APPENDIX B.

NUMERICAL MODELING AND PROGRAMMING

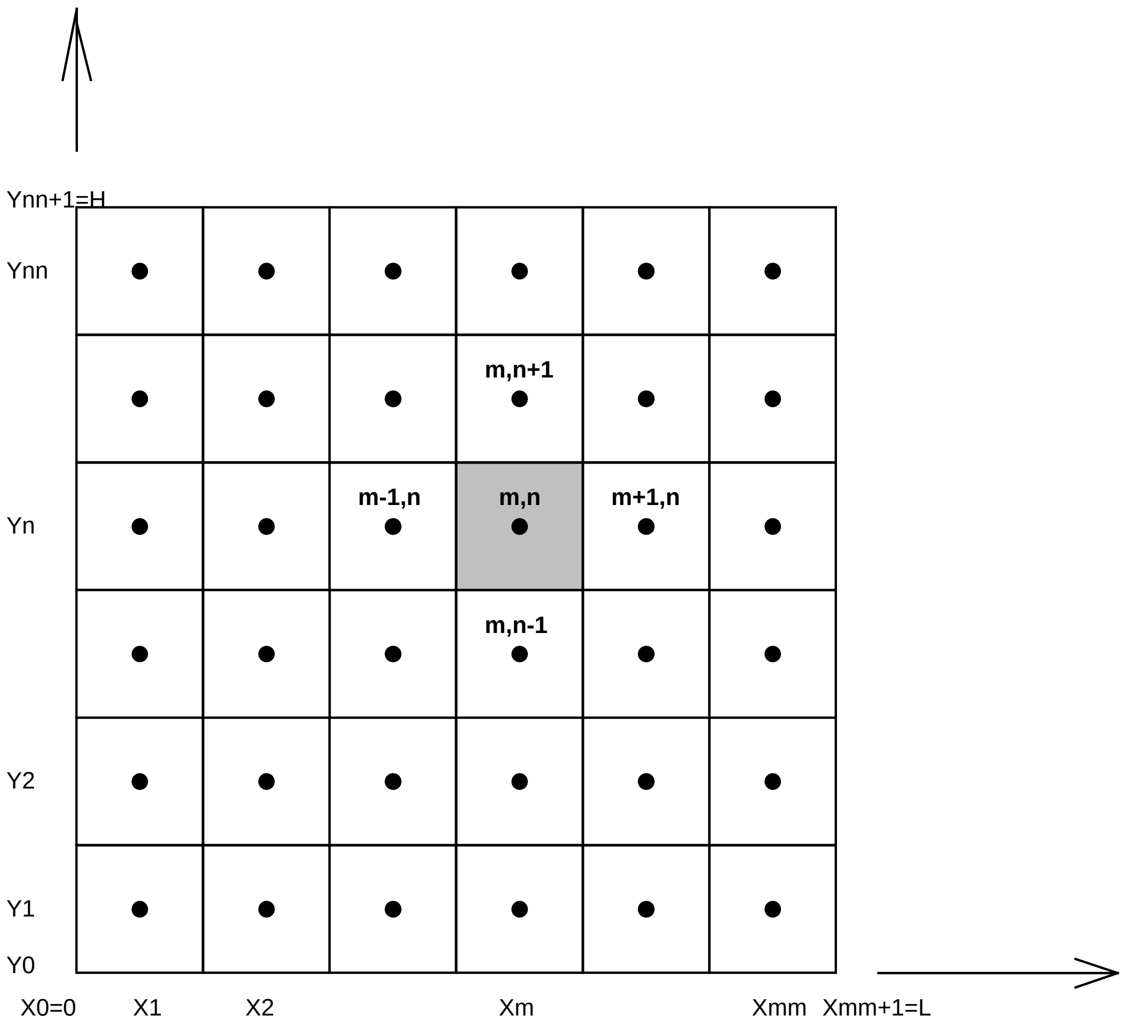
B-1. Derivation of the Finite Difference equations with the control volume for the 2D transient, anisotropic heat conduction model

Deriving the Finite Difference equation within the Control Volume for the 2-Dimensional, transient anisotropic heat conduction

