Availability Analysis for the Quasi-Renewal Process with an Age-Dependent Preventive Maintenance Policy

Boonyarit Intiyot

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Joel A. Nachlas, Chair
Ralph D. Badinelli
C. Patrick Koelling
Ebru K. Bish

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(ABSTRACT)

A quasi-renewal process is more realistic in modeling the behavior of a repairable system than traditional models such as perfect repair and minimal repair since it reflects the deterioration process of the system over time while traditional models do not. The quasi-renewal parameter is set to a value between 0 and 1 to indicate the rate of deterioration. Moreover, a quasi-renewal process can also be used to model the increasing time of maintenance actions due to the increasing difficulty of maintaining an aging system by setting the parameter to a value larger than 1. We construct a model where the operating times follow a quasi-renewal process and the corrective/preventive maintenance times follow another quasi-renewal process. A quasi-renewal function and two equivalent point availability expressions are developed for the model described by a quasi-renewal process with and age-dependent preventive maintenance policy. In addition, numerical results from various theoretical distributions are obtained to illustrate the behavior of the models. The two equivalent point availability functions each contains an infinite sum and must be truncated to obtain a numerical approximation. The two approximated point availability functions form upper and lower bounds on the real value. The bounds are useful for determining the result accuracy, which can be arbitrarily increased by adding more terms to the truncated summation. Our framework provides a new time-dependent availability model for a non-stationary process with a preventive maintenance policy without any cost structure or optimization problem.

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Chapter 1 Introduction

1.1 Background

Our everyday life, personal and business alike, has become more and more dependent on various devices. We use vehicles to make a trip to work. Manufacturers use machinery to make products. For some people, such as patients using artificial hearts, their lives literally rely on their devices. These devices do deteriorate over time and there is a chance that they fail unexpectedly, in which case a repair is necessary. Unexpected failure of these devices not only causes inconvenience but also costs us time, money, and sometimes injuries or even lives. To reduce the chance of unplanned failures, a preventive maintenance plan must be established, as it is important in ensuring that a system performs as designed without interruption.

However, maintenance can be costly, especially for a complex system. It is estimated that in a factory the total maintenance-related cost contributes about 15-40% (with approximately 25% on average) of the total expense [1]. An efficient maintenance plan will improve productivity, reduce cost, and consequently yield a profit gain. Moreover, in cases where devices are critical for maintaining or sustaining life, a good maintenance plan will also reduce the chance of unexpected tragedies.

The interest in reliability and maintenance study has been expanding as equipment become more and more sophisticated. There have been many research studies on maintenance optimization. The goal of maintenance optimization is to maximize availability with minimal cost. This can be accomplished by creating an efficient maintenance plan that reduces the amount of downtime (either from unplanned failures or preventive maintenance) and the number of unexpected failures.

Availability is a measure of system performance in terms of the system's operating state versus time. The operating state of the system is represented by a binary variable that takes the value of 1 if the system is operating and the value of 0 if the system is not. There are four measures of availability.

1. The point availability at time t is the probability that the system is functioning at time t. That is
$$A(t) = \Pr\{X(t) = 1\} = E[X(t)]$$
where $X(t)$ is a binary random variable indicating the operating state of the system at time t.
2. The limiting availability is defined by
$$A_\infty = \lim_{t \to \infty} A(t)$$

3. The average availability over an interval (t_1, t_2) is defined by
$$\bar{A}(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} A(t)dt$$
4. The limiting average availability is defined by
$$\bar{A}_\infty = \lim_{t \to \infty} \bar{A}(0, t)$$

Typically the point availability is the most informative measure but also the most difficult to obtain. The most commonly used measure is the limiting availability because it usually is the easiest to compute.

The background knowledge needed for this research includes probability theory and basic concepts in reliability and maintenance, such as reliability, availability, corrective maintenance, and preventive maintenance. Moreover, background knowledge in stochastic processes, renewal theory, and tools, such as the Laplace transform, are strongly recommended.

There are two types of maintenance, namely corrective and preventive maintenance. Corrective maintenance (CM) is performed at the time the system fails in order to fix it. This includes a repair or a replacement. Well-known types of models of CM include perfect repair, minimal repair and imperfect repair. The perfect repair model assumes that the life distribution of the system after the repair is the same as its life distribution at time 0 ("as good as new.") The minimal repair model assumes that the life distribution of the system after the repair is the same as its life distribution at the time just before it fails ("as bad as old.") The imperfect repair model is defined to be perfect repair with probability p and minimal repair with probability $q = 1 - p$. There are other types of CM models, such as Kijima and quasi-renewal models, that are more realistic but less popular than the aforementioned. We will discuss more about these models in Chapter 2.

Preventive maintenance (PM) usually involves shutting down the system, inspecting and possibly testing its components, and finally repairing or replacing components as needed. In some cases, the system is simply replaced. PM is performed at a scheduled time in order to reduce the chance of a system's unexpected failure. There are two well-known types of PM, namely age replacement and block replacement policies. In an age replacement policy, the device is replaced whenever the device fails or when it has been operating for a certain time period T_a, called the "policy age". In a block replacement policy, the device is replaced whenever the device fails or is replaced periodically with an equal time interval T_b, called the "policy time", between each scheduled replacement (regardless of the device age).

Consider a typical sample path of a system without PM which consists of the alternate intervals between the periods in which the system is functioning (T_i) and the periods in which the system is being repaired (R_i). During the repair periods, the system is down. Figure 1 depicts a representation of a sample path.

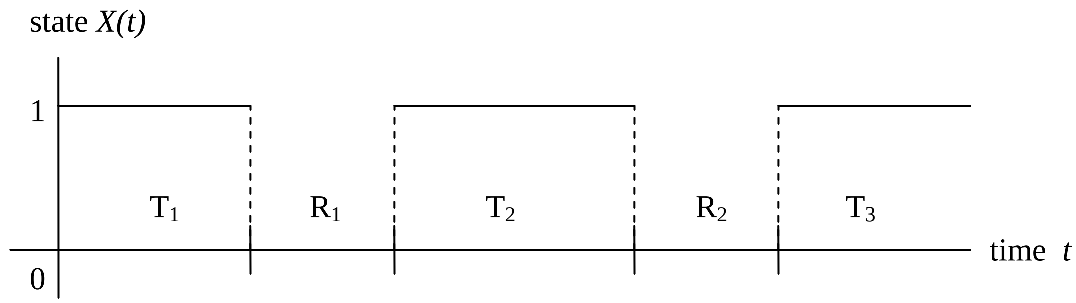

Figure 1: A Representation of a Sample Path

In a simple case, we assume that

- each repair is perfect and, consequently, the operating periods (T_i) are independent and identically distributed
- the repair periods (R_i) are also independent and identically distributed

Then the sample path is a renewal process with renewal points located at the beginning of each operating period $T_i, i = 2,3,4,\ldots$.

The point availability for this simple case can be expressed as

$$A(t) = \bar{F}(t) + \int_0^t \bar{F}(t-x) m_H(x) dx \tag{1.1.1}$$

where $F(t)$ is a distribution of system operating interval for each period T_i and $m_H(x)$ is the renewal density function, which indicates the rate of failures (renewals) at time x. The renewal density function can also be interpreted as the probability that a renewal period ends at x.

1.2 Research Motivation

Many models with various types of corrective and preventive maintenance policies have been studied and analyzed, and most of them are based on renewal processes. However, in many situations a renewal process is not a suitable model. There are a small number of studies that define the model using non-homogeneous processes; most of those studies assume imperfect repair. However, imperfect repair only assumes that the repair restores the system condition to somewhere between perfect and minimal condition. This assumption does not reflect the aging process of the device while quasi-renewal processes are more realistic in regard to aging.

Moreover, since it is often difficult to develop, not many research studies provide point availability analysis for their models. Most studies develop cost models instead. In fact, the point availability is informative and important because it describes the actual behavior of the system. It is even more important when the operating state of the system is a matter of life and death. In other cases, the point availability helps in forecasting and system utilization planning.

A point availability analysis for the quasi-renewal model has been done by Rehmert [2]. However, his model does not include any type of preventive maintenance policy. Murdock [1], has provided an expression for the point availability for an age replacement policy but his model is based on the renewal process. These two works have become our motivation to construct the point availability for the quasi-renewal model with an age dependent preventive maintenance policy.

1.3 Research Objective

Murdock [1] develops a point availability expression for a renewal model with an age replacement PM policy. The Laplace transform of his expression is given by

$$A^*(s) = \frac{R_L^*(s,T)}{1-h^*(s)} \tag{1.3.1}$$

where

T : The policy age.

$R_L^*(s,T)$: The partial or truncated Laplace transform of the system survival function

$R_L(t)$ truncated at T. It is defined by $R_L^*(s,T) = \int_0^T e^{-st} R_L(t)dt$.

$h^*(s)$: The Laplace transform of the renewal density function. It is defined by $h^*(s) = f_L^*(s,T)g_r^*(s) + e^{-sT}R_L(T)g_p^*(s)$, where $f_L^*(s,T)$ is the truncated Laplace transform of the life density function; $g_r^*(s)$ is the Laplace transform of the density function of the repair time; $R_L(T)$ is the survival function evaluated at T; and $g_p^*(s)$ is the Laplace transform of the density function for the PM time.

Rehmert [2] develops two point availability expressions for a different model which employs the quasi-renewal process notion. However, his model does not consider any PM policy. The two expressions are equivalent but are derived differently. His expressions are as follows.

The Laplace transform of the uptime-based availability function:

$$A^*(s) = \frac{1}{s}\left[\left(1 - f_{X_1}^*(s)\right) + \sum_{n=1}^{\infty}\left(1 - f_{X_{n+1}}^*(s)\right)h_{\Theta_n}^*(s)\right] \tag{1.3.2}$$

The Laplace transform of the downtime-based availability function:

$$A^*(s) = \frac{1}{s}\left[1 - \sum_{n=1}^{\infty}\left(\frac{1}{g_{Y_n}^*(s)} - 1\right)h_{\Theta_n}^*(s)\right] \tag{1.3.3}$$

where

$f^*_{X_n}(s)$: The Laplace transform of the density function for the n^{th} operating time X_n.
$g^*_{Y_n}(s)$: The Laplace transform of the density function for the n^{th} repair time Y_n.
$h^*_{\Theta_n}(s)$: The Laplace transform of the quasi-renewal density function for the first n cycles.

Rehmert [2] also develops the expression for quasi-renewal function, which represents the expected number of restarts by a given point in time. His quasi-renewal function is given by

$$Q_F(t) = \sum_{n=1}^{\infty} F_{\Omega_n}(t) \tag{1.3.4}$$

where
$F_{\Omega_n}(t) = \Pr\left[X_1 + X_2 + \cdots + X_n \le t\right]$ and X_i represents the i^{th} operating time. However, in this expression, he assumes the repair times are negligible.

The purpose of this research is to extend the work done by Murdock [1] and Rehmert [2] by developing the quasi-renewal function and the point availability function for models described by a quasi-renewal process with an age-dependent preventive maintenance policy. The term *age-dependent* is used instead of *age replacement* because we do not generally assume renewal after a PM operation. More details on these notations are described in Section 2.2.1. The differences between our availability model and the models from previous studies can be summarized in Table 1.

Modeling Process	**Preventive Maintenance Policy**	
	No PM Policy	Age-Dependent PM Policy
Renewal	Classic model (mentioned earlier on page 3)	Murdock [1] • perfect PM
Quasi-Renewal	Rehmert [2] • neglects repair times in the quasi-renewal function	Our research • imperfect PM • includes repair times in the quasi-renewal function

Table 1 Summary of the differences between our availability model and the models from the previous studies.

By using the quasi-renewal process in our models, we hope to obtain more realistic system representation than from other traditional models. The quasi-renewal function can be used in a cost optimization model while the point availability function can be used to find an optimal policy age that maximizes the availability of the system.

In this research we develop a new availability model assuming the operating times occur according to the quasi-renewal process with an age-dependent preventive maintenance policy incorporated into the model. Some numerical results are also obtained to illustrate the behavior of the model. No optimization problem is considered in this research but one may construct such a problem using our quasi-renewal function and/or the point availability function. It appears that this is the first time that the availability analysis of a non-stationary model with an age-dependent policy is developed. Since a non-stationary model is more realistic than traditional models in many situations and the preventive maintenance becomes increasingly important, this research can make a significant contribution in the reliability arena and may spur more new reliability applications.

Our model assumes that the operating intervals form a quasi-renewal process while the repair intervals and PM intervals also follow a quasi-renewal process. The quasi-renewal function and the point availability expressions are constructed for this model.

The quasi-renewal function, the availability function, and numerical examples are obtained for the following cases:

- Exponential operating intervals and exponential CM and PM intervals
- Gamma operating intervals and gamma CM and PM intervals
- Gamma operating intervals and exponential CM and PM intervals
- Normal operating intervals and normal CM and PM intervals
- Normal operating intervals and exponential CM and PM intervals

We chose the above distributions for our cases because they are widely used in the reliability field. The distributions for the operating intervals have increasing failure rate property, which indicates that the system unit is deteriorating over time. The distribution parameters correspond to those used by Rehmert [2] so that we can compare our results with his results. Evaluating the expressions for these models is done analytically, if possible, and numerically otherwise.

Chapter 2 Literature Review

This chapter provides an overview of the literature related to maintenance models of a single-unit system with the emphasis on the literature related to non-homogeneous process models.

Viewing a complex system as a single entity is a natural way of thinking when we are interested in the behavior of the system as a whole rather than that of the individual components. Moreover, the maintenance analysis for a single entity is more tractable than that of a multi-unit system and can be applied to any component of a composite system.

As mentioned in Chapter 1, maintenance can be categorized into two types, namely corrective maintenance (CM) and preventive maintenance (PM). A CM action is performed when the system fails in order to bring the system back to an operational state. A PM action is performed at a scheduled time while the system is still operating in order to reduce the chance of unexpected failure at least for a certain time period. Literature in maintenance has evolved since 1960. Most recent survey literature on this subject can be found in Valdez-Flores and Feldman [3] and Wang [4]. Murdock[1] and Rehmert [2] also provide extensive literature reviews of research in maintenance.

2.1 Corrective Maintenance

Corrective maintenance can be categorized into perfect, minimal, and general repair.

2.1.1 Perfect Repair

A perfect repair brings back the failed system's operating condition to "as good as new". An example of a perfect repair is replacing a failed system with a brand-new one. Perfect repair models utilize renewal theory and queueing theory results in their analysis. The study of this type of model has been well established and can be found in many textbooks such as Barlow and Proschan [5, 6], and Nachlas [7].

The perfect repair assumption is used in many classic models. However, this concept is not realistic in many situations and this is one of the reasons our model does not assume perfect repair.

2.1.2 Minimal Repair

A minimal repair restores the system's working state to the state just before it fails ("as bad as old"). An example of a minimal repair is replacing a failed minor component of a large composite system. Replacing such a component is not likely to affect the overall operating state of the whole system. The minimal repair model was first mentioned by

Barlow and Hunter [8]. The hazard function (failure rate) of the system is not affected after the minimal repair and therefore does not imply renewal process. In fact, the failures occur according to the non-homogeneous Poison Process (NHPP). The discussion on the properties of NHPP can be found in many probability textbooks such as Ross [9] and Beichelt and Fatti [10]. The implication of NHPP for the minimal repair models is discussed in Barlow and Hunter [8], Nachlas [7], and Beichelt and Fatti [10]. Valdez-Flores and Feldman [3] provide an extensive literature survey on minimal repair models.

The minimal repair assumption is also used in many traditional models. However, many real world situations cannot be modeled using minimal repair assumption. Our model does not assume minimal repair as we prefer a more realistic model.

2.1.3 General Repair Models

The term *general repair* is used by Kijima, Morimura, and Suzuki [11] to refer to a repair that restores the system's operating condition to a certain state between "as bad as old" and "as good as new." This is an attempt to create a more realistic model because, for most systems in real-world situations, the working condition after repair does not fall into the two extreme states but rather somewhere in between.

Some authors refer to the term *general repair* as *imperfect repair* or *imperfect maintenance*. (e.g. Pham and Wang [12], Wang and Pham [13-16], Wang [4].) The term *imperfect repair* was first used by Brown and Proschan [17] to describe their (p,q) model. (See section 2.1.3.1.) Therefore some authors use the term *imperfect repair* to refer to the (p,q) model specifically (e.g. Nachlas [7], Nakagawa [18]) rather than in the general repair sense.

Rehmert[2] extends the definition of *general repair* to include the models where the repair can make the restored working state of the system worst than bad-as-old or better than good-as-new. He suggests that the repair models that bring the system condition back to strictly between minimally repaired and brand-new conditions are classified as *imperfect repair* while any other repair models in which the restored system condition is not limited to that restriction are classified as *general repair* models.

In this paper, the models that can make the state of the system worse than just before it fails or better than renewal are not of our interest. Therefore, we will use the term *general repair* in the sense that is defined by Kijima, Morimura, and Suzuki [11] and use the term *imperfect repair* synonymously with Brown and Proschan[17]'s (p,q) model.

The major types of general repair model are described in the following sections.

2.1.3.1 Imperfect Repair (p,q) Model and Its Variants

The imperfect repair (p,q) model is proposed by Brown and Proschan [17]. In this model the repair restores the system operating condition to a brand-new condition with probability p and to a minimally-repaired condition with probability $q = 1 - p$. One can easily see that the perfect and minimal repair models are special cases of the imperfect repair model when $p = 1$ and $p = 0$, respectively.

Block, Borges, and Savits [19] generalize the (p,q) model by making the probabilities time-dependent. In their models, the failed system is perfectly repaired with probability $p(t)$ and minimally repaired with probability $q(t) = 1 - p(t)$.

Makis and Jardine [20] generalizes $(p(t), q(t))$ model further by making the probabilities depend on both time t and the number of failures n and also adding another system state when the repair is unsuccessful. In this model, the system operating condition after repair is in good-as-new condition with probability $p(n,t)$, in bad-as-old condition with probability $q(n,t)$, and still in failed condition (unsuccessful repair) with probability $s(n,t) = 1 - p(n,t) - q(n,t)$.

These models assume good-as-new or bad-as-old condition after repair with some probability. However, in reality the condition of the system after repair is more likely to be somewhere in between. Besides, these models do not reflect any kind of aging/deteriorating process of the system.

2.1.3.2 Virtual Age Models

Kijima [21] presents two models based on the concept of the *virtual age*, which represents the equivalent age of the system. We refer to these models as *Kijima I* and *Kijima II*.

Kijima I model (first mentioned in Kijima, Morimura, and Suzuki [11]) can be explained as follows. Consider the operating times T_n, $n = 1,2,3,...$ At the n^{th} failure, the n^{th} repair does not affect the $(n-1)^{\text{th}}$ virtual age (i.e. any damage prior to $(n-1)^{\text{th}}$ failure is not fixed by the n^{th} repair) but it improves the n^{th} virtual age by reducing the additional age T_n to $\pi_n T_n$ where $0 \le \pi_n \le 1$. Let A_n be the virtual age after the n^{th} repair. Then

$$A_n = A_{n-1} + \pi_n T_n. \tag{2.1.1}$$

π_n is called the *degree* of the n^{th} repair or *repair effectiveness factor* [7].

In a special case, if $\pi_n = \theta$ for all n, then the virtual age after the n^{th} repair is

$$A_n = \theta S_n \tag{2.1.2}$$

where $S_n = \sum_{i=1}^{n} T_i$ is the real age of the system after the n^{th} repair.

Kijima II model is similar to Kijima I except that the n^{th} repair affects both the $(n-1)^{\text{th}}$ virtual age and the additional age T_n. This model can be described as

$$A_n = \pi_n \left(A_{n-1} + T_n\right). \tag{2.1.3}$$

We usually assume that $A_0 = 0$ i.e. the system is new at time 0. Typically, $0 \le \pi_n \le 1$ but in reality there might be some cases where $\pi_n > 1$ ("a clumsy repair" [22, p. 147]).
If $\pi_n = 0$ for all n, the repairs are perfect for both models and this is reduced to a model with renewal process. If $\pi_n = 1$ for all n, the repairs are minimal for both models. In Kijima II model, if π_n are i.i.d. Bernoulli random variables, this model becomes the imperfect repair (p,q) model.

The life distribution of the system in each period conditioned on the virtual age from the previous period has the following form:

$$\Pr[T_n \le t \mid A_{n-1} = u] = \frac{F_T(t+u) - F_T(u)}{\bar{F}_T(u)} \tag{2.1.4}$$

where $F_T(t)$ is the life distribution of the new system.

Kijima [21] provides a sophisticated analysis on both models and derives many interesting properties including the upper bound for the expected real age. Finkelstein [23] discusses more properties of Kijima models using the *generalized renewal processes* or *g-renewal processes* mathematical concept introduced by Kijima and Sumita [24].

We do not use the Kijima virtual age concepts in our model although these concepts are more realistic than other models mentioned earlier. However, availability analysis for models using Kijima virtual age concepts would make an interesting future research study.

Stadje and Zuckerman [25] creates another virtual age model where the virtual age is subtracted by the *maintenance action* or the *repair action*, which indicates the degree of the repair. Let t_n $(n = 1, 2, \ldots)$ be the times that failures occur and let $X(t)$ be the virtual age at time t. Then the virtual age continuously increases during the time between failures but has discontinuities at the time of failures:

$$X\left(t_n -\right) = X\left(t_{n-1}\right) + t_n - t_{n-1} \tag{2.1.5}$$

and

$$X\left(t_n\right) = X\left(t_n +\right) = X\left(t_{n-1}\right) + t_n - t_{n-1} - d, \tag{2.1.6}$$

where $0 \le d \le X(t_n -)$ is the *maintenance action* used at time t_n. If $d = 0$, this repair process is a minimal repair. If $d = X(t_n -)$, the repair is perfect. Let $R(x)$ be the remaining time until next failure given that the current virtual age is x, then

$$\Pr[R(x) \le r] = \frac{F(x+r) - F(x)}{\bar{F}(x)}. \tag{2.1.7}$$

We do not use this virtual age concept in our model either but it would make an interesting research topic to explore in the future.

2.1.3.3 Geometric Process Model and Its Variants

Lam [26, 27] proposes a model where the operating time after each failure gets shorter and shorter and finally the system is impossible to operate while the repair time gets longer and longer and finally becomes infinity (meaning the system is impossible to be repaired). The model can be formulated using a geometric process. The random variables $T_1, T_2, T_3, \ldots$ follow a geometric process if, for some $a > 0$, $\{a^{i-1}T_i \mid i = 1, 2, \ldots,\}$ forms a renewal process. If $a > 1$, T_i are stochastically decreasing and this can be used to model the sequence of operating times. On the other hand, if $a < 1$, T_i are stochastically increasing and this can be used to model the sequence of repair times. The long-run average cost analysis is performed in his paper.

Finkelstein [28] introduces a generalization to Lam [26, 27]'s model. First he defines the *general deteriorating renewal process (GDRP)* as the process where $T_{i+1} \le T_i$ for $i = 1, 2, 3, \ldots$. The inequality is interpreted in the stochastic sense. That is, for $i = 1, 2, 3, \ldots$, $T_{i+1} \le T_i$ if and only if $\bar{F}_{T_{i+1}}(x) \le \bar{F}_{T_i}(x)$ for all $x \ge 0$. GDRP reflects the deteriorating condition of the system after each repair. Then he defines a *particular deteriorating renewal process (PDRP)* as the process where $F_{T_i}(x) = F(a_i x)$ for $i = 1, 2, 3, \ldots$ and $1 = a_1 \le a_2 \le a_2 \le \ldots$. If $a_i = a^{i-1}$ for $i = 1, 2, 3, \ldots,$ and $a > 1$, this PDRP model becomes a geometric process model described in Lam [26, 27].

Our model does not assume the geometric process per se but rather assumes the quasi-renewal process, which is equivalent to the geometric process. More details are explained in the next section.

2.1.3.4 Quasi-Renewal Process Model

Wang and Pham [14, 15] introduces another model using the same concept as Lam [27]'s. That is, the operating times get shorter and shorter while the repair times get longer and longer. However, their model is defined slightly differently and is based on the quasi-renewal process:

A sequence of non negative random variables $T_1, T_2, T_3, \ldots$ forms a *quasi-renewal process* if

$$T_n = \alpha^{n-1} X_n \text{ for } n = 1, 2, 3, \ldots$$

where the parameter $\alpha > 0$ is a constant and X_n are i.i.d. random variables.

If $\alpha = 1$, this process simply becomes a renewal process. If $\alpha < 1$, the sequence $T_1, T_2, T_3, \ldots$ is stochastically decreasing and can be used to represent the deteriorating process of the device following repair. If $\alpha > 1$, the sequence $T_1, T_2, T_3, \ldots$ is stochastically increasing and can be used to represent the increasing length for the repair times due to increasing difficulty of repair.

For $n = 1, 2, 3, \ldots$:

The density function and distribution function for T_n are $f_{T_n}(t) = \dfrac{1}{\alpha^{n-1}} f_{T_1}\left(\dfrac{t}{\alpha^{n-1}}\right)$ and $F_{T_n}(t) = F_{T_1}\left(\dfrac{t}{\alpha^{n-1}}\right)$ respectively. The hazard function for T_n is $h_{T_n}(t) = \dfrac{1}{\alpha^{n-1}} h_{T_1}\left(\dfrac{t}{\alpha^{n-1}}\right)$.

The mean and the variance for T_n are $E[T_n] = \alpha^{n-1} E[T_1]$ and $Var(T_n) = \alpha^{2n-2} Var(T_1)$ respectively.

Wang and Pham [14, 15] uses the quasi-renewal sequence $\{X_n\}$ with parameter $\alpha < 1$ to model the operating times and the quasi-renewal sequence $\{Y_n\}$ with parameter $\beta > 1$ to model the repair times. This model is also known as the (α, β) model (Pham and Wang [12]). Wang and Pham [14] formulates the limiting average availability and obtains the optimal maintenance policy based on the limiting average availability.

Rehmert [2] performs a point availability analysis for the (α, β) model and formulates two types of point availability, namely the uptime-based availability function and the downtime-based availability function. The two functions are equivalent and each contains an infinite sum, which is not easily computed in most cases. An approximation for the point availability can be obtained by truncating the infinite sum. After truncation, the two approximated point availability expressions form the upper and lower bounds for the point availability.

Some applications using the quasi-renewal process model are presented in the literature. Pham and Wang [29] uses the quasi-renewal process model to describe software reliability and testing costs. Yang and Kuo [30] provides an algorithm for optimizing spares provisioning using Rehmert [2]'s availability functions to predict when the system needs to be phased out due to its availability requirement. Then they use Rehmert [2]'s quasi-renewal function to determine the expected number of failures of the system until that time and calculate the spare requirement. Yang and Lin [31] provides a procedure for optimizing PM policies based on the expected maintenance cost using Rehmert [2]'s quasi-renewal function to compute expected number of failures, which is used in the cost function.

The quasi-renewal process is equivalent to the geometric process. In fact, Chukova and Hayakawa [32] observe that a quasi-renewal process with parameter α is equivalent to a geometric process with parameter $a = \frac{1}{\alpha}$. In other words, using our notations, a quasi-renewal process $T_1, T_2, T_3, \ldots$ with parameter α and the i.i.d. random variables $\{X_n\}$ is equivalent to a geometric process $T_1, T_2, T_3, \ldots$ with parameter $a = \frac{1}{\alpha}$ and consequently the sequence of random variables $\{X_i = a^{i-1}T_i \mid i = 1, 2, ...,\}$ forms a renewal process. Moreover, Wang and Pham [33] mention Lam [26, 27]'s geometric process as a quasi-renewal process but note the difference in the approach of the two studies. According to Wang and Pham [33, p. 56], "Lam (1988,1996) [which refers to Lam [27, 34]] studies the geometric process by means of the ordinary renewal process" while "the quasi-renewal process is investigated from defining the quasi-renewal function, not from the ordinary renewal process."

Our model uses the quasi-renewal process to model the operating times after repair, making it more realistic than traditional models in many cases.

2.2 Preventive Maintenance

Preventive maintenance is performed at a scheduled time while the unit is still functional. A PM action is done in order to improve the system condition so that the chance of unexpected failure at a certain time period is reduced. In other words, a PM action prolongs the residual life of the system.

The system is usually interrupted while a PM action is performed. Therefore, PM can reduce the system availability. However, it is still worthwhile as PM is usually less costly than CM and can be performed at convenient times.

In the literature, a PM action is assumed to be either perfect or imperfect. A *perfect PM action*, like a perfect repair, restores the system condition to good-as-new. A perfect PM operation is usually a replacement of the system with a brand new one. Many classic PM models assume perfect PM such as Morse[35], Barlow and Hunter [8], Kijima, Morimura, and Suzuki [11], and Lam [27]. Our model does not consider a perfect PM action.

