TESTING FOR CHANGES IN TREND IN WATER QUALITY DATA

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ABSTRACT

Time series of water quality variables typically possess many of several characteristics which complicate analysis. Of interest to researchers is often the trend over time of the water quality variable. However, sometimes water quality variable levels appear to increase or decrease monotonically for a period of time then switch direction after some intervention affects the factors which have a causal relationship with the level of the variable. Naturally, when analyzed for trend as a whole, these time series usually do not provide significant results. The problem of testing for a change in trend is addressed, and a method for performing this test based on a test of equivalence of two modified Kendall's Tau nonparametric correlation coefficients (neither necessarily equal to zero) is presented. The test is made valid for use with serially correlated data by use of a new bootstrap method titled the effective sample size bootstrap. Further issues involved in applying this test to water quality variables are also addressed.

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CHAPTER 1

Introduction

Water quality data have many characteristics that complicate statistical analyses. Water quality variables such as nitrogen, phosphorus, biological oxygen demand, fecal coliforms, pH, total and non-filterable residue, and trace metals are measured over time and space as part of water quality monitoring programs. The principal research question of interest is often whether or not a trend over time is present in the water quality variable and, if there is, to characterize and quantify this trend. Other research questions of interest include comparing trends and median levels across seasons and comparing sites. The main purpose of this research is to develop a significance test for change in trend of water quality variables.

Features of these data that complicate analysis often include non-normality, censoring, unequal sampling intervals (which often can be viewed as equal sampling intervals with missing data), seasonality, serial correlation, covariate effects, and the presence of outliers. Common influential covariates are stream flow and rainfall.

Censoring occurs when laboratories report observations as being beyond a certain value called a detection limit. Laboratories do this because of the imprecision in the measurement of small quantities or sometimes even of large quantities. In fact, with measurement error, it is not always certain that a lower detection limited value is actually less than the reported value [*Dakins et al.* 1996]. Detection limits further complicate the situation because over time the detection limits can change as new laboratory procedures are developed or as different laboratories are used. Often, all values at or below a lower detection limit are treated as being tied. For some variables at some sites, the percentage of ties induced by the censoring can exceed 80%. For further discussion of detection limits in water quality variables, see *Dakins et al.* [1996] and *Gilliom et al.* [1984].

The complicating features of water quality data described above make classical parametric methods like regression, analysis of covariance, and traditional time series approaches difficult or impossible to implement appropriately. Consequently, such data are often analyzed using nonparametric methods, e.g. *Mann* [1945], *Hirsch et al.* [1982], *Hirsch and Slack* [1984], *Dietz and Kileen* [1981], *Lettenmaier* [1988], and *Smith et al.* [1993]. More recently, semi-nonparametric methods have come into consideration, e.g. *van Belle and Hughes* [1984], *Berryman et al.* [1988], and *Loftis et al.* [1991]. These nonparametric and semi-nonparametric methods are described in Chapter 2 along with their problems, shortcomings, and some related issues. Chapter 3 presents background statistical work for a change in trend test that is described in Chapter 4. In addition to presenting an hypothesis test for change in trend, Chapter 4 also details a test for change in slope if the underlying trend is assumed to be linear. Chapter 5 describes a method for estimating autoregressive parameters in water quality data. Finally, Chapter 6 summarizes future research opportunities.

CHAPTER 2

Trend Analysis and Related Issues

2.1 Two Types of Water Quality Trends

Two types of trend are commonly considered in the water quality literature, monotonic and step. A step trend occurs when the time series experiences a change point in location. In that case, it is assumed that the water-quality variable changes at one point in time from one constant level to another constant level. *Hirsch et al.* [1991] said that step trend procedures should only be used when the data are separated into two groups by a relatively large gap in time or when the time of an event suspected to have an effect on water quality is known.

One nonparametric method for testing the step trend hypothesis which is commonly cited in the water quality literature, e.g. *Lettenmaier* [1976] and *Hirsch* [1988], is the Mann-Whitney-Wilcoxon Rank Sum test [*Wilcoxon* 1945]. Using this test is simply treating the hypothesis as a two-sample location problem which has several shortcomings. These include (i) not taking the temporal order of the data into account, (ii) sometimes erroneously assuming independence of the observations, and (iii) assuming that changes are sudden rather than gradual. Seasonality, censoring, and covariate effects should somehow be taken into account as well.

There is a large body of literature on the nonparametric change-point in location problem which could be used to test a step trend hypothesis. See for example *Brodsky and Darkhovsky* [1993], *Wolfe and Schechtman* [1984], *Lombard* [1987], *Csorgo and Horvath* [1987], or *Pettitt* [1979]. The problem for analysis of water-quality data is that these methods have some of the same shortcoming as the Mann-Whitney-Wilcoxon Rank Sum test. Namely, they entail the assumption that the observations are independent, and they ignore the seasonal nature of the data and the effect of covariates.

Jaruskova [1997] studied the problem of applying change-point detection methods to hydrological data. Jaruskova recommended using yearly averages to adjust for seasonality. However, to deal with dependence, she assumed that the observations could be transformed to normality. She then used inequalities and simulations to obtain critical values. Other methods for dealing with dependence also hinge on normality, e.g. *Tang and MacNeill* [1993]. They rely on the convergence of sums to normality and use common least squares estimation.

Usually, monotonic trend is the hypothesis of interest. Several nonparametric approaches exist for testing the hypothesis of monotonic trend. These include the Mann-Kendall test for trend on deseasonalized data [*Man*n 1945], the seasonal Kendall analysis [*Hirsch et al.* 1982], and the seasonal Kendall analysis with the covariance sum test [*Hirsch and Slack* 1984], with the covariance inversion test [*Dietz and Kileen* 1981], or with the covariance eigenvalue test [*Lettenmaier* 1988]. *Smith et al.* [1993] provide a method for performing these tests in a multivariate manner.

2.2 Seasonal Kendall Analysis and Other Nonparametric Trend Analyses

The seasonal Kendall analysis works as follows. The data are first blocked by some time division so that values from different time divisions will not be compared. This blocking prevents the seasonality of the data from invoking a trend and reduces variability when there is a seasonal effect. Often months are used as blocks. Within each season k, a Mann-Kendall statistic, S_k, and its variance are computed.

$$S_k = \sum_{i=1}^{n_k-1} \sum_{j=i+1}^{n_k} sign[(y_j - y_i)(x_j - x_i)],\ k = 1, 2, ..., m \tag{2.1}$$

where x_i is time of the i^{th} observation, y_i is the value of the i^{th} observation of the water quality variable, m is the number of seasons, and n_k is the number of observations in season k. The variance of S_k computed under the nul hypothesis is given by

$$Var(S_k) = \frac{n_k \times (n_k - 1) \times (2n_k + 5) - \sum_{h=1}^{g} t_h \times (t_h - 1) \times (2t_h + 5)}{18} \tag{2.2}$$

where t_h is the number of observations tied in the h^{th} group out of the g groups of tied observations.

The information from all the seasons is then aggregated by summing the individual S_k statistics and their variances. The test statistic is defined as follows:

$$S = \sum_{k=1}^{m} S_k, \tag{2.3}$$

$$Var(S) = \sum Var(S_k) \tag{2.4}$$

as seasons are assumed to be independent, and finally,

$$Z = \frac{S}{\sqrt{Var(S)}} \text{ is the Wald type test statistic.} \tag{2.5}$$

The trend hypothesis is then tested by invoking a fairly good normality approximation to the distribution of the standardized test statistic [*Mann* 1945 and *Hirsch et al.* 1982]. Another good quality of this test is that missing values do not invalidate the test as long as the pattern of missing values is random. Two important values related to the statistic S are the nonparametric correlation coefficients: Kendall's tau and modified Kendall's tau. These are both defined and discussed in the next chapter.

The other way to use the Mann-Kendall test for trend with seasonality is to first adjust for seasonality, i.e. deseasonalize the data, then perform the Mann-Kendall test on the resulting time series. Deseasonalization is accomplished by subtracting seasonal means or medians from all the observations in that season and also, sometimes, dividing by seasonal standard deviations [*Hirsch et al.* 1982]. This method has a tendency to be liberal for small samples because of negative correlation added to the data by deseasonalization [*Harcum et al.* 1992]. It is also not as easily adjusted to handle serial correlation as is the blocking-on-seasons approach.

In power studies with independent seasons, the seasonal Kendall analysis has been shown to be a powerful choice for a general procedure when overall monotonic trend is of interest [*Hirsch et al.* 1982, *Harcum et al.* 1992, *Loftis et al.* 1989, and *Taylor and Loftis* 1989]. However, when the seasons are not independent, this method does not maintain appropriate alpha levels. The covariance sum test [*Hirsch and Slack* 1984], the covariance inversion test [*Dietz and Kileen* 1981], and the covariance eigenvalue test [*Lettenmaier* 1988] all take this dependence into account. The covariance sum test is the most powerful of these for overall monotonic trend when analyzing a single variable [*Hirsch and Slack* 1984, *Lettenmaier* 1988, *Thas et al.* 1998]. Unfortunately, when the observations are in fact independent and especially for small sample sizes (e.g. ten years or less of data), it is much less powerful than the seasonal Kendall method described above which assumes independence [*Loftis et al.* 1991]. The covariance inversion test and the covariance eigenvalue test are capable of detecting trends that have varying signs among seasons with the covariance eigenvalue being the more powerful of the two [*Lettenmaier* 1988]. All three of these methods have low power with small sample sizes [*Hirsch and Slack* 1984, *Taylor and Loftis* 1989, *Loftis et al.* 1991, and *Harcum et al.* 1992]. The covariance sum, inversion, and eigenvalue tests can also be applied in a multivariate manner [*Smith et al.* 1993, and *Dietz and Killeen* 1981]. Thus, a multivariate trend could be detected that would not be detected by looking at any one of the individual water quality variables in a univariate manner. The statistic based on the covariance sum test is asymptotically chi-squared. Approximate p-values can be obtained for the covariance inversion and eigenvalue tests via a three parameter gamma distribution [*Rheem* 1992].

An alternative to using Kendall's tau based tests for trend is the use of Spearman's rho as a basis for tests. *Alvo and Cabilio* [1994] give a method for using

Spearman's rho to test for trend that allows missing observation from a regularly sampled time series with independent observations. In *Alvo and Cabilio* [1995] they extend their ideas to Kendall's tau, However, they once again require observations to be missing randomly from a regularly sampled time series and that the observations be independent.

2.3 The Covariance Sum Test

To preform the seasonal Kendall analysis with serially correlated data, the covariance sum test [*Hirsch and Slack* 1984] modifies the variance of the sum S from Equation 2.3 to include covariances because the seasons are no longer independent. Let $\Sigma = \{\sigma_{gh}\}$, $g, h = 1, ..., m$, denote the covariance matrix of the vector of S_k statistics. Then the variance of S used to form the asymptotically normal test statistic is given by the following formula:

$$\mathrm{Var}(S) = \vec{1}'_m \times \Sigma \times \vec{1}_m \tag{2.6}$$

where $\vec{1}_m$ is a vector of ones of length m. *Dietz and Killeen* [1981] provide a method for estimating the elements of this matrix. σ_{gg} is simply the variance of S_g as given above in equation 2.2. The covariances, σ_{gh}, where $g \neq h$, are found by the following algorithm.

Let X be an $n \times m$ matrix containing all the observations where n is the number of years the data span and m is the number of seasons. Further, let R be the $n \times m$ matrix containing the ranks of the observations in X where the observations in each season, i.e. elements within each column of X, are ranked separately. Further, let n_k be the number of years with non-missing values in season k, and let

$$K_{gh} = \sum_{i<j} \text{sign}((X_{jg} - X_{ig})(X_{jh} - X_{ih})), \quad (2.7)$$

where the sign function takes the values -1,0, or 1, and is zero if any one of the values is missing. Then, as show by *Dietz and Killeen* [1981],

$$\sigma_{gh} = [K_{gh} + 4\sum_{i=1}^{n} R_{ig}R_{ih} - n(n_g + 1)(n_h + 1)]/3. \quad (2.8)$$

2.4 Dealing with Detection Limits

When some of the values are detection limited, the S_k statistic in Equation 2.1 needs to be modified. *Hirsch, et al.* [1982] and *Gillion, et al.* [1984] suggested treating as tied all the observations smaller than the largest reported lower detection limit. This is consistent with "Kendall's tau, Breslow-type" [*Brown et al.* 1974] from survival analysis.

Hughes and Millard [1988] give a method for dealing with detection limits that does not throw out as much information and has better power properties. They compute an "expected rank vector" by averaging the ranks for each observation from all possible rankings consistent with the observed data. Lower detection limits are considered to be upper ends of the intervals which could possibly contain the true data values. They then use these ranks to compute the Mann-Kendall statistic as described above. A problem is that these type of conditional statistics no longer have the same null distribution as the standard case. With Hughes and Millard's method, the new statistic has less variability. A further complication is that their method of obtaining an approximate normal test requires that the pattern of censoring be random. This is unlikely to be the case. When the censoring mechanism is dependent on time, the null distribution for the test statistic must be estimated by generating random permutations of the expected rank vector, storing the value of the statistic computed from that configure of ranks if it agrees with the censoring pattern, and repeating a large number of times to estimate the null distribution. This quickly grows in computational difficulty. One final problem with this method is the unlikely assumption that all possible rankings are equally likely.

Another method for dealing with censored data is tobit regression [*Judge et al. 1985,* and *Helsel and Hirsch 1992*]. This is a maximum likelihood parametric method and requires the assumption that the dependent variable is normally distributed. Thus it is not an ideal method for water-quality data. Tobit estimates are slightly biased so bias corrections should be used [*Cohn* 1988]. With normal data this method can be used to estimate and test a linear trend. Perhaps some modification can be made to tobit regression to make it more applicable in the water quality data context.

2.5 The Multiple-Observation-Problem

All the nonparametric methods for trend analysis named above in Section 2.2 and 2.3 require one observation per season in every year. This is not always the case. When there are no observations in a particular season within a given year, the above methods are all still valid assuming that values are missing randomly.

Another problem occurs when a season within a year has multiple observations. In this case, *Helsel and Hirsch* [1992] suggested taking the median of the observations within the same season when the variations in the sampling frequency are random, and subsampling by using the value closest to the center of the season when there is a systematic trend in the sampling frequency. They recommended subsampling in the latter case because using the median would induce a trend in the variance. Using the median can also induce a trend in the data if many values are lower detection limited. This recommendation was verified by a Monte Carlo study performed by the author [*Darken* 1998].

Other possible solutions to the multiple-observations-per-season problem exist: (i) using a mean, (ii) treating observations in the same season and month as tied in time with an adjustment to the variance formula [*van Belle and Hughes* 1984, and *Gilbert* 1987], (iii) expanding the time variable to include days so as to eliminate ties in time, or (iv) using a weighted median. The mean is inferior to the median because of skewness and detection limits. Using all the original data weights years with more observations more heavily, and values within the same season in the same year are quite likely to be highly correlated. Because these observations are assumed to be independent for the adjustment to the variance and the hypothesis test, implementing (ii) is a bad idea. The use of a weighted median seems like a compromise between using the value closest to the center and using the median, but it is probably unnecessarily complicated. In the simulation study of *Darke*n [1998], using a Gaussian kernel for the weights, the weighted median was better at maintaining alpha levels then the median but not as good as the subsampling method. The weighted median solution was generally more powerful than the subsampling method but less powerful than using the median. So, the weighted median was in fact a compromise between the use of the median and the subsampling method, but for optimal results, it appeared that the extreme cases of the weighted median were needed. Namely, with a systematic variation in the sampling frequency, the weighted median that assigns all the weight to the value closest to the center appeared to be best, while with random variation in the sampling frequency, the weighted median with uniform weights performed best.

An interesting side note from this Monte Carlo study was that even under extremely mild levels of serial correlation, like .05 for observations one month apart, when months were used as seasons, the seasonal Kendall analysis assuming independence did not maintain appropriate alpha levels. This justifies the need for tests that allow dependence between seasons.

2.6 Testing Homogeneity of Trend and Median Levels Between Seasons

To test for homogeneity of trend between seasons, *van Belle and Hughes* [1984] gave a method that requires seasons to be independent. Letting m be equal to the number of seasons, an approximately chi-squared statistic with m degrees of freedom is decomposed into a one degree of freedom overall-trend component and an $m-1$ degree of freedom remainder. This remainder is used to test the hypothesis that the trend is homogeneous across seasons. *Smith et al.* [1993] extended this test to allow dependent seasons by using the covariance matrix of the test statistics from the different seasons.

Testing for homogeneity of median levels between seasons is not as easy as it sounds. To test this hypothesis, years need to be blocks to eliminate the effect of an overall trend. This suggests the use of a nonparametric ANOVA-like analysis, specifically, Friedman's test [*Friedman* 1937]. Even with independence, however, Friedman's test can't be used with missing values. Although several ad hoc solutions come to mind, the missing-value-problem with Friedman's test has not been solved because of the countless ways to assign ranks to missing values and the lack of a good method for concluding which is best. If season effects can be ordered in the alternative hypothesis, then *Alvo and Cabilio* [1995] give a method to handle missing values that outperforms simply ignoring blocks (years) with missing values. However, their method still requires independent errors. In reality seasons are likely to be dependent, so the problem gets more difficult.

A new idea of how to approach this problem is to analyze the problem as repeated measures in a split-plot design. Year would be the whole plot treatment while season would be the split-plot treatment. Due to the typical nature of the data, a nonparametric analysis should probably be used. Older nonparametric methods do not seem optimal. Some of these methods are based on reducing the longitudinal part (the observations within each given year in the set up here) to one or two numbers. *Ghosh et al.* [1973] fit polynomials to the longitudinal part of the data and used those parameter estimates as the raw data for performing a nonparametric analysis. This is really a semi-nonparametric method and might be hard to implement in the water quality context because of the dependence between observations one season apart and the detection limits. Perhaps a sinusoid would work better then a polynomial in this context. With some sort of robust fit, this is one possible solution.

Other methods, like those presented in *Koch* [1970], would not work without replication which generally does not exist in the water-quality data context. Replication is necessary to estimate the covariance matrix. Koch's methods were called nonparametric because the analysis is performed on rank transformed data.

More recent ideas are not ideal either. *Agresti and Pendergast* [1986] gave a method that requires compound symmetry, an assumption that does hold in the water-quality data context. Possibly the best methods currently available are those given in *Akritas and Arnold* [1994] which are based on performing rank-transformations to the data. Even in this context, the issues created by missing values and censoring still need to be dealt with. Currently, an optimal method has not been found.

2.7 Estimating the Magnitude of the Trend

If the trend is assumed to be linear, i.e. if the data can be modeled as $y_{ik} = \alpha_k + x_{ik} \times \beta + \varepsilon_{ik}$, where $i = 1, ..., n$ and $k = 1, ..., m$, then the magnitude of the trend can be estimated by the method of *Theil* [1950] and *Sen* [1968a]. Their estimate of the slope, β, is the median of the pairwise slope estimates ignoring the subscript k,

$$\hat{\beta} = med_{i<j}\{\frac{y_j - y_i}{x_j - x_i}\}, \tag{2.9}$$

where y is the water-quality variable and x is time.

Gilbert [1987] modified this estimate to adjust for seasonality and called the new estimator the seasonal Kendall slope estimator. This estimator takes the median of all the pairwise slope estimates where only the pairwise slope estimates from within seasons are considered. In other words, within every season, all the pairwise slopes are calculated where $i < j$. Then, all the pairwise slope estimates are considered at once, and the median of this set of estimates is taken to be the seasonal Kendall slope estimate. Thus, Gilbert's estimator is

$$\hat{\beta}_{SK} = med_{i<j,\ k=1,\ldots,m}\{\frac{y_{jk} - y_{ik}}{x_{jk} - x_{ik}}\}, \tag{2.10}$$

where none of the four values in the slope estimate are missing and k stands for the season.

Confidence intervals for the Theil-Sen slope estimate are based on inverting a Kendall tau hypothesis test [*Sen* 1968a, and *Hollander and Wolfe* 1973]; this is also true for the seasonal Kendall slope estimate [*Gilbert* 1987]. All β's that would not yield a rejection of the null hypothesis with the seasonal Kendall approach at the desired alpha level using adjusted water quality values are considered to be in the 100(1-α)% confidence interval. Adjusted water quality values, y_i^*, $i = 1, ..., n$, are $y_i^* = y_i - x_i \times \beta$ where x is time. Gilbert's method is based on the asymptotic normality property of Kendall's test of $\mathrm{H}_0 : \tau = 0$. The pairwise slope estimates are ordered. An approximate 100(1-α)% confidence interval for β is given by the

m_1^{th} largest to the $(m_2+1)^{\text{th}}$ largest pairwise slope estimates where m_1 and m_2 are defined as follows.

$$C_\alpha = Z_{1-\frac{\alpha}{2}}\sqrt{Var(S)}, \tag{2.11}$$

$$m_1 = (N - C_\alpha)/2, \text{ and} \tag{2.12}$$

$$m_2 = (N + C_\alpha)/2 \tag{2.13}$$

where N is the total number of pairwise slope estimates, Z_i is the i^{th} percentile of the standard normal distribution, and S is as in equation 2.3. The formula for the variance of S is given in equation 2.4.

This method for finding a confidence interval can be extended to allow serial dependence in the data by inverting the covariance sum test of *Hirsch and Slack* [1984]. This extension requires that observations a year apart or more be independent, but allows for dependence among all pairs of observations that are less than a year apart. The derivation is given in Appendix D and is based on the work of *Sen* [1968] and *Zetterqvist* [1988]. The new method works the same as Gilbert's method with the exception that the Var(S) is now given by Equation 2.6.

Another method for getting a confidence interval on the slope for the non-seasonal and independent data case, was given by *Taylor and Conover* [1988]. Instead of using the Kendall tau hypothesis test, they used the test of Spearman's rho. While confidence intervals based on either of the two tests yield intervals of comparable widths, the use of the Spearman's rho test allows intervals closer to the desired level of confidence since there are more points in the null distribution [*Taylor and Conover* 1988]. Naturally, the flexibility in interval sizes occurs at the cost of being more computationally expensive. Another way to get closer to the desired confidence level would be to use interpolation.

2.8 Semi-nonparametric Trend Analysis

If the hypothesis of monotonic trend is reduced to an hypothesis of linear trend, a semi-nonparametric method which could be applied is the rank-based linear models approach. For a review, see *Hettmansperger and McKean* [1998]. Applying this method for the detection of trends in water quality variables was studied by *van Belle and Hughes* [1984]. They based their work on *Farell* [1980] who used an aligned rank test of *Sen* [1968b]. This method requires estimation of missing values, independence, and no censoring, and it does not adjust for flow. However, for ideal conditions this method slightly outperforms the seasonal Kendall method of *Hirsch et al.* [1982] in terms of asymptotic efficiency because aligned rank tests generally have better asymptotic power properties than intrablock procedures [*van Belle and Hughes* 1984]. *Berryman et. al.* [1988] and *Loftis et al.* [1991] have also suggested applying rank-based linear models approaches to water-quality data.

A lot of work has been done using semi-nonparametric methods with censored data in survival studies. For example, see *Harrington and Fleming* [1982] and *Wei et. al.* [1990]. This work could possibly be applied in the water quality context. Solutions still need to be extended to allow dependent data although some yet to be published work of McKean's may shed some light on this subject.

2.9 Comparing Water Quality at Different Sites

Millard and Deveral [1988] used the rank-based linear models approach to compare water-quality variable levels at different sites. They studied several possibilities under lognormal errors and concluded that the best behaved method was the use of normal scores with the score statistic of *Prentice* [1978] and a permutation estimate of the variance of the test statistic. This variance estimate assumes that the censoring mechanism is the same for both samples. When the censoring mechanism differs, they recommended using an asymptotic variance estimator. The main shortfall of their method is that observations have to be independent. A second possible shortcoming is that they treated the problem as a two sample location problem. If the data were treated as paired by time, the variability could be substantially reduced. *Albers* [1988], *Dabrowska* [1987], and *O'Brien and Fleming* [1987] all proposed rank-based methods for performing tests in the paired data situation with random censoring.

Another question of interest that the Millard and Deveral method does not address is whether the two sites are *currently* different? Their method weights all points in time equally. Perhaps a method that compares two sites' current levels of different water quality variables can be developed, maybe based on exponentially weighted moving averages or rank-based linear model predicted values. The rank-based linear models approach would suit well most features of the data except for the serial dependent nature, while the use of an exponentially weighted moving average handles serial dependence fine but needs some corrections for asymmetry and detection limits.

2.10 Flow Adjustment

An important feature of water quality data is the effect of flow. Flow is the volume of the water passing a point in a river or a stream in a given unit of time and thus is often measured in cubic feet per second. Sometimes flow is simply recorded on a 1 to 4 scale, however, this situation will not be considered here. This covariate affects levels of water quality variables because of dilution and wash-off [*Hirsch et al.* 1991]. If the periods of high and low flow are not evenly dispersed over the range of the time series, a trend over time can be induced by flow. Variability in the water quality variable is always increased, regardless of any pattern in flow. Hence, flow should be accounted for if possible when performing an analysis for trend. All the ideas presented here for adjusting for flow are based on the ideas of analysis of covariance.

First of all, a parametric regression of the water quality variable on flow could be performed [*Helsel and Hirsch* 1992]. The residuals from this fit would then be used with one of the methods for detecting trend described above. This method is problematic because of the non-normal distribution of most water quality variables and the common occurrence of detection limits. An advantage of this methods occurs when the scientist has a good idea how flow and the water quality variable are related. The investigator can then impose his or her idea on the fit as a model. For example, the scientist may think that the relationship should be linear or quadratic.

When interest is in a step trend in water quality, an analysis of covariance can be performed to test for trend in water-quality adjusted for flow [*Helsel and Hirsch* 1992]. Once again, the problem is complicated by the non-normality and detection limits. Perhaps a rank-based linear models ANOCOVA could be used when step trend is of interest.

Another possibility is the use of nonparametric regression. The water quality variable would be regressed on flow, with the residuals being used for trend detection as above in the parametric regression method. *Helsel and Hirsch* [1992] suggested the use of locally weighted scatterplot smoothing [*Cleveland* 1979], hereafter abbreviated LOWESS. This choice seems reasonable because of the robustness of LOWESS to outliers.

A final idea is to analyze the data using a multivariate rank-based linear model approach. The response variables could be just one water quality variable and flow, or it could include all the water quality variables. This would take into account the correlation of the water quality variable with flow and with the other variables if they were included. A disadvantage is the assumption that the trend is linear, but this is often not all that intolerable. Complicating factors that need to be dealt with are the existence of detection limits and serial correlation. Serial correlation is a real problem that has not been dealt with in the rank-based linear model context. See *Hettmansperger and McKean* [1998] for a complete treatment of the multivariate rank-based linear models approach.

CHAPTER 3

Modified Kendall's Tau

3.1 Introduction

Modified Kendall's tau is a nonparametric correlation coefficient. It is often used to describe trends in water quality variables and is the basis for the trend tests described in the previous chapter. This is because no underlying distributional assumptions are needed and modified Kendall's tau measures monotonic trend not just linear trend. The purpose of this chapter is to build the foundation necessary for some of the material in the next chapter where changes in trend will be discussed by defining modified Kendall's tau and exploring the variance of its estimator and the two sample hypothesis test. Several possible estimators of the variance were evaluated via Monte Carlo studies. The process of choosing the best variance estimator was three stage with new variance estimators being added at each stage because of the lack of optimal performance of all the estimators considered in the previous stage. The final result of the studies was a recommendation to use three different variance formulas depending on the presence of dependence and the value of the estimator of modified Kendall's tau. Each of the selected variance formulas were compared in a power study of the test of H_0:$\tau_{\text{mod1}} = \tau_{\text{mod2}}$, and further recommendations were made based on that study.

Kendall's tau, τ, and modified Kendall's tau, τ_{mod}, are defined as follows.

$$\tau = \pi_c - \pi_d, \text{ and} \tag{3.1}$$

$$\tau_{\text{mod}} = \frac{\pi_c - \pi_d}{\pi_c + \pi_d} = \frac{\pi_c - \pi_d}{1 - \pi_t} \tag{3.2}$$

where π_c =probability that two pairs are concordant, π_d =probability that two pairs are discordant, and π_t =probability that two pairs are tied. The modification to Kendall's tau allows modified Kendall's tau to attain -1 or 1 when there are ties. A consistent estimator of τ_{mod} is

$$\hat{\tau}_{\text{mod}} = \frac{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} sign[(y_j - y_i)(x_j - x_i)]}{\frac{1}{2}n(n-1) - \frac{1}{2}\sum_{i=1}^{g} t_i(t_i - 1)} \tag{3.3}$$

where t_i is the number of y values in the group of y values which are tied at the i^{th} unique value, $i = 1, ..., g$. It is assumed that there are no ties in the x's, which will always be time for purposes here. The only difference between this formula and the one that would be used to calculate $\hat{\tau}$ is the subtraction of the number of tied pairs from the denominator.

3.2 The Variance of $\hat{\tau}_{\text{mod}}$

Unfortunately, the variance of $\hat{\tau}_{\text{mod}}$ is not known. It has to be estimated. The variance of $\hat{\tau}_{\text{mod}}$ is known only under the assumptions that observations are independent and that $\tau_{\text{mod}} = 0$. Even if independence is assumed, τ_{mod} is still not assumed to be equal to 0. The null hypothesis is simply H_0: $\hat{\tau}_{\text{mod}\,1} = \hat{\tau}_{\text{mod}\,2}$, i.e., no change in trend.

Kendall [1970] gave a bound for the variance of $\hat{\tau}$ along with an estimate of the variance. Notice these formulas were derived for $\hat{\tau}$, not $\hat{\tau}_{\text{mod}}$. However, since the only difference between $\hat{\tau}$ and $\hat{\tau}_{\text{mod}}$ is in the denominator, a rough estimate for the variance of $\hat{\tau}_{\text{mod}}$ can be obtained by a simple transformation. This estimate is called rough because the denominator of $\hat{\tau}_{\text{mod}}$ is in fact random. The bound is given in Equation 3.4 while the formula for the estimate is given below in Equations 3.5-3.8. Both will be included in the Monte Carlo study with the formulas to be given later.

$$Var(\hat{\tau}_{\text{mod}}) \leq \frac{2}{n} \times (1 - \hat{\tau}^2_{\text{mod}}) \tag{3.4}$$

$$\widehat{Var}(\hat{\tau}_{\text{mod}}) = D \times [4 \sum C_i^2 - \frac{2(2n-3)}{n(n-1)} C^2 - 2n(n-1)] \tag{3.5}$$

where

$$D = \frac{n(n-1)}{(n-2)(n-3)[n(n-1) - \sum t_i(t_i - 1)]^2} \tag{3.6}$$

$$C_i = \sum_{j=1}^{n} sign[(y_j - y_i)(x_j - x_i)] \tag{3.7}$$

$$C = 2S = \sum_{i=1}^{n} \sum_{j=1}^{n} sign[(y_j - y_i)(x_j - x_i)] \tag{3.8}$$

Noether [1967] derived an asymptotic formula for the variance of $\hat{\tau}_{\text{mod}}$ based on the work of *Goodman and Kruskal* [1963]. That formula is given below in Equation 3.21, followed by two new formulas. The derivations of the new formulas are given in Appendices 1 and 2. The first formula, given in Equation 3.22, is derived along the same lines as the derivation given by Noether, but then takes a different direction without making as many simplifications in the name of asymptopia. The second new formula, given in Equation 3.23, is derived using the delta method [*Seber* 1982] which is based on a Taylor series expansion.

A similar notation to the one appearing in Noether's book will be used. Let (X_1,Y_1), (X_2,Y_2), and (X_3,Y_3) be three independent pairs of variables randomly drawn from the population of all possible pairs of variables (X,Y). Then,

$$\begin{aligned} \pi_c &= \text{probability that } (X_1,Y_1) \text{ and } (X_2,Y_2) \text{ are concordant} \\ &= P((Y_2\text{-}Y_1)(X_2\text{-}X_1)>0). \end{aligned} \tag{3.9}$$

$$\begin{aligned} \pi_d &= \text{probability that } (X_1,Y_1) \text{ and } (X_2,Y_2) \text{ are discordant} \\ &= P((Y_2\text{-}Y_1)(X_2\text{-}X_1)<0). \end{aligned} \tag{3.10}$$

$$\begin{aligned} \pi_{cc} &= \text{probability that } (X_1,Y_1) \text{ is concordant with both } (X_2,Y_2) \text{ and } (X_3,Y_3) \\ &= P\{[(Y_2\text{-}Y_1)(X_2\text{-}X_1)>0] \ \& \ [(Y_3\text{-}Y_1)(X_3\text{-}X_1)>0]\}. \end{aligned} \tag{3.11}$$

$$\begin{aligned} \pi_{dd} &= \text{probability that } (X_1,Y_1) \text{ is discordant with both } (X_2,Y_2) \text{ and } (X_3,Y_3) \\ &= P\{[(Y_2\text{-}Y_1)(X_2\text{-}X_1)<0] \ \& \ [(Y_3\text{-}Y_1)(X_3\text{-}X_1)<0]\}. \end{aligned} \tag{3.12}$$

$$\begin{aligned} \pi_{cd} &= \text{probability that } (X_1,Y_1) \text{ is concordant with } (X_2,Y_2) \text{ and} \\ &\quad\ \text{discordant with } (X_3,Y_3) \\ &= P\{[(Y_2\text{-}Y_1)(X_2\text{-}X_1)>0] \ \& \ [(Y_3\text{-}Y_1)(X_3\text{-}X_1)<0]\}. \end{aligned} \tag{3.13}$$

Note that since in actuality the underlying probability of concordance (or discordance) can depend on the difference in time between two observations, the assumption that (X_1,Y_1), (X_2,Y_2), and (X_3,Y_3) are randomly drawn from the population is important. Recall that it is not assumed that $\tau_{\text{mod}} = 0$ under the null hypothesis. The assumption of random selection makes the π's constants that don't depend on the distance between two pairs in one of the variables. Now, define

$$\begin{aligned} C_k &= \text{the \# of points } (\mathrm{x}_i,\mathrm{y}_i) \text{ in the sample that are concordant with } (\mathrm{x}_k,\mathrm{y}_k), \\ k &\in \{1,2,...,n\},\ i = 1,...,k-1,k+1,...,n, \end{aligned} \tag{3.14}$$

$$\begin{aligned} D_k &= \text{the \# of points } (\mathrm{x}_i,\mathrm{y}_i) \text{ in the sample that are discordant with } (\mathrm{x}_k,\mathrm{y}_k), \\ k &\in \{1,2,...,n\},\ i = 1,...,k-1,k+1,...,n, \end{aligned} \tag{3.15}$$

$$p_c = \hat{\pi}_c = \frac{1}{n(n-1)} \sum_k C_k, \tag{3.16}$$

$$p_d = \hat{\pi}_d = \frac{1}{n(n-1)} \sum_k D_k, \tag{3.17}$$

$$p_{cc} = \hat{\pi}_{cc} = \frac{1}{n(n-1)(n-2)} \sum_k C_k(C_k - 1), \tag{3.18}$$

$$p_{dd} = \hat{\pi}_{dd} = \frac{1}{n(n-1)(n-2)} \sum_k D_k(D_k - 1), \text{ and} \tag{3.19}$$

$$p_{cd} = \hat{\pi}_{cd} = \frac{1}{n(n-1)(n-2)} \sum_k C_k D_k. \tag{3.20}$$

Note that the p's are consistent estimates of the π's [*Noether* 1967]; in fact they are unbiased estimates as well. The proof that these estimates are unbiased is in Appendix C. Now, Goodman and Kruskal's formula is

$$\widehat{Var}(\hat{\tau}_{\text{mod}}) = \frac{16}{n(p_c + p_d)^4}(p_d^2 p_{cc} - 2p_c p_d p_{cd} + p_c^2 p_{dd}). \tag{3.21}$$

The modification of this formula is

$$\begin{aligned} \widehat{Var}(\hat{\tau}_{\text{mod}}) &= \frac{8}{n(n-1)(p_c + p_d)^4} \times \\ &\quad \{(p_c + p_d)p_c p_d + 2(n-2)(p_d^2 p_{cc} - 2p_c p_d p_{cd} + p_c^2 p_{dd})\}. \end{aligned} \tag{3.22}$$

The formula derived using the delta method is

$$\widehat{Var}(\hat{\tau}_{\text{mod}}) = \frac{8}{n(n-1)(p_c+p_d)^4} \times \{(p_c+p_d)p_cp_d + 2(n-2) \times [(p_d^2(p_{cc}-p_{cd}) + p_cp_d(p_d^2-p_c^2-2p_{cd}) + p_c^2(p_{dd}+p_{cd})]\}. \quad (3.23)$$

Note that both of the latter two formulas add new terms to the first formula given in Equation 3.21.

3.3 Simulation Studies to Determine Which Variance Formula is Best

The var($\hat{\tau}_{\text{mod}}$) estimators considered in the first study were the Goodman and Kruskal formula (Equation 3.21), the modification of Kendall's formula (Equation 3.5), the permutation variance derived under the assumption that $\tau_{\text{mod}} = 0$ (Equation 2.2 modified by dividing by the square of the denominator in Equation 3.3), the modification to Goodman and Kruskal's formula (Equation 3.22), and the formula derived using the delta method (Equation 3.23). Several factors were considered when studying the behavior of the variance estimators. Pairs of (x,y) data were generated in the following manner. For each of the sample sizes 10, 20, and 40, observations of x were generated both as fixed and evenly spaced over the interval [0,10] and as a random sample of Uniformly distributed observations on the interval [0,10] using the uniform random number generator in SAS. Hence there were no ties in the x's. For assurances of true pseudo-randomness, the random numbers generated by SAS were checked for lag 1, 2, and 3 correlations using the serial correlation test from p.170-171 of *Kennedy and Gentle* [1980]. No correlations were found.

The y observations were then generated as gamma-autoregressive from three different gamma distributions, $f(y) = \frac{\lambda^r}{\Gamma(r)}(y-\alpha)^{r-1}\exp[-\lambda(y-\alpha)]$, each having a location parameter, α, which changes with the x variable in either a linear or exponential manner. Figure 3.1 displays the location parameter functions used to impose positive modified Kendall's tau coefficients. Figure 3.2 and Figure 3.3 show the different distributions chosen, and Figure 3.4 displays the joint distribution of x and y for the case of no serial correlation, $\alpha = .7x$, x˜Uniform(0,10), and r=2 case.

The method of *Lawrence* [1980] was used to generate the gamma-autoregressive series with AR(1) correlation coefficients of 0, .2, and .4 for pairs whose x values are $10/(n-1)$ x units apart. When the x data are randomly generated, in order to account for the varying x intervals between consecutive observations, the AR(1) correlation parameter is raised to the power $(x_i - x_{i-1})/[10/(n-1)]$ to determine the correlation coefficient used to generate the i^{th} y value. 2000 samples were generated for each combination of the factors.

Ties were induced in the data by truncating the data at fixed y values. After the y observations were generated as gamma AR(1), all y observations less than or equal to the truncation value were set equal to that value. Hence the probability of a pair having a tied y value can be estimated as the square of the proportion of the observations that were truncated. A program was written in SAS to search for the truncation value with each distribution, location parameter function, and desired percentage of tied pairs combination. Hence, each case has a different

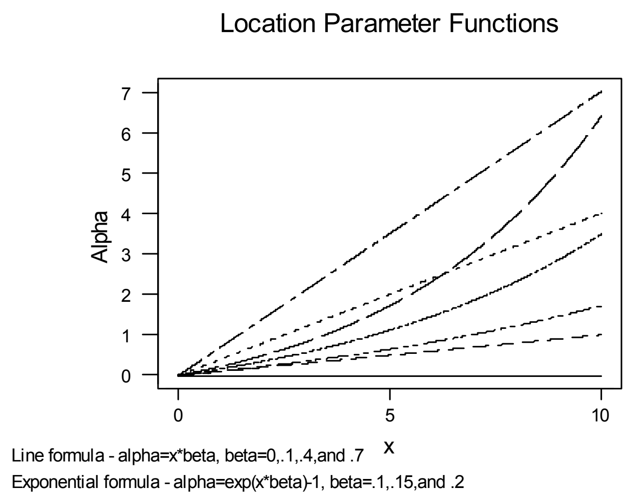

Figure 3.1: Location Parameter Functions

Distribution	λ=1,r=2	λ=1,r=3	λ=1,r=5
Mean	2+α	3+α	5+α
Variance	2	3	5
Skewness	1.414	1.155	.894

Figure 3.2: Distributions Used in Simulation Study

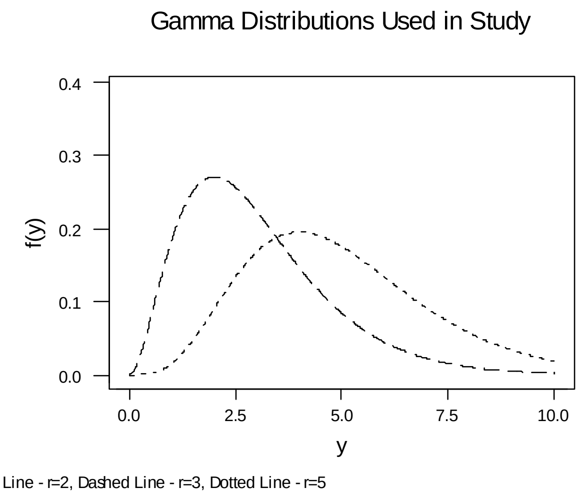

Figure 3.3: Plot of Gamma Distribution Used in Study

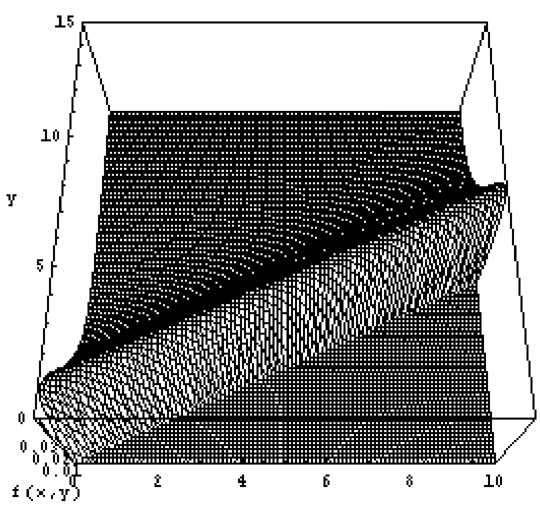

Figure 3.4: Distribution of (X,Y)(r=2, ρ=0,α=.7x case)

value where truncation occurs, but approximately 0, 5, 10, and 25 percent tied pairs. The percentage of ties induced was fairly accurate with the worst out of the 1512 cases occurring when $r = 2$, $n = 40$, $\rho = .2$, α moving linearly with $\beta = 0$, random generation of x data, and desired percentage of ties=25%, where the actual percentage of tied pairs turned out to be 18.0%. The worst case for the desired percentage of 5% was 8.1% which occurred when $r = 3$, $n = 10$, $\rho = .2$, α moving linearly with $\beta = .7$, and the x data were randomly generated. The worst case for the desired percentage of 10% was 14.5% which occurred when $r = 5$, $n = 10$, $\rho = .4$, α moving linearly with $\beta = .7$, and the x data were uniformly generated. Hence, there was no clear pattern to what caused inaccurate percentages of ties. In later studies, a better method was used to keep the percentage of ties constant. This problem is not suspected to have significantly affected the results or conclusions.

The study did yield some answers. All of the factors considered had significant effects on the mean squared errors for all the different formulas when the resulting mean squared errors were analyzed in an ANOVA setting. Nevertheless, the results could be summarized as follows. With independent data, for smaller τ_{mod}, i.e. $|\tau_{\text{mod}}| < 0.2$, the null case formula performed the best in terms of mean squared error. For the larger values of τ_{mod}, i.e. $|\tau_{\text{mod}}| > 0.4$, the delta method formula generally performed the best. For the middle values of τ_{mod}, the null case formula usually performed the best, but this is because no formula did exceptionally well. The delta method formula was negatively biased, while the null case formula was positively biased. With serial correlation, the null case formula performs better

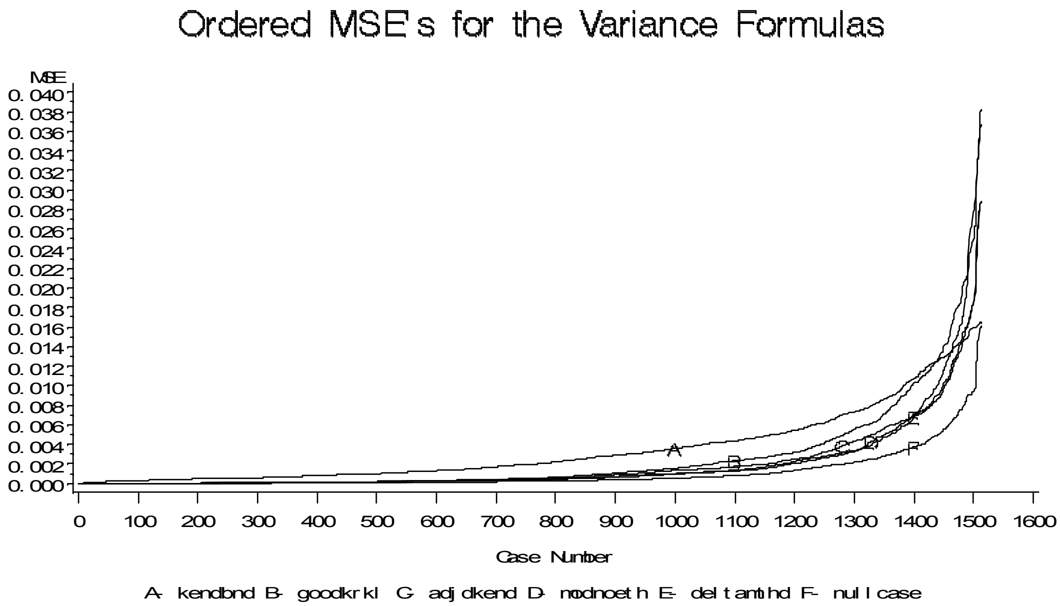

Figure 3.5: Ordered MSE's - Study 1 (All 1512 Cases)

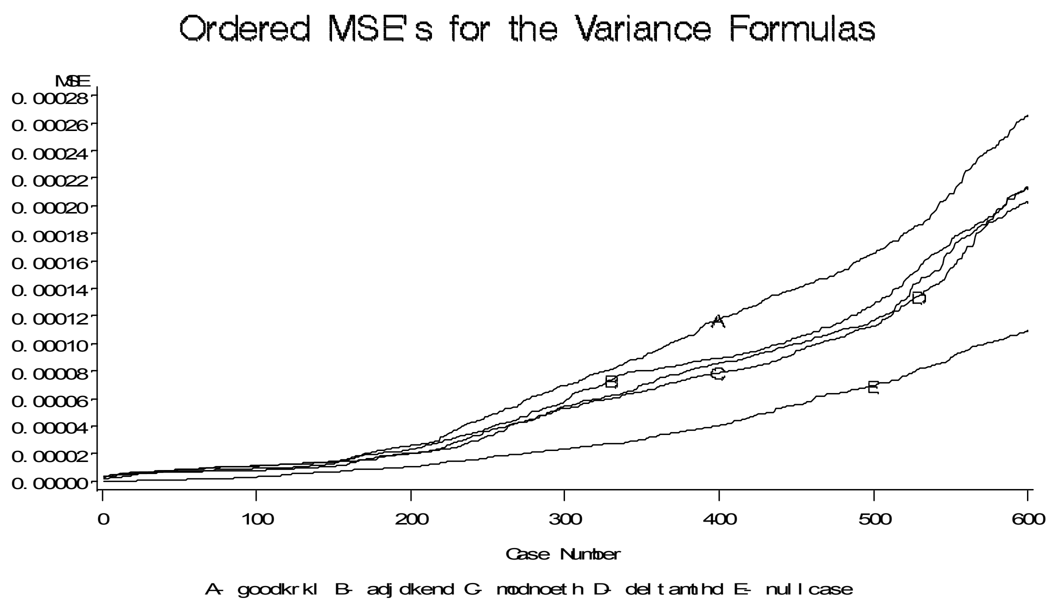

Figure 3.6: Ordered MSE's - Study 1 (First 600 Cases)

respective to the other formulas considered for a wider range of τ_{mod} because dependence increases the variance of $\hat{\tau}_{\text{mod}}$. However, none of the formulas appear to properly adjust estimates upward in the presence of serial correlation. Figures 3.5 and 3.6 display the results by plotting the ordered MSE estimates for each formula individually and then putting them on the same plot. It is clear that the null case formula performed the best for most of the cases considered. Figure 3.6 is just the first 600 cases of Figure 3.5 enlarged with the Kendall bound formula eliminated.