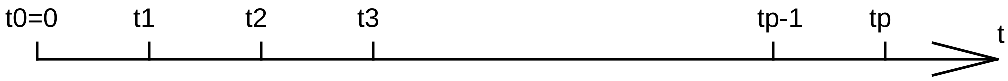
The Energy Equation for the 2-Dimensional transient heat conduction problem is:

$$\frac{\partial}{\partial x}\left(k_T \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_R \frac{\partial T}{\partial y}\right) = \rho C_p \frac{\partial T}{\partial t}$$

The 2 Dimensional domain is discretized as shown:



The time variable is discretized into a sequence of steps:



The node (m,n) is chosen for deriving the Finite Difference Equation. The heat conduction equation is integrated over the Control Volume and over the time interval $_{tp-1}$ to t_p.

$$\int_{t_{p-1}}^{t_p}\int_S^N\int_W^E\left[\frac{\partial}{\partial x}\left(k_T\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_R\frac{\partial T}{\partial y}\right)-\rho C_p\frac{\partial T}{\partial t}\right]dx\,dy\,dt=0$$

Evalute each term in a discretized sence:

$$I=\int_{t_{p-1}}^{t_p}\int_S^N\int_W^E\left[\frac{\partial}{\partial x}\left(k_T\frac{\partial T}{\partial x}\right)\right]dx\,dy\,dt$$

$$=\int_{t_{p-1}}^{t_p}\int_S^N\left[\left(k_T\frac{\partial T}{\partial x}\right)_E-\left(k_T\frac{\partial T}{\partial x}\right)_W\right]dy\,dt$$

$$=\int_{t_{p-1}}^{t_p}\int_S^N\left[k_T\left(\frac{T_{m+1,n}-T_{m,n}}{\Delta x_E}\right)_E-k_T\left(\frac{T_{m,n}-T_{m-1,n}}{\Delta x_W}\right)_W\right]dy\,dt$$

$$=\int_{t_{p-1}}^{t_p}\left[k_T\left(\frac{T_{m+1,n}-T_{m,n}}{\Delta x_E}\right)-k_T\left(\frac{T_{m,n}-T_{m-1,n}}{\Delta x_W}\right)\right]\Delta y_n\,dt$$

For the time domain integral, there are 3 logical options:

$$\int_{t_{p-1}}^{t_p}T\,dt=\begin{cases}T^p\,\Delta t_p & \text{fully implicit}\\ \dfrac{T^p+T^{p-1}}{2}\,\Delta t_p & \text{center difference}\\ T^{p-1}\,\Delta t_p & \text{explicit}\end{cases}$$

The fully implicit method is chosen since it is the most reliable method because of its unconditional stability. So the temperature over each time interval is evaluated at the end of the interval, which is, the temperature T^p at the time level t_p represents the average over the time interval Δt_p preceding time t_p.

Integration of term "I" in this fully implicit fashion produces:

$$I=\left[k_T\left(\frac{T^p{}_{m+1,n}-T^p{}_{m,n}}{\Delta x_E}\right)_E-k_T\left(\frac{T^p{}_{m,n}-T^p{}_{m-1,n}}{\Delta x_W}\right)_W\right]\Delta y_n\,\Delta t_p$$

Integrate term “II”:

$$II = \int_{t_{p-1}}^{t_p}\int_{W}^{E}\int_{S}^{N}\left[\frac{\partial}{\partial y}\left(k_R\frac{\partial T}{\partial y}\right)\right] dy\,dx\,dt$$

$$= \int_{t_{p-1}}^{t_p}\int_{E}^{W}\left[\left(k_R\frac{\partial T}{\partial y}\right)_N - \left(k_R\frac{\partial T}{\partial y}\right)_S\right] dx\,dt$$

$$= \int_{t_{p-1}}^{t_p}\int_{E}^{W}\left[k_R\left(\frac{T_{m,n+1} - T_{m,n}}{\Delta y_N}\right) - k_R\left(\frac{T_{m,n} - T_{m,n-1}}{\Delta y_S}\right)\right] dx\,dt$$

$$= \int_{t_{p-1}}^{t_p}\left[k_R\left(\frac{T_{m,n+1} - T_{m,n}}{\Delta y_N}\right) - k_R\left(\frac{T_{m,n} - T_{m,n-1}}{\Delta y_S}\right)\right] \Delta x_m\,dt$$

$$= \left[k_R\left(\frac{T^p{}_{m,n+1} - T^p{}_{m,n}}{\Delta y_N}\right) - k_R\left(\frac{T^p{}_{m,n} - T^p{}_{m,n-1}}{\Delta y_S}\right)\right] \Delta x_m\,\Delta t_p$$