On the other hand, an *imperfect PM action* improves the system condition to a certain better state than the pre-PM state but not as good as new. For example, a car undergoes an oil-change-and-lube routine after 3,000 miles of operation. The maintenance routine rejuvenates the car to a certain degree but not all the way to the brand-new condition.

This assumption gains more interest in the recent literature because it provides a more realistic model to many real-world situations.

Nakagawa [36] introduces a PM action where the system after PM is as good as new with probability p and as bad as old with probability $q = 1 - p$. Sheu, Lin, and Liao [37] extend this idea by making the probabilities p and q for each PM operation depend on the number of previous bad-as-old PM operations. Our model does not include this type of PM.

Malik [38] presents the concept of *improvement factor* to model an imperfect PM action. In his model, a system whose age is t should be younger after a PM operation. He proposes that after the PM, the age should be t/β where $\beta \geq 1$. And the reliability function $r(t)$ becomes $r\left(t/\beta\right)$. If $\beta = 1$, the PM action does not improve the system. If $\beta > 1$, the PM action reduces the age to a certain degree. If $\beta = \infty$, $t/\beta = 0$ and the PM operation is perfect. Malik suggests that a maintenance expert should be able to estimate the factor β to designate how much the system is improved after the PM action. Lie and Chun [39] extends this concept by proposing that the improvement factor should be a function of maintenance cost and system age. This PM assumption is suitable in many real world situations. Our model does not include this PM assumption.

Nakagawa [40] develops a simpler model for imperfect PM where he assumes that after each PM operation the unit will be younger by x time units. Later, Nakagawa [41] proposes another model where the hazard function of the system increases after each PM operations i.e. if $h_n(t)$ is the hazard function in the n^{th} PM period ($n = 1, 2, \ldots, N$), then $h_n(t) < h_{n+1}(t)$ for all $t > 0$. Nakagawa [42] presents a more specific model where the hazard function after n^{th} PM operation becomes $a_n h(t)$ where $1 = a_0 < a_1 \leq \ldots \leq a_{N-1}$ and $h(t)$ is the original hazard function. Nakagawa [42] also proposes another model where the age of the system is reduced by a fraction after each PM. More specifically, after the n^{th} PM operation, the system of age t becomes younger and the age is reduced to $b_n t$, where $0 = b_0 < b_1 < \ldots < b_N < 1$. The system is replaced at the N^{th} PM operation.

Canfield [43] proposes that a PM activity at time t restores the degradation rate to the rate at time $t - \tau$ but the accumulated degradation does not change. Therefore the hazard function is still monotone. Chan and Shaw [44] proposes two types of failure rate reduction due to the PM operation: (1) a PM activity reduces the failure rate by a fixed amount; and (2) a PM activity reduces the failure rate by an amount which is proportional to the current failure rate. The more failure rate is the more the reduction is made.

Our general model does not assume any type of PM mentioned above. In fact, we develop a new PM model inspired by the quasi-renewal process. More specifically, our general model can be described with equations (3.1.1) and (3.1.2) in Chapter 3. The idea is that normally after a regular repair the system operating time will be stochastically reduced by a factor α_r according to the quasi-renewal process. However, after a PM operation, the next operating time will be stochastically scaled by a different factor α_p. One may choose $\alpha_p > 1$ to represent the assumption that PM improves the life length of the system.

In reality, it is also possible that a PM action can make the system worse than before it undergoes the PM operation. This could happen due to some unfortunate factors such as human errors, accidents, unexpected events, etc. However, this situation is out of the scope of our interest and will not be discussed although it can be modeled by choosing $\alpha_p \le \alpha_r$.

After an assumption about the PM action is made, the next important question is when to schedule the PM operation. Many PM policies have been made regarding to when and which type of the PM actions should be performed. Two major types of policies are age replacement and block replacement. Other types of policies do exist and will be discussed in the following sections.

2.2.1 Age-Dependent PM Policy

In an age-dependent policy, the system undergoes a PM operation whenever its age (elapsed operating time) has reached T_a (also called *policy age*). The system is also repaired whenever it fails. For most models, the system age becomes 0 only after it undergoes a perfect maintenance operation. For some models, the system age (for the purpose of PM scheduling) is reset following a CM or PM operation regardless of the perfection of the operation.

Many models, especially the classic ones, assume renewal after the PM operation. Hence this PM policy is better known as an *age replacement* policy. Many newer models assume imperfect PM, meaning the PM operation does not necessarily imply renewal. Therefore, we will use the term *age replacement* strictly only when the PM action is assumed to be perfect and use the term *age-dependent PM policy* in general.

There are many interesting age-dependent PM models in the literature and we will discuss more about them in Section 2.3.3.

As the title of the dissertation implies, we use an age-dependent PM policy in our model.

2.2.2 Periodic PM Policy

A system under a periodic PM policy will undergo a PM operation regularly at equal PM intervals regardless of its age. In other words, a PM action takes place at times kT_b, where T_b (also called *policy time*) is a constant and $k = 1, 2, 3, \ldots$. The system is also repaired whenever it fails.

If PM is perfect, this policy is better known as a *block replacement PM policy*. We will use the term *block replacement* strictly only when the PM operation is perfect and use the term *periodic PM* in general.

In fact, if under an age replacement policy with the policy age T_a we assume that CM is not perfect (e.g. minimal) and CM time is negligible, then the age replacement policy is equivalent to a block replacement policy with the same CM assumptions and policy time is $T_b = T_a$.

Periodic PM models appear often the literature and we will discuss more about them in Section 2.3.4.

We do not use a periodic PM policy in our model.

2.2.3 Other PM Policies

Pham and Wang [12] and Wang [4] discuss other PM policies which include the *failure limit policy* and the *sequential PM policy*.

The *failure limit policy* describes the PM schedule where the PM action is performed as soon as the failure rate (hazard rate) or other reliability measure starts to reach an unacceptable point.

The *sequential PM policy* is similar to the block replacement (or periodic) PM policy except that the interval lengths between consecutive PM actions are not equal. Typically, the intervals become shorter and shorter because an aging system needs PM more frequently than a younger system. Obviously, the block replacement policy is a special case of the sequential PM policy.

Some models with these PM policies will be discussed in Section 2.3.5.

Our model does not include any PM policy mentioned in this section.

2.3 Maintenance Models

Every comprehensive maintenance plan includes both CM and PM. Therefore, most maintenance models in the literature include both CM and PM. It is unrealistic to discuss PM without mentioning CM, but it is possible that some models do not consider any PM policy. There is so much diversity among maintenance models in the literature. To understand them, there are some common elements of which we should be aware. These elements are presented in the next section.

2.3.1 Elements of a Maintenance Model

A typical maintenance model contains the following elements:

1. ***Assumption on aging/deteriorating process of the system.*** All maintenance models assume that the system is deteriorating. More specifically, the system lifetime has increasing failure rate (IFR) [4]. While most models use the failure rate (hazard rate) to describe the deteriorating process, some models use the shocks to which the unit is exposed. In the shock model, it is assumed that shocks cause a random amount of damage to the system and the damage is accumulated. The system fails when the accumulated damage exceeds a certain amount. Examples for the shock models are presented in Kijima and Nakagawa [45, 46].

   Our model assumes that the system has increasing failure rate.

2. ***Assumption on the CM operation.*** The CM operation may be perfect, minimal, or one of the general repairs. The CM times may be instantaneous or have a probability distribution.

   Our model assumes that CM operation is a general repair which represented by a quasi-renewal process. The CM times have a probability distribution.

3. ***Assumption on the PM operation.*** The PM operation may be perfect or one of the imperfect PM operations. The PM times may be instantaneous or have a probability distribution.

   Our model assumes that the PM action does not improve the system condition. In fact, we assume the PM actions create cycles in the quasi-renewal process for the operating times. Since the quasi-renewal process probabilistically reduces the operating times, the hazard rate of the system is also reduced after each CM/PM action. The PM times have a probability distribution.

4. ***Assumption on the CM policy.*** Typically, maintenance models assume that the system is repaired whenever the system fails with the same CM operation assumption. However, sometimes the assumption on each CM operation is not the

same. Most models with mixed types of CM create some conditions that indicate when the unit is repaired (but not perfectly) and when the unit is replaced upon failures. The motive behind the mixed CM types within the same CM policy often involved economic reasons or reliability requirements. Wang[4] discusses some of these models in terms of *repair limit policy*, *repair number counting policy*, and *reference time policy*.

There are two types of *repair limit policy*, namely *repair cost limit policy* and *repair time limit policy*. Under the *repair cost limit policy*, the system is repaired (but not replaced) only if the estimated repair cost does not exceed a fixed limit. If it does, the system is replaced with a new one. This model is introduced by Gardent and Nonant [47] and Drinkwater and Hasting [48]. Under the *repair time limit policy*, the system is repaired (but not replaced) whenever it fails. If the repair time exceeds a predetermined time limit, the system is replaced with a new one. This model is introduced by Nakagawa and Osaki [49].

Under the *repair number counting policy*, the unit will be repaired (but not replaced) at failures before the n^{th} failure and will be replaced only at the n^{th} failure where n is a predetermined parameter. The counting process is reset after the system is replaced. This policy is introduced by Morimura and Makabe [50].

The *reference time policy* has a reference time T that is used to determine the type of the CM operation. The system will be repaired (but not replaced) at any failure that occurs before T. The system will be replaced at the first failure that occurs after T and the process is renewed. Muth [51] presents this policy. However, the concept is originally introduced by Morimura [52] as an extension to the Morimura and Makabe [50]'s *repair number counting policy*. In Morimura [52], the system is repaired (but not replaced) for any failure before the n^{th} failure. If the n^{th} occurs before the reference time T, the system is still repaired. If the n^{th} occurs after T, the system is replaced and the whole process is renewed. The reference time policy is very similar to an age replacement PM policy except that the replacement (CM) operation needs not be exactly at the time T but rather at the first failure after T.

Our model assumes that the system is repaired whenever it fails with the same CM quasi-renewal assumption.

5. ***Assumption on the PM policy.*** The model may adopt an age-dependent PM policy, a periodic PM policy, or any other PM policy described in the previous section. Some models may not have a PM policy.

   Our model includes an age-dependent PM policy.

6. ***Objective function for the optimization problem.*** While some models are constructed just to present the new ideas and new mathematical theories/properties, many models are constructed as a part of an optimization

problem, in which case the objective function must be clearly specified. Normally, the objective function is either based on maintenance cost or availability. The cost-based objective function may be the average cost per unit time (cost rate) or the total cost for the given (finite) time horizon. The availability-based objective function may be the average availability, the limiting availability, or the limiting average availability. Mostly the limiting availability measures are used. When a cost function and an availability measure are both considered at the same time, we usually see an optimization problem whose objective is to minimize the cost while keeping the availability above a certain requirement or maximize the availability measure while keeping the cost under a certain budget.

Our study presents only a new maintenance model and does not include any optimization problem.

7. ***Other assumptions.*** The rest of the assumptions are mostly related to the optimization problem. For example, the assumption on the time horizon (finite or infinite) and the assumption on the CM/PM cost (constant, time-dependent, age-dependent, etc.) It is usually assumed that the PM cost is smaller the CM cost.

There are many different choices for each element listed above. The combinations of these choices create a vast variety of models in the literature. According to Wang[4], thousands maintenance models have been created in the past several decades.

2.3.2 Models with No PM Policy

Normally, models with mixed types of CM operations do not have a PM policy. Such maintenance models include the repair limit policy, the repair number counting policy, and the reference time policy models. The decision variables for the optimization problem are usually the variables that determine the type of CM operation. For example, in the repair number counting policy, the number of failure n is the decision variable. In the reference time policy, the reference time T is the decision variable.

Moreover, Stadje and Zuckerman [25]'s virtual age model (described in Section 2.1.3.2 on page 10) can also be viewed as a CM policy with mixed types of CM actions when the maintenance action (degree of repair) d is used as a decision variable to determine the type of CM operation for each failure. Stadje and Zuckerman considers the maintenance action d as a decision variable and constructs a cost expression as a function of the virtual age and the degree of repair d. An optimal maintenance policy that minimizes the expected discounted cost over in infinite planning horizon is derived.

Rehmert [2]'s model does not have mixed types of CM. In fact, the repair affects the system condition according to the quasi-renewal process. However, his model does not consider any PM policy or any optimization problem.

Our model is similar to Rehmert's [2] but we have incorporated a PM policy and therefore our model does not fall into this model category.

2.3.3 Models with Age-Dependent PM Policy

The age-dependent PM policy is probably the most commonly used PM policy in the literature (Wang [4]). The decision variable for the optimization problem, if any, is the policy age T_a . Our model falls into this category of model but we do not provide any optimization problem.

Age-dependent PM policy was discussed as early as 1958 in Morse [35]. Morse assumes that the repair and PM operations are perfect. An attempt to find the optimal policy is presented based on fraction of time the machine is working, fraction of time in preventive maintenance, and fraction of time the machine is being repaired. This is different from our model because ours does not assume perfect PM operations.

Barlow and Hunter [8] assume the perfect repair on both PM and CM and consider an age replacement policy. They present the optimal policy age based on "the limiting efficiency of the system," which is the limiting availability in our notation. Our model, however, does not assume perfect PM or CM operations.

Murdock [1] provides the point availability function for an age replacement policy assuming infinite time horizon and the perfect repair on both PM and CM. Murdock uses the point availability function to calculate the average availability for a finite time horizon model and finds that the optimal policy age T_∞^* for an infinite time horizon model described in Balow and Hunter [8] does not maximize the average availability for a finite time horizon model. Our model, however, does not assume perfect PM or CM operations.

There have been some attempts to extend the age replacement policy by incorporating a CM policy with mixed CM types. Nakagawa [53] presents a PM policy where the system is replaced at age T or at the N^{th} of failure, whichever occurs first. Any failures before that are minimally repaired. The age and the number of failures are reset after the system is replaced. If $N = 1$, the model is reduced to an age replacement PM policy with perfect CM. Sheu, Kuo, and Nakagawa [54] introduce an age-dependent PM policy combining with a reference time (CM) policy. Given a reference time T_0 and a policy age $T_a \geq T_0$, if the unit fails at age $y < T_0$, the unit is perfectly repaired with probability $p(y)$ and minimally repaired with probability $q(y) = 1 - p(y)$. When the age has reached T_a or at the first failure after T_0, the system is replaced, whichever occurs first.

Sheu, Griffith, and Nakagawa [55] propose another extension of the age replacement policy by adding a repair number counting (CM) policy. They categorize the failures into Type I ("minor failure") and Type II ("catastrophic failure"). If the unit fails at age z , the failure is a Type I with probability $p(z)$ and a Type II with probability $q(z) = 1 - p(z)$.

Given the parameter n and policy age T_a, the system is replaced at age T_a, at the n^{th} Type I failure, or at the first Type II failure, whichever occurs first. All other Type I failures are treated with minimal repair.

Our model does not have any kind of mixed-CM-type policy.

Normally the age of the system is measured from the last replacement. Block, Langberg, and Savits [56] propose a new policy called a *repair replacement policy*, where the age is measured from the last replacement or the last CM repair regardless of the perfection of the CM operation. If the CM action is perfect, this repair replacement policy is a regular age replacement policy. Similarly, the age in our model is measured from the last CM or PM operation, whichever occurs latest. In other words, the age used in our PM policy is defined as an elapsed uninterrupted operating time. However, none of our CM or PM operations are perfect.

Some examples for age-dependent policy with imperfect PM operation are presented in Nakagawa [36]((p,q) PM action) and Malik [38] (improvement factor). Our general model also has an age-dependent policy with imperfect PM actions. However, our assumption on PM actions is unique as described on page 15.

2.3.4 Models with Periodic PM Policy

This policy is easy to implement because it requires no record keeping of the system history. The decision variable for the optimization problem is the policy time T_b. However, our model does not consider periodic PM policy.

The most basic model of this type is a block replacement PM policy with minimal repair at failures. Boland and Proschan [57] investigate such model and construct an optimization problem whose objective is to minimize the total expected cost over a finite time horizon and another problem whose objective is to minimize the expected costs per unit time over an infinite time horizon. Our model does not use minimal repair assumption and periodic PM policy.

Kijima, Morimura, and Suzuki [11] develop a block replacement model with general repair at failures. More specifically, the general repair model in use is the virtual age Kijima I model described in Section 2.1.3.2. The repair times are negligible. Using the virtual age model, they derive the expected number of failures as a function of time. The expected number of failures is used in the cost function in an optimization problem whose objective is to find T_b that minimizes the expected average cost over an infinite time horizon. Our model does not use the virtual age to model the operating times and does not use periodic PM policy.

There are many extensions to the block replacement PM policy. For example, Nakagawa [58] assumes the system is replaced at times kT_b, $k=1,2,3,\ldots$ and any failures occur in between will have to wait until the next PM replacement. Our model assumes the failures are taken care right away under the quasi-renewal process assumption.

Examples for periodic PM policy extensions with imperfect PM are the following. Nakagawa [41] combines a block replacement PM with the idea from the repair number counting policy. In his model, the imperfect PM operation is performed at times kT_b, $k = 1, 2, 3, \ldots, (N-1)$ with minimal repair at failures. The PM is imperfect in the sense that it increases the hazard rate of the system. The system is replaced at the time NT_b. N and T_b are decision variables based on a cost function. A similar idea is presented in Liu, Makis, and Jardine [59]. Our model does not use minimal repair and periodic PM policy. Our imperfect PM assumption is also different.

Wang and Pham [60] presents a quasi-renewal model with a periodic PM policy. In their model, the failures are treated with a general repair which causes the operating intervals follow the quasi-renewal process. There is no PM for the first $N-1$ repairs. After the N^{th} repair, the system is preventatively maintained at times kT_b, $k = 1, 2, 3, \ldots$ The PM operation is perfect with probability p and minimal with probability $q = 1 - p$. The process is renewed when the PM is perfect. Our model also uses the quasi-renewal process to model the operating times but the PM policy is not a periodic policy. Moreover, our assumption on PM action is different.

Sheu, Lin, and Liao [37] construct a periodic PM policy model with imperfect PM in the sense that after the PM operation at time kT_b, the system is as bad as old with probability q_k and is as good as new with probability $\theta_k = 1 - q_k$. The probability q_k is formulated based on the number of previous bad-as-old PM operations. Our model has a different imperfect PM assumption and does not use a periodic PM policy.

2.3.5 Models with Other PM Policies

Models mentioned in this section include models with *failure limit policy* and *sequential PM policy* described in Section 2.2.3. However, our model does not fall into any of these categories.

The following are examples for the failure limit policy model. Malik [38], who introduces the concept of improvement factor for imperfect PM as described on page 14, creates a model where the PM operation takes place as soon as the reliability of the system starts to get below the minimum requirement. Lie and Chun [39] developed a cost model where PM is scheduled whenever the failure rate reaches a maximum acceptable level. More failure limit policy models are listed in Wang[4]. Our model does not use a failure limit policy.

Some examples for the sequential PM policy are as follows. Nguyen and Murthy [61] propose two sequential PM policies. In the first PM policy, they assume a new type of general repair. That is, after each repair (not a replacement), the hazard rate is greater than or equal to the hazard rate just before the repair. Hence the current hazard rate will depend on the number of previous repairs. The PM operation is also counted as a repair and hence falls into the same assumption (i.e. increase the hazard rate.) The system is replaced at the k^{th} repair. For the system subjected to $(i-1)$ repairs, $1 \leq i \leq k$, it is repaired at failures or at time T_i (T_i is the number of hours since the last repair or replacement), whichever occurs first. Another policy is the same except that the system is minimally repaired at failures (i.e. no increment in the hazard rate) but it is undergone a PM operation with the same general repair assumption (i.e. increase the hazard rate) at times T_i, $i = 1, 2, \ldots, k$. The decision variables are k and T_i, which are chosen to minimize the expected cost rate. Nakagawa [41, 42] studies a similar model where the interval between PM times are x_k, $i = 1, 2, \ldots, N$. The PM operation is imperfect in the sense that it increases the hazard rate (similar to Nguyen and Murthy [61]) or reduces the system age. These PM assumptions are described in detail on page 14. The replacement takes place at the N^{th} PM operation. Any failure occurs in between are removed with minimal repair. The decision variables are N and x_k. The objective is to minimize the cost rate. Interestingly, the numerical examples shows that the optimal solution $\left\{x_k^*\right\}$ satisfies $x_1^* \geq x_2^* \geq \ldots \geq x_N^*$. More sequential PM policy models are described in Wang[4]. Our model does not use a sequential PM policy.

2.3.6 Note on Maintenance Models

Age replacement and block replacement policies are the two major PM policies in the literature. Age replacement policy is more economical because under a block replacement policy it is possible that the system is preventatively replaced at a young age. Hence the PM replacement under the block replacement policy occurs more frequently on average and makes the PM cost more expensive. However, the block replacement policy does not require a history record of the system usage. Moreover, Barlow and Proschan [6] proves that the block replacement policy causes fewer failures than the age replacement policy. More extensive comparisons between the two PM policies can be found in Barlow and Proschan [6]. In our study, we choose an age-dependent PM policy for a quasi-renewal process model. A study of a quasi-renewal model with a periodic PM policy has been done by Wang and Pham [60] although the availability analysis is not provided.

Many newer models are the generalization of the previously known models. For example, Makis and Jardine [20] proposes a general model where a replacement (either PM or CM) can take place any time with a fixed cost c_0. At failures, the system is repaired according to the $(p(n,t), q(n,t), s(n,t))$ model (described in Section 2.1.3.1) and the repair cost is defined as a function of the number of failures n and the age of the system t. If the repair is unsuccessful, the system is replaced with a fixed cost c_0. This includes as special cases many previously known models in both age and block replacement types such as Barlow and Hunter [8], Muth [51], Boland and Proschan[57], Brown and Proschan [17], and Block, Borges, and Savits [19]. They presented algorithm for finding optimal replacement policy that minimize the average cost rate where n and t are decision variables. Wang [4] points out that these general models may provide a "globally" optimal solution but the models are usually more complex and may be difficult to implement in reality. Our model, however, is not a special case of these general models.

Most papers on maintenance models present cost analysis and/or limiting availability analysis. Only a few papers (e.g. Murdock[1] and Rehmert[2]) provide the point availability analysis, which gives the most informative data for analyzing the model behavior. Our study intends to provide the point availability analysis for our new model.

Chapter 3 Problem Statement

This chapter provides details about the availability models we use in this research as well as the results from the availability analysis.

3.1 Model Development

The main purpose of this research is to perform the availability analysis for the quasi-renewal process with an age dependent preventive maintenance policy. To model a device behavior with an aging process, we use a quasi-renewal process which consists of operating intervals whose length are stochastically shortened by a parameter $\alpha < 1$ every time the device fails. Rehmert [2] includes in his system behavior model the repair periods which form another quasi-renewal process with parameter $b \geq 1$ to indicate that the repair time gets longer as the device ages due to its increasing level of difficulty to repair. Figure 2 illustrates Rehmert's system behavior where $X_1, X_2, X_3, \ldots$ are operating periods which form a quasi-renewal with parameter $a < 1$ and $Y_1, Y_2, Y_3, \ldots$ are repair periods which form another quasi-renewal with parameter $b \geq 1$.

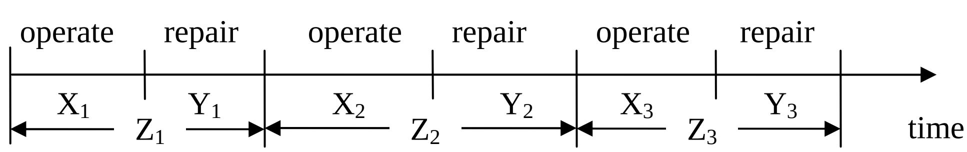

Figure 2: Rehmert's Notional System Behavior [2]

We extend this model to incorporate an age dependent preventive maintenance policy. First, we assume that a policy age $T_a > 0$ has already chosen. In our notation, let $T_1, T_2, T_3, \ldots$ be the operating intervals of the device and $R_1, R_2, R_3, \ldots$ be the repair or the PM intervals. Similar to Murdock's model [1], we define two types of cycle, namely repair cycle and a PM cycle. A *repair cycle* k consists of T_k and R_k where $T_k < T_a$, and R_k is called a repair interval. A *PM cycle* k consists of T_k and R_k where $T_k \geq T_a$ but the operating interval is truncated to T_a and R_k is called a PM interval. Figure 3 displays the sample path for each type of cycle.

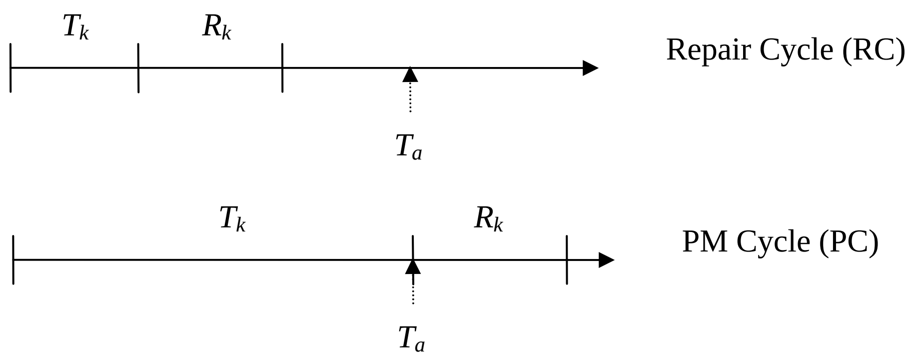

Figure 3: Types of Cycles in Our Model

The system behavior can be modeled as a random sequence comprised of both types of cycles. Figure 4 depicts a possible scenario of the system behavior.

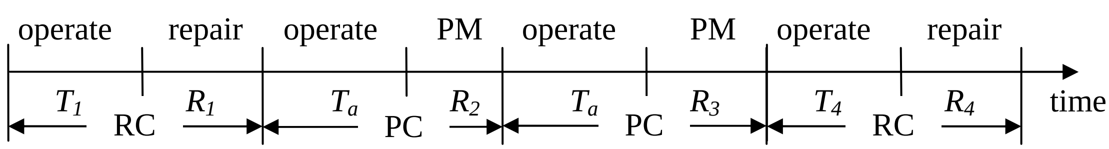

Figure 4: A Possible Scenario of the System Behavior

Furthermore, we also assume that $T_1, T_2, T_3, \ldots$ and $R_1, R_2, R_3, \ldots$ follow a process defined by

$$T_{k+1} = \begin{cases} \alpha_r T_k & \text{if } T_k < T_a \\ \alpha_p T_k & \text{if } T_k \geq T_a \end{cases} \qquad (3.1.1)$$

$$R_{k+1} = \begin{cases} \beta_r R_k & \text{if } T_k < T_a \\ \beta_p R_k & \text{if } T_k \geq T_a \end{cases} \qquad (3.1.2)$$

where $0 < \alpha_r < 1, \alpha_p > 0$, $\beta_r \geq 1$, and $\beta_p \geq 1$. The parameters α_r and α_p may be different to reflect the different aging rates due to the effect of the repair or PM from the previous interval. Similarly, the parameters β_r and β_p may be different to reflect the different difficulty increasing rates.

In a special case where $\alpha_r = \alpha_p = \alpha$, the sequence $T_1, T_2, T_3, \ldots$ is a quasi-renewal process as defined on page 12. All reliability-related functions for each operating interval can be expressed nicely in terms of the same functions for the first interval. (See page 12.) If $\alpha_r \neq \alpha_p$, the reliability-related functions are no longer in a nice compact form and will require extensive enumeration for every possible sample path.

3.2 Availability Analysis

We simplify the general model of (3.1.1) and (3.1.2) to a case in which we assume $\alpha_r = \alpha_p = \alpha$. This is our base-case model with the assumption that the PM action does not improve the system working condition after the maintenance. The availability analysis based on this base-case model is provided in this dissertation.

Let
$F_{T_k}(t)$ be the distribution function for the length of the operating interval T_k,
$G_{r,k}(t)$ be the distribution function for the repair time for the k^{th} cycle, and
$G_{p,k}(t)$ be the distribution function for the PM time for the k^{th} cycle.
Also, let $f_{T_k}(t)$, $g_{r,k}(t)$, and $g_{p,k}(t)$ be their associated density functions, respectively.