A better formula was desired, so another study was performed with essentially all the same factors as above but with three new formulas. To shrink the number of cases needed to be run, only one level of the distributional parameter, $r = 3$, was considered because r had only non-significant and marginally significant effects on the mean squared errors in the first study. The delta method derivation was extended by using a second order Taylor series to derive a new formula and hopefully reduce the bias in the old formula. The formula that resulted from this derivation was extremely long and involved three layers of estimation, so it was shortened by considering only the simplest terms which did not require any extraordinary means to estimate. A summary of this derivation is in Appendix F, and the formula is

$$\begin{aligned}\widehat{Var}(\hat{\tau}_{\text{mod}}) &= \frac{8}{n(n-1)(p_c+p_d)^4} \times \{4(p_c+p_d)p_cp_d + 3(n-2)(n+1)p_c^2p_d^2 \\ &\quad +(n-2)\times[(2p_d^2(4p_{cc} - 3p_c^2 - p_{cd}) + p_cp_d(2p_d^2 - 2p_c^2 - 16p_{cd} \\ &\quad -3(n-3)p_cp_d) + 2p_c^2(4p_{dd} - 3p_d^2 + p_{cd})]\}\end{aligned} \tag{3.24}$$

Secondly, a bootstrap estimate of the variance was considered based on 50 bootstrap resamples of the same size as the original generated data sets. Recent work based on using a conditional coefficient of variation suggests that approximately 800 resamples are needed to sufficiently reduce resampling variability [*Booth and Sarkar* 1998]. This contradicts previous beliefs that as few as 25 bootstrap resamples are necessary to sufficiently reduce the resampling variability [*Efron* 1987]. Two cases were run with resample sizes of 25, 50, 100, and 800 to ensure that 50 was a reasonable number. From these two cases, it appeared that 50 is a large enough number of resamples with very little improvement obtained when more resamples are used. In fact, using only 25 resamples did not perform too badly relative to other sizes. If the bootstrap method outperforms the others, a practitioner can always use more resamples, but in the interest of time, only 50 resamples were used

Finally, a conditional formula was derived and is displayed below in Equation 3.25 (see Appendix E for the derivation). Conditional refers to the formula being conditional on the number of ties; hence, the denominator of $\hat{\tau}_{\text{mod}}$ is no longer random. The null case formula and the delta method formula were also left in the study. The other formulas considered in the original study were eliminated from consideration. The conditional formula is

$$\begin{aligned}\widehat{Var}(\hat{\tau}_{\text{mod}}) \quad &= \quad \frac{N}{(N-N_t)^2} \times [p_c(1-p_c) + p_d(1-p_d) + \\ &\quad 2(n-2)(p_{cc} - p_c^2 + p_{dd} - p_d^2 - 2p_{cd}) + 2(2n-3)p_c p_d] \quad (3.25)\end{aligned}$$

where $N = n(n-1)/2$ is the number of pairwise comparisons made, and N_t is the number of tied pairwise comparisons.

The Goodman-Kruskal formula, the adjusted Kendall formula, and the formula modifying Noether's derivation were all discarded because they generated a much greater percentage of negative estimates than the delta method formula and generally had higher mean squared errors. As evident in Figure 3.5 and Figure 3.6, the modified Noether formula performed about as well as the delta formula in terms of mean squared error, but it had a much larger percentage of negative estimates and there is no simple way to improve it. Kendall's bound was also eliminated from consideration as it generally performed poorly in terms of MSE.

Ties were generated in an improved manner for this second study. Instead of having a constant cutoff value for each beta, r, and desired percentage of ties, the number of tied values in the data set were kept constant across different samples by simply giving the appropriate number of observations the value of the largest observation desired to be tied. For example, with samples of size 10 the smallest 3 generated observations were all given the value of the third smallest in each Monte Carlo sample. This yields a constant $\left(\frac{3\times 2}{10\times 9}\right)$ probability of a tied pairwise comparison. The number of observations to be tied were selected to yield probabilities close to each other for the 3 different sample sizes and to yield probabilities of ties close to 0, .05, .10, and .25. The selected numbers of tied observations and their resulting probabilities are displayed below in Figure 3.7.

The study does not cover the entire range of possible tau values. Because of symmetry, there was no need to study both positive and negative tau values.

Sample Size	Number Tied/Resulting Probability of Tie		
10	5/.2222	4/.1333	3/.0667
20	10/.2368	7/.1105	5/.0526
40	20/.2436	14/.1167	10/.0577

Figure 3.7: Number of Tied Observations Used

Hence, only positive or near zero negative tau values were generated. Additionally, slope and exponential parameter values that would generate tau values near 1 were not used. That is because these values are rare in practice and var($\hat{\tau}_{\text{mod}}$) $\rightarrow 0$ as $\tau_{\text{mod}} \rightarrow 1$. With $r = 3$, the mean generated tau values ranged from -0.050 to 0.716, and median generated tau values ranged from -0.029 to 0.771 for the different cases.

The resulting MSE's from this study are displayed in the following figures. The MSE for method j is calculated as $\sum_{i=1}^{2000}(Var^j(\hat{\tau}_{\text{mod}})_i - Var^{MonteCarlo}(\hat{\tau}_{\text{mod}}))^2/1999$.

The formula based on the second order Taylor series expansion was not competitive as it always overestimated the variance. Hence, its performance will not be displayed in any of the figures except for Figure 3.8 which contains all of the cases' mean squared errors lumped together. Figures 3.9-3.27 all display various combinations of cases based on one or two factors which were considered in the study. Figure 3.28 details the bias of each method for the different levels of serial correlation. Figure 3.29 displays the scaled biases.

Some of the factors considered did not affect the relative performance of the variance estimators,even though they did affect the underlying variability being estimated. These factors are the percentage of ties (Figures 3.11 and 3.12) and the sampling scheme (Figures 3.13 and 3.14). The nature of the mean relationship between X and Y, linear or exponential, did not make much difference in the relative performance of the formulas or the underlying variability (See Figures 3.15 and 3.16). The random scheme had higher MSEs than the uniform scheme, and, as expected, ties inflated the variance. Sample size and correlation both had effects on the underlying variance and on the relative performance of the various formulas.

No one formula outperforms the rest in all cases. An argument will be made for the following recommendation. When no serial correlation is present, for small values of $\hat{\tau}_{\text{mod}}$, small being roughly $|\hat{\tau}_{\text{mod}}| < 0.3$, use the null case formula, and for large values of $\hat{\tau}_{\text{mod}}$, use the delta method. When serial correlation is present, use the bootstrap.

The null case formula performed surprisingly well for several reasons. The variance of $\hat{\tau}_{\text{mod}}$ is largest when τ_{mod} is near 0 and decreases to 0 as $\tau_{\text{mod}} \to 1$. This partially explains the surprising result that the null case formula performed well in terms of mean squared error. Since the probability of ties was held fixed, the null case formula only depended on sample size that was held fixed at three different levels in the study. Thus, MSE for the null case formula consisted only of bias squared. As τ_{mod} moves away from 0, the variability of its estimate decreases, and with no correlation, the null case formula becomes increasingly biased. However, serial correlation increases the variance of $\hat{\tau}_{\text{mod}}$ causing the null case formula to be less biased or even biased in the other direction. Looking at the MSE plots (Figures 3.22 and 3.23), the null case formula performs well when τ_{mod} is near zero, outperforming every formula by a large margin when no serial correlation is present, and has roughly the same performance as the bootstrap when serial correlation is present. The null case formula also performs well for some values of τ_{mod} away from zero when serial correlation is present but not when serial correlation is absent. The value of τ_{mod} at which the null case formula performs well moves with the degree of serial correlation however, and it is not a good idea to rely on a formula that does not vary with the serial correlation. For an example of the null case performing poorly when there is no serial correlation, see Figure 3.24 and Figure 3.25.

Looking at the biases displayed in Figures 3.28 and 3.29, in terms of worst case scenario, the bootstrap appears to be the most robust estimator in the presence of serial correlation. Observing the aggregate results, the null case appears to be competitive, but this is believed to be caused by the reasons stated above. As one would expect, serial correlation does adversely affect the performance of all the variance estimators.

Another well known feature of the bootstrap estimator can be seen in Figures 3.17-3.19. Here there is some evidence that the bootstrap performed better at samples sizes 40 and 20 than at sample size 10. Perhaps the recommendation should be amended to require a minimum sample size for use with the bootstrap.

In addition to their generally poor performance in terms of MSE, the conditional formula and the delta method formula occasionally yielded negative estimates of the variance with the sample size 10 cases. The percentage of negative estimates for the conditional formula was especially bad for larger values of the slope and exponential parameters, i.e., larger values of tau. The bootstrap and null case formulas can never yield a negative estimate.

The fact that the bootstrap appears to perform the best of the formulas considered suggested trying to improve its performance by taking serial dependence into account. This problem has been considered in recent years. The following section will discuss these ideas along with some new ones.

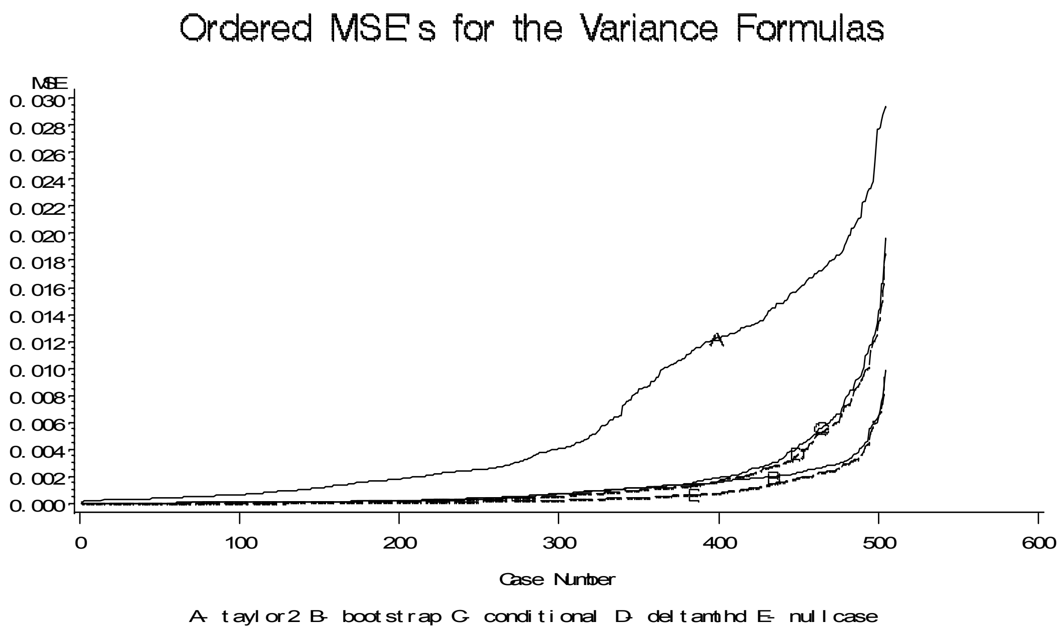

Figure 3.8: Ordered MSE's - Study 2 (All Cases and Formulas)

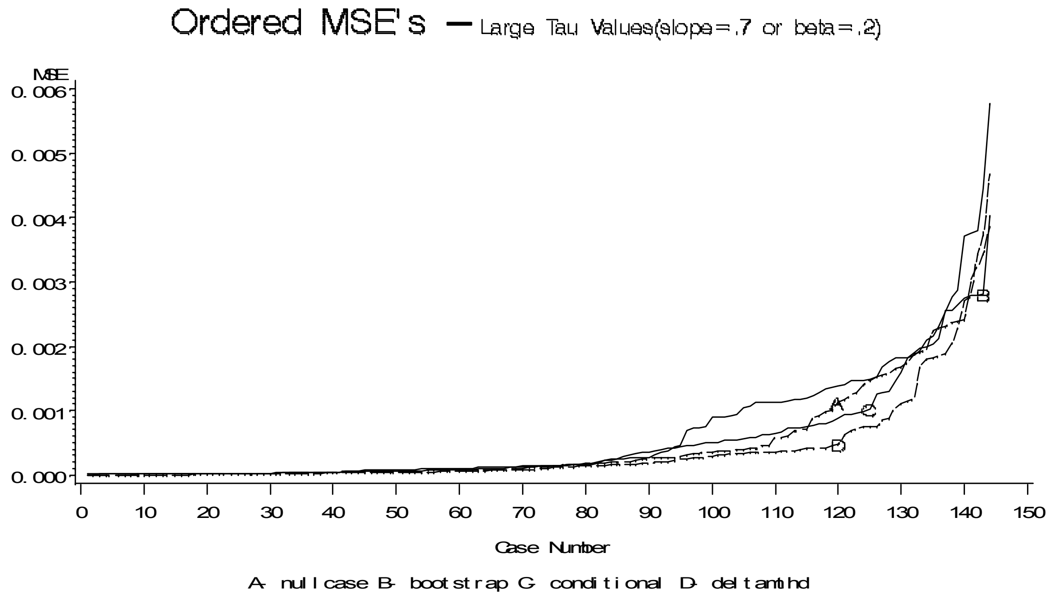

Figure 3.9: MSE's for Large Tau Values

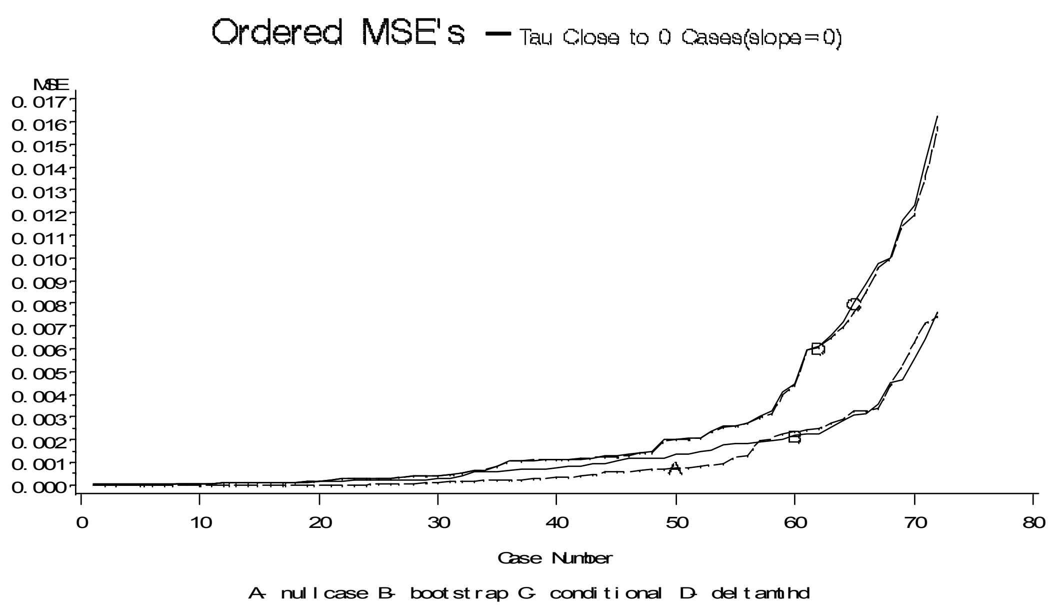

Figure 3.10: MSE's for Zero Tau Values

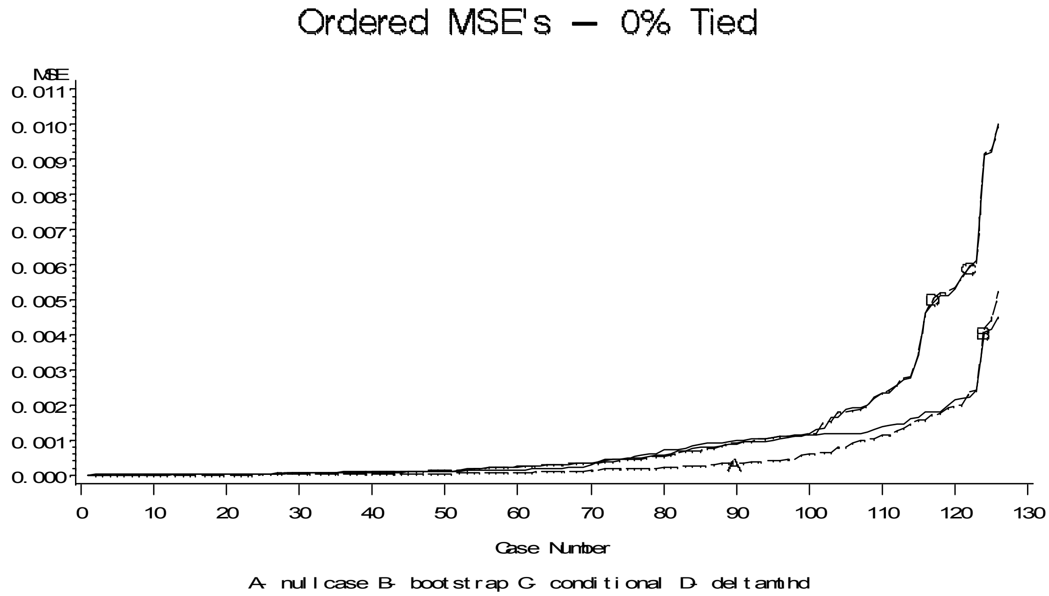

Figure 3.11: MSE's for No Ties

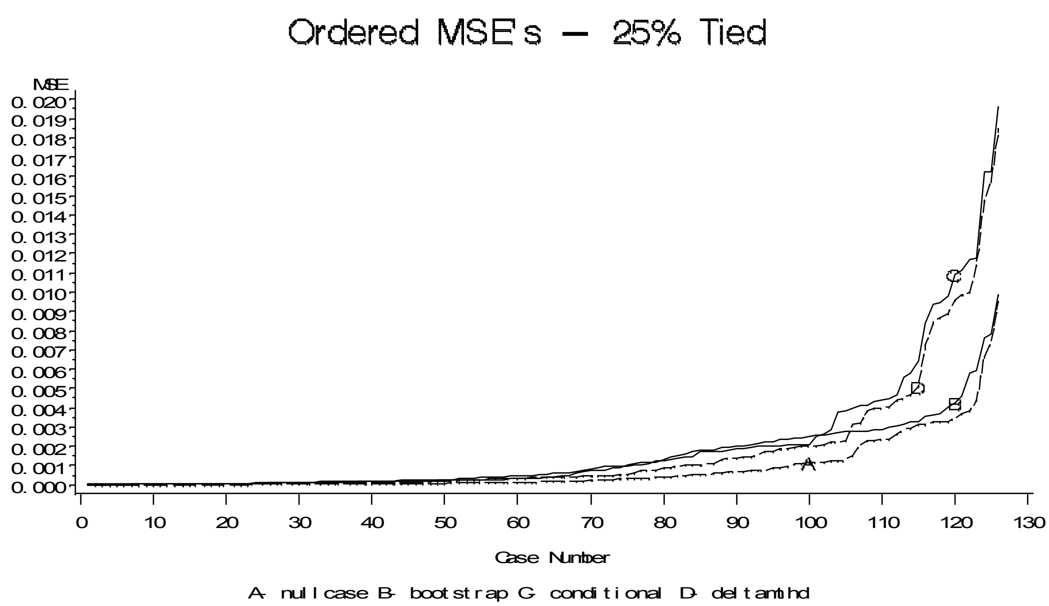

Figure 3.12: MSE's for Approximately 25% Tied Comparisons

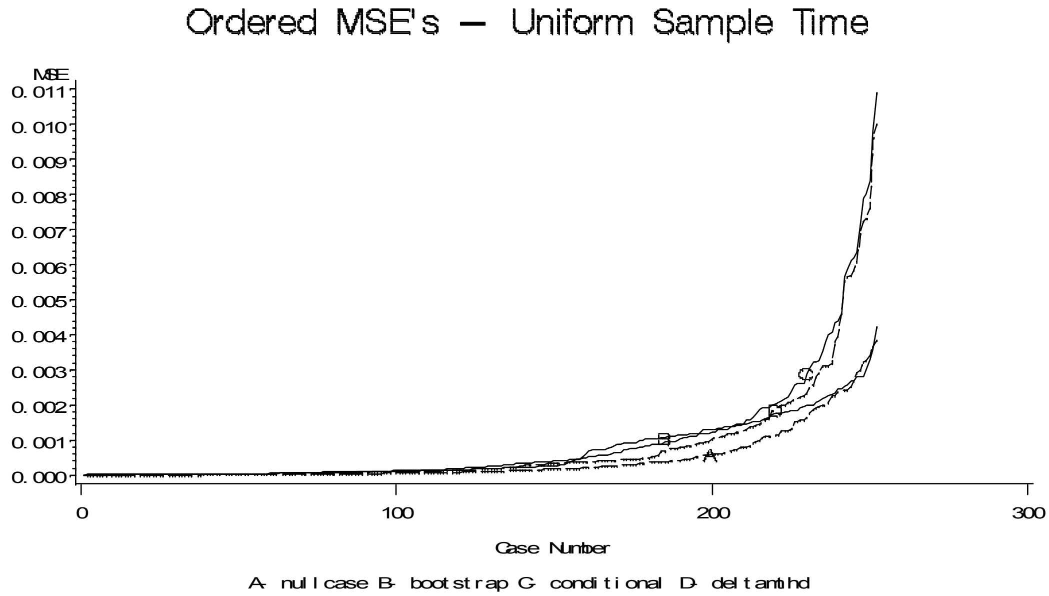

Figure 3.13: MSE's for Uniform Sampling Scheme

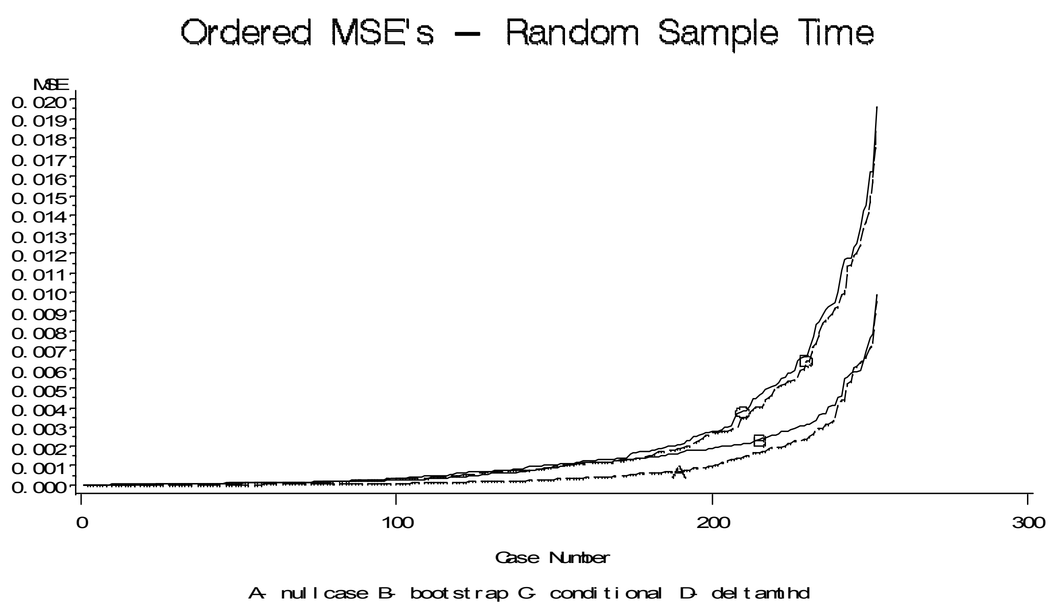

Figure 3.14: MSE's for Random Sampling Scheme

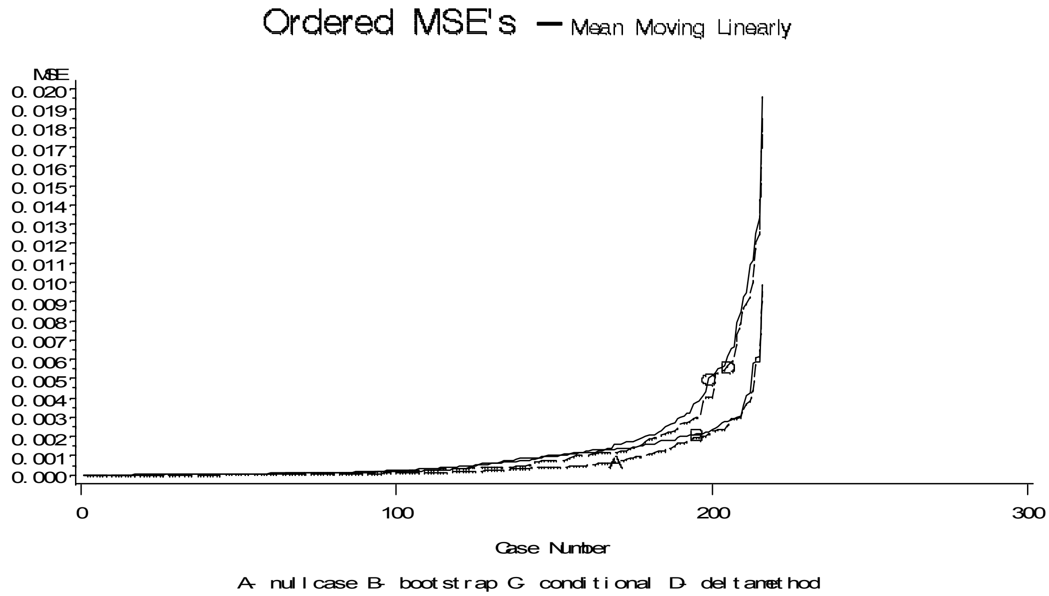

Figure 3.15: MSE's Alpha Linear in X

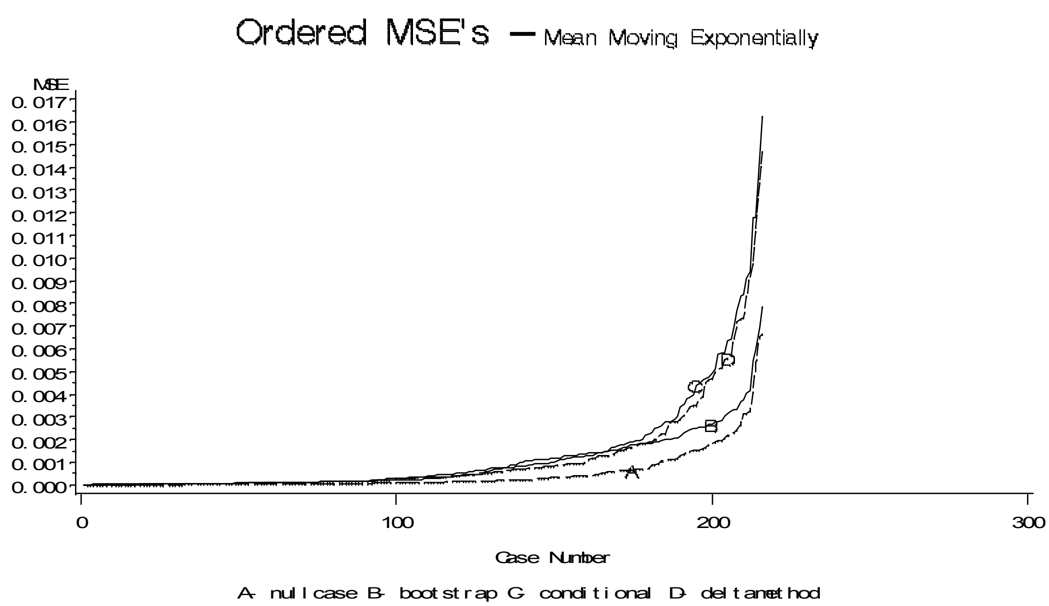

Figure 3.16: MSE's Alpha Exponential in X

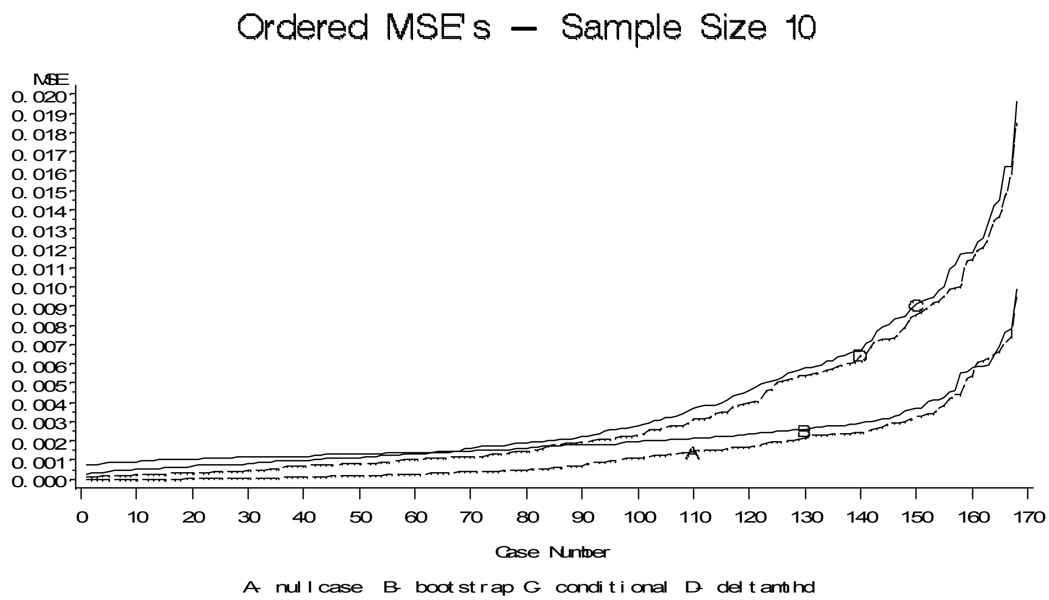

Figure 3.17: MSE's Sample Size 10

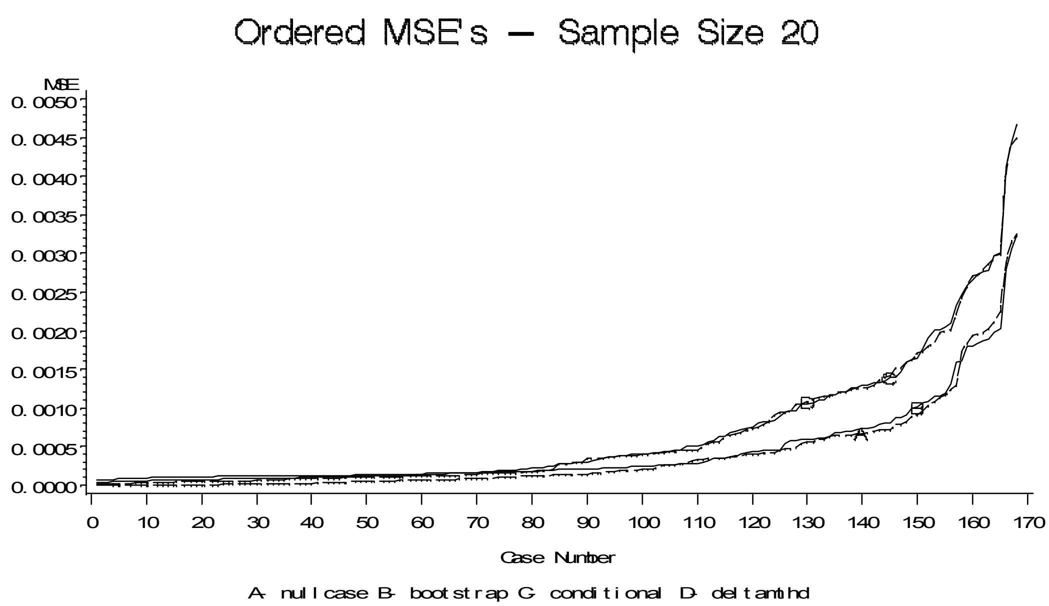

Figure 3.18: MSE's Sample Size 20

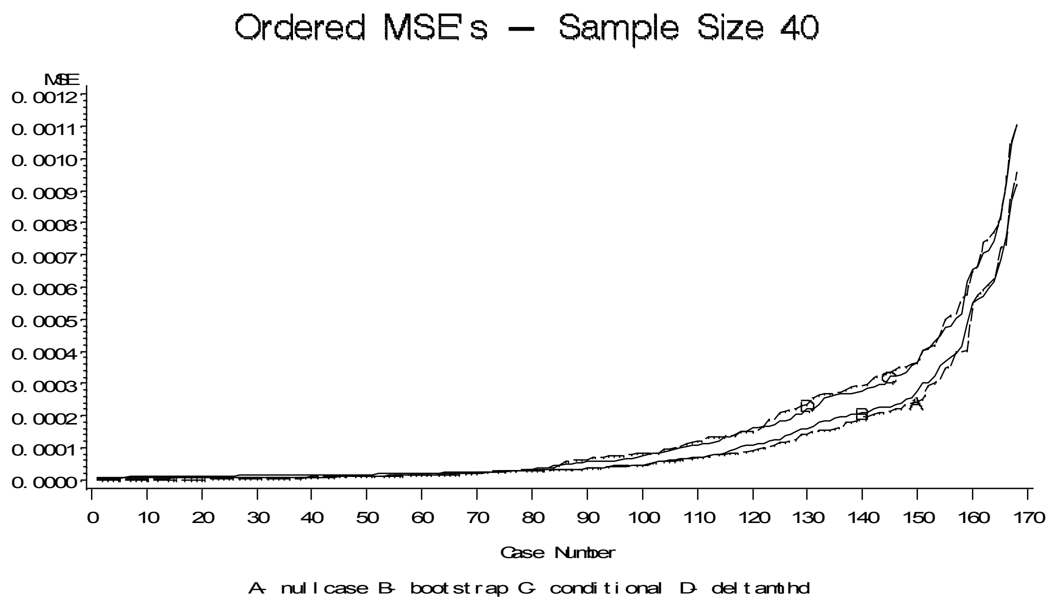

Figure 3.19: MSE's Sample Size 40

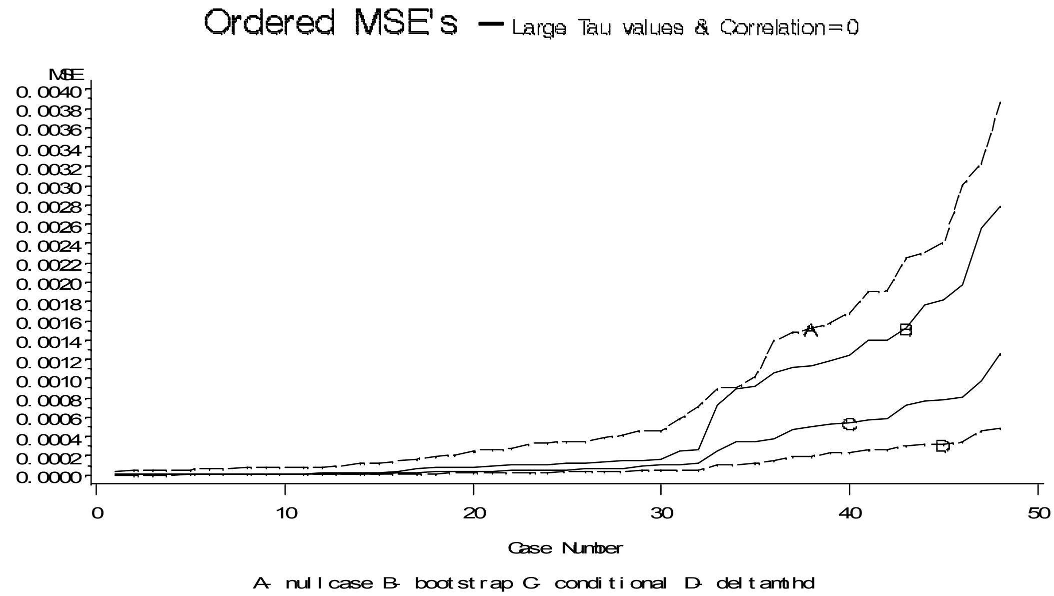

Figure 3.20: MSE's for Large Tau Values and No Correlation

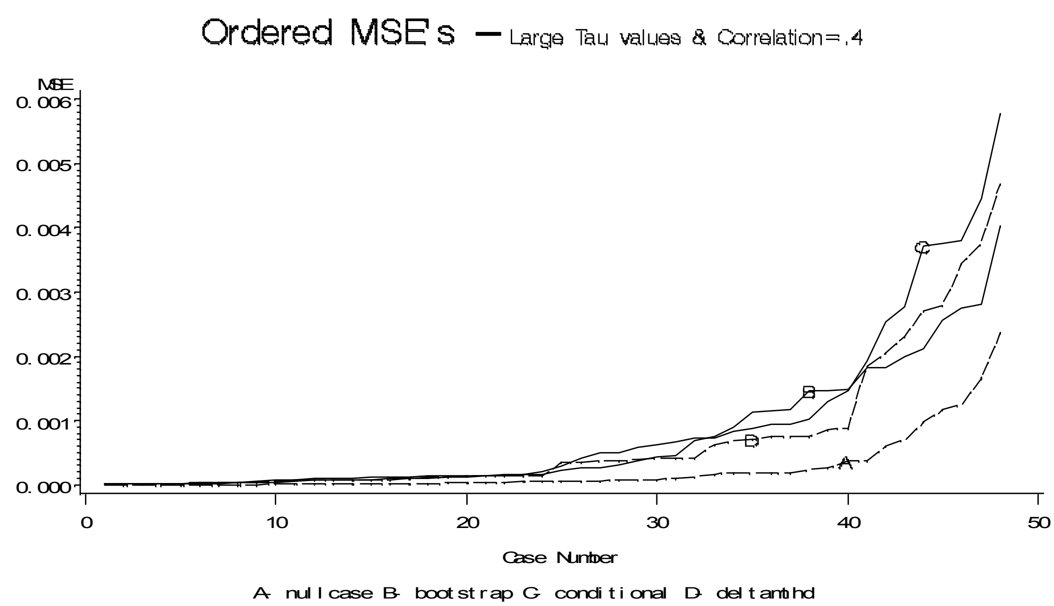

Figure 3.21: MSE's for Large Tau Values and High Correlation

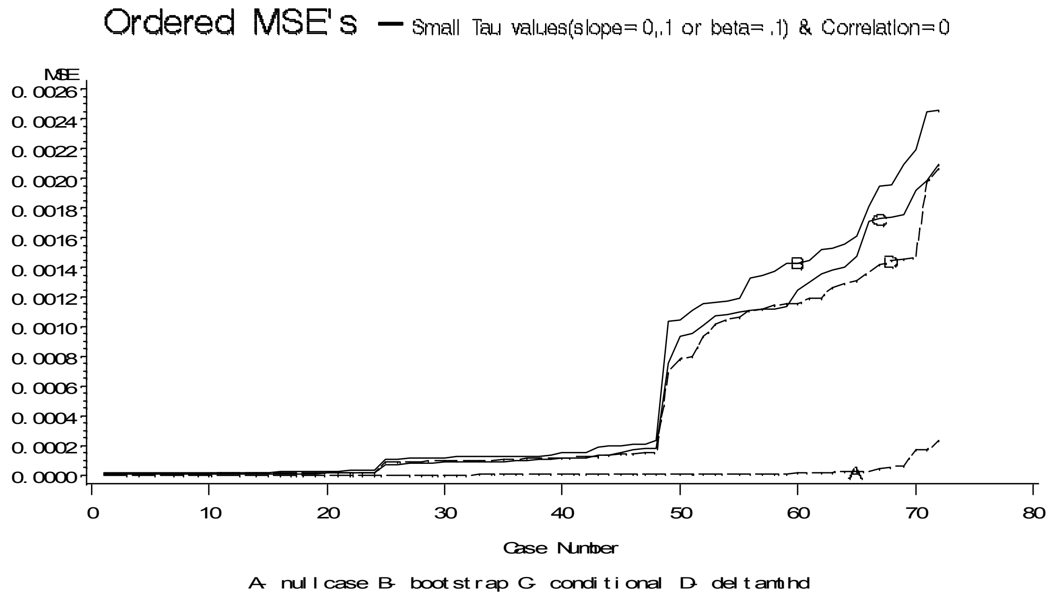

Figure 3.22: MSE's for Small Tau Values and No Correlation

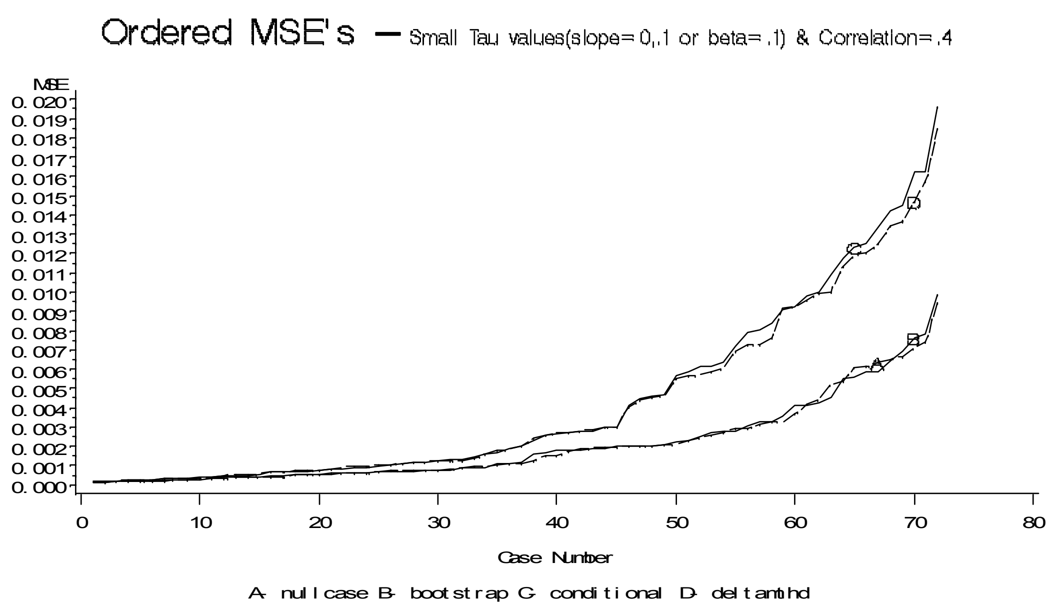

Figure 3.23: MSE's for Small Tau Values and High Correlation

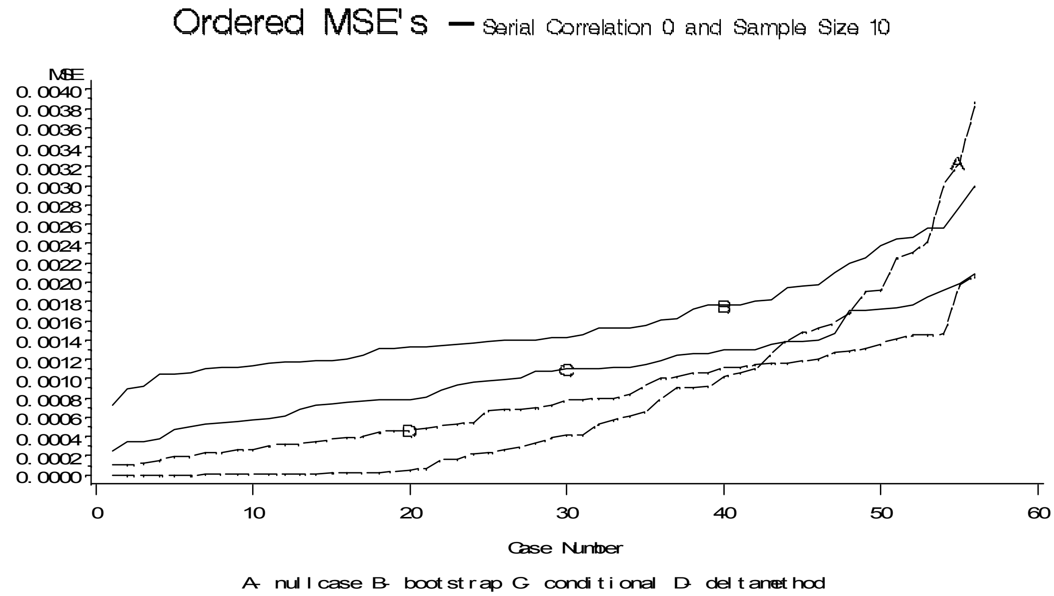

Figure 3.24: MSE's for Small Samples and No Correlation

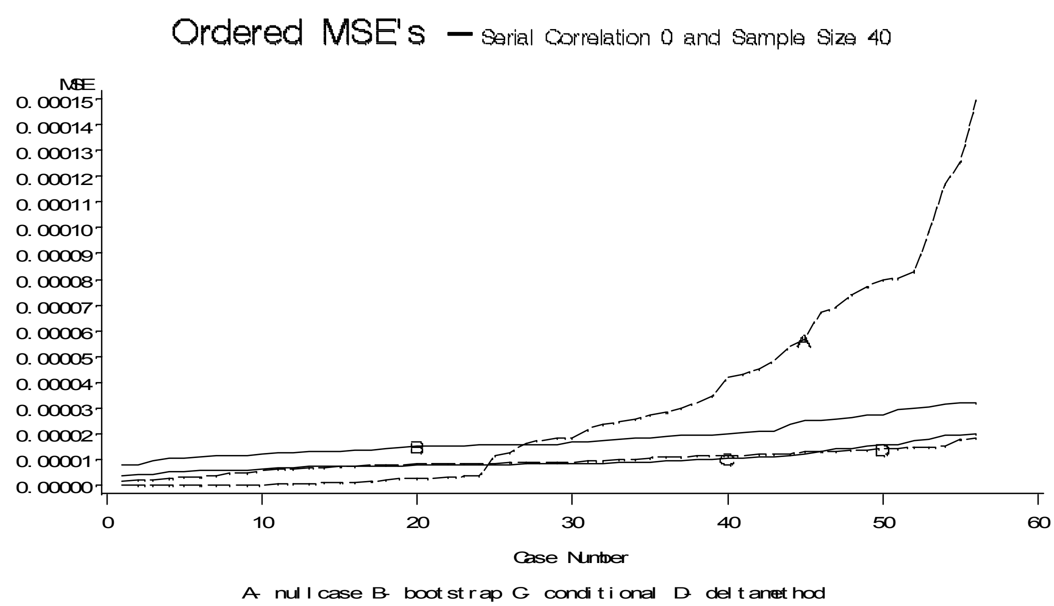

Figure 3.25: MSE's for Large Samples and No Correlation

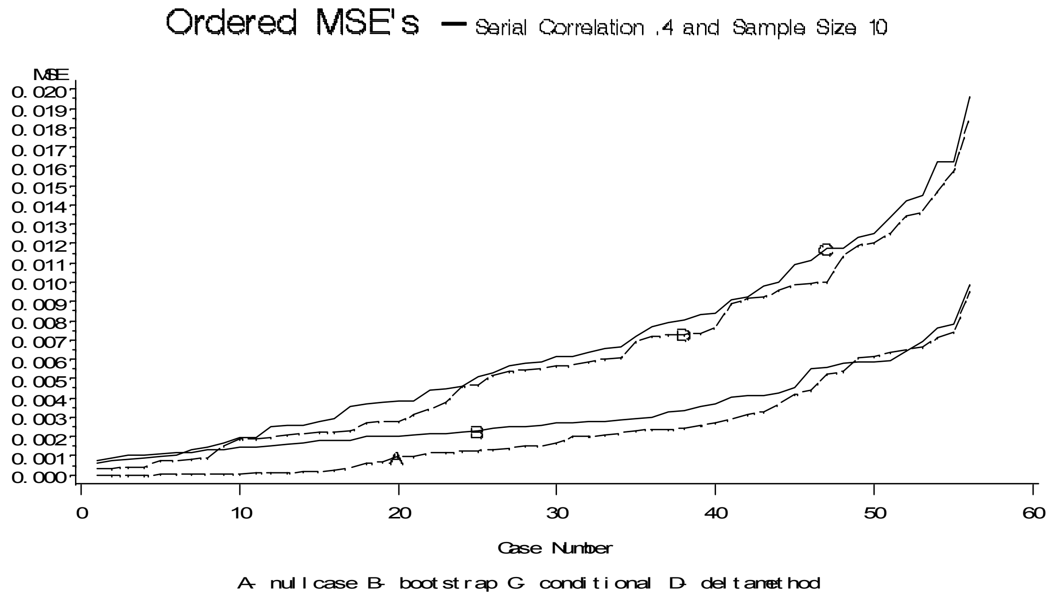

Figure 3.26: MSE's for Small Samples and High Correlation

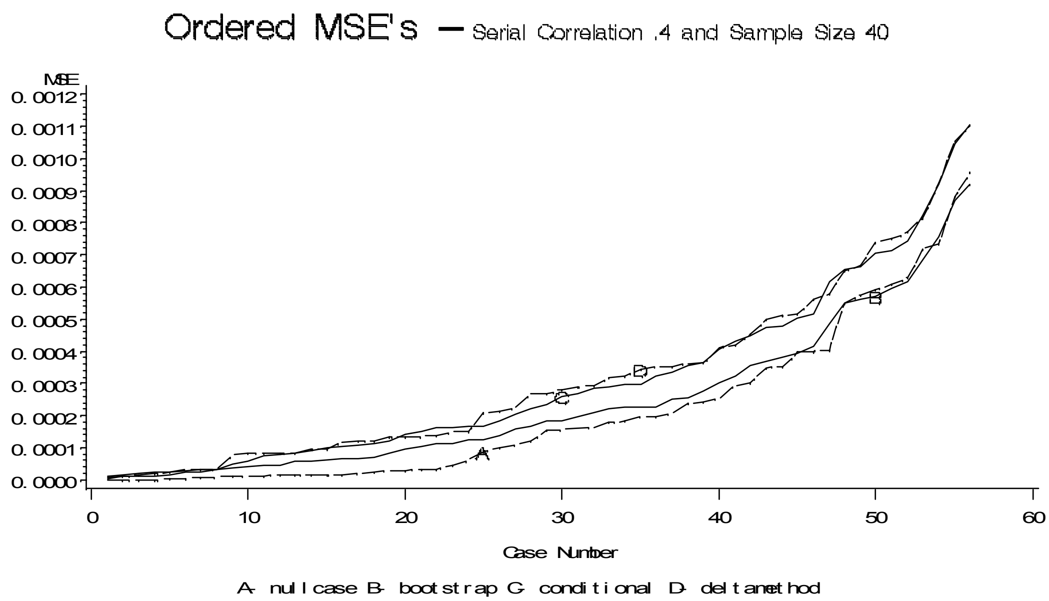

Figure 3.27: MSE's for Large Samples and High Correlation

Correlation=0					
TYPE BIAS AND METHOD	MEAN	MAX	MEDIAN	MIN	RANGE
Bias of Null Case	0.011276	0.06210	0.006627	-0.002636	0.06473
Bias of Bootstrap	0.009335	0.03060	0.005001	0.000527	0.03007
Bias of Conditional	-0.008415	0.00267	-0.004536	-0.036250	0.03892
Bias of Delta Method	-0.008027	0.00575	-0.005186	-0.036188	0.04193
Bias of Taylor 2	0.055082	0.13315	0.048382	0.009480	0.12367
Absolute Bias of Null Case	0.011404	0.06210	0.006627	0.000024	0.06207
Absolute Bias of Bootstrap	0.009335	0.03060	0.005001	0.000527	0.03007
Absolute Bias of Conditional	0.008493	0.03625	0.004536	0.000006	0.03624
Absolute Bias of Delta Method	0.008355	0.03619	0.005300	0.000051	0.03614
Absolute Bias of Taylor 2	0.055082	0.13315	0.048382	0.009480	0.12367

