(T^*T)^{-1}\mathcal{H} = \mathcal{H}$, $T\mathcal{H} = L^*\mathcal{H}$, and therefore $\mathcal{H} \ominus L^*\mathcal{H} = \mathcal{H} \ominus T\mathcal{H} = \mathcal{E}$, which justifies having only one $\mathcal{E}$ in the above equalities. The second equality will follow from the first by interchanging the roles of T and L^* since $T = (L_{L^*})^*$ as shown at the end of the previous section.

Suppose that $x \in (\bigvee_{i=0}^{\infty} T^i\mathcal{E})^{\perp}$. Since $\mathcal{E} = \mathcal{H} \ominus L^*\mathcal{H}$ and $L^*\mathcal{H}$ is a closed subset of $\mathcal{H}$, $\mathcal{H} = L^*\mathcal{H} \oplus \mathcal{E}$. Therefore, there are $x_1 \in L^*\mathcal{H}$ and $x_2 \in \mathcal{E}$ such that $x = x_1 + x_2$. Since $x \perp \bigvee_{i=0}^{\infty} T^i\mathcal{E}$, $x \perp \mathcal{E}$, and since $x_1 \in L^*\mathcal{H} \perp \mathcal{E}$ it must be that also $x_2 \perp \mathcal{E}$ but since $x_2 \in \mathcal{E}$, $x_2 = 0$, and $x \in L^*\mathcal{H}$. Applying L^* to both sides of $\mathcal{H} = L^*\mathcal{H} \oplus \mathcal{E}$ we obtain $L^*\mathcal{H} = L^{*2}\mathcal{H} \dot{+} L^*\mathcal{E}$ because L^* is one-to-one, where $Y = A \dot{+} B$ means that for every $y \in Y$ there is a unique pair $\{a, b\} \in A \times B$ such that $y = a + b$ (this is equivalent to $Y = A + B$ and $A \cap B = \{0\}$). So, again there are $x_1 \in L^{*2}\mathcal{H}$ and $x_2 \in L^*\mathcal{E}$ such that $x = x_1 + x_2$. Since $(T\varepsilon, L^{*2}y) = (LT\varepsilon, L^*y) = (\varepsilon, L^*y) = 0\ \forall\{\varepsilon, y\} \in \mathcal{E} \times \mathcal{H}$, $L^{*2}\mathcal{H} \perp T\mathcal{E}$ and $x_1 \perp T\mathcal{E}$. Since $x \perp \bigvee_{i=0}^{\infty} T^i\mathcal{E}$, $x \perp T\mathcal{E}$, and it must be that $x_2 \perp T\mathcal{E}$. Since $x_2 \in L^*\mathcal{E}$, $\exists \varepsilon \in \mathcal{E}$ such that $x_2 = L^*\varepsilon$ and $(T\varepsilon, x_2) = (T\varepsilon, L^*\varepsilon) = (LT\varepsilon, \varepsilon) = (\varepsilon, \varepsilon) = \|\varepsilon\|^2 = 0$ since $x_2 \perp T\mathcal{E}$, so again $x_2 = 0$ and $x \in L^{*2}\mathcal{H}$. We can continue this inductively by applying L^{*j} to $\mathcal{H} = L^*\mathcal{H} \oplus \mathcal{E}$ to obtain $L^{*j}\mathcal{H} = L^{*(j+1)}\mathcal{H} \dot{+} L^{*j}\mathcal{E}$, then show that $L^{*(j+1)}\mathcal{H} \perp T^j\mathcal{E}$, and finally that $x_2 \in L^{*j}\mathcal{E}$ and $x_2 \perp T^j\mathcal{E}$ implies that $x_2 = 0$, so that $x \in L^{*j}\mathcal{H}\ \forall j \geq 0$ and $x \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$. We have then shown that $(\bigvee_{i=0}^{\infty} T^i\mathcal{E})^{\perp} \subset \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$.

Suppose that $x \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$, then there exists a sequence $\{x_j\}_{j=0}^{\infty}$ such that $x = L^{*j}x_j$. Let $\varepsilon \in \mathcal{E}$, then $(T^i\varepsilon, x) = (T^i\varepsilon, L^{*(i+1)}x_{i+1}) = (L^iT^i\varepsilon, L^*x_{i+1}) = (\varepsilon, L^*x_{i+1}) = 0$ since $L^*\mathcal{H} \perp \mathcal{E}$, and hence $x \perp T^i\mathcal{E}\ \forall i \geq 0$, and since $\bigvee_{i=0}^{\infty} T^i\mathcal{E}$ is the closure of all finite linear combinations of elements in $T^i\mathcal{E}$, it follows that $x \perp \bigvee_{i=0}^{\infty} T^i\mathcal{E}$, so $\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H} \subset (\bigvee_{i=0}^{\infty} T^i\mathcal{E})^{\perp}$, and combining this with the relation from above yields $\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H} = (\bigvee_{i=0}^{\infty} T^i\mathcal{E})^{\perp}$, so that $\mathcal{H} = \bigvee_{i=0}^{\infty} T^i\mathcal{E} \oplus \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$.

Definition 2.2.5. *A left-invertible operator T is said to have the* Wandering Subspace Property *if $\mathcal{H} = \bigvee_{i=0}^{\infty} T^i\mathcal{E}$ for $\mathcal{E} = \mathcal{H} \ominus T\mathcal{H}$.*

From the above theorem we see that an operator T is pure if and only if L^* has the *Wandering Subspace Property*, and T has the *Wandering Subspace Property* if and only if L^* is pure.

2.3 Isometries

Definition 2.3.1. *An operator V is an isometry if $\|Vx\| = \|x\|\ \forall x \in \mathcal{H}$. An operator U is a unitary operator if it is an isometry that is onto $\mathcal{H}$.*

Theorem 2.3.2. *The following are equivalent:*
(i) V is an isometry
(ii) $(Vx, Vy) = (x, y)\ \forall\{x, y\} \in \mathcal{H} \times \mathcal{H}$
(iii) $V^*V = I$
(iv) VV^* is an orthogonal projection onto Im(V) and Ker$(V) = \{0\}$.

Proof (i)$\Rightarrow$(ii): Since V is an isometry, $(V^*Vx, x) = (Vx, Vx) = \|Vx\|^2 = \|x\|^2 = (x, x)$ $\forall x \in \mathcal{H}$, which implies that $V^*V = I$, so that $(Vx, Vy) = (V^*Vx, y) = (x, y)$ $\forall \{x, y\} \in \mathcal{H} \times \mathcal{H}$.
(ii)$\Rightarrow$(iii): Since $(Vx, Vy) = (V^*Vx, y) = (x, y)$ $\forall \{x, y\} \in \mathcal{H} \times \mathcal{H}$, $V^*V = I$.
(iii)$\Rightarrow$(iv): $(VV^*)^2 = VV^*VV^* = V(V^*V)V^* = VV^*$, and since VV^* is self-adjoint, VV^* is an orthogonal projection. If $x \in \mathrm{Im}(V)$, then $x = Vy$ and $VV^*x = VV^*Vy = Vy = x$ since $V^*V = I$, and if $x \perp \mathrm{Im}(V)$ then $x \in \mathrm{Ker}(V^*)$ and $VV^*x = 0$, so VV^* is the orthogonal projection onto $\mathrm{Im}(V)$. Also, $V^*V = I$ implies that $\mathrm{Ker}(V) = \{0\}$.
(iv)$\Rightarrow$(i): If $x \in \mathrm{Ker}(VV^*)$ then $\|V^*x\|^2 = (V^*x, V^*x) = (VV^*x, x) = 0$ so that $x \in \mathrm{Ker}(V^*)$. If $x \perp \mathrm{Ker}(VV^*)$, then $\|V^*x\|^2 = (V^*x, V^*x) = (VV^*x, x) = (x, x) = \|x\|^2$ since VV^* is an orthogonal projection onto $(\mathrm{Ker}(VV^*))^\perp$, so $\mathrm{Ker}(V^*) = \mathrm{Ker}(VV^*)$ and V^* is an isometry on $(\mathrm{Ker}(VV^*))^\perp = (\mathrm{Ker}(V^*))^\perp = \mathrm{Im}(V)$ since V^* has a closed image and hence so does V (see Theorem 4.4.1 in [7]) and therefore VV^* is the identity on $\mathrm{Im}(V)$. Therefore, $(V^*V)^2 = V^*VV^*V = V^*(VV^*)V = V^*V$ and V^*V is an orthogonal projection onto $(\mathrm{Ker}(V^*V))^\perp = (\mathrm{Ker}(V))^\perp = \mathcal{H}$, so that $\|Vx\|^2 = (Vx, Vx) = (V^*Vx, x) = (x, x) = \|x\|^2$, and V is an isometry.

We shall need a more general version of *(ii)* of Theorem 2.3.2:

Proposition 2.3.3. *Suppose that span$\{e_1, e_2, e_3, \ldots\}$ is dense in $\mathcal{H}$ and $\{f_i\}_{i=1}^\infty$ is a collection of vectors in $\mathcal{H}$ such that $(f_i, f_j) = (e_i, e_j)$ $\forall i, j \geq 1$. Then the formula $Ve_i = f_i$ $\forall i \geq 1$ extends by linearity and continuity to define a uniquely determined isometry (also called V) from $\mathcal{H}$ into $\mathcal{H}$.*

Proof Define V on the dense set $\mathrm{span}\{e_1, e_2, e_3, \ldots\}$ by $V\left(\sum_{i=1}^n c_i e_i\right) = \sum_{i=1}^n c_i f_i$, where V is well defined because

$$\begin{aligned}\|\sum_{i=1}^n c_i e_i\|^2 &= (\sum_{i=1}^n c_i e_i, \sum_{i=1}^n c_i e_i) = \sum_{i,j=1}^n (c_i e_i, c_j e_j) = \sum_{i,j=1}^n (c_i f_i, c_j f_j) \\ &= (\sum_{i=1}^n c_i f_i, \sum_{i=1}^n c_i f_i) = \|\sum_{i=1}^n c_i f_i\|^2\end{aligned}$$

so that $\sum_{i=1}^n c_i e_i = 0 \Rightarrow \sum_{i=1}^n c_i f_i = 0$. Since $\|\sum_{i=1}^n c_i e_i\|^2 = \|\sum_{i=1}^n c_i f_i\|^2$ and $V\left(\sum_{i=1}^n c_i e_i\right) = \sum_{i=1}^n c_i f_i$, V satisfies $\|Vx\| = \|x\|$ on a dense set. If we extend V by continuity to the whole space, then this extension is unique, and it can be seen that $\|Vx\| = \|x\|$ will hold for all $x \in \mathcal{H}$, so that V is an isometry.

Theorem 2.3.4. *The von Neumann-Wold Decomposition[24][14, Theorem 23.7][40, Pages 15-16][19, Section 1][20, Theorems 1-3]*

Let V be an isometry, then $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where both $\mathcal{H}_0$ and $\mathcal{H}_1$ are reducing for V, V restricted to $\mathcal{H}_0$ is unitary, and $\mathcal{H}_1 = [\mathcal{E}]_V$. Furthermore, $\mathcal{H}_0 = \bigcap_{i=0}^{\infty} V^i\mathcal{H}$ and every subspace on which V is unitary is contained in $\mathcal{H}_0$.

Proof Let V be an isometry, then V is left-invertible with left-inverse $(V^*V)^{-1}V^* = (I)^{-1}V^* = V^*$. As was shown in Proposition 2.5.5, any left-invertible operator T restricted to $\bigcap_{i=0}^{\infty} T^i\mathcal{H}$ is invertible with inverse L, so V is invertible on $\mathcal{H}_0$ with inverse $L = V^*$. Therefore V is onto $\mathcal{H}_0$ so it is unitary there, and V and V^* map $\mathcal{H}_0$ onto $\mathcal{H}_0$, so $\mathcal{H}_0$ and hence $(\mathcal{H}_0)^{\perp} = \mathcal{H}_1$ are reducing for V. Suppose that V is unitary on $\mathcal{H}_2$, then $V\mathcal{H}_2 = \mathcal{H}_2$ so that $V^i\mathcal{H}_2 = \mathcal{H}_2$, and hence $\mathcal{H}_2 = \bigcap_{i=0}^{\infty} V^i\mathcal{H}_2 \subset \bigcap_{i=0}^{\infty} V^i\mathcal{H} \subset \mathcal{H}_0$, so that every subspace on which V is unitary is contained in $\mathcal{H}_0$. From the decomposition given in Theorem 2.2.4, $\mathcal{H}_1 = (\mathcal{H}_0)^{\perp} = (\bigcap_{i=0}^{\infty} V^i\mathcal{H})^{\perp} = \bigvee_{j=0}^{\infty} L^{*j}\mathcal{E}$, but since $L = V^*$, $L^* = V$, and hence $\mathcal{H}_1 = \bigvee_{j=0}^{\infty} V^j\mathcal{E} = [\mathcal{E}]_V$.

The proof above uses previous results for left-invertible operators, and since an isometry has much more structure than an arbitrary left-invertible operator, it will be instructive to give an alternative proof of the second part of the above proof that depends only on the structure of isometries.

Definition 2.3.5. *An operator S is a shift (also called a unilateral shift or forward shift) of multiplicity α if there exists a subspace $\mathcal{H}_1$ with $dim(\mathcal{H}_1) = \alpha$ and a sequence of subspaces $\{\mathcal{H}_i\}_{i=2}^{\infty}$ such that the $\mathcal{H}_i$ are pairwise orthogonal ($\mathcal{H}_i \perp \mathcal{H}_j\ \forall i \neq j$), $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \dots$ and S maps $\mathcal{H}_i$ isometrically onto $\mathcal{H}_{i+1}$ for all $i \geq 1$.*

Note that because the $\mathcal{H}_i$'s are mapped isometrically onto each other, it follows that they all must have the same dimension.

Theorem 2.3.6. *See [24]. Let S be a pure isometry. Then S is a shift of multiplicity α where α is given by $dim(Ker(S^*)) = dim(\mathcal{E})$ with $\mathcal{H}_1 = \mathcal{E}$ and $\mathcal{H}_i = S^{i-1}\mathcal{E}\ \forall i \geq 1$.*

Proof Let $\mathcal{H}_1 = \mathcal{E} = \mathcal{H} \ominus S\mathcal{H}$ and $\mathcal{H}_i = S^{i-1}\mathcal{E}\ \forall i \geq 2$, then S maps $\mathcal{H}_i$ isometrically onto $\mathcal{H}_{i+1}$ for all $i \geq 1$ by the definition of the spaces, and $\mathcal{H}_1 \perp S^i\mathcal{H}_1\ \forall i \geq 1$ (it is a wandering subspace). Therefore, $(\varepsilon_1, S^i\varepsilon_2) = 0\ \forall i \geq 1, \{\varepsilon_1, \varepsilon_2\} \in \mathcal{E} \times \mathcal{E}$, and since $(Sx, Sy) = (x, y)$, and hence $(S^jx, S^jy) = (x, y)\ \forall j \geq 0$, it follows that $(S^j\varepsilon_1, S^jS^i\varepsilon_2) = 0\ \forall i \geq 1, j \geq 0, \{\varepsilon_1, \varepsilon_2\} \in \mathcal{E} \times \mathcal{E}$ so that $S^i\mathcal{E} \perp S^j\mathcal{E}\ \forall i \neq j$ and hence $\mathcal{H}_i \perp \mathcal{H}_j\ \forall i \neq j$. Similar to the preceding computation, if $\mathcal{M}_1 \perp \mathcal{M}_2$, then $S^j\mathcal{M}_1 \perp S^j\mathcal{M}_2\ \forall j \geq 0$. Since $\mathcal{H} = \mathcal{E} \oplus S\mathcal{H}$, applying S to both sides yields $S\mathcal{H} = S\mathcal{E} \oplus S^2\mathcal{H}$, which when plugged into the equation before yields $\mathcal{H} = \mathcal{E} \oplus S\mathcal{E} \oplus S^2\mathcal{H}$. Applying S^2 to this yields $S^2\mathcal{H} = S^2\mathcal{E} \oplus S^3\mathcal{E} \oplus S^4\mathcal{H}$, which when plugged into the equation before yields $\mathcal{H} = \mathcal{E} \oplus S\mathcal{E} \oplus S^2\mathcal{E} \oplus S^3\mathcal{E} \oplus S^4\mathcal{H}$. Continuing in this manner it can be shown inductively that $\mathcal{H} = \mathcal{E} \oplus S\mathcal{E} \oplus S^2\mathcal{E} \oplus S^3\mathcal{E} \oplus \dots \oplus S^{i-1}\mathcal{E} \oplus S^i\mathcal{H}\ \forall i \geq 1$. Therefore if $x \in \mathcal{H}$ is perpendicular to $\mathcal{H}_i = S^{i-1}\mathcal{E}\ \forall i \geq 1$, then $x \in S^i\mathcal{H}\ \forall i \geq 0$, so $x \in \bigcap_{i=0}^{\infty} S^i\mathcal{H} = \{0\}$ since S is pure, so $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \ldots = \mathcal{E} \oplus S\mathcal{E} \oplus S^2\mathcal{E} \oplus S^3\mathcal{E} \oplus \ldots$, and S is a shift of multiplicity $\alpha = \dim(\mathcal{H}_1) = \dim(\mathcal{E})$.

Note that since in the definition of a shift $\mathcal{H}_i \subset S\mathcal{H}\ \forall i \geq 2$, it must be that $\mathcal{E} \subset \mathcal{H}_1$, and since $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \ldots$, $S\mathcal{H} = S\mathcal{H}_1 \oplus S\mathcal{H}_2 \oplus S\mathcal{H}_3 \oplus \ldots = \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \oplus \ldots$, so that $\mathcal{H}_1 \subset \mathcal{E}$. Therefore for any shift it must be that $\mathcal{H}_i = S^{i-1}\mathcal{E}\ \forall i \geq 1$.

Definition 2.3.7. *An operator S and an operator T are said to be unitarily equivalent if there exists a unitary operator U such that $S = UTU^{-1}$.*

Proposition 2.3.8. *A shift S_1 and a shift S_2 are unitarily equivalent if and only if they have the same multiplicity.*

Proof Suppose that the shifts S_1 and S_2 are unitarily equivalent and U is such that $S_2 = US_1U^{-1}$. Let $e \in \mathcal{H} \ominus S_1\mathcal{H}$, then $(Ue, S_2x) = (S_2^*Ue, x) = (U^{-*}S_1^*U^*Ue, x) = (US_1^*e, x) = (e, S_1U^*x) = 0\ \forall x \in \mathcal{H}$, so that $Ue \in \mathcal{H}\ominus S_2\mathcal{H}$ and $U(\mathcal{H}\ominus S_1\mathcal{H}) \subset (\mathcal{H}\ominus S_2\mathcal{H})$. Since $S_1 = U^{-1}S_2U$ and U^{-1} is also a unitary operator, reversing the roles of S_1 and S_2 yields $(\mathcal{H}\ominus S_2\mathcal{H}) \subset U(\mathcal{H}\ominus S_1\mathcal{H})$ and $U(\mathcal{H}\ominus S_1\mathcal{H}) = (\mathcal{H}\ominus S_2\mathcal{H})$. Since U is invertible it must be that $\dim(\mathcal{H} \ominus S_1\mathcal{H}) = \dim(U(\mathcal{H} \ominus S_1\mathcal{H})) = \dim(\mathcal{H} \ominus S_2\mathcal{H})$ and S_1 and S_2 have the same multiplicity.

Conversely, suppose that S_1 and S_2 have the same multiplicity equal to α. Let $\{e_{1i}\}_{i=1}^{\alpha}$ be an orthonormal set in $\mathcal{H} \ominus S_1\mathcal{H}$ and $\{e_{2i}\}_{i=1}^{\alpha}$ an orthonormal set in $\mathcal{H} \ominus S_2\mathcal{H}$, then defining $US_1^je_{1i} = S_2^je_{2i}\ \forall j \geq 0, 1 \leq i \leq \alpha$, U is a unitary operator and $US_1U^{-1}S_2^je_{2i} = US_1S_1^je_{1i} = US_1^{(j+1)}e_{1i} = S_2^{(j+1)}e_{2i} = S_2S_2^je_{2i}\ \forall j \geq 0, 1 \leq i \leq \alpha$, and since the $S_2^je_{2i}\ \forall j \geq 0, 1 \leq i \leq \alpha$ span a dense set, $US_1U^{-1} = S_2$ and S_1 and S_2 are unitarily equivalent.

Theorem 2.3.9. *[14, Proposition 23.15][19, Page 17][24, Lemma4] Let S be a shift of multiplicity α, then for every closed, invariant subspace $\mathcal{M}$ of S, S restricted to $\mathcal{M}$ is a shift of multiplicity $\beta \leq \alpha$.*

2.4 Set-Theoretic Conditions for the *Wandering Subspace Property*

Theorem 2.4.1. *Let T be left-invertible with $\mathcal{E} = \mathcal{H} \ominus T\mathcal{H}$. The following are equivalent ($\mathcal{M}$, $\mathcal{A}$, and $\mathcal{B}$ are closed subspaces):*
(i) $\bigvee_{i=0}^{\infty} T^i\mathcal{E} \neq \mathcal{H}$
(ii) $\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H} \neq \{0\}$

(iii) $\exists \mathcal{M} \neq \{0\}$ *such that* $L^*\mathcal{M} \supset \mathcal{M}$
(iv) $\exists \mathcal{M} \subset T\mathcal{H}$, $\mathcal{M} \neq \{0\}$, *such that* $T^*\mathcal{M} \subset \mathcal{M}$
(v) $\exists \mathcal{M} \supset \mathcal{E}$, $\mathcal{M} \neq \mathcal{H}$, *such that* $T\mathcal{M} \subset \mathcal{M}$
(vi) $\exists \mathcal{M} \supset \mathcal{E}$, $\mathcal{M} \neq \mathcal{H}$, *such that* $L\mathcal{M} \supset \mathcal{M}$
(vii) $\exists \mathcal{A}, \mathcal{B} \neq \{0\}$ *such that* $T\mathcal{A} \subset \mathcal{A}$, $L\mathcal{A} \subset \mathcal{A} \oplus \mathcal{E}$, $P_{\mathcal{B}}T\mathcal{B} = \mathcal{B}$ *and* $T\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$
($P_{\mathcal{B}}$ is the orthogonal projection onto $\mathcal{B}$)

Proof The equivalence of (i) and (ii) is the statement of Theorem 2.2.4.

(ii)⇒(iii): Let $\mathcal{M} = \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$, then $\mathcal{M} \neq \{0\}$. Let $x \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$, then there exists a sequence $\{x_j\}_{j=0}^{\infty}$ such that $x = L^{*j}x_j$, and since $T^*L^* = I$, $y = T^*x = L^{*(j-1)}x_j$ $\forall j \geq 1$, so that $y \in L^{*j}\mathcal{H}$ $\forall j \geq 0$ and $y \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$. Since $L^*y = L^*T^*x = x$, we have that $x \in L^* \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$. Since x was arbitrary, $L^*\mathcal{M} \supset \mathcal{M}$.

(ii)⇒(iv): Again, let $\mathcal{M} = \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$, then $\mathcal{M} \neq \{0\}$ and $\mathcal{M} \subset L^*\mathcal{H} = T\mathcal{H}$. Let $x \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$. As was shown above, $y = T^*x \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$, so that $T^*\mathcal{M} \subset \mathcal{M}$. The condition $\mathcal{M} \subset T\mathcal{H}$ is necessary because otherwise one could choose $\mathcal{M} = \mathcal{E}$, and since $\mathcal{E} = \mathrm{Ker}(T^*)$, $T^*\mathcal{M} = \{0\} \subset \mathcal{M}$, so the condition would be satisfied for any left-invertible operator which is not invertible.

(i)⇒(v): Let $\mathcal{M} = \bigvee_{i=0}^{\infty} T^i\mathcal{E}$, then $\mathcal{E} \subset \mathcal{M} \neq \mathcal{H}$, and $T\mathcal{M} = T\bigvee_{i=0}^{\infty} T^i\mathcal{E} \subset \bigvee_{i=1}^{\infty} T^i\mathcal{E} \subset \bigvee_{i=0}^{\infty} T^i\mathcal{E} = \mathcal{M}$, where the first inclusion follows because T is continuous. The condition $\mathcal{E} \subset \mathcal{M}$ is necessary because for any left-invertible operator which is not invertible, $T\mathcal{H} \neq \mathcal{H}$ and $TT\mathcal{H} = T^2\mathcal{H} \subset T\mathcal{H}$.

(i)⇒(vi): Again, let $\mathcal{M} = \bigvee_{i=0}^{\infty} T^i\mathcal{E}$, then $\mathcal{E} \subset \mathcal{M} \neq \mathcal{H}$. Let $x \in \bigvee_{i=0}^{\infty} T^i\mathcal{E}$, then as above $y = Tx \in \bigvee_{i=0}^{\infty} T^i\mathcal{E}$, so that $Ly = LTx = x \in L\bigvee_{i=0}^{\infty} T^i\mathcal{E}$, and since x was arbitrary, $L\mathcal{M} \supset \mathcal{M}$. As above the condition $\mathcal{E} \subset \mathcal{M}$ is necessary because $T\mathcal{H} \subset \mathcal{H} = LT\mathcal{H}$.

(i) and (ii)⇒(vii): Let $\mathcal{A} = \bigvee_{i=1}^{\infty} T^i\mathcal{E}$ and $\mathcal{B} = \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$, then $\mathcal{A} \perp \mathcal{B}$ by Theorem 2.2.4, and since $\mathrm{Im}(T)$ is closed $\mathrm{Im}(T) = \mathcal{E}^{\perp} = \left(\bigvee_{i=0}^{\infty} T^i\mathcal{E} \oplus \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}\right) \ominus \mathcal{E} = \bigvee_{i=1}^{\infty} T^i\mathcal{E} \oplus \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$ by Theorem 2.2.4 and the conditions $\mathcal{E} \perp T^i\mathcal{E}$ $\forall i \geq 1$ and $\mathcal{E} \perp L^*\mathcal{H} = T\mathcal{H}$ (so that $T\mathcal{H} = \mathcal{A} \oplus \mathcal{B}$). By (i) and (ii) $\mathcal{A}, \mathcal{B} \neq \{0\}$. $T\mathcal{A} = T\bigvee_{i=1}^{\infty} T^i\mathcal{E} \subset \bigvee_{i=2}^{\infty} T^i\mathcal{E} \subset \bigvee_{i=1}^{\infty} T^i\mathcal{E} = \mathcal{A}$ where the first inclusion follows because T is continuous. Since $\mathcal{E} \perp T^i\mathcal{E}$ $\forall i \geq 1$, $\mathcal{A} \oplus \mathcal{E} = \bigvee_{i=1}^{\infty} T^i\mathcal{E} \oplus \mathcal{E} = \bigvee_{i=0}^{\infty} T^i\mathcal{E}$, so that by Theorem 2.2.4, $(\mathcal{A} \oplus \mathcal{E})^{\perp} = \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$. If $x \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$, then $x \in L^{*i}\mathcal{H}$ $\forall i \geq 0$ so that $L^*x \in L^{*i}\mathcal{H}$ $\forall i \geq 1$, and $L^*x \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$, so that $L^*\left(\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}\right) \subset \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$. This implies that $L\left(\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}\right)^{\perp} \subset \left(\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}\right)^{\perp}$ or that $L(\mathcal{A}\oplus\mathcal{E}) \subset \mathcal{A} \oplus \mathcal{E}$, which implies that $L\mathcal{A} \subset \mathcal{A} \oplus \mathcal{E}$.

Note that we used the fact that for a closed subspace $\mathcal{M}$ and an operator T, $T\mathcal{M} \subset \mathcal{M}$ if and only if $T^*\mathcal{M}^\perp \subset \mathcal{M}^\perp$ (see Proposition 4.42 in [16]).

Since $P_\mathcal{B}$ maps into $\mathcal{B}$, $P_\mathcal{B}T\mathcal{B} \subset \mathcal{B}$. Let $x \in \mathcal{B} = \bigcap_{j=0}^\infty L^{*j}\mathcal{H} \subset L^*\mathcal{H} = T\mathcal{H}$, then there is a $y \in \mathcal{H}$ such that $x = Ty$, where $y = y_1 + y_2$ with $y_1 \in \bigvee_{i=0}^\infty T^i\mathcal{E}$ and $y_2 \in \bigcap_{j=0}^\infty L^{*j}\mathcal{H} = \mathcal{B}$. Since $Ty_1 \in \bigvee_{i=0}^\infty T^i\mathcal{E} \perp \mathcal{B}$, $x = P_\mathcal{B}x = P_\mathcal{B}T(y_1+y_2) = P_\mathcal{B}Ty_2$. As $y_2 \in \mathcal{B}$, $x \in P_\mathcal{B}T\mathcal{B}$, and since $x \in \mathcal{B}$ was arbitrary, it follows that $P_\mathcal{B}T\mathcal{B} = \mathcal{B}$.

(iii)⇒(ii): Applying L^{*j} to the relation $\mathcal{M} \subset L^*\mathcal{M}$ we obtain $L^{*j}\mathcal{M} \subset L^{*(j+1)}\mathcal{M}$ for all j greater than or equal to zero, which when strung together implies that $\mathcal{M} \subset L^{*j}\mathcal{M}$ $\forall j \geq 0$ so that $\mathcal{M} \subset \bigcap_{j=0}^\infty L^{*j}\mathcal{M} \subset \bigcap_{j=0}^\infty L^{*j}\mathcal{H}$ and since $\mathcal{M} \neq \{0\}$, it must be that $\bigcap_{j=0}^\infty L^{*j}\mathcal{H} \neq \{0\}$.

(iv)⇒(iii): If $x \in T\mathcal{H}$, then $x = Ty$ for some $y \in \mathcal{H}$ so that $TLx = TLTy = Ty = x$, and if $x \perp \mathrm{Im}(T)$ then $x \in \mathrm{Ker}(T^*) = \mathrm{Ker}(L)$ so that $TLx = 0$, and TL must be the orthogonal projection onto $\mathrm{Im}(T)$. Since orthogonal projections are self-adjoint, $(TL)^* = L^*T^*$ is also the orthogonal projection onto $\mathrm{Im}(T)$, so if $x \in T\mathcal{H}$, then $L^*T^*x = x$. Since $\mathcal{M} \subset T\mathcal{H}$, $L^*T^*\mathcal{M} = \mathcal{M}$ and applying L^* to the relation $T^*\mathcal{M} \subset \mathcal{M}$ we obtain $L^*T^*\mathcal{M} = \mathcal{M} \subset L^*\mathcal{M}$, where by (iv) we assumed that $\mathcal{M} \neq \{0\}$.

(v)⇒(iv): Since $\mathcal{E} \subset \mathcal{M} \neq \mathcal{H}$, $\{0\} \neq \mathcal{M}^\perp \subset T\mathcal{H}$, and $T\mathcal{M} \subset \mathcal{M}$ implies $T^*\mathcal{M}^\perp \subset \mathcal{M}^\perp$, so (iv) is satisfied with the closed subspace $\mathcal{M}^\perp$.

(vi)⇒(iii): Since $\mathcal{E} \subset \mathcal{M} \neq \mathcal{H}$, $\{0\} \neq \mathcal{M}^\perp \subset T\mathcal{H}$. Let $x \in \mathcal{M}^\perp \subset T\mathcal{H} = L^*\mathcal{H}$, then there exists a $y \in \mathcal{H}$ such that $x = L^*y$ and $y = m_1 + m_2$ where $m_1 \in \mathcal{M}$ and $m_2 \in \mathcal{M}^\perp$. Suppose that $m_1 \neq 0$, then $(x, m) = (L^*y, m) = (L^*(m_1 + m_2), m) = 0$ $\forall m \in \mathcal{M}$ since $x \in \mathcal{M}^\perp$, which implies that $(m_1 + m_2, Lm) = 0$ $\forall m \in \mathcal{M}$ so that $m_1 + m_2 \in (L\mathcal{M})^\perp \subset \mathcal{M}^\perp$ because $\mathcal{M} \subset L\mathcal{M}$, but $(m_1 + m_2, m_1) = (m_1, m_1) = \|m_1\|^2 \neq 0$, which contradicts the fact that $m_1 + m_2 \in \mathcal{M}^\perp$, so it must be that $m_1 = 0$, and $x = L^*m_2$, and since x was arbitrary, $\mathcal{M}^\perp \subset L^*\mathcal{M}^\perp$, so (iii) is satisfied with the closed subspace $\mathcal{M}^\perp$.

(vii)⇒(v): Let $\varepsilon \in \mathcal{E}$, then $T\varepsilon \in T\mathcal{H}$ so there exists a pair $\{a, b\} \in \mathcal{A} \times \mathcal{B}$ such that $T\varepsilon = a + b$. Since $P_\mathcal{B}T\mathcal{B} = \mathcal{B}$, $\exists b_1 \in \mathcal{B}$ such that $P_\mathcal{B}Tb_1 = b$, or $Tb_1 = a_2 + b$ where $a_2 \in \mathcal{A}$. Let $a_1 = a - a_2 \in \mathcal{A}$, then $T\varepsilon = a + b = a + Tb_1 - a_2 = a_1 + Tb_1$, and $\varepsilon = La_1 + b_1$. Since $La_1 \in L\mathcal{A} \subset \mathcal{A} \oplus \mathcal{E} = \mathcal{B}^\perp$ and $\varepsilon \in \mathcal{E} \subset \mathcal{B}^\perp$, it must be that $b_1 \in \mathcal{B}^\perp$, which implies that $b_1 = 0$ and $P_\mathcal{B}Tb_1 = b = 0$, so $T\varepsilon = a \in \mathcal{A}$, and since ε was arbitrary, $T\mathcal{E} \subset \mathcal{A}$. Since also $T\mathcal{A} \subset \mathcal{A}$, $T(\mathcal{A} \oplus \mathcal{E}) \subset \mathcal{A} \subset \mathcal{A} \oplus \mathcal{E}$, so if $\mathcal{M} = \mathcal{A} \oplus \mathcal{E}$, then $T\mathcal{M} \subset \mathcal{M}$ and $\mathcal{E} \subset \mathcal{M} \neq \mathcal{H}$ since $\mathcal{M} \perp \mathcal{B} \neq \{0\}$. ∎

2.5 Conditions for the *Wandering Subspace Property* Based on the Norm

Theorem 2.5.1. *Let T be a left-invertible operator with $\mathcal{E} = \mathcal{H} \ominus T\mathcal{H}$. The following are equivalent:*
(i) $\bigvee_{i=0}^{\infty} T^i\mathcal{E} = \mathcal{H}$
(ii) $\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H} = \{0\}$
(iii) $X_1 = \{x \in \mathcal{H} : \exists i$ *such that* $L^i x = 0\} = \bigcup_{i=0}^{\infty} KerL^i$ *is dense in* $\mathcal{H}$
(iv) $X_2 = \{x \in \mathcal{H} : \limsup_{i\to\infty} \|T^iL^ix\| = 0\}$ *is dense in* $\mathcal{H}$
(v) $X_3 = \{x \in \mathcal{H} : \liminf_{i\to\infty} \|T^iL^ix\| = 0\}$ *is dense in* $\mathcal{H}$
(vi) $\exists c < 1$ *such that* $X_c = \{x \in \mathcal{H} : \liminf_{i\to\infty} \|T^iL^ix\| \le c\|x\|\}$ *is dense in* $\mathcal{H}$.

Proof The equivalence of (i) and (ii) is the statement of Theorem 2.2.4.

(i)⇒(iii): Since $\mathrm{Ker}(L) = \mathcal{E}$ and $LT = I$, $\mathrm{Ker}(L^i) = \mathrm{span}\{\mathcal{E}, T\mathcal{E}, T^2\mathcal{E}, ..., T^{(i-1)}\mathcal{E}\}$, so that $X_1 = \cup_{i=1}^{\infty}\mathrm{Ker}(L^i)$ is dense in $\mathcal{H}$ since $\mathcal{H} = \bigvee_{i=0}^{\infty} T^i\mathcal{E}$ is the closure of finite linear combinations of elements in $T^i\mathcal{E}$, $i \in \mathbb{N}$

(iii)⇒(iv): If $x \in X_1$, then there is an i such that $L^ix = 0$ so that $\|T^jL^jx\| = 0 \ \forall j \ge i$, and $\lim_{i\to\infty} \|T^iL^ix\| = 0$ and hence $\limsup_{i\to\infty} \|T^iL^ix\| = 0$ so that $x \in X_2$. This implies that $X_1 \subset X_2$, and since X_1 was assumed dense, so is X_2.

(iv)⇒(v): If $x \in X_2$, then $\limsup_{i\to\infty} \|T^iL^ix\| = 0$, and since $\|T^iL^ix\| \ge 0 \ \forall i$ and $\liminf_{i\to\infty} \|T^iL^ix\| \le \limsup_{i\to\infty} \|T^iL^ix\|$, it must be that $\liminf_{i\to\infty} \|T^iL^ix\| = 0$, so that $x \in X_3$. This implies that $X_2 \subset X_3$, and since X_2 was assumed dense, so is X_3.

(v)⇒(vi): If $x \in X_3$, then $\liminf_{i\to\infty} \|T^iL^ix\| = 0 < c$, so $x \in X_c$, $0 < c < 1$. This implies that $X_3 \subset X_c$, and since X_3 was assumed dense, so is X_c, $0 < c < 1$.