From (3.1.1), $T_k = \alpha T_{k-1} = \alpha^{k-1} T_1$ and therefore the probability functions can be written as

$$F_{T_k}(t) = F_{\alpha^{k-1}T_1}(t) = F_{T_1}\left(t \middle/ \alpha^{k-1} \right), \tag{3.2.1}$$

$$f_{T_k}(t) = \frac{1}{\alpha^{k-1}} f_{T_1}\left(t \middle/ \alpha^{k-1} \right). \tag{3.2.2}$$

For an age dependent policy, the distribution function for the operating time must be truncated at T_a and hence is defined by

$$F_{c,T_k}(t) = \begin{cases} \dfrac{F_{T_k}(t)}{F_{T_k}(T_a)} & \text{if } 0 \le t < T_a \\ 1 & \text{if } t \ge T_a \end{cases} . \tag{3.2.3}$$

Then the truncated density function corresponding to $F_{c,T_k}(t)$ is

$$f_{c,T_k}(t) = \frac{f_{cc,T_k}(t)}{F_{T_k}(T_a)} = \begin{cases} \dfrac{f_{T_k}(t)}{F_{T_k}(T_a)} & \text{if } 0 \le t < T \\ 0 & \text{if } t \ge T_a \end{cases} \tag{3.2.4}$$

where

$$f_{cc,T_k}(t) = \begin{cases} f_{T_k}(t) & \text{if } 0 \le t < T_a \\ 0 & \text{otherwise} \end{cases} . \tag{3.2.5}$$

Notice that the subscript c indicates scaled truncation while the subscript cc indicates truncation without scaling.

Furthermore, we also assume that:
If R_k is a repair interval (i.e. $T_k < T_a$), its distribution function can be written as

$$G_{R_k}(t) = G_{r,k}(t) = G_{r,1}\left(t \big/ \beta_r^{k-1} \right). \tag{3.2.6}$$

If R_k is a PM interval (i.e. $T_k \geq T_a$), its distribution function can be written as

$$G_{R_k}(t) = G_{p,k}(t) = G_{p,1}\left(t \big/ \beta_p^{k-1} \right). \tag{3.2.7}$$

Consequently, the density functions can be expressed as

$$g_{r,k}(t) = \frac{1}{\beta_r^{k-1}} g_{r,1}\left(t \big/ \beta_r^{k-1} \right) \tag{3.2.8}$$

$$g_{p,k}(t) = \frac{1}{\beta_p^{k-1}} g_{p,1}\left(t \big/ \beta_p^{k-1} \right). \tag{3.2.9}$$

Note that if there is no PM (i.e. $T_a = \infty$), $R_1, R_2, R_3, \ldots$ are all repair intervals and form a quasi-renewal process which is equivalent to Rehmert's model [2].

Let $V_k = T_k + R_k$ be a random variable representing the length of the k^{th} cycle. Clearly, there are two possible realizations of the k^{th} cycle.

Case 1: the operating time does not exceed T_a. So V_k is a repair cycle. Hence the distribution on its length is

$$H_{r,V_k}(t) = f_{c,T_k} * G_{r,k}(t) = \int_0^t f_{c,T_k}(u) G_{r,k}(t-u)\,du \tag{3.2.10}$$

and the corresponding density function is

$$h_{r,V_k}(t) = f_{c,T_k} * g_{r,k}(t) = \int_0^t f_{c,T_k}(u) g_{r,k}(t-u)\,du \tag{3.2.11}$$

Case 2: the operating time equals T_a. So V_k is a PM cycle. Hence the distribution on its length is

$$H_{p,V_k}(t) = \begin{cases} G_{p,k}(t - T_a) & \text{where } t \geq T_a \\ 0 & \text{otherwise} \end{cases} \tag{3.2.12}$$

and the corresponding density function is

$$h_{p,V_k}(t) = \begin{cases} g_{p,k}(t - T_a) & \text{where } t \geq T_a \\ 0 & \text{otherwise} \end{cases}. \tag{3.2.13}$$

Therefore the distribution on the length of k^{th} cycle $H_{m,V_k}(t)$ is a mixture of the two cycle length distributions. Consequently:

$$H_{m,V_k}(t) = F_{T_k}(T_a)H_{r,V_k}(t) + \bar{F}_{T_k}(T_a)H_{p,V_k}(t) \tag{3.2.14}$$

and the corresponding density function is

$$h_{m,V_k}(t) = F_{T_k}(T_a)h_{r,V_k}(t) + \bar{F}_{T_k}(T_a)h_{p,V_k}(t)\,. \tag{3.2.15}$$

Next consider a series of intervals

$$\Theta_n = \sum_{k=1}^{n} V_k\ . \tag{3.2.16}$$

The distribution on the length of the series of intervals is defined by

$$\begin{aligned} &H_{m,\Theta_n}(t) = h_{m,\Theta_{n-1}} * H_{m,V_n}(t) \quad \text{for} \quad n = 2,3,\ldots \\ &H_{m,\Theta_1}(t) = H_{m,V_1}(t) \end{aligned} \tag{3.2.17}$$

and the associated density function is defined by

$$\begin{aligned} &h_{m,\Theta_n}(t) = h_{m,\Theta_{n-1}} * h_{m,V_n}(t) \quad \text{for} \quad n = 2,3,\ldots \\ &h_{m,\Theta_1}(t) = h_{m,V_1}(t) \end{aligned}\ . \tag{3.2.18}$$

The quasi-renewal function based on the intervals Θ_n is

$$Q_{H_{m,\Theta}}(t) = \sum_{n=1}^{\infty} H_{m,\Theta_n}(t) \tag{3.2.19}$$

and the associated quasi-renewal density function is

$$q_{H_{m,\Theta}}(t) = \sum_{n=1}^{\infty} h_{m,\Theta_n}(t)\,. \tag{3.2.20}$$

The Laplace transforms for the two functions are

$$Q^*_{H_{m,\Theta}}(s) = \sum_{n=1}^{\infty} H^*_{m,\Theta_n}(s) \tag{3.2.21}$$

and

$$q^*_{H_{m,\Theta}}(s) = \sum_{n=1}^{\infty} h^*_{m,\Theta_n}(s)\,. \tag{3.2.22}$$

Consider a special case where all cycles are repair cycles. We define

$$U_n = \sum_{k=1}^{n} V'_k \quad \left(V'_k \text{ are all repair cycles}\right). \tag{3.2.23}$$

Since PM does not have any effect on these cycles, the distribution for each cycle V'_k should be defined as if PM did not exist, i.e. the original unscaled density function for the operating time must be used:

$$H_{r,V'_k}(t) = f_{T_k} * G_{r,k}(t) = \int_0^t f_{T_k}(u)G_{r,k}(t-u)du\,. \tag{3.2.24}$$

The corresponding density function is

$$h_{r,V_k'}(t) = f_{T_k} * g_{r,k}(t) = \int_0^t f_{T_k}(u) g_{r,k}(t-u)du \ . \tag{3.2.25}$$

Hence the distribution on U_n is defined by

$$\begin{aligned} &H_{r,U_n}(t) = h_{r,U_{n-1}} * H_{r,V_n'}(t) \quad \text{for} \quad n = 2,3,\ldots \\ &H_{r,U_1}(t) = H_{r,V_1'}(t) \end{aligned} , \tag{3.2.26}$$

and the associated density function is defined by

$$\begin{aligned} &h_{r,U_n}(t) = h_{r,U_{n-1}} * h_{r,V_n'}(t) \quad \text{for} \quad n = 2,3,\ldots \\ &h_{r,U_1}(t) = h_{r,V_1'}(t) \end{aligned} . \tag{3.2.27}$$

The quasi-renewal function based on the intervals U_n is

$$Q_{H_{m,U}}(t) = \sum_{n=1}^{\infty} H_{m,U_n}(t) \tag{3.2.28}$$

and the associated quasi-renewal density function is

$$q_{H_{m,U}}(t) = \sum_{n=1}^{\infty} h_{m,U_n}(t) \tag{3.2.29}$$

3.2.1 Quasi-Renewal Function

The quasi-renewal function indicates the expected number of restarts as a function of time. For our model, it can expressed as

$$Q_{H_m}(t) = \begin{cases} \sum_{n=1}^{\infty} H_{r,U_n}(t), & t < T_a \\ \sum_{n=1}^{\infty} H_{m,\Theta_n}(t), & t \geq T_a \end{cases} \tag{3.2.30}$$

When $t < T_a$, all cycles must be repair cycles. The quasi-renewal function is equivalent to that of the model without PM. Hence only $H_{m,U_n}(t)$ are used in the summation. The Laplace transform for this expression is

$$Q^*_{H_m, t<T_a}(s) = \sum_{n=1}^{\infty} \left(\frac{1}{s} h^*_{r,U_n}(s) \right) = \frac{1}{s} \sum_{n=1}^{\infty} h^*_{r,U_n}(s) \tag{3.2.31}$$

When $t \geq T_a$, each cycle may be a repair cycle or a PM cycle. The distribution $H_{m,\Theta_n}(t)$ is then used in the summation. The Laplace transform for this expression is

$$Q^*_{H_m,t\geq T_a}(s) = \sum_{n=1}^{\infty}\left(\frac{1}{s}h^*_{m,\Theta_n}(s)\right) = \frac{1}{s}\sum_{n=1}^{\infty}h^*_{m,\Theta_n}(s) \tag{3.2.32}$$

To evaluate (3.2.30), we apply the inverse Laplace transform to the appropriate expression ($Q^*_{H_m,t<T_a}(s)$ or $Q^*_{H_m,t\geq T_a}(s)$) according to the value t.

3.2.2 Point Availability Functions

A point availability function describes the behavior of the model in more details than any other types of availability function. This information is very crucial especially for a system whose failure could be translated to a matter of life and death. Unfortunately, not many research studies provide a point availability analysis for their models. This is largely due to the intractability of the point availability function. In this section, we develop two equivalent point availability functions based on uptime and downtime point of view.

A system is available at time t if it has never failed by time t or a new cycle has started before t and the system has not failed since that time. Using the notations we have previously established, we construct the availability function as

$$A(t) = \begin{cases} \bar{F}_{T_1}(t) + \sum_{n=1}^{\infty}\int_0^t h_{r,U_n}(u)\bar{F}_{T_{n+1}}(t-u)du & t < T_a \\ \sum_{n=1}^{\infty}\int_{t-T_a}^t h_{m,\Theta_n}(u)\bar{F}_{T_{n+1}}(t-u)du & t \geq T_a \end{cases} \tag{3.2.33}$$

When $t < T_a$, all cycles must be repair cycles. The availability function is equivalent to that of the model without PM. Hence the expression is similar to Rehmert [2]'s uptime-based availability function. The first term indicates the probability that the system survives t time units in the first cycle (i.e. the system has never failed by the time t). The following integral terms in the summation indicate that the n^{th} cycle ends at time u and the $(n+1)^{\text{st}}$ operating time is longer than $(t-u)$ time units.

Suppose $t \geq T_a$. There is a chance that some cycles are a PM cycle and hence $h_{m,\Theta_n}(u)$ is used to represent the density function of n cycles. The availability function for $t \geq T_a$ indicates that the n^{th} cycle ends at time u and the $(n+1)^{\text{st}}$ cycle operating time is longer than $(t-u)$. However, the operating time cannot exceed T_a and thus the domain of integration must be adjusted accordingly to be $[t-T_a, t]$. This yields the expression

$$\sum_{n=1}^{\infty}\int_{t-T_a}^t h_{m,\Theta_n}(u)\bar{F}_{T_{n+1}}(t-u)du\,. \tag{3.2.34}$$

To evaluate the expression (3.2.34), we utilize the Laplace transform. However, this expression is not in a convolution form. Applying the Laplace transform to this expression directly using integration definition requires a more complicated algebraic manipulation. (For example, see Murdock [1].)

In our practice, to make the Laplace transform of (3.2.33) easier to obtain, we define

$$\bar{F}_{cc,T_n}(t) = \begin{cases} \bar{F}_{T_n}(t) & \text{if } 0 \le t < T_a \\ 0 & \text{if } t \ge T_a \end{cases} . \tag{3.2.35}$$

Again, note that the subscript cc indicates truncation without scaling.

Therefore, the expression (3.2.34) can be written as

$$\sum_{n=1}^{\infty} \int_0^t h_{m,\Theta_n}(u) \bar{F}_{cc,T_{n+1}}(t-u)\,du \tag{3.2.36}$$

Expression (3.2.36) is now in a convolution form. The Laplace transform can be applied normally, making it more tractable than (3.2.34).

Therefore, the Laplace transform for the availability function where $t < T_a$ is

$$A^*_{t<T_a}(s) = \bar{F}^*_{T_1}(s) + \sum_{n=1}^{\infty} \left[\bar{F}^*_{T_{n+1}}(s) h^*_{r,U_n}(s) \right] \tag{3.2.37}$$

and the Laplace transform for the availability function where $t \ge T_a$ is

$$A^*_{t\ge T_a}(s) = \sum_{n=1}^{\infty} \left[\bar{F}^*_{cc,T_{n+1}}(s) h^*_{m,\Theta_n}(s) \right] \tag{3.2.38}$$

According to (3.2.27), $h^*_{r,U_n}(s)$ can be expressed by

$$h^*_{r,U_n}(s) = \prod_{i=1}^{n} h^*_{r,V_i'}(s) = \prod_{i=1}^{n} \left(f^*_{T_i}(s) g^*_{r,i}(s) \right) = \prod_{i=1}^{n} \left(f^*_{T_1}(\alpha^{i-1}s) g^*_{r,1}(\beta_r^{i-1}s) \right) \tag{3.2.39}$$

Similarly, $h^*_{m,\Theta_n}(s)$ can be expressed as

$$\begin{aligned}
h^*_{m,\Theta_n}(s) &= \prod_{i=1}^{n} h^*_{m,V_i}(s) \\
&= \prod_{i=1}^{n}\left(F_{T_i}(T_a)h^*_{r,V_i}(s) + \bar{F}_{T_k}(T_a)h^*_{p,V_i}(s)\right) \\
&= \prod_{i=1}^{n}\left(F_{T_i}(T_a)\, f^*_{c,T_i}(s)g^*_{r,i}(s) + \bar{F}_{T_k}(T_a)e^{-sT_a}\, g^*_{p,i}(s)\right) \\
&= \prod_{i=1}^{n}\left(F_{T_i}(T_a)\frac{f^*_{cc,T_i}(s)}{F_{T_i}(T_a)}\, g^*_{r,i}(s) + \bar{F}_{T_k}(T_a)e^{-sT_a}\, g^*_{p,i}(s)\right) \\
&= \prod_{i=1}^{n}\left(f^*_{cc,T_i}(s)g^*_{r,1}(\beta_r^{i-1}s) + \bar{F}_{T_k}(T_a)e^{-sT_a}\, g^*_{p,1}(\beta_p^{i-1}s)\right)
\end{aligned} \tag{3.2.40}$$

Hence we can write (3.2.37) as

$$\begin{aligned}
A^*_{t<T_a}(s) &= \frac{1}{s}\left(1 - f^*_{T_1}(s)\right) + \sum_{n=1}^{\infty}\left[\frac{1}{s}\left(1 - f^*_{T_{n+1}}(s)\right)h^*_{r,U_n}(s)\right] \\
&= \frac{1}{s}\left[\left(1 - f^*_{T_1}(s)\right) + \sum_{n=1}^{\infty}\left[\left(1 - f^*_{T_1}(\alpha^n s)\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right]
\end{aligned} \tag{3.2.41}$$

Also, equation (3.2.38) can be expressed as

$$\begin{aligned}
A^*_{t\ge T_a}(s) &= \sum_{n=1}^{\infty}\left[\bar{F}^*_{cc,T_{n+1}}(s)h^*_{m,\Theta_n}(s)\right] \\
&= \sum_{n=1}^{\infty}\left[\bar{F}^*_{cc,T_{n+1}}(s)\left(\prod_{i=1}^{n}\left(\begin{array}{l} f^*_{cc,T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s) \\ \quad + \bar{F}_{T_k}(T_a)e^{-sT_a}\, g^*_{p,1}(\beta_p^{i-1}s)\end{array}\right)\right)\right]
\end{aligned} \tag{3.2.42}$$

If $\beta_r = \beta_p = 1$,

$$A^*_{t<T_a}(s) = \frac{1}{s}\left[\left(1 - f^*_{T_1}(s)\right) + \sum_{n=1}^{\infty}\left[\left(1 - f^*_{T_1}(\alpha^n s)\right)\left(g^*_{r,1}(s)\right)^n\left(\prod_{i=1}^{n} f^*_{T_1}(\alpha^{i-1}s)\right)\right]\right] \tag{3.2.43}$$

and

$$A^*_{t\ge T_a}(s) = \sum_{n=1}^{\infty}\left[\bar{F}^*_{cc,T_1}(\alpha^n s)\left(\prod_{i=1}^{n}\left(\begin{array}{l} f^*_{cc,T_1}(\alpha^{i-1}s)g^*_{r,1}(s) \\ \quad + \bar{F}_{T_k}(T_a)e^{-sT_a}\, g^*_{p,1}(s)\end{array}\right)\right)\right]. \tag{3.2.44}$$

To evaluate the expression (3.2.33), we apply the inverse to the appropriate Laplace transform ($A^*_{t<T_a}(s)$ or $A^*_{t\ge T_a}(s)$) according to the value of t.

The point availability function (3.2.33) is called the *uptime-based availability function* because it is constructed based on the probability that the system is still operating at a point in time.

Another point availability function is called the *downtime-based availability*, which is constructed from an equivalent definition of the point availability based on the probability that the system is not operating at a point in time, namely $A(t) = 1 - \Pr\{\text{The system is down at time } t\}$.

Let $\bar{A}(t) = \Pr\{\text{The system is down at time } t\}$. A system is down at time t if a repair or a PM operation is happening at time t. Using the notations we have previously established, $\bar{A}(t)$ can be expressed as

$$\bar{A}(t) = \begin{cases} \int_0^t f_{T_1}(u)\bar{G}_{r,1}(t-u)du \\ \quad + \sum_{n=1}^{\infty} \int_0^t \int_0^u h_{r,U_n}(w) f_{T_{n+1}}(u-w)\bar{G}_{r,n+1}(t-u)dwdu & t < T_a \\ F_{T_1}(T_a)\int_0^{T_a} f_{c,T_1}(u)\bar{G}_{r,1}(t-u)du + \bar{F}_{T_1}(T_a)\bar{G}_{p,1}(t-T_a) \\ \quad + \sum_{n=1}^{\infty} F_{T_{n+1}}(T_a)\int_0^t \int_{u-T_a}^u \begin{pmatrix} h_{m,\Theta_n}(w) f_{c,T_{n+1}}(u-w) \\ \bar{G}_{r,n+1}(t-u) \end{pmatrix} dwdu & t \ge T_a \\ \quad + \sum_{n=1}^{\infty} \bar{F}_{T_{n+1}}(T_a)\int_0^{t-T_a} h_{m,\Theta_n}(u)\bar{G}_{p,n+1}(t-T_a-u)du \end{cases} \tag{3.2.45}$$

The expression represents two cases. When $t < T_a$, all cycles must be repair cycles. The unavailability function is equivalent to that of the model without PM. The first convolution term indicates that during the first cycle the system fails at time u and the repair period takes longer than $(t-u)$ time units. The following convolution terms in the summation indicate that the total time of n cycles plus the $(n+1)^{\text{st}}$ operating time is u and the following repair takes longer than $(t-u)$ time units. In fact, this expression is similar to the unavailability expression used in Rehmert [2]'s downtime-based point availability function.

When $t \ge T_a$, there is a possibility of having some PM cycles. The term $F_{T_1}(T_a)\int_0^{T_a} f_{c,T_1}(u)\bar{G}_{r,1}(t-u)du + \bar{F}_{T_1}(T_a)\bar{G}_{p,1}(t-T_a)$ indicates the probability of system unavailability within the first cycle. If $T_1 < T_a$ (which occurs with probability $F_{T_1}(T_a)$), the term $\int_0^{T_a} f_{c,T_1}(u)\bar{G}_{r,1}(t-u)du$ represents the unavailability probability of the first cycle, which suggests that within the first cycle the system fails at time $u < T_a$ and the repair period takes longer than $(t-u)$ time units. If $T_1 \ge T_a$ (which occurs with probability $\bar{F}_{T_1}(T_a)$), the system has survived T_a and so it is interrupted by a PM operation. The term $\bar{G}_{p,1}(t-T_a)$ represents the unavailability probability of the first cycle, which indicates the PM action takes longer than $(t-T_a)$ time units.

The term $F_{T_{n+1}}(T_a)\int_0^t \int_{u-T_a}^u \left(h_{m,\Theta_n}(w) f_{c,T_{n+1}}(u-w)\bar{G}_{r,n+1}(t-u)\right) dwdu$ in the second summation indicates the probability of system unavailability after n cycles given $T_{n+1} < T_a$ (which occurs with probability $F_{T_{n+1}}(T_a)$). The term $\int_0^t \int_{u-T_a}^u \left(h_{m,\Theta_n}(w) f_{c,T_{n+1}}(u-w)\bar{G}_{r,n+1}(t-u)\right) dwdu$ represents the unavailability, which suggests that the total time of n cycles plus the $(n+1)^{\text{st}}$ operating time is u and the repair period takes longer than $(t-u)$ time units. The operating time cannot exceed T_a and therefore the domain of integration must be adjusted to be $[u-T_a, u]$.

The term $\bar{F}_{T_{n+1}}(T_a)\int_0^{t-T_a} h_{m,\Theta_n}(u)\bar{G}_{p,n+1}(t-T_a-u)du$ in the third summation represents the unavailability after n cycles given $T_{n+1} \geq T_a$ (which occurs with probability $\bar{F}_{T_{n+1}}(T_a)$). The $(n+1)^{\text{st}}$ operating time is truncated at T_a due to the PM operation. The convolution term $\int_0^{t-T_a} h_{m,\Theta_n}(u)\bar{G}_{p,n+1}(t-T_a-u)du$ indicates that the total time of n cycles plus the ongoing PM time is $(t-T_a)$ time units.

Since $A(t) = 1-\bar{A}(t)$, we can construct the downtime-based availability function as

$$A(t) = \begin{cases} 1-\int_0^t f_{T_1}(u)\bar{G}_{r,1}(t-u)du \\ \quad -\sum_{n=1}^{\infty}\int_0^t\int_0^u h_{r,U_n}(w) f_{T_{n+1}}(u-w)\bar{G}_{r,n+1}(t-u)dwdu & t < T_a \\ 1-F_{T_1}(T_a)\int_0^{T_a} f_{c,T_1}(u)\bar{G}_{r,1}(t-u)du - \bar{F}_{T_1}(T_a)\bar{G}_{p,1}(t-T_a) \\ \quad -\sum_{n=1}^{\infty} F_{T_{n+1}}(T_a)\int_0^t\int_{u-T_a}^u \begin{pmatrix} h_{m,\Theta_n}(w) f_{c,T_{n+1}}(u-w) \\ \bar{G}_{r,n+1}(t-u) \end{pmatrix} dwdu & t \geq T_a \\ \quad -\sum_{n=1}^{\infty} \bar{F}_{T_{n+1}}(T_a)\int_0^{t-T_a} h_{m,\Theta_n}(u)\bar{G}_{p,n+1}(t-T_a-u)du \end{cases} \tag{3.2.46}$$

The Laplace transform for the availability function where $t < T_a$ is

$$
\begin{aligned}
A^*_{t<T_a}(s) &= \frac{1}{s} - f^*_{T_1}(s)\bar{G}^*_{r,1}(s) - \sum_{n=1}^{\infty} h^*_{r,U_n}(s)\, f^*_{T_{n+1}}(s)\bar{G}^*_{r,n+1}(s) \\
&= \frac{1}{s} - \frac{1}{s} f^*_{T_1}(s)\left(1-g^*_{r,1}(s)\right) - \frac{1}{s}\sum_{n=1}^{\infty} h^*_{r,U_n}(s)\, f^*_{T_{n+1}}(s)\left(1-g^*_{r,n+1}(s)\right) \\
&= \frac{1}{s}\left[1 - f^*_{T_1}(s)\left(1-g^*_{r,1}(s)\right) - \sum_{n=1}^{\infty} h^*_{r,U_n}(s)\, f^*_{T_{n+1}}(s)\left(1-g^*_{r,n+1}(s)\right)\right] \\
&= \frac{1}{s}\left[1 - f^*_{T_1}(s)\left(1-g^*_{r,1}(s)\right) - \sum_{n=1}^{\infty} h^*_{r,U_n}(s)\, f^*_{T_1}(\alpha^n s)\left(1-g^*_{r,1}(\beta_r^n s)\right)\right]
\end{aligned}
\tag{3.2.47}
$$

From (3.2.39), we can write (3.2.47) as

$$
\begin{aligned}
A^*_{t<T_a}(s) &= \frac{1}{s}\left[\begin{array}{l} 1 - f^*_{T_1}(s)\left(1-g^*_{r,1}(s)\right) \\ -\sum_{n=1}^{\infty}\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right) f^*_{T_1}(\alpha^n s)\left(1-g^*_{r,1}(\beta_r^n s)\right)\end{array}\right] \\
&= \frac{1}{s}\left[\begin{array}{l} 1 - \left(\frac{1}{g^*_{r,1}(s)} - 1\right) f^*_{T_1}(s)g^*_{r,1}(s) \\ -\sum_{n=1}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^n s)} - 1\right)\left(\prod_{i=1}^{n+1}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\end{array}\right] \\
&= \frac{1}{s}\left[1 - \sum_{n=0}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^n s)} - 1\right) h^*_{r,U_{n+1}}(s)\right]\right] \\
&= \frac{1}{s}\left[1 - \sum_{n=1}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^{n-1} s)} - 1\right) h^*_{r,U_n}(s)\right]\right]
\end{aligned}
$$

Hence,

$$
A^*_{t<T_a}(s) = \frac{1}{s}\left[1 - \sum_{n=1}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^{n-1} s)} - 1\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right] \tag{3.2.48}
$$

The Laplace transform for the downtime-based availability function (3.2.46) where $t \ge T_a$ is

$$
\begin{aligned}
A^*_{t\ge T_a}(s) &= \frac{1}{s} - \left(F_{T_1}(T_a) f^*_{c,T_1}(s)\bar{G}^*_{r,1}(s) + \bar{F}_{T_1}(T_a) e^{-sT_a}\bar{G}^*_{p,1}(s) \right) \\
&\quad - \sum_{n=1}^{\infty} \begin{pmatrix} F_{T_{n+1}}(T_a) h^*_{m,\Theta_n}(s) f^*_{c,T_{n+1}}(s)\bar{G}^*_{r,n+1}(s) \\ + \bar{F}_{T_{n+1}}(T_a) h^*_{m,\Theta_n}(s) e^{-sT_a}\bar{G}^*_{p,n+1}(s) \end{pmatrix} \\
&= \frac{1}{s} - \left(\frac{1}{s} F_{T_1}(T_a) \frac{f^*_{cc,T_1}(s)}{F_{T_1}(T_a)} \left(1 - g^*_{r,1}(s)\right) + \frac{1}{s}\bar{F}_{T_1}(T_a) e^{-sT_a}\left(1 - g^*_{p,1}(s)\right) \right) \\
&\quad - \frac{1}{s}\sum_{n=1}^{\infty} \begin{pmatrix} F_{T_{n+1}}(T_a) h^*_{m,\Theta_n}(s) \dfrac{f^*_{cc,T_{n+1}}(s)}{F_{T_{n+1}}(T_a)} \left(1 - g^*_{r,n+1}(s)\right) \\ + \bar{F}_{T_{n+1}}(T_a) h^*_{m,\Theta_n}(s) e^{-sT_a}\left(1 - g^*_{p,n+1}(s)\right) \end{pmatrix}
\end{aligned}
$$

Therefore

$$
A^*_{t\ge T_a}(s) = \frac{1}{s} \begin{bmatrix} 1 - \left(f^*_{cc,T_1}(s)\left(1 - g^*_{r,1}(s)\right) + \bar{F}_{T_1}(T_a) e^{-sT_a}\left(1 - g^*_{p,1}(s)\right) \right) \\ - \displaystyle\sum_{n=1}^{\infty} \begin{pmatrix} h^*_{m,\Theta_k}(s) f^*_{cc,T_{n+1}}(s)\left(1 - g^*_{r,1}(\beta_r^n s)\right) \\ + \bar{F}_{T_{n+1}}(T_a) h^*_{m,\Theta_n}(s) e^{-sT_a}\left(1 - g^*_{p,1}(\beta_p^n s)\right) \end{pmatrix} \end{bmatrix} \tag{3.2.49}
$$

To evaluate the expression (3.2.46), we apply the inverse to the appropriate Laplace transform ($A^*_{t<T_a}(s)$ or $A^*_{t\ge T_a}(s)$) according to the value of t.