Correlation=.2					
TYPE BIAS AND METHOD	MEAN	MAX	MEDIAN	MIN	RANGE
Bias of Null Case	-0.004933	0.05858	-0.005487	-0.06614	0.12471
Bias of Bootstrap	-0.007106	0.02207	-0.004739	-0.05130	0.07337
Bias of Conditional	-0.024272	0.00021	-0.016036	-0.10270	0.10291
Bias of Delta Method	-0.023844	0.00345	-0.016502	-0.10194	0.10538
Bias of Taylor 2	0.040268	0.11227	0.036965	0.00792	0.10435
Absolute Bias of Null Case	0.015688	0.06614	0.011552	0.00017	0.06597
Absolute Bias of Bootstrap	0.010426	0.05130	0.006869	0.00021	0.05109
Absolute Bias of Conditional	0.024274	0.10270	0.016036	0.00021	0.10248
Absolute Bias of Delta Method	0.023930	0.10194	0.016502	0.00040	0.10154
Absolute Bias of Taylor 2	0.040268	0.11227	0.036965	0.00792	0.10435

Correlation=.4					
TYPE BIAS AND METHOD	MEAN	MAX	MEDIAN	MIN	RANGE
Bias of Null Case	-0.019469	0.04866	-0.014895	-0.09749	0.14615
Bias of Bootstrap	-0.021465	0.01198	-0.018022	-0.08679	0.09877
Bias of Conditional	-0.038418	-0.00280	-0.027305	-0.13551	0.13271
Bias of Delta Method	-0.037834	-0.00085	-0.027918	-0.13232	0.13147
Bias of Taylor 2	0.027467	0.07963	0.024259	0.00579	0.07385
Absolute Bias of Null Case	0.024628	0.09749	0.019091	0.00011	0.09739
Absolute Bias of Bootstrap	0.022337	0.08679	0.018022	0.00001	0.08678
Absolute Bias of Conditional	0.038418	0.13551	0.027305	0.00280	0.13271
Absolute Bias of Delta Method	0.037834	0.13232	0.027918	0.00085	0.13147
Absolute Bias of Taylor 2	0.027467	0.07963	0.024259	0.00579	0.07385

Figure 3.28: Biases for the 5 Variance Formulas

Correlation=0					
TYPE BIAS AND METHOD	MEAN	MAX	MEDIAN	MIN	RANGE
Scaled bias- null case	0.60941	2.53732	0.38415	-0.05441	2.59173
Scaled bias- bootstrap	0.31192	1.01071	0.26691	0.02810	0.98261
Scaled bias- conditional	-0.21550	0.28799	-0.20660	-0.49151	0.77949
Scaled bias– delta method	-0.23037	0.54261	-0.26053	-0.50263	1.04524
Scaled bias– taylor 2	2.22448	3.86238	2.23230	1.31535	2.54704
Scaled absolute bias– null case	0.61258	2.53732	0.38415	0.00082	2.53650
Scaled absolute bias– bootstrap	0.31192	1.01071	0.26691	0.02810	0.98261
Scaled absolute bias– conditional	0.22487	0.49151	0.20908	0.00091	0.49060
Scaled absolute bias– delta method	0.25856	0.54261	0.26811	0.00427	0.53835
Scaled absolute bias– taylor 2	2.22448	3.86238	2.23230	1.31535	2.54704
Correlation=.2					
TYPE BIAS AND METHOD	MEAN	MAX	MEDIAN	MIN	RANGE
Scaled bias- null case	0.07947	1.96167	-0.18588	-0.55136	2.51303
Scaled bias- bootstrap	-0.12425	0.69710	-0.18898	-0.51583	1.21293
Scaled bias- conditional	-0.47227	0.02293	-0.51698	-0.67589	0.69882
Scaled bias– delta method	-0.48045	0.27369	-0.50751	-0.69235	0.96604
Scaled bias– taylor 2	1.16898	2.88196	0.99738	0.35674	2.52522
Scaled absolute bias– null case	0.43688	1.96167	0.30160	0.00529	1.95638
Scaled absolute bias– bootstrap	0.24949	0.69710	0.24853	0.00327	0.69382
Scaled absolute bias– conditional	0.47254	0.67589	0.51698	0.02293	0.65296
Scaled absolute bias– delta method	0.48750	0.69235	0.50751	0.02841	0.66394
Scaled absolute bias– taylor 2	1.16898	2.88196	0.99738	0.35674	2.52522
Correlation=.4					
TYPE BIAS AND METHOD	MEAN	MAX	MEDIAN	MIN	RANGE
Scaled bias- null case	–0.20855	1.22325	-0.39216	-0.66761	1.89085
Scaled bias- bootstrap	–0.34612	0.26972	-0.39571	-0.63994	0.90967
Scaled bias- conditional	-0.60267	-0.25184	-0.62347	-0.73992	0.48808
Scaled bias– delta method	-0.60672	-0.04473	-0.63480	-0.75897	0.71424
Scaled bias– taylor 2	0.63374	1.90754	0.53798	0.09966	1.80788
Scaled absolute bias– null case	0.41561	1.22325	0.44319	0.00469	1.21856
Scaled absolute bias– bootstrap	0.36877	0.63994	0.39571	0.00039	0.63955
Scaled absolute bias– conditional	0.60267	0.73992	0.62347	0.25184	0.48808
Scaled absolute bias– delta method	0.60672	0.75897	0.63480	0.04473	0.71424
Scaled absolute bias– taylor 2	0.63374	1.90754	0.53798	0.09966	1.80788

Figure 3.29: Scaled Biases for the 5 Variance Formulas

3.4 The Bootstrap and Jackknife with Dependent Data

Two methods exist in the literature for bootstraping dependent data. The first method, called the recursive bootstrap, would be hard to apply in the water quality context and works by assuming a model including the structure of the dependence, fitting the model, estimating residuals, then resampling the residuals [*Veall* 1986]. The second method, which was included in the power study of the next section, is the moving block bootstrap. In the moving blocks bootstrap, instead of points being resampled, blocks of observations are resampled. *Hall* [1985] and *Carlstein* [1986] suggested non-overlapping blocks, while *Hall* [1985] and *Kunsch* [1989] suggested overlapping blocks. *Hall et. al.* [1995] gives a method for estimating the optimal length of the block, and his suggestion of using $n^{\frac{1}{3}}$ was used in the study of the following section to choose the size of the block for the overlapping moving block bootstrap.

A third method proposed here is the use of an effective sample size. An early occurrence of the idea of an effective sample size in the literature is *Bayley and Hammersley* [1946]. Loosely speaking, an effective sample size is the number of independent observations that possess the same amount of information as exists in the dependent sample. To apply this idea to the bootstrap, instead of taking resamples of the same size as the original sample, in the presence of dependence, take resamples the size of the effective sample size of the original sample.

The concept of using an effective sample size for inferences when dependence is present has been explored. See, for example, *Zwiers and von Storch* [1995] and

Wilks [1997]. *Wilks* [1997] used the related idea of variance inflation factors and the moving blocks bootstrap to estimate variances and perform two-sample tests of multivariate mean equivalence in the presence of both serial correlation and cross correlation. The variance inflation factor is a related idea because it also can be used to calculate an effective sample size. *Wilks* [1997] gives formulas for estimating the variance inflation factor for AR(1), AR(2), and ARMA(1,1) models as well as for bias-reducing transformations. He then used these formulas to estimate the block size for the moving blocks bootstrap using formulas he developed empirically. *Zwiers and von Storch* [1995] used the same effective sample size as *Wilks* [1997] for autoregressive processes in formulating a test for the univariate means comparison case. However, their test is outperformed by the bootstrap version given by *Wilks* [1997]. Unfortunately, these formulas pertain "only to inferences concerning the means", and involve plugging in estimates of the autoregressive parameters which are estimated using normal theory methods [*Wilks* 1997]. For the AR(p) case, the formula is

$$n_{\text{eff}} = \frac{n}{1 + 2\sum_{k=1}^{n-1}(1 - \frac{k}{n})\rho_k} \tag{3.26}$$

where ρ_k is the lag-k autocorrelation. See *Thiebaux and Zwiers* [1984] for a proof of this formula. It is obtained simply by equating variances. The AR(1) simplification of this given in *Zwiers and von Storch* [1995] is

$$n_{\text{eff}} = \frac{n(1-\rho)}{(1+\rho)}. \tag{3.27}$$

Unfortunately, derivations of these quantities depend on the fact that the effective sample size is for inferences on the mean, and is derived by equating variances.

This method does easily transfer over to the case of the modified Kendall's tau. For this reason and the fact that the above formulas are for inferences about the mean, another method which is intuitively appealing was tried as well. The formula for AR(1) with positive autocorrelation parameter is

$$n_{\text{eff}} = 1 + (n-1)(1-\rho). \tag{3.28}$$

This formula is based on the first observation counting as 1 observation and each additional observation adding $(1-\rho)$, and is slightly less conservative than the *Thiebaux and Zwiers* [1984] formula given above. Along the same lines, the effective sample size for a stationary ($\rho_1 + \rho_2 < 1$ for ρ_1 and $\rho_2 > 0$ [*Box et. al.* 1994]) AR(2) would be

$$\begin{aligned} n_{\text{eff}} &= 1 + 1 - \rho_1 + (1 - \rho_1 - \rho_2)(n-2) \\ &= n + \rho_1 + 2\rho_2 - n\rho_1 - n\rho_2. \end{aligned} \tag{3.29}$$

In small scale simulation studies, this new method performed fairly well in estimating the variance of $\hat{\tau}_{\text{mod}}$ but appeared to slightly overestimate the variance. This is most likely due to the increased percentage of ties in the bootstrap resamples. Hence, the test based on this estimator is suspected to hold its level.

It should be noted that some experimenting with effective sample size ideas was done with jackknife as well. However, the jackknife did not perform as well as the bootstrap for up to sample size 40, the largest sample size considered. The jackknife, in its original form deleting one observation at a time and in the form considered where the number of observations deleted were based on the effective sample size, underestimated the variance. Both jackknife methods were examined with two different types of standardization, the regular sample variance way and the standardization given by *Kunsch* [1989].

To show that the effective sample size bootstrap is a reasonable idea, a Monte Carlo study was performed where 5000 samples of equally spaced in time AR(1) normally distributed data were generated. The variance of the mean was calculated using both effective sample size bootstraps, the moving blocks bootstrap with blocks of size $n^{\frac{1}{3}}$, the standard bootstrap, and exactly by

$$\mathrm{Var}(\bar{\mathrm{y}})=\frac{\sigma^2}{n}. \tag{3.30}$$

This is the exact formula because in the model $y_i = \rho y_{i-1} + \varepsilon$ with the way the autocorrelated data were generated, the variance of ε is $(1-\rho)^2\sigma^2$ and the variance of the mean of an AR(1) process is accurately approximated by

$$\mathrm{Var}(\bar{\mathrm{y}})=\frac{\mathrm{Var}(\varepsilon)}{(1-\rho)^2 n}. \tag{3.31}$$

The results are displayed below in Figures 3.30 - 3.32. It is clear that the effective sample size bootstrap using the effective sample size for the mean (Equation 3.27) is the best of the methods considered. An interesting side note is that in the $\rho = 0$ cases when the standard bootstrap and the effective sample size bootstraps are equivalent, the bootstrap is slightly biased. In fact, the effective sample size bootstrap with the effective sample size from Thiebaux-Zwiers does not appear to be any more biased than the bootstrap is in the independent cases.

A similar study was run to study the properties of the bootstrap methods for estimating the variance of $\hat{\tau}_{\mathrm{mod}}$. Autocorrelated normal time series were gener-

# Boot-strap Re-samples	# Obs	Ar(1) Param-eter	Theor-etical Std Dev	Empir-ical Std Dev	Effective Sample Size Theibaux-Zwiers	Effective Sample Size Ignorance Method	Moving Blocks Bootstrap	Standard Bootstrap
50	50	0	.1414	.1423	.1388	.1385	.1353	.1389
50	50	.2	.1414	.1427	.1377	.1255	.1276	.1126
50	50	.4	.1414	.1403	.1371	.1147	.1169	.0894
100	50	0	.1414	.1401	.1387	.1389	.1362	.1385
100	50	.2	.1414	.1402	.1381	.1259	.1276	.1126
100	50	.4	.1414	.1403	.1365	.1148	.1169	.0895
50	100	0	.1000	.0988	.0986	.0986	.0959	.0985
50	100	.2	.1000	.0999	.0983	.0898	.0915	.0805
50	100	.4	.1000	.0979	.0983	.0827	.0858	.0642
100	100	0	.1000	.1005	.0990	.0987	.0964	.0989
100	100	.2	.1000	.1013	.0989	.0900	.0924	.0807
100	100	.4	.1000	.1001	.0981	.0826	.0857	.0642

Figure 3.30: Monte Carlo Means of Estimated Standard Deviations

# Boot-strap Re-samples	# Obs	Ar(1) Param-eter	Theor-etical Std Dev	Empir-ical Std Dev	Effective Sample Size Theibaux-Zwiers	Effective Sample Size Ignorance Method	Moving Blocks Bootstrap	Standard Bootstrap
50	50	0	.1414	.1423	.1378	.1380	.1333	.1378
50	50	.2	.1414	.1427	.1366	.1246	.1256	.1117
50	50	.4	.1414	.1403	.1358	.1139	.1147	.0887
100	50	0	.1414	.1401	.1377	.1381	.1345	.1380
100	50	.2	.1414	.1402	.1369	.1253	.1257	.1120
100	50	.4	.1414	.1403	.1356	.1137	.1150	.0888
50	100	0	.1000	.0988	.0981	.0981	.0949	.0977
50	100	.2	.1000	.0999	.0978	.0892	.0905	.0800
50	100	.4	.1000	.0979	.0976	.0822	.0847	.0639
100	100	0	.1000	.1005	.0987	.0983	.0957	.0988
100	100	.2	.1000	.1013	.0985	.0896	.0913	.0804
100	100	.4	.1000	.1001	.0979	.0820	.0847	.0640

Figure 3.31: Monte Carlo Medians of Estimated Standard Deviations

# Bootstrap Re-samples	# Obs	Ar(1) Parameter	Theoretical Std Dev	Empirical Std Dev	Effective Sample Size Theibaux-Zwiers	Effective Sample Size Ignorance Method	Moving Blocks Bootstrap	Standard Bootstrap
50	50	0	.1414	.1423	.404	.406	.773	.405
50	50	.2	.1414	.1427	.398	.584	.914	1.099
50	50	.4	.1414	.1403	.503	1.041	1.299	2.901
100	50	0	.1414	.1401	.308	.310	.680	.309
100	50	.2	.1414	.1402	.321	.502	.809	1.038
100	50	.4	.1414	.1403	.368	.957	1.215	2.849
50	100	0	.1000	.0988	.152	.152	.281	.150
50	100	.2	.1000	.0999	.156	.232	.328	.481
50	100	.4	.1000	.0979	.176	.420	.448	1.353
100	100	0	.1000	.1005	.097	.099	.229	.098
100	100	.2	.1000	.1013	.104	.186	.280	.441
100	100	.4	.1000	.1001	.117	.389	.411	1.323

Figure 3.32: Monte Carlo MSE's($\times$1000) of Estimated Standard Deviations

ated with linear trend to induce positive tau values. Since from the previous study, 50 resamples appears to be sufficient, all estimates were calculated using 50 resamples. Figures 3.33-3.35 display the results. The moving blocks bootstrap and the effective sample size bootstrap with Equation 3.28 to estimate the effective sample size both underestimate or fairly accurately estimate the variance for the smaller values of $\hat{\tau}_{\mathrm{mod}}$ and overestimate the variance for the larger values. The effective sample size bootstrap with the effective sample size estimated using Equation 3.27 always seems to overestimate the variance. The effective sample size bootstrap with Equation 3.28 does the best job of estimating the variance in terms of mean squared error. One further note, it appears that $\hat{\tau}_{\mathrm{mod}}$ increases as the AR(1) parameter increases.

# Obs	Ar(1) Param -eter	Beta	Median Tau Value	Empir-ical Std Dev	Effective Sample Size Theibaux-Zwiers	Effective Sample Size Ignorance Method	Moving Blocks Bootstrap	Standard Bootstrap
50	0	.02	.187	.0931	.0955	.0959	.0993	.0959
50	0	.04	.355	.0806	.0853	.0854	.0958	.0854
50	.2	.02	.228	.1081	.1172	.1060	.1108	.0943
50	.2	.04	.416	.0806	.1001	.0908	.1026	.0802
50	.4	.02	.278	.1299	.1461	.1194	.1222	.0909
50	.4	.04	.484	.0913	.1201	.0980	.1090	.0744
100	0	.01	.187	.0637	.0655	.0655	.0661	.0652
100	0	.02	.349	.0547	.0581	.0580	.0644	.0581
100	.2	.01	.227	.0736	.0793	.0722	.0755	.0642
100	.2	.02	.408	.0598	.0668	.0609	.0695	.0540
100	.4	.01	.276	.0882	.0974	.0808	.0851	.0624
100	.4	.02	.484	.0625	.0776	.0639	.0728	.0486

Figure 3.33: Monte Carlo Means of Estimated Standard Deviations of $\hat{\tau}_{\text{mod}}$

# Obs	Ar(1) Param -eter	Beta	Median Tau Value	Empir-ical Std Dev	Effective Sample Size Theibaux-Zwiers	Effective Sample Size Ignorance Method	Moving Blocks Bootstrap	Standard Bootstrap
50	0	.02	.187	.0931	.0954	.0953	.0974	.0957
50	0	.04	.355	.0806	.0848	.0849	.0946	.0849
50	.2	.02	.228	.1081	.1170	.1058	.1086	.0939
50	.2	.04	.416	.0806	.0993	.0902	.1011	.0795
50	.4	.02	.278	.1299	.1460	.1188	.1205	.0908
50	.4	.04	.484	.0913	.1192	.0966	.1079	.0736
100	0	.01	.187	.0637	.0651	.0653	.0649	.0652
100	0	.02	.349	.0547	.0579	.0578	.0637	.0580
100	.2	.01	.227	.0736	.0791	.0720	.0746	.0639
100	.2	.02	.408	.0598	.0666	.0603	.0690	.0535
100	.4	.01	.276	.0882	.0972	.0805	.0838	.0621
100	.4	.02	.484	.0625	.0763	.0634	.0714	.0482

Figure 3.34: Monte Carlo Medians of Estimated Standard Deviations of $\hat{\tau}_{\text{mod}}$

# Obs	Ar(1) Param -eter	Beta	Median Tau Value	Empir- ical Std Dev	Effective Sample Size Theibaux- Zwiers	Effective Sample Size Ignorance Method	Moving Blocks Bootstrap	Standard Bootstrap
50	0	.02	.187	.0931	.182	.181	.340	.180
50	0	.04	.355	.0806	.198	.200	.564	.202
50	.2	.02	.228	.1081	.376	.243	.509	.383
50	.2	.04	.416	.0806	.464	.266	.674	.250
50	.4	.02	.278	.1299	.822	.475	.676	1.760
50	.4	.04	.484	.0913	1.361	.410	.915	.513
100	0	.01	.187	.0637	.065	.069	.143	.064
100	0	.02	.349	.0547	.072	.074	.212	.072
100	.2	.01	.227	.0736	.130	.082	.184	.158
100	.2	.02	.408	.0598	.150	.084	.251	.098
100	.4	.01	.276	.0882	.262	.183	.261	.745
100	.4	.02	.484	.0625	.392	.114	.321	.260

Figure 3.35: Monte Carlo MSE's($\times$1000) of Estimated Standard Deviations of $\hat{\tau}_{\text{mod}}$

3.5 Power Study of $\mathrm{H}_0 : \tau_{\text{mod}\,1} = \tau_{\text{mod}\,2}$

Although this specific test has not been considered in the literature, a related test has. That is the test of H_0:$\tau_1 = \tau_2$. *Schemper* [1987] considered this test of equivalence of Kendall taus with the permutation variance estimator (Equation 2.2 scaled by the square of the denominator in Equation 3.3), Kendall's upper bound (Equation 3.4), a jackknife variance estimator, and a bootstrap variance estimator. He did not consider the test in the presence of serial correlation or ties. Schemper found the bootstrap to be the best of these choices. He also considered the percentile bootstrap method [*Efron* 1982] as well as an approach that fit a four parameter beta distribution to the first four bootstrapped moments of the distribution of $\hat{\tau}$. Neither of these added substantially to the power of the bootstrap.

The methods considered for testing the hypothesis of equality in the study run here were all z tests with each of the following methods of estimating the variances: the delta method, the null case formula, the standard bootstrap, the effective sample size bootstrap with both of the above given formulas for estimating the effective sample size, and the moving blocks bootstrap with blocks of size $[n^{\frac{1}{3}}+1]$. In addition, the hypothesis test of equality of two Pearson's product moment correlation coefficients was included for comparison. All tests were performed at the .05 level. Therefore, with a Monte Carlo size of 1000, the 95% confidence interval into which 95% of the empirical alpha levels should fall for tests which truly have the correct .05 alpha level is (.0365,.0635).

The data were generated in the same manner as the second variance of $\hat{\tau}_{\mathrm{mod}}$ study reported above. The x values were all generated as evenly spaced, and the y values were generated as independent and AR(1) with correlation parameters 0.2 and 0.4 for observations 1.0 x-units apart. Three different distributions were considered: gammas with $\lambda = 1$ and $r = 2$ or 3 as above, and a normal with $\sigma^2 = 3$. Two levels of ties were considered: percent of tied comparisons= 5 and 25. The sample sizes considered for the two samples were 10 and 10, 20 and 20, 40 and 40, 10 and 30, and 30 and 10. Finally, both linear and exponential mean relationships were considered with every combination of the parameter values that were used above.

The results are displayed in Figures 6.1-6.36 in Appendix G. The following conclusions can be drawn based on this study. As can be seen by observing the cases where $\beta_1 = \beta_2$, in the presence of serial correlation, the only method of estimating the variance that consistently yielded an asymptotic z-test that held its level was the effective sample size bootstrap using Equation 3.27 to estimate the effective sample size. This method was, in fact, generally conservative with the largest empirical alpha level among all the cases considered being .058 occurring in the case $\rho = .2$, normal distribution, 25% ties, and both τ_{mod}'s$\doteq .27$. The test is called generally conservative because more than half of the empirical alpha levels were below the lower confidence limit of .0365 for a true alpha value of .05 and 1000 trials.

When truly equivalent τ_{mod} values were tested at values of τ_{mod} larger than about 0.4, all the dependent bootstrap methods generated tests generally held their level. However, both the effective sample size bootstrap with Equation 3.28 and the moving blocks bootstrap were sometimes slightly aggressive in the unequal sample size cases. In the water quality arena, τ_{mod} values are generally less then .4. Thus, the effective sample size bootstrap using Equation 3.27 to estimate the effective sample size can be recommended as the method to use when testing $\mathrm{H}_0 : \tau_{\mathrm{mod}\,1} = \tau_{\mathrm{mod}\,2}$ in the water quality arena where serial correlation generally exists and τ_{mod} values are generally less then 0.4.

In the independent cases, the bootstrap and effective sample size bootstraps are equivalent because, with no serial correlation, the original sample size is used as the resample size. Thus, looking at the power levels of these methods in the tables gives an idea of the variability involved when using the bootstrap. The three entries always agree fairly closely. Somewhat surprisingly, Pearson's holds its level consistently for both the normal and gamma distributions, both levels of ties, unequal sample sizes, and both linear and exponential true underlying relationships. The bootstrap and null case formula also hold their levels with the one exception being that the bootstrap was sometimes slightly liberal for the unequal sample size cases considered. The use of the delta method formula provided a liberal test almost all the time. As expected, Pearson's was generally the most powerful test for the normally distributed cases. It was also generally the most powerful test for the 5% tied comparison cases when at least one of the τ_{mod}'s were larger than .5. When both τ_{mod}'s were less than roughly .5 the bootstrap was generally the most powerful. With 25% tied comparisons, gamma distributions, and independence, the bootstrap generally appeared to be the most powerful test.

The unequal sample size cases can be compared to the equal sample size cases where n_1 and n_2 are 20. Noting that $0 \leq \tau_{\text{mod}\,1} \leq \tau_{\text{mod}\,2}$, regardless of the degree of serial correlation, distribution, or percentage of tied comparisons, it is a bad idea for sample 1 to possess the smaller sample size where the estimator has more variability. The unequal sample size cases where n_2 was bigger than n_1 were never more powerful than their corresponding unequal sample size cases where n_2 was smaller than n_1 and their corresponding equal sample size of 20 cases. On the flip side, there appears to be an advantage to have 30 observations in group 1, as opposed to having equal sample sizes, in the cases where $\tau_{\text{mod}\,1} \neq 0$. When $\tau_{\text{mod}\,1} = 0$ this advantage occurred only with the 25% tied comparison cases. With the more common 5% tied comparisons situation, the equal sample size approach appeared to be more powerful.

Serial correlation had the expected reducing effect on power. Dramatic differences in power occur between cases with identical parameters except for degree of serial correlation. The percentage of tied comparisons also had a reducing effect on power but not quite as much as serial correlation.

3.6 Robustness to Misspecification of Structure of Dependence

One potential drawback of using the effective sample size bootstrap to estimate the variance of $\hat{\tau}_{\text{mod}}$ for purposes of testing the hypothesis of equivalence is that the effective sample size has to be estimated. This requires knowledge of the structure of the serial correlation and the ability to estimate the parameters of that structure. However, there is hope that the method is somewhat robust to misspecification of the serial correlation structure as well as robust to the incorrect estimation of the serial correlation parameters. This is because (i) the use of the effective sample size bootstrap method for testing the hypothesis of equivalence appears to be conservative, and (ii) any positive correlation structure will inflate the variance of $\hat{\tau}_{\text{mod}}$ as compared to the independent observations case. In other words, a structure other than AR(1) might require the same effective sample size as the calculated effective sample size assuming an AR(1) structure.

To explore this potential robustness, some simulations similar to the ones of the previous section were run. However, instead of generating observations as AR(1), observations were generated as AR(2). 500 normal, mean moving linearly, with 5% tied comparisons observations were generated for each of the following 4 combinations of AR(2) parameters: 0.2, 0.1; 0.2, 0.2; 0.4, 0.1; and 0.4, 0.2. The samples were then analyzed, assuming an AR(1) structure with the first AR(2) parameter used for purposes of estimating effective sample size. Figures 3.36-3.39 display the results. Empirical alpha values falling within the 95% confidence interval for tests of true 0.05 size are in bold print. This confidence interval based on 500 observations is (.031,.069).

Happily, the effective sample bootstrap method using Equation 3.27 to estimate the effective sample size appears to be holding its level. The other methods examined did not always hold their level. This suggests that the test of the hypothesis of equivalence of two modified Kendall's tau correlation coefficients based on the effective sample size bootstrap with Equation 3.27 is fairly robust to misspecification of the serial correlation structure in terms of maintaining correct alpha levels.

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.196/.080/.080 **.046/.036/.038** .092	.156/.094/.094 **.058/.052/**.072 .102	.122/.094/.094 **.068/.060/**.078 .102	.178/.084/.070 **.050/.038/.042** .088	.214/.096/.086 **.054/.048/.050** .090
β_1=0, β_2=.2	.300/.146/.138 .090/.064/.082 .154	.400/.258/.262 .196/.158/.170 .302	.476/.408/.404 .342/.304/.334 .430	.340/.164/.186 .118/.084/.144 .192	.404/.234/.204 .164/.140/.128 .238
β_1=0, β_2=.4	.538/.334/.314 .236/.212/.218 .402	.738/.566/.548 .486/.420/.434 .636	.914/.846/.836 .796/.762/.786 .872	.526/.346/.404 .256/.206/.254 .446	.758/.552/.482 .474/.420/.366 .576
β_1=0, β_2=.7	.822/.588/.524 .452/.390/.440 .730	.986/.908/.854 .832/.800/.752 .964	1/.998/.992 .994/.990/.992 1	.826/.606/.630 .472/.410/.504 .768	.986/.906/.810 .868/.812/.724 .934
β_1=.2, β_2=.2	.174/**.054/.044** **.044/.026/.038** .078	.138/.074/**.064** **.050/.040/.044** .086	.148/.090/.082 **.052/.034/.054** .100	.168/.074/**.040** **.054/.042/.042** **.068**	.216/.082**/.052** **.058/.046/.056** .092
β_1=.2, β_2=.4	.262/.084/.058 .046/.038/.040 .138	.330/.186/.134 .124/.090/.086 .240	.474/.344/.246 .262/.216/.214 .356	.196/.074/.086 .056/.038/.052 .110	.402/.216/.094 .160/.114/.100 .204
β_1=.2, β_2=.7	.484/.208/.140 .154/.106/.140 .358	.758/.538/.352 .440/.334/.306 .652	.950/.856/.708 .798/.750/.708 .884	.406/.150/.198 .084/.066/.116 .330	.796/.604/.314 .500/.466/.362 .602
β_1=.4, β_2=.4	.108/**.024/.008** **.010/.004/.008** **.048**	.164/**.040/.018** **.020/.016/.024** .072	.148/**.064/.020** **.040/.030/.012** .074	.232/.104/**.016** **.062/.044/.034** .082	.168/.074/**.012** **.042/.038/.018** **.068**
β_1=.4, β_2=.7	.246/.056/.020 .018/.018/.028 .110	.352/.152/.040 .094/.072/.042 .234	.540/.326/.102 .234/.182/.132 .394	.128/.026/.014 .008/.010/.008 .080	.464/.278/.012 .202/.150/.098 .244
β_1=.7, β_2=.7	.078/**.006/.000** **.000/.000/.000** **.010**	.072/**.018/.000** **.006/.004/.000** **.020**	.076/**.014/.000** **.006/.002/.000** **.024**	.152/.086/**.000** **.068/.050/.018** **.044**	.124/.070/**.000** **.046/.038/.018** **.042**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure 3.36: Linear Mean Relationship, ρ_1=0.2, ρ_2=0.1, %ties=5, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.226/.094/.082 **.056/.042/.054** .080	.166/.100/.098 **.058/.054/.060** .108	.132/.098/.108 **.068/.056/**.072 .114	.242/.122/.110 .074/**.060/**.072 .112	.226/.112/.100 .080/**.064/.064** .110
β_1=0, β_2=.2	.342/.180/.170 .118/.090/.116 .206	.386/.262/.246 .200/.156/.170 .302	.480/.384/.380 .300/.262/.300 .400	.300/.174/.188 .116/.100/.136 .190	.408/.238/.226 .180/.166/.168 .270
β_1=0, β_2=.4	.572/.340/.338 .248/.206/.242 .410	.748/.582/.546 .462/.422/.382 .624	.918/.850/.832 .778/.732/.746 .870	.542/.366/.414 .286/.230/.288 .470	.750/.566/.496 .478/.436/.380 .566
β_1=0, β_2=.7	.806/.554/.516 .456/.414/.416 .706	.940/.856/.800 .792/.750/.722 .902	1/.992/.980 .986/.984/.968 .996	.804/.590/.636 .474/.408/.494 .792	.960/.864/.758 .808/.766/.668 .886
β_1=.2, β_2=.2	.208/**.066/.046** **.032/.024/.036** .084	.168/.076/**.060** **.046/.028/.034** .084	.154/.094/**.062** **.054/.038/.042** .108	.222/.094/**.048** **.056/.048/.052** .102	.216/.100/.076 .070/**.050/.050** .104
β_1=.2, β_2=.4	.216/.070/.048 .052/.024/.044 .098	.274/.128/.096 .086/.066/.076 .174	.450/.316/.246 .256/.220/.216 .354	.226/.090/.096 .060/.034/.056 .118	.380/.212/.078 .146/.102/.080 .200
β_1=.2, β_2=.7	.480/.192/.132 .124/.102/.122 .312	.708/.472/.358 .406/.324/.286 .600	.922/.820/.662 .760/.716/.654 .878	.374/.120/.142 .074/.042/.098 .296	.786/.586/.252 .484/.422/.344 .610
β_1=.4, β_2=.4	.108/**.024/.014** **.016/.008/.014** **.042**	.144/**.058/.020** **.036/.032/.022** .074	.140/**.056/.014** **.032/.018/.022** .070	.222/.078/**.018** **.052/.026/.036** .080	.164/**.056/.012** **.032/.026/.022** **.046**
β_1=.4, β_2=.7	.198/.052/.012 .016/.004/.022 .088	.302/.110/.024 .054/.040/.026 .188	.460/.260/.072 .186/.146/.086 .338	.132/.036/.020 .024/.012/.016 .082	.428/.222/.014 .160/.134/.092 .198
β_1=.7, β_2=.7	**.058/.008/.000** **.008/.006/.008** **.020**	.078/**.010/.000** **.002/.000/.002** **.032**	.094/**.034/.000** **.002/.002/.000** **.044**	.176/.098/**.000** .074/**.048/.026** **.038**	.134/**.064/.002** **.046/.038/.018** **.038**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure 3.37: Linear Mean Relationship, ρ_1=0.2, ρ_2=0.2, %ties=5, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.280/.136/.130 **.030/.020/.044** .148	.212/.140/.164 **.058/.042**/.072 .176	.240/.184/.202 .074/**.050**/.090 .210	.278/.136/.136 **.064/.058**/.076 .154	.290/.140/.136 **.062/.046/.068** .154
β_1=0, β_2=.2	.428/.230/.222 .086/.074/.118 .286	.458/.334/.340 .202/.170/.204 .384	.628/.540/.552 .396/.360/.414 .574	.412/.252/.266 .130/.102/.158 .302	.548/.366/.320 .208/.160/.214 .390
β_1=0, β_2=.4	.660/.450/.414 .266/.244/.318 .550	.818/.698/.678 .496/.454/.510 .770	.958/.934/.914 .822/.810/.808 .942	.642/.464/.516 .266/.218/.324 .596	.834/.670/.582 .470/.424/.450 .686
β_1=0, β_2=.7	.880/.710/.614 .426/.386/.506 .816	.984/.928/.866 .820/.750/.770 .976	1/.998/.990 .988/.982/.980 1	.822/.602/.648 .400/.326/.454 .804	.984/.938/.864 .844/.802/.782 .956
β_1=.2, β_2=.2	.202/**.066/.048** **.018/.006/.026** .100	.232/.112/.092 **.034/.026/.032** .156	.244/.118/.094 **.046/.040/.044** .162	.270/.114/.086 **.050/.042/.052** .136	.306/.144/.100 **.062/.034/.068** .176
β_1=.2, β_2=.4	.298/.112/.082 .030/.016/.056 .172	.400/.216/.158 .106/.080/.092 .324	.524/.382/.264 .206/.174/.188 .412	.260/.114/.106 .036/.024/.066 .164	.480/.298/.116 .146/.124/.144 .288
β_1=.2, β_2=.7	.536/.260/.164 .094/.062/.126 .398	.766/.530/.314 .282/.232/.260 .680	.944/.874/.668 .698/.658/.606 .914	.450/.194/.208 .066/.044/.120 .446	.824/.652/.254 .434/.390/.364 .656
β_1=.4, β_2=.4	.148/**.028/.022** **.004/.004/.008** **.064**	.166/**.060/.022** **.008/.008/.008** .102	.156/**.062/.008** **.010/.006/.014** .102	.232/.110/**.006** **.034/.024/.028** .096	.196/.112/**.024** **.046/.038/.030** .098
β_1=.4, β_2=.7	.226/.072/.012 .004/.002/.020 .106	.348/.152/.050 .036/.032/.028 .264	.570/.354/.066 .144/.120/.082 .478	.150/.040/.028 .014/.014/.012 .094	.530/.332/.020 .144/.120/.088 .272
β_1=.7, β_2=.7	**.068/.004/.000** **.002/.002/.002** **.016**	.082/**.018/.002** **.006/.002/.000** **.036**	.080/**.026/.000** **.006/.000/.002** **.044**	.194/.126/**.000** .072/**.048/.034** **.038**	.184/.088/**.000** **.042/.034/.022** **.018**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure 3.38: Linear Mean Relationship, ρ_1=0.4, ρ_2=0.1, %ties=5, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.296/.114/.114 **.042/.030/.054** .156	.242/.160/.172 **.066/.052**/.080 .192	.204/.168/.176 .070/.070/.102 .194	.294/.138/.148 .070/**.042**/.072 .164	.304/.146/.140 **.056/.044**/.082 .164
β_1=0, β_2=.2	.376/.210/.206 .100/.064/.126 .234	.516/.380/.384 .206/.174/.226 .420	.584/.526/.530 .360/.334/.366 .552	.420/.278/.300 .138/.108/.174 .316	.550/.374/.332 .200/.178/.202 .378
β_1=0, β_2=.4	.618/.420/.380 .206/.178/.260 .484	.830/.664/.654 .468/.440/.486 .762	.928/.902/.884 .814/.808/.790 .920	.636/.456/.506 .246/.222/.332 .580	.840/.706/.654 .502/.452/.486 .742
β_1=0, β_2=.7	.826/.648/.578 .408/.354/.504 .770	.970/.912/.868 .782/.748/.792 .956	1/.996/.988 .992/.988/.986 .996	.868/.640/.704 .386/.320/.502 .878	.978/.932/.850 .846/.808/.794 .954
β_1=.2, β_2=.2	.246/.100/.088 **.026/.018/.042** .152	.206/.114/.096 **.040/.036/.052** .156	.250/.158/.116 **.054/.048/.058** .184	.266/.146/.088 **.066/.054/.068** .156	.252/.124/**.064** **.048/.030/.050** .132
β_1=.2, β_2=.4	.292/.116/.070 .028/.016/.060 .184	.388/.222/.182 .096/.078/.094 .300	.476/.324/.254 .186/.170/.172 .396	.240/.094/.094 .024/.014/.044 .166	.430/.268/.102 .136/.112/.112 .262
β_1=.2, β_2=.7	.530/.230/.158 .088/.072/.128 .416	.766/.502/.312 .252/.220/.258 .686	.930/.854/.650 .660/.636/.568 .896	.404/.182/.176 .068/.048/.116 .372	.830/.638/.236 .408/.338/.326 .632
β_1=.4, β_2=.4	.150/**.032/.014** **.006/.002/.014** **.066**	.186/**.050/.016** **.014/.006/.008** .096	.172/.070/**.022** **.024/.022/.022** .102	.230/.112/**.008** **.038/.030/.030** .070	.190/**.068/.018** **.030/.026/.026** .078
β_1=.4, β_2=.7	.226/.020/.010 .002/.000/.012 .114	.320/.112/.028 .020/.016/.024 .214	.498/.286/.062 .116/.096/.068 .380	.144/.028/.014 .008/.002/.010 .114	.494/.316/.008 .142/.138/.106 .246
β_1=.7, β_2=.7	.098/**.010/.002** **.000/.002/.002** **.026**	**.066/.012/.000** **.000/.004/.002** **.040**	.098/**.030/.000** **.004/.000/.000** **.048**	.224/.140/**.000** .070/**.064/.040** **.044**	.150/.082/**.002** **.030/.034/.018** **.016**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure 3.39: Linear Mean Relationship, ρ_1=0.4, ρ_2=0.2, %ties=5, dist=normal

CHAPTER 4

Change in Trend and Slope Change-Points

4.1 Introduction

Sometimes the appearance of a water quality time series suggests that at the start of the time series, the variable exhibited a particular trend, then switched direction, or at least changed in magnitude. The situation where the trend switches signs or the probability of concordance changes will be called a *change in trend*, and the point in time where such a change takes place is called a *change-point*. Note that a change in the probability of concordance does not require that the trend be linear. The situation where the slope of the linear portion of the trend changes in magnitude but not necessarily in sign will simply be called a *slope change*. The change-point may or may not be known a priori. For instance, if some new regulation affecting water-quality went into effect or if a new dam was built at a known time, then the potential change-point can be considered to be known a priori. In other cases, the potential change-point would be considered unknown. So, it is of interest to test if given points are change-points whether an hypothesized potential change-point is known a priori or not.

Tang and MacNeill [1992] considered the problem of a change in trend. Their method requires the assumption of an underlying linear or quadratic trend. Additionally, their method assumes independence and normality. These assumptions are not realistic in the water quality context. *Miao* [1988] considered a change-point in slope in the linear regression context. Miao presented a method that uses a CUSUM-like statistic. The method relies on convergence of partial sums to normality and requires independence for distributional purposes. However, this method does not seem to be well suited for many water-quality variables that have seasonal components, detection limited values, serial correlation, and, often, severly non-normal distributions whose sums would not converge quickly to normality.

Sen [1980] developed a test to find a slope change-point in a simple linear regression context using a rank-based linear models approach which he called an *aligned rank approach*. His semi-nonparametric method is based on the residuals of the fit. The method does not allow serial correlation and assumes no ties. This last point is not so much a problem because ties can be given average scores, but serial correlation is a more difficult problem. Seasonality and covariate effects need to be taken into account as well.

To test if a particular point is a change-point in trend, two procedures based on Kendall's tau will be explained. The reason a new procedure is needed is that no existing nonparametric change-point estimation or testing method considers the problem of a change in trend. As referred to in Chapter 2, many methods exist for the detection of a change in location, even though these methods need to be adapted to handle detection limits and the other characteristics that most water quality variables possess. Moreover, it is often unrealistic to assume that water quality time series are stationary, even when some sort of adjustment for seasonality and flow has been made. A nonparametric method is needed because of the common characteristics of water-quality data described in Chapter 1.

4.2 Methodology for a Fixed Point

First, let us assume that only one fixed time is to be considered as a possible change-point. The data are divided into two parts by this point, and an estimate or measure of the trend in each half of the time series is calculated. These estimates are then compared. If they are significantly different, then the point can be considered a change-point.

There are several possible ways to estimate or measure the trend. Two which seem particularly appropriate are *modified* Kendall's tau [*Kendall* 1938, *Sillitto* 1947, and *Adler* 1957] for the change in trend situation and the seasonal Kendall slope estimator as defined in Equation 2.10 for the slope change situation.

4.3 Change in Trend

The use of $\hat{\tau}_{\text{mod}}$ to test for a change in trend in location is valid only under the homogeneous variance assumption. Under homogeneous variance across the span of the time series, changes in $\hat{\tau}_{\text{mod}}$ reflect differences in the speed with which the location of the variable is changing. Since the measure of this change is based on the proportion of concordance, this change does not have to be linear. If the homogeneous variance assumption is not met, a change in $\hat{\tau}_{\text{mod}}$ in magnitude (but not sign) might simply reflect a change in the underlying variance of the variable. A further advantage of using modified Kendall's tau is its ability to handle a large percentage of ties.

To use modified Kendall's tau to test for a change-point, the following statistic will be considered. The use of this statistic requires the assumption that the estimates of τ_{mod} from each half are independent. The rapid convergence of Kendall's tau to normality is well known [*Kendall* 1970], so there is hope that the distribution of this statistic converges rapidly to normality.

$$Z = \frac{\hat{\tau}_{\text{mod}\,1} - \hat{\tau}_{\text{mod}\,2}}{\sqrt{Var(\hat{\tau}_{\text{mod}\,1}) + Var(\hat{\tau}_{\text{mod}\,2})}} \tag{4.1}$$

4.4 Change in Slope

To use the seasonal Kendall slope estimator (Equation 2.10) to test for a change in slope, a statistic like the one used with the method based on modified Kendall's tau given above in Equation 4.1 could be calculated. If the relationship between the variable and time is in fact linear, then the Theil-Sen slope estimate is asymptotically normal with a mean of β, the true underlying linear slope, and a variance that depends on the underlying distribution [Sen 1968]. However, estimating the variance of the seasonal Kendall slope estimator involves estimating some complex quantities that depend on the underlying distribution [*Zetterqvist* 1988]. A possible way around this is to compare appropriately sized confidence intervals similar to the ones given above in Chapter 2, Equations 2.11-2.13. Intervals based on the Spearman's rho test for the slope could perhaps be extended to the seasonal case and compared as an alternative. If these intervals overlap, then the null hypothesis of no change-point is not rejected, otherwise the tested point where the data were divided is considered to be a change-point. The appropriate confidence levels could be derived under asymptotic normality for the $\hat{\beta}$'s.