Integrate term “III”:

$$III = \int_{t_{p-1}}^{t_p}\int_{W}^{E}\int_{S}^{N}\rho C_p\frac{\partial T}{\partial t}\,dx\,dy\,dt$$

$$= \int_{W}^{E}\int_{S}^{N}\rho C_p\,(T^p{}_{m,n} - T^{p-1}{}_{m,n})\,dx\,dy$$

$$= \rho C_p\,(T^p{}_{m,n} - T^{p-1}{}_{m,n})\,\Delta x_m\,\Delta y_n$$

Substitute these three terms into the energy equation:

$$\left[k_T\left(\frac{T^p{}_{m+1,n} - T^p{}_{m,n}}{\Delta x_E}\right)_E - k_T\left(\frac{T^p{}_{m,n} - T^p{}_{m-1,n}}{\Delta x_W}\right)_W\right]\Delta y_n$$

$$\Delta t_p + \left[k_R\left(\frac{T^p{}_{m,n+1} - T^p{}_{m,n}}{\Delta y_N}\right) - k_R\left(\frac{T^p{}_{m,n} - T^p{}_{m,n-1}}{\Delta y_S}\right)\right]\Delta x_m\,\Delta t_p + \rho C_p\,(T^{p-1}{}_{m,n} - T^p{}_{m,n})\,\Delta x_m\,\Delta y_n = 0$$

next divide by $\Delta t_{p:}$

$$\left[k_T\left(\frac{T^p{}_{m+1,n} - T^p{}_{m,n}}{\Delta x_E}\right)_E - k_T\left(\frac{T^p{}_{m,n} - T^p{}_{m-1,n}}{\Delta x_W}\right)_W\right]\Delta y_n + \left[k_R\left(\frac{T^p{}_{m,n+1} - T^p{}_{m,n}}{\Delta y_N}\right) - k_R\left(\frac{T^p{}_{m,n} - T^p{}_{m,n-1}}{\Delta y_S}\right)\right]\Delta x_m +$$

$$\rho C_p\,(T^{p-1}{}_{m,n} - T^p{}_{m,n})\,\frac{\Delta x_m\,\Delta y_n}{\Delta t_p} = 0$$

Then rearrange into the form:

$$a_{m,n}\,T^{p}_{m,n} - aW_{m,n}\,T^{p}_{m-1,n} - aE_{m,n}\,T^{p}_{m+1,n} - aS_{m,n}\,T^{p}_{m,n-1} - aN_{m,n}\,T^{p}_{m,n+1} = b_{m,n}$$

where

$$aW_{m,n} = \frac{k_T}{\Delta x_W}\,\Delta y_n = \Delta y_n \left(\frac{k_T}{\frac{\Delta x_m + \Delta x_{m-1}}{2}}\right);$$

$$aE_{m,n} = \frac{k_T}{\Delta x_W}\,\Delta y_n = \Delta y_n \left(\frac{k_T}{\frac{\Delta x_m + \Delta x_{m+1}}{2}}\right);$$

$$aS_{m,n} = \frac{k_R}{\Delta y_S}\,\Delta x_m = \Delta x_m \left(\frac{k_R}{\frac{\Delta y_n + \Delta y_{n-1}}{2}}\right);$$

$$aN_{m,n} = \frac{k_R}{\Delta y_N}\,\Delta x_m = \Delta x_m \left(\frac{k_R}{\frac{\Delta y_n + \Delta y_{n+1}}{2}}\right);$$

$$aO_{m,n} = (\rho C_p)\,\frac{\Delta x_m\,\Delta y_n}{\Delta t_p};$$

$$a_{m,n} = aO_{m,n} + aW_{m,n} + aE_{m,n} + aS_{m,n} + aN_{m,n};$$

$$b_{m,n} = aO_{m,n}\,T^{p-1}_{m,n}$$

To solve the Finite Difference Equation for the typical control volume at time level p, Line-by-Line method is used. This method is a combination of direct tridiagonal algorithm used for one-dimensional problems and the Gauss-Siedel method. The Line-by-Line method choose one direction (y-direction for example) and assume neighbors at other x-locations are known from the previous iteration, and solve this line of nodes using the tridiagonal algorithm, then move or sweep to the other x-direction. This method is by alternating the sweeps in the x direction and then the y-direction to accelerate convergence.