(vi)⇒(ii) (Proof by contradiction): Let $x \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H} \ne \{0\}$, then there exists a sequence $\{x_j\}_{j=0}^{\infty}$ such that $x = L^{*j}x_j$, and hence $L^{*i}T^{*i}x = L^{*i}T^{*i}L^{*i}x_i = L^{*i}x_i = x$, so that $L^{*i}T^{*i}$ is the identity on $\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$ for every i, and $(T^iL^ix - x, y) = (x, L^{*i}T^{*i}y) - (x, y) = (x, y) - (x, y) = 0 \ \forall y \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$, so that $P_{\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}} T^iL^ix = x$. Since x was arbitrary, $\forall x \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H} \ \forall i \ge 0 \ T^iL^ix = x + z$ where $z \in \left(\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}\right)^{\perp} = \bigvee_{i=0}^{\infty} T^i\mathcal{E}$. Suppose that for some $0 < c < 1$, X_c is dense and pick an arbitrary $l \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$ with $\|l\| = 1$. Let $x \in X_c$ be such that $\|l - x\| \le \frac{1-c}{2+2c}$, then $\|x\| \le \frac{3+c}{2+2c}$. Write $x = y + w$ where $y \in \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$ and $w \in \bigvee_{i=0}^{\infty} T^i\mathcal{E}$, then $\|l - x\|^2 = \|l - y - w\|^2 = \|l - y\|^2 + \|w\|^2$, so that $\|l - y\| \le \frac{1-c}{2+2c}$ and $\|y\| \ge \frac{1+3c}{2+2c}$. Since from above $P_{\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}} T^iL^iy = y \ \forall i \ge 0$, $\exists \{w_i\}_{i=1}^{\infty}$ with $w_i \in \bigvee_{j=0}^{\infty} T^j\mathcal{E} \ \forall i \ge 1$ such that $T^iL^iy = y + w_i$. Since $L^*\left(\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}\right) \subset \bigcap_{j=1}^{\infty} L^{*j}\mathcal{H} \subset \bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}$, we have that $L\bigvee_{i=0}^{\infty} T^i\mathcal{E} \subset \bigvee_{i=0}^{\infty} T^i\mathcal{E}$ since $\left(\bigcap_{j=0}^{\infty} L^{*j}\mathcal{H}\right)^{\perp} = \bigvee_{i=0}^{\infty} T^i\mathcal{E}$. Since also $T\bigvee_{i=0}^{\infty} T^i\mathcal{E} \subset \bigvee_{i=0}^{\infty} T^i\mathcal{E}$, $T^iL^iw \in \bigvee_{i=0}^{\infty} T^i\mathcal{E}$ $\forall i \geq 0$. Therefore $\|T^iL^ix\|^2 = \|y + w_i + T^iL^iw\|^2 = \|y\|^2 + \|w_i + T^iL^iw\|^2 \geq \|y\|^2 \geq \frac{(1+3c)^2}{(2+2c)^2} > c^2\frac{(3+c)^2}{(2+2c)^2} \geq c^2\|x\|^2$, which contradicts the fact that $\liminf_{i\to\infty} \|T^iL^ix\| \leq c\|x\|$, so X_c is not dense for any c such that $0 < c < 1$. ∎

Note: There is another way to see (v)⇒(i) by decomposing the space based on the action of the powers of T on $\mathcal{E} = \mathcal{H}\ominus T\mathcal{H}$. By definition, $\mathcal{H} = \mathcal{E}\oplus T\mathcal{H}$. Applying T to this yields $T\mathcal{H} = T\mathcal{E}\dot{+}T^2\mathcal{H}$, where $T\mathcal{E}\cap T^2\mathcal{H} = \{0\}$ because $\mathcal{E}\cap T\mathcal{H} = \{0\}$ and T is one-to-one. Plugging this into $\mathcal{H} = \mathcal{E}\oplus T\mathcal{H}$ yields $\mathcal{H} = \mathcal{E}\oplus\left(T\mathcal{E}\dot{+}T^2\mathcal{H}\right)$. Applying T to this yields $T\mathcal{H} = T\mathcal{E}\dot{+}T^2\mathcal{E}\dot{+}T^3\mathcal{H}$ so that $\mathcal{H} = \mathcal{E}\oplus\left(T\mathcal{E}\dot{+}T^2\mathcal{E}\dot{+}T^3\mathcal{H}\right)$. Continuing in this way we obtain

$$\mathcal{H} = \mathcal{E}\oplus\left(T\mathcal{E}\dot{+}T^2\mathcal{E}\dot{+}T^3\mathcal{E}\dot{+}\ldots\dot{+}T^{i-1}\mathcal{E}\dot{+}T^i\mathcal{H}\right)\ \ \forall i \geq 1.$$

Due to the definition of L, we also have the facts that $T^jL^jT^i\mathcal{E} = \{0\}$ $\forall j > i$ and $T^jL^j|_{T^i\mathcal{H}} = I$ $\forall j \leq i$. So for any $x \in \mathcal{H}$, T^iL^ix is the element in $T^i\mathcal{H}$ from the decomposition above, and $\|T^iL^ix\| = \|x - y\|$, where y is the rest of the decomposition and is contained in $\bigvee_{j=0}^{i-1} T^j\mathcal{E}$. Therefore $\liminf_{i\to\infty}\|T^iL^ix\| = 0$ implies the existence of $\{i_k\}_{k=1}^{\infty}$ such that $x - T^{i_k}L^{i_k}x \in \bigvee_{j=0}^{\infty} T^j\mathcal{E}$ and $\|x - (x - T^{i_k}L^{i_k}x)\| \to 0$, so that $x \in \bigvee_{j=0}^{\infty} T^j\mathcal{E}$, and $\bigvee_{j=0}^{\infty} T^j\mathcal{E}$ is dense in $\mathcal{H}$ and hence must be all of $\mathcal{H}$.

Included for later reference we state the following proposition, proven above, which can also be seen from equation (2.2) in [42].

Proposition 2.5.2. *For a left-invertible operator T, arbitrary $x \in \mathcal{H}$ and i, $x - T^iL^ix \in \bigvee_{j=0}^{i-1} T^j\mathcal{E}$.*

Lemma 2.5.3. *Let T be an expansive operator ($\|Tx\| \geq \|x\|$ $\forall x \in \mathcal{H}$), then*

$$\|x\|^2 = \sum_{i=0}^{n-1}\|(I - TL)L^ix\|^2 + \|L^nx\|^2 + \sum_{i=1}^{n}\|DL^ix\|^2\ \ \forall x \in \mathcal{H}, \tag{2.1}$$

where D is the positive square root of $(T^*T - I)$.

Proof First note that $((T^*T - I)x, x) = (Tx, Tx) - (x, x) = \|Tx\|^2 - \|x\|^2 \geq 0$ since T is expansive, which shows that $T^*T - I$ is a positive operator so that it has a positive square root. Also, since $\|Dx\|^2 = (Dx, Dx) = (D^2x, x) = ((T^*T - I)x, x)$, it follows that $\|Dx\|^2 = \|Tx\|^2 - \|x\|^2$ $\forall x \in \mathcal{H}$. Plugging L^ix into this equation for x yields

$$\begin{aligned}\|DL^ix\|^2 &= \|TL^ix\|^2 - \|L^ix\|^2 = \|TLL^{i-1}x\|^2 - \|L^ix\|^2\\ &= \|(I - (I - TL))L^{i-1}x\|^2 - \|L^ix\|^2\\ &= \|L^{i-1}x\|^2 - \|(I - TL)L^{i-1}x\|^2 - \|L^ix\|^2\ \ \forall i \geq 1,\end{aligned}$$

since in general $\|(I-(I-TL))x\|^2 = \|x\|^2 - \|(I-TL)x\|^2 \ \forall x \in \mathcal{H}$ because TL is the orthogonal projection onto $\mathrm{Im}(T)$ (see the proof of (iv)$\Rightarrow$(iii) in Theorem 2.4.1) so that $(I-TL)$ is also an orthogonal projection (onto $\mathrm{Ker}(T^*)$). Summing this equality over i from 1 to n yields $\sum_{i=1}^n \|DL^ix\|^2 = \|x\|^2 - \sum_{i=0}^{n-1} \|(I-TL)L^ix\|^2 - \|L^nx\|^2$. Since x was arbitrary, equation (2.1) holds for all x in $\mathcal{H}$.

The following are slight generalizations of Theorem 2.1, Proposition 2.2 and Corollary 2.1, respectively, in [35], which did not incorporate the lower bound of m into the next three results. Note that if T satisfies the hypotheses of these statements, then T restricted to any of its closed, invariant subspaces also satisfies the hypotheses. Therefore if the hypotheses of Theorem 2.5.4 are satisfied, then T restricted to any of its closed, invariant subspaces satisfies the *Wandering Subspace Property.*

Theorem 2.5.4. *Let T be a pure, expansive operator with $\|Tx\| \geq m\|x\|, \ m \geq 1$ such that*

$$\|T^ix\|^2 \leq c_i(\|Tx\|^2 - \|x\|^2) + cm^{2i}\|x\|^2 \ \ \forall x \in \mathcal{H}, \ \ i \geq 2, \tag{2.2}$$

for some positive constants $\{c_i\}_{i=2}^\infty$ and c such that $\sum_{i=2}^\infty \frac{1}{c_i} = \infty$, then $\mathcal{H} = [\mathcal{E}]_T$.

Proof Let $x \in \mathcal{H}$ be arbitrary. Plugging L^ix into equation (2.2) yields

$$\|T^iL^ix\|^2 - cm^{2i}\|L^ix\|^2 \leq c_i(\|TL^ix\|^2 - \|L^ix\|^2) = c_i\|DL^ix\|^2.$$

Therefore,

$$\begin{aligned}\min_{m\leq i\leq n}\{\|T^iL^ix\|^2 - cm^{2i}\|L^ix\|^2\}\sum_{i=m}^n \frac{1}{c_i} &\leq \sum_{i=m}^n \frac{1}{c_i}\{\|T^iL^ix\|^2 - cm^{2i}\|L^ix\|^2\} \\ &\leq \sum_{i=m}^n \|DL^ix\|^2 \leq \|x\|^2,\end{aligned}$$

where the last inequality comes from equation (2.1). Since $\sum_{i=2}^\infty \frac{1}{c_i} = \infty$,

$$\min_{m\leq i\leq n}\{\|T^iL^ix\|^2 - cm^{2i}\|L^ix\|^2\}$$

must shrink to zero as $n \to \infty$, and hence $\liminf_{i\to\infty}\{\|T^iL^ix\|^2 - cm^{2i}\|L^ix\|^2\} \leq 0$. Since $LT = I$, $mL\frac{1}{m}T = I$, and mL is the left-inverse for $\frac{1}{m}T$ with $\mathrm{Ker}(mL) = \mathrm{Ker}(L) = \mathrm{Ker}(T^*) = \mathrm{Ker}(\frac{1}{m}T^*)$. Since $\|Tx\| \geq m\|x\| \ \forall x \in \mathcal{H}$, $\frac{1}{m}T$ is still expansive, and for any expansive operator $\tilde{T}$, $\|\tilde{L}\tilde{T}x\| = \|x\| \leq \|\tilde{T}x\|$ so that $\tilde{L}$ is contractive on $\mathrm{Im}(\tilde{T})$. Since $\tilde{L}$ is zero on the orthogonal complement of $\mathrm{Im}(\tilde{T})$, it follows that $\tilde{L}$ is a contractive operator. Therefore mL is a contractive operator, and since

$$\liminf_{i\to\infty}\{\|T^iL^ix\|^2 - c\|(mL)^ix\|^2\} \leq 0,$$

it must be that $\{T^iL^ix\}_{i=0}^{\infty}$ has a bounded subsequence $\{T^{i_j}L^{i_j}x\}_{j=0}^{\infty}$. By taking a subsubsequence we can assume that the sequence $\{T^{i_j}L^{i_j}x\}_{j=0}^{\infty}$ is weakly convergent to some $y \in \mathcal{H}$. Since $\{T^{i_j}L^{i_j}x\}_{j=k}^{\infty}$ also converges to y, and $T^{i_k}\mathcal{H}$ is weakly closed, it follows that $y \in T^{i_k}\mathcal{H}$ $\forall k \geq 0$, so that, since $i_k \to \infty$, $y \in T^i\mathcal{H}$ $\forall i \geq 0$ and hence $y \in \bigcap_{i=0}^{\infty} T^i\mathcal{H} = \{0\}$. Therefore $x - T^{i_k}L^{i_k}x$ converges weakly to x. Since $x - T^{i_k}L^{i_k}x \in \bigvee_{i=0}^{\infty} T^i\mathcal{E}$ from Proposition 2.5.2 and $\bigvee_{i=0}^{\infty} T^i\mathcal{E}$ is weakly sequentially closed, $x \in \bigvee_{i=0}^{\infty} T^i\mathcal{E} = [\mathcal{E}]_T$. As x was arbitrary, $\mathcal{H} = [\mathcal{E}]_T$. $\square$

Proposition 2.5.5. *Let T satisfy $\|Tx\| \geq m\|x\|$ $\forall x \in \mathcal{H}$ for some $m \geq 1$ and be such that equation (2.2) holds with $c = 1$ and $\sum_{i=2}^{\infty} \frac{1}{c_i} = \infty$, then $\bigcap_{i=0}^{\infty} T^i\mathcal{H}$ is reducing for T, and T restricted to this subspace is m times a unitary operator.*

Proof Let $\mathcal{H}_0 = \bigcap_{i=0}^{\infty} T^i\mathcal{H}$. Since $TT^i\mathcal{H} \subset T^{i+1}\mathcal{H}$, $\mathcal{H}_0$ is invariant for T. Since $x \in \mathcal{H}_0$ implies that there is a sequence $\{y_i\}_{i=0}^{\infty}$ such that $x = T^iy_i$, $Lx = LT^iy_i = T^{i-1}y_i$, so that $Lx \in T^{i-1}\mathcal{H}$ $\forall i \geq 1$, and hence $Lx \in \mathcal{H}_0$, so $\mathcal{H}_0$ is also invariant for L. Also $TLx = x$ since $x \in T\mathcal{H}$, so T maps $\mathcal{H}_0$ onto $\mathcal{H}_0$. Since the restriction of any left-invertible operator to one of its invariant subspaces is also left-invertible (as can be seen from (iv) of Theorem 2.1.3), and T restricted to $\mathcal{H}_0$ is onto and hence right-invertible, it follows that L must be not only the left-inverse of T on $\mathcal{H}_0$, but also the inverse. Let $x \in \mathcal{H}_0$ be arbitrary. Plugging L^ix into inequality (2.2) yields $\|x\|^2 - m^{2i}\|L^ix\|^2 \leq c_i\|DL^ix\|^2$. Proceeding as in the last theorem we have

$$\min_{m \leq i \leq n}\{\|x\|^2 - m^{2i}\|L^ix\|^2\} \sum_{i=m}^{n} \frac{1}{c_i} \leq \sum_{i=m}^{n} \frac{1}{c_i}\{\|x\|^2 - m^{2i}\|L^ix\|^2\} \leq \sum_{i=m}^{n} \|DL^ix\|^2 \leq \|x\|^2,$$

so that as before we conclude that $\liminf_{i\to\infty}\{\|x\|^2 - \|(mL)^ix\|^2\} \leq 0$ and $\|x\|^2 \leq \lim_{i\to\infty} \|(mL)^ix\|^2$ $\forall x \in \mathcal{H}_0$ where the limit exists because mL is a contraction. Plugging Tx into the last inequality yields $\|Tx\|^2 \leq \lim_{i\to\infty} \|m(mL)^{i-1}x\|^2 = m^2 \lim_{i\to\infty} \|(mL)^ix\|^2 \leq m^2\|x\|$, and since $\|Tx\| \geq m\|x\|$, it must be that $\|Tx\| = m\|x\|$ $\forall x \in \mathcal{H}_0$. Therefore T is m times an isometry on $\mathcal{H}_0$, and since T is onto, it must also be m times a unitary operator. Let $D_{\frac{T}{m}} = (\frac{T^*}{m}\frac{T}{m} - I)^{1/2}$ be the positive square root, which exists because $\|\frac{T}{m}x\| \geq \|x\|$ $\forall x \in \mathcal{H}$, then $\|D_{\frac{T}{m}}x\|^2 = \|\frac{T}{m}x\|^2 - \|x\|^2 = 0$ $\forall x \in \mathcal{H}_0$, so that $\mathcal{H}_0 \subset \mathrm{Ker}(D_{\frac{T}{m}}) \subset \mathrm{Ker}(D^2_{\frac{T}{m}})$ and therefore $\frac{T^*}{m}\frac{T}{m}x = x$ $\forall x \in \mathcal{H}_0$. Let $x \in \mathcal{H}_0$, then $T^*x = T^*TLx = m^2Lx \in \mathcal{H}_0$, which shows that $T^*\mathcal{H}_0 \subset \mathcal{H}_0$, and hence $\mathcal{H}_0$ is reducing for T. $\square$

Corollary 2.5.6. *Let T satisfy the hypotheses of Proposition 2.5.5, then the space $\mathcal{H}$ has the following decomposition into orthogonal sums:*

$$\mathcal{H} = \bigcap_{i=0}^{\infty} T^i\mathcal{H} \oplus [\mathcal{E}]_T. \tag{2.3}$$

Proof Let $\mathcal{H}_1 = \mathcal{H}_0^\perp$, where $\mathcal{H}_0 = \bigcap_{i=0}^\infty T^i\mathcal{H}$ as in the previous proposition. Then $\mathcal{H}_0$ is reducing so that $\mathcal{H}_1$ is an invariant subspace which is pure (since $\bigcap_{i=0}^\infty T^i\mathcal{H}_1 \subset \bigcap_{i=0}^\infty T^i\mathcal{H} = \mathcal{H}_0$, so $x \in \bigcap_{i=0}^\infty T^i\mathcal{H}_1$ implies $x \in \mathcal{H}_1$ and $x \in \mathcal{H}_0$ so that $x = 0$). Applying Theorem 2.5.4 to $\mathcal{H}_1$ yields that $\mathcal{H}_1 = [\mathcal{H}_1 \ominus T\mathcal{H}_1]_T$. Since $T\mathcal{H}_0 = \mathcal{H}_0$, $\mathcal{E} \perp \mathcal{H}_0$, and hence $\mathcal{E} \subset \mathcal{H}_1$. Since $\mathcal{E} \perp T\mathcal{H}_1$, $\mathcal{E} \subset \mathcal{H}_1 \ominus T\mathcal{H}_1$. Since $T\mathcal{H} = T\mathcal{H}_1 \oplus \mathcal{H}_0$ and $\mathcal{H}_0 \perp \mathcal{H}_1$, $(\mathcal{H}_1 \ominus T\mathcal{H}_1) \perp T\mathcal{H}$, so that $\mathcal{H}_1 \ominus T\mathcal{H}_1 \subset \mathcal{E}$, and hence $\mathcal{E} = \mathcal{H}_1 \ominus T\mathcal{H}_1$. We conclude that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 = \bigcap_{i=0}^\infty T^i\mathcal{H} \oplus [\mathcal{H}_1 \ominus T\mathcal{H}_1]_T = \bigcap_{i=0}^\infty T^i\mathcal{H} \oplus [\mathcal{E}]_T$. $\square$

Remark 2.5.1. Combining Proposition 2.5.5 and Corollary 2.5.6 we have another proof of the von Neumann-Wold Decomposition Theorem 2.3.4, as any isometry satisfies $\|T^i x\|^2 \leq \|x\|^2 \ \forall x \in \mathcal{H}$, so that one can choose $c_i = 1$, $c = 1$, and $m = 1$ and have $\sum_{i=2}^\infty \frac{1}{c_i} = \infty$.

Examples of operators that satisfy the hypotheses of Proposition 2.5.5 are the operators T_β for $0 \leq \beta \leq 1$ (see Remark 4.3.1) [38]

The problem with using equation (2.2) is as follows. If equation (2.2) is satisfied for an operator T, then the conclusions of Theorem 2.5.4 are also satisfied by $mT \ \forall m \geq 1$, but it is not clear how to change the c_i's so that equation (2.2) with $\sum_{i=2}^\infty \frac{1}{c_i} = \infty$ is still satisfied for mT.

2.6 Introduction to Invariant Subspaces

As we will be studying the restriction of certain operators to their closed, invariant subspaces in the next chapters, we will need to know how the attributes of an operator are passed to its restriction. First we would like to know how the operators that are related to T change when T is restricted.

Proposition 2.6.1. *Let T be a left-invertible operator with a closed, invariant subspace $\mathcal{M}$, then $(T|_\mathcal{M})^* = P_\mathcal{M}T^*|_\mathcal{M}$, $L_{T|_\mathcal{M}} = LP_{T\mathcal{M}}|_\mathcal{M}$, and $(L_{T|_\mathcal{M}})^* = P_{T\mathcal{M}}L^*|_\mathcal{M}$.*

Proof $(T|_\mathcal{M})^* = P_\mathcal{M}T^*|_\mathcal{M}$ since $((T|_\mathcal{M})^*x, y) = (x, T|_\mathcal{M}y) = (x, Ty) = (T^*x, y) = (P_\mathcal{M}T^*x, y) \ \forall\{x, y\} \in \mathcal{M} \times \mathcal{M}$ and $P_\mathcal{M}T^*|_\mathcal{M}$ maps $\mathcal{M}$ to $\mathcal{M}$.
$L_{T|_\mathcal{M}} = LP_{T\mathcal{M}}|_\mathcal{M}$ since $LP_{T\mathcal{M}}|_\mathcal{M}Tx = LP_{T\mathcal{M}}Tx = LTx = x \ \forall x \in \mathcal{M}$ and $LP_{T\mathcal{M}}|_\mathcal{M}\mathcal{E}_\mathcal{M} = LP_{T\mathcal{M}}\mathcal{E}_\mathcal{M} = \{0\}$ and $LP_{T\mathcal{M}}|_\mathcal{M}$ maps $\mathcal{M}$ to $\mathcal{M}$.
$(L_{T|_\mathcal{M}})^* = P_{T\mathcal{M}}L^*|_\mathcal{M}$ since $(P_{T\mathcal{M}}L^*|_\mathcal{M}x, y) = (P_{T\mathcal{M}}L^*x, y) = (x, LP_{T\mathcal{M}}y) =$
$(x, LP_{T\mathcal{M}}|_\mathcal{M}y) = (x, L_{T|_\mathcal{M}}y) \ \forall\{x, y\} \in \mathcal{M} \times \mathcal{M}$ and $P_{T\mathcal{M}}L^*|_\mathcal{M}$ maps $\mathcal{M}$ to $\mathcal{M}$.

So if T is restricted to one of its invariant subspaces, then T^*, L and L^* are changed by composition with a projection onto an invariant subspace of T.

As we will be working with operators with a closed range and would like to know when their restrictions are left-invertible, by (*iii*) of Theorem 2.1.3 we need to know when the restrictions have a closed range. Before proving a more general theorem, we state what will be necessary for our purposes as two corollaries of what will come next.

Corollary 2.6.2. *Let T be an operator with a closed range. If $\mathcal{M}$ is a closed, invariant subspace of T with dim$\left(\text{Ker } T \cap (\mathcal{M} \cap \text{Ker } T)^{\perp}\right) < \infty$, then $T|_{\mathcal{M}}$ has a closed range.*

Corollary 2.6.3. *Let T be an operator with a closed range such that dim(Ker T) $< \infty$. Then $T|_{\mathcal{M}}$ has a closed range for any closed, invariant subspace $\mathcal{M}$ of T.*

Definition 2.6.4. *[15, Definition 9.4] The angle between two subspaces $\mathcal{M}$ and $\mathcal{N}$ is the angle $\theta(\mathcal{M}, \mathcal{N})$ between 0 and $\frac{\pi}{2}$ whose cosine, $c(\mathcal{M}, \mathcal{N}) = \cos\theta(\mathcal{M}, \mathcal{N})$ is defined by $c(\mathcal{M}, \mathcal{N}) = \sup\{|(x, y)| : x \in \mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^{\perp},\ \|x\| = 1,\ y \in \mathcal{N} \cap (\mathcal{M} \cap \mathcal{N})^{\perp},\ \|y\| = 1\}$.*

Lemma 2.6.5. *[15, Lemma 9.5 (6)] Let $\mathcal{M}$ and $\mathcal{N}$ be closed subspaces. Then $|(x, y)| \leq c(\mathcal{M}, \mathcal{N})\|x\|\|y\|$ whenever $x \in \mathcal{M}$, $y \in \mathcal{N}$, and at least one of x or y is in $(\mathcal{M} \cap \mathcal{N})^{\perp}$.*

Proposition 2.6.6. *[15, Lemma 9.36] Let $\mathcal{M}$ be a closed subspace and $\mathcal{N}$ a subspace with a finite dimension. Then $\mathcal{M} + \mathcal{N}$ is a closed subspace.*

Theorem 2.6.7. *[15, Theorem 9.35] Let $\mathcal{M}$ and $\mathcal{N}$ be closed subspaces. Then the following are equivalent:*
(i) $c(\mathcal{M}, \mathcal{N}) < 1$
(ii) $\mathcal{M} + \mathcal{N}$ is closed
(iii) $\mathcal{M}^{\perp} + \mathcal{N}^{\perp}$ is closed

Theorem 2.6.8. *Let $\mathcal{M}$ be a closed, invariant subspace for the operator T which has a closed range. Then $T|_{\mathcal{M}}$ has a closed range if and only if $c = c(\mathcal{M}, \text{Ker } T) < 1$.*

Proof Suppose that $\mathcal{M}$ is a closed, invariant subspace for T, where T has a closed range and $c = c(\mathcal{M}, \text{Ker } T) < 1$. As T has a closed range, from Theorem 2.1.6 we know that there is a $d > 0$ such that $\|Tx\| \geq d\|x\|\ \forall x \in (\text{Ker } T)^{\perp}$. Let $y \in (\text{Ker } T|_{\mathcal{M}})^{\perp} = \mathcal{M} \cap (\mathcal{M} \cap \text{Ker } T)^{\perp}$, then from Lemma 2.6.5 $|(y, k)| \leq c\|y\|$ for every $k \in \text{Ker } T$ with $\|k\| = 1$. Therefore $\|P_{\text{Ker } T}y\| \leq c\|y\|$, and hence $y = y_1 + y_2$ with $y_1 \in (\text{Ker } T)^{\perp}$, $y_2 \in \text{Ker } T$ and $\|y_2\| \leq c\|y\|$, and hence $\|y_1\| \geq \sqrt{1 - c^2}\|y\|$. This yields $\|Ty\| = \|T(y_1 + y_2)\| = \|Ty_1\| \geq d\|y_1\| \geq d\sqrt{1 - c^2}\|y\|$. Since $y \in (\text{Ker } T|_{\mathcal{M}})^{\perp}$ was arbitrary, we have $\|Tx\| \geq d\sqrt{1 - c^2}\|x\|\ \ \forall x \in (\text{Ker } T|_{\mathcal{M}})^{\perp}$, so that by Theorem 2.1.6, $T|_{\mathcal{M}}$ has a closed range.

Conversely, suppose that $\mathcal{M}$ is a closed, invariant subspace for T with $c = c(\mathcal{M}, \text{Ker } T) = 1$. Then there are two sequences $\{m_i\}_{i=1}^{\infty}$ and $\{k_i\}_{i=1}^{\infty}$ with $m_i \in \mathcal{M} \cap$

$(\mathcal{M} \cap \text{Ker } T)^{\perp}$ $\forall i \geq 1$, $k_i \in \text{Ker } T \cap (\mathcal{M} \cap \text{Ker } T)^{\perp}$ $\forall i \geq 1$, $\|m_i\| = \|k_i\| = 1$ $\forall i \geq 1$, and $\lim_i (m_i, k_i) = 1$. Then $\|m_i - k_i\|^2 = \|m_i\|^2 - 2\text{Re}(m_i, k_i) + \|k_i\|^2$, so that $\lim_i \|m_i - k_i\| = 0$ and hence $\lim_i \|T(m_i - k_i)\| = 0$. Since $k_i \in \text{Ker } T$, $T(m_i - k_i) = Tm_i$ so that also $\lim_i \|Tm_i\| = 0$. But $\|P_{(\text{Ker } T|_{\mathcal{M}})^{\perp}} m_i\| = \|P_{\mathcal{M} \cap (\mathcal{M} \cap \text{Ker } T)^{\perp}} m_i\| = \|m_i\| = 1$. Combining this with $\lim_i \|Tm_i\| = 0$ yields that there is no $d > 0$ such that $\|Tm\| \geq d\|P_{(\text{Ker } T|_{\mathcal{M}})^{\perp}} m\|$ $\forall m \in \mathcal{M}$ so that by *(iii)* of Theorem 2.1.6, $T|_{\mathcal{M}}$ does not have a closed range.

Note that Corollary 2.6.2 follows from Theorem 2.6.8 since $\mathcal{M} + \text{Ker } T = \mathcal{M} + \text{Ker T} \cap (\mathcal{M} \cap \text{Ker } T)^{\perp}$, so that by Proposition 2.6.6 and Theorem 2.6.7, $c(\mathcal{M}, \text{Ker } T) < 1$. Also, Corollary 2.6.3 follows from Corollary 2.6.2 since $\dim \left(\text{Ker T} \cap (\mathcal{M} \cap \text{Ker } T)^{\perp}\right) \leq \dim (\text{Ker } T) < \infty$.

Chapter 3

Weighted Shifts and Wandering Subspaces

3.1 Weighted Shifts

Let S be the unilateral shift of multiplicity one, then by Definition 2.3.5 and Theorem 2.3.6, $H_i = S^{(i-1)}\mathcal{E}$ where $\mathcal{E} = \mathcal{H} \ominus S\mathcal{H}$ and each H_i has a dimension equal to one. Since $SH_i = H_{i+1}$ and $H_i \perp H_j$ $i \neq j$, S can be viewed as an operator that shifts the complex number (since the dimension is one) in H_i to H_{i+1}, so if $\mathcal{E} = \text{span}\{e\}$ and $x = \sum_{i=1}^{\infty} c_i S^{(i-1)} e$ where each c_i is a complex number, one can write H as $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \ldots$ and x as $(c_1, c_2, c_3, \ldots)$ and S will simply shift the components of x, so that $Sx = (0, c_1, c_2, c_3, \ldots)$. This motivates the following:

Definition 3.1.1. *Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of complex numbers, then the operator T which maps $x = (x_1, x_2, x_3, \ldots)$ to $Tx = (0, \alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \ldots)$ is a weighted shift with the sequence of weights $\{\alpha_i\}_{i=1}^{\infty}$. Alternatively, an operator T is a weighted shift if there exist subspaces $\mathcal{H}_i$, each with dimension equal to one, such that the $\mathcal{H}_i$ are pairwise orthogonal ($\mathcal{H}_i \perp \mathcal{H}_j$ $\forall i \neq j$), $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \ldots$ and T maps $\mathcal{H}_i$ to $\mathcal{H}_{i+1}$ for all $i \geq 1$.*

Proposition 3.1.2. *Let T be a weighted shift with the sequence of weights $\{\alpha_i\}_{i=1}^{\infty}$, then $\|T\| = \sup_i |\alpha_i|$ and $\|Tx\| \geq c\|x\|$ $\forall x \in \mathcal{H}$ where $c = \inf_i |\alpha_i|$ is the largest value for which this inequality holds for the whole space.*

Proof Let $x = (x_1, x_2, x_3, \ldots)$, then

$$\begin{aligned}\|Tx\|^2 = \|(0, \alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \ldots)\|^2 = \sum_{i=1}^{\infty} |\alpha_i x_i|^2 \\ \leq \sum_{i=1}^{\infty} (\sup_i |\alpha_i|)^2 |x_i|^2 = (\sup_i |\alpha_i|)^2 \|x\|^2,\end{aligned}$$

so that $\|T\| \leq \sup_i |\alpha_i|$. Let $\epsilon > 0$ be arbitrary and j be such that $|\alpha_j| > \sup_i |\alpha_i| - \epsilon$, then $Te_j = \alpha_j e_{j+1}$ so that $\|Te_j\| > \sup_i |\alpha_i| - \epsilon$ and $\|T\| = \sup_i |\alpha_i|$. Also, $\|Tx\|^2 = \|(0, \alpha_1 x_1, \alpha_2 x_2, \alpha_3 x_3, \ldots)\|^2 = \sum_{i=1}^{\infty} |\alpha_i x_i|^2 \geq \sum_{i=1}^{\infty} (\inf_i |\alpha_i|)^2 |x_i|^2 = (\inf_i |\alpha_i|)^2 \|x\|^2$ and if $c > \inf_i |\alpha_i|$ then there is a j such that $|\alpha_j| < c$ and $\|Te_j\| = |\alpha_j| < c$ and $\|Tx\| \geq c\|x\|$ does not hold for all x in $\mathcal{H}$.

Definition 3.1.3. *Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of complex numbers, then the operator T which maps $x = (x_1, x_2, x_3, \ldots)$ to $Tx = (\alpha_1 x_2, \alpha_3 x_2, \alpha_4 x_3, \ldots)$ is a backward, weighted shift with the sequence of weights $\{\alpha_i\}_{i=1}^{\infty}$.*

Proposition 3.1.4. *Let T be a left-invertible, weighted shift with the sequence of weights $\{\alpha_i\}_{i=1}^{\infty}$, then T^* is a backward, weighted shift with the sequence of weights $\{\bar{\alpha}_i\}_{i=1}^{\infty}$, L is a backward, weighted shift with the sequence of weights $\{\frac{1}{\alpha_i}\}_{i=1}^{\infty}$ and L^* is a weighted shift with the sequence of weights $\{\frac{1}{\bar{\alpha}_i}\}_{i=1}^{\infty}$.*

Proof This follows from the facts that $(T^* e_i, e_j) = (e_i, Te_j) = (e_i, \alpha_j e_{j+1}) = \bar{\alpha}_j (e_i, e_{j+1})$, and $(e_i, e_j) = (LTe_i, e_j) = (L\alpha_i e_{i+1}, e_j) = \alpha_i (Le_{i+1}, e_j)$ and $e_1 \perp T\mathcal{H} = \text{span}\{e_2, e_3, e_4, \ldots\}$.

Note that we will only be concerned with left-invertible, weighted shifts. Since L^* is a weighted shift if and only if T is, and every weighted shift is pure, by Theorem 2.2.4 every weighted shift satisfies the *Wandering Subspace Property*.

Theorem 3.1.5. *[41, Theorems 1,2] Let T be a weighted shift with the sequence of weights $\{\alpha_i\}_{i=1}^{\infty}$ and S a weighted shift with the sequence of weights $\{\beta_i\}_{i=1}^{\infty}$, then T and S are unitarily equivalent if and only if $|\alpha_i| = |\beta_i| \ \forall i \geq 1$ and similar if and only if $0 < \inf_i \frac{|\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_i|}{|\beta_1 \beta_2 \beta_3 \cdots \beta_i|} \leq \sup_i \frac{|\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_i|}{|\beta_1 \beta_2 \beta_3 \cdots \beta_i|} < \infty$.*

The following corollary can be seen by picking $\beta_i = u_i \alpha_i$ where $|u_i| = 1 \ \forall i \geq 1$ where the u_i's are suitably chosen.

Corollary 3.1.6. *[41, Corollary 1] Let T be a weighted shift with the sequence of weights $\{\alpha_i\}_{i=1}^{\infty}$. Then T is unitarily equivalent to the weighted shift with the sequence of weights $\{|\alpha_i|\}_{i=1}^{\infty}$.*

From Theorem 3.1.5 we know that a weighted shift is similar to m times the unweighted shift if and only if $0 < \inf_i \frac{|\alpha_1\alpha_2\alpha_3\cdots\alpha_i|}{|m|^i} \leq \sup_i \frac{|\alpha_1\alpha_2\alpha_3\cdots\alpha_i|}{|m^i|} < \infty$. One would hope that for every weighted shift there would be an m such that the weighted shift were similar to m times the unweighted shift since these are the simplest kind of weighted shift, and have as much structure as an isometry. However, this is not the case as can be seen by considering the weighted shift with the sequence of weights of one m followed by two $\frac{1}{m}$'s followed by three m's followed by four $\frac{1}{m}$'s et cetera for any m with modulus not equal to one. So every weighted shift is not similar to a weighted shift whose weights only have one value, but according to the next theorem every weighted shift which is bounded below is similar to a weighted shift whose weights only have two values:

Proposition 3.1.7. *Let T be a weighted shift with the sequence of weights$\{\alpha_i\}_{i=1}^{\infty}$ and $M = \max\{\sup_i |\alpha_i|, \frac{1}{\inf_i |\alpha_i|}\} < \infty$, then T is similar to a weighted shift whose weights are either M or $\frac{1}{M}$.*

Proof According to the definition of M, $0 < c_1 = \frac{\inf_i |\alpha_i|}{M} \leq \frac{|\alpha_i|}{M} \leq 1$ and $1 \leq \frac{|\alpha_i|}{\frac{1}{M}} \leq M \sup_i |\alpha_i| = c_2 < \infty$ for every i. We will now construct a nonunique, weighted shift S with the sequence of weights $\{\beta_i\}_{i=1}^{\infty}$ such that β_i is M or $\frac{1}{M}$ for every i and T and S are similar. Let $v_i = \frac{|\alpha_1\alpha_2\alpha_3\cdots\alpha_i|}{|\beta_1\beta_2\beta_3\cdots\beta_i|}$ and $\beta_1 = M$ so that $c_1 \leq v_1 \leq 1$. For each $i \geq 2$, if $v_{i-1}M|\alpha_i| \leq 1$ let $\beta_i = \frac{1}{M}$, else let $\beta_i = M$. Since $\frac{|\alpha_i|}{M} \leq 1$ $\forall i \geq 1$ and β_i is $\frac{1}{M}$ only when $v_{i-1}M|\alpha_i| \leq 1$ so that $v_i \leq 1$ it must be that $v_i \leq 1$ $\forall i$. Since β_i is M only when $v_{i-1} > \frac{1}{M|\alpha_i|}$ and $1 \leq \frac{|\alpha_i|}{\frac{1}{M}}$, $v_i \leq v_{i-1}$ only if $v_{i-1} > \frac{1}{M|\alpha_i|}$ so that $v_i \geq \frac{1}{M^2 \sup_i |\alpha_i|}$ $\forall i$ and hence T and S are similar. ∎

The author hopes that the previous result can be used to better classify weighted shifts into appropriate groups. It has served usefully in deriving counterexamples as one knows as long as at least two weights are considered, all weighted shifts are considered, at least to equivalence by similarity.

3.2 The Wandering Subspace Problem

Note that S will always mean an unweighted shift (of multiplicity equal to one). Two important questions for S have already been answered in Theorems 2.3.4 and 2.3.9: Is the *Wandering Subspace Property* satisfied for S restricted to an arbitrary, closed, invariant subspace? (Yes) and: What values can the index of a closed, invariant subspace (index($\mathcal{M}$)=dim($\mathcal{M} \ominus S\mathcal{M}$)) of S be? (The index is always one) This follows from the fact that S restricted to any of its closed, invariant subspaces is unitarily equivalent to S (Proposition 2.3.8 and Theorem 2.3.9).