This method of using confidence interval avoids having to estimate the variance of $\hat{\beta}_{SK}$ because confidence intervals for β using $\hat{\beta}_{SK}$ are simply the appropriate order statistics. The method works as follows. Letting σ_1^2 be the variance of $\hat{\beta}_{SK}^{(1)}$ and σ_2^2 be the variance of $\hat{\beta}_{SK}^{(2)}$, the problem is to solve for z such that if the intervals $\hat{\beta}_{SK}^{(1)} \pm z\sigma_1$ and $\hat{\beta}_{SK}^{(2)} \pm z\sigma_2$ overlap the null hypothesis should not be rejected at level α and if they do not overlap then the null hypothesis of equivalence should be rejected. This method should be equivalent to performing the hypothesis test in the usual $\frac{\hat{\beta}_{SK}^{(1)} - \hat{\beta}_{SK}^{(2)}}{\sqrt{\sigma_1^2+\sigma_2^2}}$ format. Now, the solution for an α level test is

$$z = z_{1-\frac{\alpha}{2}} \frac{\sqrt{\sigma_1^2 + \sigma_2^2}}{\sigma_1 + \sigma_2}. \tag{4.2}$$

This can be rewritten in terms of the ratio k=$\frac{\sigma_2}{\sigma_1}$ as

$$z = z_{1-\frac{\alpha}{2}} \frac{\sqrt{1+k^2}}{1+k}. \tag{4.3}$$

Now, the ratio of the standard deviations, k, can be estimated based on the asymptotic variance formula for the variance of $\hat{\beta}_{SK}$ valid for serial dependence of observations up to 1 year apart given in *Zetterqvist* [1988]. Assume that the data obey the following model:

$$y_{ik} = \alpha_k + x_{ik}\beta + \varepsilon_{ik,}\ i = 1, ..., n,\ \ k = 1, ..., m, \tag{4.4}$$

where n is the number of years of data, m is the number of seasons, y is the variable of interest, and x is time in the scale of 1 unit=1 year. The variance of $\hat{\beta}_{SK}$ is

$$\mathrm{Var}(\hat{\beta}_{SK}) = \frac{1 + \sum\limits_{h=1}^{g} \gamma_h}{12[\int\limits_{-\infty}^{\infty} f(y)dy]^2 B_n^2} \tag{4.5}$$

where g is the number of seasons over which the serial dependence extends,

$$\gamma_h = Cov\{F(\varepsilon_t), F(\varepsilon_{t+h})\}, \tag{4.6}$$

and

$$B_n = \sqrt{mn(n^2-1)/12} \tag{4.7}$$

when there are no missing values. B_n is modified slightly here from the formula given in *Zetterqvist* [1988] because of the difference in scale of $\hat{\beta}_{SK}$. *Zetterqvist* [1988] does not say how to handle missing values. Since *Zetterqvist* [1988] derived Equation 4.5 by computing $B_n^2 = m\sum\limits_{j=1}^{n}(j - \frac{n+1}{2})^2$, which is the sum of the denominators of the pairwise slope estimates divided by 2, a way to calculate B_n in the presence of missing observations is the following:

$$B_n^* = \sqrt{\frac{1}{2}\sum_{k=1}^{m}\sum_{i<j}^{n} d_{ij}^{(k)}}, \tag{4.8}$$

where $d_{ij}^{(k)} = (x_{jk} - x_{ik})$ if both are non-missing, otherwise $d_{ij}^{(k)} = 0$. Finally, under the assumption that the dependence has the same structure in both samples, the ratio k is as follows

$$k = \frac{\sigma_2}{\sigma_1} = \frac{[1+\sum\limits_{h=1}^{g}\gamma_h]\sqrt{12}[\int\limits_{-\infty}^{\infty} f(y)dy]B_{n_1}}{\sqrt{12}[\int\limits_{-\infty}^{\infty} f(y)dy]B_{n_2}[1+\sum\limits_{h=1}^{g}\gamma_h]} = \frac{B_{n_1}}{B_{n_2}}. \tag{4.9}$$

For example, to obtain a .05 level test, if $n_1 = n_2$, $z = 1.96 \times \sqrt{2}/2 = 1.386$, and 83.4% confidence intervals need to be formed. P-values could be obtained by determining what level of confidence, i.e. what z value, yields intervals which abut but do not overlap, then solving backwards for the p-value. Confidence intervals for β that allow serial dependence are described in Section 2.7.

A disadvantage of using $\hat{\beta}_{SK}$ to test for a slope change occurs when there are a lot of ties in the data. In this case, the seasonal Kendall slope estimator is often 0 because so many of the pairwise slope estimates are 0. Hence, the test will not be very powerful in this case.

4.5 Adjusting for Seasonality, Serial Correlation, and Flow

To handle seasonality with the modified Kendall's tau-based solution, seasonal medians could be subtracted from every y-value in a given season. A more complicated but more accurate method for adjusting for seasonal effects is presented in Chapter 5 and involves estimating the seasonal effects at each point using LOWESS. These methods are recommended instead of the method of blocking on seasons (as in the seasonal Kendall analysis) because of sample size considerations. Bootstrap methods for estimating variance require a large enough sample size to provide reasonable approximations. The bootstrap could be used in an interblock manner by estimating variances separately in each season, then summing over the different seasons for overall estimates. With dependent seasons, however, the covariances would also have to be estimated and this leads to a further complication.

The method based on the seasonal Kendall slope estimator does not have as extreme a problem with blocking on seasons. Only two years or so at the start and end of fairly complete time series cannot be considered for testing as potential change-points. All the pairwise slope estimates are taken within seasons and then lumped together to find the overall estimate and confidence interval. So seasonality can be accounted for by blocking instead of adjusting individual observations. The original Theil-Sen estimate was based on the assumption that observations are independent. When using the seasonal Kendall slope estimator, observations used to calculate pairwise slope estimates are independent as long as observations one year apart are independent. This is because only observations within a season are used to calculate pairwise slope estimates and it is assumed that there is only one observation per season. Assuming observations that are one year apart are independent is fairly reasonable. With an autoregressive correlation structure, if observations one month apart have a serial correlation of 0.6, then observations one year apart have a correlation of only $0.6^{12} = 0.002$. The problem is that consecutive observations are not independent, and the seasonal Kendall test which is inverted to find the confidence interval requires independent seasons. A new idea is to find confidence intervals for the slope by inverting a covariance sum test rather than the seasonal Kendall test, which assumes independence. The derivation of this confidence interval is shown in Appendix D. Using this method for obtaining confidence intervals, the data can be serially dependent as long as observations one year apart are independent.

An adjustment for serial dependence could be made to the change-point-in-trend detection method based on modified Kendall's tau. By using the effective sample size bootstrap, the variance of each estimate of τ_{mod} has already taken into account the serial dependence within that period of observations. The problem comes in assuming that the estimates from each of the two periods are independent to form the test statistic in Equation 4.1. Adding a covariance term to the denominator of the test statistic would solve this problem. However, this covariance is not easy to derive, and depending on the structure of the serial correlation, is probably negligible. Since it can be expected that serial correlation in water quality will always be positive, the test statistic without an adjustment becomes conservative. This is because the covariance of the two statistics will be positive, and hence, the variance of the difference of the two statistics will be decreased. For now, this covariance will be ignored and listed as an area for future research.

Flow adjustment could be handled by performing a LOWESS fit to the water quality variable with flow as the regressor, or by performing a parametric regression of flow on the water quality variable. The residuals from that fit would then be used in the above procedure.

4.6 Trend Change-Point Unknown

Either one of the two methods discussed above give a method for testing if a point is a change-point when the point is known a priori. If the time of a suspected change in trend or change in slope is not known, a method needs to be developed to estimate that time. Here are two potential methods.

One idea to estimate the time of a change in trend with the modified Kendall's tau-based method is the following. Since the variance formulas will require at least a certain small number of observations to be trustworthy, only points that number of observations or more from the start and end of the time series should be considered. For every point in the interior of the time series, calculate the z-statistic in Equation 4.1. The point that generates the greatest in magnitude value of the statistic is then the estimate of the change-point if the associated p-value is "low enough". This maximum type statistic is common in the nonparametric change-point in location literature. Some Bonferroni- or Sidak-type adjustment to the requirement for significance of the p-value will be needed to perform an alpha level test since so many points are being considered. It is also possible that a formula to adjust the p-value can be developed empirically through a Monte Carlo study. Maybe a searching algorithm could be employed to limit the number of tests performed.

Another method would be to first specify an interval in the interior of the time series where a change might have taken place. Then, for every possible break, calculate $|\hat{\tau}^{(1)}_{\text{mod}} - \hat{\tau}^{(2)}_{\text{mod}}|$ and find the maximum value. The change-point is then estimated as the average of the two time points where the data were split to generate this maximum value. To obtain a confidence interval for this change-point, the following method could be used. First, fit the model

$$|\hat{\tau}^{(1)}_{\text{mod}} - \hat{\tau}^{(2)}_{\text{mod}}|_i = \beta_0 + \beta_1(x_i - x_{cp}) + \beta_2(x_i - x_{cp})^2 \tag{4.10}$$

where i=1,..,#breaks, x_i are the break points, and x_{cp} is the estimated change point. A confidence interval for x_{cp} is then given by the two points where

$$|\hat{\tau}^{(1)}_{\text{mod}} - \hat{\tau}^{(2)}_{\text{mod}}| = max\{|\hat{\tau}^{(1)}_{\text{mod}} - \hat{\tau}^{(2)}_{\text{mod}}|_i\} - 1.96\sqrt{\widehat{\text{Var}}(|\hat{\tau}^{(1)}_{\text{mod}} - \hat{\tau}^{(2)}_{\text{mod}}|_{\text{max}})}. \tag{4.11}$$

The variance of $|\hat{\tau}^{(1)}_{\text{mod}} - \hat{\tau}^{(2)}_{\text{mod}}|_{\text{max}}$ can be estimated by using the effective sample size bootstrap. After splitting the data into two periods by the estimated change-point, draw a resample of size $\text{n}^{(1)}_{\text{eff}}$ from the first period and estimate $\tau^{(1)}_{\text{mod}}$. Then draw a resample of size $n_{\text{eff}}^{(2)}$ from the second period, estimate $\tau_{\text{mod}}^{(2)}$, and calculate $|\hat{\tau}_{\text{mod}}^{(1)} - \hat{\tau}_{\text{mod}}^{(2)}|$. Repeat this process m times and estimate $\text{Var}(|\hat{\tau}_{\text{mod}}^{(1)} - \hat{\tau}_{\text{mod}}^{(2)}|)$ at the max as the sample variance of these values.

This second method and the associated method of calculating a confidence interval were evaluated using simulation. Either 1000 or 200 gamma AR(1) and normal AR(1) time series with time coordinates randomly drawn from a uniform distribution were generated for every combination of the following parameter values. The values were percentage of tied comparisons: 5% and 25%, relationship between y variable and time variable: linear and exponential, number of observations generated at an average rate of 12 per year: 120, distribution: normal and gamma, location of change-point: early, middle, and late, and AR(1) parameter: 0, 0.2 , and 0.4. Tau values were generated by using the following beta values. For the linear series, 0 and 0.24 were used in all possible combinations, and for the exponential series, 0 and 0.18 were used in all possible combinations. These parameter values were chosen to yield realistic modified tau values. Only 200 samples were generated for some cases because of computational limitations. With 200 samples, the 95% confidence interval for alpha in which empirical alpha levels of appropriate tests should fall 95% of the time is (.020,.080). With 1000, the interval shrinks to (.0365,.0635).

To more closely simulate true water quality variables, the means of the gamma distribution were moved both exponentially and linearly by holding $\alpha = 0$ while varying λ and r, as parameterized in Section 3.3, in such a manner that the variance, r/λ^2, was held constant. This allowed individual generated observations to approach zero for the entire series. Measures were taken to ensure that the mean did not get too close to 0 and hence cause extremely large skewness, $2/\sqrt{r}$, by forcing $r > 1$ and $r > .2\lambda$ for all but short intervals. An example of a generated time series is shown in Figure 4.1, where the time series is gamma autoregressive with percent tied comparisons=5%, AR(1) parameter=0.2, and change-point=2.5 with linear parameters of $\beta_1 = 0$ for the first half and $\beta_2 = 0.18$ for the second half.

Observing the y-axis of the plot shows why the seasonal Kendall estimator-based method yielded biased estimates. The pseudo lower detection limits in this study were generally much farther from 0 than actual detection limits in reality. This fact gave the seasonal Kendall estimator a little too much of a disadvantage in this study.

The results of this study are displayed in Figures 4.2-4.9. The modified tau hypothesis test method generally outperformed the seasonal Kendall based method in terms of power and worst case empirical alpha levels. However, in the presence of serial correlation, the test of equivalence of two modified Kendall's taus did tend to be slightly liberal. This is assumed to be due to the dependence between observations on opposite sides of the change-point. Additionally, the fact that time values were generated randomly from a uniform distribution creates some diabolical cases likely never encountered in real data where the method of calculating the effective sample size becomes too aggressive. Correlation and percentage of tied

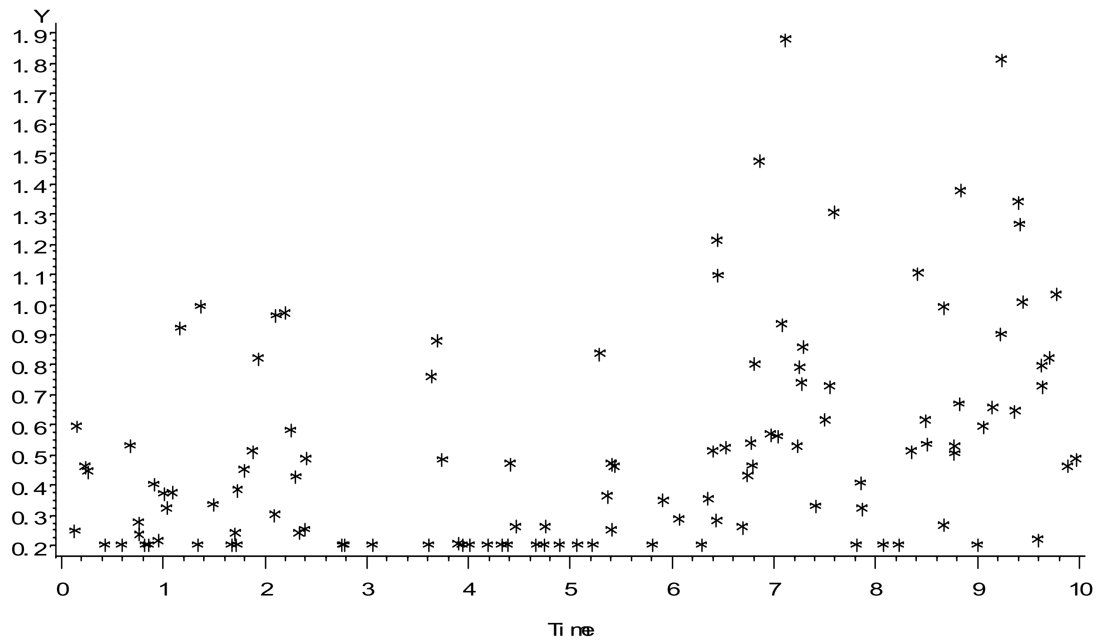

Figure 4.1: Example Generated Data Set

comparisons had the expected effects of reducing power.

Occasionally, the seasonal Kendall slope estimator based method had more power when 25% of the comparisons were tied because this actually tightened the bounds for the confidence interval around $\beta = 0$. In other words, the variability in the data was reduced substantially by truncation at the artificially high lower detection limit. Thus, power is increased, but at the cost of the test not holding its level. Empirical alpha values were quite high in these situations. This effect is probably not going to be reproduced in real data sets where the lower detection limits are not so high. Further evidence that this is what is happening is given by observing the estimates of the slope. Where the percentage of ties is lower, the estimates are only slightly biased, but the Kendall's tau based method is more powerful.

The method of estimating the change-point performed respectably, but generally not exceptionally well. The change-point was estimated most accurately in the linear cases where there was a complete reversal in trend. The method was generally not as accurate for the exponential cases.

The explored confidence interval method generally performed poorly, yielding wide intervals with too small coverage probabilities. The only case where the method performed at all reasonably were the complete reversal in linear trend cases that had change-points directly in the middle of the time series. In practice, the percentile bootstrap confidence interval method [*Efron* 1982] with at least 1000 bootstrap resamples is suggested. This method could not be evaluated in a simulation due to its computational complexity. The percentile method is known to be biased [*Hall* 1988], but it is probably the best alternative known to the author at this time. Better methods of forming a confidence interval using bootstrap methodology exist, but they require good estimates of the variance or knowledge of a transformation to normality. The exception is the double bootstrap [*Beran* 1987]. The double bootstrap reduces the level error of the percentile method, but it is highly computationally intensive [*Beran* 1987]. For each of the 1000 original bootstrap resamples, another 1000 are required.

ρ	β_1	β_2	Monte Carlo Size	Location of Change-point	Power (tau hyp. test/seas. Kendall)	Estimate Change-point (mean/ median)	Median 95% CI	Coverage %	Median Seas. Kendall Est. of β_1/β_2
0	0	0	1000	2.5	**.040/.010**	3.01/1.70	(0.9, 8.3)	73.8	.000/.000
0	0	0	1000	5.0	**.053/.018**	4.99/4.83	(1.1, 9.0)	90.1	.000/.000
0	0	0	1000	7.5	**.061/.006**	7.03/8.33	(1.2, 8.9)	72.6	.000/-.008
0	.24	0	1000	2.5	.122/.012	2.95/1.75	(1.0, 8.8)	73.8	.117/.000
0	.24	0	1000	5.0	.411/.048	5.90/6.57	(2.8, 9.2)	90.0	.144/-.006
0	.24	0	1000	7.5	.571/.006	7.56/8.37	(3.8, 10.8)	76.0	.169/.024
0	0	.24	1000	2.5	.459/.043	2.35/1.63	(-0.2, 6.3)	73.2	.000/.208
0	0	.24	1000	5.0	.399/.177	3.80/3.18	(0.7, 7.0)	88.8	.000/.214
0	0	.24	1000	7.5	.119/.023	6.57/7.88	(1.4, 8.9)	75.2	.000/.194
0	-.24	.24	200	2.5	.780/.050	.211/.182	(-0.5, 5.8)	80.0	-.143/.204
0	-.24	.24	200	5.0	.940/.350	5.02/4.99	(2.5, 7.5)	**97.5**	-.190/.194
0	-.24	.24	200	7.5	.770/.070	.775/.796	(4.4, 10.5)	83.5	-.191/.094
.2	0	0	1000	2.5	.115/**.018**	3.25/1.84	(1.3, 8.4)	71.5	.000/.000
.2	0	0	1000	5.0	.095/**.032**	4.97/4.84	(1.6, 8.2)	89.1	.000/.000
.2	0	0	1000	7.5	.110/**.004**	6.86/8.19	(1.8, 8.8)	70.5	.000/.000
.2	.24	0	1000	2.5	.177/.015	3.20/1.86	(1.3, 8.3)	74.1	.094/.000
.2	.24	0	1000	5.0	.476/.058	5.75/6.46	(2.9, 8.7)	87.7	.133/-.004
.2	.24	0	1000	7.5	.543/.013	7.22/8.24	(3.4, 10.0)	75.9	.156/-.031
.2	0	.24	1000	2.5	.466/.118	2.41/1.69	(0.3, 6.8)	75.7	.000/.210
.2	0	.24	1000	5.0	.419/.155	3.89/3.28	(1.1, 7.1)	88.5	.000/.218
.2	0	.24	1000	7.5	.165/.017	6.59/7.87	(1.6, 8.6)	72.0	.000/.206
.2	-.24	.24	200	2.5	.715/.095	2.24/1.85	(-0.4, 5.4)	83.5	-.096/.201
.2	-.24	.24	200	5.0	.885/.345	4.83/4.82	(2.4, 7.5)	**98.0**	-.172/.181
.2	-.24	.24	200	7.5	.710/.140	7.84/8.03	(3.8, 9.0)	74.0	-.200/.069
.4	0	0	1000	2.5	.081/**.010**	3.28/1.89	(1.0, 8.5)	73.3	.000/.000
.4	0	0	1000	5.0	.080/**.024**	5.08/5.40	(1.3, 8.6)	89.3	.000/.000
.4	0	0	1000	7.5	.073/**.014**	6.82/8.17	(1.3, 8.8)	71.1	.000/.000
.4	.24	0	1000	2.5	.140/.025	3.13/1.83	(1.0, 8.6)	74.6	.090/.000
.4	.24	0	1000	5.0	.391/.050	5.70/6.37	(2.5, 9.0)	89.4	.121/.000
.4	.24	0	1000	7.5	.436/.023	7.18/8.20	(3.2, 10.0)	75.3	.160/.027
.4	0	.24	1000	2.5	.339/.149	2.55/1.81	(0.5, 6.8)	74.9	.000/.217
.4	0	.24	1000	5.0	.365/.190	3.99/3.33	(1.1, 7.2)	88.5	.000/.220
.4	0	.24	200	7.5	.110/.040	6.37/7.68	(1.5, 8.8)	71.5	.000/.209
.4	-.24	.24	200	2.5	.595/.150	2.35/2.00	(-0.1, 6.0)	79.5	-.064/.200
.4	-.24	.24	200	5.0	.895/.445	5.07/5.12	(2.4, 7.7)	**98.0**	-.167/.196
.4	-.24	.24	200	7.5	.655/.150	7.53/7.68	(4.4, 9.8)	82.5	-.200/.112

Figure 4.2: Linear and Normally Distributed with 5% Tied Comparisons

ρ	β_1	β_2	Monte Carlo Size	Location of Change-point	Power (tau hyp. test/seas. Kendall)	Estimate Change-point (mean/ median)	Median 95% CI	Coverage %	Median Seas. Kendall Est. of β_1/β_2
0	0	0	1000	2.5	**.057/.035**	3.07/1.64	(1.0, 8.6)	73.8	.000/.000
0	0	0	1000	5.0	**.045**/.177	5.01/4.99	(0.9, 8.9)	86.2	.000/.000
0	0	0	1000	7.5	**.056/.028**	7.03/8.37	(1.2, 9.1)	70.6	.000/.000
0	.24	0	1000	2.5	.112/.080	2.73/1.59	(0.9, 8.9)	76.0	.000/.000
0	.24	0	1000	5.0	.381/.177	5.35/5.82	(2.6, 9.2)	90.8	.000/.000
0	.24	0	1000	7.5	.625/.014	6.92/8.15	(3.4, 10.3)	78.0	.032/.000
0	0	.24	1000	2.5	.290/.399	2.22/1.70	(0.0, 6.77)	77.4	.000/.106
0	0	.24	1000	5.0	.350/.441	3.81/3.17	(0.8, 7.2)	89.5	.000/.114
0	0	.24	1000	7.5	.107/.033	6.40/7.72	(1.7, 8.7)	75.1	.000/.067
0	-.24	.24	200	2.5	.645/.410	2.08/1.79	(0.0, 5.7)	83.0	.000/.090
0	-.24	.24	200	5.0	.865/.680	5.09/5.05	(2.3, 7.8)	**99.0**	-.052/.039
0	-.24	.24	200	7.5	.590/.355	7.94/8.28	(4.4, 10.4)	88.5	-.092/.000
.2	0	0	1000	2.5	.077/.**061**	3.09/1.77	(1.2, 8.3)	73.3	.000/.000
.2	0	0	1000	5.0	.096/.201	5.09/5.33	(1.5, 8.3)	88.8	.000/.000
.2	0	0	1000	7.5	.087/.**062**	6.77/8.24	(1.4, 8.7)	72.1	.000/.000
.2	.24	0	1000	2.5	.180/.096	2.71/1.66	(1.0, 8.7)	77.6	.000/.000
.2	.24	0	1000	5.0	.411/.177	4.88/4.88	(2.6, 8.8)	90.1	.000/.000
.2	.24	0	1000	7.5	.583/.015	6.48/7.92	(2.6, 9.9)	72.5	.007/.000
.2	0	.24	200	2.5	.300/.500	2.44/1.81	(0.1, 7.8)	82.5	.000/.107
.2	0	.24	200	5.0	.295/.525	3.80/3.12	(1.2, 7.4)	93.0	.000/.114
.2	0	.24	200	7.5	.175/.035	6.21/7.37	(2.1, 8.7)	73.5	.000/.057
.2	-.24	.24	200	2.5	.640/.495	2.38/2.21	(0.1, 5.5)	84.0	.000/.097
.2	-.24	.24	200	5.0	.900/.715	5.04/4.96	(2.4, 7.7)	**98.5**	-.047/.032
.2	-.24	.24	200	7.5	.585/.565	7.58/7.89	(4.3, 9.8)	85.0	-.095/.000
.4	0	0	1000	2.5	.066/.079	3.16/1.75	(1.2, 8.6)	73.4	.000/.000
.4	0	0	1000	5.0	.070/.198	5.11/5.49	(1.2, 8.6)	90.4	.000/.000
.4	0	0	1000	7.5	.075/.**056**	6.77/8.20	(1.3, 8.9)	72.7	.000/.000
.4	.24	0	1000	2.5	.138/.112	2.93/1.64	(0.9, 9.1)	79.1	.000/.000
.4	.24	0	1000	5.0	.379/.200	4.73/4.59	(2.4, 8.7)	90.1	.000/.000
.4	.24	0	200	7.5	.445/.025	6.59/7.84	(2.4, 10.0)	77.0	.010/-.009
.4	0	.24	200	2.5	.315/.495	2.94/2.07	(0.4, 8.2)	74.5	.000/.112
.4	0	.24	200	5.0	.305/.580	3.78/3.13	(1.2, 7.8)	88.0	.000/.143
.4	0	.24	200	7.5	.110/.070	6.22/7.53	(1.3, 8.9)	75.0	.000/.063
.4	-.24	.24	200	2.5	.570/.625	2.70/2.39	(0.3, 6.0)	82.0	.000/.101
.4	-.24	.24	200	5.0	.810/.745	4.92/5.08	(2.3, 7.9)	**97.0**	-.043/.033
.4	-.24	.24	200	7.5	.455/.540	7.48/7.86	(3.8, 10.0)	83.5	-.109/.000

Figure 4.3: Linear and Normally Distributed with 25% Tied Comparisons

ρ	β_1	β_2	Monte Carlo Size	Location of Change-point	Power (tau hyp. test/seas. Kendall)	Estimate Change-point (mean/ median)	Median 95% CI	Coverage %	Median Seas. Kendall Est. of β_1/β_2
0	0	0	200	2.5	**.060/.005**	2.93/1.65	(0.7, 8.5)	74.0	.006/.000
0	0	0	200	5.0	**.060/.025**	5.06/4.93	(0.7, 9.0)	**93.5**	.002/.000
0	0	0	200	7.5	**.065/.025**	7.10/8.30	(0.9, 9.4)	72.0	.000/.000
0	.24	0	200	2.5	.130/.000	3.25/1.98	(1.0, 9.2)	74.5	.117/.000
0	.24	0	200	5.0	.505/.060	6.03/6.63	(3.0, 9.2)	90.0	.146/-.011
0	.24	0	200	7.5	.645/.010	7.65/8.42	(4.0, 10.4)	74.0	.164/-.042
0	0	.24	200	2.5	.425/.095	2.20/1.64	(-0.3, 6.8)	75.0	.000/.222
0	0	.24	200	5.0	.460/.120	3.65/3.19	(0.6, 6.9)	91.0	.000/.221
0	0	.24	200	7.5	.100/.015	6.85/7.96	(1.1, 8.9)	71.5	.000/.228
0	-.24	.24	200	2.5	.765/.065	2.10/1.84	(-0.7, 5.4)	84.5	-.103/.206
0	-.24	.24	200	5.0	.965/.340	5.09/5.05	(2.4, 7.5)	**98.5**	-.193/.192
0	-.24	.24	200	7.5	.815/.105	7.81/7.96	(4.6, 10.4)	86.5	-.210/.083
.2	0	0	1000	2.5	.094/**.012**	3.16/1.77	(1.2, 7.9)	70.9	-.010/.000
.2	0	0	1000	5.0	.091/**.027**	5.06/5.34	(1.3, 8.2)	87.7	.000/.000
.2	0	0	1000	7.5	.091/**.009**	6.98/8.30	(1.5, 8.3)	68.6	.000/.000
.2	.24	0	1000	2.5	.102/.006	3.27/1.85	(1.3, 8.5)	72.4	.002/.000
.2	.24	0	1000	5.0	.150/.035	5.49/5.61	(2.0, 8.6)	87.6	.016/.000
.2	.24	0	1000	7.5	.150/.015	6.94/8.24	(2.0, 9.0)	69.7	.025/-.002
.2	0	.24	1000	2.5	.195/.026	2.78/1.77	(0.8, 8.0)	74.7	.000/.031
.2	0	.24	1000	5.0	.196/.061	4.52/3.79	(1.2, 7.8)	87.9	.000/.028
.2	0	.24	1000	7.5	.107/.008	6.65/8.03	(1.7, 8.7)	72.1	.000/.031
.2	-.24	.24	200	2.5	.385/.040	2.64/2.10	(0.5, 6.8)	74.5	-.030/.028
.2	-.24	.24	200	5.0	.450/.115	5.20/5.21	(2.1, 7.7)	**94.0**	-.035/.021
.2	-.24	.24	200	7.5	.290/.015	7.56/8.10	(2.7, 9.3)	78.5	-.035/.002
.4	0	0	1000	2.5	.077/**.018**	3.28/1.91	(0.9, 8.5)	72.9	-.018/.000
.4	0	0	1000	5.0	.088/**.029**	4.98/4.96	(1.3, 8.4)	89.1	.000/.000
.4	0	0	1000	7.5	.089/**.009**	6.79/8.13	(1.4, 8.8)	71.7	.000/.000
.4	.24	0	1000	2.5	.063/.010	3.25/1.86	(1.2, 8.3)	72.3	.007/.000
.4	.24	0	1000	5.0	.138/.058	5.26/5.84	(1.9, 8.9)	89.2	.026/.000
.4	.24	0	1000	7.5	.130/.025	6.75/8.11	(1.9, 9.2)	71.1	.037/.000
.4	0	.24	1000	2.5	.213/.037	2.77/1.79	(0.8, 8.0)	72.7	.000/.049
.4	0	.24	1000	5.0	.181/.069	4.51/3.79	(1.2, 8.2)	90.1	.000/.051
.4	0	.24	1000	7.5	.099/.022	6.48/7.73	(1.6, 8.7)	72.1	.000/.041
.4	-.24	.24	200	2.5	.290/.045	2.41/1.78	(0.1, 7.3)	74.0	-.039/.047
.4	-.24	.24	200	5.0	.475/.220	4.76/4.85	(1.7, 8.4)	**94.5**	-.044/.043
.4	-.24	.24	200	7.5	.305/.060	7.38/7.94	(2.8, 9.1)	73.5	-.057/.016

Figure 4.4: Linear and Gamma Distributed with 5% Tied Comparisons

ρ	β_1	β_2	Monte Carlo Size	Location of Change-point	Power (tau hyp. test/seas. Kendall)	Estimate Change-point (mean/ median)	Median 95% CI	Coverage %	Median Seas. Kendall Est. of β_1/β_2
0	0	0	200	2.5	.075/**.025**	2.79/1.67	(1.0, 9.4)	73.0	.000/.000
0	0	0	200	5.0	**.045**/.135	4.93/4.61	(0.2, 9.1)	87.0	.000/.000
0	0	0	200	7.5	**.050**/.065	7.08/8.30	(1.7, 9.2)	75.0	.000/.000
0	.24	0	200	2.5	.055/.065	2.65/1.56	(1.4, 8.7)	77.5	.000/.000
0	.24	0	200	5.0	.310/.165	5.56/6.33	(2.9, 9.5)	90.5	.000/.000
0	.24	0	200	7.5	.610/.020	7.01/8.27	(3.1, 10.6)	72.0	.016/-.015
0	0	.24	200	2.5	.325/.430	2.05/1.65	(0.1, 7.3)	80.0	.000/.108
0	0	.24	200	5.0	.420/.490	3.71/3.03	(0.7, 7.0)	91.5	.000/.127
0	0	.24	200	7.5	.105/.030	6.39/7.46	(1.5, 9.0)	78.5	.000/.053
0	-.24	.24	200	2.5	.575/.395	2.19/1.90	(0.2, 5.6)	83.0	.000/.101
0	-.24	.24	200	5.0	.910/.690	4.94/4.91	(2.4, 7.6)	**98.5**	-.047/.048
0	-.24	.24	200	7.5	.530/.375	7.82/8.10	(4.2, 10.0)	80.0	-.096/.000
.2	0	0	1000	2.5	.096/.065	3.24/1.84	(1.1, 8.2)	75.7	.000/.000
.2	0	0	1000	5.0	.102/.221	5.03/5.30	(1.6, 8.2)	89.4	.000/.000
.2	0	0	1000	7.5	.099/.074	6.71/8.08	(1.7, 8.9)	74.9	.000/.000
.2	.24	0	1000	2.5	.087/.091	3.02/1.75	(1.2, 8.6)	75.8	.000/.000
.2	.24	0	1000	5.0	.147/.221	4.74/4.18	(2.0, 8.9)	90.3	.000/.000
.2	.24	0	1000	7.5	.172/.030	6.43/7.79	(2.3, 9.3)	73.7	.000/.000
.2	0	.24	1000	2.5	.188/.187	2.48/1.71	(0.8, 8.4)	77.6	.000/.000
.2	0	.24	1000	5.0	.172/.310	4.32/3.51	(1.2, 7.8)	90.2	.000/.000
.2	0	.24	1000	7.5	.122/.059	6.60/7.84	(1.6, 8.7)	72.1	.000/.000
.2	-.24	.24	200	2.5	.265/.150	2.67/1.97	(0.7, 7.4)	79.5	.000/.000
.2	-.24	.24	200	5.0	.405/.405	4.84/4.94	(1.6, 7.8)	90.5	.000/.000
.2	-.24	.24	200	7.5	.295/.260	7.40/7.91	(2.6, 9.1)	78.0	-.004/.000
.4	0	0	1000	2.5	.074/.065	3.38/1.91	(1.1, 8.5)	73.1	.000/.000
.4	0	0	200	5.0	.105/.175	4.96/5.11	(2.1, 9.1)	91.0	.000/.000
.4	0	0	1000	7.5	.070/.074	6.78/8.20	(1.4, 9.0)	73.0	.000/.000
.4	.24	0	1000	2.5	.090/.086	3.00/1.75	(0.8, 8.8)	74.6	.000/.000
.4	.24	0	1000	5.0	.138/.239	4.78/4.34	(1.9, 8.9)	88.1	.000/.000
.4	.24	0	1000	7.5	.139/.038	6.24/7.83	(1.8, 9.1)	70.1	.000/.000
.4	0	.24	200	2.5	.170/.270	2.63/1.82	(0.3, 7.7)	83.0	.000/.009
.4	0	.24	200	5.0	.155/.375	4.33/3.55	(1.0, 8.3)	86.5	.000/.006
.4	0	.24	200	7.5	.105/.050	5.83/6.58	(1.2, 9.0)	74.0	.000/.000
.4	-.24	.24	200	2.5	.260/.250	2.61/1.88	(0.1, 7.5)	82.0	.000/.005
.4	-.24	.24	200	5.0	.420/.490	4.84/4.93	(1.8, 8.2)	**95.0**	.000/.000
.4	-.24	.24	200	7.5	.285/.320	7.39/8.01	(1.8, 8.9)	71.0	-.010/.000

Figure 4.5: Linear and Gamma Distributed with 25% Tied Comparisons

ρ	β_1	β_2	Monte Carlo Size	Location of Change-point	Power (tau hyp. test/seas. Kendall)	Estimate Change-point (mean/ median)	Median 95% CI	Coverage %	Median Seas. Kendall Estimates
0	.18	0	1000	2.5	.111/.018	4.05/3.77	(2.5, 8.5)	51.0	.000/.000
0	.18	0	1000	5.0	.557/.054	6.38/7.16	(4.9, 11.0)	52.4	.117/-.004
0	.18	0	1000	7.5	.863/.015	6.97/8.47	(-5.0, 7.7)	58.1	.264/.001
0	0	.18	1000	2.5	.558/.373	1.94/1.56	(2.1, 12.4)	64.3	.000/.355
0	0	.18	1000	5.0	.471/.242	3.20/2.68	(-1.2, 5.2)	56.2	.000/.284
0	0	.18	1000	7.5	.108/.019	5.12/5.05	(1.4, 7.3)	41.2	.000/.208
0	-.18	.18	1000	2.5	.750/.417	1.80/1.61	(2.1, 13.1)	61.0	.000/.355
0	-.18	.18	1000	5.0	.791/.340	3.08/2.90	(-0.4, 10.6)	61.1	-.022/.279
0	-.18	.18	1000	7.5	.339/.015	5.38/5.74	(2.4, 7.4)	44.4	-.062/.208
.2	.18	0	1000	2.5	.111/.018	4.05/3.78	(2.5, 8.5)	51.0	.000/.001
.2	.18	0	1000	5.0	.574/.050	6.16/7.00	(4.6, 10.6)	58.7	.105/.009
.2	.18	0	1000	7.5	.805/.029	6.84/8.26	(-3.4, 7.8)	58.5	.264/-.032
.2	0	.18	200	2.5	.480/.535	2.10/1.77	(2.0, 12.2)	64.5	.000/.360
.2	0	.18	200	5.0	.505/.375	3.41/2.85	(-0.1, 5.6)	60.5	.000/.294
.2	0	.18	200	7.5	.215/.020	5.09/4.80	(1.4, 7.3)	39.0	.000/.300
.2	-.18	.18	200	2.5	.630/.540	2.08/1.77	(1.9, 12.1)	69.0	.000/.361
.2	-.18	.18	200	5.0	.760/.415	3.15/2.99	(0.3, 5.3)	60.0	-.019/.294
.2	-.18	.18	200	7.5	.395/.020	5.52/5.74	(2.4, 7.3)	39.0	-.052/.238
.4	.18	0	1000	2.5	.156/.036	3.70/3.25	(2.2, 8.6)	58.9	.000/.000
.4	.18	0	1000	5.0	.463/.042	6.11/6.96	(4.2, 10.3)	68.7	.112/.017
.4	.18	0	1000	7.5	.634/.032	6.86/8.26	(-2.4, 7.6)	50.2	.269/.028
.4	0	.18	200	2.5	.365/.560	1.98/1.58	(1.6, 11.3)	73.5	.000/.369
.4	0	.18	200	5.0	.425/.345	3.67/3.04	(0.3, 6.1)	75.0	.000/.290
.4	0	.18	200	7.5	.180/.055	5.36/5.41	(1.7, 7.6)	54.5	.000/.263
.4	-.18	.18	200	2.5	.535/.540	2.11/1.70	(2.0, 11.6)	63.0	.000/.363
.4	-.18	.18	200	5.0	.715/.450	3.27/3.06	(0.7, 5.9)	74.5	-.015/.295
.4	-.18	.18	200	7.5	.330/.040	5.29/5.70	(2.4, 7.5)	53.5	-.052/.244

Figure 4.6: Exponential and Normally Distributed with 5% Tied Comparisons

ρ	β_1	β_2	Monte Carlo Size	Location of Change-point	Power (tau hyp. test/seas. Kendall)	Estimate Change-point (mean/ median)	Median 95% CI	Coverage %	Median Seas. Kendall Estimates
0	.18	0	1000	2.5	.171/.152	2.74/1.82	(1.7, 9.3)	68.8	.000/.000
0	.18	0	1000	5.0	.542/.107	5.73/6.50	(4.1, 10.0)	70.0	.000/.000
0	.18	0	200	7.5	.935/.010	6.56/8.22	(-1.9, 7.9)	64.5	.015/-.001
0	0	.18	200	2.5	.575/.600	2.13/1.88	(1.2, 11.0)	88.0	.000/.248
0	0	.18	200	5.0	.255/.540	3.49/2.72	(0.0, 5.9)	63.0	.000/.222
0	0	.18	200	7.5	.075/.020	4.66/4.15	(1.2, 7.3)	38.5	.000/.173
0	-.18	.18	200	2.5	.550/.685	2.10/1.67	(1.1, 11.3)	85.0	.000/.238
0	-.18	.18	200	5.0	.695/.635	3.15/2.94	(0.1, 7.8)	82.5	.000/.206
0	-.18	.18	200	7.5	.345/.015	4.83/5.08	(1.9, 7.4)	46.0	.000/.216
.2	.18	0	1000	2.5	.315/.156	2.65/2.08	(1.2, 10.0)	67.1	.000/.000
.2	.18	0	200	5.0	.450/.110	5.66/6.15	(4.4, 8.8)	69.0	.000/.000
.2	.18	0	200	7.5	.925/.025	6.92/8.17	(-0.2, 8.3)	70.5	.010/-.006
.2	0	.18	200	2.5	.760/.340	2.17/1.25	(1.4, 10.8)	84.0	.000/.230
.2	0	.18	200	5.0	.270/.600	3.79/3.08	(0.5, 6.4)	68.0	.000/.238
.2	0	.18	200	7.5	.160/.020	4.84/4.22	(1.1, 7.3)	45.0	.000/.186
.2	-.18	.18	200	2.5	.755/.485	2.13/1.36	(1.4, 11.4)	85.5	.000/.248
.2	-.18	.18	200	5.0	.560/.665	3.32/3.15	(0.3, 5.6)	69.5	.000/.235
.2	-.18	.18	200	7.5	.435/.035	4.85/5.05	(2.3, 7.3)	41.0	.000/.162
.4	.18	0	1000	2.5	.305/.121	3.02/2.58	(1.9, 10.0)	58.3	.000/.000
.4	.18	0	200	5.0	.520/.115	6.04/6.32	(4.4, 8.8)	72.0	.000/.007
.4	.18	0	200	7.5	.725/.010	7.01/8.37	(0.1, 8.1)	59.5	.023/.079
.4	0	.18	200	2.5	.840/.215	1.97/1.09	(1.6, 11.5)	85.0	.000/.256
.4	0	.18	200	5.0	.255/.610	4.10/3.52	(0.7, 6.4)	72.5	.000/.250
.4	0	.18	200	7.5	.125/.070	4.70/4.27	(1.5, 7.7)	57.0	.000/.271
.4	-.18	.18	200	2.5	.845/.350	2.11/1.30	(1.6, 11.1)	82.5	.000/.257
.4	-.18	.18	200	5.0	.460/.640	3.37/3.22	(0.5, 5.8)	74.5	.000/.241
.4	-.18	.18	200	7.5	.370/.075	5.01/5.14	(2.1, 7.5)	51.0	.000/.225

Figure 4.7: Exponential and Normally Distributed with 25% Tied Comparisons

ρ	β_1	β_2	Monte Carlo Size	Location of Change-point	Power (tau hyp. test/seas. Kendall)	Estimate Change-point (mean/ median)	Median 95% CI	Coverage %	Median Seas. Kendall Estimates
0	.18	0	200	2.5	.135/.015	4.02/3.81	(2.4, 8.3)	52.0	.000/.002
0	.18	0	200	5.0	.650/.055	6.17/6.91	(4.9, 10.5)	52.5	.113/-.008
0	.18	0	200	7.5	.890/.035	7.16/8.41	(-7.0, 7.7)	55.5	.276/-.019
0	0	.18	200	2.5	.470/.460	2.08/1.58	(1.8, 11.8)	70.5	.000/.365
0	0	.18	200	5.0	.595/.345	2.98/2.59	(-1.8, 5.1)	52.0	.000/.302
0	0	.18	200	7.5	.155/.020	5.18/4.96	(1.5, 7.3)	37.5	.000/.236
0	-.18	.18	200	2.5	.730/.520	1.81/1.68	(2.0, 12.4)	74.5	.000/.364
0	-.18	.18	200	5.0	.905/.480	3.02/2.88	(-0.2, 5.1)	53.5	-.033/.303
0	-.18	.18	200	7.5	.440/.030	5.43/5.77	(2.6, 7.2)	35.0	-.060/.219
.2	.18	0	1000	2.5	.124/.008	3.33/1.96	(1.9, 8.7)	67.0	.000/.000
.2	.18	0	1000	5.0	.245/.055	5.75/6.73	(3.2, 9.7)	81.8	.016/.000
.2	.18	0	1000	7.5	.318/.019	6.73/8.18	(0.6, 8.1)	62.4	.042/-.001
.2	0	.18	1000	2.5	.236/.111	2.53/1.60	(1.9, 10.3)	67.3	.000/.058
.2	0	.18	1000	5.0	.233/.112	3.88/3.03	(0.6, 6.5)	75.6	.000/.046
.2	0	.18	1000	7.5	.129/.009	5.37/5.08	(1.4, 7.5)	50.8	.000/.032
.2	-.18	.18	1000	2.5	.382/.151	2.29/1.64	(2.0, 10.4)	64.6	.000/.056
.2	-.18	.18	200	5.0	.505/.165	3.37/3.07	(0.5, 6.2)	73.5	-.008/.044
.2	-.18	.18	200	7.5	.265/.000	5.74/5.80	(2.2, 7.7)	54.5	-.007/.036
.4	.18	0	1000	2.5	.082/.017	3.31/1.92	(1.8, 8.9)	71.1	.000/.001
.4	.18	0	1000	5.0	.212/.061	5.49/6.41	(3.3, 9.6)	82.5	.019/.006
.4	.18	0	1000	7.5	.273/.029	6.36/7.98	(0.1, 7.9)	57.0	.062/.000
.4	0	.18	200	2.5	.240/.305	2.28/1.65	(2.1, 10.7)	60.5	.000/.093
.4	0	.18	200	5.0	.195/.180	4.10/3.24	(0.7, 6.7)	80.0	.000/.070
.4	0	.18	200	7.5	.145/.020	5.04/4.92	(1.6, 7.8)	58.5	.000/.063
.4	-.18	.18	200	2.5	.315/.215	2.33/1.69	(2.1, 10.0)	61.0	.000/.086
.4	-.18	.18	200	5.0	.475/.175	3.69/3.29	(0.5, 6.3)	77.0	-.009/.078
.4	-.18	.18	200	7.5	.265/.035	5.69/5.95	(2.2, 7.9)	68.5	-.013/.054

Figure 4.8: Exponential and Gamma Distributed with 5% Tied Comparisons

ρ	β_1	β_2	Monte Carlo Size	Location of Change-point	Power (tau hyp. test/seas. Kendall)	Estimate Change-point (mean/ median)	Median 95% CI	Coverage %	Median Seas. Kendall Estimates
0	.18	0	200	2.5	.145/.145	2.59/1.80	(1.6, 9.2)	75.0	.000/.000
0	.18	0	200	5.0	.335/.130	5.68/6.66	(4.0, 10.5)	66.5	.000/.002
0	.18	0	200	7.5	.915/.010	6.50/8.15	(-1.8, 7.6)	54.0	.003/-.069
0	0	.18	200	2.5	.450/.615	1.94/1.63	(0.8, 10.7)	88.0	.000/.239
0	0	.18	200	5.0	.285/.590	3.40/2.91	(0.0, 5.9)	68.0	.000/.246
0	0	.18	200	7.5	.110/.020	4.46/3.73	(1.2, 7.2)	38.0	.000/.176
0	-.18	.18	200	2.5	.415/.715	1.93/1.74	(0.5, 10.8)	88.0	.000/.229
0	-.18	.18	200	5.0	.610/.650	3.17/2.92	(-0.3, 5.5)	65.0	.000/.217
0	-.18	.18	200	7.5	.420/.025	5.13/5.44	(2.4, 7.3)	39.0	.000/.202
.2	.18	0	1000	2.5	.168/.139	2.60/1.68	(1.4, 9.5)	76.4	.000/.000
.2	.18	0	1000	5.0	.251/.223	5.30/6.04	(2.9, 9.2)	82.6	.000/.000
.2	.18	0	1000	7.5	.358/.026	6.01/7.73	(1.3, 8.2)	61.9	.000/.000
.2	0	.18	200	2.5	.315/.530	2.39/1.71	(0.9, 10.0)	81.5	.000/.023
.2	0	.18	200	5.0	.130/.355	4.10/3.37	(0.8, 6.5)	80.5	.000/.032
.2	0	.18	200	7.5	.135/.030	5.01/4.04	(1.4, 7.8)	59.5	.000/.054
.2	-.18	.18	200	2.5	.315/.530	2.29/1.73	(1.0, 10.0)	77.0	.000/.026
.2	-.18	.18	200	5.0	.380/.465	3.78/3.19	(0.5, 6.5)	78.5	.000/.030
.2	-.18	.18	200	7.5	.270/.045	5.36/5.41	(1.9, 7.9)	62.0	.000/.018
.4	.18	0	1000	2.5	.162/.164	2.83/1.81	(1.3, 9.7)	74.4	.000/.000
.4	.18	0	1000	5.0	.174/.213	5.02/5.46	(2.9, 9.2)	82.3	.000/.000
.4	.18	0	200	7.5	.260/.020	6.01/7.88	(0.1, 8.0)	55.5	.000/.005
.4	0	.18	200	2.5	.345/.530	2.56/1.96	(0.3, 10.0)	76.5	.000/.045
.4	0	.18	200	5.0	.125/.380	4.09/3.34	(0.2, 6.4)	80.5	.000/.051
.4	0	.18	200	7.5	.065/.015	4.96/4.04	(1.2, 7.5)	51.0	.000/.046
.4	-.18	.18	200	2.5	.380/.570	2.54/2.04	(1.4, 10.0)	73.5	.000/.046
.4	-.18	.18	200	5.0	.345/.410	3.54/2.94	(0.4, 6.7)	85.0	.000/040.
.4	-.18	.18	200	7.5	.205/.035	5.24/5.29	(1.5, 8.0)	66.5	.000/.019

Figure 4.9: Exponential and Gamma Distributed with 25% Tied Comparisons

4.7 Summary of Analysis Procedure

After obtaining a water quality time series, the procedure for testing for a change in trend at a given point in time is outlined here.

1. If flow data is available, a flow adjustment should be performed so that variability explained by flow can be accounted for. Flow adjustment can be accomplished without making any model assumptions and in a robust manner by using LOWESS.

2. To estimate modified Kendall's tau for both halves of the series, the data need to first be deseasonalized. Section 5.2 gives a method based on LOWESS for accomplishing this goal. After the time series is deseasonalized, it is split in two by the hypothesized change-point, and modified Kendall's tau is estimated for both halves.