3.2.3 Bounds on the Point Availability

The uptime-based and downtime-based point availability functions (3.2.33) and (3.2.46) are equivalent and exact. However, inverting these functions back to the time domain requires the truncation of the infinite sums to obtain their approximations. The truncated uptime-based availability function gives a lower bound to the actual point availability because each omitted term in the sum is positive and would have been added to the approximation if it was included. Similarly, the truncated downtime-based availability function gives an upper bound to the actual point availability because each omitted term in the sums is positive but it would have been subtracted from the approximated value if it was included. Obviously, the more we include terms in the summations for approximation, the lesser the gap between the upper and lower bound will be.

The following sections present specific expressions for the quasi-renewal function and the two point availability functions for the numerical cases mentioned on page 6.

3.3 Quasi-Renewal Model under Exponential Operating Intervals and Exponential Repair and PM Intervals

The density function for the first operating time interval is denoted by

$$f_{T_1}(t) = \lambda e^{-\lambda t} \tag{3.3.1}$$

Hence the density function for the k th operating interval is

$$f_{T_k}(t) = \frac{1}{\alpha^{k-1}} f_{T_1}\left(t \big/ \alpha^{k-1} \right) = \lambda \alpha^{-k+1} e^{-\lambda \alpha^{-k+1} t}. \tag{3.3.2}$$

And the distribution function for the k th operating interval is

$$F_{T_k}(t) = F_{T_1}\left(t \big/ \alpha^{k-1} \right) = 1 - e^{-\lambda \alpha^{-k+1} t} \tag{3.3.3}$$

And the Laplace transform for the density function is

$$f_{T_k}^*(s) = \frac{\lambda}{\alpha^{k-1} s + \lambda}. \tag{3.3.4}$$

The corresponding function $f_{cc,T_k}(t)$ is expressed as

$$\begin{aligned} f_{cc,T_k}(t) &= \lambda \alpha^{-k+1} e^{-\lambda \alpha^{-k+1} t} \text{ where } 0 \le t < T_a \\ &= \left(1 - u_{T_a}(t)\right) \lambda \alpha^{-k+1} e^{-\lambda \alpha^{-k+1} t} \end{aligned}. \tag{3.3.5}$$

The Laplace transform for $f_{cc,T_k}(t)$ is

$$\begin{aligned} f_{cc,T_k}^*(s) &= \frac{\lambda}{\alpha^{k-1}} \mathcal{L}\left(\left(1 - u_{T_a}(t)\right) e^{-\lambda \alpha^{-k+1} t} \right) \\ &= \frac{\lambda}{\alpha^{k-1}} \left(\frac{1 - e^{-\left(s + \lambda \alpha^{-k+1}\right) T_a}}{s + \lambda \alpha^{-k+1}} \right) \\ &= \frac{\lambda \left(1 - e^{-\left(s + \lambda \alpha^{-k+1}\right) T_a} \right)}{\alpha^{k-1} s + \lambda} \end{aligned}. \tag{3.3.6}$$

The Laplace transform for the truncated density function $f_{c,T_k}(t)$ is

$$f_{c,T_k}^*(s) = \frac{f_{cc,T_k}^*(s)}{F_{T_k}(T_a)} = \frac{\lambda \left(1 - e^{-\left(s + \lambda \alpha^{-k+1}\right) T_a} \right)}{\left(1 - e^{-\lambda \alpha^{-k+1} T_a}\right)\left(\alpha^{k-1} s + \lambda\right)} \tag{3.3.7}$$

The truncated survival function $\bar{F}_{cc,T_k}(t)$ is

$$\bar{F}_{cc,T_k}(t) = \left(1 - u_{T_a}(t)\right) e^{-\lambda\alpha^{-k+1}t} \tag{3.3.8}$$

Hence the Laplace transform for $\bar{F}_{cc,T_k}(t)$ is

$$\bar{F}^*_{cc,T_k}(s) = \frac{1 - e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}}{s + \lambda\alpha^{-k+1}} \tag{3.3.9}$$

The density function for the k th repair time interval is denoted by

$$g_{r,k}(t) = \frac{1}{\beta_r^{k-1}} \rho e^{-\frac{\rho t}{\beta_r^{k-1}}}, t > 0\,. \tag{3.3.10}$$

And the density function for the k th first PM time interval is the denoted by

$$g_{p,k}(t) = \frac{1}{\beta_p^{k-1}} \gamma e^{-\frac{\gamma t}{\beta_p^{k-1}}}, t > 0\,. \tag{3.3.11}$$

Hence, the Laplace transform for the repair and PM density functions are

$$g^*_{r,k}(s) = \frac{\rho}{\beta_r^{k-1}s + \rho} \tag{3.3.12}$$

and

$$g^*_{p,k}(s) = \frac{\gamma}{\beta_p^{k-1}s + \gamma}\,. \tag{3.3.13}$$

The term $h^*_{m,\Theta_n}(s)$ is the Laplace transform of the renewal density based on Θ_n. And according to (3.2.18), it can be expressed as

$$h^*_{m,\Theta_n}(s) = \prod_{k=1}^{n} h^*_{m,V_k}(s)\,. \tag{3.3.14}$$

The linearity property of the Laplace transform operation and (3.2.15) allow us to express $h^*_{m,V_k}(s)$ as

$$h^*_{m,V_k}(s) = F_{T_k}(T_a) h^*_{r,V_k}(s) + \bar{F}_{T_k}(T_a) h^*_{p,V_k}(s)\,. \tag{3.3.15}$$

According to (3.2.11), the term $h^*_{r,V_k}(s)$ can be written as follows.

$h^*_{r,V_k}(s) = f^*_{c,T_k}(s) g^*_{r,k}(s)$

$$=\left(\frac{\lambda\left(1-e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}\right)}{\left(1-e^{-\lambda\alpha^{-k+1}T_a}\right)\left(\alpha^{k-1}s+\lambda\right)}\right)\left(\frac{\rho}{\beta_r^{k-1}s+\rho}\right).$$

Therefore,

$$h_{r,V_k}^*(s)=\frac{\lambda\rho\left(1-e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}\right)}{\left(1-e^{-\lambda\alpha^{-k+1}T_a}\right)\left(\alpha^{k-1}s+\lambda\right)\left(\beta_r^{k-1}s+\rho\right)} \tag{3.3.16}$$

According to (3.2.13), the term $h_{p,V_k}^*(s)$ can be written as follows.

$$h_{p,V_k}^*(s)=e^{-sT_a}g_{p,k}^*(s)=e^{-sT_a}\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right) \tag{3.3.17}$$

Therefore $h_{m,V_k}^*(s)$ is written as

$$h_{m,V_k}^*(s)=F_{T_k}(T_a)h_{r,V_k}^*(s)+\bar{F}_{T_k}(T_a)h_{p,V_k}^*(s)$$

$$=\frac{\lambda\rho\left(1-e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}\right)}{\left(\alpha^{k-1}s+\lambda\right)\left(\beta_r^{k-1}s+\rho\right)}+\left(e^{-\lambda\alpha^{-k+1}T_a}\right)e^{-sT_a}\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right).$$

Hence,

$$h_{m,V_k}^*(s)=\frac{\lambda\rho\left(1-e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}\right)}{\left(\alpha^{k-1}s+\lambda\right)\left(\beta_r^{k-1}s+\rho\right)}+\frac{\gamma e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}}{\beta_p^{k-1}s+\gamma} \tag{3.3.18}$$

Therefore,

$$h_{m,\Theta_n}^*(s)=\prod_{k=1}^{n}h_{m,V_k}^*(s)=\prod_{k=1}^{n}\left(\frac{\lambda\rho\left(1-e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}\right)}{\left(\alpha^{k-1}s+\lambda\right)\left(\beta_r^{k-1}s+\rho\right)}+\frac{\gamma e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}}{\beta_p^{k-1}s+\gamma}\right) \tag{3.3.19}$$

Similarly, the term $h_{r,U_n}^*(s)$ is the Laplace transform of the renewal density based on U_n, where every cycle is a repair cycle. Hence, it can be expressed as

$$h_{r,U_n}^*(s)=\prod_{i=1}^{n}\left(f_{T_i}^*(s)g_{r,i}^*(s)\right)=\prod_{i=1}^{n}\frac{\lambda\rho}{\left(\alpha^{i-1}s+\lambda\right)\left(\beta_r^{i-1}s+\rho\right)} \tag{3.3.20}$$

3.3.1 Quasi-renewal Function

From (3.2.31) and (3.2.32), the Laplace transform for the quasi-renewal function can be expressed as

$$Q^*_{H_m,t<T_a}(s) = \frac{1}{s}\sum_{n=1}^{\infty}\left[\prod_{i=1}^{n}\frac{\lambda\rho}{\left(\alpha^{i-1}s+\lambda\right)\left(\beta_r^{i-1}s+\rho\right)}\right] \tag{3.3.21}$$

and

$$Q^*_{H_m,t\geq T_a}(s) = \frac{1}{s}\sum_{n=1}^{\infty}\prod_{k=1}^{n}\left(\frac{\lambda\rho\left(1-e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}\right)}{\left(\alpha^{k-1}s+\lambda\right)\left(\beta_r^{k-1}s+\rho\right)}+\frac{\gamma e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}}{\beta_p^{k-1}s+\gamma}\right) \tag{3.3.22}$$

Evaluating the inverse Laplace transform of $Q^*_{H_m,t<T_a}(s)$ or $Q^*_{H_m,t\geq T_a}(s)$ is too difficult to do algebraically. Thus evaluating this quantity requires numerical inversion.

3.3.2 Uptime-based Point Availability Function

Depending on the value of t, we know the Laplace transform for the uptime-based point availability expression can be written as:

For $0 \leq t < T_a$:

$$A^*_{t<T_a}(s) = \frac{1}{s}\left[\left(1-f^*_{T_1}(s)\right)+\sum_{n=1}^{\infty}\left[\left(1-f^*_{T_1}(\alpha^n s)\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right] \tag{3.2.41}$$

For $t \geq T_a$:

$$A^*_{t\geq T_a}(s) = \sum_{n=1}^{\infty}\left[\bar{F}^*_{cc,T_{n+1}}(s)\left(\prod_{i=1}^{n}\left(\begin{array}{l} f^*_{cc,T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s) \\ \quad +\bar{F}_{T_k}(T_a)e^{-sT_a}g^*_{p,1}(\beta_p^{i-1}s)\end{array}\right)\right)\right] \tag{3.2.42}$$

For $n = 0,1,2,3\ldots$, the term $\frac{1}{s}\left(1-f^*_{T_1}(\alpha^n s)\right)$ can be expressed as

$$\begin{aligned}\frac{1}{s}\left(1-f^*_{T_1}(\alpha^n s)\right) &= \frac{1}{s}\left(1-\frac{\lambda}{\alpha^n s+\lambda}\right)\\ &= \frac{1}{s}\left(\frac{\alpha^n s}{\alpha^n s+\lambda}\right) \qquad (3.3.23)\\ &= \frac{1}{s+\lambda\alpha^{-n}}\end{aligned}$$

Hence, from (3.2.41) and (3.2.42), the Laplace transform for the uptime-based point availability expression is:

For $0 \le t < T_a$:

$$A^*_{t<T_a}(s) = \frac{1}{s+\lambda} + \sum_{n=1}^{\infty}\left[\left(\frac{1}{s+\lambda\alpha^{-n}}\right)\left(\prod_{i=1}^{n}\frac{\lambda\rho}{\left(\alpha^{i-1}s+\lambda\right)\left(\beta_r^{i-1}s+\rho\right)}\right)\right] \tag{3.3.24}$$

For $t \ge T_a$:

$$A^*_{t\ge T_a}(s) = \sum_{n=1}^{\infty}\left[\begin{array}{l}\left(\dfrac{1-e^{-\left(s+\lambda\alpha^{-n}\right)T_a}}{s+\lambda\alpha^{-n}}\right)\\ \displaystyle\prod_{k=1}^{n}\left(\frac{\lambda\rho\left(1-e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}\right)}{\left(\alpha^{k-1}s+\lambda\right)\left(\beta_r^{k-1}s+\rho\right)}+\frac{\gamma e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}}{\beta_p^{k-1}s+\gamma}\right)\end{array}\right] \tag{3.3.25}$$

The inverse Laplace transform of the term $\dfrac{1}{s+\lambda}$ in (3.3.24) is exactly

$$\mathcal{L}^{-1}\left\{\frac{1}{s+\lambda}\right\} = e^{-\lambda t} = \bar{F}_{T_1}(t)$$

However the inverse Laplace transform of the other terms are too complicated to be computed exactly. Therefore, the numerical inversion is used instead.

3.3.3 Downtime-based Point Availability Function

From Section 3.2.1, the Laplace transform for the downtime-based point availability expression is:

For $t < T_a$:

$$A^*_{t<T_a}(s) = \frac{1}{s}\left[1-\sum_{n=1}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^{n-1}s)}-1\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right] \tag{3.2.48}$$

For $t \ge T_a$:

$$A^*_{t\ge T_a}(s) = \frac{1}{s}\left[\begin{array}{l}1-\left(f^*_{cc,T_1}(s)\left(1-g^*_{r,1}(s)\right)+\bar{F}_{T_1}(T_a)e^{-sT_a}\left(1-g^*_{p,1}(s)\right)\right)\\ \displaystyle -\sum_{n=1}^{\infty}\left(\begin{array}{l}h^*_{m,\Theta_n}(s)f^*_{cc,T_{n+1}}(s)\left(1-g^*_{r,1}(\beta_r^n s)\right)\\ \quad+\bar{F}_{T_{n+1}}(T_a)h^*_{m,\Theta_n}(s)e^{-sT_a}\left(1-g^*_{p,1}(\beta_p^n s)\right)\end{array}\right)\end{array}\right] \tag{3.2.49}$$

To write these expressions for the quasi-renewal model under exponential operating, repair, and PM intervals, we need to express each term explicitly.

$A^*_{t<T_a}(s)$ can be expressed as

$$A^*_{t<T_a}(s) = \frac{1}{s}\left[1 - \sum_{n=1}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^{n-1}s)} - 1\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right]$$

$$= \frac{1}{s} - \frac{1}{s}\sum_{n=1}^{\infty}\left[\left(\frac{\beta_r^{n-1}s+\rho}{\rho} - 1\right)\left(\prod_{i=1}^{n}\frac{\lambda\rho}{\left(\alpha^{i-1}s+\lambda\right)\left(\beta_r^{i-1}s+\rho\right)}\right)\right]$$

$$= \frac{1}{s} - \sum_{n=1}^{\infty}\left[\beta_r^{n-1}\rho^{n-1}\lambda^{n}\left(\prod_{i=1}^{n}\frac{1}{\left(\alpha^{i-1}s+\lambda\right)\left(\beta_r^{i-1}s+\rho\right)}\right)\right]$$

Hence

$$A^*_{t<T_a}(s) = \frac{1}{s} - \frac{\lambda}{(s+\lambda)(s+\rho)} - \sum_{n=1}^{\infty}\left[\beta_r^{n}\rho^{n}\lambda^{n+1}\left(\prod_{i=1}^{n+1}\frac{1}{\left(\alpha^{i-1}s+\lambda\right)\left(\beta_r^{i-1}s+\rho\right)}\right)\right] \quad (3.3.26)$$

The term $\frac{1}{s} - \frac{\lambda}{(s+\rho)(s+\lambda)}$ can be inverted exactly:

If $\lambda \neq \rho$,

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\lambda}{(s+\rho)(s+\lambda)}\right\} = 1 + \frac{\lambda\left(e^{-\rho t} - e^{-\lambda t}\right)}{\rho - \lambda}. \quad (3.3.27)$$

If $\lambda = \rho$,

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\lambda}{(s+\rho)(s+\lambda)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\lambda}{(s+\lambda)^2}\right\} = 1 - \lambda t e^{-\lambda t}. \quad (3.3.28)$$

The case $\lambda = \rho$ is very unrealistic because it implies the operating and repair times have the same mean.

The subsequent Laplace transform terms are too complicated to be inverted exactly. Therefore, the numerical inversion is required.

$A^*_{t\geq T_a}(s)$ can be expressed as

$$A^*_{t\geq T_a}(s) = \frac{1}{s}\left[\begin{array}{l} 1-\left(f^*_{cc,T_1}(s)\left(1-g^*_{r,1}(s)\right)+\bar{F}_{T_1}(T_a)e^{-sT_a}\left(1-g^*_{p,1}(s)\right)\right) \\ -\sum_{n=1}^{\infty}\left(\begin{array}{l} h^*_{m,\Theta_k}(s)\, f^*_{cc,T_{n+1}}(s)\left(1-g^*_{r,1}(\beta_r^n s)\right) \\ +\bar{F}_{T_{n+1}}(T_a)h^*_{m,\Theta_n}(s)e^{-sT_a}\left(1-g^*_{p,1}(\beta_p^n s)\right)\end{array}\right)\end{array}\right]$$

$$= \frac{1}{s}\left[\begin{array}{l} 1-\left(\frac{\lambda\left(1-e^{-(s+\lambda)T_a}\right)}{s+\lambda}\left(1-\frac{\rho}{s+\rho}\right)+e^{-\lambda T_a}e^{-sT_a}\left(1-\frac{\gamma}{s+\gamma}\right)\right) \\ -\sum_{n=1}^{\infty}\left(\begin{array}{l} h^*_{m,\Theta_n}(s)\left(\frac{\lambda\left(1-e^{-\left(s+\lambda\alpha^{-n}\right)T_a}\right)}{\alpha^n s+\lambda}\right)\left(1-\frac{\rho}{\beta_r^n s+\rho}\right) \\ +e^{-\lambda\alpha^{-n}T_a}h^*_{m,\Theta_n}(s)e^{-sT_a}\left(1-\frac{\gamma}{\beta_p^n s+\gamma}\right)\end{array}\right)\end{array}\right]$$

$$A^*_{t\geq T_a}(s) = \frac{1}{s}\left[\begin{array}{l} 1-\left(\frac{\lambda s\left(1-e^{-(s+\lambda)T_a}\right)}{(s+\lambda)(s+\rho)}+\frac{se^{-(s+\lambda)T_a}}{s+\gamma}\right) \\ -\sum_{n=1}^{\infty}\left(h^*_{m,\Theta_n}(s)\left(\frac{\lambda\beta_r^n s\left(1-e^{-\left(s+\lambda\alpha^{-n}\right)T_a}\right)}{\left(\alpha^n s+\lambda\right)\left(\beta_r^n s+\rho\right)}+\frac{\beta_p^n se^{-\left(s+\lambda\alpha^{-n}\right)T_a}}{\beta_p^n s+\gamma}\right)\right)\end{array}\right]$$

$$= \frac{1}{s}-\left(\frac{\lambda\left(1-e^{-(s+\lambda)T_a}\right)}{(s+\lambda)(s+\rho)}+\frac{e^{-(s+\lambda)T_a}}{s+\gamma}\right)-\sum_{n=1}^{\infty}\left(h^*_{m,\Theta_n}(s)\left(\frac{\lambda\beta_r^n\left(1-e^{-\left(s+\lambda\alpha^{-n}\right)T_a}\right)}{\left(\alpha^n s+\lambda\right)\left(\beta_r^n s+\rho\right)}+\frac{\beta_p^n e^{-\left(s+\lambda\alpha^{-n}\right)T_a}}{\beta_p^n s+\gamma}\right)\right)$$

Therefore

$$
\begin{aligned}
A^*_{t\geq T_a}(s) &= \frac{1}{s} - \left(\frac{\lambda\left(1-e^{-(s+\lambda)T_a}\right)}{(s+\lambda)(s+\rho)} + \frac{e^{-(s+\lambda)T_a}}{s+\gamma} \right) \\
&\quad - \sum_{n=1}^{\infty} \left[\left(\frac{\lambda\beta_r^n\left(1-e^{-\left(s+\lambda\alpha^{-n}\right)T_a}\right)}{\left(\alpha^n s+\lambda\right)\left(\beta_r^n s+\rho\right)} + \frac{\beta_p^n e^{-\left(s+\lambda\alpha^{-n}\right)T_a}}{\beta_p^n s+\gamma} \right) \right. \\
&\quad \left. \prod_{k=1}^{n} \left(\frac{\lambda\rho\left(1-e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}\right)}{\left(\alpha^{k-1}s+\lambda\right)\left(\beta_r^{k-1}s+\rho\right)} + \frac{\gamma e^{-\left(s+\lambda\alpha^{-k+1}\right)T_a}}{\beta_p^{k-1}s+\gamma} \right) \right]
\end{aligned} \tag{3.3.29}
$$

The term $\frac{1}{s} - \left(\frac{\lambda\left(1-e^{-(s+\lambda)T_a}\right)}{(s+\lambda)(s+\rho)} + \frac{e^{-(s+\lambda)T_a}}{s+\gamma} \right)$ can be inverted exactly:

If $\lambda \neq \rho$,

$$
\begin{aligned}
&\mathcal{L}^{-1}\left\{ \frac{1}{s} - \left(\frac{\lambda\left(1-e^{-(s+\lambda)T_a}\right)}{(s+\lambda)(s+\rho)} + \frac{e^{-(s+\lambda)T_a}}{s+\gamma} \right) \right\} \\
&= 1 + \frac{\lambda\left(e^{-\rho t}-e^{-\lambda t}\right)}{\rho-\lambda} - u_{T_a}(t)\left(\frac{\lambda e^{-\lambda T_a}\left(e^{-\rho(t-T_a)}-e^{-\lambda(t-T_a)}\right)}{\rho-\lambda} + e^{-\gamma(t-T_a)-\lambda T_a} \right)
\end{aligned} \tag{3.3.30}
$$

If $\lambda = \rho$,

$$
\mathcal{L}^{-1}\left\{ \frac{1}{s} - \left(\frac{\lambda\left(1-e^{-(s+\lambda)T_a}\right)}{(s+\lambda)^2} + \frac{e^{-(s+\lambda)T_a}}{s+\gamma} \right) \right\} = \begin{cases} 1-\lambda t e^{-\lambda t}, & t < T_a \\ 1-\lambda T_a e^{-\lambda t} - e^{-\gamma(t-T_a)-\lambda T_a}, & t \geq T_a \end{cases} \tag{3.3.31}
$$

Again the case $\lambda = \rho$ is very unrealistic.

The subsequent terms are too complex to be inverted exactly and therefore the numerical inversion is applied.

The special cases that are interesting are (1) $\rho = \gamma$ i.e. when the repair and PM times have the same mean and (2) $T_a = \infty$ i.e. when the PM policy is not considered.

3.4 Quasi-Renewal Model under Gamma Operating Intervals and Gamma Repair and PM Intervals

We assume that the first operating time interval has $\text{Gamma}(a,\lambda)$ distribution with a being the shape parameter and λ being the scale parameter. a is assumed to be an integer. Therefore the density function for is denoted by

$$f_{T_1}(t)=\frac{\lambda^a t^{a-1}e^{-\lambda t}}{(a-1)!},\quad t>0 \tag{3.4.1}$$

Hence the density function for the k^{th} operating interval is given by

$$f_{T_k}(t)=\frac{1}{\alpha^{k-1}}f_{T_1}\left(t/\alpha^{k-1}\right)=\frac{\left(\alpha^{-k+1}\lambda\right)^a t^{a-1}e^{-\alpha^{-k+1}\lambda t}}{(a-1)!}. \tag{3.4.2}$$

And the cumulative distribution function for the k^{th} operating interval is

$$\begin{aligned}F_{T_k}(t)=F_{T_1}\left(t/\alpha^{k-1}\right)&=\int_0^{\alpha^{-k+1}t}\frac{\lambda^a x^{a-1}e^{-\lambda x}}{(a-1)!}dx=\frac{\gamma\left(a,\alpha^{-k+1}\lambda t\right)}{(a-1)!}\\&=\frac{\Gamma(a)-\Gamma\left(a,\alpha^{-k+1}\lambda t\right)}{(a-1)!}\\&=\frac{(a-1)!-(a-1)!e^{-\alpha^{-k+1}\lambda t}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda t\right)^j}{j!}}{(a-1)!}\\&=1-e^{-\alpha^{-k+1}\lambda t}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda t\right)^j}{j!}\end{aligned} \tag{3.4.3}$$

where $\gamma(a,t)=\int_0^t x^{a-1}e^{-x}dx$ is the *lower incomplete gamma function* and $\Gamma(a,t)=\int_t^\infty x^{a-1}e^{-x}dx$ is the *upper incomplete gamma function*.

The Laplace transform for the density function is

$$f_{T_k}^*(s)=\left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^a. \tag{3.4.4}$$

The corresponding function $f_{cc,T_k}(t)$ is expressed as

$$\begin{aligned}f_{cc,T_k}(t)&=\frac{\left(\alpha^{-k+1}\lambda\right)^a t^{a-1}e^{-\alpha^{-k+1}\lambda t}}{(a-1)!}\quad\text{where }0\le t<T_a\\&=\left(1-u_{T_a}(t)\right)\frac{\left(\alpha^{-k+1}\lambda\right)^a t^{a-1}e^{-\alpha^{-k+1}\lambda t}}{(a-1)!}\end{aligned}. \tag{3.4.5}$$

The Laplace transform for $f_{cc,T_k}(t)$ is

$$
\begin{aligned}
f^*_{cc,T_k}(s) &= \frac{\left(\alpha^{-k+1}\lambda\right)^a}{(a-1)!}\mathcal{L}\left(\left(1-u_{T_a}(t)\right)t^{a-1}e^{-\alpha^{-k+1}\lambda t}\right) \\
&= \frac{\left(\alpha^{-k+1}\lambda\right)^a}{(a-1)!}\left(\frac{\Gamma(a)-\Gamma\left(a,\left(s+\alpha^{-k+1}\lambda\right)T_a\right)}{\left(s+\lambda\alpha^{-k+1}\right)^a}\right) \\
&= \frac{\left(\alpha^{-k+1}\lambda\right)^a}{(a-1)!}\left(\frac{(a-1)!-(a-1)!e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^j T_a{}^j}{j!}}{\left(s+\alpha^{-k+1}\lambda\right)^a}\right). \qquad (3.4.6) \\
&= \left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^a\left(1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^j T_a{}^j}{j!}\right)
\end{aligned}
$$

The Laplace transform for the truncated density function $f_{c,T_k}(t)$ is

$$
\begin{aligned}
f^*_{c,T_k}(s) &= \frac{f^*_{cc,T_k}(s)}{F_{T_k}(T_a)} \\
&= \left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^a\left(\frac{1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^j T_a{}^j}{j!}}{1-e^{-\alpha^{-k+1}\lambda T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^j}{j!}}\right) \qquad (3.4.7)
\end{aligned}
$$

The truncated survival function $\bar{F}_{cc,T_k}(t)$ is

$$
\begin{aligned}
\bar{F}_{cc,T_k}(t) &= \left(1-u_{T_a}(t)\right)\left(1-\frac{\gamma\left(a,\alpha^{-k+1}\lambda t\right)}{(a-1)!}\right) \\
&= \left(1-u_{T_a}(t)\right)\frac{\Gamma\left(a,\alpha^{-k+1}\lambda t\right)}{(a-1)!} \qquad (3.4.8) \\
&= \left(1-u_{T_a}(t)\right)e^{-\alpha^{-k+1}\lambda t}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda t\right)^j}{j!}
\end{aligned}
$$

Hence the Laplace transform for $\bar{F}_{cc,T_k}(t)$ is

$$\bar{F}^*_{cc,T_k}(s)=\sum_{j=0}^{a-1}\left(\frac{\alpha^{k-1}\lambda^j}{\left(\alpha^{k-1}s+\lambda\right)^{j+1}}\left(1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{i=0}^{j}\frac{\left(\left(s+\alpha^{-k+1}\lambda\right)T_a\right)^i}{i!}\right)\right) \qquad (3.4.9)$$

The first repair time interval has $\text{Gamma}(b,\rho)$ distribution. The density function is denoted by

$$g_{r,1}(t)=\frac{\rho^b t^{b-1}e^{-\rho t}}{(b-1)!},\quad t>0. \qquad (3.4.10)$$

Similarly, the first PM time interval is distributed according to $\text{Gamma}(c,\gamma)$ distribution. And the density function is the denoted by

$$g_{p,1}(t)=\frac{\gamma^c t^{c-1}e^{-\gamma t}}{(c-1)!},\quad t>0. \qquad (3.4.11)$$

Hence the density functions for the k^{th} repair and PM intervals are

$$g_{r,k}(t)=\frac{1}{\beta_r^{k-1}}g_{r,1}\left(t\middle/\beta_r^{k-1}\right)=\frac{\left(\beta_r^{-k+1}\rho\right)^b t^{b-1}e^{-\rho t}}{(b-1)!},\quad t>0 \qquad (3.4.12)$$

$$g_{p,k}(t)=\frac{1}{\beta_p^{k-1}}g_{p,1}\left(t\middle/\beta_p^{k-1}\right)=\frac{\left(\beta_p^{-k+1}\gamma\right)^c t^{c-1}e^{-\gamma t}}{(c-1)!},\quad t>0 \qquad (3.4.13)$$