In this section we will study whether an arbitrary, left-invertible, weighted shift also satisfies the *Wandering Subspace Property* when it is restricted to any of its closed, invariant subspaces. From Proposition 2.6.1 we know that if T is restricted to one of its invariant subspaces, then T^*, L and L^* are changed by composition with a projection onto an invariant subspace of T. This means that L^* when T is restricted may no longer be a weighted shift and may not even be pure, and hence T may no longer be a weighted shift or satisfy the *Wandering Subspace Property* when restricted.

Since we are interested in when a closed, invariant subspace is generated by its wandering subspace, it would be useful to know how the wandering subspace changes when an operator is restricted. Although this is complicated in general, there is one simple result that is always true.

Proposition 3.2.1. *Let T be left-invertible and $\mathcal{M}$ be a closed, invariant subspace of T. If $e \in \mathcal{H} \ominus T\mathcal{H} = KerT^*$, then $\varepsilon = P_{\mathcal{M}}e \in \mathcal{M} \ominus T\mathcal{M}$.*

Proof Since ε is the projection of e onto $\mathcal{M}$, $\varepsilon \in \mathcal{M}$. We must show that $\varepsilon \perp T\mathcal{M}$. Let m be an arbitrary vector in $\mathcal{M}$, then $(\varepsilon, Tm) = (P_{\mathcal{M}}e, Tm) = (e, P_{\mathcal{M}}Tm) = (e, Tm) = (T^*e, m) = 0$, so that $\varepsilon \in \mathcal{M} \ominus T\mathcal{M}$.

If T is similar to W then $W = QTQ^{-1}$ where Q is an invertible operator (we will write Q^{-*} for $(Q^{-1})^* = (Q^*)^{-1}$). If $\mathcal{M}$ is a closed, invariant subspace for T, then since $WQ\mathcal{M} = QTQ^{-1}Q\mathcal{M} = QT\mathcal{M} \subset Q\mathcal{M}$, $Q\mathcal{M}$ is a closed, invariant subspace for W. Suppose that $\varepsilon \in \mathcal{M} \ominus T\mathcal{M}$ so that $(\varepsilon, Tm) = 0 \ \forall m \in \mathcal{M}$, then $(P_{Q\mathcal{M}}Q^{-*}\varepsilon, WQm) = (P_{Q\mathcal{M}}Q^{-*}\varepsilon, QTQ^{-1}Qm) = (P_{Q\mathcal{M}}Q^{-*}\varepsilon, QTm) = (Q^{-*}\varepsilon, QTm) = (Q^*Q^{-*}\varepsilon, Tm) = 0 \ \forall m \in \mathcal{M}$ so that we have $P_{Q\mathcal{M}}Q^{-*}\varepsilon \in Q\mathcal{M} \ominus WQ\mathcal{M}$. Therefore $P_{Q\mathcal{M}}Q^{-*}(\mathcal{M} \ominus T\mathcal{M}) \subset Q\mathcal{M} \ominus WQ\mathcal{M}$, and since $W = QTQ^{-1}$ and $\mathcal{M} = Q^{-1}Q\mathcal{M}$, it follows that we can reverse the roles of T and W and obtain $P_{\mathcal{M}}Q^*(Q\mathcal{M} \ominus WQ\mathcal{M}) \subset \mathcal{M} \ominus T\mathcal{M}$. Since $(Qm, Q^{-*}x) = (Q^{-1}Qm, x) = (m, x)$, $P_{Q\mathcal{M}}Q^{-*}x = 0 \ \forall x \in \mathcal{M}^{\perp}$, so that combining the inclusions we obtain

$$
\begin{aligned}
&P_{Q\mathcal{M}}Q^{-*}P_{\mathcal{M}}Q^*(Q\mathcal{M} \ominus WQ\mathcal{M}) = P_{Q\mathcal{M}}Q^{-*}(I - P_{\mathcal{M}^{\perp}})Q^*(Q\mathcal{M} \ominus WQ\mathcal{M}) \\
=&Q\mathcal{M} \ominus WQ\mathcal{M} \subset P_{Q\mathcal{M}}Q^{-*}(\mathcal{M} \ominus T\mathcal{M}) \subset Q\mathcal{M} \ominus WQ\mathcal{M}.
\end{aligned}
$$

Therefore there must be equality throughout the inclusions, and we obtain $Q\mathcal{M} \ominus WQ\mathcal{M} = P_{Q\mathcal{M}}Q^{-*}(\mathcal{M} \ominus T\mathcal{M})$ for any similar, left-invertible operators T and W with closed, invariant subspaces related by the transformation of similarity Q. We also conclude that

$$
\dim(\mathcal{M} \ominus T\mathcal{M}) = \dim(Q\mathcal{M} \ominus WQ\mathcal{M}), \tag{3.1}
$$

so that two left-invertible operators that are similar must have the same set of possible values for the indices of their closed, invariant subspaces.

From this it follows that any weighted shift T which is similar to mS can only have invariant subspaces of index equal to one. Let $\mathcal{E}_{\mathcal{M}} = \mathcal{M} \ominus T\mathcal{M}$, then the question about the *Wandering Subspace Property* for a closed, invariant subspace $\mathcal{M}$ is equivalent to whether or not $\mathcal{M} \ominus [\mathcal{E}_{\mathcal{M}}]_T = \{0\}$. If this subspace is not the zero space then the next simplest case would be when it has a dimension equal to one. From Theorem 2.2.4 we know that $\mathcal{M} \ominus [\mathcal{E}_{\mathcal{M}}]_T = \bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M}$, which is an invariant subspace for $(T|_{\mathcal{M}})^*$ on which it is invertible. So if $\bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M}$ has dimension equal to one, then $(T|_{\mathcal{M}})^*|_{\bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M}} = P_{\mathcal{M}}T^*|_{\bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M}}$ must be a nonzero scalar times the identity, or more generally speaking, an operator with a nonzero eigenvalue. The same way that the wandering subspace for the whole space can be defined as $\varepsilon \in \mathcal{E} = \mathrm{Ker}(T^*)$ which implies that $\varepsilon \perp T^i\varepsilon \;\; \forall i \geq 1$ or $(\varepsilon, T^i\varepsilon) = 0 \;\; \forall i \geq 1$, which can then be used as the definition to find wandering subspaces inside of invariant subspaces; the definition of an eigenvector as $x_\lambda \in \mathrm{Ker}(T^* - \lambda)$ implies that $(x_\lambda, T^i x_\lambda) = (T^{*i}x_\lambda, x_\lambda) = \lambda^i \|x_\lambda\|^2 \;\; \forall i \geq 0$ which can be used to look for invariant subspaces on which the restriction of T does not satisfy the *Wandering Subspace Property*.

Theorem 3.2.2. *For a closed, invariant subspace $\mathcal{M}$ of T, $(T|_{\mathcal{M}})^*$ has an invariant subspace of span$\{f\}$ contained in $\bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M}$ only if $f \in T\mathcal{M}$ and $(T^i f, f) = \lambda^i \|f\|^2 \;\; \forall i \geq 0$ and $(Lf, f) = \frac{1}{\lambda}\|f\|^2$ for some $\lambda \neq 0$. Conversely, if $f \in T\mathcal{H}$ and $(T^i f, f) = \lambda^i \|f\|^2 \;\; \forall i \geq 0$ and $(Lf, f) = \frac{1}{\lambda}\|f\|^2$, then there is a closed, invariant subspace $\mathcal{M}$ of T such that $f \in \bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M}$ and $(T|_{\mathcal{M}})^*$span$\{f\}$ = span$\{f\}$.*

Proof Suppose that span$\{f\} \subset \bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M}$ is an invariant subspace of $(T|_{\mathcal{M}})^*$, then since $\bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M} \subset (L_{T|_{\mathcal{M}}})^*\mathcal{M} = T\mathcal{M}$ it must be that $f \in T\mathcal{M}$. Since $(T|_{\mathcal{M}})^*$ is invertible on $\bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M}$ and maps span$\{f\}$ to itself, it must be that $(T|_{\mathcal{M}})^* f = \bar{\lambda} f$ for some $\lambda \neq 0$. So $(T^i f, f) = ((T|_{\mathcal{M}})^i f, f) = (f, (T|_{\mathcal{M}})^{*i} f) = (f, \bar{\lambda}^i f) = \lambda^i \|f\|^2 \;\; \forall i \geq 0$. Since $(L_{T|_{\mathcal{M}}})^* f = (L_{T|_{\mathcal{M}}})^* \frac{1}{\bar{\lambda}}(T|_{\mathcal{M}})^* f = \frac{1}{\bar{\lambda}} f$ because $(L_{T|_{\mathcal{M}}})^*(T|_{\mathcal{M}})^*$ is the identity on $\bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M}$, it follows that

$$(Lf, f) = (LP_{T\mathcal{M}}f, f) = (f, P_{T\mathcal{M}}L^* f) = (f, (L_{T|_{\mathcal{M}}})^* f) = (f, \frac{1}{\bar{\lambda}} f) = \frac{1}{\lambda}\|f\|^2.$$

Conversely, let $\mathcal{M} = [Lf]_T$, then since $f \in T\mathcal{H}$, $TLf = f$ and $f \in T\mathcal{M}$. Let $x = \sum_{i=0}^{n} c_i T^i Lf$, then since $(T^i Lf, f) = \lambda(T^{(i-1)}Lf, f) \;\; \forall i \geq 1$, $(Tx, f) = (\sum_{i=0}^{n} c_i T^i f, f) = \sum_{i=0}^{n} c_i (T^i f, f) = \sum_{i=0}^{n} \lambda c_i (T^i Lf, f) = \lambda(\sum_{i=0}^{n} c_i T^i Lf, f) = \lambda(x, f)$. Since x's of this form are dense in $\mathcal{M}$, it must be that $(Tm, f) = \lambda(m, f)$ for every $m \in \mathcal{M}$. Let $\varepsilon \in \mathcal{E}_{\mathcal{M}} = \mathcal{M} \ominus T\mathcal{M}$, then since $f \in T\mathcal{M}$, $(\varepsilon, f) = 0$ and hence $(T^i\varepsilon, f) = \lambda^i(\varepsilon, f) = 0 \;\; \forall i \geq 0$, so $f \perp \bigvee_{i=0}^{\infty} T^i \mathcal{E}_{\mathcal{M}}$, and hence $f \in \bigcap_{i=0}^{\infty} (L_{T|_{\mathcal{M}}})^{*i}\mathcal{M}$. Since $(T|_{\mathcal{M}})^* = P_{\mathcal{M}}T^*$ and $(P_{\mathcal{M}}T^* f - \bar{\lambda} f, T^i Lf) = (f, TT^i Lf) - \bar{\lambda}(f, T^i Lf) = 0$ and $\mathcal{M} = [Lf]_T$, it must be that $(T|_{\mathcal{M}})^* f = \bar{\lambda} f$ and $(T|_{\mathcal{M}})^*$span$\{f\}$ = span$\{f\}$. ∎

Let x_λ be an eigenvector for S^* with eigenvalue $\lambda \neq 0$, then the equation $S^*x_\lambda = \lambda x_\lambda$ is equivalent to $(x_2, x_3, x_4, \ldots) = \lambda(x_1, x_2, x_3, \ldots)$ or equivalently $x_{i+1} = \lambda x_i \ \forall i \geq 1$. Thus if $x_1 = 0$ then $x = 0$, so normalizing so that $x_1 = 1$ yields that the only eigenvector for S^* with eigenvalue λ is $(1, \lambda, \lambda^2, \lambda^3, \ldots)$, which is contained in $\mathcal{H}$ if and only if $\sum_{i=0}^{\infty} |\lambda|^{2i} < \infty$ or if and only if $|\lambda| < 1$ (see Theorem 8 of [41]). Since $S^*x_\lambda = \lambda x_\lambda$, $(S^i x_\lambda, x_\lambda) = (x_\lambda, S^{*i}x_\lambda) = \bar{\lambda}^i \|x_\lambda\|^2$. However, $x_\lambda \notin S\mathcal{H}$ and $(Lx_\lambda, x_\lambda) = \bar{\lambda}\|x_\lambda\|^2 \neq \frac{1}{\lambda}\|x_\lambda\|^2$. Similarly if y_λ is an eigenvector for the adjoint of a weighted shift T, then $(T^i y_\lambda, y_\lambda) = \bar{\lambda}^i \|y_\lambda\|^2$ but it can be shown that $y_\lambda \notin T\mathcal{H}$. The question arises as to whether x_λ or more generally y_λ could be modified in some way so that an operator similar to S or some T would have a closed, invariant subspace on which its restriction does not satisfy that *Wandering Subspace Property* (from above we know that S satisfies the *Wandering Subspace Property* on all of its closed, invariant subspaces).

This will be done by modifying the inner product on $\mathcal{H}$, which will change the geometry but not the topology of the space. As it will not change the correspondence between vectors and their images under an operator, it will not change the invariant subspaces of an operator. As it does not change the topology, it will not change the *closed*, invariant subspaces of an operator. But as it does change the geometry, it will change which vectors are orthogonal to the image of an operator, so that it will change the wandering subspaces. Therefore one could hope that by changing the wandering subspaces of an operator there may be a closed, invariant subspace which is not generated by its wandering subspace.

Definition 3.2.3. *Two inner products $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ are equivalent if there are constants $m > 0$ and $M < \infty$ such that $m(x,x)_2 \leq (x,x)_1 \leq M(x,x)_2 \ \forall x \in \mathcal{H}$, or equivalently if they have the same convergent sequences.*

Theorem 3.2.4. *Let T be a left-invertible operator and D be an invertible operator. Then DTD^{-1} has a closed, invariant subspace on which the restriction of DTD^{-1} does not satisfy the* Wandering Subspace Property *if and only if T has a closed, invariant subspace $\mathcal{M}$ such that $T|_{\mathcal{M}}$ does not satisfy the* Wandering Subspace Property *when $\mathcal{H}$ is endowed with the equivalent inner product $(\cdot,\cdot)_2 = (D\cdot, D\cdot)$.*

Proof Since $D^{-1}(DTD^{-1})D = T$ and $(D^{-1}D\cdot, D^{-1}D\cdot) = (\cdot,\cdot)$, the proof of one direction will follow from the other. Let $\mathcal{M}$ be a closed, invariant subspace of $\tilde{T} = DTD^{-1}$ and $\mathcal{E}_{\mathcal{M}} = \mathcal{M} \ominus_1 \tilde{T}\mathcal{M}$ where the $\ominus_1$ means that one orthogonally removes using the inner product $(\cdot,\cdot)_1 = (\cdot,\cdot)$ (the original one). Suppose that $\mathcal{M} \ominus_1 [\mathcal{E}_{\mathcal{M}}]_{\tilde{T}} \neq \{0\}$ so that $\tilde{T}$ restricted to $\mathcal{M}$ does not satisfy the *Wandering Subspace Property.* This happens if and only if there is an $f \neq 0$ such that $f \perp_1 \tilde{T}^i \mathcal{E}_{\mathcal{M}} \ \forall i \geq 0$. Since $TD^{-1}\mathcal{M} = D^{-1}\tilde{T}DD^{-1}\mathcal{M} = D^{-1}\tilde{T}\mathcal{M} \subset D^{-1}\mathcal{M}$, $D^{-1}\mathcal{M}$ is an invariant subspace for T. Let $\varepsilon \in \mathcal{E}_{\mathcal{M}}$, then since $(D^{-1}\varepsilon, TD^{-1}m)_2 = (DD^{-1}\varepsilon, DTD^{-1}m)_1 =$

$(\varepsilon, \tilde{T}m)_1 = 0 \ \forall m \in \mathcal{M}$, it must be that $D^{-1}\varepsilon \in \mathcal{E}_{D^{-1}\mathcal{M}} = D^{-1}\mathcal{M} \ominus_2 TD^{-1}\mathcal{M}$, and reversing the roles of T and $\tilde{T}$ yields $D^{-1}\mathcal{E}_{\mathcal{M}} = D^{-1}\mathcal{M} \ominus_2 TD^{-1}\mathcal{M}$ (compare this with the case discussed after Proposition 2.6.1 where the inner product is not changed). Since $f \perp_1 \tilde{T}^i\mathcal{E}_{\mathcal{M}} \ \forall i \geq 0$, $(f, \tilde{T}^i\varepsilon)_1 = (f, (DTD^{-1})^i\varepsilon)_1 = (f, DT^iD^{-1}\varepsilon)_1 = (DD^{-1}f, DT^iD^{-1}\varepsilon)_1 = (D^{-1}f, T^iD^{-1}\varepsilon)_2 = 0 \ \forall i \geq 0, \ \varepsilon \in \mathcal{E}_{\mathcal{M}}$. So $D^{-1}f \perp_2 T^i\mathcal{E}_{D^{-1}\mathcal{M}} \ \forall i \geq 0$ and since $D^{-1}f \in D^{-1}\mathcal{M}$, $D^{-1}f \in D^{-1}\mathcal{M} \ominus_2 [\mathcal{E}_{D^{-1}\mathcal{M}}]_T \neq \{0\}$ and T does not satisfy the *Wandering Subspace Property* when restricted to its invariant subspace $D^{-1}\mathcal{M}$ under the inner product $(\cdot, \cdot)_2 = (D\cdot, D\cdot)$. ∎

Note that included in the proof was the fact that $D^{-1}\mathcal{M} \ominus_2 [\mathcal{E}_{D^{-1}\mathcal{M}}]_T$ is equal to $D^{-1}(\mathcal{M} \ominus_1 [\mathcal{E}_{\mathcal{M}}]_{\tilde{T}})$.

Corollary 3.2.5. *For any closed, invariant subspace $\mathcal{M}$ of the left-invertible operator T and invertible operator D, dim$(\mathcal{M} \ominus [\mathcal{M} \ominus_2 T\mathcal{M}]_T) =$ dim$(D\mathcal{M} \ominus [D\mathcal{M} \ominus DTD^{-1}D\mathcal{M}]_{DTD^{-1}})$.*

Definition 3.2.6. *The residual space (with respect to being generated by the wandering subspace) for a left-invertible operator is the space $\mathcal{H} \ominus [\mathcal{H} \ominus T\mathcal{H}]_T = \bigcap_{i=0}^{\infty} L^{*i}\mathcal{H}$.*

Remark 3.2.1. Since $\dim(\mathcal{M} \ominus_1 [\mathcal{M} \ominus_1 T\mathcal{M}]_T) \neq \dim(\mathcal{M} \ominus_2 [\mathcal{M} \ominus_2 T\mathcal{M}]_T)$ for two equivalent, inner products, we see that unlike the index, the dimension of the residual space of a closed, invariant subspace is not preserved under transformations of similarity. Also, the property of having only closed, invariant subspaces that are generated by their wandering subspaces is not preserved under transformations of similarity.

Thus the next question is: For which invertible operators D such that DSD^{-1} is a weighted shift does S have a closed, invariant subspace which does not satisfy the *Wandering Subspace Property* when $\mathcal{H}$ is endowed with the $(D\cdot, D\cdot)$ inner product? Let T be a weighted shift with the sequence of weights $\{\alpha_i\}_{i=1}^{\infty}$ and D the diagonal operator that maps $x = (x_1, x_2, x_3, \ldots)$ to $(d_1x_1, d_2x_2, d_3x_3, \ldots)$. D is bounded if and only if $\sup_i |d_i| < \infty$ and also invertible if and only if $\inf_i |d_i| > 0$. In this case DTD^{-1} is a weighted shift that is similar to T and has the sequence of weights $\{\beta_i\}_{i=1}^{\infty}$ with $\beta_i = \frac{d_{i+1}\alpha_i}{d_i}$ and as must be true by Theorem 3.1.5, $0 < \inf_i \frac{|d_1|}{|d_{i+1}|} = \inf_i \frac{|\alpha_1\alpha_2\alpha_3\cdots\alpha_i|}{|\beta_1\beta_2\beta_3\cdots\beta_i|} \leq \sup_i \frac{|\alpha_1\alpha_2\alpha_3\cdots\alpha_i|}{|\beta_1\beta_2\beta_3\cdots\beta_i|} = \sup_i \frac{|d_1|}{|d_{i+1}|} < \infty$. Conversely, if T and W are weighted shifts where W has the sequence of weights $\{\beta_i\}_{i=1}^{\infty}$ and $0 < \inf_i \frac{|\alpha_1\alpha_2\alpha_3\cdots\alpha_i|}{|\beta_1\beta_2\beta_3\cdots\beta_i|} \leq \sup_i \frac{|\alpha_1\alpha_2\alpha_3\cdots\alpha_i|}{|\beta_1\beta_2\beta_3\cdots\beta_i|} < \infty$, then defining $d_1 = 1$ and $d_{i+1} = \frac{\beta_i}{\alpha_i}d_i \ \forall i \geq 1$, $d_i = \frac{\beta_1\beta_2\beta_2\cdots\beta_{i-1}}{\alpha_1\alpha_2\alpha_3\cdots\alpha_{i-1}}$ so that $\inf_i |d_i| > 0$ and $\sup_i |d_i| < \infty$ and DTD^{-1}, although not necessarily equal to W, is a weighted shift with the same sequence of weights. So if we assume that W is a weighted shift that is similar to T and also shifts the same basis as T, then there exists a diagonal operator D such that $W = DTD^{-1}$. Since given any two weighted shifts T and W, there is a unitary operator such that T and UWU^{-1} shift the same basis, and according to Theorem 3.1.5 T and W are similar if and only if $0 < \inf_i \frac{|\alpha_1\alpha_2\alpha_3\cdots\alpha_i|}{|\beta_1\beta_2\beta_3\cdots\beta_i|} \leq \sup_i \frac{|\alpha_1\alpha_2\alpha_3\cdots\alpha_i|}{|\beta_1\beta_2\beta_3\cdots\beta_i|} < \infty$, from now onward when DTD^{-1} is a weighted shift similar to the weighted shift T, we will assume that D is a diagonal operator as above.

According to Theorem 3.2.4, we want to know if and when $f \perp_2 T^i\mathcal{E}_{\mathcal{M},2}\ \forall i \geq 0$ with $f \neq 0$ and $f \in \mathcal{M}$ where $\mathcal{E}_{\mathcal{M},2} = \mathcal{M} \ominus_2 T\mathcal{M}$. We next need to calculate $\mathcal{M} \ominus_2 T\mathcal{M}$; note that in the case after Proposition 2.6.1 the operator was changed but not the inner product, in the case in Theorem 3.2.4 both the operator and the inner product were changed, and this time the inner product is being changed but not the operator. Since $\|m\|_2^2 = (Dm, Dm) = (D^*Dm, m) = (P_{\mathcal{M}}D^*Dm, m)\ \forall m \in \mathcal{M}$ ($P_{\mathcal{M}}$ is defined using the original inner product) and $P_{\mathcal{M}}D^*D|_{\mathcal{M}}$ is self-adjoint, it is invertible. Let $\varepsilon \in \mathcal{E}_{\mathcal{M}} = \mathcal{M} \ominus_1 T\mathcal{M}$, then

$$\begin{aligned}((P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}\varepsilon, Tm)_2 &= (D(P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}\varepsilon, DTm)\\ &= (D^*D(P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}\varepsilon, P_{\mathcal{M}}Tm) = (\varepsilon, Tm) = 0\ \forall m \in \mathcal{M},\end{aligned}$$

and since $(P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}\varepsilon \in \mathcal{M}$, $(P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}\varepsilon \in \mathcal{M} \ominus_2 T\mathcal{M}$, and since the preceding can be reversed,

$$(P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}(\mathcal{M} \ominus_1 T\mathcal{M}) = \mathcal{M} \ominus_2 T\mathcal{M}. \tag{3.2}$$

Therefore $f \perp_2 T^i\mathcal{E}_{\mathcal{M},2}\ \forall i \geq 0$ if and only if

$$\begin{aligned}(DT^i(P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}\varepsilon, Df) &= (D^*DT^i\varepsilon_D, f)\\ &= (P_{\mathcal{M}}D^*DT^i\varepsilon_D, f) = 0\ \forall i \geq 0,\ \ \forall \varepsilon \in \mathcal{E}\end{aligned}$$

where $\varepsilon_D = (P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}\varepsilon$. This can be rewritten as $f \perp P_{\mathcal{M}}D^*DT^i(\mathcal{M} \ominus_2 T\mathcal{M})$ and since $P_{\mathcal{M}}D^*D|_{\mathcal{M}}$ is invertible there is a vector $f \neq 0$ which satisfies this if and only if $\mathcal{M} \neq \bigvee_{i=0}^{\infty} T^i(\mathcal{M} \ominus_2 T\mathcal{M})$ (since $(\cdot,\cdot)$ and $(\cdot,\cdot)_2$ are equivalent, it does not matter which inner product is used to take the closure in the $\bigvee$).

From Theorem 2.2.4 we know that $\mathcal{H}$ is generated by the wandering subspace of T if and only if $\bigcap_{i=0}^{\infty} L^{*i}\mathcal{H} = \{0\}$. Theorem 3.2.4 shows that if we are interested in when operators that are similar to the left-invertible operator T have closed, invariant subspaces on which their restrictions do not satisfy the *Wandering Subspace Property*, then we can examine the behavior under T of the wandering subspaces for T when $\mathcal{H}$ is endowed with a different norm. If we are considering a left-invertible, weighted shift, the operator D that is used to change the norm may not produce a weighted shift when T is changed to the similar operator DTD^{-1}. The next theorem shows that if T is any left-invertible weighted shift, then we can find a D such that DTD^{-1} is still a weighted shift and this new operator has a closed, invariant subspace whose residual space has a dimension of any finite, prescribed integer.

It is interesting to note that there are weighted shifts such that no invertible operator D (diagonal or not) can be used to produce a DTD^{-1} which has a closed, invariant subspace on which its restriction does not satisfy the *Wandering Subspace Property*; by the next theorem such a weighted shift must not be left-invertible. If an operator T is one-to-one but not left-invertible, one can still consider closed, invariant subspaces $\mathcal{M}$ such that $\mathcal{M} \neq \overline{T\mathcal{M}}$ and ask whether $\mathcal{M} = [\mathcal{M} \ominus \overline{T\mathcal{M}}]_T$. Theorem 3.2.4 can still be used to show that DTD^{-1} has a closed, invariant subspace $D\mathcal{M}$ such that $D\mathcal{M} \neq \overline{DTD^{-1}D\mathcal{M}}$ and $D\mathcal{M} \neq [D\mathcal{M} \ominus \overline{DTD^{-1}D\mathcal{M}}]_T$ if and only if the closed, invariant subspace $\mathcal{M}$ for T is such that $\mathcal{M} \neq \overline{T\mathcal{M}}$ and $\mathcal{M} \neq [\mathcal{M} \ominus \overline{T\mathcal{M}}]_T$.

As an example consider a unicellular, weighted shift T, that is, a weighted shift whose only closed, invariant subspaces are $S^i\mathcal{H}$ $i \geq 0$, where S is the unweighted shift. A sufficient condition for a weighted shift to be unicellular is that the α_i's are decreasing in moduli and contained in l^p for some $p < \infty$ [41, First Corollary to Proposition 38]. Since the α_i's are contained in l^p, $\lim_{i\to\infty} |\alpha_i| = 0$ so that the weighted shift is not left-invertible. Suppose that there were a D such that DTD^{-1} had a closed, invariant subspace on which its restriction did not satisfy the *Wandering Subspace Property.* Then this would imply by Theorem 3.2.4 that for some j, T restricted to $S^j\mathcal{H}$ does not satisfy the *Wandering Subspace Property* when $\mathcal{H}$ is endowed with the inner product corresponding to D. Since T is unicellular, the subspace generated by the wandering subspace for T under the new inner product must be of the form $S^k\mathcal{H}$ for some $k > j$, since changing to an equivalent inner product does not change the closed, invariant subspaces. Since $k > j$, the vectors $e_{j+1}, e_{j+2}, \ldots, e_k$ are contained in the residual space of T when restricted to $T^j\mathcal{H}$ for the new inner product, where the residual space is defined as $S^j\mathcal{H} \ominus_2 [S^j\mathcal{H} \ominus_2 TS^j\mathcal{H}]_T$, where the $\ominus_2$ is from the new inner product corresponding to D. This is a contradiction as $\operatorname{span}\{\varepsilon_D\} = S^j\mathcal{H} \ominus_2 TS^j\mathcal{H}$ so that e_{j+1} being orthogonal to ε_D (as it must be since the residual space is orthogonal to the subspace generated by the wandering subspace) implies that $e_{j+1} \in TS^j\mathcal{H}$, which is a contradiction.

We will have need of the following lemma in the next theorem. It was proven in Proposition 1 of [22] based on the properties of a quotient space. We include it here for completeness.

Lemma 3.2.7. *Let T be a left-invertible, weighted shift and $\mathcal{M}$ a closed, invariant subspace for T. Then $\mathcal{M}$ has a finite codimension if and only if $\mathcal{M} = [x]_T$ where $x = \sum_{i=0}^n c_i T^i e_1$ for some nonzero vector $\{c_i\}_{i=0}^n \in \mathbb{C}^{n+1}$, that is, $\mathcal{M} = \overline{\left(\prod_{i=1}^n (T - \lambda_i)\right)\mathcal{H}}$ for some finite set of λ_i's, $\lambda_i \in \mathbb{C}$ $1 \leq i \leq n$*

Proof Note that the n from the c_i's or the λ_i's does not have to be the same as the codimension of $\mathcal{M}$. First notice that $x = \sum_{i=0}^{n} c_i T^i e_1$ with $c_n \neq 0$ if and only if $x = \lambda_0 \left(\prod_{i=1}^{n}(T - \lambda_i)\right) e_1$ for some suitable set of λ_i's with $\lambda_0 \neq 0$, and $T^j x = \lambda_0 \left(\prod_{i=1}^{n}(T - \lambda_i)\right) T^j e_1 = \lambda_0 \left(\prod_{i=1}^{n}(T - \lambda_i)\right) \alpha_1 \alpha_2 \cdots \alpha_j e_{j+1}$. Since $[x]_T = \bigvee_{j=0}^{\infty} T^j x$ and $\{e_j\}_{j=1}^{\infty}$ is a complete set, the equivalence of $[x]_T$ and $\overline{\lambda_0 \left(\prod_{i=1}^{n}(T - \lambda_i)\right) \mathcal{H}} = \overline{\left(\prod_{i=1}^{n}(T - \lambda_i)\right) \mathcal{H}}$ follows. Suppose that $\mathcal{M} = \overline{\left(\prod_{i=1}^{n}(T - \lambda_i)\right) \mathcal{H}}$, then $\mathcal{M}^{\perp} = \operatorname{Ker} \prod_{i=1}^{n}(T^* - \bar{\lambda}_i)$, and since for any λ, $\operatorname{Ker}(T^* - \lambda)$ is either the zero space or a space of dimension equal to one, it follows that $\mathcal{M}^{\perp}$ has a finite dimension, so that $\mathcal{M}$ has a finite codimension.

Conversely, suppose that $\mathcal{M}$ is closed with a finite codimension of n. If $\mathcal{M} = \mathcal{H}$ then simply choose $x = e_1$. Else $P_{\mathcal{M}^{\perp}} e_1 \neq 0$ since e_1 is cyclic for T. Let $\delta_i = P_{\mathcal{M}^{\perp}} e_i$ $i \geq 1$, then for any $x \in \mathcal{M}^{\perp}$, $(P_{\mathcal{M}^{\perp}} T \delta_i, x) = (\delta_i, T^* x) = (e_i, T^* x) = (T e_i, x) = \alpha_i (e_{i+1}, x) = \alpha_i (\delta_{i+1}, x)$ so that $P_{\mathcal{M}^{\perp}} T \delta_i = \alpha_i \delta_{i+1}$. Since $\{e_i\}_{i=1}^{\infty}$ is a complete set for $\mathcal{H}$, $\{\delta_i\}_{i=1}^{\infty}$ is a complete set for $\mathcal{M}^{\perp}$. If $\{\delta_i\}_{i=1}^{n}$ were not linearly independent then there would be constants such that $\sum_{i=1}^{k} c_i \delta_i = 0$, where $k \leq n$ and we assume that $c_k \neq 0$; this implies that δ_k can be written in terms of δ_j $1 \leq j \leq k-1$. Applying $P_{\mathcal{M}^{\perp}} T$ to this relation yields $\sum_{i=1}^{k} c_i \alpha_i \delta_{i+1}$, or that δ_{k+1} can be written in terms of δ_j $1 \leq j \leq k-1$. Applying $P_{\mathcal{M}^{\perp}} T$ again we see that δ_{k+2} can be written in terms of δ_j $1 \leq j \leq k-1$. By iteration we see that all δ_j's with $j \geq k$ can be written in terms of this set. Since $\{\delta_i\}_{i=1}^{\infty}$ is a complete set for $\mathcal{M}^{\perp}$, this implies that $\mathcal{M}^{\perp}$ has a dimension of $k - 1 \leq n - 1$ which is a contradiction. Therefore $\{\delta_i\}_{i=1}^{n}$ is linearly independent and hence a basis for $\mathcal{M}^{\perp}$. Therefore δ_{n+1} can be written in terms of δ_i $1 \leq i \leq n$, and since $\alpha_1 \alpha_2 \cdots \alpha_i \delta_{i+1} = P_{\mathcal{M}^{\perp}} T^i \delta_1$, there is a set of λ_i's such that $P_{\mathcal{M}^{\perp}} \left(\prod_{i=1}^{n}(T - \lambda_i)\right) \delta_1 = 0$, so that $P_{\mathcal{M}^{\perp}} \left(\prod_{i=1}^{n}(T - \lambda_i)\right) \delta_j = P_{\mathcal{M}^{\perp}} \left(\prod_{i=1}^{n}(T - \lambda_i)\right) \frac{P_{\mathcal{M}^{\perp}} T^{j-1}}{\alpha_1 \alpha_2 \cdots \alpha_{j-1}} \delta_1 = \frac{P_{\mathcal{M}^{\perp}} T^{j-1}}{\alpha_1 \alpha_2 \cdots \alpha_{j-1}} P_{\mathcal{M}^{\perp}} \left(\prod_{i=1}^{n}(T - \lambda_i)\right) \delta_1 = 0$ $j \geq 1$ and $P_{\mathcal{M}^{\perp}} \left(\prod_{i=1}^{n}(T - \lambda_i)\right) \mathcal{H} = \{0\}$, and $\overline{\left(\prod_{i=1}^{n}(T - \lambda_i)\right) \mathcal{H}} \subset \mathcal{M}$. Since $\left(\prod_{i=1}^{n}(T - \lambda_i)\right) \mathcal{H}$ has a codimension of at most n (because $(T - \lambda)\mathcal{H}$ has a codimension of either zero or one for any λ), it follows that $\mathcal{M} = \overline{\left(\prod_{i=1}^{n}(T - \lambda_i)\right) \mathcal{H}}$. ∎

Theorem 3.2.8. *Let T be a left-invertible, weighted shift and n be a finite integer. Then there is an operator $\tilde{T}$ that is similar to T and also a weighted shift, and a closed, invariant subspace $\tilde{\mathcal{M}}$ for $\tilde{T}$ such that the residual space of $\tilde{T}$ restricted to $\tilde{\mathcal{M}}$ has a dimension of at least n.*

Proof Since we are only proving something about an operator that is similar to T, by Corollary 3.1.6 we can assume that T has positive weights. In order to prove the theorem according to Theorem 3.2.4 we must show that there is some diagonal operator D, closed, invariant subspace $\mathcal{M}$ for T, and n linearly independent vectors $\{\nu_i\}_{i=1}^{n}$ such that the vectors are orthogonal to $T^i \left(\mathcal{M} \ominus_2 T\mathcal{M}\right)$ $\forall i \geq 0$ in the inner product corresponding to D (so that they are contained in the residual space of T restricted to $\mathcal{M}$ with the new inner product). Note that if the ν_i's are chosen to be eigenvectors for the adjoint of T restricted to $\mathcal{M}$ with respect to the new inner product, then this will follow from $\nu_i \perp_2 \mathcal{M} \ominus_2 T\mathcal{M}$. If f is an eigenvector with eigenvalue λ for T^*, then $((P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}f, (T-\bar{\lambda})m)_2 = ((P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}f, D^*D(T-\bar{\lambda})m) = ((P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}f, P_{\mathcal{M}}D^*D(T-\bar{\lambda})m) = (f, (T-\bar{\lambda})m) = ((T^*-\lambda)f, m) = 0 \ \forall m \in \mathcal{M}$. Thus $(P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}f$ is an eigenvector with eigenvalue λ for the adjoint of T restricted to $\mathcal{M}$ with respect to the new inner product. Since these steps can be reversed, $P_{\mathcal{M}}D^*D\nu_i$ is an eigenvector for the adjoint of T restricted to $\mathcal{M}$ with respect to the original inner product. Thus $\nu_i \perp_2 \mathcal{M} \ominus_2 T\mathcal{M}$ is equivalent to $(D\nu_i, D\varepsilon_D) = (P_{\mathcal{M}}D^*D\nu_i, \varepsilon_D) = (\mu_i, \varepsilon_D) = 0 \ \forall \varepsilon_D \in \mathcal{M} \ominus_2 T\mathcal{M}$ where $\mu_i = P_{\mathcal{M}}D^*D\nu_i \ \ 1 \leq i \leq n$ are eigenvectors for the adjoint of T restricted to $\mathcal{M}$ in the original inner product. Putting all of the above together, it is sufficient to show that for some closed, invariant subspace $\mathcal{M}$ of T there are n linearly independent vectors $\{\mu_i\}_{i=1}^n$ that are eigenvectors for the adjoint of T restricted to $\mathcal{M}$ such that $\mu_i \perp \mathcal{M} \ominus_2 T\mathcal{M} \ \ 1 \leq i \leq n$.