3. To perform the change in trend test, the variances of the estimates of the two modified Kendall's tau correlation coefficients are needed. The effective sample size bootstrap (which is the standard bootstrap under independence) is recommended here for this purpose. To use the effective sample size bootstrap to estimate the variance, the effective sample size first needs to be estimated. Equation 3.27 gives the recomended formula for an AR(1) dependence structure. Chapter 5 deals with issues in estimating the parameter ρ needed to perform this calculation and recommends the use of R-estimation for non-normal data. Hence, R-estimation is used with the entire detrended

and deseasonalized time series to estimate the serial correlation parameters and these estimates are then used to estimate the effective sample size for each half of the time series.

4. Finally, the hypothesis test is carried out using the z statistic of Equation 4.1.

4.8 Example

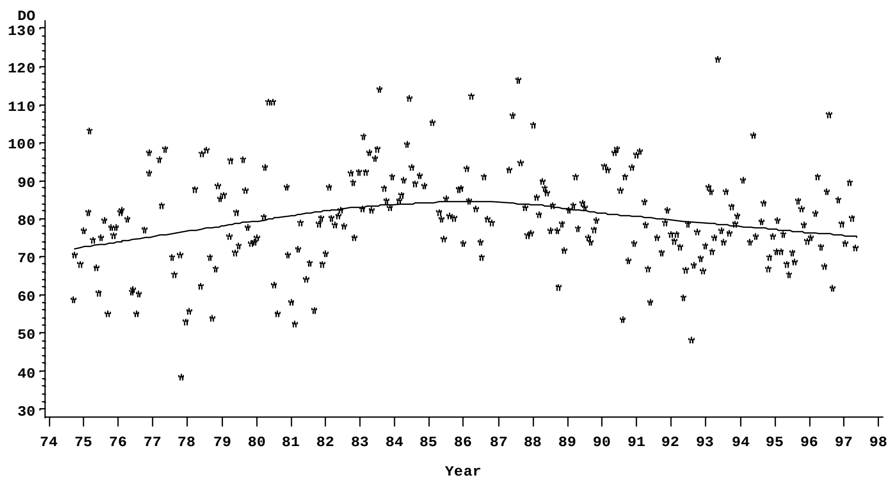

Figure 4.10: DO by Time with LOWESS fit (d=2/3)

Figure 4.10 displays a time series of 233 dissolved oxygen levels collected at a Virginia Department of Environmental Quality monitoring station in Arlington, Virginia on Fourmile Run Creek from 1974 to 1997. Fourmile Run Creek flows through Arlington and empties into the Potomac River. The station where this time series was taken is 0.19 miles from the mouth of the creek on the Potomac near National Airport. When the data are analyzed as a whole using the covariance sum test [*Hirsch and Slack* 1984] with 12 seasons, a p-value of .841 is obtained, indicating no significant overall trend. However, the appearance of the data set suggests an upper trend followed by a downward trend. It is suspected that a change in trend may have occurred in the 1980's when steady significant population growth may have added more pollutants to the creek after significant population declines throughout the 1970's. Using the change-point estimation method from above yields June 25, 1984 as the estimated change-point. This date splits the time series into 101 observations in the first half and 132 in the second half, and the resulting estimates of τ_{mod} are 0.275 for the first half and -0.224 for the second half.

Water quality time series have a strong seasonal component that ought to be taken into account to reduce variability. Hence, before estimating τ_{mod} for each half, the data were first deseasonalized by subtracting estimates of the seasonal effect from each observation. This was accomplished by the method described in the following chapter.

Another issue commonly encountered in water quality data, detection limited values, did not have to be dealt with here because this particular time series contained no censoring. If there were detection limited values, they could be handled when estimating τ_{mod} by assigning ties to some of the pairwise comparisons which involve one or two detection limited values. The same algorithm would be applied when estimating τ_{mod} repetitively in the bootstrap.

To perform the change in trend test using the effective sample size bootstrap, the serial correlation parameter, ρ, assuming an AR(1) structure, must first be estimated. This can often be a tricky issue with typical water quality time series and was also accomplished by a method detailed in the following Chapter. For this data set, the estimate of ρ turned out to be $\hat{\rho} = .307$ for observations one month apart, the average interval between observations in this data set. It should be noted that the correlation parameter needs to be scaled to the average difference in time between consecutive observations for use in estimating the effective sample size and performing the hypothesis test.

When the change in trend test was performed using this ρ value and 100 bootstrap resamples were used to estimate the variance of $\hat{\tau}_{\text{mod}}$, a p-value of .00030 was obtained. Hence, it can be concluded that the trend before June 25th,1984 is different from the trend after June 25th, 1984. It turns out that for this particular data set, the test is somewhat robust to the serial correlation coefficient used. With the effective sample size bootstrap Method 2 with 100 resamples, for input serial correlation levels of .2, .3, .4 and .5, p-values of .000009, .000013, .00082, and .0035 were obtained respectively.

CHAPTER 5

Estimating Autoregressive Parameters

5.1 Introduction

Water-quality time series are non-normal and unequally spaced. The unequal spacing is necessary because potential polluters cannot know when samples are going to be taken from the water. These aspects present a problem when estimating autoregressive parameters since traditional methods rely on equal spacing as well as normality.

If the distribution is known, an exact likelihood can be obtained using a Kalman filter, and then quasi-Newton methods can be used to obtain maximum likelihood estimates [*Jones* 1985]. This method does not work in the water-quality context when there is no known distribution. Another method which has been suggested by *Rosner* [1988] for use with medical data is simply to fit a nonlinear lag model to the data. Another method is to model the data as a diffential equation derived from a Brownian motion assuming a continuous time AR(1) process [*Jones* 1993]. This can be accomplished using Proc Mixed in SAS. In this chapter, Rosner's method, the method detailed in *Jones* 1993, and the use of R-estimation will all be explored for use with water-quality type data.

5.2 Methods

Before fitting a lag model the data must be detrended and deseasonalized. This can be accomplished by first regressing the water-quality variable on time using LOWESS to detrend the data. The residuals from this fit would then be deseasonalized before estimating the correlation parameter. From trial and error of different percentages, d, of the data to include in the LOWESS estimation window with both real and simulated data, a large percentage was found to be needed with each local linear regression. $D = .67$ seemed to work well. This provides a less flexible fit that allows serial correlation patterns to go unmasked. With larger order of fit p and/or smaller d, runs of observations above or below the true level which indicate positive serial correlation are fit more closely by the LOWESS curve. If flow data were also available, a flow adjustment could be performed with another LOWESS fit prior to estimation of the correlation parameter.

Once the data have been detrended, the following method employing LOWESS once again could be used to deseasonalize the data. First, the adjusted data from each year are overlayed on top of each other to form a dense time series spanning one year in time by ignoring the year part of the date when plotting the variable values as a function of time. A LOWESS fit with $d = 1/6$ is then going to be used to estimate the seasonal effect at each original data point, but before the estimation is done more points are added to attempt to make the estimated curve cyclic. In other words, it is suspected that seasonal effects vary over time in a manner something akin to the shape of a sinusoid with the effect at the end of the year continuing smoothing into the start of the next. To ensure that the estimated curve has this type of behavior, all the data points from the last season of the year are duplicated and placed at the start of the series. Likewise, all points from the first season of the year are duplicated and placed at the end of the series. If, for example, months are used as seasons, this would involve creating a new time series with all December observations in order of what day of the month the observations were taken on at the beginning, then all the observations from January, then February, and so on until the end when all the observations from January are tacked on. The LOWESS curve then estimates the seasonal effect at each point and the data are deseasonalized by subtracting the fit from that particular point in the year. This method seems to mimic the true nature of seasonal effects better than simply subtracting seasonal means or medians. In the remainder of this Section, it is assumed that the data have been deseasonalized and detrended.

Rosner's method and the R-estimation based method explored here estimate the autoregressive parameters by estimating the parameters in the nonlinear model

$$y_i = \rho^{s(x_i - x_{i-1})} y_{i-1} + \varepsilon_i \tag{5.1}$$

where y is the water-quality variable, x is time with 1 unit equal to 1 year, and s is a scaling parameter put into the model so that the estimate of ρ will be the estimate of the serial correlation of observations $\frac{12}{s}$ months apart. Here, s is set equal to 12 so that the correlation of observations one month apart is estimated. The estimate of ρ from this model is then like the estimate of the AR(1) parameter.

To extend this method to be similar to AR(p), simply add more lags to the model. The trick to this model is that the errors are not i.i.d. In fact, the variance of the errors depend on the time lag from the previous point. *Rosner* 1988 gives the variance of these errors under normality and uses them to weight the estimation process. Without ε in the above model, the predicted values are given by the model for the method detailed in *Jones* [1993].

To estimate the parameter ρ in this model, three methods will be considered: quasi-Newton in Proc NLIN in SAS, REML in Proc Mixed in SAS, and R-estimation. For non-normal data, it is hoped that R-estimation, which is based on linear differences instead of squared differences, will improve upon the performance of fitting the model in the classical way using quasi-Newton methods or assuming normality and using restricted maximum likelihood even though the i.i.d. error assumption is violated because the errors have different variances. Due to the nonlinear form of the model, with R-estimation a grid search was used in all the work done here to find the value of ρ which optimizes the dispersion function. This is because the convexity result detailed in *Hettmansperger* [1984] requires a linear model, and the method of steepest ascent, for example, is not guaranteed to converge. Non-convex dispersion functions have in fact been encountered by the author in actual water quality data sets proving that the dispersion function is not necessarily convex.

Figure 5.1 displays the results of a simulation study which compared R-estimation with the quasi-Newton method used by Proc NLIN in SAS and with the REML estimation performed in Proc Mixed. The SAS code for performing REML estimation with Proc Mixed for this case is given in Appendix H. Both normal and gamma distributions were considered and 3000 time series of 100 observations were generated from each case. The data were generated as unequally spaced by generating equally spaced time series of size 125, then randomly choosing 20% of the observations to remove. As can be seen in the table, the results were varied. With the most skewed distribution, gamma with r=2, R-estimation performed the best in terms of MSE. However, REML provided the most unbiased estimator in these cases as it did in all the cases. With gamma r=3 time series, REML performed the best in terms of MSE for larger values of ρ. R-estimation performed the best for ρ near 0 with the gamma r=3 time series. With normal data, R-estimation had the lowest MSE for $\rho = 0$, REML had the lowest MSE for $\rho = .2$, and when Quasi-Newton converged, it had the lowest MSE for the $\rho = .3$ and $\rho = .4$ cases. Quasi-Newton failed to converge a large percentage of the time and can not be recommended as a general procedure. The choice between the other two methods would seem to depend on the particular data set on hand. It would seem that R-estimation is better for time series with larger degrees of skewness and estimating near-zero AR(1)-like parameters, while REML is better for less skewed data that possess moderate to larger degrees of serial dependence.

5.3 Application to Real Data

The R-estimation method detailed above for estimating serial correlation was applied to water quality time series from USGS stations which had instantaneous

	Method	R-Estimation		Cont. AR(1) Fit using Proc Mixed		Non-linear Fit using Proc IML		
Distibution	AR(1) Param-eter	Median/ Mean Estimate	MSE x100 of Est.	Median/ Mean Estimate	MSE x100 of Est.	Median/ Mean Estimate	MSE x100 of Est.	Number Converged
Gamma r=2	0.0	-.008/-.006	.666	-.005/-.003	1.254	*	*	*
Gamma r=2	0.2	.155/.157	.965	.202/.201	1.228	.235/.203	1.879	1186
Gamma r=2	0.3	.251/.251	1.058	.296/.294	1.096	.350/.341	1.450	867
Gamma r=2	0.4	.371/.362	.900	.394/.393	1.027	.459/.468	1.051	883
Gamma r=3	0.0	-.006/-.003	.698	-.004/-.002	1.281	*	*	*
Gamma r=3	0.2	.135/.140	1.217	.197/.197	1.165	.223/.183	2.028	996
Gamma r=3	0.3	.223/.229	1.516	.296/.293	1.009	.344/.326	1.639	813
Gamma r=3	0.4	.334/.335	1.604	.397/.392	.948	.460/.465	1.200	846
Normal	0.0	-.009/-.007	1.225	-.001/.000	1.293	*	*	*
Normal	0.2	.185/.184	1.248	.198/.196	1.176	.244/.227	1.452	765
Normal	0.3	.284/.278	1.112	.298/.292	1.027	.342/.342	.869	651
Normal	0.4	.378/.377	1.083	.394/.390	.975	.448/.450	.645	700

* Can't use Quasi Newton here because of derivatives=0 (causes singular Hessian matrix)
Based on 3000 Monte Carlo replicates

Figure 5.1: Comparison of Estimation Methods

flow values as well as measurements of several water quality parameters. The purpose of including flow data was to examine the effect of flow adjustment on serial correlation. The effect of deseasonalization was also studied. Since seasonal effects appear to have a serially correlated pattern, if water quality time series are not deseasonalized when estimating the AR(1) parameter, it is quite possible that the estimate will be inflated.

Assuming that the seasons used as blocks are fixed when performing the seasonal Kendall analysis, looking at the estimates of ρ obtained from the deseasonalized water quality time series gives an indication of the severity of the dependence assuming a continuous time AR(1) structure. If blocks are assumed to be random, one would want to look at the estimates from the not deseasonalized series to make this decision. However, since every possible block is being used, i.e., every season or every month of the year, the assumption of fixed effects blocks seems reasonable.

Rank von Neumann's ratio tests for randomness [*Bartels* 1982] were also performed in the case studies to look for dependence. Hence, highly significant rank von Neumann's ratio test p-values with small estimated ρ parameters indicate that the non-convex dispersion functions may be occasionally causing zero estimates in cases where there is significant first order autoregressive serial correlation. The case studies indicate that serial dependence is prevalent in many water quality variables.

CHAPTER 6

Future Research Opportunities

More work is needed on estimation of the change-point for the change in trend problem, especially in developing confidence intervals. With better computing power, the bootstrap confidence interval method can be fully explored in the Monte Carlo environment.

Estimating nonlinear parameters using R-estimation also needs more research. There is potential here to outperform least squares based estimation in many situations. Asymptotic properties and tests need to be developed. Testing can quite possibly be performed with a drop in dispersion test similar to testing parameters estimated using R-estimation in linear models. The non-convexity of the dispersion function makes this a more difficult problem though. Perhaps the use of different score functions might help force convexity. Additionally, generalized R-estimation might provide a better method of fitting non-linear models with non-homogeneous errors.

The effective sample size bootstrap shows a lot of promise for estimating variances of complex statistics in dependent data situations. Asymptotic properties need to be derived for the effective sample size bootstrap for both modified Kendall's tau and for general statistics.

Further research lies in dealing with complications that arise when trying to answer other questions in the water quality arena not tackled here. Examples include the comparison of water quality at different sites. This problem could possibly be addressed by pairing observations from the different stations by time and taking differences to eliminate seasonal and possibly covariate effects. Then the differences could be used with a sign or Wilcoxon test to see if values are generally higher at one of the two stations. This method then needs to be extended for use in a multivariate manner.

Another problem is how to deal with detection limits. Perhaps simulation can be used to increase the robustness of such methods as randomly generating values for the detection limited values or considering all possible rankings for the detection limited values. The seasonal Kendall estimator is biased by the existence of detection limited values if the detection limit is simply substituted as the observation value. This was shown clearly by the study in Chapter 4. A better method needs to be developed. Perhaps work done in survival analysis can be applied here.

A. Derivation of Var($\hat{\tau}_{\mathrm{mod}}$) - Modifying Noether's derivation

The variance formula for $\hat{\tau}_{\mathrm{mod}}$ based on modifying the derivation given in Noether [1967] is detailed here. Independence is required.

Note: $\hat{\tau}_{\mathrm{mod}} = \frac{p_c - p_d}{p_c + p_d}$ and $\tau_{\mathrm{mod}} = \frac{\pi_c - \pi_d}{\pi_c + \pi_d}$ as defined in Chapter 3

$\mathrm{Var}(\hat{\tau}_{\mathrm{mod}}) = \mathrm{Var}(\hat{\tau}_{\mathrm{mod}} - \tau_{\mathrm{mod}}) = \mathrm{Var}(\sqrt{n}(\hat{\tau}_{\mathrm{mod}} - \tau_{\mathrm{mod}}))/n$

So, $\mathrm{Var}(\hat{\tau}_{\mathrm{mod}}) = \mathrm{Var}(P')/n$ where $P' = \sqrt{n}(\hat{\tau}_{\mathrm{mod}} - \tau_{\mathrm{mod}})$

Now, $P' = \sqrt{n}(\frac{p_c - p_d}{p_c + p_d} - \frac{\pi_c - \pi_d}{\pi_c + \pi_d}) = \frac{2\sqrt{n}(\pi_d p_c - \pi_c p_d)}{(p_c + p_d)(\pi_c + \pi_d)}$

which is asymptotically equivalent to $\frac{2\sqrt{n}(\pi_d p_c - \pi_c p_d)}{(\pi_c + \pi_d)^2} = P$

So, $\mathrm{Var}(\hat{\tau}_{\mathrm{mod}}) \cong \mathrm{Var}(P)/n$

This is why the derived formula is only approximate. A further degree of approximation will occur later when consistent estimates are substituted in for their population values at the end.

$\mathrm{Var}(P) = \frac{4n}{(\pi_c + \pi_d)^4}[\pi_d^2 \mathrm{Var}(p_c) - 2\pi_c \pi_d \mathrm{Cov}(p_c, p_d) + \pi_c^2 \mathrm{Var}(p_d)]$

$p_c = \frac{2}{n(n-1)} \sum_{i<j} w_{ij}$ where $w_{ij} = 1$ if (x_i,y_i) and (x_j,y_j) are concordant, 0 otherwise.

$\mathrm{Var}(p_c) = \frac{4}{n^2(n-1)^2}[\frac{n(n-1)}{2}\mathrm{Var}(w_{12}) + n(n-1)(n-2)\mathrm{Cov}(w_{12}, w_{13})]$

$= \frac{2}{n(n-1)}[\mathrm{Var}(w_{12}) + 2(n-2)\mathrm{Cov}(w_{12}, w_{13})]$

because $n(n-1)(n-2) = \binom{n}{3} \times 6 =$ (# ways to select three numbers) $\times$ (#ways to place the three numbers under the restriction $i < j$), and the other covariances are 0 under the independence assumption.

Now, $\mathrm{Var}(w_{12}) = \mathrm{E}(w_{12}^2) - [\mathrm{E}(w_{12})]^2$

$= (1 \times \pi_c + 0 \times (1 - \pi_c)) - (1 \times \pi_c + 0 \times (1 - \pi_c))^2 = \pi_c - \pi_c^2,$

and $\mathrm{Cov}(w_{12}, w_{13})$=$\mathrm{E}(w_{12} \times w_{13})$−$\mathrm{E}(w_{12})\mathrm{E}(w_{13})$

$=1\times\pi_{cc} + 0 - \pi_c^2 = \pi_{cc} - \pi_c^2$.

So, $\mathrm{Var}(p_c)=\frac{2}{n(n-1)}[\pi_c(1-\pi_c) + 2(n-2)(\pi_{cc} - \pi_c^2)]$

Similarly, $\mathrm{Var}(p_d)=\frac{2}{n(n-1)}[\pi_d(1-\pi_d) + 2(n-2)(\pi_{dd} - \pi_d^2)]$,

and $\mathrm{Var}(p_t)=\frac{2}{n(n-1)}[\pi_t(1-\pi_t) + 2(n-2)(\pi_{tt} - \pi_t^2)]$ where t denotes tied.

Now, $p_c + p_d + p_t = 1$ so, $\mathrm{Var}(p_c + p_d)$=$\mathrm{Var}(1 - p_t)$

$\Rightarrow\mathrm{Var}(p_c)+\mathrm{Var}(p_d)+2\mathrm{Cov}(p_c,p_d)=\mathrm{Var}(p_t)$

$\Rightarrow\mathrm{Cov}(p_c,p_d)=\frac{1}{2}[\mathrm{Var}(p_t)-\mathrm{Var}(p_c)-\mathrm{Var}(p_d)]$

Hence, $\mathrm{Var}(\hat{\tau}_{\mathrm{mod}})\cong \frac{4}{(\pi_c+\pi_d)^4}[\pi_d^2\mathrm{Var}(p_c) - 2\pi_c\pi_d\mathrm{Cov}(p_c, p_d) + \pi_c^2\mathrm{Var}(p_d)]$

$=\frac{4}{(\pi_c+\pi_d)^4}\{\pi_d^2\mathrm{Var}(p_c) - 2\pi_c\pi_d\frac{1}{2}[\mathrm{Var}(p_t)-\mathrm{Var}(p_c)-\mathrm{Var}(p_d)] + \pi_c^2\mathrm{Var}(p_d)\}$

$=\frac{4}{(\pi_c+\pi_d)^4}\{\pi_d(\pi_c + \pi_d)\mathrm{Var}(p_c) + \pi_c(\pi_c + \pi_d)\mathrm{Var}(p_d) - \pi_c\pi_d\mathrm{Var}(p_t)\}$

$=\frac{4}{(\pi_c+\pi_d)^3}\{\pi_d\mathrm{Var}(p_c) + \pi_c\mathrm{Var}(p_d) - \frac{\pi_c\pi_d}{\pi_c+\pi_d}\mathrm{Var}(p_t)\}$

$=\frac{4}{(\pi_c+\pi_d)^3}\{\pi_d\frac{2}{n(n-1)}[\pi_c(1-\pi_c) + 2(n-2)(\pi_{cc} - \pi_c^2)]+$

$\pi_c\frac{2}{n(n-1)}[\pi_d(1-\pi_d) + 2(n-2)(\pi_{dd} - \pi_d^2)]-$

$\frac{\pi_c\pi_d}{\pi_c+\pi_d}\frac{2}{n(n-1)}[\pi_t(1-\pi_t) + 2(n-2)(\pi_{tt} - \pi_t^2)]\}$

$=\frac{8}{(\pi_c+\pi_d)^3 n(n-1)}\{\pi_c\pi_d - \pi_c^2\pi_d + \pi_c\pi_d - \pi_c\pi_d^2 - \frac{\pi_c\pi_d\pi_t(1-\pi_t)}{\pi_c+\pi_d} + 2(n-2)\times$

$[\pi_d\pi_{cc} - \pi_c^2\pi_d + \pi_c\pi_{dd} - \pi_c\pi_d^2 - \frac{\pi_c\pi_d}{\pi_c+\pi_d}\pi_{tt} + \frac{\pi_c\pi_d}{\pi_c+\pi_d}\pi_t^2]\}$

$=\frac{8}{(\pi_c+\pi_d)^3 n(n-1)}\{\pi_c\pi_d(1 - \pi_c + 1 - \pi_d - \pi_t) + 2(n-2)\frac{1}{\pi_c+\pi_d}\times$

$[\pi_d\pi_{cc}(\pi_c + \pi_d) - \pi_c^2\pi_d(\pi_c + \pi_d) + \pi_c\pi_{dd}(\pi_c + \pi_d) - \pi_c\pi_d^2(\pi_c + \pi_d)-$

$\pi_c\pi_d\pi_{tt} + \pi_c\pi_d\pi_t^2]\}$

$=\frac{8}{(\pi_c+\pi_d)^3 n(n-1)}\{\pi_c\pi_d + 2(n-2)\frac{1}{\pi_c+\pi_d} \times [\pi_c\pi_d\pi_{cc} + \pi_d^2\pi_{cc} - \pi_c^3\pi_d - \pi_c^2\pi_d^2+$

$\pi_c^2\pi_{dd} + \pi_c\pi_d\pi_{dd} - \pi_c^2\pi_d^2 - \pi_c\pi_d^3 - \pi_c\pi_d\pi_{tt} + \pi_c\pi_d\pi_t^2]\}$

$=\frac{8}{(\pi_c+\pi_d)^4 n(n-1)}\{\pi_c\pi_d(\pi_c+\pi_d)+2(n-2)\times[\pi_c\pi_d\pi_{cc}+\pi_d^2\pi_{cc}-\pi_c^3\pi_d-$

$\pi_c^2\pi_d^2+\pi_c^2\pi_{dd}+\pi_c\pi_d\pi_{dd}-\pi_c^2\pi_d^2-\pi_c\pi_d^3-\pi_c\pi_d\pi_{tt}+\pi_c\pi_d(1-\pi_c-\pi_d)^2]\}$

$=\frac{8}{(\pi_c+\pi_d)^4 n(n-1)}\{\pi_c\pi_d(\pi_c+\pi_d)+2(n-2)\times[\pi_c\pi_d\pi_{cc}+\pi_d^2\pi_{cc}-\pi_c^3\pi_d-$

$\pi_c^2\pi_d^2+\pi_c^2\pi_{dd}+\pi_c\pi_d\pi_{dd}-\pi_c^2\pi_d^2-\pi_c\pi_d^3-\pi_c\pi_d\pi_{tt}+$

$\pi_c\pi_d(1+\pi_c^2+\pi_d^2-2\pi_c-2\pi_d+2\pi_c\pi_d)]\}$

$=\frac{8}{(\pi_c+\pi_d)^4 n(n-1)}\{\pi_c\pi_d(\pi_c+\pi_d)+2(n-2)\times[\pi_c\pi_d\pi_{cc}+\pi_d^2\pi_{cc}-\pi_c^3\pi_d-$

$\pi_c^2\pi_d^2+\pi_c^2\pi_{dd}+\pi_c\pi_d\pi_{dd}-\pi_c^2\pi_d^2-\pi_c\pi_d^3-\pi_c\pi_d\pi_{tt}+$

$\pi_c\pi_d+\pi_c^3\pi_d+\pi_c\pi_d^3-2\pi_c^2\pi_d-2\pi_c\pi_d^2+2\pi_c^2\pi_d^2]\}$

$=\frac{8}{(\pi_c+\pi_d)^4 n(n-1)}\{\pi_c\pi_d(\pi_c+\pi_d)+2(n-2)\times[\pi_c\pi_d\pi_{cc}+\pi_d^2\pi_{cc}+$

$\pi_c^2\pi_{dd}+\pi_c\pi_d\pi_{dd}-\pi_c\pi_d\pi_{tt}+\pi_c\pi_d-2\pi_c^2\pi_d-2\pi_c\pi_d^2]\}$

$=\frac{8}{(\pi_c+\pi_d)^4 n(n-1)}\{\pi_c\pi_d(\pi_c+\pi_d)+2(n-2)\times[\pi_d^2\pi_{cc}+\pi_c^2\pi_{dd}+$

$\pi_c\pi_d(\pi_{cc}+\pi_{dd}-\pi_{tt}+1-2\pi_c-2\pi_d)]\}$

$=\frac{8}{(\pi_c+\pi_d)^4 n(n-1)}\{\pi_c\pi_d(\pi_c+\pi_d)+2(n-2)\times[\pi_d^2\pi_{cc}+\pi_c^2\pi_{dd}+$

$\pi_c\pi_d(\pi_{cc}+\pi_{dd}-\pi_{tt}+1-2\pi_{cc}-2\pi_{cd}-2\pi_{ct}-2\pi_{dd}-2\pi_{cd}-2\pi_{dt})]\}$

$=\frac{8}{(\pi_c+\pi_d)^4 n(n-1)}\{\pi_c\pi_d(\pi_c+\pi_d)+2(n-2)\times[\pi_d^2\pi_{cc}+\pi_c^2\pi_{dd}+$

$\pi_c\pi_d(1-\pi_{cc}-\pi_{cd}-\pi_{ct}-\pi_{dd}-\pi_{cd}-\pi_{dt}-\pi_{tt}-\pi_{ct}-\pi_{dt}-2\pi_{cd})]\}$

$=\frac{8}{(\pi_c+\pi_d)^4 n(n-1)}\{\pi_c\pi_d(\pi_c+\pi_d)+2(n-2)\times[\pi_d^2\pi_{cc}+\pi_c^2\pi_{dd}+$

$\pi_c\pi_d(1-\pi_c-\pi_d-\pi_t-2\pi_{cd})]\}$

$=\frac{8}{(\pi_c+\pi_d)^4 n(n-1)}\{\pi_c\pi_d(\pi_c+\pi_d)+2(n-2)[\pi_d^2\pi_{cc}-2\pi_c\pi_d\pi_{cd}+\pi_c^2\pi_{dd}]\}$

So finally,

$\widehat{Var}(\hat{\tau}_{\text{mod}})=\frac{8}{n(n-1)(p_c+p_d)^4}\times\{(p_c+p_d)p_cp_d+2(n-2)(p_d^2p_{cc}-2p_cp_dp_{cd}+p_c^2p_{dd})\}.$

B. Derivation of Var($\hat{\tau}_{\text{mod}}$) - Delta method

The variance formula for $\hat{\tau}_{\text{mod}}$ derived by use of a first order Taylor series expansion is detailed here. Independence is required.

Let $X = p_c - p_d$ and $Y = p_c + p_d$.

Then E(X)= $\pi_c - \pi_d$ and E(Y)= $\pi_c + \pi_d = B$.

Let E(X)= A and E(Y)= B.

Notice $\hat{\tau}_{\text{mod}} = \frac{X}{Y}$ so let f(x, y)=$\frac{x}{y}$.

Now a first order Taylor Series expansion gives us

f$(x, y) \approx$ f(A, B)+$\frac{\partial f(x,y)}{\partial x}|_{(A,B)}(x - A) + \frac{\partial f(x,y)}{\partial y}|_{(A,B)}(y - B)$

=f(A, B)+$\frac{1}{B}(x - A) - \frac{A}{B^2}(y - B)$

$\Rightarrow$ (f(x, y)−f$(A, B))^2 \approx \frac{1}{B^2}(x - A)^2 - \frac{2A}{B^3}(x - A)(y - B) + \frac{A^2}{B^4}(y - B)^2$

$\Rightarrow$Var($\hat{\tau}_{\text{mod}}$)$\approx$E$[\frac{1}{B^2}(x - A)^2 - \frac{2A}{B^3}(x - A)(y - B) + \frac{A^2}{B^4}(y - B)^2]$

=$\frac{1}{B^2}$E$[(x - A)^2]$−$\frac{2A}{B^3}$E$[(x - A)(y - B)]$+$\frac{A^2}{B^4}$E$[(y - B)^2]$

=$\frac{1}{B^2}$Var(X)−$\frac{2A}{B^3}$Cov(X, Y) + $\frac{A^2}{B^4}$Var(Y).

Var(X) =Var($p_c - p_d$) =Var(p_c) − 2Cov(p_c, p_d)+Var(p_d).

Var(Y) =Var($p_c + p_d$) =Var(p_c) + 2Cov(p_c, p_d)+Var(p_d).

Cov(X, Y)=Cov($p_c - p_d, p_c + p_d$) =E$[(p_c - p_d)(p_c + p_d)]$-E$[p_c - p_d]$E$[p_c + p_d]$

=E(p_c^2)-2E($p_c p_d$)-E(p_d^2)-($\pi_c^2 - \pi_d^2$)

=Var(p_c)-Var(p_d)-2Cov(p_c, p_d)−$2\pi_c\pi_d$.

Hence, Var($\hat{\tau}_{\text{mod}}$)$\approx$ $\frac{1}{B^2}$[Var(p_c) − 2Cov(p_c, p_d)+Var(p_d)]−

$\frac{2A}{B^3}$[Var(p_c)-Var(p_d)-2Cov(p_c, p_d)−$2\pi_c\pi_d$]+

$\frac{A^2}{B^4}$[Var(p_c) + 2Cov(p_c, p_d)+Var(p_d)].

Now, $\text{Cov}(p_c, p_d) = \frac{1}{2}[\text{Var}(p_t) - \text{Var}(p_c) - \text{Var}(p_d)]$ from Appendix A,

so, $\text{Var}(\hat{\tau}_{\text{mod}}) \approx \frac{1}{B^2}[2\text{Var}(p_c) - \text{Var}(p_t) + 2\text{Var}(p_d)] -$

$\frac{2A}{B^3}[2\text{Var}(p_c)\text{-Var}(p_t) - 2\pi_c\pi_d] + \frac{A^2}{B^4}[\text{Var}(p_t)]$

$= [\frac{2}{B^2} - \frac{4A}{B^3}]\text{Var}(p_c) + [\frac{2}{B^2}]\text{Var}(p_d) + [\frac{2A}{B^3} - \frac{1}{B^2} + \frac{A^2}{B^4}]\text{Var}(p_t) + \frac{4A\pi_c\pi_d}{B^3}$

$= [\frac{2B-4A}{B^3}]\text{Var}(p_c) + [\frac{2}{B^2}]\text{Var}(p_d) + [\frac{2AB-B^2+A^2}{B^4}]\text{Var}(p_t) + \frac{4A\pi_c\pi_d}{B^3}$

$= \frac{1}{B^4}\{[2B(B-2A)]\text{Var}(p_c) + [2B^2]\text{Var}(p_d) + [2AB - B^2 + A^2]\text{Var}(p_t) + 4AB\pi_c\pi_d\}$

$= \frac{1}{(\pi_c+\pi_d)^4}\{[2(\pi_c+\pi_d)(\pi_c+\pi_d-2(\pi_c-\pi_d))]\text{Var}(p_c) + [2(\pi_c+\pi_d)^2]\text{Var}(p_d) +$

$[2(\pi_c-\pi_d)(\pi_c+\pi_d) - (\pi_c+\pi_d)^2 + (\pi_c-\pi_d)^2]\text{Var}(p_t) + 4(\pi_c-\pi_d)(\pi_c+\pi_d)\pi_c\pi_d\}$

$= \frac{1}{(\pi_c+\pi_d)^4}\{[2(\pi_c+\pi_d)(3\pi_d-\pi_c)]\text{Var}(p_c) + [2(\pi_c^2+2\pi_c\pi_d+\pi_d^2)]\text{Var}(p_d) +$

$[2\pi_c^2 - 2\pi_d^2 - \pi_c^2 - 2\pi_c\pi_d - \pi_d^2 + \pi_c^2 - 2\pi_c\pi_d + \pi_d^2]\text{Var}(p_t) + 4(\pi_c^2-\pi_d^2)\pi_c\pi_d\}$

$= \frac{2}{(\pi_c+\pi_d)^4}\{[3\pi_d^2 + 2\pi_c\pi_d - \pi_c^2]\text{Var}(p_c) + [\pi_c^2 + 2\pi_c\pi_d + \pi_d^2]\text{Var}(p_d) +$

$[\pi_c^2 - 2\pi_c\pi_d - \pi_d^2]\text{Var}(p_t) + 2(\pi_c^3\pi_d - \pi_c\pi_d^3)\}$.

Now substituting in the variances which were derived in Appendix A,

$\text{Var}(\hat{\tau}_{\text{mod}}) \approx \frac{4}{n(n-1)(\pi_c+\pi_d)^4}\{[3\pi_d^2 + 2\pi_c\pi_d - \pi_c^2][\pi_c(1-\pi_c) + 2(n-2)(\pi_{cc} - \pi_c^2)] +$

$[\pi_c^2 + 2\pi_c\pi_d + \pi_d^2][\pi_d(1-\pi_d) + 2(n-2)(\pi_{dd} - \pi_d^2)] +$

$[\pi_c^2 - 2\pi_c\pi_d - \pi_d^2][\pi_t(1-\pi_t) + 2(n-2)(\pi_{tt} - \pi_t^2)] + 2(\pi_c^3\pi_d - \pi_c\pi_d^3)\}$

$= \frac{4}{n(n-1)(\pi_c+\pi_d)^4}\{[3\pi_c\pi_d^2 + 2\pi_c^2\pi_d - \pi_c^3 - 3\pi_c^2\pi_d^2 - 2\pi_c^3\pi_d + \pi_c^4 + \pi_c^2\pi_d + 2\pi_c\pi_d^2 +$

$\pi_d^3 - \pi_c^2\pi_d^2 - 2\pi_c\pi_d^3 - \pi_d^4 + \pi_c^2\pi_t - \pi_d^2\pi_t - 2\pi_c\pi_d\pi_t - \pi_c^2\pi_t^2 + \pi_d^2\pi_t^2 + 2\pi_c\pi_d\pi_t^2 +$

$2\pi_c^3\pi_d - 2\pi_c\pi_d^3] + 2(n-2)(\pi_c+\pi_d)[\pi_{cc}(3\pi_d-\pi_c) - \pi_c^2(3\pi_d-\pi_c) +$

$\pi_{dd}(\pi_c+\pi_d) - \pi_d^2(\pi_c+\pi_d) + (1-\pi_c-\pi_d-\pi_{ct}-\pi_{dt})(\pi_c - \pi_d - \frac{2\pi_c\pi_d}{\pi_c+\pi_d}) -$

$(1-\pi_c-\pi_d)^2(\pi_c - \pi_d - \frac{2\pi_c\pi_d}{\pi_c+\pi_d})]\}$

$= \frac{4}{n(n-1)(\pi_c+\pi_d)^4}\{[5\pi_c\pi_d^2 + 3\pi_c^2\pi_d - \pi_c^3 - 4\pi_c^2\pi_d^2 + \pi_c^4 + \pi_d^3 - 4\pi_c\pi_d^3 - \pi_d^4 +$

$\pi_c^2(1-\pi_c-\pi_d)-\pi_d^2(1-\pi_c-\pi_d)-2\pi_c\pi_d(1-\pi_c-\pi_d)-$

$\pi_c^2(1+\pi_c^2+\pi_d^2-2\pi_c-2\pi_d+2\pi_c\pi_d)+\pi_d^2(1+\pi_c^2+\pi_d^2-2\pi_c-2\pi_d+2\pi_c\pi_d)+$

$2\pi_c\pi_d(1+\pi_c^2+\pi_d^2-2\pi_c-2\pi_d+2\pi_c\pi_d)]+$

$2(n-2)(\pi_c+\pi_d)[3\pi_d\pi_{cc}-\pi_c\pi_{cc}-3\pi_c^2\pi_d+\pi_c^3+\pi_c\pi_{dd}+\pi_d\pi_{dd}-$

$\pi_c\pi_d^2-\pi_d^3+\pi_c(1-\pi_c-\pi_d-\pi_{ct}-\pi_{dt})-\pi_d(1-\pi_c-\pi_d-\pi_{ct}-\pi_{dt})-$

$\frac{2\pi_c\pi_d}{\pi_c+\pi_d}(1-\pi_c-\pi_d-\pi_{ct}-\pi_{dt})-\pi_c(1+\pi_c^2+\pi_d^2-2\pi_c-2\pi_d+2\pi_c\pi_d)+$

$\pi_d(1+\pi_c^2+\pi_d^2-2\pi_c-2\pi_d+2\pi_c\pi_d)+\frac{2\pi_c\pi_d}{\pi_c+\pi_d}(1+\pi_c^2+\pi_d^2-2\pi_c-2\pi_d+2\pi_c\pi_d)]\}$

$=\frac{4}{n(n-1)(\pi_c+\pi_d)^4}\{[5\pi_c\pi_d^2+3\pi_c^2\pi_d-\pi_c^3-4\pi_c^2\pi_d^2+\pi_c^4+\pi_d^3-4\pi_c\pi_d^3-\pi_d^4+$

$\pi_c^2-\pi_c^3-\pi_c^2\pi_d-\pi_d^2+\pi_c\pi_d^2+\pi_d^3-2\pi_c\pi_d+2\pi_c^2\pi_d+2\pi_c\pi_d^2-\pi_c^2-$

$\pi_c^4-\pi_c^2\pi_d^2+2\pi_c^3+2\pi_c^2\pi_d-2\pi_c^3\pi_d+\pi_d^2+\pi_c^2\pi_d^2+\pi_d^4-2\pi_c\pi_d^2-2\pi_d^3+$

$2\pi_c\pi_d^3+2\pi_c\pi_d+2\pi_c^3\pi_d+2\pi_c\pi_d^3-4\pi_c^2\pi_d-4\pi_c\pi_d^2+4\pi_c^2\pi_d^2]+$

$2(n-2)(\pi_c+\pi_d)[3\pi_d\pi_{cc}-\pi_c\pi_{cc}-3\pi_c^2\pi_d+\pi_c^3+\pi_c\pi_{dd}+\pi_d\pi_{dd}-$

$\pi_c\pi_d^2-\pi_d^3+\pi_c-\pi_c^2-\pi_c\pi_d-\pi_c\pi_{ct}-\pi_c\pi_{dt}-\pi_d+\pi_c\pi_d+\pi_d^2+\pi_d\pi_{ct}+$

$\pi_d\pi_{dt}-\pi_c-\pi_c^3-\pi_c\pi_d^2+2\pi_c^2+2\pi_c\pi_d-2\pi_c^2\pi_d+\pi_d+\pi_c^2\pi_d+$

$\pi_d^3-2\pi_c\pi_d-2\pi_d^2+2\pi_c\pi_d^2+\frac{2\pi_c\pi_d}{\pi_c+\pi_d}(1+\pi_c^2+\pi_d^2-2\pi_c-2\pi_d+2\pi_c\pi_d$

$-1+\pi_{ct}+\pi_{dt}+\pi_c+\pi_d)]\}$

$=\frac{4}{n(n-1)(\pi_c+\pi_d)^4}\{[2\pi_c\pi_d^2+2\pi_c^2\pi_d]+$

$2(n-2)(\pi_c+\pi_d)[3\pi_d\pi_{cc}-\pi_c\pi_{cc}-4\pi_c^2\pi_d+\pi_c\pi_{dd}+\pi_d\pi_{dd}+$

$\pi_c^2-\pi_c\pi_{ct}-\pi_c\pi_{dt}+\pi_d\pi_{ct}+\pi_d\pi_{dt}+-\pi_d^2+$

$\frac{2\pi_c\pi_d}{\pi_c+\pi_d}(\pi_c^2+\pi_d^2-\pi_c-\pi_d+2\pi_c\pi_d+\pi_{ct}+\pi_{dt})]\}$

$=\frac{8}{n(n-1)(\pi_c+\pi_d)^4}\{\pi_c\pi_d(\pi_d+\pi_c)+(n-2)(\pi_c+\pi_d)\times$

$[2\pi_d\pi_{cc}-\pi_c\pi_{cc}+\pi_d(\pi_{cc}+\pi_{ct})+\pi_c(\pi_{dd}-\pi_{dt})+\pi_d(\pi_{dd}+\pi_{dt})+$

$\pi_c(\pi_c - \pi_{ct}) - \pi_d^2 - 4\pi_c^2\pi_d +$

$\frac{2\pi_c\pi_d}{\pi_c+\pi_d}(\pi_c^2 + \pi_d^2 - \pi_c - \pi_d + 2\pi_c\pi_d + \pi_{ct} + \pi_{dt})]\}$

$= \frac{8}{n(n-1)(\pi_c+\pi_d)^4}\{\pi_c\pi_d(\pi_d + \pi_c) + (n-2)(\pi_c + \pi_d)\times$

$[2\pi_d\pi_{cc} - \pi_c\pi_{cc} + \pi_d(\pi_c - \pi_{cd}) + \pi_c(\pi_{cd} - \pi_d + 2\pi_{dd}) + \pi_d(\pi_d - \pi_{cd}) +$

$\pi_c(\pi_{cc} + \pi_{cd}) - \pi_d^2 - 4\pi_c^2\pi_d +$

$\frac{2\pi_c\pi_d}{\pi_c+\pi_d}(\pi_c^2 + \pi_d^2 - \pi_c - \pi_d + 2\pi_c\pi_d + \pi_{ct} + \pi_{dt})]\}$

$= \frac{8}{n(n-1)(\pi_c+\pi_d)^4}\{\pi_c\pi_d(\pi_d + \pi_c) + (n-2)(\pi_c + \pi_d)\times$

$[2\pi_d\pi_{cc} - 2\pi_d\pi_{cd} + 2\pi_c\pi_{cd} + 2\pi_c\pi_{dd} - 4\pi_c^2\pi_d +$

$\frac{2\pi_c\pi_d}{\pi_c+\pi_d}(\pi_c^2 + \pi_d^2 - \pi_c - \pi_d + 2\pi_c\pi_d + \pi_{ct} + \pi_{dt})]\}$

$= \frac{8}{n(n-1)(\pi_c+\pi_d)^4}\{\pi_c\pi_d(\pi_d + \pi_c) + (n-2)\times$

$[(\pi_c + \pi_d)(2\pi_d\pi_{cc} - 2\pi_d\pi_{cd} + 2\pi_c\pi_{cd} + 2\pi_c\pi_{dd} - 4\pi_c^2\pi_d) +$

$2\pi_c\pi_d(\pi_c^2 + \pi_d^2 - \pi_c - \pi_d + 2\pi_c\pi_d + \pi_{ct} + \pi_{dt})]\}$

$= \frac{8}{n(n-1)(\pi_c+\pi_d)^4}\{\pi_c\pi_d(\pi_d + \pi_c) + (n-2)\times$

$[2\pi_c\pi_d\pi_{cc} - 2\pi_c\pi_d\pi_{cd} + 2\pi_c^2\pi_{cd} + 2\pi_c^2\pi_{dd} - 4\pi_c^3\pi_d + 2\pi_d^2\pi_{cc} -$

$2\pi_d^2\pi_{cd} + 2\pi_c\pi_d\pi_{cd} + 2\pi_c\pi_d\pi_{dd} - 4\pi_c^2\pi_d^2 + 2\pi_c^3\pi_d + 2\pi_c\pi_d^3 -$

$2\pi_c^2\pi_d - 2\pi_c\pi_d^2 + 4\pi_c^2\pi_d^2 + 2\pi_c\pi_d\pi_{ct} + 2\pi_c\pi_d\pi_{dt}]\}$

$= \frac{8}{n(n-1)(\pi_c+\pi_d)^4}\{\pi_c\pi_d(\pi_d + \pi_c) + 2(n-2)\times$

$[\pi_c\pi_d\pi_{cc} + \pi_c^2\pi_{cd} + \pi_c^2\pi_{dd} - \pi_c^3\pi_d + \pi_d^2\pi_{cc} - \pi_d^2\pi_{cd} + \pi_c\pi_d\pi_{dd} +$

$\pi_c\pi_d^3 - \pi_c^2\pi_d - \pi_c\pi_d^2 + \pi_c\pi_d\pi_{ct} + \pi_c\pi_d\pi_{dt}]\}$

$= \frac{8}{n(n-1)(\pi_c+\pi_d)^4}\{\pi_c\pi_d(\pi_d + \pi_c) + 2(n-2)\times$

$[\pi_c^2(\pi_{dd} - \pi_{cd}) + \pi_d^2(\pi_{cc} - \pi_{cd}) + \pi_c\pi_d(\pi_{cc} + \pi_{ct} + \pi_{dd} + \pi_{dt}) +$

$\pi_c\pi_d^3 - \pi_c^2\pi_d - \pi_c\pi_d^2 - \pi_c^3\pi_d]\}$

$=\frac{8}{n(n-1)(\pi_c+\pi_d)^4}\{\pi_c\pi_d(\pi_d+\pi_c)+2(n-2)\times$

$[\pi_c^2(\pi_{dd}-\pi_{cd})+\pi_d^2(\pi_{cc}-\pi_{cd})+\pi_c\pi_d(\pi_c-2\pi_{cd}+\pi_d)+$

$\pi_c\pi_d^3-\pi_c^2\pi_d-\pi_c\pi_d^2-\pi_c^3\pi_d]\}$

$=\frac{8}{n(n-1)(\pi_c+\pi_d)^4}\{\pi_c\pi_d(\pi_d+\pi_c)+2(n-2)\times$

$[\pi_c^2(\pi_{dd}-\pi_{cd})+\pi_d^2(\pi_{cc}-\pi_{cd})+\pi_c\pi_d(\pi_d^2-\pi_c^2-2\pi_{cd})]\}$

Hence, $\widehat{Var}(\hat{\tau}_{\text{mod}})=\frac{8}{n(n-1)(p_c+p_d)^4}\times\{p_cp_d(p_c+p_d)+2(n-2)\times$

$[(p_d^2(p_{cc}-p_{cd})+p_cp_d(p_d^2-p_c^2-2p_{cd})+p_c^2(p_{dd}+p_{cd})]\}$

C. Proof that the p's are Unbiased Estimators of the π's

In this appendix, it is proven that the frequentist based estimators of the probabilities of concordance, discordance, and combinations there of are unbiased.

First, some preliminaries.

Let $C_k = \sum_{i=1,\ i\neq k}^{n} 1_{\{(y_i-y_k)(x_i-x_k)>0\}}$, and similarly let $D_k = \sum_{i=1,\ i\neq k}^{n} 1_{\{(y_i-y_k)(x_i-x_k)<0\}}$.

Then E$[C_k]$=E$[\sum_{i=1,\ i\neq k}^{n} 1_{\{(y_i-y_k)(x_i-x_k)>0\}}]$=$\sum_{i=1,\ i\neq k}^{n}E[1_{\{(y_i-y_k)(x_i-x_k)>0\}}]$=

$\sum_{i=1,\ i\neq k}^{n}$P$\{(y_i - y_k)(x_i - x_k) > 0\} = (n-1) \times \pi_c$.

Similarly, E$[D_k]$=$(n-1) \times \pi_d$.