Hence, the Laplace transform for the k^{th} repair and PM density functions are

$$g^*_{r,k}(s)=\left(\frac{\rho}{\beta_r^{k-1}s+\rho}\right)^b \qquad (3.4.14)$$

and

$$g^*_{p,k}(s)=\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right)^c. \qquad (3.4.15)$$

The term $h^*_{r,V_k}(s)$ can be written as:

$$h^*_{r,V_k}(s)=f^*_{c,T_k}(s)g^*_{r,k}(s)$$

$$=\left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^a\left(\frac{\rho}{\beta_r^{k-1}s+\rho}\right)^b\left(\frac{1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^j T_a^{\,j}}{j!}}{1-e^{-\alpha^{-k+1}\lambda T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^j}{j!}}\right) \qquad (3.4.16)$$

The term $h^*_{p,V_k}(s)$ can be written as:

$$h^*_{p,V_k}(s) = e^{-sT_a} g^*_{p,k}(s) = e^{-sT_a}\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right)^c \tag{3.4.17}$$

Therefore $h^*_{m,V_k}(s)$ is written as

$$h^*_{m,V_k}(s) = F_{T_k}(T_a) h^*_{r,V_k}(s) + \bar{F}_{T_k}(T_a) h^*_{p,V_k}(s)$$

$$= \left[\begin{array}{c}\left(1-e^{-\alpha^{-k+1}\lambda T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^j}{j!}\right)\left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^a \\ \left(\frac{\rho}{\beta_r^{k-1}s+\rho}\right)^b\left(\frac{1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^j T_a{}^j}{j!}}{1-e^{-\alpha^{-k+1}\lambda T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^j}{j!}}\right)\end{array}\right]$$

$$+\left(e^{-\alpha^{-k+1}\lambda T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^j}{j!}\right)e^{-sT_a}\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right)^c$$

Hence,

$$\begin{aligned} h^*_{m,V_k}(s) &= \left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^a\left(\frac{\rho}{\beta_r^{k-1}s+\rho}\right)^b\left(1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^j T_a{}^j}{j!}\right) \\ &\quad + \left(e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^j}{j!}\right)\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right)^c \end{aligned} \tag{3.4.18}$$

Therefore,

$$h^*_{m,\Theta_n}(s) = \prod_{k=1}^{n} h^*_{m,V_k}(s)$$

$$= \prod_{k=1}^{n}\left(\begin{array}{c}\left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^a\left(\frac{\rho}{\beta_r^{k-1}s+\rho}\right)^b\left(1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^j T_a{}^j}{j!}\right) \\ +\left(e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^j}{j!}\right)\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right)^c\end{array}\right) \tag{3.4.19}$$

Similarly, the term $h^*_{r,U_n}(s)$ is the Laplace transform of the renewal density based on U_n, where every cycle is a repair cycle. According to (3.2.39), it can be expressed as

$$h^*_{r,U_n}(s) = \prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s) g^*_{r,1}(\beta_r^{i-1}s)\right) = \prod_{i=1}^{n}\left[\left(\frac{\lambda}{\lambda+\alpha^{i-1}s}\right)^a \left(\frac{\rho}{\rho+\beta_r^{i-1}s}\right)^b\right] \quad (3.4.20)$$

3.4.1 Quasi-renewal Function

From (3.2.31) and (3.2.32), the Laplace transform for the quasi-renewal function can be expressed as

$$Q^*_{H_m,t<T_a}(s) = \frac{1}{s}\sum_{n=1}^{\infty}\prod_{i=1}^{n}\left[\left(\frac{\lambda}{\lambda+\alpha^{i-1}s}\right)^a \left(\frac{\rho}{\rho+\beta_r^{i-1}s}\right)^b\right] \quad (3.4.21)$$

and

$$\begin{aligned} &Q^*_{H_m,t\geq T_a}(s) \\ &= \frac{1}{s}\sum_{n=1}^{\infty}\prod_{k=1}^{n}\left(\begin{aligned} &\left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^a \left(\frac{\rho}{\beta_r^{k-1}s+\rho}\right)^b \left(1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^j T_a{}^j}{j!}\right) \\ &+\left(e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^j}{j!}\right)\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right)^c \end{aligned}\right) \end{aligned} \quad (3.4.22)$$

Evaluating the inverse Laplace transform of $Q^*_{H_m,t<T_a}(s)$ or $Q^*_{H_m,t\geq T_a}(s)$ is too difficult to do algebraically. Thus evaluating this quantity requires numerical inversion.

3.4.2 Uptime-based Point Availability Function

The Laplace transform for the uptime-based point availability expression is:

For $0 \leq t < T_a$:

$$\begin{aligned} A^*_{t<T_a}(s) &= \frac{1}{s}\left[\left(1-f^*_{T_1}(s)\right)+\sum_{n=1}^{\infty}\left[\left(1-f^*_{T_1}(\alpha^n s)\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right] \quad (3.4.23) \\ &= \frac{1}{s}\left(1-\left(\frac{\lambda}{s+\lambda}\right)^a\right) \\ &\quad +\sum_{n=1}^{\infty}\left[\frac{1}{s}\left(1-\left(\frac{\lambda}{\alpha^n s+\lambda}\right)^a\right)\prod_{i=1}^{n}\left[\left(\frac{\lambda}{\lambda+\alpha^{i-1}s}\right)^a\left(\frac{\rho}{\rho+\beta_r^{i-1}s}\right)^b\right]\right] \end{aligned}$$

For $t \geq T_a$:

$$
\begin{aligned}
A^*_{t\geq T_a}(s) &= \sum_{n=1}^{\infty}\left[\bar{F}^*_{cc,T_{n+1}}(s)h^*_{m,\Theta_n}(s)\right] \\
&= \sum_{n=1}^{\infty}\left[\begin{array}{l}
\left(\sum_{j=0}^{a-1}\left(\frac{\alpha^{k-1}\lambda^{j}}{\left(\alpha^{k-1}s+\lambda\right)^{j+1}}\left(1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{i=0}^{j}\frac{\left(\left(s+\alpha^{-k+1}\lambda\right)T_a\right)^{i}}{i!}\right)\right)\right) \\
\prod_{k=1}^{n}\left(\begin{array}{l}
\left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^{a}\left(\frac{\rho}{\beta_r^{k-1}s+\rho}\right)^{b}\left(1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^{j}T_a{}^{j}}{j!}\right) \\
\quad +\left(e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^{j}}{j!}\right)\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right)^{c}
\end{array}\right)
\end{array}\right]
\end{aligned}
\tag{3.4.24}
$$

The inverse Laplace transform of the term $\frac{1}{s}\left(1-\left(\frac{\lambda}{s+\lambda}\right)^{a}\right)$ in (3.4.23) is exactly

$$
\mathcal{L}^{-1}\left\{\frac{1}{s}\left(1-\left(\frac{\lambda}{s+\lambda}\right)^{a}\right)\right\} = \frac{\Gamma(a,\lambda t)}{\Gamma(a)} = \bar{F}_{T_1}(t)
$$

However the inverse Laplace transform of the other terms are too complicated to be computed exactly. Therefore, the numerical inversion is used instead.

3.4.3 Downtime-based Point Availability Function

According to (3.2.48), the Laplace transform for the downtime-based point availability function $A^*_{t<T_a}(s)$ can be expressed as

$$\begin{aligned}
A^*_{t<T_a}(s) &= \frac{1}{s}\left[1-\sum_{n=1}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^{n-1}s)}-1\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right] \\
&= \frac{1}{s}\left[\begin{array}{l} 1-f^*_{T_1}(s)+f^*_{T_1}(s)g^*_{r,1}(s) \\ -\sum_{n=2}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^{n-1}s)}-1\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\end{array}\right] \\
&= \frac{1}{s}-\frac{\lambda^a}{s(s+\lambda)^a}+\frac{\lambda^a\rho^b}{s(s+\lambda)^a(s+\rho)^b} \\
&\quad -\frac{1}{s}\sum_{n=2}^{\infty}\left(\left(\frac{\left(\beta_r^{n-1}s+\rho\right)^b}{\rho^b}-1\right)\prod_{i=1}^{n}\left[\left(\frac{\lambda}{\lambda+\alpha^{i-1}s}\right)^a\left(\frac{\rho}{\rho+\beta_r^{i-1}s}\right)^b\right]\right) \\
&= \frac{1}{s}-\frac{\lambda^a}{s(s+\lambda)^a}+\frac{\lambda^a\rho^b}{s(s+\lambda)^a(s+\rho)^b} \\
&\quad -\frac{1}{s}\sum_{n=2}^{\infty}\left(\left(\frac{\left(\beta_r^{n-1}s+\rho\right)^b}{\rho^b}-1\right)\left(\lambda^a\rho^b\right)^n\prod_{i=1}^{n}\frac{1}{\left(\lambda+\alpha^{i-1}s\right)^a\left(\rho+\beta_r^{i-1}s\right)^b}\right)
\end{aligned} \tag{3.4.25}$$

If $\lambda=\rho$, the term $\frac{1}{s}-\frac{\lambda^a}{s(s+\lambda)^a}+\frac{\lambda^a\rho^b}{s(s+\lambda)^a(s+\rho)^b}$ in (3.4.25) can be inverted exactly:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{\lambda^a}{s(s+\lambda)^a}+\frac{\lambda^a\rho^b}{s(s+\lambda)^a(s+\rho)^b}\right\}=1-e^{-\lambda t}\sum_{j=a}^{a+b-1}\frac{(\lambda t)^j}{j!}. \tag{3.4.26}$$

Again, the case $\lambda=\rho$ is very unrealistic because it also implies that the operating and repair times have the same mean.

If $\lambda\neq\rho$ and $a=b=2$ (which is one of the values being used for our numerical examples), the term $\frac{1}{s}-\frac{\lambda^a}{s(s+\lambda)^a}+\frac{\lambda^a\rho^b}{s(s+\lambda)^a(s+\rho)^b}$ can be inverted exactly:

$$
\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{\lambda^2}{s\left(s+\lambda\right)^2}+\frac{\lambda^2\rho^2}{s\left(s+\lambda\right)^2\left(s+\rho\right)^2}\right\}\\
&=1+e^{-\lambda t}\left(1+\lambda t\right)+\frac{\rho^2}{\left(\lambda-\rho\right)^3}e^{-\lambda t}\left(\left(\lambda\rho-\lambda^2\right)t-3\lambda+\rho\right).\\
&-\frac{\lambda^2}{\left(\lambda-\rho\right)^3}e^{-\rho t}\left(\left(\lambda\rho-\rho^2\right)t-3\rho+\lambda\right)
\end{aligned}
\tag{3.4.27}
$$

However the subsequent Laplace transform terms are too complicated to be inverted exactly. Therefore, the numerical inversion is required.

From (3.2.49), the term $A^*_{t\geq T_a}(s)$ can be expressed as

$$
\begin{aligned}
A^*_{t\geq T_a}(s) &= \frac{1}{s}-\frac{1}{s}\left(\begin{array}{l}\left(\dfrac{\lambda}{s+\lambda}\right)^a\left(1-e^{-(s+\lambda)T_a}\displaystyle\sum_{j=0}^{a-1}\frac{\left(s+\lambda\right)^j T_a{}^j}{j!}\right)\left(1-\left(\dfrac{\rho}{s+\rho}\right)^b\right)\\[2ex] +\left(e^{-(s+\lambda)T_a}\displaystyle\sum_{j=0}^{a-1}\frac{\left(\lambda T_a\right)^j}{j!}\right)\left(1-\left(\dfrac{\gamma}{s+\gamma}\right)^c\right)\end{array}\right)\\
&-\frac{1}{s}\sum_{n=1}^{\infty}\left(h^*_{m,\Theta_n}(s)\left(\begin{array}{l}\left(\dfrac{\lambda}{\alpha^n s+\lambda}\right)^a\left(1-\left(\dfrac{\rho}{\beta_r^n s+\rho}\right)^b\right)\\[2ex] \left(1-e^{-\left(s+\alpha^{-n}\lambda\right)T_a}\displaystyle\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-n}\lambda\right)^j T_a{}^j}{j!}\right)\\[2ex] +\left(e^{-\left(s+\alpha^{-n}\lambda\right)T_a}\displaystyle\sum_{j=0}^{a-1}\frac{\left(\alpha^{-n}\lambda T_a\right)^j}{j!}\right)\left(1-\left(\dfrac{\gamma}{\beta_p^n s+\gamma}\right)^c\right)\end{array}\right)\right)
\end{aligned}
\tag{3.4.28}
$$

Assuming $a=b=c=2$, the term

$$
B(s):=\frac{1}{s}-\frac{1}{s}\left(\begin{array}{l}\left(\dfrac{\lambda}{\lambda+s}\right)^a\left(1-e^{-(s+\lambda)T_a}\displaystyle\sum_{j=0}^{a-1}\frac{\left(s+\lambda\right)^j T_a{}^j}{j!}\right)\left(1-\left(\dfrac{\rho}{s+\rho}\right)^b\right)\\[2ex] +\left(e^{-(s+\lambda)T_a}\displaystyle\sum_{j=0}^{a-1}\frac{\left(\lambda T_a\right)^j}{j!}\right)\left(1-\left(\dfrac{\gamma}{s+\gamma}\right)^c\right)\end{array}\right)
$$

can be inverted exactly as follows:

If $\lambda = \rho$,

$$\mathcal{L}^{-1}\{B(s)\} = \begin{cases} 1 - \frac{1}{6} e^{-\lambda t} \lambda^2 t^2 (3 + \lambda t), & t < T_a \\ 1 - (1 + \lambda T_a) e^{-\gamma t} \left(\gamma (t - T_a) + 1\right) \\ \quad - \frac{1}{6} T_a{}^2 \lambda^2 e^{-\lambda t} \left(3\lambda (t - T_a) + \lambda T_a + 3\right), & t \geq T_a \end{cases}$$

Again, the case $\lambda = \rho$ is very unrealistic.

If $\lambda \neq \rho$,

$$\mathcal{L}^{-1}\{B(s)\}$$

$$= \begin{cases} 1 - \frac{\lambda^2 e^{-\rho t}}{(\lambda - \rho)^3} \left(\lambda - 3\rho + \rho(\lambda - \rho) t\right) + \frac{\lambda^2 e^{-\lambda t}}{(\lambda - \rho)^3} \left(\lambda - 3\rho + (\lambda - 2\rho)(\lambda - \rho) t\right), & t < T_a \\ 1 - e^{-\lambda T_a - \gamma (t - T_a)} (1 + \lambda T_a)\left(1 + \gamma (t - T_a)\right) - \frac{\lambda^2 e^{-\rho t}}{(\lambda - \rho)^3} \left(\lambda - 3\rho + \rho(\lambda - \rho) t\right) \\ - \frac{\lambda^2 e^{-\lambda T_a - \rho (t - T_a)}}{(\lambda - \rho)^3} \left(\begin{array}{l} \rho(\rho - \lambda)\left(1 - (\rho - \lambda) T_a\right) t - (\lambda - 3\rho)\left((\lambda - \rho) T_a + 1\right) \\ + \rho (\rho - \lambda)^2 T_a{}^2 \end{array} \right), & t \geq T_a \end{cases}$$

The subsequent terms are too complex to be inverted exactly and therefore the numerical inversion is applied.

The special cases that are interesting are (1) $\rho = \gamma$ i.e. when the repair and PM times have the same mean and (2) $T_a = \infty$ i.e. when the PM policy is not considered and we have Rehmert's [2] model.

3.5 Quasi-Renewal Model under Gamma Operating Intervals and Exponential Repair and PM Intervals

The operating time distribution functions and their Laplace transform are the same as previous section. However the density functions for the repair and the PM time interval are exponential. Since exponential distribution is special case of a gamma distribution, the density functions for the repair and the PM time intervals are essentially the same as in the previous section with $b = c = 1$.

3.5.1 Quasi-renewal Function

From Section 3.4.1, we can write the Laplace transform for the quasi-renewal function for model under gamma operating intervals and exponential repair/PM intervals by substituting $b = c = 1$, i.e.

$$Q^*_{H_m, t<T_a}(s) = \frac{1}{s}\sum_{n=1}^{\infty}\prod_{i=1}^{n}\left[\left(\frac{\lambda}{\lambda+\alpha^{i-1}s}\right)^a\left(\frac{\rho}{\rho+\beta_r^{i-1}s}\right)\right] \tag{3.5.1}$$

and

$$= \frac{1}{s}\sum_{n=1}^{\infty}\prod_{k=1}^{n}\left(\begin{array}{l}\left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^a\left(\frac{\rho}{\beta_r^{k-1}s+\rho}\right)\left(1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^j T_a^{\,j}}{j!}\right)\\ \quad +\left(e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^j}{j!}\right)\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right)\end{array}\right) \tag{3.5.2}$$

Evaluating the inverse Laplace transform of $Q^*_{H_m, t<T_a}(s)$ or $Q^*_{H_m, t\ge T_a}(s)$ is too difficult to do algebraically. Thus evaluating this quantity requires numerical inversion.

3.5.2 Uptime-based Point Availability Function

From Section 0, the Laplace transform for the uptime-based point availability function for the model under gamma operating intervals and exponential repair/PM intervals can be written as follows.

For $0 \le t < T_a$:

$$\begin{aligned} A^*_{t<T_a}(s) = {} & \frac{1}{s}\left(1-\left(\frac{\lambda}{s+\lambda}\right)^a\right) \\ & +\sum_{n=1}^{\infty}\left[\frac{1}{s}\left(1-\left(\frac{\lambda}{\alpha^n s+\lambda}\right)^a\right)\prod_{i=1}^{n}\left[\left(\frac{\lambda}{\lambda+\alpha^{i-1}s}\right)^a\left(\frac{\rho}{\rho+\beta_r^{i-1}s}\right)\right]\right] \end{aligned} \tag{3.5.3}$$

For $t \ge T_a$:

$$A^*_{t\geq T_a}(s)=\sum_{n=1}^{\infty}\left[\begin{array}{l}\left(\sum_{j=0}^{a-1}\left(\frac{\alpha^{k-1}\lambda^{j}}{\left(\alpha^{k-1}s+\lambda\right)^{j+1}}\left(1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{i=0}^{j}\frac{\left(\left(s+\alpha^{-k+1}\lambda\right)T_a\right)^{i}}{i!}\right)\right)\right) \\ \prod_{k=1}^{n}\left(\begin{array}{l}\left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^{a}\left(\frac{\rho}{\beta_r^{k-1}s+\rho}\right)\left(1-e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-k+1}\lambda\right)^{j}T_a{}^{j}}{j!}\right) \\ \quad +\left(e^{-\left(s+\alpha^{-k+1}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-k+1}\lambda T_a\right)^{j}}{j!}\right)\left(\frac{\gamma}{\beta_p^{k-1}s+\gamma}\right)\end{array}\right)\end{array}\right] \quad (3.5.4)$$

Similarly, the inverse Laplace transform of the term $\frac{1}{s}\left(1-\left(\frac{\lambda}{s+\lambda}\right)^{a}\right)$ in (3.5.3) is exactly

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(1-\left(\frac{\lambda}{s+\lambda}\right)^{a}\right)\right\}=\frac{\Gamma(a,\lambda t)}{\Gamma(a)}=\bar{F}_{T_1}(t)$$

However the inverse Laplace transform of the other terms are too complicated to be computed exactly. Therefore, the numerical inversion is used instead.

3.5.3 Downtime-based Point Availability Function

From Section 3.4.3, the Laplace transform for the downtime-based point availability expression is:

$$\begin{aligned}A^*_{t<T_a}(s)=&\frac{1}{s}-\frac{1}{s}\left(\frac{\lambda}{s+\lambda}\right)^{a}+\frac{1}{s}\left(\frac{\lambda}{s+\lambda}\right)^{a}\left(\frac{\rho}{s+\rho}\right)\\&-\sum_{n=2}^{\infty}\left(\lambda^{an}\left(\beta_r\rho\right)^{n-1}\prod_{i=1}^{n}\frac{1}{\left(\lambda+\alpha^{i-1}s\right)^{a}\left(\rho+\beta_r^{i-1}s\right)}\right)\end{aligned} \quad (3.5.5)$$

If $\lambda=\rho$, the term $\frac{1}{s}-\frac{1}{s}\left(\frac{\lambda}{s+\lambda}\right)^{a}+\frac{1}{s}\left(\frac{\lambda}{s+\lambda}\right)^{a}\left(\frac{\rho}{s+\rho}\right)$ in (3.5.5) can be inverted exactly:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s}\left(\frac{\lambda}{s+\lambda}\right)^{a}+\frac{1}{s}\left(\frac{\lambda}{s+\lambda}\right)^{a}\left(\frac{\rho}{s+\rho}\right)\right\}=1-e^{-\lambda t}\frac{(\lambda t)^{a}}{a!}. \quad (3.5.6)$$

As mentioned before, the case $\lambda=\rho$ is very unrealistic.

If $\lambda \neq \rho$, the term $\frac{1}{s}-\frac{1}{s}\left(\frac{\lambda}{s+\lambda}\right)^{a}+\frac{1}{s}\left(\frac{\lambda}{s+\lambda}\right)^{a}\left(\frac{\rho}{s+\rho}\right)$ can be inverted exactly:

$$
\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s}\left(\frac{\lambda}{s+\lambda}\right)^{a}+\frac{1}{s}\left(\frac{\lambda}{s+\lambda}\right)^{a}\left(\frac{\rho}{s+\rho}\right)\right\} \\
&=1-e^{-\rho t}\left(\frac{\lambda}{\lambda-\rho}\right)^{a}\left(1-\frac{\Gamma\left(a,(\lambda-\rho)t\right)}{\Gamma(a)}\right) \\
&=1-e^{-\rho t}\left(\frac{\lambda}{\lambda-\rho}\right)^{a}\left(1-e^{-(\lambda-\rho)t}\sum_{j=0}^{a-1}\frac{\left((\lambda-\rho)t\right)^{j}}{j!}\right) \\
&=1-e^{-\rho t}\left(\frac{\lambda}{\lambda-\rho}\right)^{a}+e^{-\lambda t}\left(\frac{\lambda}{\lambda-\rho}\right)^{a}\sum_{j=0}^{a-1}\frac{\left((\lambda-\rho)t\right)^{j}}{j!}
\end{aligned}
\quad . \tag{3.5.7}
$$

However the subsequent Laplace transform terms are too complicated to be inverted exactly. Therefore, the numerical inversion is required.

On the other hand, $A^{*}_{t\geq T_a}(s)$ can be expressed as

$$
\begin{aligned}
A^{*}_{t\geq T_a}(s)=\frac{1}{s}-\frac{1}{s}&\left(\begin{aligned}&\left(\frac{\lambda}{s+\lambda}\right)^{a}\left(1-e^{-(s+\lambda)T_a}\sum_{j=0}^{a-1}\frac{(s+\lambda)^{j}\,T_a^{\,j}}{j!}\right)\left(1-\left(\frac{\rho}{s+\rho}\right)\right)\\ &+\left(e^{-(s+\lambda)T_a}\sum_{j=0}^{a-1}\frac{(\lambda T_a)^{j}}{j!}\right)\left(1-\left(\frac{\gamma}{s+\gamma}\right)\right)\end{aligned}\right) \\
&-\frac{1}{s}\sum_{n=1}^{\infty}\left(h^{*}_{m,\Theta_n}(s)\left(\begin{aligned}&\left(\frac{\lambda}{\alpha^{n}s+\lambda}\right)^{a}\left(1-\left(\frac{\rho}{\beta_r^{n}s+\rho}\right)\right)\\ &\left(1-e^{-\left(s+\alpha^{-n}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(s+\alpha^{-n}\lambda\right)^{j}\,T_a^{\,j}}{j!}\right)\\ &+\left(e^{-\left(s+\alpha^{-n}\lambda\right)T_a}\sum_{j=0}^{a-1}\frac{\left(\alpha^{-n}\lambda T_a\right)^{j}}{j!}\right)\left(1-\left(\frac{\gamma}{\beta_p^{n}s+\gamma}\right)\right)\end{aligned}\right)\right)
\end{aligned}
\tag{3.5.8}
$$

Assuming $a = 2$, the term

$$B(s) := \frac{1}{s} - \frac{1}{s}\left(\begin{array}{l} \left(\left(\frac{\lambda}{\lambda+s}\right)^a \left(1 - e^{-(s+\lambda)T_a} \sum_{j=0}^{a-1} \frac{(s+\lambda)^j T_a{}^j}{j!}\right)\left(1 - \left(\frac{\rho}{s+\rho}\right)\right)\right. \\ \left. + \left(e^{-(s+\lambda)T_a} \sum_{j=0}^{a-1} \frac{(\lambda T_a)^j}{j!}\right)\left(1 - \left(\frac{\gamma}{s+\gamma}\right)\right)\right) \end{array} \right)$$ can be inverted exactly

as follows:

If $\lambda = \rho$,

$$\mathcal{L}^{-1}\{B(s)\} = 1 - \frac{\lambda^2}{2} T_a{}^2 e^{-\lambda t} - e^{-\gamma(t-T_a)-\lambda T_a}\left(1 + \lambda T_a\right) \quad \text{for} \quad t \geq T_a$$

Again, the case $\lambda = \rho$ is very unrealistic.

If $\lambda \neq \rho$,

$$\mathcal{L}^{-1}\{B(s)\}$$

$$= 1 - e^{(\gamma-\lambda)T_a - \gamma t}\left(1 + \lambda T_a\right) - e^{-\rho t}\left(\frac{\lambda}{\lambda-\rho}\right)^2 \left(1 - e^{(\rho-\lambda)T_a}\left(1 + (\lambda-\rho)T_a\right)\right) \quad \text{for} \quad t \geq T_a$$

The subsequent terms are too complex to be inverted exactly and therefore the numerical inversion is applied.

3.6 Quasi-Renewal Model under Normal Operating Intervals and Normal Repair and PM Intervals

The density function for the first and the k th operating time intervals are respectively expressed by

$$f_{T_1}(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \tag{3.6.1}$$

and

$$f_{T_k}(t) = \frac{1}{\alpha^{k-1}\sigma\sqrt{2\pi}} e^{-\frac{\left(t-\alpha^{k-1}\mu\right)^2}{2\left(\alpha^{k-1}\sigma\right)^2}} \tag{3.6.2}$$

To properly use the normal distribution for modeling the life length of a system, we assume that the mean and the variance are chosen so that the probability mass for $t < 0$ is negligible. Thus the Laplace transform for the normal density function can be derived from the moment generating function. Hence we can write the Laplace transform for the above functions as

$$f_{T_1}^*(s) = e^{-s\mu + \frac{1}{2}s^2\sigma^2} \tag{3.6.3}$$

and

$$f_{T_k}^*(s) = e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2} \tag{3.6.4}$$

The corresponding function $f_{cc,T_k}(t)$ is expressed as

$$f_{cc,T_k}(t) = \left(1 - u_{T_a}(t)\right)\frac{1}{\alpha^{k-1}\sigma\sqrt{2\pi}}e^{-\frac{\left(t-\alpha^{k-1}\mu\right)^2}{2\left(\alpha^{k-1}\sigma\right)^2}} \tag{3.6.5}$$

The Laplace transform for $f_{cc,T_k}(t)$ is

$$\begin{aligned} f_{cc,T_k}^*(s) &= \frac{1}{2}e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}\left(1 + \operatorname{erf}\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\sqrt{2}\left(\alpha^{k-1}\sigma\right)}\right)\right) \\ &= e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}\Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right) \end{aligned} \tag{3.6.6}$$

where $\Phi(x)$ is the standard normal cdf. The construction of this expression is presented in Appendix A. 1.