Note that for a weighted shift with positive weights all of the eigenvectors for T^* are of the form $v_\lambda = (1, \frac{\lambda}{\alpha_1}, \frac{\lambda^2}{\alpha_1\alpha_2}, \frac{\lambda^3}{\alpha_1\alpha_2\alpha_3}, \ldots)$ where $|\lambda| \leq \liminf_n(\alpha_1\alpha_2\cdots\alpha_n)^{1/n}$ and if λ satisfies the inequality strictly then the corresponding vector is an eigenvector and the only one with eigenvalue λ. Let $\mathcal{M} = (\text{span}\{v_\lambda\})^\perp$ where λ will be chosen to be something positive later. Note that if $P_{\mathcal{M}}v_\mu$ is nonzero then it is an eigenvector for the adjoint of T restricted to $\mathcal{M}$ with the same eigenvalue, and it is orthogonal to $\mathcal{M} \ominus_2 T\mathcal{M}$ if and only if v_μ is. Since $\mathcal{M}$ has a finite codimension, by Lemma 3.2.7 it is generated by a single vector so that it has an index of one. Since $P_{\mathcal{M}}e_1$ is always contained in $\mathcal{M} \ominus T\mathcal{M}$ and is nonzero since $e_1 \notin \text{span}\{v_\lambda\} = \mathcal{M}^\perp$, it must span this space. Since $\mathcal{M}^\perp$ is the span of v_λ, $P_{\mathcal{M}}e_1$ is $e_1 - \left(e_1, \frac{v_\lambda}{\|v_\lambda\|}\right)\frac{v_\lambda}{\|v_\lambda\|}$ which is a constant (namely $-\frac{1}{\|v_\lambda\|^2}$) times

$$\varepsilon = \left(-\sum_{i=1}^{\infty}\left(\frac{\lambda^i}{\alpha_1\alpha_2\cdots\alpha_i}\right)^2, \frac{\lambda}{\alpha_1}, \frac{\lambda^2}{\alpha_1\alpha_2}, \frac{\lambda^3}{\alpha_1\alpha_2\alpha_3}, \ldots\right).$$

Since $\dim(\mathcal{M} \ominus T\mathcal{M}) = \dim(\mathcal{M} \ominus_2 T\mathcal{M})$, $\varepsilon_D = (P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}\varepsilon$ spans $\mathcal{M} \ominus_2 T\mathcal{M}$. Suppose that D^*D were such that $(D^*D)^{-1}\varepsilon \in \mathcal{M}$, then $P_{\mathcal{M}}D^*D(D^*D)^{-1}\varepsilon = \varepsilon$ so that $(D^*D)^{-1}\varepsilon = (P_{\mathcal{M}}D^*D)^{-1}\varepsilon = \varepsilon_D$. Since $P_{\mathcal{M}}v_\mu$ is always nonzero if $\mu \neq \lambda$ and a vector is contained in $\mathcal{M}$ if and only if it is orthogonal to v_λ, we need to find a D and n different eigenvectors $\{v_{\mu_i}\}_{i=1}^n$ such that $((D^*D)^{-1}\varepsilon, v_\lambda) = 0$ and $((D^*D)^{-1}\varepsilon, v_{\mu_i}) = 0 \ \ 1 \leq i \leq n$.

Since the D is diagonal on the e_i's, $(D^*D)^{-1}$ maps e_i to $\frac{1}{|d_i|^2}e_i$. Define $\hat{D}$ so that it is diagonal on the e_i's and such that $\hat{D}e_i = \frac{1}{|d_i|^2}e_i$ so that $\hat{d}_i = \frac{1}{|d_i|^2}$. To guarantee that $\hat{D} = (D^*D)^{-1}$ for some D we only must require that

$$\hat{d}_i > 0 \ \ \forall i \geq 1 \text{ and } 0 < \inf_i \hat{d}_i \leq \sup_i \hat{d}_i < \infty. \tag{3.3}$$

We will now work backwards. Choose n distinct, negative μ_i's and λ positive but small enough that $q(x) = (x-\lambda)\prod_{i=1}^{n}(x-\mu_i) = \sum_{i=0}^{n+1} c_i x^i$ satisfies $c_i > 0$ $1 \le i \le n$ (this is possible since all of the μ_i's are negative so that the $(x-\mu_i)$'s have positive coefficients with $\prod_{i=1}^{n}(x-\mu_i)$ having all coefficients greater than zero so that for some λ the condition is satisfied; we can also see this from the equations $c_i = (-1)^{n+1-i}\sum_{1\le k_1<k_2<\ldots<k_{n+1-i}\le n+1}\mu_{k_1}\mu_{k_2}\ldots\mu_{k_{n+1-i}}$ where $\mu_{n+1} = \lambda$). Notice that c_0 must be negative. Let $p(x)$ be the unique polynomial of degree n such that $p(\lambda^2) = 0$ and $p(\lambda\mu_i) = -\sum_{j=1}^{\infty}\frac{(\lambda\mu_i)^j}{(\alpha_1\alpha_2\cdots\alpha_j)^2} + \sum_{j=1}^{\infty}\left(\frac{\lambda^j}{\alpha_1\alpha_2\cdots\alpha_j}\right)^2$ $1 \le i \le n$. Choose c large enough that $\tilde{p}(x) = p(x) + cq(\frac{x}{\lambda}) = \sum_{i=0}^{n+1}\tilde{c}_i x^i$ also satisfies $\tilde{c}_0 < 0$ and $\tilde{c}_i > 0$ $1 \le i \le n+1$. Set $\hat{d}_1 = 1 - \tilde{c}_0/\sum_{j=1}^{\infty}\left(\frac{\lambda^j}{\alpha_1\alpha_2\cdots\alpha_j}\right)^2$, $\hat{d}_i = 1 + \tilde{c}_{i-1}(\alpha_1\alpha_2\cdots\alpha_{i-1})^2$ for $2 \le i \le n+2$ and $\hat{d}_i = 1$ $\forall i \ge n+3$. Then the sequence $\{\hat{d}_i\}_{i=1}^{\infty}$ satisfies equation (3.3) and hence defines a diagonal operator $\hat{D} = (D^*D)^{-1}$ on l^2. The equation $(\hat{D}\varepsilon, v_\lambda) = 0$ yields (using the facts that $(\varepsilon, v_\lambda) = 0$ and $\hat{d}_i = 1$ $\forall i \ge n+3$)

$$-\hat{d}_1\sum_{j=1}^{\infty}\left(\frac{\lambda^j}{\alpha_1\alpha_2\cdots\alpha_j}\right)^2 + \sum_{j=2}^{n+2}\hat{d}_j\left(\frac{\lambda^{j-1}}{\alpha_1\alpha_2\cdots\alpha_{j-1}}\right)^2 =$$
$$-\sum_{j=1}^{\infty}\left(\frac{\lambda^j}{\alpha_1\alpha_2\cdots\alpha_j}\right)^2 + \sum_{j=1}^{n+1}\left(\frac{\lambda^j}{\alpha_1\alpha_2\cdots\alpha_j}\right)^2.$$

The equations $(\hat{D}\varepsilon, v_{\mu_i}) = 0$ yield
$\left(\text{since } (\varepsilon, v_{\mu_i}) = -\sum_{j=1}^{\infty}\left(\frac{\lambda^j}{\alpha_1\alpha_2\cdots\alpha_j}\right)^2 + \sum_{j=1}^{\infty}\frac{(\lambda\mu_i)^j}{(\alpha_1\alpha_2\cdots\alpha_j)^2}\right)$

$$-\hat{d}_1\sum_{j=1}^{\infty}\left(\frac{\lambda^j}{\alpha_1\alpha_2\cdots\alpha_j}\right)^2 + \sum_{j=2}^{n+2}\hat{d}_j\frac{(\lambda\mu_i)^{j-1}}{(\alpha_1\alpha_2\cdots\alpha_{j-1})^2} =$$
$$-\sum_{j=1}^{\infty}\frac{(\lambda\mu_i)^j}{(\alpha_1\alpha_2\cdots\alpha_j)^2} + \sum_{j=1}^{n+1}\frac{(\lambda\mu_i)^j}{(\alpha_1\alpha_2\cdots\alpha_j)^2} \quad 1 \le i \le n.$$

Plugging the defining relations $\hat{d}_1 = 1 - \tilde{c}_0/\sum_{j=1}^{\infty}\left(\frac{\lambda^j}{\alpha_1\alpha_2\cdots\alpha_j}\right)^2$, and
$\hat{d}_i = 1 + \tilde{c}_{i-1}(\alpha_1\alpha_2\cdots\alpha_{i-1})^2$ for $2 \le i \le n+2$ into these equations yields

$$\tilde{c}_0 + \sum_{j=1}^{n+1}\tilde{c}_j\lambda^{2j} = 0$$

and

$$\tilde{c}_0 + \sum_{j=1}^{n+1} \tilde{c}_j (\lambda\mu_i)^j = -\sum_{j=1}^{\infty} \frac{(\lambda\mu_i)^j}{(\alpha_1\alpha_2\cdots\alpha_j)^2} + \sum_{j=1}^{\infty} \left(\frac{\lambda^j}{\alpha_1\alpha_2\cdots\alpha_j} \right)^2 \quad 1 \leq i \leq n.$$

These are equivalent to the conditions $\tilde{p}(\lambda^2) = 0$ and $\tilde{p}(\lambda\mu_i) = -\sum_{j=1}^{\infty} \frac{(\lambda\mu_i)^j}{(\alpha_1\alpha_2\cdots\alpha_j)^2} + \sum_{j=1}^{\infty} \left(\frac{\lambda^j}{\alpha_1\alpha_2\cdots\alpha_j} \right)^2 \quad 1 \leq i \leq n$, which are satisfied because they are satisfied by $p(x)$ and $q(\frac{\lambda^2}{\lambda}) = 0$ and $q(\frac{\lambda\mu_i}{\lambda}) = 0 \; 1 \leq i \leq n$. Therefore we have found a $\hat{D}$ with $\hat{d}_i$ bounded away from zero and bounded, so that $(D^*D)^{-1} = \hat{D}$ if we set $d_i = \frac{1}{\sqrt{\hat{d}_i}}$ and $((D^*D)^{-1}\varepsilon, v_\lambda) = 0$ and $((D^*D)^{-1}\varepsilon, v_{\mu_i}) = 0 \; 1 \leq i \leq n$. This guarantees as above that $(D^*D)^{-1}\varepsilon = (P_{\mathcal{M}} D^*D|_{\mathcal{M}})^{-1}\varepsilon$ so that the $P_{\mathcal{M}} v_{\mu_i}$'s are eigenvectors for the adjoint of the restriction of T to $\mathcal{M}$ with the new inner product which are contained in the residual space of T restricted to $\mathcal{M}$ with the new inner product. Also note that the $P_{\mathcal{M}} v_{\mu_i}$'s are linearly independent because the set $\{v_\lambda, v_{\mu_1}, v_{\mu_2}, \ldots, v_{\mu_n}\}$ is (as the eigenvalues $\{\lambda, \mu_1, \mu_2, \ldots, \mu_n\}$ were chosen to be distinct) and $\mathcal{M}^\perp = \text{span}\{v_\lambda\}$. Therefore it must be that T restricted to $\mathcal{M}$ with the inner product defined by D has a residual space of dimension at least n and by Corollary 3.2.5, DTD^{-1} has a closed, invariant subspace with a residual space of dimension at least n. ∎

Corollary 3.2.9. *Let T be a left-invertible, weighted shift. Then there is a weighted shift $\tilde{T}$ with the same weights as T with at most three exceptions such that $\tilde{T}$ has a closed, invariant subspace on which its restriction does not satisfy the* Wandering Subspace Property*.*

In the next section we will see that the unweighted shift only has to have one weight changed so that it will have an invariant subspace on which its restriction does not satisfy the *Wandering Subspace Property.* We have seen that not only are there weighted shifts with residuals on their invariant subspaces, but these operators form a large part of the set of all left-invertible, weighted shifts.

3.3 The Structure of the Invariant Subspaces of S

In this section we will take the theory from the previous section to try to give more precise results when the operator under consideration is the unweighted shift S.

Remember what we learned from Theorem 3.2.4: In order for an operator that is similar to S to have a closed, invariant subspace on which its restriction does not satisfy the *Wandering Subspace Property*, it necessary and sufficient that there is an equivalent inner product and a closed, invariant subspace $\mathcal{M}$ for S such that the restriction of S to $\mathcal{M}$ does not satisfy the property under the new norm. In order to study the property under the new norm, we must know what the new wandering subspace is, which according to equation (3.2) is $(P_{\mathcal{M}}D^*D|_{\mathcal{M}})^{-1}(\mathcal{M}\ominus_1 T\mathcal{M})$.

Since S restricted to any of its subspaces is unitarily equivalent to S, $\mathcal{M}\ominus S\mathcal{M}$ always has a dimension of one, and we denote a fixed representative of this space by ε. We need to change $\varepsilon\in\mathcal{E}$ to ε_D such that $\mathcal{M}\neq\bigvee_{i=0}^{\infty}T^i\varepsilon_D$. Since $P_{\mathcal{M}}D^*D\varepsilon_D=\varepsilon$, $(D^*D\varepsilon_D-\varepsilon,m)=0\ \forall m\in\mathcal{M}$. Suppose that $(\varepsilon,e_1)\neq 0$ and D^*D has $(d_1,1,1,1,\ldots)$ on the diagonal where it must be that $d_1>0$ and suppose that $\varepsilon_D=\sum_{i=0}^{\infty}c_iS^i\varepsilon$, then $(D^*D\varepsilon_D,\varepsilon)=c_0(\|\varepsilon\|^2+(d_1-1)|(\varepsilon,e_1)|)=(\varepsilon,\varepsilon)=\|\varepsilon\|^2$ and $(D^*D\varepsilon_D,S^i\varepsilon)=c_i\|S^i\varepsilon\|^2=(\varepsilon,S^i\varepsilon)=0\ \forall i\geq 1$, therefore ε_D is a constant times ε, so that DSD^{-1} must always possess the *Wandering Subspace Property* on all of its invariant subspaces just as S does.

Now suppose that D^*D has $(1,d_2,1,1,1,\ldots)$ on the diagonal where it must be that $d_2>0$. Then $(D^*D\varepsilon_D,\varepsilon)=c_0(\|\varepsilon\|^2+(d_2-1)|(\varepsilon,e_2)|^2)+c_1(d_2-1)(S\varepsilon,e_2)(e_2,\varepsilon)=\|\varepsilon\|^2$, $(D^*D\varepsilon_D,S\varepsilon)=c_0(d_2-1)(\varepsilon,e_2)(e_2,S\varepsilon)+c_1(\|S\varepsilon\|^2+(d_2-1)|(S\varepsilon,e_2)|^2)=0$ and $c_i\|S^i\varepsilon\|^2=0\ \forall i\geq 2$. Thus $\varepsilon_D=c_0\varepsilon+c_1S\varepsilon$, with the solution being unique because $P_{\mathcal{M}}D^*D|_{\mathcal{M}}$ is invertible, and if ε_D is normalized so that $c_0=1$, then $\varepsilon_D=\varepsilon+\frac{(1-d_2)(\varepsilon,e_2)(e_1,\varepsilon)}{\|\varepsilon\|^2+(d_2-1)|(\varepsilon,e_1)|^2}S\varepsilon$. Since $[e_1+\beta e_2]_S\neq\mathcal{H}$ if $|\beta|>1$ as shown in the remark below, and S and S restricted to one of its invariant subspaces are unitarily equivalent, if $|c_1|>1$ then $[\varepsilon+c_1S\varepsilon]_S\neq\mathcal{M}$ so that $P_{\mathcal{M}}D^*D[\varepsilon+c_1S\varepsilon]_S\neq\mathcal{M}$ so there is an $f\neq 0$ such that $f\perp P_{\mathcal{M}}D^*DT^i(\mathcal{M}\ominus_2 T\mathcal{M})$, and hence $Df\in D\mathcal{M}$ but $Df\notin\bigvee_{i=0}^{\infty}(DTD^{-1})^i(D\mathcal{M}\ominus DTD^{-1}D\mathcal{M})$. Note that multiple D's can be chosen with the same D^*D, but for any D_1 and D_2 with $D_1^*D_1=D_2^*D_2$, $D_1TD_1^{-1}$ and $D_2TD_2^{-1}$ are unitarily equivalent. Thus the question is: For which d_2 is there a ε such that $\left|\frac{(1-d_2)(\varepsilon,e_2)(e_1,\varepsilon)}{\|\varepsilon\|^2+(d_2-1)|(\varepsilon,e_1)|^2}\right|>1$? In order to answer this we will need to know more about the structure of the invariant subspaces of S.

Remark Let $x_\beta=(1,\beta,0,0,0,\ldots)$. Then $[x_\beta]_S$ is the closure of the ranges of all polynomials in S applied to x_β. For any polynomial in S, $p=\sum_{i=0}^{n}c_iS^i$, $px_\beta=p(1,\beta,0,0,0,\ldots)=p(e_1+\beta Se_1)=p(I+\beta S)e_1=(I+\beta S)pe_1$. Since the set of pe_1 is dense in $\mathcal{H}$ and all steps can be reversed, the question of whether $[x_\beta]_S$ is all of $\mathcal{H}$ is the same as whether $I+\beta S$ has a dense range. $I+\beta S$ has a dense range if and only if $I+\bar{\beta}S^*$ is one-to-one or if $\frac{1}{\bar{\beta}}I+S^*$ does not have a nonzero kernel. As shown after Theorem 3.2.2, $\frac{1}{\bar{\beta}}I+S^*$ has an eigenvalue equal to zero if and only if $\left|\frac{1}{\bar{\beta}}\right|<1$ or $|\beta|>1$, so $I+\beta S$ does not have a dense range and hence $[x_\beta]_S$ is not all of $\mathcal{H}$ if and only if $|\beta|>1$.

In general, if $x=\sum_{i=0}^{n}c_ie_{i+1}=\sum_{i=0}^{n}c_iS^ie_1$, then the question of whether $[x]_S$ is all of $\mathcal{H}$ is the same as whether $\sum_{i=0}^{n}c_iS^i$ has a dense range, or whether $\sum_{i=0}^{n}\bar{c}_iS^{*i}$

has a nontrivial kernel.

In general, one can consider a diagonal operator D that eventually has only ones on the diagonal. Again ε_D can be represented as $\sum_{i=0}^{\infty} c_i S^i \varepsilon$ because the unweighted shift satisfies the *Wandering Subspace Property* when restricted to any of its closed, invariant subspaces and because it is an isometry it satisfies the stronger condition that $S^i\varepsilon \perp S^j\varepsilon \;\; i \neq j$. The equations to solve for ε_D based on $(D^*D\varepsilon_D - \varepsilon, S^i\varepsilon) = 0 \;\; \forall i \geq 0$ will be finite since the D operator eventually only has ones on the diagonal. This leads to a system of equations that must be solved to find the c_i's. Once one has the solution, ε_D will be of the form $\sum_{i=0}^{k} c_i S^i \varepsilon$ for some k depending on how many d_i's there are until they are all one, and c_i's depending on the d_i's and the values of $(\varepsilon, e_i) \;\; 1 \leq i \leq k$ and $\|\varepsilon\|^2$.

As shown in Lemma 3.2.7, since ε_D has a finite representation in terms of the $S^i\varepsilon$'s and the restriction of S to any of its closed, invariant subspaces is again an unweighted shift, the residual space must be finite dimensional, so that it only consists of eigenvectors for the adjoint of the restriction. As shown in Theorem 3.2.8, in this case we only need to solve for when there is an eigenvector v_λ for the adjoint of the restriction such that $(\varepsilon_D, v_\lambda) = 0$. Since all eigenvectors of the adjoint of the restriction are of the form $\sum_{i=0}^{\infty} \lambda^i S^i \varepsilon$ with $|\lambda| < 1$, this leads to $\sum_{i=0}^{k} c_i \bar{\lambda}^i = 0$ where the c_i's are from $\varepsilon_D = \sum_{i=0}^{k} c_i S^i \varepsilon$.

Putting all of this together, we need to know for which d_i's that are eventually all ones is there a wandering vector ε such that the solution to the equations $(D^*D \sum_{i=0}^{k} c_i S^i \varepsilon - \varepsilon, S^j\varepsilon) = 0$ for $0 \leq j \leq k$ is orthogonal to the vector $(1, \lambda, \lambda^2, \lambda^3, \ldots)$ for some λ whose modulus is less than one. In order to do this we have the Carathéodory Interpolation Theorem at our disposal [12, 13, 39, 21].

Theorem 3.3.1. *Let $p = (c_1, c_2, \ldots, c_n, 0, 0, 0, \ldots)$ be arbitrary. Then for any $M \geq \sup\{\|(I - P_{S^n\mathcal{H}})p * f\| : \; f \in \mathcal{H}, \; \|f\| = 1\}$ where $p * f$ is defined by the coefficients of $p(x)f(x)$ where $p(x) = \sum_{i=1}^{n} c_i x^{i-1}$ and $f(x) = \sum_{i=1}^{\infty} f_i x^{i-1}$, there is a wandering vector ε for S such that $(\varepsilon, e_i) = c_i \;\; 1 \leq i \leq n$ and $\|\varepsilon\| = M$.*

In the general case we can use Theorem 3.3.1 to know that no matter what values of (ε, e_i) we pick in the equations for ε_D, there is a wandering vector ε that produces those values. The only other unknown in the equations for ε_D is $\|\varepsilon\|$, and the theorem states that we can choose any value greater than or equal to the M from the theorem. So given a D that is eventually ones on the diagonal, we would have to compute what the possible values of ε_D can be as ε ranges over all wandering vectors (but this is a space whose dimension is finite since there are only $k+1$ variables) and see if any ε_D's are orthogonal to any v_λ's.

We now demonstrate by returning to the case where d_2 is the only coefficient that is not one. We need to know when $\left| \frac{(1-d_2)(\varepsilon, e_2)(e_1, \varepsilon)}{\|\varepsilon\|^2 + (d_2 - 1)|(\varepsilon, e_1)|^2} \right|$ is greater than one. Note that another way to look at it that is consistent with the previous paragraph is that $\varepsilon_D = \varepsilon + cS\varepsilon$ where c is the quantity that we want to know when its modulus is greater than one. In order for $(1, c, 0, 0, 0, \ldots)$ to be orthogonal to some v_λ, it is necessary and sufficient that $|c| > 1$, as the unique eigenvector that would be orthogonal to $(1, c, 0, 0, 0, \ldots)$ must have $\lambda = -\frac{1}{c}$. Since $|\lambda|$ must be less than one, we need $|c|$ to be greater than one.

To simplify the notation let $\varepsilon_1 = (\varepsilon, e_1)$ and $\varepsilon_2 = (\varepsilon, e_2)$. Since $\|\varepsilon\|^2$ is always greater than or equal to $|\varepsilon_1|^2$, we would want to use Theorem 3.3.1 with the smallest possible value of M^2, which is (see Appendix A)

$$|\varepsilon_1|^2 + \frac{1}{2}(\sqrt{4|\varepsilon_1|^2 + |\varepsilon_2|^2}|\varepsilon_2| + |\varepsilon_2|^2). \tag{3.4}$$

This leads to the equation

$$|1 - d_2||\varepsilon_2||\varepsilon_1| > \frac{1}{2}(\sqrt{4|\varepsilon_1|^2 + |\varepsilon_2|^2}|\varepsilon_2| + |\varepsilon_2|^2) + d_2|\varepsilon_1|^2,$$

which implies that $d_2 > 1$. Therefore we must have

$$d_2(|\varepsilon_2||\varepsilon_1| - |\varepsilon_1|^2) > \frac{1}{2}(\sqrt{4|\varepsilon_1|^2 + |\varepsilon_2|^2}|\varepsilon_2| + |\varepsilon_2|^2) + |\varepsilon_1||\varepsilon_2|.$$

So that $d_2 > 0$ we must have $|\varepsilon_2| > |\varepsilon_1|$. Rewriting the inequality we obtain

$$d_2(\frac{|\varepsilon_2|}{|\varepsilon_1|} - 1) > \frac{|\varepsilon_2|}{|\varepsilon_1|} + \frac{1}{2}(\sqrt{4 + \left(\frac{|\varepsilon_2|}{|\varepsilon_1|}\right)^2}\frac{|\varepsilon_2|}{|\varepsilon_1|} + \left(\frac{|\varepsilon_2|}{|\varepsilon_1|}\right)^2).$$

And finally

$$d_2 > \frac{\frac{|\varepsilon_2|}{|\varepsilon_1|} + \frac{1}{2}(\sqrt{4 + \left(\frac{|\varepsilon_2|}{|\varepsilon_1|}\right)^2}\frac{|\varepsilon_2|}{|\varepsilon_1|} + \left(\frac{|\varepsilon_2|}{|\varepsilon_1|}\right)^2)}{\frac{|\varepsilon_2|}{|\varepsilon_1|} - 1}.$$

The right-hand side of this expression has a minimum of $\frac{32}{5}$ on the domain $\frac{|\varepsilon_2|}{|\varepsilon_1|} \in [1, \infty)$, so that according to what was shown above, if $d_2 \leq \frac{32}{5}$ then there are no wandering vectors that work, so that every closed, invariant subspace of DSD^{-1} is generated by its wandering subspace, and if $d_2 > \frac{32}{5}$, then there is at least one closed, invariant subspace of DSD^{-1} which is not generated by its wandering subspace.

3.3.1 The Carathéodory Interpolation Theorem

In this subsection we offer a new proof of the Carathéodory Interpolation Theorem based on the theory of operators, by building an extension of the unweighted shift that is again an unweighted shift such that certain properties are satisfied. In order to know how to build an extension, we must first know the structure of a closed, invariant subspace in relation to the whole space.

Proposition 3.3.2. *Let T be a pure, left-invertible operator such that $\mathcal{H} \ominus T\mathcal{H} = span\{e_1\}$, and $\mathcal{M}$ be a closed, invariant subspace of T such that $P_{\mathcal{M}}e_1 \neq 0$. Then the vectors $\varepsilon_{T^i} = P_{\mathcal{M}\cap T^i\mathcal{H}}e_{i+1}$ $i \geq 0$ are mutually orthogonal, wandering vectors (with ε_{T^i} never zero) such that $\mathcal{M} = span\{\varepsilon\} \oplus span\{\varepsilon_T\} \oplus span\{\varepsilon_{T^2}\} \oplus \ldots$. Also, there are mutually orthogonal vectors $\tilde{\varepsilon}_{T^i}$ $i \geq 0$ (with $\tilde{\varepsilon}_{T^i}$ allowed to be zero) such that $\mathcal{M}^{\perp} = span\{\tilde{\varepsilon}\} \oplus span\{\tilde{\varepsilon}_T\} \oplus span\{\tilde{\varepsilon}_{T^2}\} \oplus \ldots$ and $e_i = \sum_{j=0}^{i-1}(c_{i,\varepsilon_{T^j}}\varepsilon_{T^j} + c_{i,\tilde{\varepsilon}_{T^j}}\tilde{\varepsilon}_{T^j})$ $\forall i \geq 1$ for some constants $c_{i,\varepsilon_{T^j}}$ and $c_{i,\tilde{\varepsilon}_{T^j}}$ for $0 \leq j \leq i-1$ $\forall i \geq 1$. Moreover, if $dim(\mathcal{M} \ominus T\mathcal{M}) = 1$, then $\varepsilon_{T^i} = P_{T^i\mathcal{M}}e_{i+1} \in T^i\mathcal{M} \ominus T^{i+1}\mathcal{M}$.*

Proof Let $\mathcal{M}$ be a closed, invariant subspace of the pure, left-invertible operator T such that $\mathcal{H} \ominus T\mathcal{H} = \text{span}\{e_1\}$ and $\mathcal{M} \not\subset T\mathcal{H}$. As already shown in Proposition 3.2.1, $P_{\mathcal{M}}e_1 = \varepsilon \in \mathcal{M} \ominus T\mathcal{M}$ (note that usually we only assume that $\varepsilon \in \mathcal{M} \ominus T\mathcal{M}$, but now we are assuming that ε is the unique $P_{\mathcal{M}}e_1$). Since $(e_1, m) = (e_1, P_{\mathcal{M}}m) = (\varepsilon, m)$ $\forall m \in \mathcal{M}$, a vector $m \in \mathcal{M}$ is orthogonal to ε if and only if $m \in T\mathcal{H}$ (this also shows that $(\varepsilon, e_1) = (\varepsilon, P_{\mathcal{M}}e_1) = (\varepsilon, \varepsilon) = \|\varepsilon\|^2$). Thus $\mathcal{M} \ominus \text{span}\{\varepsilon\} = \mathcal{M} \cap T\mathcal{H}$. Since $\varepsilon = P_{\mathcal{M}}e_1$, e_1 can be written as $e_1 = \varepsilon + \tilde{\varepsilon}$ where $\tilde{\varepsilon} = P_{\mathcal{H}\ominus\mathcal{M}}e_1$ so that $\tilde{\varepsilon} \perp \mathcal{M}$. Since $\varepsilon \in \mathcal{M} \ominus T\mathcal{M}$, if $\dim(\mathcal{M} \ominus T\mathcal{M}) = 1$, then $\mathcal{M} \ominus \text{span}\{\varepsilon\} = T\mathcal{M}$.

Since $(\varepsilon, e_1) \neq 0$, $\varepsilon \notin T\mathcal{H}$, so $T\varepsilon \in T\mathcal{H}$ but $T\varepsilon \notin T^2\mathcal{H}$ (since else there is a $y \in \mathcal{H}$ such that $T\varepsilon = T^2y$ and then $\varepsilon = LT\varepsilon = LT^2y = Ty \in T\mathcal{H}$) and hence $(T\varepsilon, e_2) \neq 0$. Since $T\varepsilon \in T\mathcal{H}$, $\varepsilon_T = P_{\mathcal{M}\cap T\mathcal{H}}e_2 \neq 0$ and the same as above a vector $m \in \mathcal{M} \cap T\mathcal{H}$ is orthogonal to ε_T if and only if $m \in T^2\mathcal{H}$, $(\varepsilon_T, e_2) = \|\varepsilon_T\|^2$, and $\mathcal{M} \cap T\mathcal{H} \ominus \text{span}\{\varepsilon_T\} = \mathcal{M} \ominus \text{span}\{\varepsilon\} \ominus \text{span}\{\varepsilon_T\} = \mathcal{M} \cap T^2\mathcal{H}$. Since $\mathcal{M} \cap T\mathcal{H} \subset T\mathcal{H}$ and $\text{span}\{e_2\} = T\mathcal{H} \ominus T^2\mathcal{H}$, as in Proposition 3.2.1, $\varepsilon_T \in (\mathcal{M} \cap T\mathcal{H}) \ominus T(\mathcal{M} \cap T\mathcal{H})$ so that ε_T is a wandering vector. If $\dim(\mathcal{M} \ominus T\mathcal{M}) = 1$ then since $\mathcal{M} \cap T\mathcal{H} = \mathcal{M} \ominus \text{span}\{\varepsilon\} = T\mathcal{M}$, then $\varepsilon_T \in T\mathcal{M} \ominus T^2\mathcal{M}$. Also, since then $\dim(T\mathcal{M} \ominus T^2\mathcal{M}) = 1$, $\mathcal{M} \ominus \text{span}\{\varepsilon\} \ominus \text{span}\{\varepsilon_T\} = T^2\mathcal{M}$.

Also, $e_2 = \varepsilon_T + P_{\mathcal{H}\ominus(\mathcal{M}\cap T\mathcal{H})}e_2$ where $P_{\mathcal{H}\ominus(\mathcal{M}\cap T\mathcal{H})}e_2 = (P_\varepsilon + P_{\tilde{\varepsilon}} + (I - P_\varepsilon - P_{\tilde{\varepsilon}}))P_{\mathcal{H}\ominus(\mathcal{M}\cap T\mathcal{H})}e_2$, so that $\tilde{\varepsilon}_T = (I - P_\varepsilon - P_{\tilde{\varepsilon}})P_{\mathcal{H}\ominus(\mathcal{M}\cap T\mathcal{H})}e_2$ is contained in $\mathcal{H} \ominus (\text{span}\{\varepsilon\} \oplus \text{span}\{\tilde{\varepsilon}\} \oplus \text{span}\{\varepsilon_T\} \oplus (\mathcal{M} \cap T^2\mathcal{H}))$ and $e_2 = c_{2,\varepsilon}\varepsilon + c_{2,\tilde{\varepsilon}}\tilde{\varepsilon} + \varepsilon_T + \tilde{\varepsilon}_T$ where ε, $\tilde{\varepsilon}$, ε_T, and $\tilde{\varepsilon}_T$ are mutually orthogonal, $\tilde{\varepsilon}, \tilde{\varepsilon}_T \in \mathcal{H} \ominus \mathcal{M}$, $\varepsilon \in \mathcal{M} \ominus T\mathcal{M}$, and $\varepsilon_T \in (\mathcal{M} \cap T\mathcal{H}) \ominus (\mathcal{M} \cap T^2\mathcal{H})$.

Continuing in this fashion we can define $\varepsilon_{T^i} = P_{\mathcal{M}\cap T^i\mathcal{H}}e_{i+1} \neq 0$ and $\tilde{\varepsilon}_{T^i} = (I - \sum_{j=0}^{i-1}\left(P_{\varepsilon_{T^j}} + P_{\tilde{\varepsilon}_{T^j}}\right))P_{\mathcal{H}\ominus(\mathcal{M}\cap T^i\mathcal{H})}e_{i+1}$ where the ε_{T^i} and $\tilde{\varepsilon}_{T^i}$ are mutually orthogonal and the ε_{T^i} are wandering vectors, $\tilde{\varepsilon}_{T^i} \in \mathcal{H} \ominus \mathcal{M}$ $\forall i \geq 0$, $\varepsilon_{T^i} \in (\mathcal{M} \cap T^i\mathcal{H}) \ominus (\mathcal{M} \cap T^{i+1}\mathcal{H})$ $\forall i \geq 0$ and e_i can be written as a linear combination of ε_{T^j} and $\tilde{\varepsilon}_{T^j}$ with $j \leq i-1$. If $\dim(\mathcal{M} \ominus T\mathcal{M}) = 1$ then $\mathcal{M} \ominus \text{span}\{\varepsilon\} \ominus \text{span}\{\varepsilon_T\} \ominus \ldots \ominus \text{span}\{\varepsilon_{T^{i-1}}\} = T^i\mathcal{M}$ so that $\varepsilon_{T^i} \in T^i\mathcal{M} \ominus T^{i+1}\mathcal{M}$ and $T^i\mathcal{M} \ominus \text{span}\{\varepsilon_{T^i}\} = T^{i+1}\mathcal{M}$.

Since $e_1 = \varepsilon+\tilde{\varepsilon}$, any $x \in \mathcal{H}$ which is orthogonal to both ε and $\tilde{\varepsilon}$ must be contained in $T\mathcal{H}$, and since $e_2 = c_{2,\varepsilon}\varepsilon + c_{2,\tilde{\varepsilon}}\tilde{\varepsilon} + \varepsilon_T + \tilde{\varepsilon}_T$, any $x \in \mathcal{H}$ which is orthogonal to ε, $\tilde{\varepsilon}$, ε_T, and $\tilde{\varepsilon}_T$ must be contained in $T^2\mathcal{H}$, and since each e_i can be written as a finite linear combination of ε_{T^j} and $\tilde{\varepsilon}_{T^j}$, any vector which is orthogonal to all ε_{T^j}'s and $\tilde{\varepsilon}_{T^j}$'s must be contained in $T^i\mathcal{H}$ for all i, and since T is pure, this implies that $x = 0$. Therefore $\mathcal{H} = \mathrm{span}\{\varepsilon\} \oplus \mathrm{span}\{\tilde{\varepsilon}\} \oplus \mathrm{span}\{\varepsilon_T\} \oplus \mathrm{span}\{\tilde{\varepsilon}_T\} \oplus \ldots$. Since $\varepsilon_{T^i} \in \mathcal{M}$ and $\tilde{\varepsilon}_{T^i} \in \mathcal{M}^\perp$ $\forall i \geq 0$, it must be that $\mathcal{M} = \mathrm{span}\{\varepsilon\} \oplus \mathrm{span}\{\varepsilon_T\} \oplus \mathrm{span}\{\varepsilon_{T^2}\} \oplus \ldots$ and $\mathcal{M}^\perp = \mathrm{span}\{\tilde{\varepsilon}\} \oplus \mathrm{span}\{\tilde{\varepsilon}_T\} \oplus \mathrm{span}\{\tilde{\varepsilon}_{T^2}\} \oplus \ldots$.

There is one special property in the above for weighted shifts. It is possible that one or more of the $\tilde{\varepsilon}_{T^j}$'s are zero; for a weighted shift, if $\tilde{\varepsilon}_{T^i} = 0$ for some i then $\tilde{\varepsilon}_{T^j} = 0$ $\forall j \geq i$. To see this, note that $T\tilde{\varepsilon} = T(e_1 - \varepsilon) = (\alpha_1 e_2 - T\varepsilon) \in \mathcal{M} \oplus \mathrm{span}\{\tilde{\varepsilon}\} \oplus \mathrm{span}\{\tilde{\varepsilon}_T\}$ and $T\tilde{\varepsilon}_T = T(e_2 - c_{2,\varepsilon}\varepsilon - c_{2,\tilde{\varepsilon}}\tilde{\varepsilon} - \varepsilon_T) = (\alpha_2 e_3 - T\left(c_{2,\varepsilon}\varepsilon + c_{2,\tilde{\varepsilon}}\tilde{\varepsilon} + \varepsilon_T\right)) \in \mathcal{M} \oplus \mathrm{span}\{\tilde{\varepsilon}\} \oplus \mathrm{span}\{\tilde{\varepsilon}_T\} \oplus \mathrm{span}\{\tilde{\varepsilon}_{T^2}\}$ and in general $T\tilde{\varepsilon}_{T^i} \in \mathcal{M} \oplus \mathrm{span}\{\tilde{\varepsilon}\} \oplus \mathrm{span}\{\tilde{\varepsilon}_T\} \oplus \mathrm{span}\{\tilde{\varepsilon}_{T^2}\} \oplus \ldots \oplus \mathrm{span}\{\tilde{\varepsilon}_{T^{i+1}}\}$. Suppose that $\tilde{\varepsilon}_{T^i} = 0$, then since e_{i+1} can be written as a combination of ε_{T^j} $j \leq i$ and $\tilde{\varepsilon}_{T^j}$ $j \leq i-1$, $e_{i+2} = \frac{1}{\alpha_{i+1}} T e_{i+1}$ can be written as a combination of ε_{T^j} $j \leq i+1$ and $\tilde{\varepsilon}_{T^j}$ $j \leq i$, and since $\tilde{\varepsilon}_{T^{i+1}} = (I - \sum_{j=0}^{i}\left(P_{\varepsilon_{T^j}} + P_{\tilde{\varepsilon}_{T^j}}\right))P_{\mathcal{H}\ominus(\mathcal{M}\cap T^{i+1}\mathcal{M})}e_{i+2} = (I - \sum_{j=0}^{i}\left(P_{\varepsilon_{T^j}} + P_{\tilde{\varepsilon}_{T^j}}\right) - P_{\varepsilon_{T^{i+1}}})e_{i+2}$, it follows that $\tilde{\varepsilon}_{T^{i+1}} = 0$, and continuing in this fashion $\tilde{\varepsilon}_{T^j} = 0$ $\forall j \geq i$.