<u>Boundary Conditions :</u>

The boundary node shown on the 2-Dimensional grid are surrounded by imaginary control volume with zero volume. The equations at the Boundaries remain the same in the transient problem as in the steady state case except that everything is time dependent.

All but one neighbor coefficient is zero on each boundary.

Boundary at x=x0=0 (m=0)

The Boundary condition for some combination of specified heat flux, *qx0*, and convective conditions, *T∞x0* and *hx0*, can be written in the form,

$$-k\,\frac{\partial T}{\partial x} = qx0 + hx0\,(T\infty x0 - T), \text{ at } x = 0$$

The discretized form of this boundray condition is,

$$k_{1,n}\,\frac{T^{p}_{0,n} - T^{p}_{1,n}}{\Delta x_1 / 2} = qx0 + hx0_n\,(T\infty x0_n - T^{p}_{0,n})$$

The discretized equation can be expressed in term of coefficients as:

$$a_{0,n} T^p_{0,n} - aE_{0,n} T^p_{1,n} = b_{0,n}$$

Note that *aW* $_{0,n}$ is not needed at this boundary since there is no west neighbor. The coefficients are,

$$aE_{0,n} = \frac{k_{1,n}}{\triangle x_1 / 2};$$

$$a_{0,n} = aE_{0,n} + hx0;$$

$$b_{0,n} = qx0 + hx0_n * T\infty x0_n$$

If the Boundary condition is a specified surface temperature, $Tx0_n$, we have the special case

$$aE_{0,n} = 0,\ a_{0,n} = 1,\ b_{0,n} = Tx0_n$$

We use an indicator to keep a record of the type of boundary condition we have:

BCx0= **1**, Specified temperature
2, Specified flux
3, Specified convection

Boundary at $x=x_{mm+1}=L$ ***(m=mm+1)***

The similar derivation is for the coefficients at Boundary x=L as it is for the coefficients at Boundary of x=0, but with different subscripts and change *aE* $_{0,n}$ with *aW* $_{0,n}$, because there is no east neighbor at this boundary.

$$k \frac{\partial T}{\partial x} = qxL + hxL\,(T\infty xL - T),\ \text{at } x = L$$

Discretized as

$$k_{mm+1,n} \frac{T^p_{mm+1,n} - T^p_{mm,n}}{\triangle x_{mm} / 2} = qxL_n + hxL_n\,(T\infty xL_n - T^p_{mm+1,n})$$

rearrange as

$$a_{mm+1,n} T^p_{mm,n} - aW_{mm+1,n} T^p_{mm+1,n} = b_{mm+1,n}$$

$$aW_{mm+1,n} = \frac{k_{mm,n}}{\triangle x_{mm} / 2};$$

$$a_{mm+1,n} = aW_{mm+1,n} + hxL_n;$$

$$b_{mm+1,n} = qxL_n + hxL_n * T\infty xL_n$$

This is for the Boundary condition of Specified flux and convection. For the specified temperature boundary condition:

$aW_{mm+1,n} = 0,\ a_{mm+1,n} = 1,\ b_{mm+1,n} = Tx0_n$

Boundary at y=y₀=0 (n=0)

Similar as x=x0, so the coefficients for the specified flux and convection are:

$$aS_{m,0} = \frac{k_{m,1}}{\Delta y_1 / 2};$$
$$a_{m,0} = aS_{m,0} + hy0_m;$$
$$b_{m,0} = qy0_m + hy0_m * T\infty y0_m$$

The coefficients for the specified temperature are:

$aS_{m,0} = 0,\ a_{m,0} = 1,\ b_{m,0} = Ty0_m$

Boundary at y=y_{nn+1}=H (n=nn+1)

Similar as x=xL, so the coefficients for the specified flux and convection are:

$$aN_{m,nn+1} = \frac{k_{m,nn}}{\Delta y_{nn} / 2};$$
$$a_{m,nn+1} = aN_{m,nn+1} + hyL_m;$$
$$b_{m,nn+1} = qyL_m + hyL_m * T\infty yL_m$$

The coefficients for the specified temperature are:

$aN_{m,nn+1} = 0,\ a_{m,nn+1} = 1,\ b_{m,nn+1} = Ty0_m$

B-2. Mathematica Code for calculating the thermal conductivity values in radial and tangential direction by the geometric models and solving the 2D transient heat conduction model in the Mathematica software. *

(* Not available here. Contact the author if you would like to see it)