E$[C_k^2]$=E$[\sum_{i=1,\ i\neq k}^{n} 1_{\{(y_i-y_k)(x_i-x_k)>0\}} \times \sum_{i=1,\ i\neq k}^{n} 1_{\{(y_i-y_k)(x_i-x_k)>0\}}]$=

E$[\sum_{i=1,\ i\neq k}^{n} \sum_{j=1,\ j\neq k}^{n} 1_{\{(y_i-y_k)(x_i-x_k)>0\ \&\ (y_j-y_k)(x_j-x_k)>0\}}]$=

$\sum_{i=1,\ i\neq k}^{n} \sum_{j=1,\ j\neq k}^{n}P\{(y_i - y_k)(x_i - x_k) > 0\ \&\ (y_j - y_k)(x_j - x_k) > 0\}$=

$(n-1)(n-2) \times \pi_{cc} + (n-1) \times \pi_c$.

Similarly, E$[D_k^2]$=$(n-1)(n-2) \times \pi_{dd} + (n-1) \times \pi_d$.

E$[C_k D_k]$=E$[\sum_{i=1,\ i\neq k}^{n} 1_{\{(y_i-y_k)(x_i-x_k)>0\}} \times \sum_{i=1,\ i\neq k}^{n} 1_{\{(y_i-y_k)(x_i-x_k)<0\}}]$=

$\sum_{i=1,\ i\neq k}^{n} \sum_{j=1,\ j\neq k}^{n}P\{(y_i - y_k)(x_i - x_k) > 0\ \&\ (y_j - y_k)(x_j - x_k) < 0\} =$

$(n-1)(n-2) \times \pi_{cd}$.

Now, E$[p_c]$=E$[\frac{1}{n(n-1)} \sum_k C_k]$=$\frac{1}{n-1}\ (n-1) \times \pi_c = \pi_c$,

E$[p_d]$=E$[\frac{1}{n(n-1)} \sum_k D_k]$=$\frac{1}{n-1}\ (n-1) \times \pi_d = \pi_d$,

E$[p_{cc}]$=E$[\frac{1}{n(n-1)(n-2)} \sum_k C_k(C_k - 1)]$=

$\frac{1}{(n-1)(n-2)}[(n-1)(n-2) \times \pi_{cc} + (n-1) \times \pi_c - (n-1) \times \pi_c]$=$\pi_{cc}$,

E$[p_{dd}]$=E$[\frac{1}{n(n-1)(n-2)} \sum_k D_k(D_k - 1)]$=

$\frac{1}{(n-1)(n-2)}[(n-1)(n-2) \times \pi_{dd} + (n-1) \times \pi_d - (n-1) \times \pi_d]$=$\pi_{dd}$, and

$$\mathrm{E}[p_{cd}]=\mathrm{E}[\tfrac{1}{n(n-1)(n-2)}\sum_{k} C_k D_k]=\tfrac{1}{(n-1)(n-2)}(n-1)(n-2)\times\pi_{cd}=\pi_{cd}.$$

D. Derivation of a Confidence Interval for β Which Allows Serial Dependence

A confidence interval for the slope which allows dependent data is derived in this appendix by inverting the covariance sum hypothesis test. The derivation is based on the work of *Sen* [1968] and *Zetterqvist* [1988].

Assume that the data obey the following model: $y_{ik} = \gamma_k + x_{ik} \times \beta + \epsilon_{ik}$ where $i = 1, ..., n$, $k = 1, ..., m$, y is the water-quality variable, x is time, the γ_k represent the intercept and seasonal effects, β is the linear slope, and the ϵ_{ik} ordered by time are a strictly stationary p-dependent process (0≤p≤ $[(m-1)/2]$) with mean zero and a continuous marginal distribution. In other words, observation one year apart need to be independent.

Let N=the number of pairwise slope estimates in the set M, where M=$\{\mathrm{m}_{ijk}\}=$ $\{\frac{y_{jk}-y_{ik}}{x_{jk}-x_{ik}}\}_{0\leq i<j\leq n,\ k=1,...,m}$. Further, following along the same lines as *Sen* [1968], let $\mathrm{U}_n(\mathrm{b})=\frac{\sum_{k=1}^{m}\sum_{1\leq i<j\leq n}\mathrm{sign}\left[\left(\mathrm{x}_{jk}-\mathrm{x}_{ik}\right)\left(\mathrm{z}_{jk(b)}-\mathrm{z}_{ik(b)}\right)\right]}{N}$ where $\mathrm{z}_{ik}(\mathrm{b})=y_{ik}-x_{ik}b$. Note that $\mathrm{U}_n(\mathrm{b})$ is non-increasing in b.

Zetterqvist [1988] showed that β_{SK} from Equation 2.10 is asymtotically normal, that $\mathrm{U}_n(\beta)$ has a distribution that is symmetric about 0, and that $\mathrm{U}_n(\beta)$ has an asymptotic normal distribution.

Since $\mathrm{U}_n(\beta)$ has a distribution that is symmetric about 0, $(\mathrm{U}_n^*, \alpha_n)$ can always be selected such that $\mathrm{P}\{-\mathrm{U}_n^* \leq U_n(\beta) \leq \mathrm{U}_n^* | \beta\}=1-\alpha_n$. Furthermore, since $\mathrm{U}_n(\beta)$ has an asymptotic normal distribution, U_n^* can be approximated by $\hat{\mathrm{U}}_n^* = Z_{1-\frac{\alpha}{2}}\frac{\sqrt{\widehat{Var}(S)}}{N}$ where $\widehat{Var}(S)$ is as given in Equation 2.6. Finally, following the method of hypothesis test inversion to find confidence intervals the bounds of the $(1-\alpha_n)100\%$

confidence interval are, $\hat{\beta}_u^* = Sup\{b : U_n(b) \geq -\hat{U}_n^*\}$ and $\hat{\beta}_l^* = Inf\{b : U_n(b) \leq \hat{U}_n^*\}$.

Now, exact expressions for $\hat{\beta}_u^*$ and $\hat{\beta}_l^*$ will be obtained. Let $m_{(1)}, ..., m_{(N)}$ be the ordered (from smallest to largest) pairwise slope estimates m_{ijk}. Note that $U_n(m_{(r)}) = \frac{r-1-(N-r)}{N} = \frac{2r-N-1}{N}$ because (N-r) of the $[z_{jk}(m_{(r)})-z_{ik}(m_{(r)})]$ will be negative while (r-1) will be positive. (Recall U_n(b) is non-increasing in b.)

Letting $S^* = N \times \hat{U}_n^* = Z_{1-\frac{\alpha}{2}}\sqrt{\widehat{Var}(S)}$, c=$\frac{1}{2}(N - S^*)$, and d=$\frac{1}{2}(N + S^*)$ notice the following. $U_n(m_{(c)}) = \frac{2c-N-1}{N} = \frac{-S^*-1}{N} < -\hat{U}_n^*$, but $U_n(m_{(c)} - \varepsilon) = \frac{-S^*}{N} = -\hat{U}_n^*$ for small enough positive ε. Hence $\hat{\beta}_l^*$ =$m_{(c)}$. Similarly, $U_n(m_{(d+1)}) = \frac{2d+2-N-1}{N} = \frac{S^*+1}{N} > \hat{U}_n^*$, but $U_n(m_{(c)} + \varepsilon) = \frac{S^*}{N} = \hat{U}_n^*$ for small enough positive ε. Hence $\hat{\beta}_u^*$ =$m_{(d+1)}$ and a (1-α_n)100% confidence interval on β is ($m_{(c)}$, $m_{(d+1)}$).

E. Derivation of Var($\hat{\tau}_{\text{mod}}$) - Conditional Formula

The variance formula for $\hat{\tau}_{\text{mod}}$ derived by conditioning on the percentage of tied comparisons is detailed here. Independence is required. Condition on the fact that N_t out of $N = n(n-1)/2$ pairwise comparisons are tied. Hence, the probability of two pairs not being tied is $\frac{N-N_t}{N}$ and $\hat{\tau}_{\text{mod}} = \frac{p_c-p_d}{\frac{N-N_t}{N}} = \frac{N(p_c-p_d)}{N-N_t}$.

Thus, Var($\hat{\tau}_{\text{mod}}$)=$\frac{N^2}{(N-N_t)^2}$Var(p_c−p_d) $= \frac{N^2}{(N-N_t)^2}\times$[Var($p_c$) + Var($p_d$) − 2Cov($p_c$,$p_d$)]. Recall from Appendix A that Var(p_c) $= \frac{2}{n(n-1)} \times [\pi_c(1-\pi_c) + 2(n-2)(\pi_{cc} - \pi_c^2)]$ and Var(p_d) $= \frac{2}{n(n-1)} \times [\pi_d(1-\pi_d) + 2(n-2)(\pi_{dd} - \pi_d^2)]$.

Now, Cov(p_c,p_d)=E[p_cp_d]-E[p_c]E[p_d]. As seen in Appendix C, E[p_c]=π_c and E[p_d]=π_d, but E[p_cp_d] has yet to be derived. E[p_cp_d]=E$\left[\frac{1}{n(n-1)}\sum_{k=1}^{n} C_k \times \frac{1}{n(n-1)}\sum_{k=1}^{n} D_k\right] = \frac{1}{n^2(n-1)^2}\sum_{i=1}^{n}\sum_{j=1}^{n}$E[$C_iD_j$]. There are n expectations of type E[$\mathrm{C}_1\mathrm{D}_1$] and $n(n-1)$ of type E[$\mathrm{C}_1\mathrm{D}_2$]. Hence, E[p_cp_d]=$\frac{1}{n^2(n-1)^2} \times$ (nE[$\mathrm{C}_1\mathrm{D}_1$]+$n(n-1)$E[$\mathrm{C}_1\mathrm{D}_2$]). Now, E[$\mathrm{C}_1\mathrm{D}_1$]=$\sum_{i=2}^{n}\sum_{j=2}^{n}$E[$w_{1i}u_{1j}$] $= (n-1)(n-2)\pi_{cd}$ where $w_{1i} = 1_{\{(y_i-y_1)(x_i-x_1)>0\}}$ and $u_{1j} = 1_{\{(y_j-y_1)(x_j-x_1)<0\}}$, and E[$\mathrm{C}_1\mathrm{D}_2$]=$\sum_{i=2}^{n}\sum_{j=1,j\neq 2}^{n}$E[$w_{1i}u_{2j}$] $= 3(n-2)\pi_{cd} + (n-2)(n-3)\pi_c\pi_d$ since there are $3(n-2)$ways to obtain an expectation of the form E[$w_{12}u_{13}$] and $(n-2)(n-3)$ ways to obtain expectations of the form E[$w_{12}u_{34}$]. Thus, E[p_cp_d]=$\frac{1}{n^2(n-1)^2} \times \{n(n-1)(n-2)\pi_{cd} + n(n-1)[3(n-2)\pi_{cd} + (n-2)(n-3)\pi_c\pi_d]\}= \frac{1}{n(n-1)} \times \{4(n-2)\pi_{cd} + (n-2)(n-3)\pi_c\pi_d\}$, and Cov($p_c$,$p_d$)=$\frac{4(n-2)}{n(n-1)}\pi_{cd} + \frac{(n-2)(n-3)}{n(n-1)}\pi_c\pi_d$-$\pi_c\pi_d = \frac{1}{n(n-1)}\{4(n-2)\pi_{cd}$ - $(n^2-5n+6-n^2+n)\pi_c\pi_d\} = \frac{2}{n(n-1)}[2(n-2)\pi_{cd} - (2n-3)\pi_c\pi_d]$.

Substituting the variances and covariances into the above equation for the variance of $\hat{\tau}_{\text{mod}}$, the following formula is obtained: Var($\hat{\tau}_{\text{mod}}$)=$\frac{N}{(N-N_t)^2} \times \{\pi_c(1-$

$\pi_c) + 2(n-2)(\pi_{cc} - \pi_c^2) + \pi_d(1-\pi_d) + 2(n-2)(\pi_{dd} - \pi_d^2)$ - $4(n-2)\pi_{cd} + 2(2n - 3)\pi_c\pi_d\} = \frac{N}{(N-N_t)^2} \times \{\pi_c(1-\pi_c) + \pi_d(1-\pi_d) + 2(n-2)(\pi_{cc} - \pi_c^2 + \pi_{dd} - \pi_d^2 - 2\pi_{cd})$ $+ 2(2n-3)\pi_c\pi_d\}$. Replace the π's with their unbiased estimates and the above formula yields a consistent estimate of the conditional variance.

F. Summary of the Derivation of Var($\hat{\tau}_{\text{mod}}$) - Based on Second Order Taylor Series Expansion

Some of the details of the derivation of the variance formula for $\hat{\tau}_{\text{mod}}$ derived by use of a second order Taylor series expansion is given here. Independence is required.

Using the same set up as in Appendix B, the derivation based on a fist order Taylor Series expansion, let f(X,Y)=$\frac{X}{Y}$ where X=p$_c$-p$_d$ and Y=p$_c$+p$_d$. Further let E[X]=$\pi_c - \pi_d = A$ and E[Y]=$\pi_c + \pi_d = B$. Now a second order Taylor Series yields the following equation: f(X,Y) $\approx$ f(A,B) + $\frac{\partial f(X,Y)}{\partial X}|_{(A,B)}(X-A)$ + $\frac{\partial f(X,Y)}{\partial Y}|_{(A,B)}(Y-B)$ + $\frac{1}{2}\frac{\partial^2 f(X,Y)}{\partial X^2}|_{(A,B)}(X-A)^2$ + $\frac{1}{2}\frac{\partial^2 f(X,Y)}{\partial X^2}|_{(A,B)}(X-A)^2$ + $\frac{1}{2}\frac{\partial^2 f(X,Y)}{\partial Y^2}|_{(A,B)}(Y-B)^2$ + $\frac{1}{2}\frac{\partial^2 f(X,Y)}{\partial Y \partial X}|_{(A,B)}(Y-B)(X-A)$ + $\frac{1}{2}\frac{\partial^2 f(X,Y)}{\partial Y \partial X}|_{(A,B)}(Y-B)(X-A)$. Substituting in the derivatives gives f(X,Y) $\approx$ f(A,B) + $\frac{1}{B}(X-A)$ - $\frac{A}{B^2}(Y-B)$ - $\frac{1}{B^2}(X-A)(Y-B)$ + $\frac{A}{B^3}(Y-B)^2$.

Hence, (f(X,Y) - f(A,B))^2 $\approx$ $\frac{1}{B^2}\{(X-A)^2$ - $\frac{2A}{B}(X-A)(Y-B)$ - $\frac{2}{B}(X-A)^2(Y-B)$ + $\frac{4A}{B^2}(X-A)(Y-B)^2$ + $\frac{A^2}{B^2}(Y-B)^2$ - $\frac{2A^2}{B^3}(Y-B)^3$ + $\frac{1}{B^2}(X-A)^2(Y-B)^2$ - $\frac{2A}{B^3}(X-A)(Y-B)^3$ + $\frac{A^2}{B^4}(Y-B)^4\}$, and Var($\hat{\tau}_{\text{mod}}$) = E[(f(X,Y) - f(A,B))^2] . After some algebra, the formula looks as follows: Var($\hat{\tau}_{\text{mod}}$) = the delta method formula + estimated squared bias formula, where the estimated squared bias formula = $\frac{1}{(\pi_c+\pi_d)^4} \times \{$E($p_c^4$)[$\frac{4\pi_d^2}{(\pi_c+\pi_d)^2}$] + E($p_c^3$)[$\frac{-16\pi_d^2}{(\pi_c+\pi_d)}$] + E($p_c^2$)[$12\pi_d^2$] + E($p_c^3 p_d$)[$\frac{8\pi_d(\pi_d-\pi_c)}{(\pi_c+\pi_d)^2}$] + E($p_c^2 p_d$)[$\frac{16\pi_d(2\pi_c-\pi_d)}{(\pi_c+\pi_d)}$] + E($p_c^2 p_d^2$)[$\frac{4(\pi_c^2-4\pi_c\pi_d+\pi_d^2)}{(\pi_c+\pi_d)^2}$] + E($p_c p_d$)[-$24\pi_c\pi_d$] + E($p_c p_d^2$)[$\frac{16\pi_c(2\pi_d-\pi_c)}{(\pi_c+\pi_d)}$] + E($p_c p_d^3$)[$\frac{8\pi_c(\pi_c-\pi_d)}{(\pi_c+\pi_d)^2}$] + E($p_d^4$)[$\frac{4\pi_c^2}{(\pi_c+\pi_d)^2}$] + E($p_d^3$)[$\frac{-16\pi_c^2}{(\pi_c+\pi_d)}$] + E($p_d^2$)[$12\pi_c^2$]$\}$.

Now, it is possible to estimate every term in the above equation, however the equations get very large and messy. Furthermore, they require estimates of probabilities such as that of drawing 5 pairs, having the first pair concordant with the second and third, and the third pair being concordant with the fourth and fifth. These type of probabilities can be estimated in a frequentist manner by simply counting the number of times this occurs out of all possible enumerations. However, this type of estimation quickly becomes computationally infeasible. For example, with a sample size of 50 pairs, there are 2,118,760 distinct ways to draw 5 pairs. A method of counting the number of times a particular combination of concordances and discordances arises in a specific data set was developed using indicator matrices that required fairly little computational time to estimate these probabilities. However, this adds another of complexity to the problem and seems a little extreme. Thus, a simplification of this formula is proposed. The easiest terms to estimate in the above equation are the second order terms which also appear to be the largest. Hence, following the elimination of higher order terms, the formula becomes estimated squared bias = $\frac{1}{(\pi_c+\pi_d)^4} \times \{\mathrm{E}(p_c^2)[12\pi_d^2] + \mathrm{E}(p_d^2)[12\pi_c^2] + \mathrm{E}(p_c p_d)[\text{-}24\pi_c\pi_d]\}$. Inclusion of the two cubed terms was also considered, but the resulting formula consistently yielded a negative estimate.

Now, $\mathrm{E}(p_c^2) = \mathrm{Var}(\mathrm{p}_c) + [\mathrm{E}(\mathrm{p}_c)]^2 = \frac{2}{n(n-1)} \times [\pi_c(1-\pi_c) + 2(n-2)(\pi_{cc} - \pi_c^2)] + \pi_c^2$, and $\mathrm{E}(p_d^2) = \mathrm{Var}(\mathrm{p}_d) + [\mathrm{E}(\mathrm{p}_d)]^2 = \frac{2}{n(n-1)} \times [\pi_d(1-\pi_d) + 2(n-2)(\pi_{dd} - \pi_d^2)] + \pi_d^2$ so all that needs to be derived is $\mathrm{E}(p_c p_d)$. $\mathrm{E}(p_c p_d) = \frac{1}{n^2(n-1)^2}\mathrm{E}\left[\sum_{k=1}^{n} C_k \sum_{k=1}^{n} D_k\right] = \frac{1}{n^2(n-1)^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}[C_i D_j] = \frac{1}{n^2(n-1)^2}[n\mathrm{E}(\mathrm{C}_1\mathrm{D}_1) + n(n-1)\mathrm{E}(\mathrm{C}_1\mathrm{D}_2)]$. Now, $\mathrm{E}(\mathrm{C}_1\mathrm{D}_1)$

$= \sum_{i=2}^{n}\sum_{j=2}^{n}\mathrm{E}[w_{1i}u_{1j}]$ = (n-1)(n-2)π_{cd}, and E(C$_1$D$_2$)] $= \sum_{i=2}^{n}\sum_{j=1,j\neq 2}^{n}\mathrm{E}[w_{1i}u_{2j}]$ = 3(n-2) π_{cd} + (n-2)(n-3)$\pi_c\pi_d$, so E(p_cp_d) $= \frac{1}{n(n-1)}[4(n-2)\pi_{cd}$ + (n-2)(n-3)$\pi_c\pi_d]$. Hence, estimated squared bias $= \frac{24}{(\pi_c+\pi_d)^4} \times \{\frac{\pi_d^2}{n(n-1)} \times [\pi_c(1-\pi_c) + 2(n-2)(\pi_{cc} - \pi_c^2)] + \pi_c^2\pi_d^2 + \frac{\pi_c^2}{n(n-1)} \times [\pi_d(1-\pi_d) + 2(n-2)(\pi_{dd} - \pi_d^2)]$ - $\frac{\pi_c\pi_d}{n(n-1)}[4(n-2)\pi_{cd}$ + (n-2)(n-3)$\pi_c\pi_d]\}$

Finally, substituting in the unbiased estimates of the π's and combining the above squared bias formula with the variance formula, $\widehat{Var}(\hat{\tau}_{\mathrm{mod}}) = \frac{8}{n(n-1)(p_c+p_d)^4} \times \{4(p_c+p_d)p_cp_d + 3(n-2)(n+1)p_c^2p_d^2 + (n-2) \times [(2p_d^2(4p_{cc} - 3p_c^2 - p_{cd}) + p_cp_d(2p_d^2 - 2p_c^2 - 16p_{cd} - 3(n-3)p_cp_d) + 2p_c^2(4p_{dd} - 3p_d^2 + p_{cd})]\}$.

G. Tables of Power of H$_0$:$\tau_{\text{mod}\,1} = \tau_{\text{mod}\,2}$

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.293/.156/.153 **.061/.018/**.078 .168	.218/.161/.168 .069/**.035**/.083 .180	.223/.182/.190 .086/**.032**/.090 .187	.300/.163/.147 **.062/.021**/.074 .145	.302/.161/.156 .076/**.020**/.088 .154
β_1=0, β_2=.2	.387/.214/.218 .094/.040/.117 .239	.479/.342/.351 .198/.107/.211 .336	.567/.497/.492 .324/.221/.349 .418	.378/.245/.273 .124/.058/.158 .246	.461/.293/.255 .167/.076/.159 .295
β_1=0, β_2=.4	.594/.405/.386 .213/.076/.240 .448	.728/.598/.591 .409/.259/.401 .632	.911/.855/.843 .732/.580/.695 .805	.598/.404/.482 .230/.111/.307 .462	.732/.592/.531 .426/.200/.403 .601
β_1=0, β_2=.7	.812/.610/.565 .386/.168/.427 .724	.952/.877/.824 .718/.480/.690 .914	.995/.975/.966 .952/.908/.942 .981	.785/.591/.632 .377/.167/.451 .730	.938/.868/.783 .714/.535/.661 .892
β_1=.2, β_2=.2	.228/.093/.095 **.032/.005/.056** .135	.225/.105/.099 **.047/.016/.045** .161	.235/.147/.133 **.060/.023**/.068 .175	.240/.109/.087 **.039/.012/.039** .139	.259/.139/.106 **.055/.021**/.064 .168
β_1=.2, β_2=.4	.290/.119/.097 .036/.006/.048 .190	.362/.222/.175 .094/.028/.088 .296	.485/.362/.282 .199/.106/.193 .410	.254/.109/.124 .039/.017/.059 .175	.432/.264/.135 .157/.062/.131 .323
β_1=.2, β_2=.7	.509/.259/.180 .099/.027/.128 .405	.688/.522/.366 .285/.108/.259 .660	.872/.766/.621 .570/.379/.532 .835	.441/.218/.230 .087/.027/.138 .398	.753/.597/.294 .400/.226/.339 .673
β_1=.4, β_2=.4	.167/**.050/.030** **.009/.001/.013** .102	.188/.069/**.035** **.020/.002/.017** .142	.200/.106/**.044** **.033/.007/.024** .163	.243/.106/**.030** **.045/.017/.035** .150	.195/.079/**.029** **.032/.009/.029** .117
β_1=.4, β_2=.7	.267/.078/.035 .016/.001/.022 .160	.357/.175/.080 .056/.006/.054 .301	.504/.341/.148 .158/.070/.124 .471	.202/.067/.053 .017/.003/.018 .161	.462/.320/.044 .176/.095/.131 .337
β_1=.7, β_2=.7	.135/**.029/.000** **.000/.000/.002** **.058**	.129/**.040/.003** **.004/.000/.003** .096	.140/.064/**.007** **.013/.004/.009** .129	.229/.135/**.001** .068/**.046/.034** .090	.222/.132/**.002** **.055/.038/.031** .094

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.1: Linear Mean Relationship, ρ=.4, %ties=5, dist=gamma(r=2)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.285/.133/.130 **.059/.057**/.079 .136	.228/.160/.172 .072/.065/.088 .162	.182/.140/.147 .068/**.051**/.080 .150	.262/.148/.133 .077/.064/.080 .120	.286/.160/.158 .085/.074/.094 .139
β_1=0, β_2=.2	.365/.208/.204 .099/.080/.111 .192	.393/.304/.310 .158/.130/.175 .258	.435/.397/.391 .242/.229/.267 .320	.335/.219/.248 .123/.102/.135 .189	.413/.287/.249 .166/.138/.155 .249
β_1=0, β_2=.4	.505/.357/.336 .201/.180/.235 .347	.610/.525/.500 .340/.304/.332 .458	.802/.765/.750 .620/.587/.621 .639	.508/.358/.398 .207/.185/.260 .326	.669/.569/.464 .393/.362/.345 .500
β_1=0, β_2=.7	.709/.558/.486 .344/.290/.383 .541	.863/.798/.754 .622/.578/.609 .778	.968/.961/.929 .898/.881/.879 .914	.694/.527/.578 .327/.281/.399 .570	.889/.835/.701 .702/.659/.637 .787
β_1=.2, β_2=.2	.209/.114/.102 **.045/.034/.057** .134	.203/.141/.130 **.050/.038**/.064 .150	.198/.169/.141 .070/**.061**/.080 .156	.232/.146/.094 .064/**.052/.061** .126	.225/.160/.105 .074/.069/.073 .133
β_1=.2, β_2=.4	.244/.119/.103 .038/.029/.064 .158	.256/.194/.151 .090/.067/.091 .196	.333/.309/.230 .169/.145/.176 .298	.207/.123/.128 .056/.052/.067 .142	.353/.297/.130 .173/.129/.148 .255
β_1=.2, β_2=.7	.408/.240/.181 .099/.080/.140 .311	.493/.443/.314 .220/.184/.208 .450	.724/.722/.548 .526/.487/.490 .685	.370/.204/.238 .088/.071/.139 .283	.595/.569/.243 .391/.355/.338 .500
β_1=.4, β_2=.4	.123/**.052/.040** **.017/.012/.033** .092	.106/.074/**.047** **.020/.013/.017** .128	.115/.105/**.051** **.038/.031/.038** .130	.187/.144/**.038** **.060/.056/.054** .114	.147/.128/**.031** **.057/.040/.041** .119
β_1=.4, β_2=.7	.151/.065/.033 .020/.013/.027 .102	.177/.150/.063 .047/.039/.043 .212	.262/.292/.127 .140/.110/.105 .315	.120/.071/.057 .025/.014/.029 .098	.330/.337/.041 .206/.181/.169 .217
β_1=.7, β_2=.7	.066/**.036/.002** **.012/.012/.017** **.024**	**.038/.043/.004** **.002/.002/.005** .070	**.035/.054/.002** **.011/.011/.009** .083	.114/.141/**.003** .082/.076/**.047** **.051**	.132/.160/**.003** .083/**.063/.038** **.054**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.2: Linear Mean Relationship, ρ=.4, %ties=25, dist=gamma(r=2)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.292/.154/.145 **.060/.008**/.074 .164	.246/.154/.172 .071/**.032**/.082 .173	.213/.182/.182 .069/**.037**/.082 .177	.280/.150/.151 **.062/.026**/.075 .151	.285/.151/.130 **.062/.019**/.076 .139
β_1=0, β_2=.2	.340/.200/.199 .090/.022/.107 .219	.374/.267/.284 .137/.067/.145 .276	.460/.396/.390 .257/.131/.259 .352	.380/.234/.247 .122/.040/.131 .238	.420/.259/.217 .135/.051/.137 .260
β_1=0, β_2=.4	.497/.299/.301 .163/.050/.185 .349	.638/.507/.500 .322/.171/.326 .505	.812/.745/.719 .564/.430/.569 .695	.508/.355/.400 .217/.096/.263 .377	.592/.435/.381 .262/.150/.256 .454
β_1=0, β_2=.7	.709/.502/.493 .302/.145/.352 .620	.863/.768/.732 .577/.385/.580 .810	.981/.957/.941 .880/.798/.860 .947	.721/.515/.570 .314/.149/.394 .642	.866/.746/.678 .578/.344/.549 .791
β_1=.2, β_2=.2	.251/.108/.121 **.043/.009/.056** .139	.247/.148/.137 **.052/.018/.055** .178	.259/.185/.159 .071/**.028**/.074 .186	.284/.145/.122 .060/**.012**/.073 .167	.275/.137/.103 **.055/.027/.051** .153
β_1=.2, β_2=.4	.303/.118/.109 .037/.011/.055 .186	.320/.204/.163 .090/.033/.093 .249	.402/.295/.236 .149/.099/.148 .320	.294/.166/.165 .071/.023/.104 .212	.399/.240/.124 .114/.045/.105 .254
β_1=.2, β_2=.7	.489/.277/.225 .124/.023/.162 .379	.619/.437/.346 .233/.119/.232 .550	.823/.715/.612 .525/.352/.486 .755	.422/.209/.232 .096/.041/.146 .335	.688/.510/.293 .296/.172/.279 .576
β_1=.4, β_2=.4	.199/.066/**.050** **.011/.003/.021** .118	.221/.099/.076 **.022/.012/.028** .170	.210/.116/.077 **.053/.014/.055** .163	.244/.116/**.052** **.046/.017/.050** .137	.222/.094/**.041** **.032/.014/.035** .129
β_1=.4, β_2=.7	.251/.094/.061 .025/.005/.036 .173	.342/.170/.091 .058/.007/.056 .280	.473/.329/.184 .159/.081/.132 .410	.213/.085/.071 .031/.004/.037 .138	.449/.281/.074 .144/.051/.105 .308
β_1=.7, β_2=.7	.129/**.025/.006** **.003/.000/.008** .075	.158/**.054/.010** **.008/.001/.007** .109	.160/.064**/.013** **.016/.003/.014** .125	.212/.106/**.006** **.048/.022/.024** .097	.209/.112/**.010** **.047/.013/.031** .104

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.3: Linear Mean Relationship, ρ=.4, %ties=5, dist=gamma (r=3)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.281/.133/.130 **.057/.016**/.069 .132	.223/.158/.162 .065/**.026**/.075 .154	.211/.168/.173 .074/**.031**/.093 .172	.269/.151/.130 .081/**.028**/.084 .127	.269/.152/.133 .078/**.022**/.082 .138
β_1=0, β_2=.2	.315/.186/.175 .076/.023/.110 .161	.307/.230/.230 .115/.042/.128 .206	.374/.315/.319 .177/.077/.206 .278	.293/.188/.201 .100/.048/.115 .163	.370/.253/.199 .127/.060/.139 .207
β_1=0, β_2=.4	.432/.268/.263 .149/.072/.174 .303	.537/.450/.442 .271/.144/.291 .417	.687/.638/.612 .470/.318/.464 .501	.439/.306/.338 .173/.086/.208 .281	.530/.402/.333 .257/.163/.242 .348
β_1=0, β_2=.7	.644/.470/.430 .279/.131/.318 .474	.785/.725/.681 .511/.305/.505 .670	.936/.926/.897 .839/.697/.841 .874	.607/.451/.498 .259/.153/.319 .477	.819/.729/.614 .586/.354/.533 .669
β_1=.2, β_2=.2	.217/.114/.099 **.042/.016/.063** .119	.182/.120/.115 **.045/.011/.060** .134	.199/.152/.143 .064/**.025**/.084 .164	.243/.153/.112 .079/**.036**/.075 .139	.231/.161/.115 .077/**.042**/.091 .136
β_1=.2, β_2=.4	.265/.145/.125 .057/.015/.075 .159	.258/.192/.154 .070/.023/.085 .187	.346/.317/.264 .164/.082/.178 .280	.226/.149/.131 .066/.027/.081 .149	.306/.223/.137 .147/.060/.132 .199
β_1=.2, β_2=.7	.375/.234/.188 .108/.044/.135 .282	.458/.404/.326 .215/.080/.222 .403	.677/.651/.528 .457/.270/.440 .591	.325/.188/.218 .086/.048/.119 .246	.561/.496/.242 .329/.217/.301 .443
β_1=.4, β_2=.4	.159/.074/**.047** **.017/.006/.028** .097	.152/.119/.069 **.029/.009/.039** .135	.140/.110/.071 **.052/.018/.045** .130	.182/.140/**.055** **.059/.021**/.067 .125	.174/.129/**.049** **.055/.026/.045** .122
β_1=.4, β_2=.7	.188/.098/.057 .028/.005/.041 .132	.244/.199/.116 .079/.016/.066 .217	.281/.280/.156 .140/.061/.110 .268	.155/.098/.083 .039/.013/.048 .114	.286/.269/.044 .134/.081/.117 .217
β_1=.7, β_2=.7	.072/**.031/.010** **.007/.002/.014** **.037**	.068/**.063/.017** **.009/.000/.008** .082	.067/.082/**.016** **.019/.005/.013** .108	.125/.121/**.009** **.061/.029/.035** **.064**	.108/.127/**.005** **.053/.022/.032** **.059**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.4: Linear Mean Relationship, ρ=.4, %ties=25, dist=gamma (r=3)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.278/.117/.117 **.041/.009**/.069 .148	.221/.147/.157 **.060/.026**/.075 .182	.218/.190/.197 .089/**.032**/.102 .196	.291/.170/.157 .063/**.022**/.093 .177	.281/.146/.146 **.045/.026/.061** .153
β_1=0, β_2=.2	.408/.231/.234 .104/.034/.141 .300	.478/.361/.354 .203/.091/.231 .404	.625/.536/.534 .365/.216/.380 .564	.421/.269/.300 .138/.036/.185 .337	.498/.334/.301 .169/.063/.185 .360
β_1=0, β_2=.4	.634/.426/.406 .229/.072/.287 .546	.823/.689/.655 .501/.289/.506 .758	.953/.915/.893 .819/.673/.815 .940	.625/.430/.491 .255/.105/.328 .562	.829/.673/.598 .482/.270/.459 .718
β_1=0, β_2=.7	.842/.661/.602 .447/.167/.483 .797	.976/.919/.865 .787/.572/.790 .969	1/.999/.992 .990/.958/.983 .999	.838/.633/.688 .399/.183/.518 .838	.981/.934/.866 .838/.631/.808 .963
β_1=.2, β_2=.2	.238/.103/.082 **.027/.005/.049** .134	.246/.104/.086 **.034/.009/.033** .153	.226/.139/.106 **.049/.019/.053** .170	.268/.127/.087 **.050/.018/.061** .150	.260/.129/.080 **.045/.019/.062** .147
β_1=.2, β_2=.4	.324/.136/.105 .046/.005/.067 .197	.398/.221/.154 .089/.026/.093 .310	.562/.406/.284 .216/.104/.187 .450	.267/.104/.110 .040/.012/.060 .173	.523/.324/.121 .174/.062/.136 .339
β_1=.2, β_2=.7	.550/.252/.164 .086/.026/.138 .417	.765/.600/.375 .324/.117/.298 .708	.956/.885/.717 .733/.519/.663 .922	.470/.238/.250 .079/.024/.152 .456	.851/.668/.282 .466/.248/.391 .681
β_1=.4, β_2=.4	.153/**.037/.013** **.005/.000/.012** .076	.155/**.051/.019** **.011/.001/.013** .079	.193/.081/**.020** **.022/.006/.016** .114	.223/.107/**.020** **.040/.008/.028** .093	.201/.098/**.017** **.038/.011/.025** .074
β_1=.4, β_2=.7	.217/.056/.015 .009/.001/.015 .093	.359/.151/.043 .048/.006/.037 .277	.571/.381/.076 .146/.058/.087 .471	.147/.028/.019 .011/.002/.018 .099	.511/.359/.009 .194/.083/.119 .274
β_1=.7, β_2=.7	.067/**.007/.000** **.004/.000/.003** **.008**	.070/**.015/.000** **.001/.000/.000** **.033**	.081/**.027/.000** **.005/.000/.001** **.048**	.158/.090/**.000** **.040/.025/.018** **.022**	.183/.114/**.000** **.044/.013/.021** **.032**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.5: Linear Mean Relationship, ρ=.4, %ties=5, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.265/.131/.119 .070/**.021**/.082 .150	.232/.163/.165 .068/**.024**/.075 .174	.186/.157/.159 .063/**.021**/.084 .160	.257/.164/.136 .083/**.033**/.085 .150	.263/.138/.121 .070/**.041**/.082 .139
β_1=0, β_2=.2	.359/.225/.209 .116/.034/.126 .228	.414/.329/.325 .192/.094/.206 .336	.541/.490/.476 .323/.166/.337 .478	.358/.248/.259 .120/.058/.150 .254	.454/.354/.287 .211/.117/.202 .316
β_1=0, β_2=.4	.576/.402/.368 .230/.117/.259 .425	.748/.665/.629 .466/.236/.474 .666	.917/.881/.852 .767/.632/.757 .866	.565/.399/.465 .239/.121/.294 .487	.758/.644/.517 .461/.292/.447 .601
β_1=0, β_2=.7	.775/.618/.522 .385/.195/.452 .651	.911/.882/.789 .705/.485/.709 .858	.991/.989/.978 .964/.930/.951 .985	.760/.566/.617 .370/.178/.468 .690	.951/.918/.780 .801/.635/.757 .876
β_1=.2, β_2=.2	.173/.071/**.052** **.021/.009/.037** .080	.151/.101/.083 **.033/.010/.043** .107	.144/.116/.073 **.045/.018/.042** .107	.213/.140/.078 **.062/.028**/.066 .114	.203/.129/.077 **.049/.025/.056** .104
β_1=.2, β_2=.4	.199/.096/.066 .034/.012/.047 .104	.257/.208/.136 .083/.027/.092 .206	.386/.364/.224 .180/.077/.174 .311	.200/.109/.099 .053/.014/.069 .122	.374/.321/.091 .181/.092/.165 .224
β_1=.2, β_2=.7	.397/.229/.146 .085/.024/.128 .245	.554/.481/.293 .229/.081/.252 .480	.794/.815/.562 .602/.380/.534 .757	.309/.173/.176 .064/.024/.118 .246	.697/.662/.219 .467/.252/.409 .489
β_1=.4, β_2=.4	.095/**.030/.011** **.007/.003/.015** **.028**	.071/**.052/.019** **.012/.000/.014** **.058**	.081/.073/**.015** **.021/.005/.013** .066	.124/.110/**.019** **.049/.025/.036** **.051**	.113/.103/**.006** **.049/.027/.031** **.046**
β_1=.4, β_2=.7	.112/.044/.011 .013/.007/.015 .032	.144/.131/.022 .025/.004/.015 .121	.257/.311/.064 .115/.034/.077 .238	.095/.053/.021 .019/.005/.015 .053	.303/.339/.009 .177/.082/.120 .120
β_1=.7, β_2=.7	**.039/.022/.000** **.014/.012/.015** **.005**	**.007/.011/.000** **.001/.000/.000** **.011**	**.005/.025/.000** **.001/.000/.000** **.015**	.082/.124/**.000** .067/**.023/.016** **.007**	.081/.129/**.000** .067/**.021/.020** **.006**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.6: Linear Mean Relationship, ρ=.4, %ties=25, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.311/.150/.149 .067/**.049**/.085 .170	.226/.165/.169 .066/**.054**/.075 .162	.242/.195/.211 .089/.079/.095 .175	.295/.183/.171 .081/.068/.093 .178	.275/.165/.146 .078/**.057**/.077 .151
β_1=0, β_2=.1	.356/.198/.206 .092/.073/.117 .225	.382/.285/.290 .150/.138/.144 .293	.469/.406/.400 .247/.226/.247 .341	.366/.240/.261 .136/.109/.147 .232	.383/.255/.233 .127/.111/.143 .247
β_1=0, β_2=.15	.520/.334/.339 .177/.138/.206 .371	.650/.502/.498 .336/.299/.327 .535	.819/.752/.725 .583/.563/.559 .707	.536/.347/.395 .191/.153/.232 .369	.647/.469/.429 .288/.258/.283 .492
β_1=0, β_2=.2	.685/.488/.469 .270/.227/.305 .575	.849/.736/.714 .570/.522/.541 .795	.981/.952/.943 .891/.864/.862 .957	.704/.496/.560 .303/.256/.349 .610	.845/.714/.662 .530/.494/.485 .750
β_1=.1, β_2=.1	.248/.107/.107 **.036/.033/.050** .166	.251/.142/.138 .064**/.054/.063** .194	.240/.152/.128 .066/**.055**/.065 .194	.293/.146/.116 **.053/.046/.058** .177	.266/.130/.105 **.049/.042/.057** .148
β_1=.1, β_2=.15	.274/.121/.106 .046/.030/.053 .190	.318/.202/.160 .078/.065/.080 .260	.443/.330/.269 .177/.147/.152 .350	.313/.158/.162 .063/.044/.089 .213	.366/.206/.124 .103/.084/.098 .256
β_1=.1, β_2=.2	.461/.239/.202 .084/.064/.127 .333	.600/.392/.322 .197/.172/.192 .503	.798/.657/.552 .469/.422/.417 .749	.389/.186/.225 .087/.068/.114 .331	.646/.456/.276 .279/.242/.210 .515
β_1=.15, β_2=.15	.187/**.063/.048** **.013/.011/.030** .123	.196/.078/**.056** **.023/.019/.030** .152	.229/.106/**.057** **.025/.026/.029** .156	.214/.090/**.043** **.035/.026/.033** .118	.243/.106/**.052** **.040/.038/.044** .137
β_1=.15, β_2=.2	.239/.080/.060 .026/.018/.035 .148	.326/.157/.094 .047/.040/.052 .258	.471/.304/.163 .135/.110/.104 .412	.209/.064/.073 .016/.013/.024 .138	.394/.211/.058 .109/.087/.079 .226
β_1=.2, β_2=.2	.139/**.024/.011** **.001/.002/.003** .065	.168/**.047/.013** **.003/.003/.003** .094	.170/**.063/.014** **.014/.009/.005** .113	.195/.075/**.011** **.037/.028/.020** .068	.221/.085**/.015** **.039/.032/.022** .073

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.7: Exponential Mean Relationship, ρ=.4, %ties=5, dist=gamma (r=2)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.258/.131/.129 .**054/.018**/.064 .126	.228/.152/.164 .068/.**019**/.078 .145	.195/.175/.166 .079/.**035**/.099 .154	.268/.153/.140 .070/**.037**/.073 .140	.272/.162/.148 .068/**.036**/.073 .125
β_1=0, β_2=.1	.337/.199/.202 .102/.027/.109 .192	.338/.257/.256 .136/.068/.157 .226	.385/.341/.332 .175/.107/.201 .263	.340/.222/.228 .114/.049/.117 .179	.357/.240/.203 .129/.071/.130 .200
β_1=0, β_2=.15	.462/.315/.303 .179/.062/.187 .315	.556/.464/.447 .271/.142/.283 .406	.717/.671/.643 .491/.360/.497 .578	.440/.317/.351 .174/.099/.206 .283	.561/.422/.347 .286/.153/.260 .409
β_1=0, β_2=.2	.660/.489/.452 .290/.142/.329 .479	.791/.730/.678 .522/.325/.506 .719	.954/.934/.914 .839/.719/.836 .910	.648/.492/.543 .308/.171/.357 .550	.800/.702/.618 .562/.397/.498 .685
β_1=.1, β_2=.1	.211/.107/.101 **.043/.015/.053** .123	.202/.134/.125 **.052/.017/.057** .134	.214/.168/.150 **.060/.032**/.078 .156	.241/.149/.114 **.063/.024/.063** .150	.229/.146/.097 **.056/.034/.061** .139
β_1=.1, β_2=.15	.242/.121/.111 .050/.022/.063 .164	.251/.187/.164 .085/.031/.078 .200	.320/.286/.242 .158/.094/.156 .283	.218/.141/.125 .053/.026/.076 .144	.309/.238/.124 .136/.066/.118 .216
β_1=.1, β_2=.2	.377/.211/.172 .094/.035/.116 .254	.507/.399/.333 .225/.098/.214 .461	.736/.688/.577 .481/.288/.460 .702	.338/.193/.222 .088/.031/.112 .286	.569/.481/.244 .320/.206/.266 .408
β_1=.15, β_2=.15	.142/.069/**.054** **.023/.003/.031** .103	.125/.071/**.046** **.020/.002/.013** .130	.132/.108/**.063** **.035/.015/.028** .153	.191/.134/**.052** .065/**.016/.057** .132	.170/.118/**.055** **.059/.020/.045** .136
β_1=.15, β_2=.2	.183/.069/.048 .014/.010/.029 .105	.178/.139/.075 .047/.011/.043 .186	.296/.281/.162 .138/.056/.114 .336	.131/.065/.059 .025/.007/.024 .101	.328/.297/.043 .175/.090/.119 .189
β_1=.2, β_2=.2	.084/**.027/.011** **.006/.005/.009** **.028**	**.054/.033/.009** **.000/.000/.001** **.050**	.070/.070/**.016** **.019/.000/.009** .079	.122/.112/**.006** **.049/.018/.023** **.025**	.147/.142/**.009** .067/**.025/.034** **.049**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.8: Exponential Mean Relationship, ρ=.4, %ties=25, dist=gamma (r=2)

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β1=0, β2=0	.287/.133/.135 **.049/.019**/.069 .152	.225/.155/.166 .067/**.030**/.084 .180	.209/.171/.174 .080/**.037**/.094 .172	.291/.150/.143 .067/**.025**/.077 .165	.307/.163/.149 **.058/.011**/.073 .164
β1=0, β2=.1	.331/.160/.156 .060/.022/.081 .184	.368/.240/.265 .109/.053/.134 .264	.429/.364/.362 .216/.091/.214 .316	.351/.208/.211 .105/.033/.127 .207	.369/.224/.213 .104/.025/.120 .232
β1=0, β2=.15	.452/.276/.266 .109/.034/.146 .314	.554/.439/.430 .247/.128/.246 .455	.711/.633/.614 .458/.314/.458 .590	.431/.285/.322 .155/.059/.192 .313	.520/.334/.317 .181/.085/.194 .371
β1=0, β2=.2	.612/.406/.394 .219/.080/.231 .472	.756/.652/.624 .473/.298/.481 .700	.937/.881/.873 .785/.607/.761 .896	.654/.454/.518 .283/.121/.354 .555	.766/.605/.543 .406/.256/.382 .661
β1=.1, β2=.1	.254/.118/.122 **.051/.006/.063** .166	.253/.162/.157 **.062/.016**/.066 .183	.229/.172/.163 .067/**.032**/.071 .173	.278/.141/.117 **.059/.018/.062** .152	.264/.131/.109 **.047/.016/.055** .154
β1=.1, β2=.15	.291/.134/.125 .051/.014/.069 .179	.341/.202/.188 .092/.030/.099 .263	.404/.291/.239 .151/.068/.149 .315	.258/.135/.142 .056/.018/.084 .170	.313/.162/.110 .076/.034/.066 .204
β1=.1, β2=.2	.422/.212/.202 .086/.019/.125 .311	.513/.356/.304 .192/.080/.191 .446	.757/.628/.563 .421/.296/.399 .692	.408/.205/.239 .086/.028/.130 .311	.565/.378/.259 .222/.109/.196 .442
β1=.15, β2=.15	.208/**.059/.063** **.016/.003/.026** .123	.204/.109/.094 **.043/.009/.045** .153	.210/.108/.085 **.041/.023/.035** .163	.237/.101/**.054** **.029/.007/.034** .137	.251/.100/.068 **.034/.014/.033** .135
β1=.15, β2=.2	.237/.097/.066 .026/.003/.046 .151	.321/.172/.122 .055/.015/.059 .254	.429/.279/.196 .144/.061/.112 .360	.229/.085/.082 .031/.008/.041 .137	.402/.223/.083 .108/.037/.091 .247
β1=.2, β2=.2	.140/**.040/.020** **.007/.000/.010** .084	.174/**.044/.024** **.005/.000/.006** .102	.168/**.063/.024** **.018/.005/.007** .119	.179/.080/**.017** **.037/.009/.027** .089	.198/.081/**.017** **.035/.004/.019** .091