Theorem 3.3.3. *Let $\hat{S}$ be the unweighted shift on the space $\mathcal{K}$ which shifts the basis $\{e_1, e_2, e_3 \ldots\}$. Let $\{c_i\}_{i=1}^n$ be a nonzero, finite vector of complex numbers. Then there is a $\varepsilon \in \mathcal{K}$ such that $\varepsilon \perp \hat{S}^i\varepsilon$ $\forall i \geq 1$, $(\varepsilon, e_i) = c_i$ $1 \leq i \leq n$ and $[\varepsilon]_{\hat{S}}$ has a codimension of n.*

Proof If $c_1 = 0$ then let c_j be the first nonzero c_i, and since $\hat{S}|_{\hat{S}^{j-1}\mathcal{H}}$ is unitarily equivalent to $\hat{S}$ we can construct ε as below using $\{c_i\}_{i=j}^n$ and then append $j-1$ zeros to the front of ε, so we can assume that $c_1 \neq 0$. We will construct the required ε by starting with a restriction of $\hat{S}$, which we will call S, to a closed, invariant subspace $\mathcal{H}$ and extending it to be $\hat{S}$ such that the $\varepsilon = \mathcal{H}\ominus\hat{S}\mathcal{H} = \mathcal{H}\ominus S\mathcal{H}$ from the restriction has the required properties. Define $\mathcal{K} = \mathrm{span}\{\tilde{\varepsilon}\} \oplus \mathrm{span}\{\tilde{\varepsilon}_S\} \oplus \ldots \oplus \mathrm{span}\{\tilde{\varepsilon}_{S^{n-1}}\} \oplus \mathrm{span}\{\varepsilon\} \oplus \mathrm{span}\{\varepsilon_S\} \oplus \mathrm{span}\{\varepsilon_{S^2}\} \oplus \ldots$ for two mutually orthogonal sets of orthonormal vectors $\{\tilde{\varepsilon}_{S^i}\}_{i=0}^{k-1}$ and $\{\varepsilon_{S^i}\}_{i=0}^{\infty}$. Define $\mathcal{H} = \mathrm{span}\{\varepsilon\} \oplus \mathrm{span}\{\varepsilon_S\} \oplus \mathrm{span}\{\varepsilon_{S^2}\} \oplus \ldots$ and $\hat{S}\varepsilon_{S^i} = \varepsilon_{S^{i+1}}$ so that $\hat{S}|_{\mathcal{H}} = S$ is an unweighted shift. Therefore $\varepsilon \in \mathcal{H} \ominus S\mathcal{H}$ with $\|\varepsilon\| = 1$ and $\varepsilon_{S^i} = S^i\varepsilon$ $\forall i \geq 1$. Note that the $\tilde{\varepsilon}_{S^i}$'s will become the ones from Proposition 3.3.2. We will use the notation e_i for what will become a vector in $\hat{S}^{i-1}\mathcal{K} \ominus \hat{S}^i\mathcal{K}$ with $\|e_i\| = 1$ and $e_i = \hat{S}^{i-1}e_1$ $\forall i \geq 2$. According to what was shown above, we need to construct each e_i as a linear combination of ε_{S^j}'s and $\tilde{\varepsilon}_{S^j}$'s with $0 \leq j \leq i-1$ (with other requirements) so that $(\varepsilon, e_i) = cc_i$ $1 \leq i \leq n$ for some $c > 0$. Unlike above, we will assume that $|\varepsilon_{S^i}| = |\tilde{\varepsilon}_{S^i}| = 1$ $\forall i \geq 0$.

Since $S\varepsilon_{S^i} = \varepsilon_{S^{i+1}}$ $\forall i \geq 0$ and $\hat{S}|_{\mathcal{H}} = S$, it remains to define $\hat{S}$ for each $\tilde{\varepsilon}_{S^i}$ for $0 \leq i \leq n-1$. If $|c_1| \geq 1$, pick $c > 0$ so that $|cc_1| < 1$. Define $e_1 = c_{1,\varepsilon}\varepsilon + c_{1,\tilde{\varepsilon}}\tilde{\varepsilon}$ where $c_{1,\varepsilon} = \bar{c}c_1$ and $c_{1,\tilde{\varepsilon}}$ is any constant so that $|c_{1,\varepsilon}|^2 + |c_{1,\tilde{\varepsilon}}|^2 = 1$ (possible since $|cc_1| < 1$), then $\|e_1\| = 1$. We will define e_2 and then set $\hat{S}\tilde{\varepsilon}$ so that $\hat{S}e_1 = e_2$. Define $e_2 = c_{2,\varepsilon}\varepsilon + c_{2,\tilde{\varepsilon}}\tilde{\varepsilon} + c_{2,\varepsilon_S}\varepsilon_S + c_{2,\tilde{\varepsilon}_S}\tilde{\varepsilon}_S$ with $c_{2,\varepsilon_S} = c_{1,\varepsilon}$. Set $c_{2,\varepsilon} = \bar{c}c_2$ and define $c_{2,\tilde{\varepsilon}}$ so that $(e_2, e_1) = |c|^2\bar{c}_2c_1 + c_{2,\tilde{\varepsilon}}\bar{c}_{1,\tilde{\varepsilon}} = 0$. Pick $c_{2,\tilde{\varepsilon}_S} \neq 0$ so that $|c_{2,\varepsilon}|^2 + |c_{2,\tilde{\varepsilon}}|^2 + |c_{2,\varepsilon_S}|^2 + |c_{2,\tilde{\varepsilon}_S}|^2 = 1$. If this is not possible then pick a new c which is smaller, then since any smaller c will work in the definition of e_1, $c_{2,\varepsilon_S} = \bar{c}c_1$, $c_{2,\varepsilon} = \bar{c}c_2$ and $c_{2,\tilde{\varepsilon}} = \frac{-|c|^2\bar{c}_2c_1}{\bar{c}_{1,\tilde{\varepsilon}}}$ all go to zero as c does ($\bar{c}_{1,\tilde{\varepsilon}}$ goes to one in modulus as c goes to zero), there must be a smaller c which works. Notice that we have $(\hat{S}e_1, \hat{S}e_1) = (e_2, e_2) = (e_1, e_1)$ as required for $\hat{S}$ to be isometric. Also, since $e_2 = \hat{S}e_1$ and $\hat{S}\varepsilon = \varepsilon_S$ are orthogonal to e_1 and $e_1 = c_{1,\varepsilon}\varepsilon + c_{1,\tilde{\varepsilon}}\tilde{\varepsilon}$, it must be that $\hat{S}\tilde{\varepsilon}$ is orthogonal to e_1.

Define $e_3 = c_{3,\varepsilon}\varepsilon + c_{3,\tilde{\varepsilon}}\tilde{\varepsilon} + c_{3,\varepsilon_S}\varepsilon_S + c_{3,\tilde{\varepsilon}_S}\tilde{\varepsilon}_S + c_{3,\varepsilon_{S^2}}\varepsilon_{S^2} + c_{3,\tilde{\varepsilon}_{S^2}}\tilde{\varepsilon}_{S^2}$ with $c_{3,\varepsilon_{S^2}} = c_{2,\varepsilon_S} = c_{1,\varepsilon}$ and $c_{3,\varepsilon_S} = c_{2,\varepsilon}$. Set $c_{3,\varepsilon} = \bar{c}c_3$ and define $c_{3,\tilde{\varepsilon}}$ so that $(e_3, e_1) = |c|^2\bar{c}_3c_1 + c_{3,\tilde{\varepsilon}}\bar{c}_{1,\tilde{\varepsilon}} = 0$ and define $c_{3,\tilde{\varepsilon}_S}$ so that $(e_3, e_2) = |c|^2\bar{c}_3c_2 + |c|^2\bar{c}_2c_1 + c_{3,\tilde{\varepsilon}}\bar{c}_{2,\tilde{\varepsilon}} + c_{3,\tilde{\varepsilon}_S}\bar{c}_{2,\tilde{\varepsilon}_S} = 0$. Pick $c_{3,\tilde{\varepsilon}_{S^2}} \neq 0$ so that $|c_{3,\varepsilon}|^2 + |c_{3,\tilde{\varepsilon}}|^2 + |c_{3,\varepsilon_S}|^2 + |c_{3,\tilde{\varepsilon}_S}|^2 + |c_{3,\varepsilon_{S^2}}|^2 + |c_{3,\tilde{\varepsilon}_{S^2}}|^2 = 1$. If this is not possible then pick a new c which is smaller, and as before since $c_{3,\varepsilon_{S^i}} = \bar{c}c_{3-i}$, they all go to zero as c goes to zero and due to the equations for the e_i's to be orthogonal the $c_{3,\tilde{\varepsilon}_{S^i}}$'s all go to zero if $i \neq 2$ and the $c_{i,\tilde{\varepsilon}_{S^{i-1}}}$ $i = 1, 2$ go to one in modulus, so there must be a smaller c that works (for e_1, e_2 and e_3). Notice that we have $(\hat{S}e_2, \hat{S}e_2) = (e_3, e_3) = (e_2, e_2)$ and $(\hat{S}e_2, \hat{S}e_1) = (e_3, e_2) = (e_2, e_1)$ as required for $\hat{S}$ to be isometric on $\mathrm{span}\{e_1, e_2\}$. Also, since $e_3 = \hat{S}e_2$ and $\hat{S}\varepsilon_S = \varepsilon_{S^2}$ are orthogonal to e_1 and $e_2 = c_{2,\varepsilon}\varepsilon + c_{2,\tilde{\varepsilon}}\tilde{\varepsilon} + c_{2,\varepsilon_S}\varepsilon_S + c_{2,\tilde{\varepsilon}_S}\tilde{\varepsilon}_S$, it must be that $\hat{S}\tilde{\varepsilon}_S$ is orthogonal to e_1.

Continuing in this fashion, e_i can be defined as $\sum_{j=0}^{i-1}\left(c_{i,\varepsilon_{S^j}}\varepsilon_{S^j} + c_{i,\tilde{\varepsilon}_{S^j}}\tilde{\varepsilon}_{S^j}\right)$ with $c_{i,\varepsilon_{S^j}} = c_{i-1,\varepsilon_{S^{j-1}}}$ $1 \leq j \leq i-2$ and $c_{i,\varepsilon} = \bar{c}c_i$. Then define the other constants so that $(e_i, e_j) = 0$ $\forall j < i$ and $\|e_i\|^2 = 1$ with $c_{i,\tilde{\varepsilon}_{S^{i-1}}} \neq 0$, which is always possible for some c since $c_{i,\varepsilon_{S^j}} = \bar{c}c_{i-j}$ so that they all go to zero as c goes to zero and due to the equations for the e_i's to be orthogonal the $c_{i,\tilde{\varepsilon}_{S^j}}$'s all go to zero if $j \neq i-1$ and the $c_{j,\tilde{\varepsilon}_{S^{j-1}}}$ $1 \leq j \leq i-1$ go to one in modulus. As before the $\hat{S}\tilde{\varepsilon}_{S^{i-1}}$'s will be orthogonal to e_1. Do this until e_n is defined. Since there are no more c_i's to constrain the e_i's, we are free to choose the remaining $c_{i,\varepsilon_{S^j}}$'s and $c_{i,\tilde{\varepsilon}_{S^j}}$'s.

Define $\hat{S}\tilde{\varepsilon}_{S^{n-1}} = c_{n+1,\varepsilon}\varepsilon + c_{n+1,\tilde{\varepsilon}}\tilde{\varepsilon} + c_{n+1,\tilde{\varepsilon}_S}\tilde{\varepsilon}_S + c_{n+1,\tilde{\varepsilon}_{S^2}}\tilde{\varepsilon}_{S^2} + \ldots + c_{n+1,\tilde{\varepsilon}_{S^{n-1}}}\tilde{\varepsilon}_{S^{n-1}} + c_{n+1,\tilde{\varepsilon}_{S^n}}\tilde{\varepsilon}_{S^n}$ (notice that we are extending the shift in a different way than before and that the constants are now associated with the definition of the $\hat{S}\tilde{\varepsilon}_{S^i}$'s instead of the $\hat{S}e_i$'s). Set $c_{n+1,\varepsilon}$ equal to one for now, and pick $c_{n+1,\tilde{\varepsilon}}$ so that $\hat{S}\tilde{\varepsilon}_{S^{n-1}}$ is orthogonal to e_1 (possible since $c_{1,\tilde{\varepsilon}} \neq 0$), then pick $c_{n+1,\tilde{\varepsilon}_S}$ independently of the chosen values of the other $c_{n+1,\tilde{\varepsilon}_{S^i}}$'s so that $\hat{S}\tilde{\varepsilon}_{S^{n-1}}$ is orthogonal to

$\hat{S}\tilde{\varepsilon}$ (possible since $(e_2, \varepsilon_S) = d(\hat{S}\tilde{\varepsilon}, \varepsilon_S)$ for some nonzero d and $c_{2,\tilde{\varepsilon}_S} \neq 0$), then pick $c_{n+1,\tilde{\varepsilon}_{S^2}}$ (again independently) so that $\hat{S}\tilde{\varepsilon}_{S^{n-1}}$ is orthogonal to $\hat{S}\tilde{\varepsilon}_S$ (possible since $(e_3, \varepsilon_{S^2}) = d(\hat{S}\tilde{\varepsilon}_S, \varepsilon_{S^2})$ for some nonzero d and $c_{3,\tilde{\varepsilon}_{S^2}} \neq 0$), and continue in this fashion until all of the constants are chosen and $\hat{S}\tilde{\varepsilon}_{S^{n-1}}$ is orthogonal to $\hat{S}\tilde{\varepsilon}_{S^i}$ for $i \leq n-2$. Now scale the constants so that $\hat{S}\tilde{\varepsilon}_{S^{n-1}}$ will have a norm of one, this will force what would have been $c_{n+1,\tilde{\varepsilon}_{S^n}}$ to be zero, and since we are only scaling, $\hat{S}\tilde{\varepsilon}_{S^{n-1}}$ will still be orthogonal to the vectors from before. We have now defined $\hat{S}$ on all of $\mathcal{K} = \text{span}\{e_1, e_2, \ldots, e_{n-1}, \tilde{\varepsilon}_{S^{n-1}}, \varepsilon, \varepsilon_S, \varepsilon_{S^2}, \ldots\}$. Since $(\hat{S}e_i, \hat{S}e_j) = (e_i, e_j)\ \forall i, j \leq n-1$, $(\hat{S}e_j, \hat{S}\varepsilon_{S^i}) = (e_j, \varepsilon_{S^i})\ i \geq 0,\ j \leq n-1$, $(\hat{S}\varepsilon_{S^j}, \hat{S}\varepsilon_{S^i}) = (\varepsilon_{S^j}, \varepsilon_{S^i})\ \forall i, j \geq 0$, and $\tilde{\varepsilon}_{S^i} \in \text{span}\{e_1, e_2, \ldots, e_{n-1}, \varepsilon, \varepsilon_S, \varepsilon_{S^2}, \ldots\}$ for $0 \leq i \leq n-2$, it must be that $(\hat{S}\tilde{\varepsilon}_{S^i}, \hat{S}\tilde{\varepsilon}_{S^j}) = (\tilde{\varepsilon}_{S^i}, \tilde{\varepsilon}_{S^j})\ \forall i, j \leq n-2$ and $(\hat{S}\tilde{\varepsilon}_{S^j}, \hat{S}\varepsilon_{S^i}) = (\tilde{\varepsilon}_{S^j}, \varepsilon_{S^i})\ \forall i \geq 0,\ j \leq n-2$. Since we defined $\hat{S}\tilde{\varepsilon}_{S^{n-1}}$ so that $(\hat{S}\tilde{\varepsilon}_{S^{n-1}}, \hat{S}\tilde{\varepsilon}_{S^{n-1}}) = (\tilde{\varepsilon}_{S^{n-1}}, \tilde{\varepsilon}_{S^{n-1}}) = 1$, $(\hat{S}\tilde{\varepsilon}_{S^{n-1}}, \hat{S}\tilde{\varepsilon}_{S^j}) = (\tilde{\varepsilon}_{S^{n-1}}, \tilde{\varepsilon}_{S^j}) = 0$ for $0 \leq j \leq n-2$ and $(\hat{S}\tilde{\varepsilon}_{S^{n-1}}, \hat{S}\varepsilon_{S^i}) = (\tilde{\varepsilon}_{S^{n-1}}, \tilde{\varepsilon}_{S^i}) = 0\ \forall i \geq 0$ and $\mathcal{K}$ is spanned by $\{\varepsilon_{S^i}\}_{i=0}^{\infty}$ and $\{\tilde{\varepsilon}_{S^j}\}_{j=0}^{n-1}$, according to Proposition 2.3.3, $\hat{S}$ is an isometry on $\mathcal{K}$. Note that $(\frac{1}{c}\varepsilon, e_i) = \frac{1}{c}cc_i = c_i\ \ 1 \leq i \leq n$ due to the way that the e_i's were constructed, so $\frac{1}{c}\varepsilon$ works as the desired wandering vector. Also, $[\varepsilon]_{\hat{S}}$ will have a codimension of n since $([\varepsilon]_{\hat{S}})^{\perp} = \text{span}\{\tilde{\varepsilon}, \tilde{\varepsilon}_S, \ldots, \tilde{\varepsilon}_{S^{n-1}}\}$. We still must show that $\hat{S}$ has a multiplicity of one and that it is pure so that it is a shift with a multiplicity of one.

Since the $\hat{S}\varepsilon_{S^i}$'s and $\hat{S}\tilde{\varepsilon}_{S^j}$'s are orthogonal to e_1, $e_1 \in \mathcal{K} \ominus \hat{S}\mathcal{K}$. Suppose that $x \in \mathcal{K} \ominus \hat{S}\mathcal{K}$ is a vector not in the span of e_1; we can assume that e_1 and x are orthogonal since $\mathcal{K} \ominus \hat{S}\mathcal{K}$ is a subspace. Since $\hat{S}\varepsilon_{S^i} = \varepsilon_{S^{i+1}}\ \forall i \geq 0$, x must be orthogonal to $\varepsilon_{S^i}\ \forall i \geq 1$ and hence $x \in \mathcal{N} = \text{span}\{\varepsilon\} \oplus \text{span}\{\tilde{\varepsilon}\} \oplus \text{span}\{\tilde{\varepsilon}_S\} \oplus \text{span}\{\tilde{\varepsilon}_{S^2}\} \oplus \ldots \oplus \text{span}\{\tilde{\varepsilon}_{S^{n-1}}\}$, but e_1 and $\hat{S}\varepsilon_{S^i}\ \ 0 \leq i \leq n-1$ are orthogonal and in the subspace $\mathcal{N}$, and since their number is the same as the dimension of $\mathcal{N}$, x must be the zero vector. Therefore $\hat{S}$ has multiplicity (at least its shift part) equal to one.

We will now show that $\hat{S}$ is pure, or that no vector can be orthogonal to $\hat{S}^i e_1\ \forall i \geq 0$. First notice that $\hat{S}$ does not have any eigenvectors. If $x \neq 0$ were an eigenvector, then since $\hat{S}$ is an isometry it must be an eigenvector with eigenvalue of modulus one. If $(x, \varepsilon_{S^i}) = d \neq 0$ for some i, then since $\hat{S}x = \lambda x$ with $|\lambda| = 1$ and $\hat{S}^*\hat{S} = I$ since $\hat{S}$ is an isometry, $d = (x, \varepsilon_{S^i}) = (\hat{S}^{*j}\hat{S}^j x, \varepsilon_{S^i}) = (\lambda^j x, \hat{S}^j \varepsilon_{S^i}) = \lambda^j (x, \varepsilon_{S^{i+j}})\ \forall j \geq 0$ so that $\|x\|^2 \geq \sum_{j=i}^{\infty} |(x, \varepsilon_{S^j})|^2 = \sum_{j=i}^{\infty} \frac{|d|^2}{|\lambda|^{2j}} = \sum_{j=i}^{\infty} |d|^2 = \infty$, so it must be that x is contained in the span of the $\tilde{\varepsilon}_{S^j}$'s. Also, $x = \hat{S}^*\hat{S}x = \lambda\hat{S}^* x$ so that x is an eigenvector with eigenvalue of modulus one for $\hat{S}^*$. Since $\hat{S}\hat{S}^*$ is the projection onto the range of $\hat{S}$, x must be orthogonal to e_1, and likewise $\hat{S}^{*j}x$ must be orthogonal to e_1 for every j so that x must be orthogonal to $\hat{S}^j e_1$ for every j. Since x is orthogonal to both e_1 and ε, it must be orthogonal to $\tilde{\varepsilon}$. Since x is orthogonal to e_2, ε, $\tilde{\varepsilon}$ and ε_S, it must be orthogonal to $\tilde{\varepsilon}_S$. Likewise for analogous reasons x must be orthogonal to all $\tilde{\varepsilon}_{S^j}$'s, and hence x must be zero. We conclude that $\hat{S}$ has no eigenvectors as asserted.

Suppose that $0 \neq f \in \cap_{i=0}^{\infty} \hat{S}^i \mathcal{K}$. Then $\hat{S}^{*j} f \in \cap_{i=0}^{\infty} \hat{S}^i \mathcal{K}$ $\forall j \geq 0$ and $\hat{S}^{*j} f \perp e_1$ $\forall j \geq 0$. If the set $\{\hat{S}^{*j} f\}_{j=0}^{n+1}$ were not linearly independent, then there would be some constants such that $\sum_{j=0}^{n+1} d_j \hat{S}^{*j} f = 0$. Applying $\hat{S}^{n+1}$ to this relation and using the fact that $\hat{S}\hat{S}^*$ is the projection onto the range of $\hat{S}$ and $\hat{S}^{*j} f \perp e_1$ $\forall j \geq 0$, we obtain $\sum_{j=0}^{n+1} d_j \hat{S}^{n+1-j} f = 0$, which cannot happen because $\hat{S}$ does not have any eigenvectors. Since $\mathrm{span}\{f, \hat{S}^* f, \hat{S}^{*2} f, \ldots, \hat{S}^{*(n+1)} f\}$ has a dimension of $n+2$, there must be a nonzero g contained in this subspace which is orthogonal to all $\tilde{\varepsilon}_{S^j}$ $0 \leq j \leq n$ so that $g \in \mathcal{H}$. Since $g \in \cap_{i=0}^{\infty} \hat{S}^i \mathcal{K}$, by Theorem 2.3.4 $\hat{S}^{*j} g$ must be orthogonal to e_1 for every $j \geq 0$ but since g is contained in the span of the ε_{S^j}'s, this means that $\hat{S}^{*j} g$ must be orthogonal to ε for every $j \geq 0$ or g must be orthogonal to $\hat{S}^j \varepsilon = S^j \varepsilon$ for every $j \geq 0$, and hence $g \perp \mathcal{H}$, which contradicts the fact that g was assumed to be nonzero. Therefore $\cap_{i=0}^{\infty} \hat{S}^i \mathcal{K} = \{0\}$, and again by Theorem 2.3.4 $\hat{S}$ is pure. ∎

Remark 3.3.1. The proof above involves choosing the parameter $c > 0$ smaller and smaller. The largest such $c > 0$ that can be used is $\frac{1}{M}$ where $M = \inf\{\|\varepsilon\| : (\varepsilon, e_i) = c_i \ 1 \leq i \leq n, \ \varepsilon \text{ is wandering}\}$. From the Theory of Interpolation (see for example [21]) it is known that

$$M = \|(I - P_{S^n\mathcal{H}})p(S)|_{\mathcal{H}\ominus S^n\mathcal{H}}\| = \left\| \begin{bmatrix} c_1 & 0 & 0 & 0 \\ c_2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ c_n & \cdots & c_2 & c_1 \end{bmatrix} \right\|.$$

We will use the next general proposition in the following theorem:

Proposition 3.3.4. *Given vectors $\{w_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ and constants $\{c_i\}_{i=1}^n$, there is a unique $V \in \mathrm{span}\{w_1, w_2, \ldots, w_n\}$ such that $(V, v_i) = c_i$ $1 \leq i \leq n$ if and only if the matrix*

$$A = \begin{bmatrix} (w_1, v_1) & (w_2, v_1) & \cdots & (w_n, v_1) \\ (w_1, v_2) & (w_2, v_2) & \cdots & (w_n, v_2) \\ \vdots & \vdots & & \vdots \\ (w_1, v_n) & (w_2, v_n) & \cdots & (w_n, v_n) \end{bmatrix}$$

is invertible.

Theorem 3.3.5. *Let $\hat{S}$ be the unweighted shift on the space $\mathcal{K}$. Let $\{c_i\}_{i=1}^n$ be a nonzero, finite vector of complex numbers, then there is a $\varepsilon \in \mathcal{K}$ such that $\varepsilon \perp \hat{S}^i \varepsilon$ $\forall i \geq 1$, $(\varepsilon, e_i) = c_i$ $1 \leq i \leq n$ and $[\varepsilon]_{\hat{S}}$ has a codimension of k, where k is any integer such that $k \geq n$ or $k = \infty$.*

Proof If $k \geq n$ is finite, then append $n-k$ zeros to $\{c_i\}_{i=1}^n$ and apply the previous theorem.

We now consider the case where $k = \infty$. Define $\mathcal{K} = \text{span}\{\varepsilon\} \oplus \text{span}\{\tilde{\varepsilon}\} \oplus \text{span}\{\varepsilon_S\} \oplus \text{span}\{\tilde{\varepsilon}_S\} \oplus \ldots$, for two mutually orthogonal sets of orthonormal vectors $\{\tilde{\varepsilon}_{S^i}\}_{i=0}^\infty$ and $\{\varepsilon_{S^i}\}_{i=0}^\infty$. Define $\mathcal{H} = \text{span}\{\varepsilon\} \oplus \text{span}\{\varepsilon_S\} \oplus \text{span}\{\varepsilon_{S^2}\} \oplus \ldots$ and $\hat{S}\varepsilon_{S^i} = \varepsilon_{S^{i+1}}$ so that $\hat{S}|_{\mathcal{H}} = S$ is an unweighted shift. We will now continue the proof as above in the proof of Theorem 3.3.3 where we assume that $c_{i,\tilde{\varepsilon}_{S^{i-1}}}$ is never zero so that $[\varepsilon]_{\hat{S}}$ will not have a finite codimension. Let V_i be defined by $V_i \in \text{span}\{\varepsilon\} \oplus \text{span}\{\tilde{\varepsilon}\} \oplus \text{span}\{\tilde{\varepsilon}_S\} \oplus \ldots \oplus \text{span}\{\tilde{\varepsilon}_{S^i}\}$, $(V_i, \varepsilon) = 1$, $V_i \perp e_1$ and $V_i \perp \hat{S}\tilde{\varepsilon}_{S^j}$ $0 \leq j \leq i-1$; this is equivalent to solving $Ax = b$ with

$$A = \begin{bmatrix} (\varepsilon,\varepsilon) & (\tilde{\varepsilon},\varepsilon) & \cdots & (\tilde{\varepsilon}_{S^i},\varepsilon) \\ (\varepsilon,e_1) & (\tilde{\varepsilon},e_1) & \cdots & (\tilde{\varepsilon}_{S^i},e_1) \\ (\varepsilon,\hat{S}\tilde{\varepsilon}) & (\tilde{\varepsilon},\hat{S}\tilde{\varepsilon}) & \cdots & (\tilde{\varepsilon}_{S^i},\hat{S}\tilde{\varepsilon}) \\ \vdots & \vdots & & \vdots \\ (\varepsilon,\hat{S}\tilde{\varepsilon}_{S^{i-2}}) & (\tilde{\varepsilon},\hat{S}\tilde{\varepsilon}_{S^{i-2}}) & \cdots & (\tilde{\varepsilon}_{S^i},\hat{S}\tilde{\varepsilon}_{S^{i-2}}) \\ (\varepsilon,\hat{S}\tilde{\varepsilon}_{S^{i-1}}) & (\tilde{\varepsilon},\hat{S}\tilde{\varepsilon}_{S^{i-1}}) & \cdots & (\tilde{\varepsilon}_{S^i},\hat{S}\tilde{\varepsilon}_{S^{i-1}}) \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

Then set $V_i = d_\varepsilon \varepsilon + d_{\tilde{\varepsilon}}\tilde{\varepsilon} + \ldots + d_{\tilde{\varepsilon}_{S^i}}\tilde{\varepsilon}_{S^i}$ where $x = (d_\varepsilon, d_{\tilde{\varepsilon}}, \ldots, d_{\tilde{\varepsilon}_{S^i}})$ is the solution of $Ax = b$. This is always possible since $\hat{S}$ was constructed so that this matrix would be lower-triangular, and $(\tilde{\varepsilon}_{S^{j+1}}, \hat{S}\tilde{\varepsilon}_{S^j}) \neq 0$ $\forall j \geq 0$ so that the entries on the diagonal are all nonzero. Therefore, by Proposition 3.3.4, since A is invertible there is a unique V_i that satisfies the conditions.

For $i \geq n$ define $\hat{S}\tilde{\varepsilon}_{S^{i-1}} = c_{i+1,V_{i-1}}V_{i-1} + c_{i+1,\tilde{\varepsilon}_{S^i}}\tilde{\varepsilon}_{S^i}$ where $c_{i+1,V_{i-1}}$ and $c_{i+1,\tilde{\varepsilon}_{S^i}} \neq 0$ are any constants so that $\|\hat{S}\tilde{\varepsilon}_{S^{i-1}}\| = 1$. As before by Proposition 2.3.3, $\hat{S}$ will be an isometry on $\mathcal{K} = \overline{\text{span}\{\varepsilon, \tilde{\varepsilon}, \varepsilon_S, \tilde{\varepsilon}_S, \varepsilon_{S^2}, \tilde{\varepsilon}_{S^2}, \ldots\}}$ since we start with $(\hat{S}\varepsilon_{S^j}, \hat{S}\varepsilon_{S^i}) = (\varepsilon_{S^j}, \varepsilon_{S^i})$ $\forall i,j \geq 0$, $(\hat{S}\tilde{\varepsilon}_{S^i}, \hat{S}\tilde{\varepsilon}_{S^j}) = (\tilde{\varepsilon}_{S^i}, \tilde{\varepsilon}_{S^j})$ $\forall i,j \leq n-2$, and $(\hat{S}\tilde{\varepsilon}_{S^j}, \hat{S}\varepsilon_{S^i}) = (\tilde{\varepsilon}_{S^j}, \varepsilon_{S^i})$ $\forall i \geq 0$, $j \leq n-2$ and we are defining $\hat{S}\tilde{\varepsilon}_{S^j}$ $\forall j \geq n-1$ so that $(\hat{S}\tilde{\varepsilon}_{S^j}, \hat{S}\tilde{\varepsilon}_{S^j}) = 1 = (\tilde{\varepsilon}_{S^j}, \tilde{\varepsilon}_{S^j})$ $\forall j \geq n-1$,

$$\begin{aligned}(\hat{S}\tilde{\varepsilon}_{S^i}, \hat{S}\tilde{\varepsilon}_{S^j}) &= (c_{i+2,V_i}V_i + c_{i+2,\tilde{\varepsilon}_{S^{i+1}}}\tilde{\varepsilon}_{S^{i+1}}, \hat{S}\tilde{\varepsilon}_{S^j}) = (c_{i+2,\tilde{\varepsilon}_{S^{i+1}}}\tilde{\varepsilon}_{S^{i+1}}, \hat{S}\tilde{\varepsilon}_{S^j}) \\ &= 0 = (\tilde{\varepsilon}_{S^i}, \tilde{\varepsilon}_{S^j}) \ \forall i > j \geq n-1\end{aligned}$$

and $(\hat{S}\tilde{\varepsilon}_{S^j}, \hat{S}\varepsilon_{S^i}) = (\tilde{\varepsilon}_{S^j}, \varepsilon_{S^i})$ $\forall i \geq 0$, $j \geq n-1$ and $\mathcal{K}$ is spanned by $\{\varepsilon_{S^i}\}_{i=0}^\infty$ and $\{\tilde{\varepsilon}_{S^j}\}_{j=0}^\infty$.

Since $\hat{S}\tilde{\varepsilon}_{S^i}$ and $\hat{S}\varepsilon_{S^j}$ are orthogonal to e_1 for all $i,j \geq 0$, $e_1 \in \mathcal{K} \ominus \hat{S}\mathcal{K}$. We must show that for some choice of constants $c_{i+1,V_{i-1}}$, $c_{i+1,\tilde{\varepsilon}_{S^i}}$ $i \geq n$, $\mathcal{K} \subset \hat{\mathcal{K}} = \text{span}\{e_1\} \oplus \text{span}\{\hat{S}e_1\} \oplus \text{span}\{\hat{S}^2 e_1\} \oplus \ldots$ so that $\hat{S}$ is a shift with a multiplicity of one. Notice that if $\varepsilon \in \hat{\mathcal{K}}$, then $\varepsilon_{S^i} \in \hat{\mathcal{K}}$ $\forall i \geq 0$ because $\hat{\mathcal{K}}$ is invariant for $\hat{S}$. Also, then $\tilde{\varepsilon} \in \hat{\mathcal{K}}$ because e_1 and ε are contained in $\hat{\mathcal{K}}$ and $\tilde{\varepsilon}$ is a linear combination of them. Similarly, since $\tilde{\varepsilon}_S$ is a linear combination of e_1, e_2, ε and ε_S, and they would be contained in the space, $\tilde{\varepsilon}_S$ would be, and for analogous reasons, $\tilde{\varepsilon}_{S^i}$ $\forall i \geq 0$ would be. We conclude that to show that $\mathcal{K} \subset \hat{\mathcal{K}}$, it suffices to show that $\varepsilon \in \hat{\mathcal{K}}$.

Since

$$\begin{aligned}\|P_{\mathcal{H}}\hat{S}^i e_1\|^2 &= \sum_{j=0}^{\infty} |(\hat{S}^i e_1, \varepsilon_{S^j})|^2 = \sum_{j=0}^{i} |(\hat{S}^i e_1, \varepsilon_{S^j})|^2 = \sum_{j=0}^{i} |(\hat{S}^{i-j} e_1, \hat{S}^{*j}\varepsilon_{S^j})|^2 \\ &= \sum_{j=0}^{i} |(\hat{S}^{i-j} e_1, \varepsilon)|^2 = \sum_{j=0}^{i} |(\hat{S}^j e_1, \varepsilon)|^2\end{aligned}$$

and $\|P_{\tilde{\mathcal{K}}}\varepsilon\|^2 = \sum_{j=0}^{\infty} |(\hat{S}^j e_1, \varepsilon)|^2$, we see that $\varepsilon \in \hat{\mathcal{K}}$ if and only if $\lim_{i\to\infty} \|P_{\mathcal{H}}\hat{S}^i e_1\|^2 = 1$.

Let $m = n$. Since we know that if at any time we pick $\hat{S}\tilde{\varepsilon}_i$ so that $c_{i+2,\tilde{\varepsilon}_{S^{i+1}}} = 0$ we will have $\epsilon \in \text{span}\{e_1\} \oplus \text{span}\{\hat{S}e_1\} \oplus \text{span}\{\hat{S}^2 e_1\} \oplus \ldots$, we will pick the $c_{i,\tilde{\varepsilon}_{S^i}}$'s to mimic this behavior. Note that due to the form of $\hat{S}$, $\|P_{\mathcal{H}}\hat{S}^i e_1\|^2$ cannot decrease with i. Pick a δ between zero and one. Let l be such that $\|P_{\mathcal{H}}\bar{S}^{m+l} e_1\|^2 > 1 - \frac{\delta}{2}$ where $\bar{S}$ represents the operator had we set $c_{m+1,\tilde{\varepsilon}_{S^m}}$ equal to zero. Note that as $c_{m+1,\tilde{\varepsilon}_{S^m}}$ goes to zero, $\hat{S}^{m+l} e_1$ converges to $\bar{S}^{m+l} e_1$ since $m + l$ is finite and $\hat{S}$ is bounded. Therefore there must be a $c_{m+1,\tilde{\varepsilon}_{S^m}}$ such that $\|P_{\mathcal{H}}\hat{S}^{m+l} e_1\|^2 > 1 - \delta$. Pick such a $c_{m+1,\tilde{\varepsilon}_{S^m}}$ and then choose a smaller δ_1 and larger m_l and repeat the above so that for some l_1 $\|P_{\mathcal{H}}\hat{S}^{m_1+l_1} e_1\|^2 > 1 - \delta_1$. Continue this process, iteratively picking smaller δ_i's and larger m_i's. Then we know that $\varepsilon \in \hat{\mathcal{K}}$ so that from above $\hat{S}$ is a pure shift with a multiplicity of one. As we never change any $c_{i,\varepsilon}$'s once they are chosen, as before we have $(\frac{1}{c}\varepsilon, e_i) = c_i$ $1 \leq i \leq n$, so that $\frac{1}{c}\varepsilon$ is the desired wandering vector, and since there are an infinite number of $\tilde{\varepsilon}_{S^i}$'s, $[\varepsilon]_{\hat{S}}$ has an infinite codimension. ∎

As an application of Theorem 3.3.3, consider the Hardy Space, that is, the space of all analytic functions on the unit disk $f = \sum a_n z^n$ such that

$$\sup_{0<r<1} \int |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

By means of radial limits, this space can also be identified with the space of all analytic functions in L^2 on the boundary of the unit disk: $\mathcal{H}^2 = \{f \in L^2(\partial\mathbb{D}) : \int f\bar{z}^n \frac{|dz|}{2\pi} = 0 \ \forall n < 0\}$. On this space the operator of multiplication by z is unitarily equivalent to the unweighted shift operator of multiplicity one, and a function ϕ is a wandering vector if and only if $(\phi, z^n\phi) = \int |\phi|^2 \bar{z}^n dm = 0$ $\forall n \geq 0$, which is equivalent to $|\phi(e^{i\theta})|$ being a constant almost everywhere on the boundary of the unit circle. In the context of spaces of functions, wandering vectors with a norm of one are usually called inner functions. We now have the following corollary:

Corollary 3.3.6. *Let $p(z)$ be an arbitrary polynomial of degree n, then there is an analytic function $q(z)$ and a constant $c > 0$ such that $\phi(z) = cp(z) + z^{n+1}q(z)$ is an inner function. Moreover, the constant c and the function $q(z)$ can be chosen so that dim$(\mathcal{H}^2 \ominus [\phi]_z)$ is any integer greater than n or is equal to infinity.*

Chapter 4

The Index of Invariant Subspaces

4.1 Hereditariness of L^*

Theorem 4.1.1. *A weighted shift T has an invariant subspace $\mathcal{M}$ of index equal to one such that $\mathcal{M} \not\subset T\mathcal{H}$, $\mathcal{M} \neq \mathcal{H}$ and $\left(L_{T|_{\mathcal{M}}}\right)^* = L^*|_{\mathcal{M}}$ if and only if T has weights whose moduli are periodic (that is, there is a k such that $|\alpha_i| = |\alpha_{i+k}|$ $\forall i \geq 1$).*

Remark Note that the condition $\mathcal{M} \not\subset T\mathcal{H}$ can be disregarded if we consider weighted shifts that eventually are periodic (we must then assume that $\mathcal{M} \neq T^i\mathcal{H}$ $\forall i \geq 0$ since these satisfy $\left(L_{T|_{\mathcal{M}}}\right)^* = L^*|_{\mathcal{M}}$ for any weighted shift).