Hence the Laplace transform for the truncated density function $f_{c,T_k}(t)$ is

$$f_{c,T_k}^*(s) = \frac{f_{cc,T_k}^*(s)}{F_{T_k}(T_a)} = \frac{e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}}{\Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)}\Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right) \tag{3.6.7}$$

The truncated survival function $\bar{F}_{cc,T_k}(t)$ is

$$\bar{F}_{cc,T_k}(t) = \left(1 - u_{T_a}(t)\right)\left(1 - \Phi\left(\frac{t - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)\right) \tag{3.6.8}$$

Hence the Laplace transform for $\bar{F}_{cc,T_k}(t)$ is

$$\begin{aligned}\bar{F}^{*}_{cc,T_k}(s) &= \frac{1-e^{-sT_a}}{s} \\ &\quad +\frac{1}{2s}\left(\begin{array}{l} e^{-sT_a}\left(1+\operatorname{erf}\left(\dfrac{T_a-\alpha^{k-1}\mu}{\sqrt{2}\left(\alpha^{k-1}\sigma\right)}\right)\right) \\ -e^{-s\alpha^{k-1}\mu+\frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}\left(1+\operatorname{erf}\left(\dfrac{T_a-\alpha^{k-1}\mu+s\left(\alpha^{k-1}\sigma\right)^2}{\sqrt{2}\left(\alpha^{k-1}\sigma\right)}\right)\right)\end{array}\right) \\ &= \frac{1-e^{-sT_a}}{s} \\ &\quad +\frac{1}{s}\left(\begin{array}{l} e^{-sT_a}\Phi\left(\dfrac{T_a-\alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right) \\ -e^{-s\alpha^{k-1}\mu+\frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}\Phi\left(\dfrac{T_a-\alpha^{k-1}\mu+s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right)\end{array}\right)\end{aligned} \tag{3.6.9}$$

The density function for the k^{th} repair time interval is denoted by

$$g_{r,k}(t)=\frac{1}{\beta_r^{\,k-1}\sigma_r\sqrt{2\pi}}e^{-\frac{\left(t-\beta_r^{\,k-1}\mu_r\right)^2}{2\left(\beta_r^{\,k-1}\sigma_r\right)^2}}. \tag{3.6.10}$$

And the density function for the k^{th} PM time interval is the denoted by

$$g_{p,k}(t)=\frac{1}{\beta_p^{\,k-1}\sigma_p\sqrt{2\pi}}e^{-\frac{\left(t-\beta_p^{\,k-1}\mu_p\right)^2}{2\left(\beta_p^{\,k-1}\sigma_p\right)^2}}. \tag{3.6.11}$$

Hence, the Laplace transform for the k^{th} repair and PM density functions are

$$g^{*}_{r,k}(s)=e^{-s\beta_r^{\,k-1}\mu_r+\frac{1}{2}s^2\left(\beta_r^{\,k-1}\sigma_r\right)^2} \tag{3.6.12}$$

and

$$g^{*}_{p,k}(s)=e^{-s\beta_p^{\,k-1}\mu_p+\frac{1}{2}s^2\left(\beta_p^{\,k-1}\sigma_p\right)^2}. \tag{3.6.13}$$

Note that the parameters μ, σ, μ_r, σ_r, μ_p, and σ_p are assumed to be chosen so that the probability mass for $t<0$ is insignificant and can be ignored.

Next we express the term $h^{*}_{m,\Theta_n}(s)$, the Laplace transform of the renewal density based on Θ_n, under the normal distribution model.

First, the term $h^*_{r,V_k}(s)$ can be written as follows.

$$h^*_{r,V_k}(s) = f^*_{c,T_k}(s) g^*_{r,k}(s)$$
$$= \left(\frac{e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}}{\Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)} \Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right)\right)\left(e^{-s\beta_r{}^{k-1}\mu_r + \frac{1}{2}s^2\left(\beta_r{}^{k-1}\sigma_r\right)^2}\right).$$

Therefore,

$$h^*_{r,V_k}(s) = \frac{e^{-s\left(\alpha^{k-1}\mu + \beta_r{}^{k-1}\mu_r\right) + \frac{1}{2}s^2\left(\left(\alpha^{k-1}\sigma\right)^2 + \left(\beta_r{}^{k-1}\sigma_r\right)^2\right)}}{\Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)} \Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right) \quad (3.6.14)$$

According to (3.2.13), the term $h^*_{p,V_k}(s)$ can be written as follows.

$$h^*_{p,V_k}(s) = e^{-sT_a} g^*_{p,k}(s) = e^{-s\left(\beta_p{}^{k-1}\mu_p + T_a\right) + \frac{1}{2}s^2\left(\beta_p{}^{k-1}\sigma_p\right)^2} \quad (3.6.15)$$

Therefore $h^*_{m,V_k}(s)$ is written as

$$h^*_{m,V_k}(s) = F_{T_k}(T_a) h^*_{r,V_k}(s) + \bar{F}_{T_k}(T_a) h^*_{p,V_k}(s)$$
$$= \Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right) \frac{e^{-s\left(\alpha^{k-1}\mu + \beta_r{}^{k-1}\mu_r\right) + \frac{1}{2}s^2\left(\left(\alpha^{k-1}\sigma\right)^2 + \left(\beta_r{}^{k-1}\sigma_r\right)^2\right)}}{\Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)} \Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right)$$
$$+ \left(1 - \Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)\right) e^{-s\left(\beta_p{}^{k-1}\mu_p + T_a\right) + \frac{1}{2}s^2\left(\beta_p{}^{k-1}\sigma_p\right)^2}.$$

Hence,

$$h^*_{m,V_k}(s) = e^{-s\left(\alpha^{k-1}\mu + \beta_r{}^{k-1}\mu_r\right) + \frac{1}{2}s^2\left(\left(\alpha^{k-1}\sigma\right)^2 + \left(\beta_r{}^{k-1}\sigma_r\right)^2\right)} \Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right)$$
$$+ \left(1 - \Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)\right) e^{-s\left(\beta_p{}^{k-1}\mu_p + T_a\right) + \frac{1}{2}s^2\left(\beta_p{}^{k-1}\sigma_p\right)^2} \quad (3.6.16)$$

Therefore,

$$
\begin{aligned}
h^*_{m,\Theta_n}(s) &= \prod_{k=1}^{n} h^*_{m,V_k}(s) \\
&= \prod_{k=1}^{n}\left(\begin{array}{l} e^{-s\left(\alpha^{k-1}\mu+\beta_r^{k-1}\mu_r\right)+\frac{1}{2}s^2\left(\left(\alpha^{k-1}\sigma\right)^2+\left(\beta_r^{k-1}\sigma_r\right)^2\right)}\Phi\left(\dfrac{T_a-\alpha^{k-1}\mu+s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right) \\ +\left(1-\Phi\left(\dfrac{T_a-\alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)\right)e^{-s\left(\beta_p^{k-1}\mu_p+T_a\right)+\frac{1}{2}s^2\left(\beta_p^{k-1}\sigma_p\right)^2} \end{array}\right)
\end{aligned}
\tag{3.6.17}
$$

Similarly, the term $h^*_{r,U_n}(s)$ is the Laplace transform of the renewal density based on U_n, where every cycle is a repair cycle. Hence, it can be expressed as

$$
\begin{aligned}
h^*_{r,U_n}(s) &= \prod_{k=1}^{n} f^*_{T_k}(s)g^*_{r,k}(s) \\
&= \prod_{k=1}^{n}\left(e^{-s\alpha^{k-1}\mu+\frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}\right)\left(e^{-s\beta_r^{k-1}\mu_r+\frac{1}{2}s^2\left(\beta_r^{k-1}\sigma_r\right)^2}\right) \\
&= \left(e^{\sum_{k=1}^{n}\left[-s\alpha^{k-1}\mu+\frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2\right]}\right)\left(e^{\sum_{k=1}^{n}\left[-s\beta_r^{k-1}\mu_r+\frac{1}{2}s^2\left(\beta_r^{k-1}\sigma_r\right)^2\right]}\right) \\
&= e^{\sum_{k=1}^{n}\left[-s\alpha^{k-1}\mu+\frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2\right]+\sum_{k=1}^{n}\left[-s\beta_r^{k-1}\mu_r+\frac{1}{2}s^2\left(\beta_r^{k-1}\sigma_r\right)^2\right]}
\end{aligned}
$$

If $\beta_r = 1$,

$$
\begin{aligned}
h^*_{r,U_n}(s) &= e^{\left(-s\mu\frac{1-\alpha^n}{1-\alpha}+\frac{1}{2}s^2\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}\right)+\left(-ns\mu_r+\frac{1}{2}ns^2\sigma_r^2\right)} \\
&= e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha}+n\mu_r\right)+\frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+n\sigma_r^2\right)}
\end{aligned}
\tag{3.6.18}
$$

If $\beta_r \neq 1$,

$$
\begin{aligned}
h^*_{r,U_n}(s) &= e^{\left(-s\mu\frac{1-\alpha^n}{1-\alpha}+\frac{1}{2}s^2\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}\right)+\left(-s\mu_r\frac{1-\beta_r^{\,n}}{1-\beta_r}+\frac{1}{2}s^2\sigma_r^2\frac{1-\beta_r^{\,2n}}{1-\beta_r^{\,2}}\right)} \\
&= e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha}+\mu_r\frac{1-\beta_r^{\,n}}{1-\beta_r}\right)+\frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+\sigma_r^2\frac{1-\beta_r^{\,2n}}{1-\beta_r^{\,2}}\right)}
\end{aligned}
\tag{3.6.19}
$$

3.6.1 Quasi-renewal Function

From (3.2.31) and (3.2.32), the Laplace transform for the quasi-renewal function can be expressed as

$$Q^*_{H_m,t<T_a}(s)=\begin{cases}\dfrac{1}{s}\displaystyle\sum_{n=1}^{\infty}\left[e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha}+n\mu_r\right)+\frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+n\sigma_r^2\right)}\right] & \text{if } \beta_r=1,\\ \dfrac{1}{s}\displaystyle\sum_{n=1}^{\infty}\left[e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha}+\mu_r\frac{1-\beta_r{}^n}{1-\beta_r}\right)+\frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+\sigma_r^2\frac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}\right)}\right] & \text{if } \beta_r\neq 1\end{cases} \tag{3.6.20}$$

and

$$Q^*_{H_m,t\geq T_a}(s)=\frac{1}{s}\sum_{n=1}^{\infty}\prod_{k=1}^{n}\left(\begin{array}{l} e^{-s\left(\alpha^{k-1}\mu+\beta_r{}^{k-1}\mu_r\right)+\frac{1}{2}s^2\left(\left(\alpha^{k-1}\sigma\right)^2+\left(\beta_r{}^{k-1}\sigma_r\right)^2\right)} \\ \Phi\left(\dfrac{T_a-\alpha^{k-1}\mu+s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right) \\ +\left(1-\Phi\left(\dfrac{T_a-\alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)\right)e^{-s\left(\beta_p{}^{k-1}\mu_p+T_a\right)+\frac{1}{2}s^2\left(\beta_p{}^{k-1}\sigma_p\right)^2}\end{array}\right). \tag{3.6.21}$$

In fact, the Laplace transform in (3.6.20) can be inverted exactly as:

$$Q_{H_m,t<T_a}(t)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}\Phi\left(\frac{t-\left(\mu\frac{1-\alpha^n}{1-\alpha}+n\mu_r\right)}{\sqrt{\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+n\sigma_r^2}}\right) & \text{if } \beta_r=1,\\ \displaystyle\sum_{n=1}^{\infty}\Phi\left(\frac{t-\left(\mu\frac{1-\alpha^n}{1-\alpha}+\mu_r\frac{1-\beta_r{}^n}{1-\beta_r}\right)}{\sqrt{\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+\sigma_r^2\frac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}}}\right) & \text{if } \beta_r\neq 1\end{cases} \tag{3.6.22}$$

Evaluating the inverse Laplace transform of $Q^*_{H_m,t\geq T_a}(s)$ is too difficult to do algebraically. Thus evaluating this quantity requires numerical inversion.

3.6.2 Uptime-based Point Availability Function

Depending on the value of t, we know the Laplace transform for the uptime-based point availability expression can be written as:

For $0 \le t < T_a$:

$$A^*_{t<T_a}(s) = \frac{1}{s}\left[\left(1 - f^*_{T_1}(s)\right) + \sum_{n=1}^{\infty}\left[\left(1 - f^*_{T_1}(\alpha^n s)\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right] \qquad (3.2.41)$$

For $t \ge T_a$:

$$A^*_{t\ge T_a}(s) = \sum_{n=1}^{\infty}\left[\bar{F}^*_{cc,T_{n+1}}(s)h^*_{m,\Theta_n}(s)\right] \qquad (3.2.38)$$

For $n = 1,2,3...$, the term $\frac{1}{s}\left(1 - f^*_{T_1}(\alpha^n s)\right)$ can be expressed as

$$\frac{1}{s}\left(1 - f^*_{T_1}(\alpha^n s)\right) = \frac{1}{s}\left(1 - e^{-s\alpha^n\mu + \frac{1}{2}s^2\left(\alpha^n\sigma\right)^2}\right) = \bar{F}^*_{T_{n+1}}(s) \qquad (3.6.23)$$

Hence the Laplace transform for the uptime-based point availability expression is:

For $0 \le t < T_a$:

$$\begin{aligned}
A^*_{t<T_a}(s) &= \frac{1}{s}\left[\left(1 - f^*_{T_1}(s)\right) + \sum_{n=1}^{\infty}\left[\left(1 - f^*_{T_1}(\alpha^n s)\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right] \\
&= \frac{1}{s}\left(1 - e^{-s\mu + \frac{1}{2}s^2\sigma^2}\right) + \sum_{n=1}^{\infty}\left[\begin{array}{l} \frac{1}{s}\left(1 - e^{-s\alpha^n\mu + \frac{1}{2}s^2\left(\alpha^n\sigma\right)^2}\right) \\ \left(\prod_{i=1}^{n}\left(e^{-s\alpha^{i-1}\mu + \frac{1}{2}s^2\left(\alpha^{i-1}\sigma\right)^2}\right)\left(e^{-s\beta_r^{i-1}\mu_r + \frac{1}{2}s^2\left(\beta_r^{i-1}\sigma_r\right)^2}\right)\right) \end{array}\right]
\end{aligned}$$

If $\beta_r = 1$,

$$\begin{aligned}
A^*_{t<T_a}(s) &= \frac{1}{s}\left(1 - e^{-s\mu + \frac{1}{2}s^2\sigma^2}\right) \\
&\quad + \sum_{n=1}^{\infty}\left[\frac{1}{s}\left(1 - e^{-s\alpha^n\mu + \frac{1}{2}s^2\left(\alpha^n\sigma\right)^2}\right)e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha} + n\mu_r\right) + \frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2} + n\sigma_r^2\right)}\right]
\end{aligned}$$

If $\beta_r \neq 1$,

$$A^*_{t<T_a}(s) = \frac{1}{s}\left(1 - e^{-s\mu + \frac{1}{2}s^2\sigma^2}\right) + \sum_{n=1}^{\infty}\left[\frac{1}{s}\left(1 - e^{-s\alpha^n\mu + \frac{1}{2}s^2\left(\alpha^n\sigma\right)^2}\right)e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha} + \mu_r\frac{1-\beta_r{}^n}{1-\beta_r}\right) + \frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2} + \sigma_r^2\frac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}\right)}\right]$$

Therefore:

$$A^*_{t<T_a}(s) = \begin{cases} \frac{1}{s}\left(1 - e^{-s\mu + \frac{1}{2}s^2\sigma^2}\right) & \text{if } \beta_r = 1, \\ \quad + \sum_{n=1}^{\infty}\left[\frac{1}{s}\left(e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha} + n\mu_r\right) + \frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2} + n\sigma_r^2\right)} - e^{-s\left(\frac{\mu\left(1-\alpha^n\right)}{1-\alpha} + \alpha^n\mu + n\mu_r\right) + \frac{s^2}{2}\left(\frac{\sigma^2\left(1-\alpha^{2n}\right)}{1-\alpha^2} + \left(\alpha^n\sigma\right)^2 + n\sigma_r^2\right)}\right)\right] & \\ \frac{1}{s}\left(1 - e^{-s\mu + \frac{1}{2}s^2\sigma^2}\right) & \text{if } \beta_r \neq 1 \\ \quad + \sum_{n=1}^{\infty}\left[\frac{1}{s}\left(e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha} + \mu_r\frac{1-\beta_r{}^n}{1-\beta_r}\right) + \frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2} + \sigma_r^2\frac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}\right)} - e^{-s\left(\frac{\mu\left(1-\alpha^n\right)}{1-\alpha} + \alpha^n\mu + \mu_r\frac{1-\beta_r{}^n}{1-\beta_r}\right) + \frac{s^2}{2}\left(\frac{\sigma^2\left(1-\alpha^{2n}\right)}{1-\alpha^2} + \left(\alpha^n\sigma\right)^2 + \sigma_r^2\frac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}\right)}\right)\right] & \end{cases} \tag{3.6.24}$$

Hence the Laplace transform (3.6.24) can be inverted exactly to the time domain as

$$A_{t<T_a}(t)=\begin{cases}1-\Phi\left(\dfrac{t-\mu}{\sigma}\right) & \text{if } \beta_r=1,\\ +\displaystyle\sum_{n=1}^{\infty}\left[\Phi\left(\dfrac{t-\left(\dfrac{\mu\left(1-\alpha^n\right)}{1-\alpha}+n\mu_r\right)}{\sqrt{\dfrac{\sigma^2\left(1-\alpha^{2n}\right)}{1-\alpha^2}+n\sigma_r^2}}\right)-\Phi\left(\dfrac{t-\left(\dfrac{\mu\left(1-\alpha^n\right)}{1-\alpha}+\alpha^n\mu+n\mu_r\right)}{\sqrt{\dfrac{\sigma^2\left(1-\alpha^{2n}\right)}{1-\alpha^2}+\left(\alpha^n\sigma\right)^2+n\sigma_r^2}}\right)\right] & \\ 1-\Phi\left(\dfrac{t-\mu}{\sigma}\right) & \text{if } \beta_r\neq 1\\ +\displaystyle\sum_{n=1}^{\infty}\left[\Phi\left(\dfrac{t-\left(\dfrac{\mu\left(1-\alpha^n\right)}{1-\alpha}+\mu_r\dfrac{1-\beta_r{}^n}{1-\beta_r}\right)}{\sqrt{\dfrac{\sigma^2\left(1-\alpha^{2n}\right)}{1-\alpha^2}+\sigma_r^2\dfrac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}}}\right)-\Phi\left(\dfrac{t-\left(\dfrac{\mu\left(1-\alpha^n\right)}{1-\alpha}+\alpha^n\mu+\mu_r\dfrac{1-\beta_r{}^n}{1-\beta_r}\right)}{\sqrt{\dfrac{\sigma^2\left(1-\alpha^{2n}\right)}{1-\alpha^2}+\left(\alpha^n\sigma\right)^2+\sigma_r^2\dfrac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}}}\right)\right] & \end{cases} \tag{3.6.25}$$

For $t \geq T_a$:

$$A^*_{t\geq T_a}(s) = \sum_{n=1}^{\infty}\left[\bar{F}^*_{cc,T_{n+1}}(s)h^*_{m,\Theta_n}(s)\right]$$

$$= \sum_{n=1}^{\infty}\left[\left(\frac{1-e^{-sT_a}}{s} + \frac{1}{s}\left(\begin{array}{l} e^{-sT_a}\Phi\left(\frac{T_a-\alpha^n\mu}{\alpha^n\sigma}\right) \\ -e^{-s\alpha^n\mu+\frac{1}{2}s^2\left(\alpha^n\sigma\right)^2}\Phi\left(\frac{T_a-\alpha^n\mu+s\left(\alpha^n\sigma\right)^2}{\alpha^n\sigma}\right)\end{array}\right)\right) \prod_{k=1}^{n}\left(\begin{array}{l} e^{-s\left(\alpha^{k-1}\mu+\beta_r^{k-1}\mu_r\right)+\frac{1}{2}s^2\left(\left(\alpha^{k-1}\sigma\right)^2+\left(\beta_r^{k-1}\sigma_r\right)^2\right)}\Phi\left(\frac{T_a-\alpha^{k-1}\mu+s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right) \\ +\left(1-\Phi\left(\frac{T_a-\alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)\right)e^{-s\left(\beta_p^{k-1}\mu_p+T_a\right)+\frac{1}{2}s^2\left(\beta_p^{k-1}\sigma_p\right)^2}\end{array}\right)\right] \quad (3.6.26)$$

However the inverse Laplace transform of each term in (3.6.26) is too complicated to be computed exactly. Therefore, the numerical inversion is used instead.

3.6.3 Downtime-based Point Availability Function

From Section 3.2.1, the Laplace transform for the downtime-based point availability expression is:

For $t < T_a$:

$$A^*_{t<T_a}(s) = \frac{1}{s}\left[1-\sum_{n=1}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^{n-1}s)}-1\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right] \quad (3.2.48)$$

For $t \geq T_a$:

$$A^*_{t\geq T_a}(s) = \frac{1}{s}\left[\begin{array}{l} 1-\left(f^*_{cc,T_1}(s)\left(1-g^*_{r,1}(s)\right)+\bar{F}_{T_1}(T_a)e^{-sT_a}\left(1-g^*_{p,1}(s)\right)\right) \\ -\sum_{n=1}^{\infty}\left(\begin{array}{l} h^*_{m,\Theta_k}(s)f^*_{cc,T_{n+1}}(s)\left(1-g^*_{r,1}(\beta_r^n s)\right) \\ \quad +\bar{F}_{T_{n+1}}(T_a)h^*_{m,\Theta_n}(s)e^{-sT_a}\left(1-g^*_{p,1}(\beta_p^n s)\right)\end{array}\right)\end{array}\right] \quad (3.2.49)$$

$A^*_{t<T_a}(s)$ can be expressed as

$$
\begin{aligned}
A^*_{t<T_a}(s) &= \frac{1}{s}\left[1-\sum_{n=1}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^{n-1}s)}-1\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right] \\
&= \frac{1}{s}\left[1-\sum_{n=1}^{\infty}\left[\left(\frac{1}{e^{-s\beta_r{}^{n-1}\mu_r+\frac{1}{2}s^2\left(\beta_r{}^{n-1}\sigma_r\right)^2}}-1\right)\left(\prod_{i=1}^{n}\left(e^{-s\alpha^{i-1}\mu+\frac{1}{2}s^2\left(\alpha^{i-1}\sigma\right)^2}\right)\left(e^{-s\beta_r{}^{i-1}\mu_r+\frac{1}{2}s^2\left(\beta_r{}^{i-1}\sigma_r\right)^2}\right)\right)\right]\right]
\end{aligned}
$$

If $\beta_r = 1$,

$$
A^*_{t<T_a}(s) = \frac{1}{s}-\sum_{n=1}^{\infty}\left(\frac{1}{s}\left(e^{s\mu_r-\frac{1}{2}s^2\sigma_r^2}-1\right)e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha}+n\mu_r\right)+\frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+n\sigma_r^2\right)}\right)
$$

If $\beta_r \neq 1$,

$$
A^*_{t<T_a}(s) = \frac{1}{s}-\sum_{n=1}^{\infty}\left(\frac{1}{s}\left(e^{s\beta_r{}^{n-1}\mu_r-\frac{1}{2}s^2\left(\beta_r{}^{n-1}\sigma_r\right)^2}-1\right)e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha}+\mu_r\frac{1-\beta_r{}^n}{1-\beta_r}\right)+\frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+\sigma_r^2\frac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}\right)}\right)
$$

Hence

$$
A^*_{t<T_a}(s) = \begin{cases}
\dfrac{1}{s}-\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{s}\left(e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha}+(n-1)\mu_r\right)+\frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+(n-1)\sigma_r^2\right)} - e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha}+n\mu_r\right)+\frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+n\sigma_r^2\right)}\right)\right) & \text{if } \beta_r = 1, \\
\dfrac{1}{s} -\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{s}\left(e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha}+\mu_r\left(\frac{1-\beta_r{}^n}{1-\beta_r}-\beta_r{}^{n-1}\right)\right)+\frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+\sigma_r^2\left(\frac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}-\beta_r{}^{2n-2}\right)\right)} - e^{-s\left(\mu\frac{1-\alpha^n}{1-\alpha}+\mu_r\frac{1-\beta_r{}^n}{1-\beta_r}\right)+\frac{1}{2}s^2\left(\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+\sigma_r^2\frac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}\right)}\right)\right) & \text{if } \beta_r \neq 1
\end{cases}
\qquad (3.6.27)
$$

The expression (3.6.27) can be inverted exactly:

$$A_{t<T_a}(t)=\begin{cases} 1-\displaystyle\sum_{n=1}^{\infty}\left(\Phi\left(\frac{t-\left(\mu\frac{1-\alpha^n}{1-\alpha}+(n-1)\mu_r\right)}{\sqrt{\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+(n-1)\sigma_r^2}}\right)-\Phi\left(\frac{t-\left(\mu\frac{1-\alpha^n}{1-\alpha}+n\mu_r\right)}{\sqrt{\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+n\sigma_r^2}}\right)\right) & \text{if } \beta_r=1, \\ 1-\displaystyle\sum_{n=1}^{\infty}\left(\Phi\left(\frac{t-\left(\mu\frac{1-\alpha^n}{1-\alpha}+\mu_r\left(\frac{1-\beta_r{}^n}{1-\beta_r}-\beta_r{}^{n-1}\right)\right)}{\sqrt{\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+\sigma_r^2\left(\frac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}-\beta_r{}^{2n-2}\right)}}\right)-\Phi\left(\frac{t-\left(\mu\frac{1-\alpha^n}{1-\alpha}+\mu_r\frac{1-\beta_r{}^n}{1-\beta_r}\right)}{\sqrt{\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}+\sigma_r^2\frac{1-\beta_r{}^{2n}}{1-\beta_r{}^2}}}\right)\right) & \text{if } \beta_r\neq 1 \end{cases} \tag{3.6.28}$$

On the other hand, $A^*_{t\geq T_a}(s)$ can be expressed as

$$A^*_{t\geq T_a}(s)=\frac{1}{s}\left[1-\left(f^*_{cc,T_1}(s)\left(1-g^*_{r,1}(s)\right)+\bar{F}_{T_1}(T_a)e^{-sT_a}\left(1-g^*_{p,1}(s)\right)\right)-\sum_{n=1}^{\infty}\left(h^*_{m,\Theta_k}(s)f^*_{cc,T_{n+1}}(s)\left(1-g^*_{r,1}(\beta_r^n s)\right)+\bar{F}_{T_{n+1}}(T_a)h^*_{m,\Theta_n}(s)e^{-sT_a}\left(1-g^*_{p,1}(\beta_p^n s)\right)\right)\right]$$

$$=\frac{1}{s}\left[1-\left(e^{-s\mu+\frac{1}{2}s^2\sigma^2}\Phi\left(\frac{T_a-\mu+s\sigma^2}{\sigma}\right)\left(1-e^{-s\mu_r+\frac{1}{2}s^2\sigma_r^2}\right)+\left(1-\Phi\left(\frac{T_a-\mu}{\sigma}\right)\right)e^{-sT_a}\left(1-e^{-s\mu_p+\frac{1}{2}s^2\sigma_p^2}\right)\right)-\sum_{n=1}^{\infty}\left(h^*_{m,\Theta_n}(s)\left(e^{-s\alpha^n\mu+\frac{1}{2}s^2\left(\alpha^n\sigma\right)^2}\Phi\left(\frac{T_a-\alpha^n\mu+s\left(\alpha^n\sigma\right)^2}{\alpha^n\sigma}\right)\right)\left(1-e^{-s\beta_r^n\mu_r+\frac{1}{2}s^2\left(\beta_r^n\sigma_r\right)^2}\right)+\left(1-\Phi\left(\frac{T_a-\alpha^n\mu}{\alpha^n\sigma}\right)\right)h^*_{m,\Theta_n}(s)e^{-sT_a}\left(1-e^{-s\beta_p^n\mu_p+\frac{1}{2}s^2\left(\beta_p^n\sigma_p\right)^2}\right)\right)\right]$$

Therefore

$$
\begin{aligned}
A^*_{t\geq T_a}(s) = {} & \frac{1}{s} - \frac{1}{s} e^{-s\mu + \frac{1}{2}s^2\sigma^2} \Phi\left(\frac{T_a - \mu + s\sigma^2}{\sigma}\right)\left(1 - e^{-s\mu_r + \frac{1}{2}s^2\sigma_r^2}\right) \\
& - \frac{1}{s}\left(1 - \Phi\left(\frac{T_a - \mu}{\sigma}\right)\right) e^{-sT_a}\left(1 - e^{-s\mu_p + \frac{1}{2}s^2\sigma_p^2}\right) \\
& - \frac{1}{s}\sum_{n=1}^{\infty} h^*_{m,\Theta_n}(s) \left(\begin{array}{l}
e^{-s\alpha^n\mu + \frac{1}{2}s^2(\alpha^n\sigma)^2} \Phi\left(\frac{T_a - \alpha^n\mu + s(\alpha^n\sigma)^2}{\alpha^n\sigma}\right) \\
\left(1 - e^{-s\beta_r^n\mu_r + \frac{1}{2}s^2(\beta_r^n\sigma_r)^2}\right) \\
\quad + \left(1 - \Phi\left(\frac{T_a - \alpha^n\mu}{\alpha^n\sigma}\right)\right) \\
\quad e^{-sT_a}\left(1 - e^{-s\beta_p^n\mu_p + \frac{1}{2}s^2(\beta_p^n\sigma_p)^2}\right)
\end{array}\right)
\end{aligned}
\tag{3.6.29}
$$

where $h^*_{m,\Theta_n}(s)$ is expressed in (3.6.17).