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.9: Exponential Mean Relationship, ρ=.4, %ties=5, dist=gamma (r=3)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.258/.131/.129 .**054/.018**/.064 .126	.228/.152/.164 .068/.**019**/.078 .145	.195/.175/.166 .079/.**035**/.099 .154	.268/.153/.140 .070/**.037**/.073 .140	.272/.162/.148 .068/**.036**/.073 .125
β_1=0, β_2=.1	.337/.199/.202 .102/.027/.109 .192	.338/.257/.256 .136/.068/.157 .226	.385/.341/.332 .175/.107/.201 .263	.340/.222/.228 .114/.049/.117 .179	.357/.240/.203 .129/.071/.130 .200
β_1=0, β_2=.15	.462/.315/.303 .179/.062/.187 .315	.556/.464/.447 .271/.142/.283 .406	.717/.671/.643 .491/.360/.497 .578	.440/.317/.351 .174/.099/.206 .283	.561/.422/.347 .286/.153/.260 .409
β_1=0, β_2=.2	.660/.489/.452 .290/.142/.329 .479	.791/.730/.678 .522/.325/.506 .719	.954/.934/.914 .839/.719/.836 .910	.648/.492/.543 .308/.171/.357 .550	.800/.702/.618 .562/.397/.498 .685
β_1=.1, β_2=.1	.211/.107/.101 **.043/.015/.053** .123	.202/.134/.125 **.052/.017/.057** .134	.214/.168/.150 **.060/.032**/.078 .156	.241/.149/.114 **.063/.024/.063** .150	.229/.146/.097 **.056/.034/.061** .139
β_1=.1, β_2=.15	.242/.121/.111 .050/.022/.063 .164	.251/.187/.164 .085/.031/.078 .200	.320/.286/.242 .158/.094/.156 .283	.218/.141/.125 .053/.026/.076 .144	.309/.238/.124 .136/.066/.118 .216
β_1=.1, β_2=.2	.377/.211/.172 .094/.035/.116 .254	.507/.399/.333 .225/.098/.214 .461	.736/.688/.577 .481/.288/.460 .702	.338/.193/.222 .088/.031/.112 .286	.569/.481/.244 .320/.206/.266 .408
β_1=.15, β_2=.15	.142/.069/**.054** **.023/.003/.031** .103	.125/.071/**.046** **.020/.002/.013** .130	.132/.108/.**063** **.035/.015/.028** .153	.191/.134/.**052** .065/.**016/.057** .132	.170/.118/**.055** **.059/.020/.045** .136
β_1=.15, β_2=.2	.183/.069/.048 .014/.010/.029 .105	.178/.139/.075 .047/.011/.043 .186	.296/.281/.162 .138/.056/.114 .336	.131/.065/.059 .025/.007/.024 .101	.328/.297/.043 .175/.090/.119 .189
β_1=.2, β_2=.2	.084/.**027/.011** **.006/.005/.009** **.028**	**.054/.033/.009** **.000/.000/.001** **.050**	.070/.070/.**016** **.019/.000/.009** .079	.122/.112/.**006** **.049/.018/.023** **.025**	.147/.142/**.009** .067/.**025/.034** **.049**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.10: Exponential Mean Relationship, ρ=.4, %ties=25, dist=gamma (r=3)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.276/.125/.134 **.043/.008**/.064 .162	.239/.164/.180 .074/**.021**/.083 .198	.190/.141/.157 .064/**.026**/.084 .166	.275/.139/.143 .064/**.023**/.082 .157	.309/.167/.151 .069/**.028**/.076 .174
β_1=0, β_2=.1	.367/.184/.182 .082/.022/.111 .245	.380/.279/.285 .161/.068/.173 .319	.520/.436/.434 .270/.181/.285 .465	.389/.245/.264 .121/.038/.159 .277	.471/.292/.262 .143/.069/.166 .322
β_1=0, β_2=.15	.548/.356/.363 .183/.059/.228 .467	.705/.576/.564 .367/.193/.379 .651	.886/.820/.807 .681/.490/.662 .851	.538/.358/.417 .195/.094/.249 .468	.675/.500/.465 .336/.156/.322 .555
β_1=0, β_2=.2	.761/.531/.495 .318/.144/.374 .675	.939/.837/.798 .676/.451/.662 .910	.995/.981/.971 .945/.854/.920 .997	.756/.545/.612 .320/.145/.427 .751	.911/.803/.727 .625/.388/.580 .862
β_1=.1, β_2=.1	.244/.097/.097 **.035/.004/.053** .148	.229/.128/.117 **.059/.012**/.068 .173	.242/.148/.127 .073/**.032**/.073 .178	.254/.127/.088 **.044/.019/.058** .150	.277/.145/.107 **.055/.024**/.065 .155
β_1=.1, β_2=.15	.280/.130/.095 .031/.007/.071 .173	.335/.189/.154 .082/.030/.094 .266	.471/.311/.252 .174/.082/.162 .381	.260/.115/.123 .038/.012/.065 .174	.419/.231/.126 .126/.048/.107 .241
β_1=.1, β_2=.2	.466/.213/.171 .091/.025/.121 .339	.667/.441/.315 .219/.096/.206 .562	.893/.785/.640 .574/.386/.520 .855	.440/.202/.240 .086/.025/.143 .390	.716/.508/.263 .321/.186/.270 .521
β_1=.15, β_2=.15	.178/**.051/.037** **.010/.003/.022** .086	.211/**.057/.036** **.016/.006/.015** .108	.211/.101/**.042** **.023/.005/.017** .103	.222/.096/**.033** **.035/.008/.032** .107	.234/.095/**.024** **.036/.007/.032** .091
β_1=.15, β_2=.2	.271/.085/.051 .017/.001/.033 .125	.341/.144/.067 .039/.004/.041 .193	.523/.346/.156 .157/.050/.106 .404	.204/.059/.053 .018/.003/.028 .121	.454/.266/.035 .134/.063/.095 .182
β_1=.2, β_2=.2	.147/**.012/.004** **.000/.001/.003** **.027**	.120/**.024/.003** **.002/.000/.001** **.042**	.166/**.042/.001** **.006/.001/.000** **.048**	.206/.084/**.002** **.039/.022/.023** **.038**	.197/.092/**.002** **.034/.011/.021** **.027**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.11: Exponential Mean Relationship, ρ=.4, %ties=5, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.283/.136/.135 .**056**/**.016**/.078 .136	.220/.154/.158 .066**/.014/.**080 .158	.202/.169/.163 .080/.**029**/.091 .186	.270/.156/.142 .071/**.028**/.083 .137	.283/.167/.159 .085/.**043**/.097 .164
β_1=0, β_2=.1	.321/.178/.162 .086/.048/.097 .184	.379/.292/.301 .153/.082/.177 .313	.485/.421/.424 .284/.131/.306 .425	.319/.193/.208 .109/.054/.116 .212	.384/.284/.237 .154/.086/.150 .265
β_1=0, β_2=.15	.503/.347/.325 .181/.068/.217 .371	.659/.549/.536 .360/.174/.370 .612	.843/.807/.786 .664/.457/.666 .814	.541/.393/.446 .219/.087/.281 .455	.662/.531/.445 .369/.198/.347 .509
β_1=0, β_2=.2	.720/.525/.463 .301/.148/.361 .553	.877/.813/.765 .645/.453/.635 .850	.994/.986/.971 .952/.850/.937 .987	.739/.556/.614 .345/.187/.434 .680	.914/.811/.706 .669/.521/.615 .810
β_1=.1, β_2=.1	.207/.099/.086 **.038/.012/.050** .124	.200/.131/.118 **.046/.018/.058** .125	.169/.141/.112 **.052/.025/**.069 .134	.218/.151/.101 .073/.**020**/.078 .121	.222/.143/.088 .066/.**024**/.073 .116
β_1=.1, β_2=.15	.243/.135/.110 .045/.012/.079 .142	.261/.186/.145 .077/.030/.095 .206	.372/.320/.237 .165/.067/.173 .315	.222/.131/.127 .052/.016/.073 .146	.343/.272/.108 .154/.073/.137 .200
β_1=.1, β_2=.2	.410/.222/.151 .076/.036/.119 .224	.559/.450/.311 .232/.097/.233 .457	.800/.771/.587 .577/.378/.535 .752	.379/.219/.238 .097/.028/.141 .290	.671/.581/.249 .418/.248/.378 .419
β_1=.15, β_2=.15	.124/.**040/.023** **.007/.004/.020** **.046**	.111/.073/.**037** **.015/.003/.017** .072	.088/.072**/.027** **.020/.003/.015** .064	.157/.132**/.032** **.053/.012/.054** **.062**	.164/.122/.**026** **.054/.019/.051** **.053**
β_1=.15, β_2=.2	.164/.053/.032 .011/.004/.031 .044	.202/.136/.052 .044/.012/.044 .103	.316/.310/.105 .135/.060/.101 .228	.138/.068/.052 .020/.009/.033 .080	.338/.297/.026 .171/.072/.111 .087
β_1=.2, β_2=.2	.066**/.016/.001** **.013/.009/.013** **.003**	**.036/.022/.002** **.001/.000/.000** **.010**	**.023/.032/.000** **.001/.000/.001** **.010**	.117/.123**/.003** **.051/.009/.018** **.005**	.107/.116/.**003** **.052/.011/.013** **.004**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.12: Exponential Mean Relationship, ρ=.4, %ties=25, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.218/.094/.079 **.053/.026/.057** .083	.150/.090/.094 **.056/.026/.055** .089	.135/.105/.104 .070/**.050**/.065 .098	.229/.100/.089 .065/**.036**/.068 .104	.220/.111/.103 .075/**.037**/.074 .101
β_1=0, β_2=.2	.351/.172/.164 .111/.081/.094 .175	.439/.314/.319 .242/.170/.219 .298	.581/.504/.493 .414/.325/.402 .388	.345/.184/.210 .131/.090/.135 .179	.461/.292/.265 .227/.156/.185 .258
β_1=0, β_2=.4	.606/.371/.360 .283/.189/.250 .434	.776/.637/.606 .556/.472/.486 .629	.949/.900/.893 .854/.808/.831 .852	.599/.396/.453 .319/.230/.308 .434	.764/.594/.520 .512/.408/.407 .576
β_1=0, β_2=.7	.845/.595/.555 .491/.343/.450 .727	.971/.906/.850 .846/.792/.774 .912	1/.999/.996 .994/.992/.989 .997	.804/.590/.637 .491/.369/.499 .736	.963/.884/.817 .822/.734/.736 .888
β_1=.2, β_2=.2	.180/**.056/.046** **.032/.012/.027** .093	.175/.076/**.059** **.043/.034/.028** .098	.149/.085/**.061** **.047/.024/.046** .106	.209/.097/**.063** .065/**.030/.050** .102	.180/.079/**.047** **.049/.039/.041** .088
β_1=.2, β_2=.4	.236/.083/.068 .057/.025/.048 .141	.331/.172/.120 .112/.097/.077 .240	.465/.334/.235 .255/.196/.207 .355	.201/.076/.073 .048/.036/.052 .121	.421/.264/.098 .175/.122/.127 .288
β_1=.2, β_2=.7	.491/.229/.159 .157/.064/.138 .388	.682/.479/.295 .371/.292/.270 .628	.926/.810/.642 .732/.692/.640 .864	.436/.185/.207 .115/.061/.137 .373	.780/.598/.259 .500/.377/.347 .664
β_1=.4, β_2=.4	.108/**.028/.014** **.012/.003/.013** **.053**	.126/**.038/.013** **.016/.010/.011** .082	.135/**.054/.013** **.034/.022/.019** .104	.180/.077/**.009** **.046/.041/.020** .087	.199/.082/**.012** **.050/.026/.032** .089
β_1=.4, β_2=.7	.203/.058/.019 .028/.013/.025 .120	.324/.125/.034 .085/.036/.037 .257	.462/.290/.086 .199/.147/.118 .428	.147/.039/.027 .021/.009/.012 .103	.432/.286/.020 .208/.174/.109 .312
β_1=.7, β_2=.7	.093/**.011/.000** **.003/.001/.002** **.029**	.099/**.021/.000** **.011/.002/.001** .064	.099/**.036/.001** **.015/.005/.006** .076	.178/.096/**.000** **.059/.035/.027** .069	.187/.107/**.002** .072/**.036/.023** .068

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.13: Linear Mean Relationship, ρ=.2, %ties=5, dist=gamma(r=2)

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β₁=0, β₂=0	.227/.095/.087 **.061/.045/.054** .091	.128/.079/.078 **.048/.035/.038** .064	.127/.092/.090 **.057/.047**/.073 .086	.215/.123/.096 .086/.068/.075 .087	.226/.113/.086 .079/**.062/.058** .081
β₁=0, β₂=.2	.291/.158/.148 .114/.084/.095 .127	.334/.237/.240 .173/.144/.144 .185	.433/.368/.360 .293/.249/.291 .262	.300/.173/.179 .131/.112/.129 .151	.373/.247/.194 .189/.170/.152 .176
β₁=0, β₂=.4	.475/.305/.274 .234/.185/.213 .296	.622/.538/.491 .437/.389/.366 .410	.862/.811/.781 .748/.695/.718 .647	.471/.306/.350 .227/.201/.245 .286	.641/.539/.436 .458/.407/.359 .407
β₁=0, β₂=.7	.718/.528/.456 .417/.359/.416 .532	.881/.830/.773 .763/.700/.684 .794	.989/.982/.966 .966/.955/.956 .945	.706/.486/.542 .382/.319/.416 .530	.902/.837/.698 .769/.735/.670 .771
β₁=.2, β₂=.2	.150/.064/**.051** **.044/.030/.045** **.059**	.110/.074/**.063** **.043/.037/.035** .086	.110/.081/**.059** **.050/.034/.045** .079	.155/.093/**.045** .066/**.053/.041** .071	.172/.103/**.052** .079/**.062/.055** .085
β₁=.2, β₂=.4	.199/.091/.079 .056/.045/.062 .105	.211/.161/.113 .110/.084/.079 .158	.305/.269/.206 .218/.171/.182 .218	.159/.075/.084 .042/.036/.060 .083	.295/.228/.083 .175/.147/.117 .182
β₁=.2, β₂=.7	.361/.188/.123 .125/.097/.123 .253	.516/.450/.298 .351/.286/.258 .455	.795/.778/.604 .691/.652/.577 .713	.304/.157/.179 .109/.081/.126 .240	.616/.548/.205 .477/.408/.338 .476
β₁=.4, β₂=.4	.098/**.032/.017** **.016/.008/.023** **.046**	.082/**.053/.021** **.028/.019/.012** .069	**.053/.057/.020** **.032/.026/.023** .069	.111/.093/**.018** .068/**.044/.036** .074	.132/.095/**.013** .072/**.059/.040** .065
β₁=.4, β₂=.7	.130/.056/.015 .020/.014/.023 .069	.154/.130/.044 .079/.051/.044 .157	.257/.278/.076 .194/.150/.105 .297	.109/.049/.033 .025/.019/.029 .082	.281/.300/.014 .243/.200/.140 .187
β₁=.7, β₂=.7	**.035/.011/.002** **.007/.004/.006** **.017**	**.028/.027/.001** **.010/.004/.000** **.034**	**.008/.029/.002** **.011/.007/.002** **.048**	.072/.098/**.000** .074/.065/**.026** **.020**	.084/.113/**.001** .084/.070/**.034** **.020**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.14: Linear Mean Relationship, ρ=.2, %ties=25, dist=gamma (r=2)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.235/.088/.087 **.054/.022/.052** .092	.176/.101/.104 **.056/.035/.055** .109	.131/.104/.106 .073/**.053**/.080 .093	.232/.110/.096 .079/**.048/.058** .102	.239/.113/.110 .085/**.030**/.070 .105
β_1=0, β_2=.2	.325/.144/.148 .094/.049/.083 .167	.337/.226/.238 .167/.115/.140 .215	.419/.334/.336 .260/.232/.267 .286	.304/.174/.180 .127/.072/.123 .169	.375/.205/.170 .139/.108/.123 .193
β_1=0, β_2=.4	.484/.284/.269 .213/.125/.183 .303	.656/.507/.501 .408/.320/.369 .492	.849/.773/.758 .699/.640/.664 .690	.514/.310/.363 .230/.150/.249 .343	.651/.461/.386 .372/.270/.299 .424
β_1=0, β_2=.7	.734/.499/.450 .392/.275/.348 .589	.916/.802/.758 .726/.643/.646 .848	.997/.983/.969 .967/.955/.947 .970	.722/.511/.572 .416/.297/.423 .630	.887/.782/.718 .721/.614/.602 .795
β_1=.2, β_2=.2	.211/.079/.067 **.047/.015/.041** .094	.163/.078/.075 **.049/.032/.051** .106	.165/.090/.074 **.052/.031/.052** .105	.198/.081/**.052** **.060/.031/.044** .084	.214/.088/.065 **.054/.027/.043** .089
β_1=.2, β_2=.4	.265/.101/.081 .061/.031/.054 .131	.294/.152/.120 .091/.080/.076 .181	.378/.244/.202 .187/.164/.156 .268	.230/.087/.095 .049/.038/.050 .125	.378/.199/.094 .158/.086/.100 .208
β_1=.2, β_2=.7	.446/.198/.141 .137/.084/.135 .312	.650/.445/.342 .343/.248/.268 .545	.881/.785/.665 .700/.582/.631 .810	.397/.198/.205 .123/.063/.138 .313	.687/.496/.278 .428/.315/.271 .548
β_1=.4, β_2=.4	.149/**.043/.025** **.027/.010/.017** .081	.146/**.058/.034** **.029/.018/.021** .096	.155/.082/**.037** **.049/.017/.036** .117	.187/.081/**.026** **.046/.027/.024** .097	.196/.086/**.031** **.057/.027/.035** .093
β_1=.4, β_2=.7	.215/.067/.037 .036/.015/.032 .132	.312/.144/.060 .094/.036/.054 .253	.471/.304/.144 .238/.173/.156 .381	.191/.057/.047 .031/.013/.033 .122	.398/.247/.034 .160/.108/.084 .264
β_1=.7, β_2=.7	.109/**.013/.001** **.009/.001/.004** **.041**	.110/**.021/.003** **.006/.001/.001** **.059**	.107/**.039/.004** **.019/.013/.005** .080	.169/.081/**.000** **.052/.026/.020** **.060**	.165/.077/**.004** **.047/.021/.026** **.061**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.15: Linear Mean Relationship, ρ=.2, %ties=5, dist=gamma(r=3)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.195/**.061**/.064 **.044/.027/.035** .070	.156/.094/.098 **.063/.032/.046** .089	.111/.083/.087 **.052/.043/.060** .080	.199/.104/.081 **.062/.044/.058** .084	.193/.089/.069 **.053/.049/.041** .088
β_1=0, β_2=.2	.282/.147/.143 .112/.061/.100 .142	.280/.184/.197 .137/.093/.125 .161	.310/.260/.262 .207/.151/.210 .216	.244/.143/.150 .092/.081/.085 .108	.276/.170/.133 .122/.106/.076 .129
β_1=0, β_2=.4	.395/.231/.216 .176/.126/.154 .218	.515/.410/.399 .327/.248/.266 .350	.708/.650/.625 .579/.499/.560 .517	.392/.244/.283 .188/.125/.166 .216	.523/.408/.318 .337/.281/.260 .329
β_1=0, β_2=.7	.637/.453/.391 .354/.222/.304 .436	.803/.714/.667 .631/.523/.544 .671	.968/.952/.929 .924/.874/.890 .887	.631/.436/.480 .343/.241/.346 .426	.818/.736/.610 .677/.580/.552 .672
β_1=.2, β_2=.2	.185/.075/.064 **.044/.027/.044** .079	.134/.085/.071 **.056/.026/.043** .088	.113/.089/.078 **.060/.025/.063** .092	.154/.090/**.053** **.059/.043/.049** .074	.147/.078/**.049** **.051/.039/.044** .068
β_1=.2, β_2=.4	.215/.105/.082 .062/.037/.064 .122	.209/.150/.113 .110/.071/.082 .145	.292/.256/.184 .179/.119/.156 .202	.175/.082/.085 .050/.031/.056 .083	.251/.194/.087 .154/.125/.106 .128
β_1=.2, β_2=.7	.334/.173/.143 .116/.077/.129 .211	.503/.411/.310 .323/.222/.255 .390	.732/.694/.557 .607/.524/.543 .611	.305/.156/.191 .086/.087/.132 .205	.578/.516/.212 .420/.345/.316 .398
β_1=.4, β_2=.4	.110/**.040/.024** **.019/.011/.021** **.053**	.080/**.049/.028** **.028/.014/.022** **.056**	.082/.069/**.029** **.037/.016/.031** .074	.115/.088/**.023** **.058/.029/.035** .064	.144/.105/**.028** **.062/.024/.045** .064
β_1=.4, β_2=.7	.146/.063/.033 .037/.008/.034 .077	.167/.120/.046 .076/.039/.048 .148	.254/.243/.105 .177/.113/.105 .242	.133/.070/.048 .043/.015/.038 .068	.294/.269/.045 .215/.146/.134 .188
β_1=.7, β_2=.7	**.043/.011/.003** **.006/.003/.006** **.021**	**.028/.032/.001** **.004/.003/.003** **.038**	**.021/.038/.003** **.017/.001/.005** **.052**	.069/.086/**.002** **.063/.046/.021** **.032**	.077/.093/**.004** **.057/.048/.028** **.047**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.16: Linear Mean Relationship, ρ=.2, %ties=25, dist=gamma (r=3)

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.231/.084/.083 **.056/.021/.054** .100	.138/.082/.085 **.053/.032/.048** .094	.117/.085/.091 **.052/.042**/.067 .085	.219/.091/.087 **.063/.028/.052** .087	.226/.104/.090 .064/**.035/.057** .096
β_1=0, β_2=.2	.338/.147/.146 .105/.069/.090 .187	.390/.282/.282 .213/.142/.176 .301	.525/.425/.430 .347/.260/.345 .461	.327/.184/.208 .146/.075/.153 .208	.424/.248/.212 .187/.125/.156 .246
β_1=0, β_2=.4	.563/.329/.325 .262/.156/.239 .410	.738/.594/.591 .518/.393/.460 .666	.908/.861/.830 .808/.743/.770 .881	.539/.349/.397 .252/.164/.266 .446	.743/.537/.464 .458/.365/.366 .574
β_1=0, β_2=.7	.804/.571/.495 .439/.335/.408 .709	.966/.887/.839 .829/.719/.757 .941	1/.996/.987 .991/.986/.981 .999	.802/.577/.621 .458/.351/.462 .791	.973/.885/.806 .833/.754/.722 .907
β_1=.2, β_2=.2	.188/**.057/.054** **.034/.016/.035** .075	.156/.066/**.050** **.040/.028/.029** .087	.155/.090/.074 **.054/.029/.056** .099	.198/.071/**.055** **.048/.041/.042** .082	.223/.095/**.048** **.056/.026/.050** .092
β_1=.2, β_2=.4	.252/.097/.082 .063/.032/.055 .147	.320/.163/.116 .119/.065/.089 .211	.447/.293/.203 .214/.166/.182 .300	.220/.071/.085 .045/.029/.058 .109	.430/.224/.095 .160/.118/.106 .218
β_1=.2, β_2=.7	.500/.225/.152 .138/.097/.139 .354	.753/.518/.361 .409/.292/.309 .643	.959/.856/.695 .786/.690/.688 .890	.427/.193/.217 .129/.075/.148 .378	.800/.600/.270 .499/.389/.334 .601
β_1=.4, β_2=.4	.139/**.029/.014** **.017/.004/.012** **.049**	.138/**.041/.008** **.016/.010/.013** **.062**	.145/**.063/.017** **.034/.012/.019** .075	.174/.069/**.017** **.041/.023/.025** .068	.191/**.061/.015** **.033/.036/.028** .069
β_1=.4, β_2=.7	.219/.058/.025 .027/.008/.022 .108	.362/.144/.037 .075/.040/.039 .238	.574/.356/.133 .269/.188/.151 .430	.164/.038/.030 .020/.009/.021 .095	.465/.270/.019 .199/.138/.094 .223
β_1=.7, β_2=.7	.069/**.009/.001** **.002/.000/.003** **.017**	.071/**.016/.000** **.009/.004/.000** **.021**	.079/**.026/.001** **.013/.004/.003** **.037**	.135/.069/**.000** **.042/.026/.013** **.039**	.148/.071/**.001** **.048/.024/.010** **.035**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.17: Linear Mean Relationship, ρ=.2, %ties=5, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.208/.084/.076 **.048/.038/.043** .087	.127/.072/.077 **.045/.035/.037** .082	.117/.094/.090 **.061/.046**/.066 .085	.208/.095/.079 .072/**.052/.050** .071	.190/.107/.082 .076/**.052/.062** .083
β_1=0, β_2=.2	.284/.150/.144 .106/.084/.094 .145	.317/.223/.226 .164/.138/.148 .230	.421/.355/.352 .290/.246/.297 .344	.297/.185/.192 .132/.110/.114 .175	.353/.222/.174 .169/.142/.128 .177
β_1=0, β_2=.4	.478/.294/.279 .229/.197/.215 .315	.599/.519/.503 .434/.367/.400 .518	.844/.796/.763 .719/.672/.707 .755	.468/.280/.345 .212/.184/.221 .334	.669/.530/.427 .461/.406/.375 .480
β_1=0, β_2=.7	.700/.501/.432 .409/.357/.384 .549	.898/.836/.767 .764/.717/.685 .839	.993/.985/.975 .973/.969/.958 .983	.703/.509/.557 .406/.357/.425 .609	.912/.853/.723 .804/.776/.697 .803
β_1=.2, β_2=.2	.159/.066/**.053** **.042/.029/.039** .065	.122/.070/**.055** **.048/.034/.039** **.062**	.113/.078/.065 **.053/.037/.052** .064	.154/.085/**.041** **.056/.041/.041** **.058**	.144/.082/**.050** **.055/.044/.050** .067
β_1=.2, β_2=.4	.218/.091/.067 .054/.047/.062 .098	.210/.134/.097 .096/.065/.077 .144	.335/.282/.206 .224/.186/.188 .235	.184/.096/.094 .054/.043/.056 .103	.305/.243/.080 .178/.153/.126 .156
β_1=.2, β_2=.7	.354/.188/.130 .120/.092/.134 .194	.536/.459/.290 .359/.284/.259 .413	.795/.773/.566 .692/.639/.572 .724	.341/.157/.172 .109/.084/.129 .214	.645/.569/.180 .454/.395/.346 .418
β_1=.4, β_2=.4	.078/**.019/.010** **.009/.007/.010** **.027**	**.061/.035/.010** **.015/.016/.009** **.037**	**.061/.058/.015** **.033/.022/.020** **.029**	.109/.076/**.011** **.053/.043/.032** **.035**	.118/.100/**.011** .070/**.055/.039** **.035**
β_1=.4, β_2=.7	.105/.039/.011 .019/.012/.022 .041	.165/.122/.035 .067/.047/.041 .099	.247/.256/.062 .178/.137/.100 .212	.086/.040/.023 .024/.015/.019 .044	.273/.274/.007 .194/.162/.107 .103
β_1=.7, β_2=.7	**.026/.007/.002** **.006/.005/.005** **.004**	**.012/.011/.000** **.003/.002/.001** **.008**	**.008/.026/.000** **.010/.006/.001** **.014**	**.060**/.069/**.000** **.056/.037/.021** **.006**	.071/.088/**.000** .070/**.061/.028** **.007**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.18: Linear Mean Relationship, ρ=.2, %ties=25, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.200/.086/.081 **.048/.032/.054** .078	.166/.088/.103 **.062/.040/.061** .101	.138/.110/.107 .070/**.045**/.077 .103	.216/.097/.090 **.060/.042/.050** .089	.249/.134/.116 .097/**.039**/.086 .120
β_1=0, β_2=.1	.332/.165/.173 .107/.070/.093 .168	.362/.258/.259 .185/.143/.155 .222	.485/.404/.412 .329/.251/.328 .313	.295/.164/.186 .106/.071/.124 .162	.372/.201/.183 .154/.111/.138 .177
β_1=0, β_2=.15	.501/.292/.281 .216/.142/.192 .343	.671/.509/.503 .417/.350/.341 .507	.881/.812/.803 .753/.675/.716 .741	.523/.323/.376 .250/.192/.248 .366	.656/.464/.429 .386/.290/.303 .471
β_1=0, β_2=.2	.761/.508/.463 .403/.274/.357 .592	.920/.816/.786 .738/.687/.651 .854	.997/.986/.974 .974/.948/.949 .985	.744/.495/.568 .396/.283/.398 .639	.917/.773/.714 .703/.578/.572 .817
β_1=.1, β_2=.1	.195/**.063/.051** **.029/.018/.032** .091	.165/.078/.066 **.052/.022/.043** .095	.163/.084/.068 **.052/.047/.048** .094	.204/.082/**.050** **.048/.026/.039** .082	.231/.093/.073 .065/**.043/.055** .092
β_1=.1, β_2=.15	.223/.076/.062 .049/.029/.035 .132	.303/.156/.119 .109/.057/.073 .216	.425/.275/.214 .213/.162/.171 .328	.222/.081/.080 .043/.038/.051 .127	.360/.198/.099 .154/.097/.087 .223
β_1=.1, β_2=.2	.413/.170/.124 .108/.072/.098 .273	.654/.425/.309 .306/.210/.239 .548	.856/.744/.609 .652/.555/.556 .800	.401/.164/.198 .100/.061/.115 .309	.689/.453/.228 .370/.260/.239 .489
β_1=.15, β_2=.15	.137/**.033/.021** **.020/.005/.014** .067	.147/**.042/.025** **.022/.005/.018** .100	.128/**.050/.023** **.029/.013/.020** .100	.163/.067/**.021** **.047/.037/.022** .088	.173/.074/**.017** **.050/.022/.024** .091
β_1=.15, β_2=.2	.168/.042/.025 .029/.008/.020 .095	.308/.132/.058 .066/.031/.035 .210	.481/.278/.113 .180/.116/.108 .388	.146/.049/.044 .029/.010/.028 .101	.387/.204/.027 .139/.094/.080 .200
β_1=.2, β_2=.2	.096/**.007/.002** **.004/.001/.004** **.028**	.116/**.026/.004** **.011/.002/.005** .064	.108/**.039/.003** **.013/.007/.005** **.063**	.166/.076/**.003** **.054/.027/.017** **.033**	.168/.079/**.000** **.051/.033/.023** **.047**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.19: Exponential Mean Relationship, ρ=.2, %ties=5, dist=gamma (r=2)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**N_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.210/.073/.068 **.050/.039/.045** **.058**	.134/.087/.097 **.045/.034/.043** .083	.145/.114/.117 .071/**.046**/.088 .103	.216/.112/.084 .074/**.046**/.065 .078	.185/.090/.082 **.056/.049/.050** .073
β_1=0, β_2=.1	.248/.130/.115 .084/.076/.080 .118	.287/.206/.215 .155/.108/.138 .174	.333/.287/.282 .228/.235/.223 .211	.240/.154/.155 .109/.080/.099 .119	.330/.205/.157 .164/.153/.123 .160
β_1=0, β_2=.15	.448/.282/.255 .194/.161/.186 .264	.559/.455/.458 .369/.367/.317 .406	.774/.721/.692 .641/.635/.613 .590	.445/.283/.350 .223/.189/.223 .288	.580/.440/.352 .367/.385/.267 .368
β_1=0, β_2=.2	.657/.454/.428 .365/.308/.322 .496	.852/.779/.736 .689/.658/.578 .774	.969/.951/.931 .931/.952/.902 .934	.663/.477/.504 .367/.308/.370 .532	.854/.738/.651 .656/.705/.535 .701
β_1=.1, β_2=.1	.161/.076/.065 **.045/.022/.040** .075	.101/.070/**.060** **.047/.029/.029** .068	.123/.098/.081 .065/**.030/.058** .086	.173/.099/**.062** .077/**.039/.058** .074	.170/.112/**.062** .073/**.042/.053** .076
β_1=.1, β_2=.15	.218/.098/.076 .059/.038/.055 .120	.190/.147/.121 .096/.058/.072 .170	.313/.260/.198 .195/.155/.173 .236	.176/.093/.090 .053/.026/.058 .109	.280/.204/.077 .169/.133/.102 .164
β_1=.1, β_2=.2	.375/.196/.150 .125/.062/.138 .243	.502/.405/.269 .287/.223/.207 .437	.765/.714/.570 .628/.586/.521 .727	.324/.165/.193 .111/.058/.114 .248	.584/.492/.214 .414/.387/.312 .412
β_1=.15, β_2=.15	.101/**.026/.021** **.015/.006/.014** **.041**	.067/**.038/.023** **.024/.013/.014** **.059**	.072/**.063/.030** **.038/.020/.022** .075	.123/.093/**.020** .067/**.035/.036** **.054**	.132/.083/**.019** **.060/.030/.033** .077
β_1=.15, β_2=.2	.137/.053/.032 .031/.013/.030 .080	.176/.142/.061 .076/.035/.033 .159	.254/.234/.098 .165/.128/.084 .295	.125/.053/.051 .021/.010/.025 .100	.304/.262/.031 .199/.170/.112 .157
β_1=.2, β_2=.2	**.050/.008/.003** **.003/.007/.006** **.012**	**.021/.015/.002** **.011/.002/.003** **.019**	**.027/.017/.002** **.011/.007/.002** **.035**	.101/.094/**.001** .067/**.049/.028** **.017**	.097/.111/**.003** .076/**.056/.038** **.030**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.20: Exponential Mean Relationship, ρ=.2, %ties=25, dist=gamma (r=2)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.245/.094/.096 .065/**.029/.048** .102	.164/.100/.099 **.058/.043/.058** .096	.135/.108/.114 .073/**.044**/.086 .106	.215/.088/.093 .067/**.036/.060** .090	.232/.120/.096 .076/.**050**/.071 .099
β_1=0, β_2=.1	.282/.128/.137 .088/.035/.080 .122	.285/.175/.178 .129/.099/.116 .178	.372/.294/.295 .217/.163/.219 .250	.287/.158/.173 .106/.064/.117 .144	.315/.168/.137 .113/.077/.088 .135
β_1=0, β_2=.15	.424/.235/.233 .169/.078/.149 .267	.551/.400/.392 .328/.237/.259 .401	.745/.658/.644 .573/.512/.548 .594	.422/.251/.294 .195/.113/.202 .268	.540/.355/.307 .288/.207/.226 .358
β_1=0, β_2=.2	.633/.417/.386 .320/.209/.295 .473	.849/.730/.695 .645/.500/.563 .770	.973/.940/.923 .896/.872/.868 .928	.671/.454/.526 .362/.259/.366 .566	.830/.664/.598 .576/.433/.467 .695
β_1=.1, β_2=.1	.200/.068/.**062** **.043/.019/.040** .093	.181/.088/.079 **.053/.024/.058** .101	.147/.085/.072 **.050/.034/.050** .097	.210/.088/.**057** **.055/.032/.053** .095	.219/.082**/.059** **.051/.040/.034** .091
β_1=.1, β_2=.15	.240/.091/.084 .058/.030/.054 .114	.233/.116/.099 .073/.067/.066 .148	.346/.218/.183 .170/.134/.152 .214	.233/.103/.105 .063/.036/.070 .126	.311/.159/.080 .106/.070/.079 .161
β_1=.1, β_2=.2	.388/.164/.142 .112/.074/.104 .252	.575/.381/.316 .296/.222/.236 .477	.805/.686/.606 .596/.431/.529 .717	.381/.180/.204 .123/.066/.131 .287	.622/.413/.244 .330/.243/.222 .436
β_1=.15, β_2=.15	.153/.**042/.027** **.023/.004/.023** .067	.160/.**056/.034** **.028/.009/.022** .084	.137/.**062/.035** **.032/.024/.028** .099	.198/.074/.**035** **.044/.022/.025** .081	.178/.067/.**033** **.047/.022/.034** .081
β_1=.15, β_2=.2	.192/.055/.034 .020/.013/.020 .105	.288/.116/.066 .073/.038/.045 .186	.402/.249/.147 .176/.112/.117 .326	.182/.067/.060 .036/.016/.027 .109	.367/.189/.037 .129/.098/.072 .200
β_1=.2, β_2=.2	.106/.**009/.002** **.005/.003/.005** **.040**	.121/.**028/.003** **.006/.007/.005** **.060**	.122/.**043/.008** **.017/.008/.006** .070	.171/.**062/.000** **.042/.029/.016** **.057**	.153/.**060/.005** **.037/.031/.022** **.055**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.21: Exponential Mean Relationship, ρ=.2, %ties=5, dist=gamma (r=3)

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.182/.080/.070 **.057/.042/.042** .074	.158/.097/.095 .072/**.063/.060** .081	.132/.107/.105 .067/**.047**/.076 .087	.221/.113/.089 .069/**.060**/.064 .095	.217/.112/.093 .076/**.055**/.074 .086
β_1=0, β_2=.1	.258/.124/.127 .087/.074/.074 .119	.228/.155/.158 .114/.094/.107 .129	.290/.244/.234 .174/.145/.185 .183	.246/.154/.155 .112/.094/.107 .115	.281/.165/.128 .109/.089/.097 .146
β_1=0, β_2=.15	.364/.217/.207 .154/.113/.125 .211	.442/.346/.343 .269/.220/.229 .303	.616/.537/.533 .461/.407/.431 .442	.356/.207/.232 .163/.134/.156 .207	.453/.319/.266 .256/.223/.200 .269
β_1=0, β_2=.2	.585/.385/.337 .290/.250/.257 .402	.746/.642/.610 .533/.489/.454 .624	.928/.896/.876 .855/.828/.817 .875	.548/.368/.428 .292/.250/.294 .423	.781/.653/.532 .562/.504/.434 .600
β_1=.1, β_2=.1	.190/.071/.071 **.050/.039/.047** .082	.146/.084/.083 **.056/.041/.048** .092	.110/.084/.075 **.042/.036/.051** .079	.163/.095/**.063** .067/**.054/.053** .083	.174/.106/.074 .070/**.055/.051** .080
β_1=.1, β_2=.15	.193/.078/.080 .050/.033/.037 .100	.201/.127/.115 .084/.053/.060 .140	.251/.197/.173 .137/.117/.138 .195	.177/.092/.102 .068/.053/.072 .110	.257/.189/.093 .134/.118/.105 .141
β_1=.1, β_2=.2	.344/.166/.135 .110/.091/.102 .197	.479/.380/.310 .269/.211/.222 .400	.699/.627/.553 .536/.487/.453 .637	.326/.176/.209 .109/.079/.125 .218	.531/.430/.205 .346/.298/.241 .371
β_1=.15, β_2=.15	.106/**.037/.035** **.021/.018/.030** .071	.085/**.054/.039** **.028/.029/.027** .080	.109/.076/**.054** **.048/.037/.039** .096	.131/.087/**.038** **.063/.061/.042** .069	.142/.087/**.033** **.048/.043/.035** .068
β_1=.15, β_2=.2	.177/.065/.048 .037/.028/.038 .093	.207/.144/.095 .091/.066/.052 .195	.305/.276/.170 .194/.167/.151 .299	.132/.056/.061 .029/.022/.035 .087	.279/.216/.049 .172/.146/.103 .142
β_1=.2, β_2=.2	.078/**.016/.009** **.007/.004/.007** **.025**	**.040/.021/.009** **.013/.006/.005** **.037**	**.036/.031/.009** **.018/.011/.007** **.032**	.104/.098/**.002** .066/**.052/.032** **.022**	.100/.077/**.006** **.050/.049/.023** **.029**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.22: Exponential Mean Relationship, ρ=.2, %ties=25, dist=gamma (r=3)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.227/.085/.085 **.047/.025/.047** .086	.147/.077/.079 **.050/.034/.049** .084	.108/.077/.078 **.044/.045/.058** .083	.210/.082/.073 **.053/.045/.054** .089	.219/.088/.091 .066/**.043/.060** .086
β_1=0, β_2=.1	.300/.137/.134 .092/.042/.085 .171	.293/.191/.189 .148/.096/.136 .216	.411/.337/.341 .271/.191/.277 .352	.286/.147/.177 .106/.073/.103 .176	.354/.185/.181 .133/.081/.120 .201
β_1=0, β_2=.15	.453/.253/.244 .184/.108/.157 .334	.642/.468/.457 .379/.281/.334 .546	.820/.733/.726 .680/.586/.643 .793	.470/.272/.327 .209/.150/.230 .383	.624/.430/.374 .350/.249/.283 .448
β_1=0, β_2=.2	.704/.466/.439 .379/.230/.334 .602	.897/.790/.746 .699/.613/.636 .873	.990/.972/.962 .957/.942/.936 .987	.708/.472/.535 .355/.300/.372 .657	.887/.740/.679 .660/.557/.553 .824
β_1=.1, β_2=.1	.195/**.061/.052** **.035/.015/.041** .083	.163/.069/**.059** **.055/.030/.042** .086	.141/.075/.072 **.047/.037/.050** .079	.206/.084/.069 **.058/.030/.043** .088	.203/.091/.068 **.058/.040/.048** .094
β_1=.1, β_2=.15	.260/.091/.083 .061/.034/.055 .125	.306/.168/.139 .107/.047/.083 .202	.359/.240/.185 .171/.151/.145 .260	.223/.095/.103 .055/.029/.072 .139	.359/.192/.104 .143/.086/.095 .196
β_1=.1, β_2=.2	.470/.204/.154 .132/.075/.119 .291	.653/.441/.356 .339/.235/.255 .568	.881/.767/.642 .672/.565/.573 .836	.416/.188/.222 .124/.069/.146 .337	.693/.463/.267 .387/.271/.265 .484
β_1=.15, β_2=.15	.146/**.027/.014** **.014/.007/.010** **.046**	.150/**.043/.024** **.022/.015/.017** **.059**	.145/**.060/.029** **.036/.013/.027** .072	.185/.068/**.026** **.043/.025/.027** .069	.191/.076/**.029** **.052/.032/.026** .086
β_1=.15, β_2=.2	.215/.054/.027 .027/.009/.022 .079	.324/.135/.059 .083/.045/.040 .183	.491/.274/.127 .181/.146/.116 .341	.158/.045/.042 .020/.009/.028 .088	.422/.203/.031 .149/.102/.081 .155
β_1=.2, β_2=.2	.101/**.009/.000** **.005/.001/.004** **.016**	.102/**.027/.002** **.005/.003/.003** **.034**	.113/**.032/.003** **.011/.002/.002** **.028**	.169/**.062/.001** **.045/.029/.014** **.028**	.152/**.056/.000** **.041/.017/.016** **.025**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.23: Exponential Mean Relationship, ρ=.2, %ties=5, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.201/.085/.087 **.054/.033/.055** .084	.123/.071/.072 **.043/.042/.042** .084	.124/.094/.092 **.062/.049**/.078 .110	.216/.103/.078 .066/**.040/.049** .090	.205/.088/.076 **.061/.033/.059** .089
β_1=0, β_2=.1	.277/.140/.136 .098/.054/.089 .137	.270/.185/.176 .140/.098/.118 .191	.369/.315/.314 .238/.117/.238 .301	.238/.137/.137 .093/.066/.090 .120	.306/.193/.152 .157/.108/.113 .167
β_1=0, β_2=.15	.439/.242/.230 .171/.130/.154 .284	.576/.469/.457 .362/.259/.312 .480	.787/.735/.714 .655/.552/.645 .728	.440/.272/.307 .186/.146/.192 .329	.585/.414/.325 .352/.283/.264 .384
β_1=0, β_2=.2	.696/.461/.429 .375/.250/.337 .509	.865/.770/.724 .681/.613/.584 .804	.980/.970/.959 .952/.930/.929 .984	.690/.495/.556 .409/.278/.412 .614	.881/.770/.651 .692/.592/.555 .744
β_1=.1, β_2=.1	.150/.065/**.051** **.033/.020/.036** **.061**	.142/.079/.078 **.051/.023/.044** .081	.120/.085/.064 **.053/.032/.051** .083	.175/.099/.065 .069/**.058/.053** .074	.158/.089/**.056** .067/**.047/.054** .077
β_1=.1, β_2=.15	.190/.077/.063 .041/.035/.039 .099	.205/.140/.123 .097/.063/.075 .153	.298/.233/.188 .177/.125/.158 .238	.194/.095/.104 .061/.027/.061 .103	.300/.207/.083 .146/.118/.095 .136
β_1=.1, β_2=.2	.373/.187/.135 .121/.075/.117 .200	.551/.434/.310 .322/.222/.245 .444	.823/.769/.627 .692/.596/.597 .769	.253/.180/.200 .126/.063/.158 .244	.643/.501/.238 .428/.333/.321 .402
β_1=.15, β_2=.15	.114/**.039/.019** **.016/.011/.021** **.035**	.102/**.052/.026** **.032/.019/.018** **.047**	.077/**.057/.026** **.029/.026/.024** **.059**	.127/.091/**.026** **.060/.035/.043** **.046**	.138/.088/**.023** **.049/.040/.038** **.041**
β_1=.15, β_2=.2	.179/.055/.028 .027/.009/.033 .053	.195/.117/.052 .072/.040/.048 .119	.317/.271/.105 .193/.130/.120 .240	.118/.054/.053 .023/.016/.021 .055	.330/.269/.027 .210/.163/.138 .103
β_1=.2, β_2=.2	**.044/.013/.002** **.004/.003/.005** **.004**	**.030/.012/.002** **.004/.003/.000** **.007**	**.024/.023/.001** **.009/.005/.005** **.008**	.085/.079/**.003** **.057/.029/.022** **.006**	.100/.089/**.001** **.059/.051/.024** **.001**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.24: Exponential Mean Relationship, ρ=.2, %ties=25, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.134/**.030/.031 .035/.030/.025 .045**	.091/**.050/.054 .051/.046/.038 .041**	.082/**.059/.056 .053/.056/.057 .053**	.149/**.059/.045 .055/.057/.042 .040**	.161/**.063/.054** .066/.064/**.050 .044**
β_1=0, β_2=.2	.323/.146/.138 .146/.145/.119 .136	.407/.250/.255 .259/.250/.203 .213	.579/.467/.470 .475/.470/.466 .349	.322/.169/.189 .164/.171/.154 .149	.430/.235/.193 .242/.236/.173 .184
β_1=0, β_2=.4	.600/.356/.322 .352/.355/.282 .393	.798/.654/.628 .661/.654/.550 .602	.969/.938/.929 .942/.939/.922 .865	.575/.346/.409 .349/.347/.324 .405	.806/.620/.545 .621/.630/.466 .579
β_1=0, β_2=.7	.858/.629/.549 .617/.620/.498 .733	.985/.927/.895 .930/.927/.866 .940	1/.999/.997 .999/.999/.997 .997	.850/.587/.656 .599/.602/.553 .764	.973/.917/.831 .911/.914/.773 .899
β_1=.2, β_2=.2	.136/**.032/.023 .029/.031/.015 .041**	.111/**.036/.027 .040/.040/.022 .056**	.090/**.042/.033 .040/.044/.040 .060**	.154/**.050/.025 .050/.050/.024** .065	.155/.066/**.026** .067/.066/**.030 .052**
β_1=.2, β_2=.4	.240/.060/.025 .062/.056/.034 .079	.305/.142/.087 .141/.146/.096 .207	.463/.306/.214 .312/.301/.231 .329	.200/.061/.061 .061/.057/.059 .106	.410/.228/.060 .221/.222/.103 .224
β_1=.2, β_2=.7	.477/.197/.117 .205/.202/.159 .389	.739/.483/.295 .498/.492/.310 .647	.943/.863/.715 .865/.867/.764 .910	.400/.141/.135 .141/.146/.133 .324	.789/.602/.220 .604/.592/.372 .632
β_1=.4, β_2=.4	.107/**.007/.002 .011/.008/.008 .047**	.100/**.028/.008 .032/.031/.011 .060**	.080/**.024/.002 .025/.024/.009 .049**	.158/**.060/.004 .059/.061/.018** .067	.146/**.057/.002 .061/.058/.023 .053**
β_1=.4, β_2=.7	.165/.039/.010 .039/.037/.020 .119	.284/.116/.020 .115/.108/.043 .216	.492/.278/.076 .282/.288/.157 .422	.115/.024/.013 .024/.020/.012 .096	.414/.249/.012 .245/.247/.100 .265
β_1=.7, β_2=.7	**.053/.005/.000 .005/.005/.001 .022**	**.055/.006/.000 .003/.009/.002 .044**	**.061/.022/.001 .019/.020/.003 .060**	.135/.073/**.000** .077/.077/**.023 .046**	.128/.070/**.000** .073/**.060/.022 .040**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.25: Linear Mean Relationship, ρ=0, %ties=5, dist=gamma (r=2)

dlt/bt/nc **ef1/ef2/mbb** **pr**	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.165/**.040/.041** **.047/.039/.029** **.050**	.080/**.044/.045** **.051/.046/.033** **.047**	.074/**.039/.044** **.045/.046/.055** **.046**	.150/**.058/.049** **.061/.059/.049** **.045**	.171/.069/**.045** .066/**.062/.046** **.045**
β_1=0, β_2=.2	.281/.124/.111 .114/.121/.093 .100	.273/.183/.191 .191/.182/.146 .144	.397/.346/.326 .339/.338/.329 .247	.234/.115/.135 .120/.122/.097 .099	.357/.209/.144 .207/.216/.125 .138
β_1=0, β_2=.4	.482/.283/.270 .297/.295/.223 .257	.657/.545/.502 .529/.544/.419 .406	.874/.840/.804 .838/.829/.799 .638	.515/.328/.380 .314/.329/.276 .316	.701/.555/.432 .549/.556/.405 .406
β_1=0, β_2=.7	.758/.552/.466 .560/.560/.440 .549	.910/.857/.795 .855/.864/.747 .791	.996/.994/.988 .992/.994/.987 .960	.721/.491/.560 .491/.491/.459 .535	.930/.866/.731 .870/.869/.723 .785
β_1=.2, β_2=.2	.127**/.044/.027** **.046/.042/.028** **.036**	.074/**.044/.031** **.042/.045/.025** **.046**	.073/**.050/.035** **.047/.050/.046** **.062**	.116/.075**/.029** .076/.081/**.042** **.049**	.144/.083/**.037** .080/.086/**.048** **.060**
β_1=.2, β_2=.4	.165/.066/.042 .069/.063/.052 .087	.194/.133/.088 .141/.141/.077 .134	.305/.266/.177 .269/.264/.193 .198	.136/.048/.052 .054/.054/.047 .057	.265/.213/.074 .217/.206/.133 .143
β_1=.2, β_2=.7	.326/.187/.109 .199/.188/.136 .209	.514/.427/.261 .419/.429/.271 .432	.804/.781/.577 .781/.772/.634 .713	.300/.116/.143 .128/.119/.117 .209	.589/.541/.169 .538/.532/.359 .400
β_1=.4, β_2=.4	.073**/.021/.009** **.022/.023/.015** **.031**	**.045/.033/.010** **.036/.029/.012** **.049**	**.038/.031/.011** **.039/.037/.019** **.048**	.092/.078**/.004** .079/.076**/.032** **.045**	.085/.078/**.005** .082/.081**/.033** **.045**
β_1=.4, β_2=.7	.131/.041/.018 .042/.040/.032 .070	.137/.110/.024 .100/.109/.044 .150	.203/.235/.057 .232/.250/.101 .243	.071/.027/.014 .025/.029/.023 .044	.281/.301/.011 .294/.299/.148 .174
β_1=.7, β_2=.7	**.031/.011/.002** **.012/.012/.005** **.014**	**.009/.016/.000** **.015/.011/.003** **.021**	**.005/.021/.001** **.023/.025/.001** **.039**	**.061/**.082/**.001** .085/.080**/.025** **.018**	**.061**/.083/**.001** .093/.096/**.032** **.025**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.26: Linear Mean Relationship, ρ=0, %ties=25, dist=gamma (r=2)