Proof Consider first the case where T is a periodic, weighted shift with period k whose weights are all positive. Let ε be a wandering vector for the unweighted shift such that $(\varepsilon, e_1) \neq 0$ and $\varepsilon \notin \text{span}\{e_1\}$. Form a new $\tilde{\varepsilon}$ by inserting k zeros in between each coefficient of ε (so $\tilde{\varepsilon}$ will be $((\varepsilon, e_1)$, k zeros, (ε, e_2), k zeros, (ε, e_3), k zeros, et cetera)). Then for any weighted shift $(\tilde{\varepsilon}, T^i\tilde{\varepsilon}) = 0$ when i is not a multiple of k, and when i is a multiple of k, because T is periodic with period k, $(\tilde{\varepsilon}, T^{mk}\tilde{\varepsilon}) = (\varepsilon, (\alpha_1\alpha_2\alpha_3 \cdots \alpha_k)^m S^{mk}\varepsilon) = 0$. Since $L^*T^{mk+i}\tilde{\varepsilon} = \frac{1}{|\alpha_{i+1}|^2}T^{mk+i+1}\tilde{\varepsilon}$ $\forall m \geq 0, 0 \leq i \leq k-1$, $L^*\mathcal{M} = T\mathcal{M}$ where $\mathcal{M} = [\tilde{\varepsilon}]_T$, which must have an index of one since it is generated by the single vector $\tilde{\varepsilon}$. Therefore $L^*|_{\mathcal{M}} = P_{T\mathcal{M}}L^*|_{\mathcal{M}} = \left(L_{T|_{\mathcal{M}}}\right)^*$.

Let T be an arbitrary weighted shift whose weights have periodic moduli, and let U be the diagonal operator such that $Ue_1 = e_1$ and $Ue_i = \frac{(Ue_{i-1}, e_{i-1})}{|\alpha_{i-1}|}\alpha_{i-1}e_i$ $i \geq 2$, then U is unitary and $U^{-1}TU = \tilde{T}$ has periodic weights that are all positive. Let $\mathcal{M}$ be an invariant subspace of $\tilde{T}$ constructed as above so that $\left(L_{\tilde{T}}\right)^*|_{\mathcal{M}} = \left(L_{\tilde{T}|_{\mathcal{M}}}\right)^*$. Then $TU\mathcal{M} = U\tilde{T}U^{-1}U\mathcal{M} = U\tilde{T}\mathcal{M} \subset U\mathcal{M}$ so that $U\mathcal{M}$ is an invariant subspace for T, and $\tilde{L}^* = \tilde{T}(\tilde{T}^*\tilde{T})^{-1} = U^{-1}TU(U^{-1}T^*UU^{-1}TU)^{-1} = U^{-1}T(T^*T)^{-1}U = U^{-1}L^*U$, so that $L^*U\mathcal{M} = U\tilde{L}^*U^{-1}U\mathcal{M} = U\tilde{T}\mathcal{M} = UU^{-1}TU\mathcal{M} = TU\mathcal{M}$ and $\left(L_{T|_{U\mathcal{M}}}\right)^* = P_{TU\mathcal{M}}L^*|_{U\mathcal{M}} = L^*|_{U\mathcal{M}}$, and $U\mathcal{M}$ is the desired invariant subspace since

$U\mathcal{M} \not\subset T\mathcal{H}$ (because U maps each e_i to the span of e_i), $U\mathcal{M} \neq \mathcal{H}$ (because U is one to one and $\mathcal{M} \neq \mathcal{H}$), and $U\mathcal{M}$ must have index equal to one since by equation (3.1) $\dim(\mathcal{M} \ominus \tilde{T}\mathcal{M}) = \dim(U\mathcal{M} \ominus TU\mathcal{M})$.

Conversely, suppose that $\mathcal{M}$ is an invariant subspace of index equal to one for the weighted shift T such $\mathcal{M} \not\subset T\mathcal{H}$, $\mathcal{M} \neq \mathcal{H}$ and $\left(L_{T|_\mathcal{M}}\right)^* = L^*|_\mathcal{M}$. Let $\varepsilon_{T^i} = P_{T^i\mathcal{M}}e_{i+1}$ $\forall i \geq 0$, then as shown in Proposition 3.3.2, since $\mathcal{M} \not\subset T\mathcal{H}$ and $\dim(\mathcal{M} \ominus T\mathcal{M}) = 1$, none of these are zero, they are all wandering vectors, $\varepsilon_{T^i} \in T^i\mathcal{M} \ominus T^{i+1}\mathcal{M}$, and $\mathcal{M} = \operatorname{span}\{\varepsilon\} \oplus \operatorname{span}\{\varepsilon_T\} \oplus \operatorname{span}\{\varepsilon_{T^2}\} \oplus \ldots$. For an arbitrary left-invertible operator on its invariant subspace $\mathcal{M}$, L^* maps $\mathcal{M}^\perp$ into $(T\mathcal{M})^\perp$, since for $m_1 \in \mathcal{M}^\perp$ and $m_2 \in \mathcal{M}$, $(L^*m_1, Tm_2) = (m_1, LTm_2) = (m_1, m_2) = 0$. Since in our case $L^*\mathcal{M} = \left(L_{T|_\mathcal{M}}\right)^*\mathcal{M} = T\mathcal{M}$, $L^*\mathcal{M} \subset \mathcal{M}$ so that $\mathcal{M}$ is an invariant subspace for L^* as well, and since $\left(L_{L^*|_\mathcal{M}}\right)^* = P_{L^*\mathcal{M}}\left(L_{L^*}\right)^*|_\mathcal{M} = P_{L^*\mathcal{M}}T|_\mathcal{M} = P_{T\mathcal{M}}T|_\mathcal{M} = T|_\mathcal{M}$, so that according to the above with the roles of T and L^* reversed, T maps $\mathcal{M}^\perp$ orthogonally to $L^*\mathcal{M} = T\mathcal{M}$. Therefore $T\tilde{\varepsilon} \perp T\mathcal{M}$, where $\tilde{\varepsilon} = P_{\mathcal{M}^\perp}e_1$.

We will have need of the following relation:

$$\alpha_i\|\varepsilon_{T^{i-1}}\|^2 = (T\varepsilon_{T^{i-1}}, \varepsilon_{T^i}) \;\; \forall i \geq 1. \tag{4.1}$$

Since $T\varepsilon_{T^{i-1}} \in T^i\mathcal{M}$ and $\varepsilon_{T^i} = P_{T^i\mathcal{M}}e_{i+1}$,

$$(T\varepsilon_{T^{i-1}}, \varepsilon_{T^i}) = (T\varepsilon_{T^{i-1}}, e_{i+1}) = \alpha_i(\varepsilon_{T^{i-1}}, e_i) = \alpha_i(\varepsilon_{T^{i-1}}, \varepsilon_{T^{i-1}}) = \alpha_i\|\varepsilon_{T^{i-1}}\|^2,$$

and equation (4.1) follows.

So $(Te_1, \varepsilon_T) = (\alpha_1 e_2, \varepsilon_T) = \alpha_1(P_{T\mathcal{M}}e_2, \varepsilon_T) = \alpha_1(\varepsilon_T, \varepsilon_T) = \alpha_1\|\varepsilon_T\|^2$. But also $(Te_1, \varepsilon_T) = (T\tilde{\varepsilon} + T\varepsilon, \varepsilon_T) = (T\varepsilon, \varepsilon_T) = \alpha_1\|\varepsilon\|^2$, so that $\|\varepsilon\| = \|\varepsilon_T\|$. Doing the above to ε_{T^i} $i \geq 2$ instead of ε_T we obtain $0 = \alpha_1(\varepsilon_T, \varepsilon_{T^i}) = (\alpha_1 e_2, \varepsilon_{T^i}) = (Te_1, \varepsilon_{T^i}) = (T\tilde{\varepsilon} + T\varepsilon, \varepsilon_{T^i}) = (T\varepsilon, \varepsilon_{T^i})$ $i \geq 2$. Since $\mathcal{M} = \operatorname{span}\{\varepsilon\} \oplus \operatorname{span}\{\varepsilon_T\} \oplus \operatorname{span}\{\varepsilon_{T^2}\} \oplus \ldots$, it must be that $T\varepsilon = (T\varepsilon, \frac{\varepsilon_T}{\|\varepsilon_T\|^2})\varepsilon_T = \alpha_1\frac{\|\varepsilon\|^2}{\|\varepsilon_T\|^2}\varepsilon_T = \alpha_1\varepsilon_T$. Since $L^*T\mathcal{M} = L^{*2}\mathcal{M} = \operatorname{span}\{\varepsilon_{L^{*2}}\} \oplus \operatorname{span}\{\varepsilon_{L^{*3}}\} \oplus \operatorname{span}\{\varepsilon_{L^{*4}}\} \oplus \ldots$ and $\varepsilon_{L^*} = P_{L^*\mathcal{M}}e_2 = P_{T\mathcal{M}}e_2 = \varepsilon_T$, $L^{*2}\mathcal{M} \subset \mathcal{M} \ominus \operatorname{span}\{\varepsilon\} \ominus \operatorname{span}\{\varepsilon_T\} = T^2\mathcal{M}$ so that $\left(L_{T|_{T\mathcal{M}}}\right)^* = P_{T^2\mathcal{M}}L^*|_{T\mathcal{M}} = L^*|_{T\mathcal{M}}$, and we can do the above to $T\mathcal{M}$ and obtain $T\varepsilon_T = \alpha_2\varepsilon_{T^2}$ and $\left(L_{T|_{T^2\mathcal{M}}}\right)^* = P_{T^3\mathcal{M}}L^*|_{T^2\mathcal{M}} = L^*|_{T^2\mathcal{M}}$, and by induction we obtain that $T\varepsilon_{T^i} = \alpha_{i+1}\varepsilon_{T^{i+1}}$ $i \geq 0$ and $\varepsilon_{T^i} = \varepsilon_{L^{*i}}$ so that reversing the roles of T and L^* yields that $L^*\varepsilon_{T^i} = \frac{1}{\bar{\alpha}_{i+1}}\varepsilon_{T^{i+1}}$ $i \geq 0$.

Since $T\varepsilon_{T^{i-1}} = \alpha_i\varepsilon_{T^i}$,

$$(\varepsilon_{T^i}, e_{j+1}) = (\frac{1}{\alpha_i}T\varepsilon_{T^{i-1}}, e_{j+1}) = (\frac{1}{\alpha_i}\varepsilon_{T^{i-1}}, \bar{\alpha}_j e_j) = \frac{\alpha_j}{\alpha_i}(\varepsilon_{T^{i-1}}, e_j) \;\; i \geq 1, j \geq 1 \tag{4.2}$$

and since $L^*\varepsilon_{T^{i-1}} = \frac{1}{\bar{\alpha}_i}\varepsilon_{T^i}$,

$$(\varepsilon_{T^i}, e_{j+1}) = (\bar{\alpha}_i L^*\varepsilon_{T^{i-1}}, e_{j+1}) = (\bar{\alpha}_i\varepsilon_{T^{i-1}}, \frac{1}{\alpha_j}e_j) = \frac{\bar{\alpha}_i}{\bar{\alpha}_j}(\varepsilon_{T^{i-1}}, e_j) \;\; i \geq 1, j \geq 1. \tag{4.3}$$

Combining equations (4.2) and (4.3) yields that $|\alpha_j| = |\alpha_i|$ for every $\{i, j\}$ such that $(\varepsilon_{T^{i-1}}, e_j) \neq 0$.

Suppose that $(\varepsilon, e_2) \neq 0$, then $|\alpha_1| = |\alpha_2|$ and

$$\begin{aligned}(\varepsilon_{T^i}, e_{i+2}) &= (\frac{1}{\alpha_1\alpha_2\cdots\alpha_{i-1}\alpha_i} T^i\varepsilon, e_{i+2}) = \frac{\alpha_2\alpha_3\cdots\alpha_i\alpha_{i+1}}{\alpha_1\alpha_2\cdots\alpha_{i-1}\alpha_i}(\varepsilon, e_2)\\ &= \frac{\alpha_{i+1}}{\alpha_1}(\varepsilon, e_2) \neq 0 \ \ i \geq 0\end{aligned}$$

so that $|\alpha_{i+2}| = |\alpha_{i+1}|$ $i \geq 0$ and hence $|\alpha_i| = |\alpha_1|$ $i \geq 1$, so T has weights which all have the same moduli, id est, it has a period of 1.

Suppose that $(\varepsilon, e_2) = 0$ and let k be the first number such that $(\varepsilon, e_k) \neq 0$ (if no such number exists then $\varepsilon \in \text{span}\{e_1\}$ and hence $\mathcal{M} = \mathcal{H}$, which is contrary to the assumptions). Then $(\varepsilon_{T^{i-1}}, e_{i+k-1}) = \frac{1}{\alpha_1\alpha_2\cdots\alpha_{i-2}\alpha_{i-1}}(T^{i-1}\varepsilon, e_{i+k-1}) = \frac{\alpha_k\alpha_{k+1}\cdots\alpha_{i+k-3}\alpha_{i+k-2}}{\alpha_1\alpha_2\cdots\alpha_{i-2}\alpha_{i-1}}(\varepsilon, e_k) \neq 0$ so that $|\alpha_i| = |\alpha_{i+k-1}|$ $i \geq 1$, so that T has weights whose moduli have a period of $k - 1$ (they could have a shorter period). ∎

4.2 The Index of Invariant Subspaces

In this section we will study the second, main problem, that of when a weighted shift has only closed, invariant subspaces with indices equal to one. We will start with a relation showing how the index of a closed, invariant subspace is related to the index of the whole space.

The following is a generalization of Proposition 1.4 in [2] (although only for Hilbert spaces).

Theorem 4.2.1. *Let $\mathcal{M}$ be a closed, invariant subspace for a left-invertible operator T on $\mathcal{H}$, then*

$$dim(\mathcal{M} \ominus T\mathcal{M}) = dim(P_{\mathcal{M}}(\mathcal{H} \ominus T\mathcal{H})) + dim(Ker(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})) \tag{4.4}$$

Proof Let $\mathcal{M} \ominus T\mathcal{M} = E$ and $\varepsilon \in \mathcal{E} = \mathcal{H} \ominus T\mathcal{H} = \text{Ker}(T^*)$, then $T^*\varepsilon = 0$ and $(P_{\mathcal{M}}\varepsilon, Tm) = (\varepsilon, Tm) = (T^*\varepsilon, m) = 0$ $\forall m \in \mathcal{M}$, so that $P_{\mathcal{M}}\varepsilon \in \mathcal{M} \ominus T\mathcal{M} = \text{Ker}(T|_{\mathcal{M}})^* = E$ and $P_{\mathcal{M}}\mathcal{E} \subset E$. Let $x \in \text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$ with $x \neq 0$, then $x \in \mathcal{H}\ominus\mathcal{M}$ and $Tx \in \mathcal{M}$. Since T is one-to-one and maps $\mathcal{M}$ onto $T\mathcal{M}$ and $x \notin \mathcal{M}$, it must be that $Tx \notin T\mathcal{M}$, so that $P_{\mathcal{M}\ominus T\mathcal{M}}Tx \neq 0$. Since $Tx \perp \mathcal{E}$ and $Tx \in \mathcal{M}$, $Tx \perp P_{\mathcal{M}}\mathcal{E}$, and since $P_{\mathcal{M}}\mathcal{E} \subset E$, $P_E Tx \perp P_{\mathcal{M}}\mathcal{E}$. Therefore we can associate to each vector $x \in \text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$ the vector $P_E Tx \in E \ominus P_{\mathcal{M}}\mathcal{E} = E \ominus P_E\mathcal{E}$. If two different vectors x_1 and x_2 in $\text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$ are associated with the same vector, then $P_E T(x_1 - x_2) = 0$ which implies that $T(x_1 - x_2) \in T\mathcal{M}$ which as stated before cannot happen since $x_1 - x_2 \neq 0$ is in $\text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$. Suppose that $y \in \text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$, then $Ty \in \mathcal{M}$ and $P_E Ty = Ty + Tz$ for some $z \in \mathcal{M}$. Let $\varepsilon \in \mathcal{E} = \mathcal{H} \ominus T\mathcal{H}$, then $P_{\mathcal{M}}\varepsilon = \varepsilon + m_\perp$ for some $m_\perp \in \mathcal{M}^\perp$. Therefore $(P_{\mathcal{M}}\varepsilon, P_E Ty) = (\varepsilon + m_\perp, Ty + Tz) = (\varepsilon, T(y+z)) = 0$ because $T(y+z) \in \mathcal{M} \cap T\mathcal{H}$, and hence $P_{\mathcal{M}}(\mathcal{H} \ominus T\mathcal{H}) \perp P_E T\text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$.

Since we have associated a unique vector in $E = \mathcal{M} \ominus T\mathcal{M}$ to each vector in $P_{\mathcal{M}}\mathcal{E} = P_{\mathcal{M}}(\mathcal{H} \ominus T\mathcal{H})$ (the vector $x = P_{\mathcal{M}}x$ for $x \in P_{\mathcal{M}}\mathcal{E}$) and to each vector in $\text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$ (the vector $P_E Tx$ for $x \in \text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$) and $P_{\mathcal{M}}(\mathcal{H} \ominus T\mathcal{H}) \cap P_E T\text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}}) = \{0\}$ since these subspace are orthogonal to each other, it follows that

$$\dim(\mathcal{M} \ominus T\mathcal{M}) \geq \dim(P_{\mathcal{M}}(\mathcal{H} \ominus T\mathcal{H})) + \dim(\text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})).$$

We now prove the reverse inequality. Associate to each vector $x \in P_{\mathcal{M}}(\mathcal{H} \ominus T\mathcal{H}) \subset \mathcal{M} \ominus T\mathcal{M}$ a vector y in $\mathcal{H} \ominus T\mathcal{H}$ such that $x = P_{\mathcal{M}}y$. Then $y_1 - y_2 = 0$ implies that $P_{\mathcal{M}}(y_1 - y_2) = x_1 - x_2 = 0$, so that this association is unique. Let $x \in E = \mathcal{M} \ominus T\mathcal{M}$ be such that $x \perp P_{\mathcal{M}}\mathcal{E} = P_{\mathcal{M}}(\mathcal{H} \ominus T\mathcal{H})$ and $x \neq 0$. Then $x \perp P_{\mathcal{M}}\mathcal{E} + P_{\mathcal{M}^\perp}\mathcal{E}$, so that $x \perp \mathcal{E}$ and hence $x \in T\mathcal{H}$. Since $x \notin T\mathcal{M}$, $Lx \notin \mathcal{M}$ and $P_{(\mathcal{H}\ominus\mathcal{M})}Lx \neq 0$ and $Lx = x_1 + x_2$ where $x_1 \neq 0$, $x_1 \in \mathcal{H} \ominus \mathcal{M}$ and $x_2 \in \mathcal{M}$. Since $Tx_1 = TLx - Tx_2 = x - Tx_2 \in \mathcal{M}$, $x_1 \in \text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$, and we can associate to each vector $x \in (\mathcal{M} \ominus T\mathcal{M}) \ominus P_{\mathcal{M}}(\mathcal{H} \ominus T\mathcal{H})$ the vector $P_{\mathcal{H}\ominus\mathcal{M}}Lx \in \text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$. The association is unique since $P_{\mathcal{H}\ominus\mathcal{M}}L(x_1 - x_2) = 0$ with $x_1 - x_2 \neq 0$ implies that $L(x_1 - x_2) \in \mathcal{M}$ which is a contradiction to the above. Suppose that $x \in \mathcal{H} \ominus T\mathcal{H}$ and $P_{\mathcal{M}}x \neq 0$, then $x \notin \mathcal{H} \ominus \mathcal{M}$ and hence $x \notin \text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$. Since we have associated a unique vector in either $\{x : x \in \mathcal{E}, P_{\mathcal{M}}x \neq 0\}$ or $\text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})$ (and these two spaces have the zero space as their intersection) to each vector in $\mathcal{M} \ominus T\mathcal{M}$, it follows that

$$\dim(\mathcal{M} \ominus T\mathcal{M}) \leq \dim(P_{\mathcal{M}}(\mathcal{H} \ominus T\mathcal{H})) + \dim(\text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})).$$

Combining this with the previous paragraph yields

$$\dim(\mathcal{M} \ominus T\mathcal{M}) = \dim(P_{\mathcal{M}}(\mathcal{H} \ominus T\mathcal{H})) + \dim(\text{Ker}(P_{\mathcal{H}\ominus\mathcal{M}}T|_{\mathcal{H}\ominus\mathcal{M}})).$$

∎

The next theorem is a list of some of the equivalent conditions for a pure, left-invertible operator with an index of one to have a larger index on a closed, invariant subspace. Condition (vii) has proven to be the most useful.

The following notation will be used in the next theorem: Let M_c^* denote the set $\{x \in \mathcal{H} : \|T^{*i}x\| \geq c\|T^{*(i-1)}x\| \ \forall i \geq 1\}$. Note that this is an invariant set for T^*, but not an invariant subspace for T^*; in fact, for a weighted shift the closure of the span of elements in M_c^* (that is, the smallest closed space containing M_c^*) is all of $\mathcal{H}$.

The following theorem gives the equivalence of condition (i), $T|_{\mathcal{M}}$ has an index of one for every closed, invariant subspace $\mathcal{M}$ of T, with a number of other conditions, (ii)–($viii$). The equivalence of (i) and (ii) is known, with (ii) usually called the division property (see Lemma 3.1 of [37]). The equivalence of conditions (i) and (iv) is known and is very close to the definition of the index not being one. All other results are new, although some of them are innately related to known results such as: Theorem 2 of [48] alluded to condition (iv); compare condition (vii) with Theorem 4.2.1 and Theorem 12 of [48]; compare condition ($viii$) with Lemma 1.2 in [2] and (2) of Proposition 10 of [48].

Theorem 4.2.2. *Let T be a pure, left-invertible operator such that $\mathcal{H} \ominus T\mathcal{H} =$ span$\{e_1\}$, then the following are equivalent:*
(i) For every closed, invariant subspace $\mathcal{M}$ of T, dim$(\mathcal{M} \ominus T\mathcal{M}) = 1$.
(ii) Let i be such that $\mathcal{M} \subset T^i\mathcal{H}$ and $\mathcal{M} \not\subset T^{i+1}\mathcal{H}$, where $\mathcal{M}$ is a closed, invariant subspace for T. Then for every f such that $T^{i+1}f \in \mathcal{M}$, $T^i f \in \mathcal{M}$.
(iii) For every $f, g \notin T^i\mathcal{H}$ such that $f + g \in T^i\mathcal{H}$, $f + g \in T([f] \vee [g])$.
(iv) For every pair of wandering vectors $\{\varepsilon, \tilde{\varepsilon}\}$, $\tilde{\varepsilon} \not\perp [\varepsilon]_T \vee [\varepsilon]_{T^*}$.
(v) For every pair of wandering vectors $\{\varepsilon, \tilde{\varepsilon}\}$ such that $(\tilde{\varepsilon}, e_1) \neq 0$, $\tilde{\varepsilon} \not\perp [\varepsilon]_{T^*}$.
(vi) For every wandering vector ε and $f \in \mathcal{H}$ such that $(f, e_1) \neq 0$, $f \not\perp [\varepsilon]_{T^*}$.
(vii) For every invariant subspace $\mathcal{M}$ of T, $T^*|_{\mathcal{M}^\perp}$ being left-invertible implies that it is right-invertible.
(viii) For every wandering vector ε, $[\varepsilon]_{T^*} \not\subset \mathcal{M}_c^*$ for every $c > 0$.

Proof All proofs will be by contrapositive.

(ii)⇒(i): Suppose that $\mathcal{M}$ is a closed, invariant subspace for T with $\dim(\mathcal{M} \ominus T\mathcal{M}) > 1$. Let i be such that $\mathcal{M} \subset T^i\mathcal{H}$ and $\mathcal{M} \not\subset T^{i+1}\mathcal{H}$. Let $\mathcal{H} = \text{span}\{e_1\} \oplus \text{span}\{e_2\} \oplus \text{span}\{e_3\} \oplus \dots$ be the decomposition of $\mathcal{H}$ as given in Corollary 2.1.9.

Since $\mathcal{M} \not\subset T^{i+1}\mathcal{H}$ but $\mathcal{M} \subset T^i\mathcal{H}$, $\varepsilon = P_{\mathcal{M}}e_{i+1}$ must be nonzero, and since $\text{span}\{e_1\} \oplus \text{span}\{e_2\} \oplus \dots \oplus \text{span}\{e_i\} \oplus \text{span}\{e_{i+1}\}$ is an invariant subspace for T^* (since its orthogonal complement is an invariant subspace for T), similar to the proof of Proposition 3.2.1, ε is a wandering vector contained in $\mathcal{M} \ominus T\mathcal{M}$. Let $\tilde{\varepsilon}$ be a nonzero vector contained in $\mathcal{M} \ominus T\mathcal{M}$ that is orthogonal to ε. Since $\tilde{\varepsilon}$ is orthogonal to ε and ε is the projection of e_{i+1} onto $\mathcal{M}$, $\tilde{\varepsilon}$ must be orthogonal to e_{i+1} and hence contained in $T^{i+1}\mathcal{H}$ (since all of $\mathcal{M}$ is orthogonal to e_j for $j \leq i$). Thus there is an f such that $\tilde{\varepsilon} = T^{i+1}f$, but $T^i f$ can not be contained in $\mathcal{M}$ as this would imply that $\tilde{\varepsilon} = TT^i f$ would be contained in $T\mathcal{M}$, which is a contradiction.

(iii)⇒(ii): Let $\mathcal{M}$ be a closed, invariant subspace for T such that $\mathcal{M} \subset T^i\mathcal{H}$ and $\mathcal{M} \not\subset T^{i+1}\mathcal{H}$, and $\tilde{f}$ be such that $T^{i+1}\tilde{f} \in \mathcal{M}$, but $T^i\tilde{f} \notin \mathcal{M}$. As before set

$\varepsilon = P_{\mathcal{M}}e_{i+1}$, so that span$\{\varepsilon\} \subset \mathcal{M} \ominus T\mathcal{M}$. Then $T^{i+1}\tilde{f}$ is contained in $T\mathcal{M}$ since it is orthogonal to e_{i+1} and hence orthogonal to ε. So $f = T^{i+1}\tilde{f} + \varepsilon$ and $g = T^{i+1}\tilde{f} - \varepsilon$ are both not contained in $T^{i+1}\mathcal{H}$ but $f + g = 2T^{i+1}\tilde{f}$ is contained in $T^{i+1}\mathcal{H}$, and since $f + g$ is not in $T\mathcal{M}$, it must not be in $T([f] \vee [g]) \subset T\mathcal{M}$.

(i)⇒(iii): Let f and g be not contained in $T^i\mathcal{H}$ but such that $f + g \in T^i\mathcal{H}$ and $f + g \notin T([f] \vee [g])$. Let $\mathcal{M} = [f] \vee [g]$, then $\mathcal{M}$ is a closed, invariant subspace for T. Since $f \notin T^i\mathcal{H}$, $P_{\mathcal{M}}e_j$ must be nonzero for some $j < i + 1$; let $\varepsilon = P_{\mathcal{M}}e_j$ be the first one that is nonzero, then span$\{\varepsilon\} \subset \mathcal{M} \ominus T\mathcal{M}$. Then $P_{\mathcal{M}\ominus T\mathcal{M}}(f + g)$ is nonzero since $f + g \notin T\mathcal{M}$, and since $f + g$ is orthogonal to ε since it is orthogonal to e_j, and $P_{\mathcal{M}\ominus T\mathcal{M}}(f + g)$ is formed by adding a vector from $T\mathcal{M}$ to $f + g$, $P_{\mathcal{M}\ominus T\mathcal{M}}(f + g)$ is orthogonal to ε and contained in $\mathcal{M} \ominus T\mathcal{M}$. Thus ε and $P_{\mathcal{M}\ominus T\mathcal{M}}(f + g)$ are linearly independent vectors in $\mathcal{M} \ominus T\mathcal{M}$, so $\dim(\mathcal{M} \ominus T\mathcal{M}) \geq 2$.

(iv)⇒(i): Suppose that $\mathcal{M}$ is a closed, invariant subspace for T with $\dim(\mathcal{M} \ominus T\mathcal{M}) > 1$. Let ε and $\tilde{\varepsilon}$ be two vectors contained in $\mathcal{M} \ominus T\mathcal{M}$ that are orthogonal. Since $\tilde{\varepsilon} \perp T\mathcal{M}$, $(\tilde{\varepsilon}, T^i\varepsilon) = 0\ \forall i \geq 0$, and since $\varepsilon \perp T\mathcal{M}$, $(\tilde{\varepsilon}, T^{*i}\varepsilon) = (T^i\tilde{\varepsilon}, \varepsilon) = 0\ \forall i \geq 0$, so that $\tilde{\varepsilon} \perp [\varepsilon]_T \vee [\varepsilon]_{T^*}$.

(v)⇒(iv): Let $\{\varepsilon, \tilde{\varepsilon}\}$ be a pair of wandering vectors such that $\tilde{\varepsilon} \perp [\varepsilon]_T \vee [\varepsilon]_{T^*}$ and $\mathcal{M} = [\varepsilon] \vee [\tilde{\varepsilon}]$. Then $(\varepsilon, T^i\varepsilon) = 0\ \forall i \geq 1$ and $(\tilde{\varepsilon}, T^i\tilde{\varepsilon}) = 0\ \forall i \geq 1$ since ε and $\tilde{\varepsilon}$ are both wandering and $(\varepsilon, T^i\tilde{\varepsilon}) = 0\ \forall i \geq 0$ since $\tilde{\varepsilon} \perp [\varepsilon]_{T^*}$, and $(\tilde{\varepsilon}, T^i\varepsilon) = 0\ \forall i \geq 0$ since $\tilde{\varepsilon} \perp [\varepsilon]_T$, so that both ε and $\tilde{\varepsilon}$ are contained in $\mathcal{M} \ominus T\mathcal{M}$.

If $\mathcal{M} \subset T\mathcal{H}$, then $L\mathcal{M}$ is invariant for T since $TLm = m = LTm \in L\mathcal{M}$ and $L\mathcal{M}$ is closed since L is right-invertible. Also, $\dim(L\mathcal{M} \ominus TL\mathcal{M}) > 1$ since $\varepsilon_1 = P_{L\mathcal{M}\ominus TL\mathcal{M}}L\varepsilon$ and $\varepsilon_2 = P_{L\mathcal{M}\ominus TL\mathcal{M}}L\tilde{\varepsilon}$ are both nonzero and linearly independent since else $L\varepsilon = c_1\varepsilon_1 + m_1$ and $L\tilde{\varepsilon} = \tilde{c}_1\varepsilon_1 + m_2$, so that $\varepsilon = c_1T\varepsilon_1 + Tm_1$ and $\tilde{\varepsilon} = \tilde{c}_1T\tilde{\varepsilon}_1 + Tm_2$, so that $\varepsilon = c_1P_{\mathcal{M}\ominus T\mathcal{M}}T\varepsilon_1$ and $\tilde{\varepsilon} = \tilde{c}_1P_{\mathcal{M}\ominus T\mathcal{M}}T\varepsilon_1$. Then linear dependence of $\{\varepsilon_1, \varepsilon_2\}$ implies linear dependence of $\{\varepsilon, \tilde{\varepsilon}\}$ which is a contradiction since ε and $\tilde{\varepsilon}$ are nonzero and orthogonal. Since T is pure, there must be a j such that $\mathcal{M} \subset T^j\mathcal{H}$ but $\mathcal{M} \not\subset T^{j+1}\mathcal{H}$, and $L^j\mathcal{M} \subset \mathcal{H}$ but $L^j\mathcal{M} \not\subset T\mathcal{H}$, and since each time that we applied L the subspace was still invariant for T and had an index greater than one, we can assume that $\mathcal{M}$ is not contained in $T\mathcal{H}$.

Then $P_{\mathcal{M}}e_1$ is not zero and contained in $\mathcal{M} \ominus T\mathcal{M}$. Let $\tilde{\varepsilon}$ be equal to $P_{\mathcal{M}}e_1$ so that it is a wandering vector and $(\tilde{\varepsilon}, e_1) \neq 0$. Let ε be a nonzero vector contained in $\mathcal{M} \ominus T\mathcal{M}$ that is orthogonal to $\tilde{\varepsilon}$, then ε is a wandering vector and $\tilde{\varepsilon} \perp [\varepsilon]_{T^*}$ since $(\varepsilon, T^i\tilde{\varepsilon}) = (T^{*i}\varepsilon, \tilde{\varepsilon}) = 0\ \forall i \geq 0$.

(vi)⇒(v): If (v) is not satisfied, then set $f = \tilde{\varepsilon}$ and (vi) is not satisfied.

(vii)⇒(vi): Let ε be a wandering vector and $f \in \mathcal{H}$ such that $(f, e_1) \neq 0$ and $f \perp [\varepsilon]_{T^*}$. Set $\mathcal{M}$ equal to $([\varepsilon]_{T^*})^\perp$, then $\mathcal{M}$ is an invariant subspace for T since $[\varepsilon]_{T^*}$ is an invariant subspace for T^*, and $f \in \mathcal{M}$ since $f \perp [\varepsilon]_{T^*}$. Since $(f, e_1) \neq 0$,

$e_1 \not\perp \mathcal{M}$ and $e_1 \notin \mathcal{M}^\perp$, so that $T^*|_{\mathcal{M}^\perp}$ is one-to-one since $\mathrm{Ker}(T^*) = \mathrm{span}\{e_1\}$. Since T^* is right invertible, it has a closed range and since Ker T^* has a dimension of one, by Corollary 2.6.3 $T^*|_{\mathcal{M}^\perp}$ has a closed range, so that $T^*|_{\mathcal{M}^\perp}$ is left-invertible. Since $(T^{*i}\varepsilon, \varepsilon) = (\varepsilon, T^i\varepsilon) = 0\ \forall i \geq 1$, $\varepsilon \perp T^*[\varepsilon]_{T^*}$, so that $T^*|_{\mathcal{M}^\perp}$ is not onto and hence not right-invertible.

(viii)$\Rightarrow$(vii): Let $\mathcal{M}$ be an invariant subspace for T such that $T^*|_{\mathcal{M}^\perp}$ is left-invertible but not right-invertible. Since $T^*|_{\mathcal{M}^\perp}$ is left invertible, there exists a $c > 0$ such that $\|T^*x\| \geq c\|x\|\ \forall x \in \mathcal{M}^\perp$, and since $T^*x \in \mathcal{M}^\perp$ for all $x \in \mathcal{M}^\perp$, $\|T^{*i}x\| \geq c\|T^{*(i-1)}x\|\ \forall i \geq 1, x \in \mathcal{M}^\perp$, and hence $\mathcal{M}^\perp \subset \mathcal{M}_c^*$. Since $T^*|_{\mathcal{M}^\perp}$ is not right-invertible, there is a $\varepsilon \in \mathcal{M}^\perp$ such that $\varepsilon \in \mathcal{M}^\perp \ominus T^*\mathcal{M}^\perp$ and hence $(\varepsilon, T^{*i}\varepsilon) = (T^i\varepsilon, \varepsilon) = 0\ \forall i \geq 1$, and ε is a wandering vector, and $[\varepsilon]_{T^*} \subset \mathcal{M}^\perp \subset \mathcal{M}_c^*$.