The term $\frac{1}{s}\left(1 - \Phi\left(\frac{T_a - \mu}{\sigma}\right)\right) e^{-sT_a}\left(1 - e^{-s\mu_p + \frac{1}{2}s^2\sigma_p^2}\right)$ in (3.6.29) can be inverted exactly:

$$
\begin{aligned}
& \mathcal{L}^{-1}\left\{\frac{1}{s}\left(1 - \Phi\left(\frac{T_a - \mu}{\sigma}\right)\right) e^{-sT_a}\left(1 - e^{-s\mu_p + \frac{1}{2}s^2\sigma_p^2}\right)\right\} \\
& = \left(1 - \Phi\left(\frac{T_a - \mu}{\sigma}\right)\right)\bar{G}_p(t - T_a) \\
& = \left(1 - \Phi\left(\frac{T_a - \mu}{\sigma}\right)\right)\left(1 - \Phi\left(\frac{t - T_a - \mu_p}{\sigma_p}\right)\right)
\end{aligned}
\tag{3.6.30}
$$

The other terms are too complex to be inverted exactly and therefore the numerical inversion is applied.

3.7 Quasi-Renewal Model under Normal Operating Intervals and Exponential Repair and PM Intervals

The operating times are normally distributed the same way we have done in Section 3.6. The repair and PM intervals, however, are exponentially distributed the same way we have done in Section 3.3. Therefore,

$$h^*_{r,V_k}(s) = f^*_{c,T_k}(s) g^*_{r,k}(s)$$

$$= \left(\frac{e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}}{\Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)} \Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right) \right) \left(\frac{\rho}{\beta_r^{k-1}s + \rho}\right)$$

And

$$h^*_{p,V_k}(s) = e^{-sT_a} g^*_{p,k}(s) = e^{-sT_a}\left(\frac{\gamma}{\beta_p^{k-1}s + \gamma}\right)$$

Therefore $h^*_{m,V_k}(s)$ is written as

$$h^*_{m,V_k}(s) = F_{T_k}(T_a) h^*_{r,V_k}(s) + \bar{F}_{T_k}(T_a) h^*_{p,V_k}(s)$$

$$= \Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right) \left(\frac{e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}}{\Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)} \Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right) \right) \left(\frac{\rho}{\beta_r^{k-1}s + \rho}\right)$$

$$+ \left(1 - \Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)\right) e^{-sT_a}\left(\frac{\gamma}{\beta_p^{k-1}s + \gamma}\right)$$

Hence,

$$\begin{aligned} h^*_{m,V_k}(s) = &\left(e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2} \Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right) \right) \left(\frac{\rho}{\beta_r^{k-1}s + \rho}\right) \\ &+ \left(1 - \Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)\right) e^{-sT_a}\left(\frac{\gamma}{\beta_p^{k-1}s + \gamma}\right) \end{aligned} \tag{3.7.1}$$

Therefore,

$$\begin{aligned} h^*_{m,\Theta_n}(s) &= \prod_{k=1}^{n} h^*_{m,V_k}(s) \\ &= \prod_{k=1}^{n} \left(\begin{aligned} &\left(e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2} \Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right) \right) \left(\frac{\rho}{\beta_r^{k-1}s + \rho}\right) \\ &+ \left(1 - \Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)\right) e^{-sT_a}\left(\frac{\gamma}{\beta_p^{k-1}s + \gamma}\right) \end{aligned} \right) \end{aligned} \tag{3.7.2}$$

Likewise, the term $h^*_{r,U_n}(s)$ is the Laplace transform of the renewal density based on U_n, where every cycle is a repair cycle. $h^*_{r,U_n}(s)$ can be expressed as

$$\begin{aligned} h^*_{r,U_n}(s) &= \prod_{k=1}^{n} f^*_{T_k}(s) g^*_{r,k}(s) \\ &= \prod_{k=1}^{n} \left(e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2} \right) \left(\frac{\rho}{\beta_r^{k-1}s + \rho} \right) \\ &= e^{\left(-s\mu\frac{1-\alpha^n}{1-\alpha} + \frac{1}{2}s^2\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}\right)} \prod_{k=1}^{n} \frac{\rho}{\beta_r^{k-1}s + \rho} \end{aligned} \tag{3.7.3}$$

3.7.1 Quasi-renewal Function

From (3.2.31) and (3.2.32), the Laplace transform for the quasi-renewal function can be expressed as

$$Q^*_{H_m,t<T_a}(s) = \frac{1}{s} \sum_{n=1}^{\infty} \left[e^{\left(-s\mu\frac{1-\alpha^n}{1-\alpha} + \frac{1}{2}s^2\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}\right)} \prod_{k=1}^{n} \frac{\rho}{\beta_r^{k-1}s + \rho} \right] \tag{3.7.4}$$

and

$$Q^*_{H_m,t\geq T_a}(s) = \frac{1}{s} \sum_{n=1}^{\infty} \prod_{k=1}^{n} \left(\begin{array}{l} e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2} \Phi\left(\dfrac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma} \right) \\ \left(\dfrac{\rho}{\beta_r^{k-1}s + \rho} \right) \\ + \left(1 - \Phi\left(\dfrac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma} \right) \right) e^{-sT_a} \left(\dfrac{\gamma}{\beta_p^{k-1}s + \gamma} \right) \end{array} \right) \tag{3.7.5}$$

Evaluating the inverse Laplace transform of $Q^*_{H_m,t<T_a}(s)$ or $Q^*_{H_m,t\geq T_a}(s)$ is too difficult to do algebraically. Thus evaluating this quantity requires numerical inversion.

3.7.2 Uptime-based Point Availability Function

For $0 \leq t < T_a$:

$$\begin{aligned} A^*_{t<T_a}(s) &= \frac{1}{s} \left[\left(1 - f^*_{T_1}(s)\right) + \sum_{n=1}^{\infty} \left[\left(1 - f^*_{T_1}(\alpha^n s)\right) \left(\prod_{i=1}^{n} \left(f^*_{T_1}(\alpha^{i-1}s) g^*_{r,1}(\beta_r^{i-1}s) \right) \right) \right] \right] \\ &= \frac{1}{s}\left(1 - e^{-s\mu + \frac{1}{2}s^2\sigma^2}\right) + \sum_{n=1}^{\infty} \left[\begin{array}{l} \dfrac{1}{s}\left(1 - e^{-s\alpha^n\mu + \frac{1}{2}s^2\left(\alpha^n\sigma\right)^2}\right) \\ \left(\prod_{i=1}^{n} \left(e^{-s\alpha^{i-1}\mu + \frac{1}{2}s^2\left(\alpha^{i-1}\sigma\right)^2} \right) \left(\dfrac{\rho}{\beta_r^{i-1}s + \rho} \right) \right) \end{array} \right] \end{aligned}$$

Therefore,

$$A^*_{t<T_a}(s) = \frac{1}{s}\left(1 - e^{-s\mu + \frac{1}{2}s^2\sigma^2}\right) + \sum_{n=1}^{\infty}\left[\begin{array}{l}\frac{1}{s}\left(1 - e^{-s\alpha^n\mu + \frac{1}{2}s^2\left(\alpha^n\sigma\right)^2}\right) \\ \left(e^{-s\mu\frac{1-\alpha^n}{1-\alpha} + \frac{1}{2}s^2\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}}\prod_{i=1}^{n}\left(\frac{\rho}{\beta_r^{i-1}s + \rho}\right)\right)\end{array}\right] \tag{3.7.6}$$

The term $\frac{1}{s}\left(1 - e^{-s\mu + \frac{1}{2}s^2\sigma^2}\right)$ in (3.7.6) can be inverted exactly as

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\left(1 - e^{-s\mu + \frac{1}{2}s^2\sigma^2}\right)\right\} = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right)$$

The inverse Laplace transform of other terms in (3.7.6) are too complicated to be computed exactly. Therefore, the numerical inversion is used to obtain the approximation.

For $t \geq T_a$:

$$A^*_{t\geq T_a}(s) = \sum_{n=1}^{\infty}\left[\bar{F}^*_{cc,T_{n+1}}(s)h^*_{m,\Theta_n}(s)\right]$$

$$= \sum_{n=1}^{\infty}\left[\begin{array}{l}\left(\frac{1-e^{-sT_a}}{s} + \frac{1}{s}\left(\begin{array}{l}e^{-sT_a}\Phi\left(\frac{T_a - \alpha^n\mu}{\alpha^n\sigma}\right) \\ -e^{-s\alpha^n\mu + \frac{1}{2}s^2\left(\alpha^n\sigma\right)^2}\Phi\left(\frac{T_a - \alpha^n\mu + s\left(\alpha^n\sigma\right)^2}{\alpha^n\sigma}\right)\end{array}\right)\right) \\ \prod_{k=1}^{n}\left(\begin{array}{l}\left(e^{-s\alpha^{k-1}\mu + \frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}\Phi\left(\frac{T_a - \alpha^{k-1}\mu + s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right)\right)\left(\frac{\rho}{\beta_r^{k-1}s + \rho}\right) \\ +\left(1 - \Phi\left(\frac{T_a - \alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right)\right)e^{-sT_a}\left(\frac{\gamma}{\beta_p^{k-1}s + \gamma}\right)\end{array}\right)\end{array}\right] \tag{3.7.7}$$

However the inverse Laplace transform of each term in (3.7.7) is too complicated to be computed exactly. Therefore, the numerical inversion is used instead.

3.7.3 Down-based Point Availability Function

For $t < T_a$:

$$A^*_{t<T_a}(s) = \frac{1}{s}\left[1 - \sum_{n=1}^{\infty}\left[\left(\frac{1}{g^*_{r,1}(\beta_r^{n-1}s)} - 1\right)\left(\prod_{i=1}^{n}\left(f^*_{T_1}(\alpha^{i-1}s)g^*_{r,1}(\beta_r^{i-1}s)\right)\right)\right]\right]$$
$$= \frac{1}{s}\left[1 - \sum_{n=1}^{\infty}\left[\left(\frac{\beta_r^{n-1}s + \rho}{\rho} - 1\right)\left(\prod_{i=1}^{n}\left(e^{-s\alpha^{i-1}\mu + \frac{1}{2}s^2\left(\alpha^{i-1}\sigma\right)^2}\right)\left(\frac{\rho}{\beta_r^{i-1}s + \rho}\right)\right)\right]\right]$$

Therefore,

$$A^*_{t<T_a}(s) = \frac{1}{s} - \sum_{n=1}^{\infty}\left[\beta_r^{n-1}\rho^{n-1}e^{-s\mu\frac{1-\alpha^n}{1-\alpha} + \frac{1}{2}s^2\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}}\prod_{i=1}^{n}\left(\frac{1}{\beta_r^{i-1}s + \rho}\right)\right] \tag{3.7.8}$$

For $t \geq T_a$:

$$A^*_{t\geq T_a}(s) = \frac{1}{s}\left[\begin{array}{l} 1 - \left(f^*_{cc,T_1}(s)\left(1 - g^*_{r,1}(s)\right) + \bar{F}_{T_1}(T_a)e^{-sT_a}\left(1 - g^*_{p,1}(s)\right)\right) \\ - \sum_{n=1}^{\infty}\left(\begin{array}{l} h^*_{m,\Theta_k}(s)f^*_{cc,T_{n+1}}(s)\left(1 - g^*_{r,1}(\beta_r^n s)\right) \\ + \bar{F}_{T_{n+1}}(T_a)h^*_{m,\Theta_n}(s)e^{-sT_a}\left(1 - g^*_{p,1}(\beta_p^n s)\right) \end{array}\right) \end{array}\right]$$

$$= \frac{1}{s}\left[\begin{array}{l} 1 - \left(e^{-s\mu + \frac{1}{2}s^2\sigma^2}\Phi\left(\frac{T_a - \mu + s\sigma^2}{\sigma}\right)\left(1 - \frac{\rho}{s+\rho}\right) + \left(1 - \Phi\left(\frac{T_a - \mu}{\sigma}\right)\right)e^{-sT_a}\left(1 - \frac{\gamma}{s+\gamma}\right)\right) \\ - \sum_{n=1}^{\infty}\left(\begin{array}{l} h^*_{m,\Theta_k}(s)\left(e^{-s\alpha^n\mu + \frac{1}{2}s^2\left(\alpha^n\sigma\right)^2}\Phi\left(\frac{T_a - \alpha^n\mu + s\left(\alpha^n\sigma\right)^2}{\alpha^n\sigma}\right)\right)\left(1 - \frac{\rho}{\beta_r^n s + \rho}\right) \\ + \left(1 - \Phi\left(\frac{T_a - \alpha^n\mu}{\alpha^n\sigma}\right)\right)h^*_{m,\Theta_n}(s)e^{-sT_a}\left(1 - \frac{\gamma}{\beta_p^n s + \gamma}\right) \end{array}\right) \end{array}\right]$$

Therefore,

$$A^*_{t\geq T_a}(s)=\frac{1}{s}\left[\begin{array}{l}1-\left(\begin{array}{l}e^{-s\mu+\frac{1}{2}s^2\sigma^2}\Phi\left(\frac{T_a-\mu+s\sigma^2}{\sigma}\right)\left(\frac{s}{s+\rho}\right)\\ \quad+\left(1-\Phi\left(\frac{T_a-\mu}{\sigma}\right)\right)e^{-sT_a}\left(\frac{s}{s+\gamma}\right)\end{array}\right)\\ \quad-\sum_{n=1}^{\infty}h^*_{m,\Theta_n}(s)\left(\begin{array}{l}\left(e^{-s\alpha^n\mu+\frac{1}{2}s^2(\alpha^n\sigma)^2}\Phi\left(\frac{T_a-\alpha^n\mu+s(\alpha^n\sigma)^2}{\alpha^n\sigma}\right)\right)\\ \left(\frac{\beta_r^n s}{\beta_r^n s+\rho}\right)\\ \quad+\left(1-\Phi\left(\frac{T_a-\alpha^n\mu}{\alpha^n\sigma}\right)\right)e^{-sT_a}\left(\frac{\beta_p^n s}{\beta_p^n s+\gamma}\right)\end{array}\right)\end{array}\right] \tag{3.7.9}$$

where $h^*_{m,\Theta_k}(s)$ is expressed in (3.7.2).

However the Laplace transform terms in both (3.7.8) and (3.7.9) are too complicated to be inverted exactly. Therefore, the numerical inversion is required.

Chapter 4 Analysis and Discussion

In the previous chapter we have developed the quasi-renewal function as well as the uptime-based and downtime-based availability functions. We use these functions to analyze the behavior of the quasi-renewal model where the first operating time period and repair/PM time periods are distributed according to various distributions. Specifically, our main attention is given to the analysis of the impact of the model parameters on the model behavior. The parameters include the deterioration rate for the operating time quasi-renewal process (α), the deceleration rate for the repair and PM time quasi-renewal process (β_r and β_p respectively), the policy age (T_a), and other parameters (depending on the distribution) that have effect on the mean of the operating times, the repair times, and the PM times. Moreover, the impact of computational limitations will also be discussed.

4.1 Computation and Limitations

Evaluating the quasi-renewal function and the two point availability functions requires inverting the infinite sum of the Laplace transform terms. While some Laplace transforms can be easily inverted algebraically, most of the terms in the expressions are too intricate and thus a numerical inversion method is required. However, the result from a numerical method will inevitably involve numerical error due to the limitations posed by the numerical inversion algorithm as well as the machine precision. Therefore, to minimize the numerical error, we find the inverse algebraically when possible and numerically otherwise.

All the computations were performed using *Mathematica* version 5.1 running on Windows XP platform. In fact, Mathematica provides a built-in function for finding the inverse Laplace transform called *inverseLaplaceTransform*, which is capable of finding the inverse symbolically as well as numerically. However, for a complicated Laplace transform expression, this routine is not time-efficient and is very impractical. Therefore, a faster numerical inversion routine is needed to obtain an approximation solution.

We chose a numerical Laplace transform inversion routine developed by Valkó and Abate [62] in our computation. The routine utilizes the Stehfest's method of numerical Laplace transform inversion [63] based on Gaver functionals [64] using the Wynn rho algorithm [65] as a convergence accelerator. Their routine is called *GWR*, which stands for Gaver-Wynn-Rho. GWR routine is written in Mathematica with multi-precision computation where the level of precision is automatically determined by the algorithm. The details of their algorithm are described in [66] and [67]. The source code for GWR routine is provided in Appendix A. 1. For a large class of Laplace transforms, the estimated error from this routine is given by

$$\left|\frac{f(t)-f(t,M)}{f(t)}\right| \approx 10^{-0.8M} \qquad (4.1.1)$$

where M is the number of terms of Gaver functional, $f(t)$ is the time function, and $f(t, M)$ is the estimation of $f(t)$ using M terms. The default value of M = 32 is used throughout the calculation.

Both the quasi-renewal function and the availability functions in the Laplace transform space contain an infinite sum, and thus a truncation is needed to obtain the approximation. The impact of the truncation of the sum will be discussed in the following sections. Some examples of the Mathematica source code used for obtaining the numerical results are presented in Appendices A. 3-0.

Every numerical computation involves some degree of numerical round-off errors due to the finite discrete state of computer physical memory. In many cases, these errors are minimal and hardly noticeable. But in some cases, these errors may be augmented and significantly change the result. This is another source of error that we may encounter in the computation.

4.2 Impact of Truncating the Quasi-renewal Function

The quasi-renewal function $Q_{H_m}(t)$ indicates the expected number of restarts by time t. From (3.2.19), the quasi-renewal is defined by

$$Q_{H_m}(t) = \sum_{n=1}^{\infty} H_{m,\Theta_n}(t)$$

where $H_{m,\Theta_n}(t)$ is the distribution function on the length of a series of n intervals. When the summation is truncated to c terms, the quasi-renewal is approximated by

$$\tilde{Q}_{H_m}(t) = \sum_{n=1}^{c} H_{m,\Theta_n}(t) \qquad (4.2.1)$$

Since each $H_{m,\Theta_n}(t)$ term corresponds to a cycle, $\tilde{Q}_{H_m}(t)$ should converge to c as t increases.

4.3 Impact of Truncating the Availability Function

The uptime-based point availability function we have developed in Section 3.2.2 is given in (3.2.33), which is

$$A_{uptime}(t)=\begin{cases}\bar{F}_{T_1}(t)+\sum_{n=1}^{\infty}\int_0^t h_{r,U_n}(u)\bar{F}_{T_{n+1}}(t-u)du & t<T_a\\ \sum_{n=1}^{\infty}\int_{t-T_a}^t h_{m,\Theta_n}(u)\bar{F}_{T_{n+1}}(t-u)du & t\geq T_a\end{cases}.$$

To evaluate this function, the infinite sum needs to be truncated to obtain an approximation. We define the approximation of the availability function using c terms as

$$\tilde{A}_{uptime}(t)=\begin{cases}\bar{F}_{T_1}(t)+\sum_{n=1}^{c-1}\int_0^t h_{r,U_n}(u)\bar{F}_{T_{n+1}}(t-u)du & t<T_a\\ \sum_{n=1}^{c-1}\int_{t-T_a}^t h_{m,\Theta_n}(u)\bar{F}_{T_{n+1}}(t-u)du & t\geq T_a\end{cases} \tag{4.3.1}$$

Obviously, we would like to include as many terms as practically possible in our approximation as far as the accuracy is concerned. However, in some special cases, especially when t is small, we might not need as many terms to obtain an approximation with high accuracy as one might think. To understand the impact of truncating the availability function, we need to know how much each term contributes to the value $A_{uptime}(t)$. Figure 5 displays the contribution from each term in (3.2.33) using an example from the exponential quasi-renewal model.

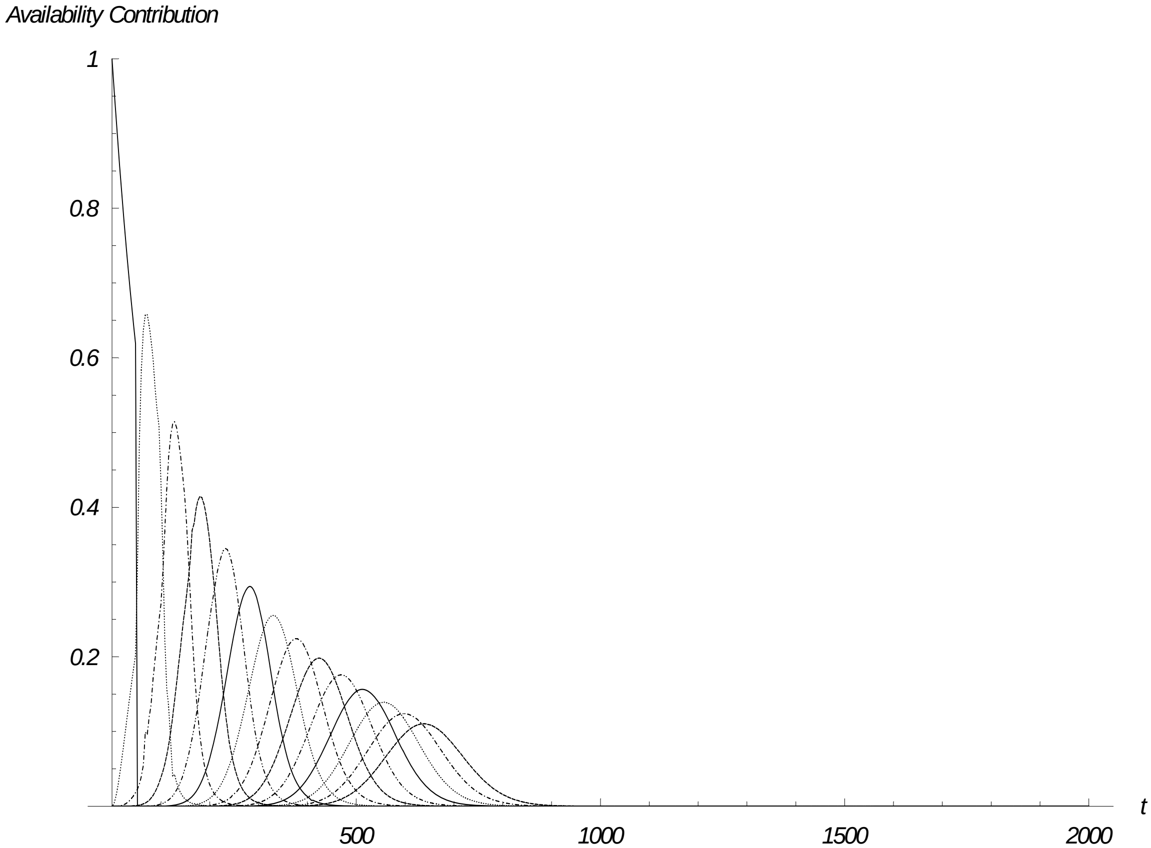

Figure 5. The contribution from each term in the uptime-based availability function using the exponential quasi-renewal model with parameters λ = 0.01, ρ = 0.05, γ = 0.1, T_a = 50, α = 0.9, β_r = 1, and β_p = 1.

Figure 5 shows that the first term $\bar{F}_{T_1}(t)$ contributes the most but its contribution ceases after $t = T_a = 50$. The subsequent terms ($\int_0^t h_{r,U_n}(u)\bar{F}_{T_{n+1}}(t-u)du$ for $t < T_a$ and $\int_{t-T_a}^t h_{m,\Theta_n}(u)\bar{F}_{T_{n+1}}(t-u)du$ for $t \geq T_a$) contribute less and less as the index n increases. However, while a term with higher index contributes less, its contribution span gets shifted further to the right and gets wider over the time domain. This means that for a small value of t, $A_{uptime}(t)$ can be evaluated with high accuracy using only the first few terms. As t gets larger, more and more terms contribute to the value of $A_{uptime}(t)$ although the contribution gets smaller as the term index gets higher. Therefore, truncating the infinite sum to c terms does not affect the accuracy of $A_{uptime}(t)$ for a sufficiently small value of t. However, the accuracy becomes greatly impaired as t gets large. Fortunately, this problem can be avoided by increasing c as t grows.

The downtime-based point availability function we have developed is defined by (3.2.46) , which is

$$A_{downtime}(t)=\begin{cases} 1-\int_0^t f_{T_1}(u)\bar{G}_{r,1}(t-u)du \\ \quad -\sum_{n=1}^{\infty}\int_0^t\int_0^u h_{r,U_n}(w)f_{T_{n+1}}(u-w)\bar{G}_{r,n+1}(t-u)dwdu & t<T_a \\ 1-F_{T_1}(T_a)\int_0^{T_a} f_{c,T_1}(u)\bar{G}_{r,1}(t-u)du-\bar{F}_{T_1}(T_a)\bar{G}_{p,1}(t-T_a) \\ \quad -\sum_{n=1}^{\infty}F_{T_{n+1}}(T_a)\int_0^t\int_{u-T_a}^u\begin{pmatrix} h_{m,\Theta_n}(w)f_{c,T_{n+1}}(u-w) \\ \bar{G}_{r,n+1}(t-u)\end{pmatrix}dwdu & t\geq T_a \\ \quad -\sum_{n=1}^{\infty}\bar{F}_{T_{n+1}}(T_a)\int_0^{t-T_a}h_{m,\Theta_n}(u)\bar{G}_{p,n+1}(t-T_a-u)du \end{cases}.$$

To evaluate this function, the infinite sum needs to be truncated to obtain an approximation. We define the approximation of the availability function using c terms as

$$\tilde{A}_{downtime}(t)=\begin{cases} 1-\int_0^t f_{T_1}(u)\bar{G}_{r,1}(t-u)du \\ \quad -\sum_{n=1}^{c-1}\int_0^t\int_0^u h_{r,U_n}(w)f_{T_{n+1}}(u-w)\bar{G}_{r,n+1}(t-u)dwdu & t<T_a \\ 1-\left(F_{T_1}(T_a)\int_0^{T_a} f_{c,T_1}(u)\bar{G}_{r,1}(t-u)du+\bar{F}_{T_1}(T_a)\bar{G}_{p,1}(t-T_a)\right) \\ \quad -\sum_{n=1}^{c-1}\left(\begin{array}{l} F_{T_{n+1}}(T_a)\int_0^t\int_{u-T_a}^u\begin{pmatrix} h_{m,\Theta_n}(w)f_{c,T_{n+1}}(u-w) \\ \bar{G}_{r,n+1}(t-u)\end{pmatrix}dwdu \\ \quad +\bar{F}_{T_{n+1}}(T_a)\int_0^{t-T_a}h_{m,\Theta_n}(u)\bar{G}_{p,n+1}(t-T_a-u)du \end{array}\right) & t\geq T_a \end{cases} \tag{4.3.2}$$

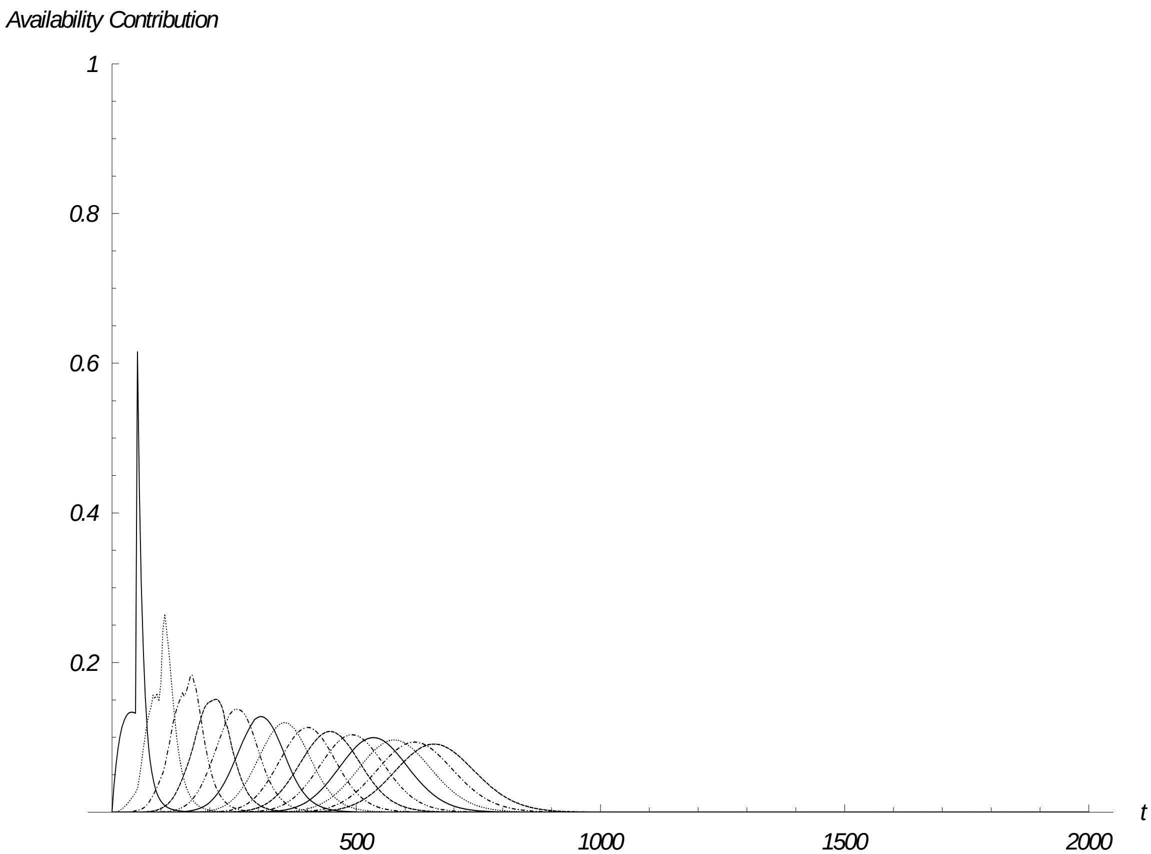

Figure 6. The contribution for each term in the downtime-based availability function using the exponential quasi-renewal model with parameters λ = 0.01, ρ = 0.05, γ = 0.1, T_a = 50, α = 0.9, β_r = 1, and β_p = 1.