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.183/**.057/.063 .058/.058/.046 .058**	.100/**.049/.051 .051/.047/.041 .051**	.064/**.045/.046 .048/.049/.050 .042**	.164/**.061/.054** .064/.066/**.042 .049**	.162/.067/**.052** .069/**.063/.043 .050**
β_1=0, β_2=.2	.252/.107/.100 .110/.099/.079 .100	.297/.174/.177 .179/.171/.152 .164	.407/.301/.299 .299/.296/.322 .230	.260/.124/.138 .116/.119/.114 .093	.366/.201/.165 .190/.201/.132 .148
β_1=0, β_2=.4	.500/.251/.229 .250/.248/.204 .281	.655/.492/.472 .485/.497/.401 .454	.889/.800/.790 .806/.807/.791 .726	.471/.266/.318 .276/.260/.260 .278	.659/.460/.385 .452/.449/.327 .405
β_1=0, β_2=.7	.743/.492/.454 .502/.488/.395 .588	.948/.848/.809 .844/.845/.765 .858	.999/.997/.994 .998/.997/.991 .988	.742/.493/.556 .492/.492/.455 .629	.933/.830/.751 .830/.827/.674 .814
β_1=.2, β_2=.2	.166/**.056/.046 .055/.058/.041** .069	.097/**.045/.033 .035/.038/.030 .046**	.104/**.049/.039 .050/.055/.054 .056**	.170/**.053/.038** .066/**.058/.042 .060**	.161/**.060/.040 .059/.054/.044 .059**
β_1=.2, β_2=.4	.205/.063/.040 .059/.057/.037 .086	.275/.126/.087 .124/.126/.088 .157	.407/.253/.192 .255/.249/.211 .263	.194/.069/.076 .068/.065/.058 .093	.334/.156/.056 .154/.162/.083 .134
β_1=.2, β_2=.7	.453/.193/.134 .187/.200/.135 .304	.690/.431/.300 .424/.429/.304 .566	.920/.806/.667 .804/.813/.711 .832	.393/.169/.190 .171/.171/.152 .314	.747/.482/.216 .493/.483/.272 .511
β_1=.4, β_2=.4	.108/**.023/.009 .024/.022/.011 .033**	.102/**.024/.013 .030/.030/.015 .037**	.098/**.027/.011 .033/.030/.016 .045**	.133/**.054/.013 .059/.052/.026 .057**	.165/**.049/.010 .054/.055/.024** .067
β_1=.4, β_2=.7	.182/.037/.010 .045/.044/.019 .105	.290/.105/.030 .111/.114/.042 .194	.457/.278/.096 .284/.276/.163 .369	.135/.030/.023 .031/.028/.020 .076	.411/.212/.014 .205/.206/.096 .230
β_1=.7, β_2=.7	.083/**.008/.001 .011/.012/.003 .029**	.080/**.014/.001 .007/.014/.002 .045**	.065/**.022/.002 .020/.015/.003 .042**	.127**/.049/.001 .049/.052/.017 .036**	.120**/.053/.000 .056/.052/.014 .044**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.27: Linear Mean Relationship, ρ=0, %ties=5, dist=gamma(r=3)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.171/**.044/.041** **.047/.045/.032** **.037**	.087/**.043/.047** **.046/.045/.033** **.038**	**.060/.039/.042** **.040/.039/.050** **.048**	.144/.064/**.058** **.062/.059/.051** **.044**	.137/**.053/.042** **.053/.055/.032** **.031**
β_1=0, β_2=.2	.219/.105/.091 .102/.096/.071 .079	.240/.156/.148 .152/.151/.106 .111	.293/.229/.239 .244/.234/.223 .169	.216/.121/.120 .122/.112/.101 .096	.284/.169/.119 .160/.153/.102 .112
β_1=0, β_2=.4	.388/.229/.207 .223/.227/.170 .184	.499/.387/.365 .390/.387/.325 .300	.761/.689/.676 .689/.693/.670 .535	.406/.219/.267 .220/.219/.205 .202	.551/.407/.312 .404/.401/.271 .259
β_1=0, β_2=.7	.616/.424/.350 .406/.408/.335 .395	.825/.751/.695 .754/.749/.627 .676	.977/.962/.943 .961/.967/.943 .897	.637/.425/.488 .414/.422/.392 .441	.844/.756/.616 .756/.753/.590 .661
β_1=.2, β_2=.2	.112/**.046/.041** **.050/.054/.033** **.048**	.089/**.046/.040** **.041/.046/.034** **.045**	**.060/.044/.033** **.036/.044/.036** **.052**	.126/.064/**.038** .070/.071/**.040** **.040**	.126/.079/**.034** .073/.071/**.039** **.047**
β_1=.2, β_2=.4	.163/.069/.049 .076/.072/.050 .070	.186/.122/.089 .126/.128/.078 .112	.240/.210/.166 .211/.204/.176 .158	.145/.075/.076 .071/.067/.058 .060	.259/.197/.075 .197/.201/.120 .121
β_1=.2, β_2=.7	.353/.183/.119 .179/.184/.135 .192	.456/.362/.246 .355/.352/.226 .344	.766/.719/.546 .716/.718/.604 .610	.287/.143/.152 .132/.142/.138 .169	.569/.477/.179 .471/.463/.306 .353
β_1=.4, β_2=.4	.086/**.023/.011** **.021/.020/.015** **.028**	.082/**.042/.021** **.052/.048/.020** **.053**	**.054/.037/.017** **.035/.040/.023** **.045**	.094/**.061/.009** .064/.064/**.034** **.034**	.109/.074/**.013** .076/.072/**.039** **.032**
β_1=.4, β_2=.7	.109/.037/.014 .033/.031/.023 .061	.149/.102/.044 .102/.102/.051 .138	.258/.243/.092 .253/.256/.146 .227	.117/.045/.041 .045/.047/.038 .044	.245/.232/.016 .226/.225/.125 .130
β_1=.7, β_2=.7	**.046/.006/.000** **.012/.008/.006** **.013**	**.016/.018/.000** **.015/.015/.000** **.014**	**.006/.022/.000** **.018/.018/.003** **.030**	**.052**/.072/**.000** .069/.065/**.025** **.018**	.069/.083/**.000** .087/.085/**.025** **.025**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.28: Linear Mean Relationship, ρ=0, %ties=25, dist=gamma (r=3)

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.169/**.043/.048 .049/.045/.040 .042**	.092/**.051/.054 .054/.053/.048 .051**	**.063/.034/.039 .038/.040/.051 .040**	.156/.070/**.050 .063/.063/.055 .050**	.136/**.044/.039 .042/.042/.031 .038**
β_1=0, β_2=.2	.252/.087/.090 .078/.087/.063 .111	.255/.138/.137 .139/.136/.124 .146	.335/.254/.258 .247/.253/.251 .270	.249/.109/.123 .106/.106/.083 .122	.341/.149/.133 .139/.146/.115 .136
β_1=0, β_2=.4	.427/.217/.199 .224/.219/.154 0.267	.624/.440/.431 .436/.438/.387 .474	.876/.793/.770 .781/.776/.764 .802	.484/.279/.322 .274/.276/.239 .356	.636/.411/.349 .403/.412/.311 .400
β_1=0, β_2=.7	.744/.472/.423 .469/.485/.366 .592	.950/.849/.802 .851/.849/.760 .903	.999/.992/.984 .993/.991/.983 .995	.734/.481/.537 .476/.475/.456 .666	.927/.799/.690 .794/.791/.646 .816
β_1=.2, β_2=.2	.160/**.033/.031 .034/.036/.032 .050**	.101/**.036/.033 .036/.041/.027 .045**	.089/**.039/.034 .043/.043/.044 .033**	.152/**.051/.032 .054/.050/.040 .044**	.164/**.056/.030 .048/.049/.035 .046**
β_1=.2, β_2=.4	.185/.047/.035 .046/.042/.031 .071	.246/.107/.085 .107/.102/.087 .137	.393/.257/.180 .247/.249/.224 .268	.162/.043/.051 .044/.047/.050 .065	.348/.161/.079 .174/.156/.091 .154
β_1=.2, β_2=.7	.414/.163/.116 .164/.170/.116 .235	.695/.437/.303 .440/.452/.319 .545	.902/.800/.667 .806/.803/.717 .844	.413/.179/.203 .176/.173/.169 .323	.739/.495/.250 .500/.497/.305 .499
β_1=.4, β_2=.4	.105/**.015/.006 .016/.016/.007 .031**	.126/**.023/.010 .029/.028/.015 .034**	.104/**.035/.012 .031/.029/.020 .032**	.148/**.036/.006 .041/.035/.015 .036**	.160/**.052/.008 .049/.054/.021 .034**
β_1=.4, β_2=.7	.206/.044/.015 .041/.042/.026 .080	.308/.111/.044 .111/.115/.049 .163	.519/.309/.138 .297/.307/.182 .352	.166/.027/.026 .028/.030/.023 .066	.397/.203/.024 .198/.198/.085 .175
β_1=.7, β_2=.7	.075/**.008/.001 .004/.008/.001 .014**	.066/**.012/.001 .016/.022/.002 .019**	**.050/.014/.001 .011/.015/.001 .023**	.118/**.053/.001 .052/.048/.013 .025**	.106/**.039/.001 .040/.043/.008 .031**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.29: Linear Mean Relationship, ρ=0, %ties=5, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.162/**.055/.045** **.054/.051/.038** **.042**	.092/**.042/.048** **.039/.047/.035** **.050**	**.055/.043/.044** **.043/.041/.049** **.040**	.137/**.054/.046** **.051/.054/.032** **.050**	.162/.067/**.050** **.061**/.066/**.046** **.052**
β_1=0, β_2=.2	.221/.103/.097 .102/.102/.074 .092	.220/.138/.136 .134/.143/.106 .160	.329/.263/.259 .262/.272/.267 .264	.225/.108/.110 .108/.107/.092 .097	.290/.170/.125 .172/.170/.125 .130
β_1=0, β_2=.4	.390/.211/.204 .211/.219/.163 .206	.535/.410/.397 .412/.410/.321 .393	.759/.692/.659 .696/.686/.663 .677	.407/.221/.281 .230/.229/.210 .258	.573/.403/.299 .404/.389/.291 .333
β_1=0, β_2=.7	.656/.429/.377 .420/.436/.336 .430	.851/.770/.690 .763/.759/.661 .728	.983/.972/.948 .973/.971/.955 .963	.653/.432/.478 .425/.427/.388 .496	.877/.767/.639 .764/.776/.622 .708
β_1=.2, β_2=.2	.120/**.046/.032** **.039/.041/.027** **.043**	.078/**.038/.037** **.042/.041/.032** **.041**	**.060/.035/.027** **.036/.041/.037** **.025**	.137/.068/**.037** .070/.075/**.042** **.040**	.132/.066/**.032** .067/.071/**.049** **.038**
β_1=.2, β_2=.4	.161/.054/.046 .057/.063/.045 .063	.190/.118/.092 .124/.123/.080 .107	.250/.209/.152 .207/.200/.183 .170	.144/.056/.053 .054/.047/.054 .055	.264/.184/.066 .178/.183/.107 .115
β_1=.2, β_2=.7	.332/.153/.092 .157/.148/.108 .164	.521/.395/.269 .395/.402/.262 .378	.811/.760/.581 .751/.762/.621 .658	.303/.134/.142 .125/.133/.139 .169	.566/.465/.182 .464/.460/.306 .310
β_1=.4, β_2=.4	.076/**.015/.009** **.014/.018/.013** **.013**	**.052/.033/.013** **.027/.027/.014** **.028**	**.051/.038/.009** **.043/.037/.019** **.013**	.093/**.052/.012** **.049/.053/.029** **.031**	.088/.064/**.013** **.058/.058/.038** **.027**
β_1=.4, β_2=.7	.139/.050/.018 .052/.056/.035 .039	.169/.118/.038 .122/.110/.061 .114	.265/.232/.072 .232/.243/.127 .177	.099/.046/.028 .035/.037/.043 .037	.241/.232/.013 .235/.233/.121 .090
β_1=.7, β_2=.7	**.036/.011/.001** **.009/.011/.005** **.002**	**.008/.006/.001** **.004/.008/.003** **.004**	**.007/.013/.000** **.017/.015/.000** **.011**	**.049/.055/.000** **.054/.056/.019** **.008**	.067/.067/**.001** .069/.064/**.031** **.008**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.30: Linear Mean Relationship, ρ=0, %ties=25, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.151/**.036/.035** **.037/.037/.035** **.044**	.110/**.056/.055** **.059/.061/.052** **.063**	.069/**.052/.050** **.052/.049/.063** **.043**	.167/.073/.067 .067/.073/**.059** **.059**	.155/**.055/.046** **.053/.053/.042** **.050**
β_1=0, β_2=.1	.288/.124/.121 .126/.122/.100 .125	.324/.205/.203 .207/.202/.166 .163	.480/.387/.380 .376/.380/.382 .264	.284/.142/.158 .141/.133/.118 .126	.370/.198/.175 .205/.199/.137 .156
β_1=0, β_2=.15	.506/.288/.272 .282/.277/.215 .292	.701/.509/.484 .502/.507/.423 .479	.922/.849/.834 .846/.847/.817 .776	.519/.303/.359 .293/.302/.279 .343	.707/.493/.412 .492/.486/.371 .474
β_1=0, β_2=.2	.734/.496/.460 .503/.497/.380 .575	.954/.855/.837 .870/.864/.762 .887	.997/.993/.991 .993/.995/.985 .991	.787/.527/.596 .519/.528/.490 .667	.917/.791/.728 .785/.783/.620 .806
β_1=.1, β_2=.1	.117/**.025/.023** **.031/.031/.021** **.039**	.100/**.030/.017** **.023/.030/.015** **.055**	.091/**.045/.028** **.043/.048/.036** **.062**	.158/**.058/.034** .065/**.062/.034** **.059**	.163/**.054/.030** **.047/.054/.029** .069
β_1=.1, β_2=.15	.210/.065/.045 .064/.062/.041 .099	.265/.125/.084 .125/.125/.075 .163	.395/.247/.168 .228/.244/.184 .262	.179/.047/.055 .049/.048/.044 .083	.367/.180/.076 .178/.176/.089 .192
β_1=.1, β_2=.2	.451/.171/.109 .171/.171/.132 .276	.643/.384/.248 .380/.393/.258 .526	.897/.763/.623 .761/.765/.626 .827	.381/.147/.172 .135/.144/.116 .306	.717/.479/.189 .485/.476/.277 .515
β_1=.15, β_2=.15	.120/**.018/.007** **.019/.016/.009** **.040**	.098/**.026/.009** **.020/.027/.010** **.046**	.088/**.022/.008** **.021/.023/.009** **.046**	.151/**.057/.008** **.054/.057/.017** **.059**	.159/**.057/.003** **.054/.054/.021** **.053**
β_1=.15, β_2=.2	.188/.039/.011 .038/.042/.023 .087	.264/.092/.033 .087/.088/.034 .175	.475/.263/.101 .258/.263/.143 .360	.129/.022/.018 .032/.029/.018 .074	.400/.205/.019 .217/.211/.091 .199
β_1=.2, β_2=.2	.068/**.013/.001** **.010/.006/.002** **.011**	.081/**.005/.001** **.010/.011/.000** **.024**	.082/**.018/.001** **.025/.015/.004** **.031**	.141/**.048/.000** **.055/.058/.011** **.016**	.162/.064/**.001** **.063/.072/.017** **.032**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.31: Exponential Mean Relationship, ρ=0, %ties=5, dist=gamma (r=2)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.167/**.054/.047** **.051/.046/.032** **.042**	.098/**.043/.044** **.043/.046/.036** **.054**	.069/**.046/.047** **.049/.047/.058** **.040**	.154/.068/**.048** .065/.067/**.049** **.053**	.146/**.054/.044** **.050/.051/.044** **.041**
β_1=0, β_2=.1	.231/.095/.096 .097/.097/.064 .087	.260/.179/.185 .179/.182/.144 .124	.326/.263/.254 .257/.271/.262 .182	.236/.128/.129 .120/.117/.094 .087	.311/.176/.134 .176/.180/.125 .126
β_1=0, β_2=.15	.438/.245/.226 .249/.249/.170 .238	.552/.428/.404 .429/.426/.341 .361	.825/.768/.747 .767/.770/.727 .627	.430/.263/.315 .265/.264/.241 .233	.603/.451/.353 .449/.448/.310 .355
β_1=0, β_2=.2	.669/.463/.413 .471/.455/.367 .461	.891/.801/.765 .804/.799/.682 .789	.991/.979/.975 .981/.984/.953 .970	.724/.470/.554 .478/.483/.439 .565	.887/.778/.668 .781/.782/.595 .756
β_1=.1, β_2=.1	.125/**.042/.035** **.043/.036/.023** **.046**	.079/**.044/.036** **.044/.043/.022** **.042**	.065/**.041/.036** **.044/.047/.041** **.058**	.126/.065/**.033** .069/.071/**.043** **.053**	.109/**.058/.032** **.060/.057/.031** **.041**
β_1=.1, β_2=.15	.178/.072/.050 .072/.069/.049 .089	.202/.136/.097 .137/.140/.082 .140	.263/.212/.155 .219/.210/.170 .201	.135/.056/.054 .053/.053/.045 .056	.281/.220/.067 .222/.213/.119 .154
β_1=.1, β_2=.2	.387/.167/.100 .171/.178/.120 .180	.521/.403/.293 .394/.409/.252 .434	.794/.729/.587 .736/.735/.616 .736	.337/.161/.189 .161/.151/.150 .249	.624/.500/.181 .479/.488/.321 .388
β_1=.15, β_2=.15	.065/**.013/.007** **.018/.018/.007** **.021**	**.052/.034/.014** **.035/.031/.018** **.035**	**.045/.035/.008** **.028/.026/.011** **.046**	.098/.065/**.005** .069/.068/**.025** **.041**	.098/.077/**.014** .073/.075/**.041** **.043**
β_1=.15, β_2=.2	.135/.046/.015 .045/.045/.025 .061	.170/.115/.033 .117/.109/.040 .128	.268/.237/.080 .235/.250/.111 .308	.093/.027/.027 .024/.024/.018 .054	.276/.252/.015 .243/.253/.123 .135
β_1=.2, β_2=.2	**.055/.007/.002** **.005/.007/.002** **.005**	**.012/.006/.000** **.005/.004/.000** **.008**	**.017/.024/.000** **.024/.025/.004** **.023**	.089/.089/**.000** .091/.090/**.021** **.012**	.081/.074/**.001** .073/.070/**.023** **.009**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.32: Exponential Mean Relationship, ρ=0, %ties=25, dist=gamma (r=2)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.161/**.048/.046 .045/.047/.034 .044**	.098/**.052/.055 .044/.046/.044 .043**	.067/**.046/.051 .045/.048/.057 .053**	.168/**.062/.055 .061/.058/.039 .049**	.156/**.058/.055** .064/**.063/.043 .059**
β_1=0, β_2=.1	.232/.082/.088 .090/.085/.059 .075	.237/.149/.147 .138/.141/.119 .123	.348/.262/.274 .262/.263/.265 .220	.226/.107/.110 .102/.096/.083 .084	.282/.125/.110 .126/.127/.090 .114
β_1=0, β_2=.15	.444/.200/.196 .187/.200/.147 .214	.564/.401/.397 .397/.392/.335 .382	.787/.682/.670 .677/.680/.665 .610	.413/.226/.264 .218/.227/.215 .251	.583/.376/.316 .374/.381/.277 .333
β_1=0, β_2=.2	.683/.412/.383 .426/.419/.301 .509	.872/.719/.697 .719/.734/.620 .752	.991/.971/.964 .971/.970/.936 .961	.655/.417/.481 .419/.418/.396 .526	.859/.705/.629 .710/.698/.529 .731
β_1=.1, β_2=.1	.130/**.030/.025 .029/.031/.027 .036**	.100/**.041/.035 .040/.045/.033 .051**	.102/**.049/.040 .047/.048/.047 .060**	.157/**.061/.039 .060/.055/.032 .053**	.151/**.058/.027 .058/.049/.028 .044**
β_1=.1, β_2=.15	.211/.060/.041 .057/.063/.044 .084	.234/.120/.084 .123/.122/.086 .145	.320/.195/.147 .192/.189/.156 .216	.153/.044/.049 .049/.044/.044 .075	.320/.153/.080 .145/.151/.085 .142
β_1=.1, β_2=.2	.394/.161/.120 .160/.158/.115 .233	.632/.367/.286 .372/.376/.269 .490	.830/.688/.569 .687/.682/.571 .736	.383/.144/.182 .148/.152/.137 .258	.662/.391/.212 .403/.395/.236 .427
β_1=.15, β_2=.15	.109/**.021/.011 .021/.026/.014 .044**	.119/**.033/.015 .035/.037/.011** .066	.088/**.024/.007 .028/.028/.014 .055**	.138/**.048/.010 .044/.049/.016 .041**	.133/**.044/.009 .045/.043/.020 .051**
β_1=.15, β_2=.2	.196/.053/.021 .053/.051/.034 .103	.255/.104/.047 .095/.103/.047 .156	.425/.241/.101 .249/.236/.145 .310	.135/.028/.024 .029/.029/.016 .075	.364/.185/.028 .187/.178/.058 .154
β_1=.2, β_2=.2	.080/**.009/.001 .009/.008/.000 .017**	.074/**.016/.002 .015/.017/.004 .034**	.075/**.018/.001 .017/.016/.002 .035**	.145/**.057/.001 .060/.058/.016 .034**	.131/**.047/.003 .043/.044/.011 .029**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.33: Exponential Mean Relationship, ρ=0, %ties=5, dist=gamma (r=3)

dlt/bt/nc ef1/ef2/mbb pr	**n_1=10, n_2=10**	**n_1=20, n_2=20**	**n_1=40, n_2=40**	**n_1=10, n_2=30**	**n_1=30, n_2=10**
β_1=0, β_2=0	.158/**.040/.039** **.042/.041/.032** **.049**	.101/**.051/.054** **.047/.044/.040** **.046**	**.057/.046/.046** **.047/.049/.054** **.051**	.149/**.059/.040** **.057/.059/.040** **.039**	.134/**.061/.042** **.061/.062/.040** **.041**
β_1=0, β_2=.1	.199/.074/.075 .085/.081/.054 .065	.175/.116/.115 .118/.115/.089 .102	.226/.185/.179 .183/.175/.197 .143	.188/.087/.083 .090/.088/.063 .078	.256/.153/.121 .144/.148/.099 .110
β_1=0, β_2=.15	.341/.188/.168 .189/.189/.129 .169	.445/.325/.321 .321/.329/.265 .287	.647/.560/.550 .571/.573/.532 .442	.356/.195/.228 .188/.195/.166 .187	.472/.332/.237 .319/.326/.214 .247
β_1=0, β_2=.2	.604/.375/.345 .382/.385/.281 .419	.790/.673/.642 .672/.670/.542 .644	.958/.934/.917 .932/.937/.904 .898	.603/.394/.452 .401/.394/.350 .442	.797/.658/.558 .655/.650/.487 .598
β_1=.1, β_2=.1	.134/**.042/.039** **.041/.042/.026** **.046**	.065/**.029/.027** **.030/.030/.018** **.039**	.064/**.043/.036** **.042/.038/.047** .064	.152/.082/**.043** .087/.081/**.051** **.060**	.115/**.061/.036** **.059**/.064/**.037** **.050**
β_1=.1, β_2=.15	.157/.060/.052 .058/.062/.048 .063	.162/.107/.086 .102/.105/.070 .122	.233/.183/.149 .180/.190/.165 .163	.143/.065/.065 .069/.069/.064 .074	.252/.161/.064 .158/.160/.087 .121
β_1=.1, β_2=.2	.347/.145/.114 .148/.156/.107 .196	.462/.335/.254 .329/.340/.223 .375	.722/.651/.555 .655/.652/.547 .642	.291/.129/.165 .126/.129/.125 .186	.566/.448/.213 .431/.442/.286 .358
β_1=.15, β_2=.15	.099/**.021/.015** **.027/.029/.014** **.049**	.074/**.040/.024** **.041/.041/.020** **.049**	.068/**.044/.016** **.045/.043/.034** **.053**	.114/.065/**.014** .067/.070/**.032** **.048**	.111/.072/**.014** .073/.078/**.039** **.045**
β_1=.15, β_2=.2	.135/.045/.020 .042/.043/.032 .055	.158/.113/.065 .110/.109/.053 .131	.258/.222/.117 .215/.224/.138 .242	.130/.058/.056 .058/.053/.050 .067	.286/.231/.027 .220/.226/.127 .143
β_1=.2, β_2=.2	**.057/.006/.002** **.011/.008/.009** **.009**	**.025/.013/.002** **.013/.015/.001** **.020**	**.021/.016/.002** **.021/.022/.002** **.023**	.088/.087/**.003** .080/.076/**.019** **.013**	**.057/.063/.004** **.058/.059/.017** **.018**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.34: Exponential Mean Relationship, ρ=0, %ties=25, dist=gamma (r=3)

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.150/**.035/.039 .039/.041/.033 .050**	.097/**.042/.052 .052/.048/.047 .053**	.068/**.043/.050 .049/.048/.057 .043**	.166/.065/**.058** .068/.075/**.050 .052**	.155/**.051/.051 .055/.048/.045 .050**
β_1=0, β_2=.1	.246/.088/.087 .088/.079/.062 .096	.237/.140/.136 .128/.138/.108 .141	.301/.221/.232 .219/.224/.240 .238	.199/.074/.088 .077/.079/.083 .092	.290/.125/.106 .128/.126/.098 .123
β_1=0, β_2=.15	.389/.194/.192 .196/.202/.144 .227	.502/.341/.349 .339/.346/.291 .408	.728/.640/.630 .641/.635/.622 .680	.384/.221/.253 .209/.214/.200 .251	.536/.307/.260 .302/.304/.218 .317
β_1=0, β_2=.2	.662/.370/.344 .374/.375/.276 .501	.871/.711/.686 .693/.714/.595 .816	.988/.965/.955 .965/.961/.941 .986	.638/.387/.453 .386/.388/.369 .580	.848/.627/.565 .637/.627/.469 .707
β_1=.1, β_2=.1	.163/**.055/.047 .048/.050/.038 .054**	.102/**.038/.038 .039/.045/.030 .042**	.087/**.045/.039 .044/.046/.051 .049**	.156/**.055/.039 .051/.058/.040 .054**	.163/**.054/.042 .048/.053/.038 .057**
β_1=.1, β_2=.15	.204/.049/.044 .053/.055/.035 .065	.212/.099/.079 .095/.096/.069 .108	.292/.184/.142 .181/.177/.158 .216	.139/.040/.046 .042/.048/.042 .057	.303/.139/.070 .146/.137/.076 .127
β_1=.1, β_2=.2	.401/.164/.129 .157/.158/.113 .238	.606/.362/.289 .367/.367/.256 .476	.843/.691/.606 .692/.703/.606 .782	.353/.146/.176 .146/.145/.153 .244	.649/.386/.222 .390/.388/.216 .387
β_1=.15, β_2=.15	.132**/.027/.021 .024/.030/.018 .028**	.109/**.026/.012 .028/.023/.015 .038**	.100/**.035/.015 .036/.030/.022 .039**	.136**/.033/.013 .032/.035/.019 .041**	.131/**.055/.015 .044/.057/.021 .034**
β_1=.15, β_2=.2	.219/.050/.032 .060/.054/.038 .085	.292/.118/.061 .119/.122/.056 .148	.451/.240/.127 .242/.246/.156 .311	.159/.040/.038 .044/.041/.032 .069	.363/.173/.034 .167/.168/.084 .137
β_1=.2, β_2=.2	.064/**.009/.002 .005/.010/.004 .012**	.090/**.013/.002 .012/.009/.001 .019**	.081/**.013/.001 .013/.013/.002 .010**	.129/**.036/.001 .034/.038/.009 .023**	.136/**.045/.002 .050/.039/.019 .022**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.35: Exponential Mean Relationship, ρ=0, %ties=5, dist=normal

dlt/bt/nc ef1/ef2/mbb pr	n_1=10, n_2=10	n_1=20, n_2=20	n_1=40, n_2=40	n_1=10, n_2=30	n_1=30, n_2=10
β_1=0, β_2=0	.158/**.052/.046** **.055/.051/.041** **.041**	.113/**.059**/.066 **.059/.062/.042** **.054**	.073/**.053/.049** **.051/.048/.051** **.062**	.156/.067/**.053** .068/.071/**.048** **.046**	.143/**.062/.040** **.057/.057/.033** **.042**
β_1=0, β_2=.1	.213/.082/.075 .083/.080/.053 .080	.192/.129/.134 .126/.128/.098 .128	.252/.197/.199 .200/.205/.202 .205	.212/.080/.079 .079/.085/.062 .078	.233/.108/.087 .105/.104/.079 .090
β_1=0, β_2=.15	.365/.176/.171 .183/.182/.131 .200	.435/.301/.303 .307/.305/.221 .311	.668/.585/.567 .579/.581/.561 .585	.380/.221/.258 .214/.219/.193 .246	.491/.314/.247 .316/.322/.225 .300
β_1=0, β_2=.2	.597/.370/.334 .373/.372/.277 .400	.799/.685/.651 .682/.679/.563 .725	.973/.944/.918 .946/.947/.903 .957	.621/.380/.448 .384/.393/.363 .498	.827/.653/.561 .655/.663/.476 .669
β_1=.1, β_2=.1	.142/**.038/.033** **.040/.042/.023** **.051**	.096/**.051/.047** **.053/.055/.037** **.045**	.070/**.047/.038** **.046/.046/.044** **.045**	.111/**.052/.029** **.053/.051/.030** **.039**	.132/.065/**.035** .064/.064/**.039** **.038**
β_1=.1, β_2=.15	.158/.065/.060 .070/.070/.041 .064	.171/.099/.083 .105/.097/.075 .098	.229/.169/.143 .177/.178/.156 .175	.158/.061/.060 .063/.064/.055 .067	.251/.151/.064 .158/.163/.096 .113
β_1=.1, β_2=.2	.341/.150/.114 .151/.163/.113 .163	.490/.338/.260 .338/.337/.227 .358	.744/.669/.555 .678/.667/.563 .677	.333/.142/.192 .140/.146/.127 .215	.566/.413/.193 .416/.412/.274 .314
β_1=.15, β_2=.15	.079/**.017/.010** **.021/.018/.012** **.025**	.067/**.032/.020** **.032/.033/.014** **.027**	.062/**.042/.019** **.039/.043/.039** **.033**	.120/.068/**.013** .068/.068/**.036** **.028**	.098/**.062/.013** .065/.069/**.033** **.030**
β_1=.15, β_2=.2	.145/.044/.023 .045/.048/.033 .036	.173/.111/.054 .121/.118/.052 .104	.317/.242/.122 .257/.246/.156 .220	.124/.045/.046 .046/.046/.032 .050	.299/.232/.030 .233/.228/.108 .083
β_1=.2, β_2=.2	**.053/.017/.005** **.012/.014/.008** **.002**	**.031/.014/.003** **.016/.016/.002** **.006**	**.022/.020/.001** **.016/.018/.004** **.012**	.084/.075/**.001** .071/.078/**.031** **.007**	.078/.071/**.002** .072/.074/**.025** **.008**

Key: dlt=delta method, bt=bootstrap, nc=null case, ef1=effective sample size bootstrap with selection method 1, ef2=effective sample size bootstrap with selection method 2, mbb= moving blocks bootstrap, and pr=Pearson.

Figure G.36: Exponential Mean Relationship, ρ=0, %ties=25, dist=normal

H. SAS Code for Performing REML Estimation of AR(1) Parameter

Here the SAS code for estimating the AR(1) parameter from a continuous time AR(1) process is given. It is assumed that y has already been detrended and deseasonalized.

```
proc mixed;
 model y=;
 repeated/subject=intercept type=sp(pow)(time);
 make 'covparms' out=mixparms;
run;
```

REFERENCES

Adler, L. M., A modification of Kendall's tau for the case of arbitrary ties in both rankings, *Journal of the American Statistical Association*, 52, 33-35, 1957.

Agresti, S. and J. Pendergast, Comparing mean ranks for repeated measures data, *Communications in Statistics A*, 15, 1417-1434, 1986.

Akritas, M. G., and S. F. Arnold, Fully nonparametric hypotheses for factorial designs, I: Multivariate repeated measures designs, *Journal of the American Statistical Association*, 89, 336-343, 1994.

Albers, W., Combined rank tests for randomly censored paired data, *Journal of the American Statistical Association*, 83(404), 1159-1162, 1988.

Alvo, M., and P. Cabilio, Rank correlation methods for missing data, *Canadian Journal of Statistics*, 23(4), 345-358, 1995.

Alvo, M., and P. Cabilio, Rank test of trend when data are incomplete, *Environmetrics*, 5, 21-27, 1994.

Alvo, M., and P. Cabilio, Testing ordered alternatives in the presence of incomplete data, *Journal of the American Statistical Association*, 90(431), 1015-1024, 1995.

Bartels, R., The rank version of von Neumann's ratio test for randomness, *Journal of the American Statistical Association*, 77, 40-46, 1982.

Bayley, G. V., and J. M. Hammersley, The "effective" number of independent observations in an autocorrelated time series, *Journal of the Royal Statistical Society*, Series B, 8, 184-197, 1946.

Beran, R., Prepivoting to reduce level error of confidence sets, *Biometrika*, 74(3), 457-468, 1987.

Berryman, D., B. Bobee, D. Cluis, and J. Haemmerli, Nonparametric tests for trend detection oin water quality time series, *Water Resources Bulletin*, 24(3), 545-556, 1988.

Booth, J. G., and S. Sarkar, Monte Carlo Approximation of Bootstrap Variances, *The American Statistician,* 52(4), 354-357, 1998.

Box, G. E. P., G. M. Jenkins, and G. C. Reinsel, *Time Series Analysis*, 3rd edition, Prentice Hall, Englewood Cliffs, 1994.

Brodsky, B. E., and B. S. Darkhovsky, *Nonparametric Methods in Change-Point Problems*, Kluer Academic Publishers, Boston,1993.

Brown, B. W., M. Hollander, and R. M. Korwar, Nonparametric tests of independence for censored data, with applications to heart transplant studies, in *Reliability and Biometry*, eds. F. Proschan, and R. J. Serfling, S.I.A.M., Philadelphia, 321-354, 1974.

Carlstein, E., The use of subseries methods for estimating the variance of a general statistic from a stationary time series, *The Annals of Statistics*, 14, 1171-1179, 1986.

Cleveland, W. S., Robust locally weighted regression and smoothing scatterplots, *Journal of the American Statistical Association*, 74, 829-836, 1979.

Cohn, T. A., Adjusted maximum likelihood estimation of the moments of lognormal populations from type I censored samples, *U. S. Geological Survey Open-File Report*, 88-350, 1988.

Csorgo, M., and L. Horvath, Nonparametric tests for the changepoint problem, *Journal of Statistical Planning and Inference*, 17, 1-9, 1987.

Dabrowska, D. M., Rank tests for matched pair experiments with censored data, Technical report 90, University of California, Berkley, 1987.

Dakins, M. E., P. S. Porter, M. West, and S. T. Rao, Using uncensored trace-level measurements to detect trends in ground water contamination, *Water Resources Bulletin*, 32(4), 799-805, 1996.

Darken, P. F., Multiple observations per season with the seasonal Kendall analysis, Technical Report #98-10, Department of Statistics, Virginia Tech, 1998.

Dietz, E. J., and T. J. Killeen, A nonparametric multivariate test for monotone trend with pharmaceutical applications, *Journal of the American Statistical Association*, 76, 169-174, 1981.

Efron, B., Better bootstrap confidence intervals, *Journal of the American Statistical Association*, 82, 171-185, 1987.

Efron, B., *The Jackknife, the Bootstrap, and Other Resampling Plans*, S. I. A. M., Philadelphia, 1982.

Farrell, R., Methods for classifying changes in environmental conditions, *Technical Report* VRF-EPA7.4-FR80-1, Vector Research Inc., Ann Arbor, Michigan, 1980.

Friedman, M., The use of ranks to avoid the assumption of normality implicit in the analysis of variance, *Journal of the American Statistical Association*, 32, 675-701, 1937.

Ghosh, M., J. E. Grizzle, and P. K. Sen, Nonparametric methods in longitudinal studies, *Journal of the American Statistical Association*, 68, 29-36, 1973.

Gilbert, R. O., *Statistical Methods for Environmental Pollution Monitoring*, Van Nostrand Reinhold, New York, 1987.

Gilliom, R. J., R. M. Hirsch, and E. J. Gilroy, Effects of censoring trace-level water-quality data on trend-detection capability, *Environmental Science Technology*, 18(7), 530-535, 1984.

Goodman L. A. and W. H. Kruskal, Measures of association for cross classifications III: approximate sampling theory, *Journal of the American Statistical Association*, 58, 310-364, 1963.

Hall, P., Resampling a coverage process, *Stochastic Processes Applications*, 20, 231-246, 1985.

Hall, P., Theoretical comparison of bootstrap confidence intervals (with discussion), *Annals of Statistics*, 16, 927-985, 1988.

Hall, P., J. L. Horowitz, and B. Jing, On blocking rules for the bootstrap with dependent data, *Biometrika*, 82(3), 561-574, 1995.

Harcum, J. B., J. C. Loftis, and R. C. Ward, Selecting trend tests for water quality series with serial correlation and missing vlaues, *Water Resources Bulletin*, 28(3), 469-478, 1992.

Harrington, D. P., and T. R. Fleming, A class of rank test procedures for censored survival data, *Biometrika*, 69(3), 553-566, 1982.

Helsel, D. H., and R. M. Hirsch, *Statistical Methods in Water Resources*, Elsevier Science Publishers, Amsterdam, 1992.

Hettmansperger, T. P., *Statistical Inference Based on Ranks*, John Wiley and Sons, New York, 1984.

Hettmansperger, T. P., and J. W. McKean, *Robust Nonparametric Statistical Methods*, Arnold, London, 1998.

Hirsch, R. M., Statistical methods and sampling design for estimating step trends in surface-water quality, *Water Resources Bulletin*, 24(3), 493-503, 1988.

Hirsch, R. M., R. B. Alexander, and R. A. Smith, Selection of methods for the detection and estimation of trends in water quality, *Water Resources Research*, 27(5), 803-813, 1991.

Hirsch, R. M., and J. R. Slack, A nonparametric trend test for seasonal data with serial dependence, *Water Resources Research*, 20(6), 727-732, 1984.

Hirsch, R. M., J. R. Slack, and R. A. Smith, Techniques of trend analysis for monthly water quality data, *Water Resources Research*, 18(1), 107-121, 1982.

Hollander, M., and D. A. Wolfe, *Nonparametric Statistical Methods*, John Wiley and Sons, New York, 1973.

Hughes, J. P., and S. P. Millard, A tau-like test for trend in the presence of multiple censoring points, *Water Resources Bulletin*, 24(3), 521-531, 1988.

Jaruskova, D., Some problems with applications of change-point detection methods to environmental data, *Environmetrics*, 8, 469-483, 1997.

Jones, R. H., *Longitudinal Data with Serial Correlation: A State-Space Approach*, Chapman and Hall, London, 1993.

Jones, R. H., Time series analysis with unequally spaced data, *Handbook of Statistics,* vol. 5, eds. E. J. Hannan, P. R. Krishnaiah, and M. M. Rao, Elsevier ,157-177, 1985.

Judge, G. G., W. E. Griffiths, R. C. Hill, H. Lutkepohl, and T. C. Lee, Qualitative and limited dependent variable models, in *The Theory and Practice of Econometrics,* John Wiley and Sons, New York, 1985.

Kendall, M. G., A new measure of rank correlation, *Biometrika*, 30, 81-93, 1938.

Kendall, M. G., *Rank Correlation Methods,* Carles Griffin and Company, London, Fourth edition, 1970.

Kennedy, W. J., and J. E. Gentle, *Statistical Computing*, Marcel Dekker, New York, 1980.

Koch, G. G., The use of nonparametric methods in the statistical analysis of a complex split plot experiment, *Biometrics*, 26(1), 105-128, 1970.

Kunsch, H. R., The jackknife and bootstrap for general stationary observations, *The Annals of Statistics,* 17, 1217-1241, 1989.

Lawrence, A. J., Some autoregressive models for point processes, in *Point Processes and Queueing Problems*, ed. P. Bartfai and J. Tomko, pp.257-275, North-Holland, Amsterdam, 1980.

Lettenmaier, D. P., Detection of trends in water quality data from records with dependent observations, *Water Resources Research*, 12(5), 1037-1046, 1976.

Lettenmaier, D. P., Multivariate nonparametric tests for trend in water quality, *Water Resour. Res.*, 24(3), 503-512, 1988.

Loftis, J. C., C. H. Taylor, A. D. Newell, and P. L. Chapman, Multivariate trend testing lake water quality, *Water Resources Bulletin*, 27(3), 461-473, 1991.

Loftis, J. C., R. C. Ward, R. D. Phillips, and C. H. Taylor, An evaluation of trend detection techniques for use in water monitoring programs, EPA Report no.600 3-89 037, March 1989.

Lombard, F., Rank tests for changepoint problems, *Biometrika*, 74, 615-624, 1987.

Mann, H. B., Non-parametric tests against trend, *Econometrica*, 13, 245-259, 1945.

Miao, B. Q., Inference in a model with at most one slope-change point, *Journal of Multivariate Analysis*, 27, 375-391, 1988.

Millard S. P., and S. J. Deveral, Nonparametric statistical methods for comparing two sites based on data with multiple nondetect limits, *Water Resources Research*, 24(12), 2087-2098, 1988.

Noether, G. E., *Elements of Nonparametric Statistics*, John Wiley and Sons, New York, 1967.

O'Brien, P. C., and T. R. Fleming, A paired Prentice-Wilcoxon test for censored paired data, *Biometrics*, 43, 169-180, 1987.

Pettitt, A. N., A nonparametric approach to the change-point problem, *Applied Statistics*, 28, 126-135, 1979.

Prentice, R. L., Linear rank tests with right censored data, *Biometrika*, 65, 167-179, 1978.

Rheem, S., Multivariate nonparametric trend assessment with environmental applications, Ph. D. dissertation, Department of Statistics, Virginia Tech, Blacksburg, Virginia, 1992.

Rosner, Autoregressive modelling for the analysis of longitudinal data with unequally spaced examinations, *Statistics in Medicine*, vol 7, 59-71, 1988.

SAS Institute Inc. SAS/IML™ Usage and Reference, Version 6, First Edition, Cary, NC: SAS Institute Inc. 1989.

Schemper, M., One- and two-sample tests of Kendall's τ, *Biometrical Journal*, 29(8), 1003-1009, 1987.

Seber, G. A. F., *The Estimation of Animal Abundance*, Macmillian Publishing, New York, 1982.

Sen, P. K., Asymptotic theory of some tests for a possible change in the regression slope occuring at an unknown time point, *Zeitschrift fur Wahrscheinlichkeitstheorie und verwandte Gebiete*, 52, 203-218, 1980.

Sen, P. K., Estimates of the regression coefficient based on Kendall's tau, *Journal of the American Statistical Association*, 63, 1379-1389, 1968a.

Sen, P. K., On a class of aligned rank order tests in two-way layouts, *Annals of Mathematical Statistics*, 39, 1115-1124, 1968b.

Sillitto, G. P., The distribution of Kendall's tau coefficient of rank correlation in rankings containing ties, *Biometrika*, 34, 36-40, 1947.

Smith, E. P., S. Rheem, and G. I. Holtzman, Multivariate assessment of trend in environmental variables, in *Multivariate Environmental Statistics*, eds. G. P. Patil and C. R. Rao, Elsevier Science Publishers, Amsterdam, 1993.

Smith, R. A., G. E. Schwartz, and R. B. Alexander, Regional interpretation of water-quality monitoring data, *Water Resources Research*, 33(12), 2781-2798, 1997.

Tang, S. M., and I. B. MacNeill, Monitoring statistics which have increased power over a reduced time range, *Environmental Monitoring and Assessment*, 23, 189-203, 1992.

Tang, S. M., and I. B. MacNeill, The effect of serial correlation on tests for parameter change at unknown time, *Annals of Statistics*, 21(1), 552-575, 1993.

Taylor, C. H., and J. C. Loftis, Testing for trend in lake and ground water quality time series, *Water Resources Bulletin*, 28(3), 715-726, 1989.

Taylor, G., and W. J. Conover, A nonparametric confidence interval for slope based on Spearman's rho, *Communications in Statistics B*, 17(3), 905-916, 1988.

Thas, O., L. Van Vooren, and J. P. Ottoy, Nonparametric test performance for trends in water quality with sampling design applications, *Water Resources Research*, 34(2), 347-357, 1998.

Thiebaux, H. J., and F. W. Zwiers, The interpretation and estimation of effective sample size, *Journal of Climate and Applied Meteorology*, 23, 800-811, 1984.

Theil, H., A rank-invariant method of linear and polynomial regression analysis, 1, 2, and 3, *Nederl. Akad. Wetensch Proc.*, 53, 386-392, 521-525, 1397-1412, 1950.

Van Belle, G., and J. P. Hughes, Nonparametric tests for trends in water quality, *Water Resources Research*, 20(1), 127-136, 1984.

Veall, M. R., Bootstrapping regresion estimators under first-order serial correlation, *Econometrics Letters*, 21, 41-44, 1986.

Wei, L. J., Z. Ling, and D. Y. Lin, Linear regression analysis of censored survival data based on rank tests, *Biometrika*, 77(4), 845-851, 1990.

Wilcoxon, F., Individual comparisons by ranking methods, *Biometrics*, 1, 80-83, 1945.

Wilks, D. S., Resampling hypothesis tests for autocorrelated fields, *Journal of Climate, 10,* 65-82, 1997.

Wolfe, D. A., and E. Schechtman, Nonparametric statistical procedures for the changepoint problem, *Journal of Statistical Planning and Inference*, 9, 389-396, 1984.

Zetterqvist, L., Asymptotic distribution of Mann's test for trend for m-dependent seasonal observations, *Scandanavian Journal of Statistics*, 15, 81-95, 1988.

Zwiers, F. W., H. von Storch, Taking serial correlation into account in tests of the mean, *Journal of Climate*, 8, 336-351, 1995.

Vita

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