(i)$\Rightarrow$(viii): Let ε be a wandering vector such that $[\varepsilon]_{T^*} \subset \mathcal{M}_c^*$ for some $c > 0$. Since $\|T^*x\| \neq 0$ for all $x \in [\varepsilon]_{T^*}$, $e_1 \notin [\varepsilon]_{T^*}$, and since $e_1 \perp [T\varepsilon]_T$, $\tilde{\varepsilon} = P_{\mathcal{H}\ominus([\varepsilon]_{T^*}\oplus[T\varepsilon]_T)}e_1 \neq 0$. Again, since $e_1 \perp [T\varepsilon]_T$, $\tilde{\varepsilon} = P_{\mathcal{H}\ominus[\varepsilon]_{T^*}}e_1$, so that $\tilde{\varepsilon}$ is a wandering vector since $\mathcal{H} \ominus [\varepsilon]_{T^*}$ is a closed, invariant subspace for T. So $(\varepsilon, T^i\varepsilon) = 0\ \forall i \geq 1$ and $(\tilde{\varepsilon}, T^i\tilde{\varepsilon}) = 0\ \forall i \geq 1$ since ε and $\tilde{\varepsilon}$ are wandering vectors, $(\varepsilon, T^i\tilde{\varepsilon}) = (T^{*i}\varepsilon, \tilde{\varepsilon}) = 0\ \forall i \geq 1$ since $\tilde{\varepsilon} \perp [\varepsilon]_{T^*}$, and $(\tilde{\varepsilon}, T^i\varepsilon) = 0\ \forall i \geq 1$ since $\tilde{\varepsilon} \perp [T\varepsilon]_T$. Therefore both ε and $\tilde{\varepsilon}$ are contained in $\mathcal{M} \ominus T\mathcal{M}$ when $\mathcal{M}$ is $[\varepsilon] \vee [\tilde{\varepsilon}]$, and since they are linearly independent, $\mathcal{M}$ is a closed, invariant subspace for T such that $\dim(\mathcal{M} \ominus T\mathcal{M}) > 1$. ∎

Note that in condition (vii) above we always have that $T^*|_{\mathcal{M}^\perp}$ being right-invertible implies that it is left-invertible (disregarding the trivial case of $\mathcal{M}^\perp = \mathcal{H}$), since if $T^*|_{\mathcal{M}^\perp}$ is not left-invertible, then it must not be one-to-one (since T^* being right-invertible implies that it has a closed range and hence also a closed range on all of its closed, invariant subspaces), so that $e_1 \in \mathcal{M}^\perp$ and hence $\mathcal{M}$ (take the closure if necessary) is contained in $T\mathcal{H}$. Let $\varepsilon \in \mathcal{M} \ominus T\mathcal{M}$ be nonzero, then $L\varepsilon \notin \mathcal{M}$ (since else $\varepsilon \in T\mathcal{M}$) so that $P_{\mathcal{M}^\perp}L\varepsilon = L\varepsilon + m \neq 0$ where $m \in \mathcal{M}$. So $TP_{\mathcal{M}^\perp}L\varepsilon = TL\varepsilon + Tm = \varepsilon + Tm \in \mathcal{M}$ and $P_{\mathcal{M}^\perp}T = (T^*|_{\mathcal{M}^\perp})^*$ is not one-to-one, and hence $T^*|_{\mathcal{M}^\perp}$ is not right-invertible. Therefore (vii) is equivalent to $T^*|_{\mathcal{M}^\perp}$ being left-invertible if and only if it is right-invertible (for a nontrivial $\mathcal{M}$).

As an application of the above theorem we have the following sufficient condition for a vector to be cyclic for a backward, weighted shift. This is a generalization of part (i) of Theorem 2.3 of [1] using different methods; see also Section 5.5 of [17]:

Corollary 4.2.3. *Let T be a left-invertible, weighted shift and $x \in \mathcal{H}$ be such that $x \notin \mathcal{M}_c^*$ for any $c > 0$ and $T^{*i}x \neq 0\ \forall i \geq 0$, then x is cyclic for T^* (that is, $[x]_{T^*} = \mathcal{H}$).*

Proof For ease of notation let $\mathcal{M}_0^* = \{x \in \mathcal{H} : \forall c > 0, x \notin \mathcal{M}_c^*\}$. Suppose that there is some $f \in \mathcal{H}$ such that $(f, e_1) \neq 0$ and a $y \in \mathcal{H}$ such that $y \perp [f]_T$. Then since $e_1 \not\perp [f]_T$, $e_1 \notin ([f]_T)^\perp$, so that $T^*|_{([f]_T)^\perp}$ is one-to-one and by Corollary 2.6.3 it is left-invertible and hence there is a $c > 0$ such that $y \in ([f]_T)^\perp \subset \mathcal{M}_c^*$. Therefore by the contrapositive of the preceding, if $y \in \mathcal{M}_0^*$ then $y \not\perp [f]_T$ for every $f \in \mathcal{H}$ such that $(f, e_1) \neq 0$ and hence $f \not\perp [x]_{T^*}$ for every $f \in \mathcal{H}$ such that $(f, e_1) \neq 0$. Let g be an arbitrary vector in $\mathcal{H}$, and let i be the first integer such that $(g, e_{i+1}) \neq 0$. Note that $P_{T^i\mathcal{H}}x \neq 0$, since this would imply that $x \in \mathrm{Ker}(T^{*i}) = \mathcal{H} \ominus T^i\mathcal{H}$, but by the hypotheses $T^{*i}x \neq 0\ \forall i \geq 0$. Since $x \in \mathcal{M}_0^*$, there is a subsequence $\{n_j\}_{j=1}^\infty$ such that $\frac{\|T^{*(j_n+i+1)}x\|^2}{\|T^{*(j_n+i)}x\|^2} \to 0$, and since L^{*i} is left-invertible, there is a $c > 0$ and an $M < \infty$ such that $c\|x\| \leq \|L^*x\| \leq M\|x\|\ \forall x \in \mathcal{M}$, and hence $\frac{\|L^{*i}T^{*i}T^{*(j_n+1)}x\|^2}{\|L^{*i}T^{*i}T^{*j_n}x\|^2} = \frac{\|L^{*i}T^{*(j_n+i+1)}x\|^2}{\|L^{*i}T^{*(j_n+i)}x\|^2} \to 0$, but $P_{T^i\mathcal{H}}T^{*j}P_{T^i\mathcal{H}}x = L^{*i}T^{*i}T^{*j}P_{T^i\mathcal{H}}x = L^{*i}T^{*(j+i)}x\ \forall j \geq i$ since $T^{*j}x = T^{*j}(P_{T^i\mathcal{H}}x + P_{\mathcal{H}\ominus T^i\mathcal{H}}x) = T^{*j}P_{T^i\mathcal{H}}x\ \forall j \geq i$ since $\mathcal{H}\ominus T^i\mathcal{H} \subset \mathcal{H}\ominus T^j\mathcal{H} = \mathrm{Ker}(T^{*j})\ \forall j \geq i$. Therefore, considering $\tilde{\mathcal{H}} = T^i\mathcal{H}$, $\tilde{T} = T|_{T^i\mathcal{H}}$, and $\tilde{x} = P_{T^i\mathcal{H}}x$, $\tilde{T}$ is a left-invertible, weighted shift on $\tilde{H}$, and $\tilde{x} \in \mathcal{M}_{0,\tilde{H}}^*$, so that by the above $g \not\perp [\tilde{x}]_{\tilde{T}^*}$ (since $\tilde{\mathcal{H}} \ominus \tilde{T}\tilde{\mathcal{H}} = \mathrm{span}\{e_{i+1}\}$) so that there is a k such that $(\tilde{T}^k g, \tilde{x}) = (T^k g, x) \neq 0$, and hence $g \not\perp [x]_{T^*}$. Since g was arbitrary, there is no vector that is orthogonal to $[x]_{T^*}$ and hence $[x]_{T^*} = \mathcal{H}$. ∎

We now state a slightly stronger version of $(i) \Leftrightarrow (vii)$ of Theorem 4.2.2 since it was stated more weakly there and will be useful later. It is not hard to prove in the pure case using arguments similar to those used in Theorem 4.2.2, but since it is actually a corollary of Corollary 2.6.3 and Theorem 4.2.1, we will not prove it.

Corollary 4.2.4. *Let T be a left-invertible operator such that $\mathcal{H} \ominus T\mathcal{H} = span\{e_1\}$ and $\mathcal{M}$ be a closed, invariant subspace for T with $P_\mathcal{M} e_1 \neq 0$. Then dim $(\mathcal{M} \ominus T\mathcal{M}) > 1$ if and only if $T^*|_{\mathcal{M}^\perp}$ is left-invertible but not right-invertible.*

The following corollary removes the constraint $P_\mathcal{M} e_1 \neq 0$ in Corollary 4.2.4 by using an appropriate strengthening of the hypothesis that $T^*|_{\mathcal{M}^\perp}$ is left but not right-invertible.

Corollary 4.2.5. *Let T be a pure, left-invertible operator such that $\mathcal{H} \ominus T\mathcal{H} = span\{e_1\}$ and $\mathcal{M}$ be a closed, invariant subspace for T. Then dim$(\mathcal{M} \ominus T\mathcal{M}) > 1$ if and only if $T^{*(i+1)}\mathcal{M}^\perp \neq T^{*i}\mathcal{M}^\perp\ \forall i \geq 0$.*

Proof Suppose that $\mathcal{M}$ is a closed, invariant subspace for the pure, left-invertible operator T such that $\dim(\mathcal{M} \ominus T\mathcal{M}) > 1$. Suppose that $\mathcal{M} \subset T\mathcal{H}$. If $x \perp L\mathcal{M}$, then since T^* is onto there is a $y \in \mathcal{H}$ such that $T^*y = x$ and $(x, Lm) = (T^*y, Lm) = (y, TLm) = (y, m) = 0\ \forall m \in \mathcal{M}$, so that $y \in \mathcal{M}^\perp$ and $x \in T^*\mathcal{M}^\perp$. Similarly, if $x \in T^*\mathcal{M}^\perp$, then $T^*y = x$ for some $y \in \mathcal{M}^\perp$, and from the above we see that $x \perp L\mathcal{M}$; hence $T^*\mathcal{M}^\perp = (L\mathcal{M})^\perp$. Since $L\mathcal{M}$ is strictly larger than $\mathcal{M}$, $T^*\mathcal{M}^\perp$

must be strictly smaller than $\mathcal{M}^\perp$. Since $\dim(L\mathcal{M}\ominus TL\mathcal{M}) = \dim(\mathcal{M}\ominus T\mathcal{M}) > 1$, if $\mathcal{M}\subset T\mathcal{H}$, we can operator by L enough times and assume that $\mathcal{M}\not\subset T\mathcal{H}$. By Corollary 4.2.4, $T^*|_{\mathcal{M}^\perp}$ is left-invertible but not right-invertible. Therefore $(T^*|_{\mathcal{M}^\perp})^i$ is left-invertible but not right-invertible for every $i \geq 0$, so that it is never onto for any power, and hence $T^{*(i+1)}\mathcal{M}^\perp$ is always strictly contained in $T^{*i}\mathcal{M}^\perp$ for every $i \geq 0$.

Conversely, suppose that $\mathcal{M}^\perp$ is such that $T^{*(i+1)}\mathcal{M}^\perp$ is always strictly contained in $T^{*i}\mathcal{M}^\perp$ for every $i \geq 0$. As above, since by Proposition 2.1.8 $\dim(T\mathcal{M}\ominus T^2\mathcal{M}) = \dim(\mathcal{M}\ominus T\mathcal{M})$ and $T^*\mathcal{M}^\perp = (L\mathcal{M})^\perp$ if $\mathcal{M}\subset T\mathcal{H}$, we can assume that $\mathcal{M}\not\subset T\mathcal{H}$. Since $e_1 \notin \mathcal{M}^\perp$, by Corollary 2.6.3 $T^*|_{\mathcal{M}^\perp}$ is one-to-one with a closed range and hence left-invertible, and since $T^*\mathcal{M}^\perp$ is strictly contained in $\mathcal{M}^\perp$, $T^*|_{\mathcal{M}^\perp}$ is not right-invertible, so that by Corollary 4.2.4, $\mathcal{M}$ has an index greater than one. ∎

The following proposition shows that there is a large class of vectors by which one can perturb a vector f that will not affect whether or not T^* is left but not right-invertible on the closed, invariant subspace for T^* generated by f. It also shows that the set of vectors which generate closed, invariant subspaces for T^* on which the restriction is left but not right-invertible is dense.

Proposition 4.2.6. *Let T be a left-invertible operator such that $\mathcal{H}\ominus T\mathcal{H} = span\{e_1\}$, f be a nonzero vector, $x = \sum_{i=0}^n c_iT^ie_1$, $\mathcal{M} = ([f]_{T^*})^\perp$ and $\tilde{\mathcal{M}} = ([f+x]_{T^*})^\perp$. Then $T^*|_{\mathcal{M}^\perp}$ is left-invertible but not right-invertible if and only if $T^*|_{\tilde{\mathcal{M}}^\perp}$ is.*

Proof We will first prove the forward direction assuming that $x = ce_1$. Assume that $f \neq 0$ is given and $T^*|_{\mathcal{M}^\perp}$ is left-invertible but not right-invertible. Since $x = ce_1 \in \mathrm{Ker}(T^*)$, $T^{*i}f = T^{*i}(f+x)$ $\forall i \geq 1$, so that $T^{*i}\mathcal{M}^\perp = T^{*i}\tilde{\mathcal{M}}^\perp$ $\forall i \geq 1$. If $T^*|_{\tilde{\mathcal{M}}^\perp}$ were right-invertible, then it would be onto, so that $T^*\mathcal{M}^\perp = T^*\tilde{\mathcal{M}}^\perp = T^{*2}\tilde{\mathcal{M}}^\perp = T^{*2}\mathcal{M}^\perp$, but this is a contradiction since if an operator $A : \mathcal{X}\to\mathcal{X}$ is one-to-one, then $\dim(\mathcal{X}\ominus A\mathcal{X}) = \dim(A\mathcal{X}\ominus A^2\mathcal{X})$, so that $T^*\mathcal{M}^\perp\ominus T^{*2}\mathcal{M}^\perp$ must be nonzero since $\mathcal{M}^\perp\ominus T^*\mathcal{M}^\perp$ is (since $T^*|_{\mathcal{M}^\perp}$ is not right-invertible and hence not onto). If $T^*|_{\tilde{\mathcal{M}}^\perp}$ were not left-invertible, then e_1 would be in $\tilde{\mathcal{M}}^\perp = \mathrm{span}\{f+x\}\dot{+}T^*\tilde{\mathcal{M}}^\perp = \mathrm{span}\{f+x\}\dot{+}T^*\mathcal{M}^\perp$, but since e_1 is not in $T^*\mathcal{M}^\perp$ there must be a nonzero constant d and a vector $g\in T^*\mathcal{M}^\perp$ such that $e_1 = d(f+x)+g$, but this implies that $0 = T^*e_1 = dT^*f + T^*g$, or that T^*f is contained in $T^{*2}\mathcal{M}^\perp$, which would imply that $T^*\mathcal{M}^\perp = T^{*2}\mathcal{M}^\perp$ since $T^*\mathcal{M}^\perp = [T^*f]_{T^*}$ which is a contradiction. Hence $T^*|_{\tilde{\mathcal{M}}^\perp}$ is left-invertible but not right-invertible.

To prove the general case, notice that if $T^*|_{\mathcal{M}^\perp}$ is left-invertible but not right-invertible, then $T^*|_{T^*\mathcal{M}^\perp}$ also is, and $T^*\mathcal{M}^\perp = [T^*f]_{T^*}$. Similarly, $T^*|_{[L^*f]_{T^*}}$ cannot be right-invertible, since this would imply that $[f]_{T^*} = T^*[L^*f]_{T^*} = T^{*2}[L^*f]_{T^*} = T^*[f]_{T^*}$. And if $T^*|_{[L^*f]_{T^*}}$ were not left-invertible, then since e_1 is not in $[f]_{T^*}$, as above there would be a nonzero constant d and a vector $g\in T^*[L^*f]_{T^*} = [f]_{T^*}$ such that $e_1 = dL^*f + g$, which would imply that f is contained in $T^*[f]_{T^*}$ which is a contradiction, so $T^*|_{[L^*f]_{T^*}}$ is also left-invertible but not right invertible.

Suppose that $f \neq 0$ and $x = \sum_{i=0}^{n} c_i T^i e_1$ are given, and $T^*|_{\mathcal{M}^\perp}$ is left-invertible but not right-invertible. Applying inductively the fact that $T^*|_{[T^*f]_{T^*}}$ is left-invertible but not right-invertible if $T^*|_{[f]_{T^*}}$ is, we can take T^* as many times as necessary until $T^{*k}f = T^{*k}(f + x)$ (which must happen due to the form of x) and still have that $T^*_{[T^{*k}f]_{T^*}}$ is left-invertible but not right-invertible. Then from above we know that $T^*|_{[L^*T^{*k}f]_{T^*}} = T^*|_{[L^*T^{*k}(f+x)]_{T^*}}$ is left-invertible but not right-invertible, and since $L^*T^{*k}(f + x) + ce_1 = T^{*(k-1)}(f + x)$ for some c, from above we know that $T^*|_{T^{*(k-1)}(f+x)]_{T^*}}$ must also be left-invertible but not right-invertible. We can repeat the last two steps $k - 1$ times, taking L^* and then adjusting by ce_1 for a suitable constant so that $L^*T^{*j}(f+x)+ce_1 = T^{*(j-1)}(f+x)$, knowing that each time we will have a space on which the restriction of T^* is left-invertible but not right-invertible, so that $T^*|_{[f+x]_{T^*}}$ must also be left-invertible but not right-invertible. Finally, since $f = (f+x)-x$, if T^* restricted to $[f+x]_{T^*}$ is left-invertible but not right-invertible, then it must also be when restricted to $[f]_T^*$, so that the proof is if and only if. ∎

Proposition 4.2.7. *Let T be a left-invertible operator such that $\mathcal{H}\ominus T\mathcal{H} = span\{e_1\}$, and $\mathcal{M}^\perp = [f]_{T^*}$ be a closed, invariant subspace for T^*. Then one of the five following conditions holds:*
(i) $e_1 \in [f]_{T^*}$ and $[f]_{T^*} = [T^*f]_{T^*}$, in which case $e_1 \in [f]_{T^*} = [f + ce_1]_{T^*} = [T^*(f + ce_1)]_{T^*}$ $\forall c \in \mathbb{C}$ and $([f]_{T^*})^\perp \subset \bigcap_{i=0}^{\infty} T^i\mathcal{H}$
(ii) $e_1 \notin [f]_{T^*}$ and $[f]_{T^*} \neq [T^*f]_{T^*}$, in which case $e_1 \notin [f + ce_1]_{T^*}$ and $[f + ce_1]_{T^*} \neq [T^*(f + ce_1)]_{T^*}$ for every $c \in \mathbb{C}$
(iii) $e_1 \notin [f]_{T^*}$ and $[f]_{T^*} = [T^*f]_{T^*}$, in which case $e_1 \in [f + ce_1]_{T^*}$ and $[f + ce_1]_{T^*} \neq [T^*(f + ce_1)]_{T^*}$ for every $c \in \mathbb{C} \setminus 0$
(iv) $e_1 \in [f]_{T^*}$, $[f]_{T^*} \neq [T^*f]_{T^*}$, and $e_1 \notin [T^*f]_{T^*}$, in which case $[f + ce_1]_{T^*}$ is in case (iii) for some c (and hence exactly one c)
(v) $e_1 \in [f]_{T^*}$, $[f]_{T^*} \neq [T^*f]_{T^*}$, and $e_1 \in [T^*f]_{T^*}$, in which case $e_1 \in [f + ce_1]_{T^*}$ and $[f + ce_1]_{T^*} \neq [T^*(f + ce_1)]_{T^*}$ for every $c \in \mathbb{C}$

Proof Since a vector being contained in a set and two sets being equal are either true or not true, only these five conditions are possible, so it remains to prove the implications.

Suppose that case (i) holds. Since $e_1 \in [f]_{T^*}$, $f+ce_1 \in [f]_{T^*}$, so that $[f+ce_1]_{T^*} \subset [f]_{T^*}$. Also, $[f]_{T^*} = [T^*f]_{T^*} = [T^*(f + ce_1)]_{T^*} \subset [f + ce_1]_{T^*}$, so that $[f]_{T^*} = [f + ce_1]_{T^*}$ $\forall c \in \mathbb{C}$. Therefore $e_1 \in [f + ce_1]_{T^*}$ $\forall c \in \mathbb{C}$ and $[f + ce_1]_{T^*} = [f]_{T^*} = [T^*f]_{T^*} = [T^*(f + ce_1)]_{T^*}$ $\forall c \in \mathbb{C}$. If $([f]_{T^*})^\perp$ were not contained in $\bigcap_{i=0}^{\infty} T^i\mathcal{H}$ then there would be a nonzero $x \in ([f]_{T^*})^\perp$ such that $x \notin \bigcap_{i=0}^{\infty} T^i\mathcal{H}$ and hence $(L^kx, e_1) \neq 0$ for some k, which we assume to be the smallest possible. For any

$y \in ([f]_{T^*})^{\perp}$ since $e_1 \in [f]_{T^*}$, $(y, e_1) = 0$, and hence $(Ly, T^{*i}f) = (TLy, T^{*(i-1)}f) = (y, T^{*(i-1)}f) = 0\ \forall i \geq 1$, so that $Ly \perp [T^*f]_{T^*} = [f]_{T^*}$. That is, $([f]_{T^*})^{\perp}$ is invariant under L. In particular, if $y \in ([f]_{T^*})^{\perp}$ then $L^i y \in ([f]_{T^*})^{\perp}\ \ \forall i \geq 0$. Hence $L^k x \in ([f]_{T^*})^{\perp}$, so that $(L^k x, e_1) = 0$ which is a contradiction, so there must be no nonzero x such that $x \perp [f]_{T^*}$ and $x \notin \bigcap_{i=0}^{\infty} T^i\mathcal{H}$, and hence $([f]_{T^*})^{\perp} \subset \bigcap_{i=0}^{\infty} T^i\mathcal{H}$.

Case (ii) was a part of Proposition 4.2.6.

Suppose that case (iii) holds. Since $f \in [f]_{T^*} = [T^*f]_{T^*} = [T^*(f+ce_1)]_{T^*}$, there is an $x \in [T^*(f+ce_1)]_{T^*}$ such that $f = x$, and hence $f + ce_1 - f = ce_1 \in \text{span}\{f + ce_1\} + [T^*(f+ce_1)]_{T^*} = [f+ce_1]_{T^*}$, so that $e_1 \in [f+ce_1]_{T^*}$ if $c \neq 0$. If $[f+ce_1]_{T^*}$ were $[T^*(f+ce_1)]_{T^*}$ then $f + ce_1$ would be contained in $[T^*(f+ce_1)]_{T^*} = [T^*f]_{T^*}$ and hence $f + ce_1 - f = ce_1$ would be contained in $[f]_{T^*}$ which is a contradiction if $c \neq 0$, so it must be that $[f+ce_1]_{T^*} \neq [T^*(f+ce_1)]_{T^*}$ when $c \neq 0$.

Suppose that case (iv) holds. Since $e_1 \notin [T^*f]_{T^*}$ but $e_1 \in [f]_{T^*}$ which is $\text{span}\{f\}\dot{+}[T^*f]_{T^*}$, there must be a nonzero constant d such that $e_1 = df + x$ where $x \in [T^*f]_{T^*}$. Let $c = -\frac{1}{d}$, then $f + ce_1 = cx \in [T^*f]_{T^*} = [T^*(f+ce_1)]_{T^*}$ so that $[f+ce_1]_{T^*} = [T^*(f+ce_1)]_{T^*}$, and $e_1 \notin [f+ce_1]_{T^*}$, else $f+ce_1-ce_1 = f \in [f+ce_1]_{T^*} = [T^*(f+ce_1)]_{T^*} = [T^*f]_{T^*}$ which is a contradiction, so $f + ce_1$ is contained in case (iii).

Suppose that case (v) holds. Since $e_1 \in [T^*f]_{T^*} = [T^*(f+ce_1)]_{T^*} \subset [f+ce_1]_{T^*}$, it follows that $e_1 \in [f+ce_1]_{T^*}\ \ \forall c \in \mathbb{C}$. According to case (i), if $[f+ce_1]_{T^*}$ were $[T^*(f+ce_1)]_{T^*}$, then $[f]_{T^*}$ would be equal to $[T^*f]_{T^*}$, but this is a contradiction, so it must be that $[f+ce_1]_{T^*} \neq [T^*(f+ce_1)]_{T^*}\ \ \forall c \in \mathbb{C}$. ∎

Corollary 4.2.8. *Let T be a left-invertible operator such that $\mathcal{H} \ominus T\mathcal{H} = span\{e_1\}$. If for a vector f there are two constants $c_1 \neq c_2$ such that $e_1 \notin [f + c_i e_1]_{T^*}\ \ i = 1, 2$, then $T^*|_{[f]_{T^*}}$ is left-invertible but not right-invertible, and hence $([f]_{T^*})^{\perp}$ has an index of two. If T is pure then a vector f is cyclic for T^* if and only if both e_1 and f are contained in $[T^*f]_{T^*}$ (or equivalently $e_1 \in [f]_{T^*}$ and $[f]_{T^*} = [T^*f]_{T^*}$), or if and only if there are two constants $c_1 \neq c_2$ such that $[f + c_i e_1]_{T^*} = [T^*(f + c_i e_1)]_{T^*}\ \ i = 1, 2$.*

Proof Let f be a vector such that there are two constants $c_1 \neq c_2$ such that $e_1 \notin [f + c_i e_1]_{T^*}\ \ i = 1, 2$. Since case (ii) is the only case of Proposition 4.2.7 with this possibility, it must be that $[f]_{T^*}$ is in case (ii), and hence $T^*|_{[f]_{T^*}}$ is left-invertible but not right-invertible, and by Theorem 4.2.1, $([f]_{T^*})^{\perp}$ has an index of two.

If T is pure and f is cyclic, then $\mathcal{H} = [f]_{T^*} = [T^*f]_{T^*}$, so that every vector is contained in $[T^*f]_{T^*}$, and hence both e_1 and f are.

Conversely, if $f \in [T^*f]_{T^*}$, then $[f]_{T^*} = \text{span}\{f\} + [T^*f]_{T^*} = [T^*f]_{T^*}$, and $e_1 \in [T^*f]_{T^*} = [f]_{T^*}$ implies that $[f]_{T^*}$ is in case (i). Therefore $([f]_{T^*})^{\perp} \subset \bigcap_{i=0}^{\infty} T^i\mathcal{H} = \{0\}$ since T is pure, so that $[f]_{T^*} = \mathcal{H}$, and hence f is cyclic for T^*.

Similarly, $[f]_{T^*}$ is in case (i) if and only if there are two constants $c_1 \neq c_2$ such that $[f + c_i e_1]_{T^*} = [T^*(f + c_i e_1)]_{T^*}$ $i = 1, 2$, so that if T is pure, f is cyclic for T^* if and only if this happens.

In Proposition 3.3.2 we saw that for any pure, left-invertible operator T and closed, invariant subspace $\mathcal{M}$ of T, $\mathcal{M}$ has a decomposition into a string of closed, invariant subspaces. There were mutually orthogonal vectors ε_{T^i} $i \geq 0$, each wandering, such that if we define $\mathcal{M}_i = \text{span}\{\varepsilon_{T^i}\} \oplus \text{span}\{\varepsilon_{T^{i+1}}\} \oplus \text{span}\{\varepsilon_{T^{i+2}}\} \oplus \ldots$, then $T\mathcal{M}_i \subset \mathcal{M}_{i+1}$ $\forall i \geq 0$. The following theorem shows that if there is a closed, invariant subspace $\mathcal{M}$ such that the decomposition holds in a bilateral way, then T has a closed, invariant subspace with an index greater than one (namely, $\mathcal{M}_i$ for any i).

Theorem 4.2.9. *Let T be a pure, left-invertible operator such that $\mathcal{H} \ominus T\mathcal{H} =$ span$\{e_1\}$, then there is an invariant subspace for T with dim$(\mathcal{M} \ominus T\mathcal{M}) > 1$ if and only if there is a bilateral sequence of orthogonal vectors $\{\varepsilon_i\}_{i=-\infty}^{\infty}$ such that each ε_i is a wandering vector and $T\mathcal{M}_i \subset \mathcal{M}_{i+1}$ $\forall i$ where $\mathcal{M}_i =$ span$\{\varepsilon_i\} \oplus$ span$\{\varepsilon_{i+1}\} \oplus$ span$\{\varepsilon_{i+2}\} \oplus \ldots$.*

Proof Suppose that there is a bilateral sequence of orthogonal wandering vectors $\{\varepsilon_i\}_{i=-\infty}^{\infty}$ such that $T\mathcal{M}_i \subset \mathcal{M}_{i+1}$ $\forall i \in \mathbb{Z}$ where $\mathcal{M}_i$ is defined as above. Let $\mathcal{M} = \overline{\text{span}\{\ldots, \varepsilon_{-2}, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots\}}$. Since T is pure, there must be a $k \geq 0$ such that $\mathcal{M} \not\subset T^{k+1}\mathcal{H}$ but $\mathcal{M} \subset T^k\mathcal{H}$. Since $T|_{T^k\mathcal{H}}$ is pure and left-invertible we can assume that $\mathcal{M} \not\subset T\mathcal{H}$ so that $P_{\mathcal{M}}e_1 \neq 0$. Let j be such that $(\varepsilon_j, e_1) \neq 0$, then $P_{\mathcal{M}_j}e_1 \neq 0$ so that $e_1 \notin \mathcal{M}_j^{\perp}$ and by Corollary 2.6.3 $T^*|_{\mathcal{M}_j^{\perp}}$ is left-invertible. Also, since $\mathcal{M}_{j-1} = \mathcal{M}_j \oplus \text{span}\{\varepsilon_{j-1}\}$, $\mathcal{M}_j^{\perp} = \mathcal{M}_{j-1}^{\perp} \oplus \text{span}\{\varepsilon_{j-1}\}$, and since ε_{j-1} is a wandering vector, $T^*\mathcal{M}_j^{\perp} \subset \mathcal{M}_{j-1}^{\perp}$ so that ε_{j-1} is not contained in the range of $T^*|_{\mathcal{M}_j^{\perp}}$ and hence this operator is left-invertible but not right-invertible, so that by Theorem 4.2.2 $T|_{T^k\mathcal{H}}$ and hence also T has an invariant subspace with index greater than one.

Conversely, suppose that T has an invariant subspace with index greater than one. We must produce a bilateral sequence $\{\varepsilon\}_{i=-\infty}^{\infty}$ of orthogonal wandering vectors as in the statement of the theorem. By Theorem 4.2.2 there is an invariant subspace $\mathcal{N}^{\perp}$ for T^* on which T^* is left-invertible but not right-invertible. Let f be a nonzero vector in $\mathcal{N}^{\perp}$ such that $f \notin T^*\mathcal{N}^{\perp}$. Then T^* is also left-invertible but not right-invertible when restricted to $[f]_{T^*}$ (it is left-invertible because it is the restriction of the left-invertible operator $T^*|_{\mathcal{N}^{\perp}}$ and it is not right invertible because $T^*[f]_{T^*} \subset T^*\mathcal{N}^{\perp}$ and $f \notin T^*\mathcal{N}^{\perp}$ implies that $f \notin T^*[f]_{T^*}$). Since $[f]_{T^*} = \bigvee_{i=0}^{\infty} T^{*i}f$ and $T^{*i}f \in T^*[f]_{T^*}$ $\forall i \geq 1$, $[f]_{T^*} \ominus T^*[f]_{T^*}$ must have a dimension of one so that by Corollary 2.1.9 $[f]_{T^*} = (\text{span}\{\delta_1\} \oplus \text{span}\{\delta_2\} \oplus \text{span}\{\delta_3\} \oplus \ldots) \oplus \bigcap_{i=0}^{\infty} T^{*i}[f]_{T^*}$ where each δ_i is a wandering vector and $[f]_{T^*} \ominus (\text{span}\{\delta_1\} \oplus \text{span}\{\delta_2\} \oplus \ldots \oplus \text{span}\{\delta_i\}) = T^{*i}[f]_{T^*}$

is an invariant subspace for T^* for every $i \geq 0$. Set $\varepsilon_{1-i} = \delta_i \ \forall i \geq 1$. In this way we have defined ε_i for every $i \leq 0$.

It remains to define ε_i for $i \geq 1$. Note that $\mathcal{M}_1 = [f]_{T^*}^{\perp}$ is an invariant subspace for T which is not the zero space since this would imply that $[f]_{T^*} = \mathcal{H}$, which contradicts the fact that T^* is left-invertible on $[f]_{T^*}$. Also, $P_{\mathcal{M}_1} e_1 \neq 0$ since $e_1 \notin [f]_{T^*} = (\mathcal{M}_1)^{\perp}$. By Proposition 3.3.2, $\mathcal{M}_1$ can be written in the form $\mathcal{M}_1 = \text{span}\{\varepsilon_1\} \oplus \text{span}\{\varepsilon_2\} \oplus \text{span}\{\varepsilon_3\} \oplus \ldots$ where each ε_i is a wandering vector and $T\mathcal{M}_i \subset \mathcal{M}_{i+1} \ \forall i \geq 1$. We now have a bilateral sequence of orthogonal wandering vectors $\{\varepsilon_i\}_{i=-\infty}^{\infty}$ since the ones with positive indices are orthogonal to the ones with nonpositive indices since the latter are contained in $[f]_{T^*}$ and the former are contained in $\mathcal{M}_1 = [f]_{T^*}^{\perp}$. We already have that $T\mathcal{M}_i \subset \mathcal{M}_{i+1} \ \forall i \geq 1$, and since $[f]_{T^*} \ominus \text{span}\{\delta_1\} \ominus \text{span}\{\delta_2\} \ominus \ldots \ominus \text{span}\{\delta_i\}$ is an invariant subspace for T^* for every i greater than or equal to zero and $[f]_{T^*}^{\perp} = \mathcal{M}_1$, $([f]_{T^*} \ominus \text{span}\{\delta_1\} \ominus \text{span}\{\delta_2\} \ominus \ldots \ominus \text{span}\{\delta_i\})^{\perp} = \text{span}\{\varepsilon_{1-i}\} \oplus \text{span}\{\varepsilon_{2-i}\} \oplus \ldots \oplus \text{span}\{\varepsilon_0\} \oplus \mathcal{M}_1 = \mathcal{M}_{1-i}$ is an invariant subspace for T, and since each ε_i is a wandering vector, it follows that $T\mathcal{M}_i \subset \mathcal{M}_{i+1} \ \forall i$ and the theorem is proved. ∎

The following proposition shows that if a left-invertible operator only has closed, invariant subspaces of dimension equal to one, then there cannot be large jumps in how close the subspace is to the vectors $\{e_i\}_{i=1}^{\infty}$.

Proposition 4.2.10. *Let T be a left-invertible operator such that $\mathcal{H} \ominus T\mathcal{H} = \mathit{span}\{e_1\}$ and $\mathcal{M}$ be a closed, invariant subspace for T with $\varepsilon = P_{\mathcal{M}} e_1 \neq 0$. If $\frac{\|\varepsilon_T\|}{\|\varepsilon\|} > \frac{|(Te_1, e_2)|}{c}$ where $\varepsilon_T = P_{\mathcal{M} \cap T\mathcal{H}} e_2 = P_{\mathcal{M} \ominus span\{\varepsilon\}} e_2$ and $c = \inf\{\|Tx\| : \|x\| = 1\}$, then $\dim(\mathcal{M} \ominus T\mathcal{M}) > 1$.*

Proof Suppose that $\frac{\|\varepsilon_T\|}{\|\varepsilon\|} > \frac{|(Te_1, e_2)|}{c}$. First notice that since ε is a wandering vector, $(T\varepsilon, \varepsilon) = 0$ so that $T\varepsilon \in \mathcal{M} \ominus \text{span}\{\varepsilon\}$. Also, $(T\varepsilon, e_2) \neq 0$ since this would imply that $T\varepsilon \in T\mathcal{H}$ and $T\varepsilon \perp T\mathcal{H} \ominus T^2\mathcal{H}$ so that $T\varepsilon \in T^2\mathcal{H}$. But this implies that $\varepsilon \in T^2\mathcal{H}$ and hence $(\varepsilon, e_1) = 0$, which contradicts $\varepsilon = P_{\mathcal{M}} e_1 \neq 0$. Therefore $\varepsilon_T \neq 0$. If $L\varepsilon_T$ were contained in $\mathcal{M}$, then since $\varepsilon_T = (\varepsilon_T, e_2)e_2 + x = \|\varepsilon_T\|^2 e_2 + x$ where x is contained in $T^2\mathcal{H}$, $(L\varepsilon_T, \varepsilon) = (L\varepsilon_T, P_{\mathcal{M}} e_1) = (\|\varepsilon_T\|^2 Le_2 + Lx, e_1) = (\|\varepsilon_T\|^2 Le_2, e_1)$. Also, $|(L\varepsilon_T, \varepsilon)| \leq \|L\|\|\varepsilon_T\|\|\varepsilon\|$. Since $\frac{1}{\|L\|} = c = \inf\{\|Tx\| : \|x\| = 1\}$, combining these two equations yields $\frac{\|\varepsilon_T\|}{\|\varepsilon\|}|(Le_2, e_1)| \leq \frac{1}{c}$. Since Le_2 can be rewritten as $Le_2 = (Le_2, e_1)e_1 + y$ where $y \in T\mathcal{H}$, rearranging to $e_1 = \frac{1}{(Le_2, e_1)}(Le_2 - y)$ shows that $(Te_1, e_2) = (\frac{1}{(Le_2, e_1)}(TLe_2 - Ty), e_2) = \frac{1}{(Le_2, e_1)}(e_2, e_2) = \frac{1}{(Le_2, e_1)}$. Substituting this in the above inequality yields $\frac{\|\varepsilon_T\|}{\|\varepsilon\|} \leq \frac{|(Te_1, e_2)|}{c}$. Since we already assumed that this is not true, it must be that $L\varepsilon_T$ is not contained in $\mathcal{M}$. By part (*ii*) of Theorem 4.2.2 (note that Theorem 4.2.2 requires that T be pure, but this was only required to preclude the case where $\mathcal{M} \subset \bigcap_{i=0}^{\infty} T^i\mathcal{H}$, which cannot happen if we assume that

$P_{\mathcal{M}}e_1 \neq 0$), since $e_1 \not\perp \mathcal{M}$ so that $\mathcal{M} \not\subset T\mathcal{H}$, if $\dim(\mathcal{M} \ominus T\mathcal{M})$ were equal to one, then since $\varepsilon_T \in T\mathcal{H}$, $L\varepsilon_T$ would have to be contained in $\mathcal{M}$. Since this is not the case, it must be that $\dim(\mathcal{M} \ominus T\mathcal{M}) > 1$. ∎

The next proposition is a kind of converse to the former one.

Proposition 4.2.11. *Let T be a left-invertible operator such that $\mathcal{H} \ominus T\mathcal{H} = span\{e_1\}$, with a closed, invariant subspace $\mathcal{M}$ of T such that $P_{\mathcal{M}}e_1 \neq 0$ and $dim(\mathcal{M} \ominus T\mathcal{M}) > 1$. Then for every $c > 0$ there is a closed, invariant subspace $\tilde{\mathcal{M}}$ of T such that $\frac{\|P_{\tilde{\mathcal{M}}}e_2\|}{\|\varepsilon\|} \geq \frac{\|\varepsilon_T\|}{\|\varepsilon\|} > c$, where $\varepsilon = P_{\tilde{\mathcal{M}}}e_1$ and $\varepsilon_T = P_{\tilde{\mathcal{M}} \cap T\mathcal{H}}e_2 = P_{\tilde{\mathcal{M}} \ominus span\{\varepsilon\}}e_2$.*

Proof Note that $\|P_{\tilde{\mathcal{M}}}e_2\| \geq \|\varepsilon_T\|$ is a consequence of the fact that $\tilde{\mathcal{M}} \supset \tilde{\mathcal{M}} \ominus \text{span}\{\varepsilon\}$. Since $\mathcal{M}$ has $P_{\mathcal{M}}e_1 \neq 0$ and $\dim(\mathcal{M} \ominus T\mathcal{M}) > 1$, by Corollary 4.2.4 $T^*|_{\mathcal{M}^\perp}$ is left-invertible but not right-invertible. Let f be a nonzero vector contained in $\mathcal{M}^\perp \ominus T^*\mathcal{M}^\perp$, then $T^*|_{[f]_{T^*}}$ must also be left-invertible but not right-invertible. By Proposition 4.2.6 $T^*|_{[f+de_1]_{T^*}}$ is also left-invertible but not right invertible. Let $\mathcal{M}_d = ([f + de_1]_{T^*})^\perp$, then $\varepsilon_d = P_{\mathcal{M}_d}e_1 \neq 0$ since $e_1 \notin [f+de_1]_{T^*}$ and $\lim_{d\to\infty} \varepsilon_d \to 0$ since $\lim_{d\to\infty} \frac{f+de_1}{\|f+de_1\|} = e_1$ and $f + de_1 \in \mathcal{M}_d^\perp$. Let $x = P_{([f]_{T^*})^\perp \cap T\mathcal{H}}e_2 \neq 0$, then since $(x, e_1) = 0$, $x \perp [f + de_1]_{T^*}$ $\forall d > 0$, so that $x \in \mathcal{M}_d$ $\forall d > 0$ and hence $\|P_{\mathcal{M}_d \cap T\mathcal{H}}e_2\| \geq \|x\| > 0$. Therefore, for any $c > 0$, there must be a d such that $\tilde{\mathcal{M}} = \mathcal{M}_d$ has $\frac{\|\varepsilon_T\|}{\|\varepsilon\|} > c$. ∎

4.3 The Index of Invariant Subspaces for Weighted Shifts

Theorem 4.3.1. *Let T be a left-invertible, weighted shift, then $\|\frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j\|$ is uniformly bounded in j if and only if the sequence of operators $\frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j$ converges weakly to zero.*

Proof Suppose that the $\|\frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j\|$ are uniformly bounded in j but the sequence $\frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j$ does not converge weakly to zero. Then there is a pair $\{x, y\} \in \mathcal{H} \times \mathcal{H}$ such that $(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*j}x, y)$ does not converge to zero, and hence there is an $\delta > 0$ and a subsequence $\{n_j\}_{j=1}^\infty$ such that $|(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*n_j}x, y)| \geq \delta$ $\forall j \geq 0$. For an arbitrary $z \in \mathcal{H}$, $|(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*n_j}x, z)| \geq |(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*n_j}x, y)| - |(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*n_j}x, y-z)| \geq |(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*n_j}x, y)| - \|\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*n_j}\|\|x\|\|y - z\|$ so that if

$$\|y - z\| \sup_j \|\frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j\|\|x\| = \|y - z\| \sup_j \|\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*j}\|\|x\| \leq \frac{\delta}{2} \qquad (4.5)$$

then $|(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*n_j}x,z)| \geq \frac{\delta}{2}$. Let k be chosen large enough that $z_k = P_{\mathcal{H}\ominus T^k\mathcal{H}}y$ satisfies inequality (4.5), and write z_k as $z_k = \sum_{i=1}^k c_i T^{i-1}e_1$, then if we write $y = (y_1, y_2, y_3, \ldots)$ then $c_i = \frac{y_i}{\alpha_1\alpha_2\cdots\alpha_{i-1}}$. So $\|\sum_{i=1}^k \bar{c}_i T^{*(i-1)}x\|$ is bounded, but $\|\sum_{i=1}^k \bar{c}_i T^{*(i-1)}x\|^2 = \sum_{j=1}^\infty |(\sum_{i=1}^k \bar{c}_i T^{*(i-1)}x, e_j)|^2 =$ $\sum_{j=0}^\infty |(\sum_{i=1}^k \bar{c}_i T^{*(i-1)}x, \frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j e_1)|^2 = \sum_{j=0}^\infty |(x, \frac{1}{\alpha_1\alpha_2\cdots\alpha_j}\sum_{i=1}^k c_i T^{(i-1)}T^j e_1)|^2 =$ $\sum_{j=0}^\infty |(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*j}x, \sum_{i=1}^k c_i T^{(i-1)}e_1)|^2 = \sum_{j=0}^\infty |(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_j}T^{*j}x, z_k)|^2 \geq$ $\sum_{j=0}^\infty |(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{n_j}}T^{*n_j}x, z_k)|^2 \geq \sum_{j=0}^\infty \frac{\delta^2}{4} = \infty$ where Bessel's inequality was used with the complete, orthonormal set $\{e_j\}_{j=1}^\infty$. Since this is a contradiction it must be that the sequence $\frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j$ converges weakly to zero.

Conversely, suppose that $\|\frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j\|$ is not uniformly bounded in j. Then by the Principle of Uniform Boundedness there is an x such that $\|\frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j x\|$ is not bounded and hence does not converge weakly to zero, so that $\|\frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j\|$ does not converge weakly to zero. ∎

Theorem 4.3.2. *Let T be a left-invertible, weighted shift such that the sequence of operators $\{\frac{1}{\alpha_1\alpha_2\cdots\alpha_j}T^j\}_{j=1}^\infty$ is uniformly bounded in norm. If there is a closed, invariant subspace $\mathcal{M}$ for T such that dim$(\mathcal{M}\ominus T\mathcal{M}) > 1$, then $\mathcal{M}$ does not contain any vectors that are in l^1.*

Proof As applying L to $\mathcal{M}$ does not change the index if $\mathcal{M} \subset T\mathcal{H}$ nor does it change whether or not $\mathcal{M}$ has vectors in l^1, we can assume that $\mathcal{M} \not\subset T\mathcal{H}$. Assume that $\dim(\mathcal{M}\ominus T\mathcal{M}) > 1$ and that there is a nonzero y such that $y \in \mathcal{M}$ and $y \in l^1$. By Corollary 4.2.4, $T^*|_{\mathcal{M}^\perp}$ is left-invertible but not right-invertible, so that there is a vector $\varepsilon \in \mathcal{M}^\perp$ such that $\varepsilon \notin T^*\mathcal{M}^\perp$. Write y as $y = (y_1, y_2, y_3, \ldots)$ and let m be the first integer such that $(y, e_i) \neq 0$. We show that $\sum_{i=1}^k \frac{\bar{y}_i}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{i-1}}T^{*(i-1)}\varepsilon =$ $\sum_{i=m}^k \frac{\bar{y}_i}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{i-1}}T^{*(i-1)}\varepsilon$ converges weakly to zero as k goes to infinity. This implies that $-\sum_{i=m+1}^k \frac{\bar{y}_i}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{i-1}}T^{*(i-1)}\varepsilon$ converges weakly to $\frac{\bar{y}_m}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{m-1}}T^{*(m-1)}\varepsilon$ as k goes to infinity, which implies that $T^{*(m-1)}\varepsilon$ is contained in $T^{*m}\mathcal{M}^\perp$ since $T^{*m}\mathcal{M}^\perp$ is a subspace so that it is weakly closed and $T^{*j}\varepsilon \in T^{*m}\mathcal{M}^\perp\ \forall j \geq m$. Since $T^*|_{\mathcal{M}^\perp}$ is left-invertible so that it is one-to-one, this also implies that ε is contained in $T^*\mathcal{M}^\perp$, which is a contradiction.

As $\|\frac{1}{\alpha_1\alpha_2\cdots\alpha_i}T^i\| = \|\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_i}T^{*i}\|$ is uniformly bounded in i and $y \in l^1$,

$$\|\sum_{i=1}^k \frac{\bar{y}_i}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{i-1}}T^{*(i-1)}\varepsilon\| \leq \sum_{i=1}^k |y_i|\|\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{i-1}}T^{*(i-1)}\varepsilon\| \leq \|y\|_{l^1}M\|\varepsilon\|, \quad (4.6)$$

where $M = \sup_i \|\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_i}T^{*i}\|$. Therefore the sequence $\sum_{i=1}^k \frac{\bar{y}_i}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{i-1}}T^{*(i-1)}\varepsilon$ is bounded and hence converges weakly. For each e_j,

$$(\sum_{i=1}^{k} \frac{\bar{y}_i}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{i-1}} T^{*(i-1)}\varepsilon, e_j) = (\sum_{i=1}^{k} \frac{\bar{y}_i}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{i-1}} T^{*(i-1)}\varepsilon, \frac{1}{\alpha_1\alpha_2\cdots\alpha_{j-1}} T^{j-1}e_1) =$$

$$(\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{j-1}} T^{*(j-1)}\varepsilon, \sum_{i=1}^{k} \frac{y_i}{\alpha_1\alpha_2\cdots\alpha_{i-1}} T^{i-1}e_1) = (\frac{1}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{j-1}} T^{*(j-1)}\varepsilon, \sum_{i=1}^{k} y_i e_i).$$

Since $\sum_{i=1}^{k} y_i e_i$ converges to y as k goes to infinity and $y \perp [\varepsilon]_{T^*}$, this shows that $x_k = \sum_{i=1}^{k} \frac{\bar{y}_i}{\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{i-1}} T^{*(i-1)}\varepsilon$ converges coordinatewise to zero. Since the sequence $\{x_i\}_{i=1}^{\infty}$ is uniformly bounded by inequality (4.6), it must converge weakly to zero, and completes the proof. ∎

Corollary 4.3.3. *Let T be a left-invertible, weighted shift such that the sequence of operators $\{\frac{1}{\alpha_1\alpha_2\cdots\alpha_j} T^j\}_{j=1}^{\infty}$ is uniformly bounded in norm and y be a nonzero vector in l^1. Then all closed, invariant subspaces of T that contain y must have an index of one.*

Remark 4.3.1. For $\beta \in \mathbb{R}$ let T_β be the weighted shift with weights $\alpha_i = (\frac{i+1}{i})^{\beta/2}$. For β equal to -1, 0, and 1 these are the weighted shifts that correspond to multiplication by z on the Bergman, Hardy and Dirichlet spaces, respectively. For $\beta \geq 0$ the weights are monotonically decreasing, so that the $\|\frac{1}{\alpha_1\alpha_2\cdots\alpha_j} T^j\|$ are uniformly bounded, and Theorem 4.3.2 yields that T_β cannot have any invariant subspaces with an index greater than one if it contains any vectors in l^1. It is known that T_β does not have any invariant subspaces with an index greater than one when $0 \leq \beta \leq 1$ (this was shown in Theorem 1 of [38] using a theorem similar to Theorem 2.5.4). For $\beta < 0$ it can be shown that the $\frac{1}{\alpha_1\alpha_2\cdots\alpha_j} T^j$ are not uniformly bounded and it was shown in [6] that for operators in a class containing these the operators have invariant subspaces with an index of any finite integer as well as infinity [6]. The following theorem also shows that T_β has a closed, invariant subspace with an index greater than one when $\beta < 0$.

Theorem 4.3.4. *Let T be a left-invertible, weighted shift. If*

$$\lim_{j} \limsup_{i\to\infty} \left| \frac{\alpha_i\alpha_{i+1}\cdots\alpha_{i+j-1}}{\alpha_1\alpha_2\cdots\alpha_j} \right| = \lim_{j} \limsup_{i\to\infty} \|\frac{1}{\alpha_1\alpha_2\cdots\alpha_j} T^j|_{T^i\mathcal{H}}\| = \infty,$$

then there is a closed, invariant subspace $\mathcal{M}$ of T such that dim$(\mathcal{M} \ominus T\mathcal{M}) > 1$.

Proof For ease of notation we assume that $\alpha_i > 0\ \forall i \geq 1$, since by Corollary 3.1.6 this is possible by using a similarity transformation, which does not change the index of the invariant subspaces. Set $M = \frac{sup_i \alpha_i}{\inf_i \alpha_i}$. Let $m \geq 2$, we will pick a sequence of integers $\{n_i\}_{i=1}^{\infty}$ that meets certain criteria and define $f = (\frac{1}{m}, 0, \ldots, 0, \frac{1}{m^2}, 0, \ldots, 0, \frac{1}{m^3}, 0, \ldots, 0, \frac{1}{m^4}, 0, \ldots)$ where there are $n_1 - 2$ zeros between $\frac{1}{m}$ and $\frac{1}{m^2}$, and $n_i - 1$ zeros between $\frac{1}{m^i}$ and $\frac{1}{m^{i+1}}$ for $i \geq 2$. We will then show that $e_1 \notin [f]_{T^*}$ and $e_1 \notin [f + ce_1]_{T^*}$ for some nonzero c, so that by Corollary 4.2.8 and Theorem 4.2.1 $([f]_{T^*})^{\perp}$ has an index of two. We will define the sequence iteratively, picking n_k after all n_i's with $i < k$ have been chosen. Note that if $n_j > \sum_{i=1}^{j-1} n_i \ \forall j \geq 2$ then for every pair $\{i, j\}$ $i \neq j$ there is at most one k such that $(T^{*i} f, e_k) \neq 0$ and $(T^{*j} f, e_k) \neq 0$, that is, $T^{*i} f$ and $T^{*j} f$ share at most one coordinate where they are both nonzero. Since we can always pick larger n_j's if necessary by what follows, we will assume that *$T^{*i} f$ and $T^{*j} f$ share at most one coordinate where they are both nonzero.* As we pick the n_i's, we will define f_i to be as above but with only i nonzero entries, so that $f = \lim_{i\to\infty} f_i$. To show that $e_1 \notin [f]_{T^*}$ we will recursively define a convergent sequence $\{x_i\}_{i=1}^{\infty}$ so that $(x_i, e_1) = 1$ and $x_i \perp T^{*j} f \ \forall j \leq M_i$ where $\lim_{i\to\infty} M_i = \infty$. Hence $x = \lim_{i\to\infty} x_i$ will satisfy $(x, e_1) = 1$ and $x \perp [f]_{T^*}$ so that $e_1 \notin [f]_{T^*}$.

So we start with $f_1 = \frac{1}{m} e_1$. Let n_1 be such that

$$\limsup_{i\to\infty} \| \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_j} T^j|_{T^i \mathcal{H}} \| > 2m \ \ \forall j \geq n_1 - 1,$$

so $f_2 = f_1 + \frac{1}{m^2} e_{n_1}$ Define $x_1 = e_1 - m e_{n_1}$ so that $(x_1, e_1) = 1$ and $x_1 \perp f$. We will never add anything else to the x_i's in any of the coordinates where f is nonzero, so $x_i \perp f \ \forall i \geq 1$. Since n_2 will be picked to be large enough, the next $T^{*i} f$ that will share a coordinate with the x_i's will be $T^{*(n_1-1)} f$ which will be nonzero in the first coordinate. Let n_2 be such that $\left| \frac{\alpha_{n_2+1} \alpha_{n_2+2} \cdots \alpha_{n_1+n_2-1}}{\alpha_1 \alpha_2 \cdots \alpha_{n_1-1}} \right| > 2m$, possible since $\limsup_{i\to\infty} \left| \frac{\alpha_i \alpha_{i+1} \cdots \alpha_{i+j-1}}{\alpha_1 \alpha_2 \cdots \alpha_j} \right| = \limsup_{i\to\infty} \| \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_j} T^j|_{T^i \mathcal{H}} \| > 2m \ \forall j \geq n_1 - 1$, and n_2 also be such that $\limsup_{i\to\infty} \| \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_j} T^j|_{T^i \mathcal{H}} \| > (2m)^2 M^{n_1-1} \ \forall j \geq n_2$ (and $n_2 > n_1$). Therefore $f_3 = f_2 + \frac{1}{m^3} e_{n_1+n_2}$. So that $x \perp T^{*(n_1-1)} f$, set $x_2 = x_1 - \frac{\alpha_1 \alpha_2 \cdots \alpha_{n_1-1} m}{\alpha_{n_2+1} \alpha_{n_2+2} \cdots \alpha_{n_1+n_2-1}} e_{n_2+1}$, where we know that the norm of what we added is less than $\frac{1}{2}$. The addition of $\frac{1}{m^3} e_{n_1+n_2}$ to f_2 has now made it so that $T^{*n_2} f$ and $T^{*(n_1+n_2-1)} f$ may not be orthogonal to x_2. Pick n_3 such that

$$\left| \frac{\alpha_{n_1+n_3} \alpha_{n_1+n_3+1} \cdots \alpha_{n_1+n_2+n_3-1}}{\alpha_1 \alpha_2 \cdots \alpha_{n_2}} \right| > (2m)^2 M^{n_1-1}$$

and $\limsup_{i\to\infty} \| \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_j} T^j|_{T^i \mathcal{H}} \| > (2m)^3 M^{n_1+n_2} \ \forall j \geq n_3$ (and $n_3 > n_1 + n_2$). Note that we automatically have

$$\left| \frac{\alpha_{n_1+n_3} \alpha_{n_1+n_3+1} \cdots \alpha_{n_1+n_2+n_3-1}}{\alpha_1 \alpha_2 \cdots \alpha_{n_2}} \right| > (2m)^2 \left| \frac{\alpha_{n_2+1} \alpha_{n_2+2} \cdots \alpha_{n_1+n_2-1}}{\alpha_{n_3+1} \alpha_{n_3+2} \cdots \alpha_{n_1+n_3-1}} \right|.$$

So that $x_3 \perp T^{*n_2} f$, set $x_3 = x_2 - \frac{\alpha_1\alpha_2\cdots\alpha_{n_2}m^2}{\alpha_{n_1+n_3}\alpha_{n_1+n_3+1}\cdots\alpha_{n_1+n_2+n_3-1}} e_{n_1+n_3}$, where we know that the part that we added has norm less than $\frac{1}{4}$. So that $x_4 \perp T^{*(n_1+n_2-1)} f$, set $x_4 = x_3 - \frac{\alpha_1\alpha_2\cdots\alpha_{n_1+n_2-1}m}{\alpha_{n_3+1}\alpha_{n_3+2}\cdots\alpha_{n_1+n_2+n_3-1}} e_{n_3+1}$, where the part that we added has norm less than $\frac{1}{4m}$.

Continue in this way, picking n_{k+1} such that $n_{k+1} > \sum_{i=1}^{k} n_i$,

$$\limsup_{i\to\infty} \|\frac{1}{\alpha_1\alpha_2\cdots\alpha_j} T^j|_{T^i\mathcal{H}}\| > (2m)^{k+1} M^{\sum_{l=1}^{k} n_l} \ \ \forall j \geq n_{k+1},$$

and such that

$$\left|\frac{\alpha_{(\sum_{l=1}^{k+1} n_l)-N_k}\alpha_{(\sum_{l=1}^{k+1} n_l)-N_k+1}\cdots\alpha_{(\sum_{l=1}^{k+1} n_l)-1}}{\alpha_1\alpha_2\cdots\alpha_{N_k}}\right| > (2m)^k M^{\sum_{l=1}^{k-1} n_l}$$

where N_k is the distance from $\frac{1}{m^{k+2}}$ (in the vector f) to the closest coefficient in the last defined x_j that has a $T^{*i}f$ not orthogonal to it. Set $f_{k+2} = f_{k+1} + \frac{1}{m^{k+1}} e_{\sum_{l=1}^{k+1} n_l}$, then define the next x_{i+1}'s by $x_{i+1} = x_i + c e_{\sum_{l=1}^{k+1} n_l - j}$ for $n_{k+1} \leq i \leq \sum_{l=1}^{k+1} n_l$, where the c is picked so that x_{i+1} is orthogonal to $T^{*j}f$'s, noting that the number of x_{i+1}'s for which $c \neq 0$ for each k is the Fibonacci sequence, which never doubles, and each of the bounds for the increases in the norms of the x_i's decreases by a factor of $\frac{1}{m}$ at each iteration where $m \geq 2$. Since we are adding to x_i at coefficients that were zero before, we are adding orthogonal vectors so that the x_i's are a Cauchy sequence (since $\sum_{i=0}^{\infty} \left(\frac{2}{m}\right)^i \leq \infty$ if $m > 2$) and hence converge to some x. Since $(x_i, e_1) = 1 \ \forall i \geq 1$, $(x, e_1) = 1$. Since for every j there is an i_j such that $x_i \perp T^{*j}f \ \forall i \geq i_j$, $x \perp [f]_{T^*}$ and hence $e_1 \notin [f]_{T^*}$.

Let $g = f + \frac{1}{m}e_1$, then we can construct a y such that $y \perp [g]_{T^*}$ as before, starting with $y_1 = e_1 - 2me_{n_1}$, then setting $y_2 = y_1 - \frac{\alpha_1\alpha_2\cdots\alpha_{n_1-1}m}{\alpha_{n_2+1}\alpha_{n_2+2}\cdots\alpha_{n_1+n_2-1}} e_{n_2+1}$, then $y_3 = y_2 - \frac{2\alpha_1\alpha_2\cdots\alpha_{n_2}m^2}{\alpha_{n_1+n_3}\alpha_{n_1+n_3+1}\cdots\alpha_{n_1+n_2+n_3-1}} e_{n_1+n_3}$ and $y_4 = y_3 - \frac{\alpha_1\alpha_2\cdots\alpha_{n_1+n_2-1}m}{\alpha_{n_3+1}\alpha_{n_3+2}\cdots\alpha_{n_1+n_2+n_3-1}} e_{n_3+1}$. We would continue to define the y_i's recursively by adding (in the same places as in x) the right amount so that $y \perp [g]_{T^*}$, where again for the same reasons y is bounded. Therefore T^* restricted to $[f]_{T^*}$ is left-invertible but not right-invertible, and hence $([f]_{T^*})^{\perp}$ has an index of two for T. ∎

Remark 4.3.2. Theorem 3.6 of [6] shows that *If a weighted shift is a contraction with a spectral radius of one such that* $\lim_{n\to\infty} \prod_{i=1}^{n} \alpha_i = 0$ (see the remarks above Theorem 3.6 of [6] for the equivalence of $\lim_{n\to\infty} \prod_{i=1}^{n} \alpha_i = 0$ and $T \in C_{00}$ for a weighted shift that is a contraction), *then it has closed, invariant subspaces with an index of any finite number or infinity.* We now show how if we specialize this result to the case where the weighted shift is left-invertible and the conclusion is that there is a closed, invariant subspace with an index of two, then this is a corollary of

Theorem 4.3.4. Since the spectral radius is one, $\|T^i\|$ must be equal to one for all i. If there were not a sequence $\{n_j\}_{j=1}^{\infty}$ such that $|\alpha_{n_j}\alpha_{n_j+1}\cdots\alpha_{n_j+i-1}|$ approached one, then there would be some k and a $0 < \delta < 1$ such that $|\alpha_j\alpha_{j+1}\cdots\alpha_{j+i-1}| < 1-\delta\ \forall j \geq k$. This implies that if $mi > k$ then $\|T^{2mi}\|^{1/2mi} < (1-\delta)^{1/2i}$. Thus if the spectral radius is one then there is such a sequence, and combining this with the requirement that $\lim_{n\to\infty}\prod_{i=1}^{n}\alpha_i = \lim_{n\to\infty}\alpha_1\alpha_2\cdots\alpha_n = 0$, yields that $\lim_j \limsup_{i\to\infty}\left|\frac{\alpha_i\alpha_{i+1}\cdots\alpha_{i+j-1}}{\alpha_1\alpha_2\cdots\alpha_j}\right| = \infty$, so that Theorem 4.3.4 applies.

Remark 4.3.3. Note that Theorem 4.3.4 assumes that T is left-invertible, whereas Theorem 3.6 of [6] does not. Consider the weighted shift T with weights $\alpha_i = 1\ \forall i \neq 2^k$ and $\alpha_i = \frac{1}{k}\ \forall i = 2^k$. Then T is not left invertible since $\inf_i |\alpha_i| = 0$ so that Theorem 4.3.4 does not apply. Since $\lim_{j\to\infty}\frac{1}{1}\frac{1}{2}\frac{1}{3}\cdots\frac{1}{j} = 0$, T satisfies $\lim_{n\to\infty}\prod_{i=1}^{n}\alpha_i = 0$. Since there are arbitrarily large gaps between 2^k and 2^{k+1}, for any j there is an n_j such that $\alpha_{n_j}\alpha_{n_j+1}\cdots\alpha_{n_j+j-1} = 1$, so that $\|T^j\| = 1$ and hence $r(T) = 1$, and in this case Theorem 3.6 of [6] does apply.

Remark 4.3.4. We now show how Theorem 4.3.4 covers examples that do not fit the hypotheses of Theorem 3.6 of [6]. To see that the hypotheses of Theorem 4.3.4 are weaker, one can take a weighted shift that satisfies the requirements of [6] and multiply all of the weights except the first one by c where $c < |\alpha_1|$. Then the spectral radius is not one, and as one of the requirements is that the weighted shift be a contraction, the largest factor by which one can multiply the new operator and have it still be a contraction is $\frac{1}{|\alpha|}$, but even then the spectral radius will be less than one by the requirement that $c < |\alpha_1|$. However, since only one weight in the normalized powers of T was changed, Theorem 4.3.4 still applies.

This example demonstrates the limitations of the hypotheses of Theorem 3.6 of [6], similar to those of Theorem 2.5.4. The main problem is that the two conditions $\lim_{n\to\infty}\prod_{i=1}^{n}\alpha_i = 0$ and the spectral radius being one are not combined as they are when one normalizes.

Remark 4.3.5. The previous case is easily recovered, however, if transforming by similarity first is allowed, since this does not change the indices of the closed, invariant subspaces. Then one can shrink the first weight by c by transforming by similarity, and then multiply the similar operator by $\frac{1}{|\alpha|}$ and still have it be a contraction. Thus the scaled operator is similar to a the original operator after it is scaled, and does satisfy the hypotheses of Theorem 3.6 of [6], so that the results can be made to still apply.

To demonstrate something more diabolical which cannot be recovered (at least the author does not know how), let T be the Bergman shift, that is the weighted shift with weights $\alpha_i = \sqrt{\frac{i}{i+1}}$. It can be seen that the Bergman shift satisfies the requirements of [6]. Then, as shown above, $\lim_j \limsup_{i\to\infty} \left|\frac{\alpha_i\alpha_{i+1}\cdots\alpha_{i+j-1}}{\alpha_1\alpha_2\cdots\alpha_j}\right| = \infty$. Let $0 < c < 1$ be arbitrary, and pick a sequence $\{n_j\}_{j=1}^{\infty}$ by defining n_j to be the first integer such that $\limsup_{i\to\infty} \left|\frac{\alpha_i\alpha_{i+1}\cdots\alpha_{i+l-1}}{\alpha_1\alpha_2\cdots\alpha_l}\right| c^j > 2^j \ \forall l \geq n_j$ is true. Let $\tilde{T}$ be the weighted shift whose weights are $\tilde{\alpha}_i = \alpha_i$ if $i = n_j$ for some j and $\tilde{\alpha}_i = c\alpha_i$ otherwise. Then $\limsup_{i\to\infty} \left|\frac{\tilde{\alpha}_i\tilde{\alpha}_{i+1}\cdots\tilde{\alpha}_{i+l-1}}{\tilde{\alpha}_1\tilde{\alpha}_2\cdots\tilde{\alpha}_l}\right| > 2^j \ \forall j : n_j \leq l$, so that the conditions of Theorem 4.3.4 are still satisfied. Also, $\lim_{n\to\infty} \prod_{i=1}^{n} \tilde{\alpha}_i = 0$ since none of the weights has grown. As can be seen from the formula for the weights, $\limsup_{i\to\infty} \left|\frac{\alpha_i\alpha_{i+1}\cdots\alpha_{i+j-1}}{\alpha_1\alpha_2\cdots\alpha_j}\right| = \frac{1}{\alpha_1\alpha_2\cdots\alpha_j} = \sqrt{j+1}$. Since $\frac{\sqrt{j+1}}{2^j}$ goes to zero as j goes to infinity, there are larger and larger spaces between the n_j's so that the spectral radius of $\tilde{T}$ is c.

Therefore the results from [6] do not apply, but there is also no way to transform by similarity, as the spectral radii of similar operators are equal, so that the requirement that the spectral radius be one means that one must multiply by $\frac{1}{c}$. Then it will be impossible to transform by similarity and have it be a contraction, as there are an infinite number of weights that are greater than $\frac{1}{c} - \delta$ for any $\delta > 0$. Therefore the results from Theorem 4.3.4 are more inclusive than those from [6].

Chapter 5

Conclusions and Future Work

In this dissertation we have studied the invariant subspaces of weighted shifts, including more general operators when possible. This has led to results concerning arbitrary, pure, left-invertible operators, and sometimes simply left-invertible operators.

We started by observing the conditions that must be satisfied for a left-invertible operator to satisfy the *Wandering Subspace Property* based on how certain subspaces are mapped by the operator and its left-inverse (Theorem 2.4.1). We then saw that it is necessary and sufficient that there be a dense set on which the operators $T^i L^i$ do not grow too large pointwise (Theorem 2.5.1). We then slightly generalized the results from [35] to see a sufficient condition for a pure, left-invertible operator based on how large the powers of T grow pointwise, where they must grow, but slowly enough (Theorem 2.5.4). This led to results about general, left-invertible operators whose powers grow pointwise slowly enough in a uniform way. It showed that this is a strong requirement as it yields that the pure part is a unitary operator and is equal to the pure part of L^* (Proposition 2.5.5 and Corollary 2.5.6).

The first, main problem that we have studied was: *Is every closed, invariant subspace of an arbitrary, left-invertible, weighted shift generated by its wandering subspace?* We saw that this is not true in general, and in a sense the class of left-invertible, weighted shifts that do not satisfy this property is dense in the set of all left-invertible, weighted shifts (Theorem 3.2.8). This showed that the first, main problem, which was an open question, had a negative answer. We saw that this property is not preserved under transformations of similarity, and that studying the invariant subspaces of all operators that are similar to a given operator is equivalent to studying that operator under all possible equivalent inner products (Theorem 3.2.4). Then we showed how one can know if a weighted shift that eventually only has weights that are one can have closed, invariant subspaces on which the restriction of the operator does not satisfy the *Wandering Subspace Property*.

The second, main problem that we studied was: *When does a weighted shift have only closed, invariant subspaces of index equal to one?* This was answered by several equivalent conditions, with the most useful one being that for any closed, invariant subspace $\mathcal{M}^\perp$ of T^*, $T^*|_{\mathcal{M}^\perp}$ being left-invertible implies that it is right-invertible (Theorem 4.2.2 and Corollary 4.2.4). We then saw that the set of vectors for which T^* restricted to the smallest closed, invariant subspace of T^* containing said vector is left-invertible but not right-invertible is either the zero-set or dense in the whole space (Proposition 4.2.6). This led to results about how the four different classes of subspaces of T^* (based on the different possibilities of being or not being left-invertible and right-invertible) change if they are cyclic and one perturbs the cyclic vector (Proposition 4.2.7). We also saw that a pure, left-invertible operator has a subspace of index greater than one if and only if there exists a bilateral chain of invariant subspaces shifted by the operator or it satisfies a jump-condition (Theorem 4.2.9 and Proposition 4.2.10). We then used the previous results to show that if the normalized powers of T are bounded then any closed, invariant subspace of T which has an index greater than two cannot contain certain vectors (4.3.2). Finally we constructed a closed, invariant subspace of T which had an index of two if the normalized powers of T were unbounded on smaller and smaller subspaces, which generalized a result from [6] (Theorem 4.3.4)

5.1 Future Work

The following is a list of some of the problems that the author would like to see researched. The difficulty of the problems or their usefulness in solving other, current work is not known (and may be trivial), and it is listed in the order that it appeared in this work.

An improvement of Theorem 2.5.4, including a better understanding of why it works, and what the innate limitations of the proof are.

In Corollary 2.5.6, not only is the space on which T is pure equal to the space on which L^* is, but they are equal operators there. The author would like to know in general when the spaces are the same, without the limitation that the operators be the same.

What are the uses of Proposition 3.1.7 and how does it show that weighted shifts are different from other operators, or how could it be generalized to arbitrary, left-invertible operators?

A better understanding of when a left-invertible, weighted shift satisfies the *Wandering Subspace Property* when it is restricted to any of its closed, invariant subspaces and why the property would not be satisfied.

A Theorem of equivalent conditions for an operator to not satisfy the *Wandering*

Subspace Property when restricted to one of its closed, invariant subspaces that is analogous to Theorem 4.2.2.

Can Theorem 3.3.3 be generalized to cover any left-invertible weighted shift? When can an operator be extended to be a weighted shift?

What does it mean for an operator that is right but not left-invertible to become left but not right-invertible when restricted to one of its closed, invariant subspaces (Corollary 4.2.4 for T^*)?

Can the fact that a vector f is cyclic for the adjoint of a left-invertible, weighted shift if and only if $[T^*f]_{T^*}$ contains f and e_1 be used ((*i*) of Proposition 4.2.7)?

What does it mean to have two vectors that are arbitrarily close to each other with one cyclic for T^* and the other generating a closed, invariant subspace on which T^* is left but not right-invertible (Propositions 4.2.6 and 4.2.7)?

When does a left-invertible, weighted shift have a closed, invariant subspace on which the restriction can be extended in a nontrivial way to be a bilateral, weighted shift (Theorem 4.2.9)?

Is it true that $\lim_{i\to\infty} \|P_{\mathcal{M}} e_i\| = 1$ for any nontrivial closed, invariant subspace if the operator is a left-invertible, weighted shift whose normalized powers are bounded? When is this true and does it have any relation with the index of a subspace?

Can Theorem 4.3.2 be strengthened to show that the index of every closed, invariant subspace is one?

Can Theorem 4.3.4 be strengthened to cover any left-invertible, weighted shift whose normalized powers are unbounded or at least a larger class of operators?

Appendix A

This appendix shows the calculations that were used to obtain equation (3.4).

We want to calculate $\sup\{\|(I - P_{S^n\mathcal{H}})p * f\|^2 : \ f \in \mathcal{H}, \ \|f\| = 1\}$, where $p = (\varepsilon_1, \varepsilon_2, 0, 0, 0, \ldots)$. For any $f = (f_1, f_2, f_3, \ldots)$, $(I - P_{S^n\mathcal{H}})p * f = (\varepsilon_1 f_1, \varepsilon_1 f_2 + \varepsilon_2 f_1, 0, 0, 0, \ldots)$. Therefore we want to maximize $\left\| \begin{bmatrix} \varepsilon_1 & 0 \\ \varepsilon_2 & \varepsilon_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\|^2 : \ \left\| \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\| = 1.$ Since for any matrix A, $(Ax, Ax) = (A^*Ax, x)$, and A^*A is self-adjoint, this will be the largest eigenvalue of

$$\begin{bmatrix} \bar{\varepsilon}_1 & \bar{\varepsilon}_2 \\ 0 & \bar{\varepsilon}_1 \end{bmatrix} \begin{bmatrix} \varepsilon_1 & 0 \\ \varepsilon_2 & \varepsilon_1 \end{bmatrix} = \begin{bmatrix} |\varepsilon_1|^2 + |\varepsilon_2|^2 & \bar{\varepsilon}_2\varepsilon_1 \\ \bar{\varepsilon}_1\varepsilon_2 & |\varepsilon_1|^2 \end{bmatrix}.$$

Taking the determinant of $\begin{bmatrix} |\varepsilon_1|^2 + |\varepsilon_2|^2 - \lambda & \bar{\varepsilon}_2\varepsilon_1 \\ \bar{\varepsilon}_1\varepsilon_2 & |\varepsilon_1|^2 - \lambda \end{bmatrix}$ we obtain

$$(|\varepsilon_1|^2 + |\varepsilon_2|^2 - \lambda)(|\varepsilon_1|^2 - \lambda) - |\varepsilon_1|^2|\varepsilon_2|^2 = \lambda^2 - (2|\varepsilon_1|^2 + |\varepsilon_2|^2)\lambda + |\varepsilon_1|^4.$$

Using the quadratic formula yields

$$\begin{aligned} \lambda &= \frac{2|\varepsilon_1|^2 + |\varepsilon_2|^2 \pm \sqrt{(2|\varepsilon_1|^2 + |\varepsilon_2|^2)^2 - 4|\varepsilon_1|^4}}{2} \\ &= |\varepsilon_1|^2 + \frac{1}{2}|\varepsilon_2|^2 \pm \frac{1}{2}\sqrt{4|\varepsilon_1|^2|\varepsilon_2|^2 + |\varepsilon_2|^4} \\ &= |\varepsilon_1|^2 + \frac{1}{2}|\varepsilon_2|^2 \pm \frac{1}{2}|\varepsilon_2|\sqrt{4|\varepsilon_1|^2 + |\varepsilon_2|^2}. \end{aligned}$$

It is also possible to show directly that $\sup\{\|(I - P_{S^n\mathcal{H}})p * f\|^2 : \ f \in \mathcal{H}, \ \|f\| = 1\} = |\varepsilon_1|^2 + \frac{1}{2}|\varepsilon_2|^2 \pm \frac{1}{2}|\varepsilon_2|\sqrt{4|\varepsilon_1|^2 + |\varepsilon_2|^2}$ (without using the connection with the eigenvalues) or by using Mathematica.

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