The contribution from each term using an example from the exponential quasi-renewal model is plotted in Figure 6. Please note that plot displays the magnitude of the contribution from each term. These contributions are collectively subtracted from 1 to obtain the downtime-based availability. The first term ($\int_0^t f_{T_1}(u)\bar{G}_{r,1}(t-u)du$ when $t < T_a$ and $F_{T_1}(T_a)\int_0^{T_a} f_{c,T_1}(u)\bar{G}_{r,1}(t-u)du + \bar{F}_{T_1}(T_a)\bar{G}_{p,1}(t-T_a)$ when $t \geq T_a$) contributes the most. Then the subsequent terms contribute less as the index n gets higher while its contribution span widens and gets shifted to the right over the time scale. The contribution pattern is similar to the one from the uptime-based function except that the contribution from downtime-based terms is much smaller for the first term and the decline in contribution from the subsequent terms is not as steep. However, the impact of truncation is still the same. That is, for a small value of t, $A_{downtime}(t)$ can be evaluated with high accuracy using only the first few terms. But as t gets larger, more and more terms contribute to the value of $A_{downtime}(t)$. Hence the accuracy suffers greatly from the truncation for large t. One may increase the number of terms c to improve accuracy as t grows.

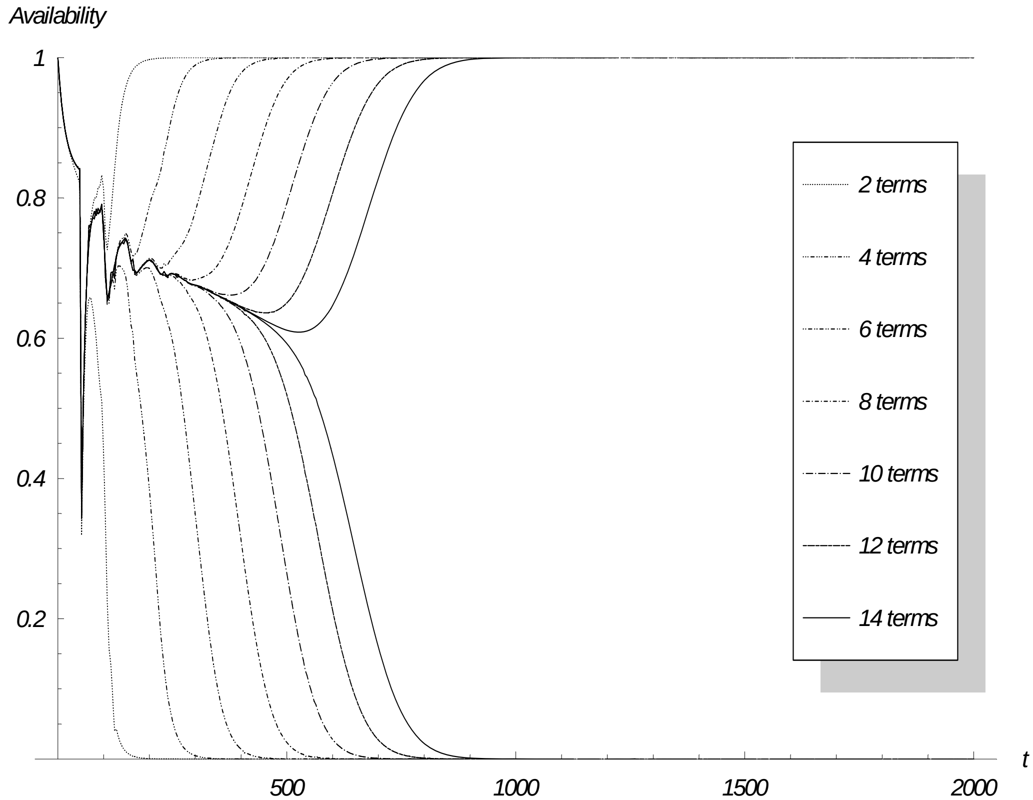

Figure 7. The uptime-based and downtime-based availability functions approximated using various number of terms in the exponential model with parameters λ = 0.01, ρ = 0.05, γ = 0.1, T_a = 50, α = 0.9, β_r = 1, and β_p = 1.

As mentioned before, the uptime-based and the downtime-based availability functions are equivalent. However, when a truncation occurs, the uptime-based availability forms a lower bound for the availability function while the downtime-based forms an upper bound. Therefore, when we plot the uptime-based and downtime-based availability functions together in the same graph (see Figure 7), we know that the availability is accurate where the bounds meet or are very close and loses its accuracy when the bounds diverge. The plot in Figure 7 indicates that the approximation with 14 terms maintain the accuracy for the longest period of time. Other approximation functions with smaller number of terms still give an accurate result when t is small enough.

4.4 Exponential Distribution Model

In our exponential quasi-renewal model with exponential repair and PM intervals, we assume the first operating interval is distributed according to $\text{Exponential}(\lambda)$ while the first repair time and the first PM time are distributed according to $\text{Exponential}(\rho)$ and $\text{Exponential}(\gamma)$ respectively. In reality, we will always use $\gamma > \rho$ to indicate that the mean PM time is less than the mean repair time. More specifically, $\rho = a\gamma$, for some $a < 1$. In our base numerical example, we set the mean operating time for the first interval to be 100 ($\lambda = \frac{1}{100}$), the mean for the first repair time to be 20 ($\rho = \frac{1}{20}$), the mean for the first PM time to be 10 ($\gamma = \frac{1}{10}$), T_a = 50, α =0.90, β_r = 1, and β_p = 1. This base example is used to compare against all other examples.

4.4.1 Quasi-renewal Function

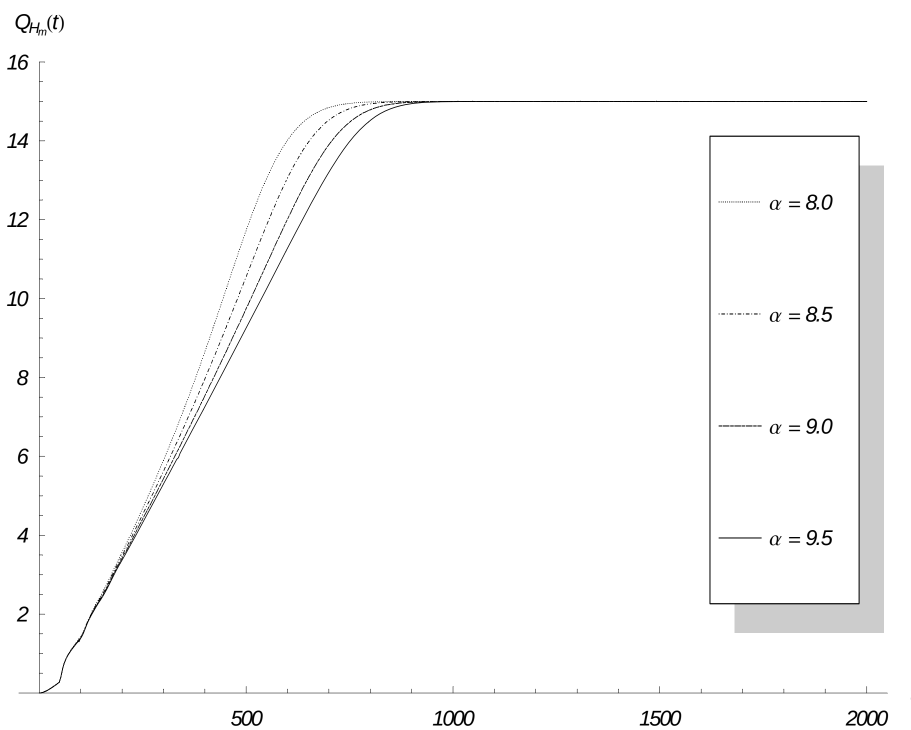

Figure 8. $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with exponential repair and PM intervals using parameters λ = 0.01, ρ = 0.05, γ = 0.1, T_a = 50, β_r = 1, β_p = 1, and α = 0.8, 0.85, 0.9, 0.95.

Figure 8 displays the quasi-renewal function plot from our exponential model with different deterioration rates. When the deterioration rate is high (α is low), the operating times are stochastically shortened quickly, resulting in more cycles occurring sooner than the model with lower deterioration rate. Therefore, the graph of $\tilde{Q}_{H_m}(t)$ with lower α value will reach the value c sooner than others with higher α values. All curves converge to 15, which is the number of terms we used in the approximation.

The value $T_a = 50$ has some effect on the quasi-renewal function around $t = 50$ and $t = 100$ but has no effect on the convergent value. In our base model, the PM process does not improve the life distribution of the system and hence it creates more cycles due to the interruption of the operating time. Consequently, we can see a trend of the plot become suddenly steeper after $t = T_a = 50$ and again after $t = 2T_a = 100$. The change at $t = 50$ is more visible than the change at $t = 100$ or at any other multiple values of T_a because a PM action is most likely to occur at $t = 50$ (which results in an upsurge in the expected number of cycles) and the possibility of PM action become lesser as t grows. This is due to the fact that, as t increases, the operating time is probabilistically shorter and the cycle is more likely to be a repair cycle.

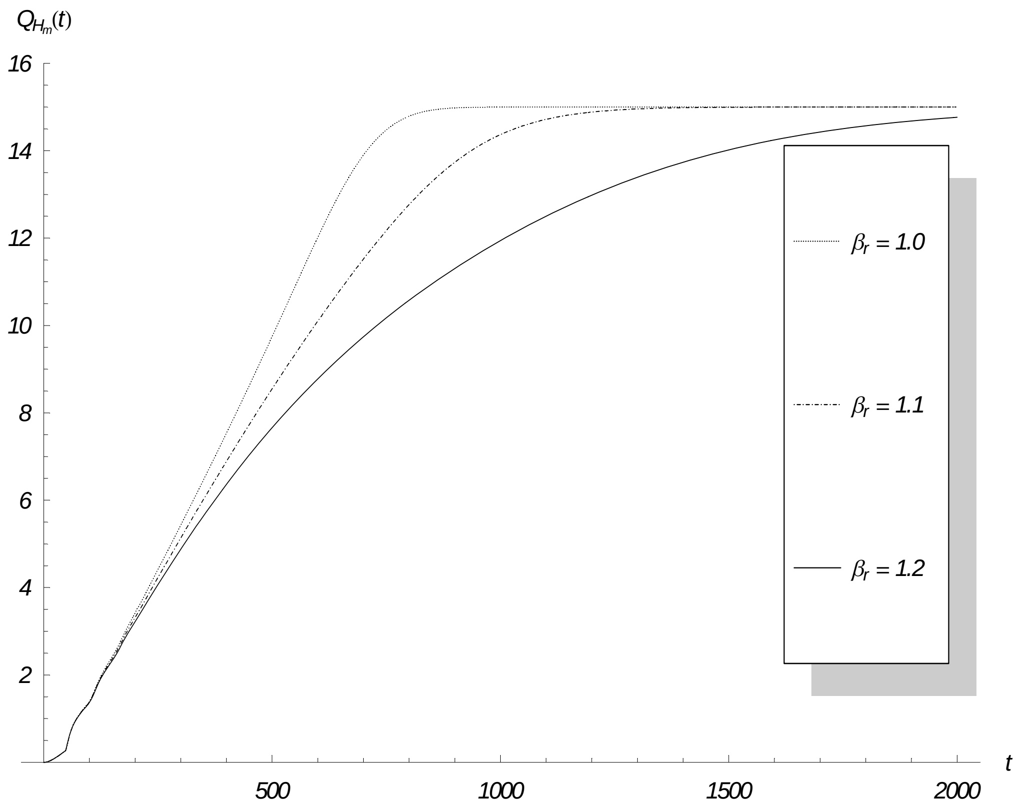

Figure 9. $\tilde{Q}_{H_m}(t)$ **truncated to 15 terms for the quasi-renewal model with exponential repair and PM intervals using parameters** λ **= 0.01,** ρ **= 0.05,** γ **= 0.1,** T_a **= 50,** α **= 0.9,** β_p **= 1, and** β_r **= 1.0, 1.1, 1.2**

Figure 9 depicts the quasi-renewal function plot for the exponential model with different deceleration rates for the repair time intervals β_r. When the deceleration rate is low (β_r is low), the repair intervals are not growing too quickly, resulting in more cycles end sooner than the model with high deceleration rate. Consequently, the graph of $\tilde{Q}_{H_m}(t)$ with lower β_r value converges to the value c sooner than others with higher β_r values. In fact, when β_r =1.2, the cycle lengths grow too long and the expected number of cycles fails to reach the value c at the end of our data collecting point.

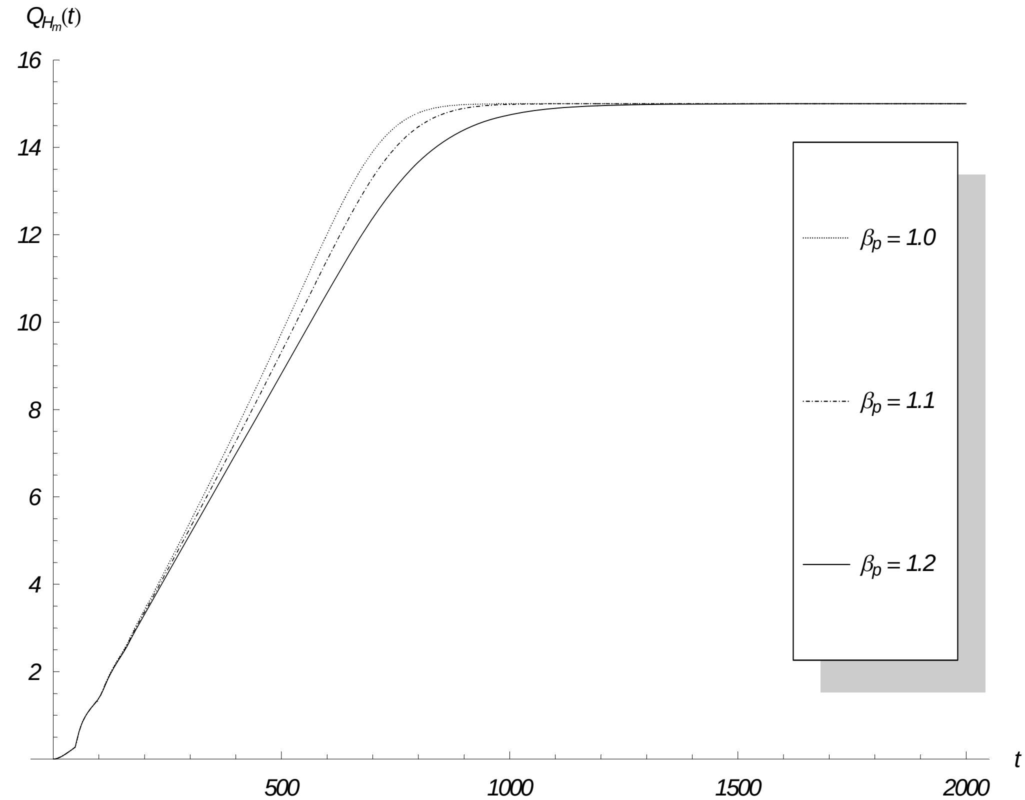

Figure 10. $\tilde{Q}_{H_m}(t)$ **truncated to 15 terms for the quasi-renewal model with exponential repair and PM intervals using parameters** λ **= 0.01,** ρ **= 0.05,** γ **= 0.1,** T_a **= 50,** α **= 0.9,** β_r **= 1, and** β_p **= 1.0, 1.1, 1.2**

Figure 10 displays the quasi-renewal function plot for the exponential model with different deceleration rates for the PM time intervals β_p. The effect of the difference values of β_p is similar to that of the difference values of β_r but less prominent. This is due to the fact that as t grows larger, a PM interval is less likely to occur and hence β_p has less the effect on the expected number of restarts than β_r.

4.4.2 Availability Function

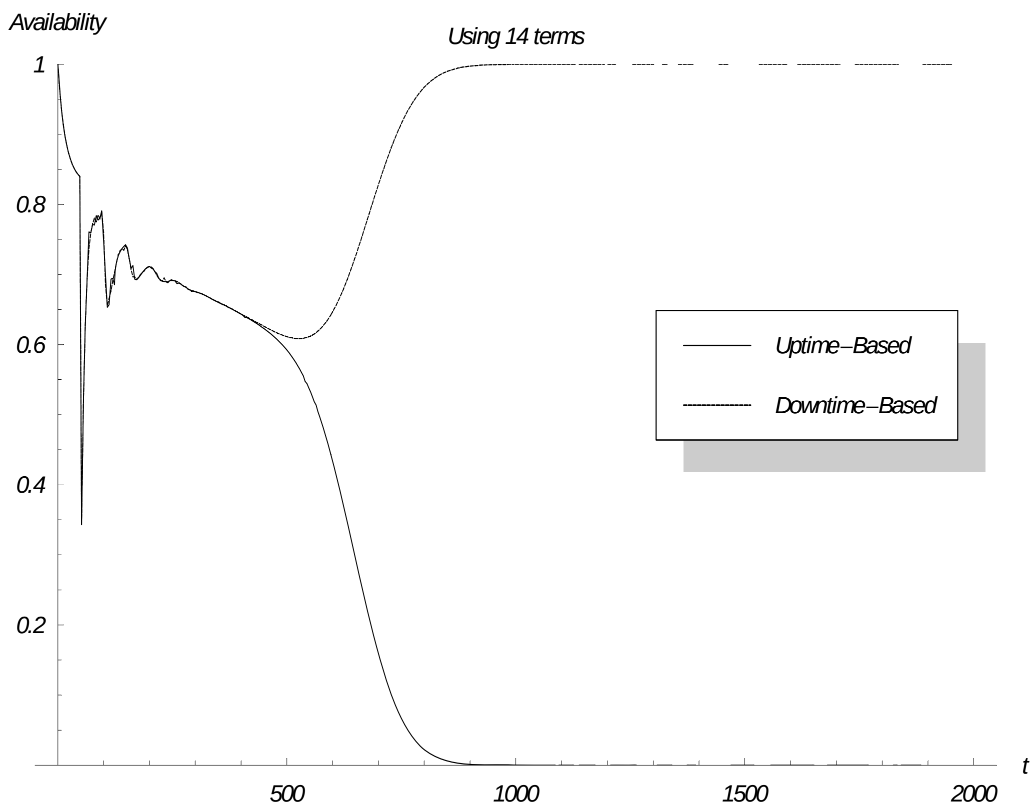

Figure 11. The uptime-based and downtime-based availability functions approximated using 14 terms for the exponential model with parameters λ = 0.01, ρ = 0.05, γ = 0.1, T_a = 50, α = 0.9, β_r = 1, and β_p = 1.

Figure 11 depicts the availability plot using 14 terms. Both uptime-based and downtime-based graphs exhibit a steep dent right after $t = T_a = 50$, indicating that the availability suddenly experiences a huge drop right after $t = T_a$ for a short while before rising back to its previous trend. The similar occurrence happens again at t equals a multiple value of T_a but the dent becomes less visible as t gets large. This phenomenon can be explained as follows.

When $t < T_a$, all cycles are repair cycles. The availability smoothly declines as the system ages. When $t = T_a$, there is a possibility that a PM cycle may occur, in which case the operating time interval has to be interrupted. This causes a sudden drop in the availability plot right after $t = T_a$. Since PM does not improve the life distribution of the system in our base model, the aging process continues normally. Hence the availability plot rises up and continues in its previous trend. The availability plot drops again when t

equals a multiple value of T_a but the drop becomes smaller as t increases. This means the operating time in each cycle is probabilistically shorter as time increases and thus the event that the operating time will exceed T_a becomes unlikely. Therefore, PM interval hardly occurs afterward.

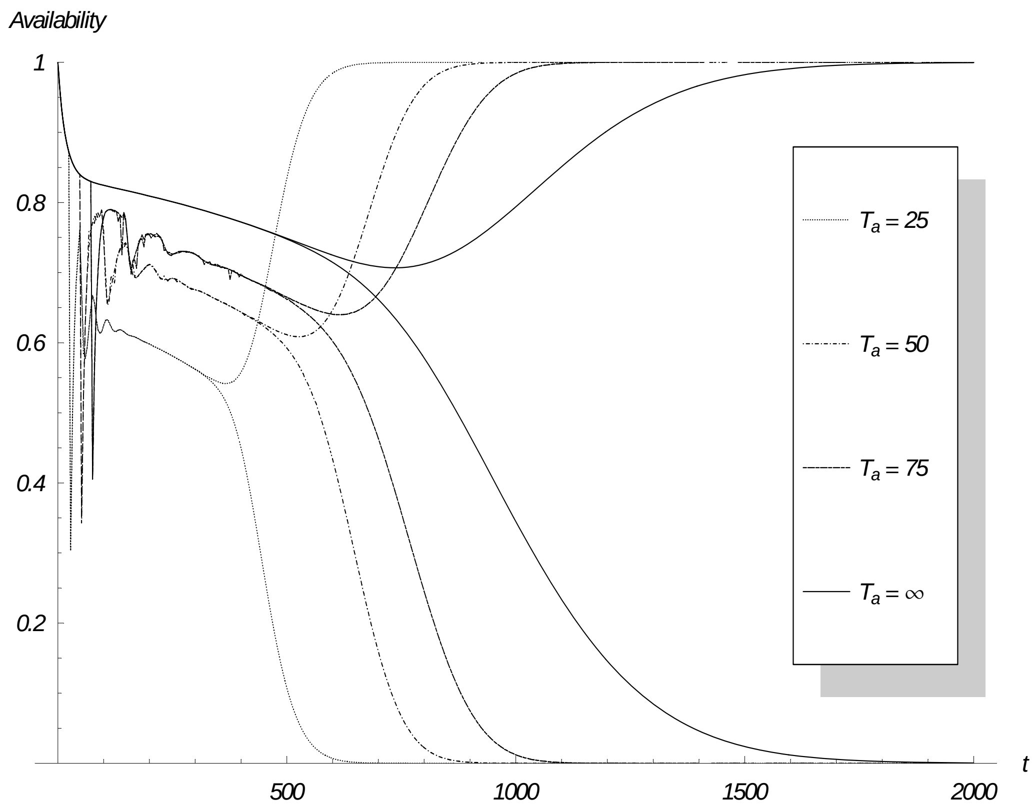

Figure 12. The uptime-based and downtime-based availability functions approximated using 14 terms for the exponential model with parameters λ = 0.01, ρ = 0.05, γ = 0.1, α = 0.9, β_r = 1, β_p = 1, and T_a = 25, 50, 75, ∞.

Figure 12 shows the plot of the uptime-based and downtime-based availability functions when the policy age T_a is set to 25, 50, 75, and ∞. When $T_a = \infty$, PM policy does not exist and therefore the availability plot matches Rehmert's [2]. For other T_a values, when $t < T_a$, all cycles that contribute to the availability functions up to this point are repair cycles, and hence the availability plot fits nicely with the plot for $T_a = \infty$. However, for each plot, there is a dent when t equals a multiple of T_a. The first dent ($t = T_a$) is the steepest one while the subsequent dents get shallower. This can be explained as follows.

When $t = T_a$, the probability that the system is still operating is high. So when it is interrupted by PM, the availability drops for a short period of time, resulting a deep dent just after $t = T_a$. The depth shortens for the subsequent dents because the probability that the PM cycle occurs quickly declines.

When T_a is low, a PM action at $t = T_a$ is more likely to occur than when T_a is high. Therefore, the dent at $t = T_a$ is deeper for lower T_a. However, the subsequent dents for a low T_a plot become less visible relatively quickly comparing with a high T_a plot. This is because for a low T_a model, the first few PM actions are more likely to occur in a short period of time and consequently this increases the number of cycles more quickly than a model with high T_a. Since each cycle stochastically shortens the life length of the unit, the probability of having a PM action declines rapidly after a few PM actions. Therefore the dents disappear sooner for the model with a lower value of T_a. Furthermore, the model with the lowest T_a loses its accuracy the soonest because all cycles occur rapidly and the effect on numerical accuracy of truncation after c terms is more pronounced.

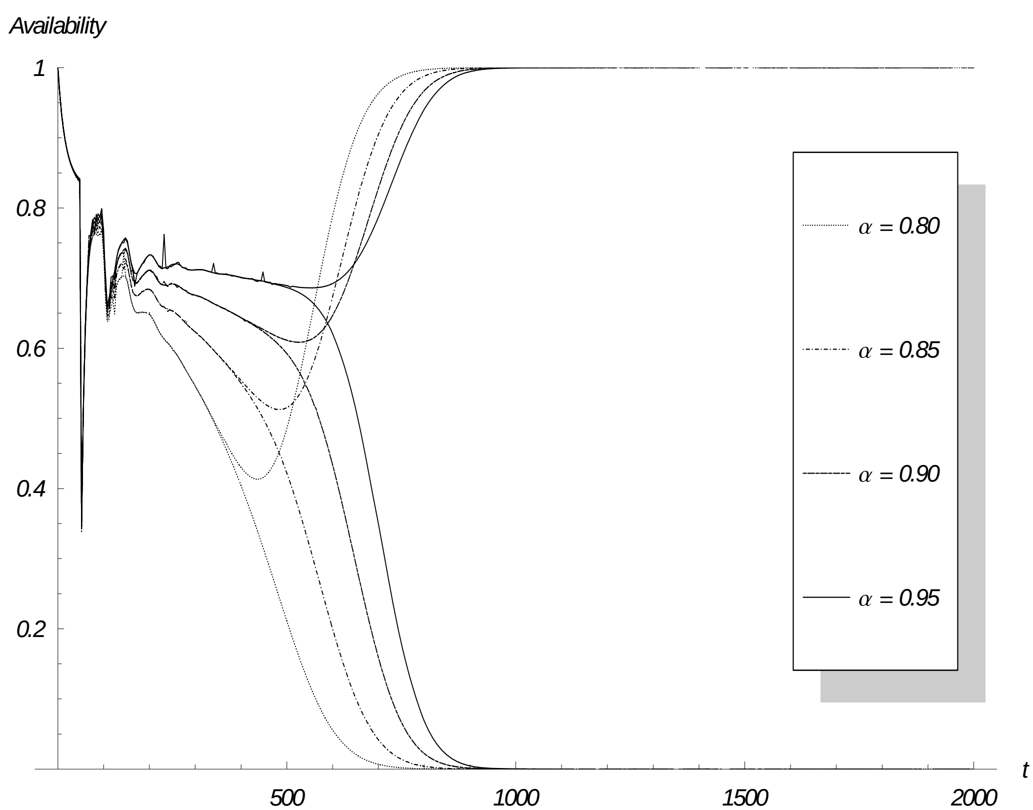

Figure 13. The plot of availability functions approximated using 14 terms for the exponential model with parameters λ = 0.01, ρ = 0.05, γ = 0.1, T_a = 50, β_r = 1, β_p = 1, and α =0.80, 0.85, 0.90, 0.95.

Figure 13 displays the plot of the availability functions for our exponential model using different deterioration rates α =0.80, 0.85, 0.90, and 0.95. When the deterioration rate is high (α is low), the life distribution of the system is probabilistically shortened at a faster rate than the model with lower deterioration rates. This creates shorter operating cycles and therefore lower availability. Moreover, the model with the highest deterioration rate also maintains the accuracy for the shortest period of time because all cycles are shorter and the effect on numerical accuracy of truncation after c terms is more prominent.

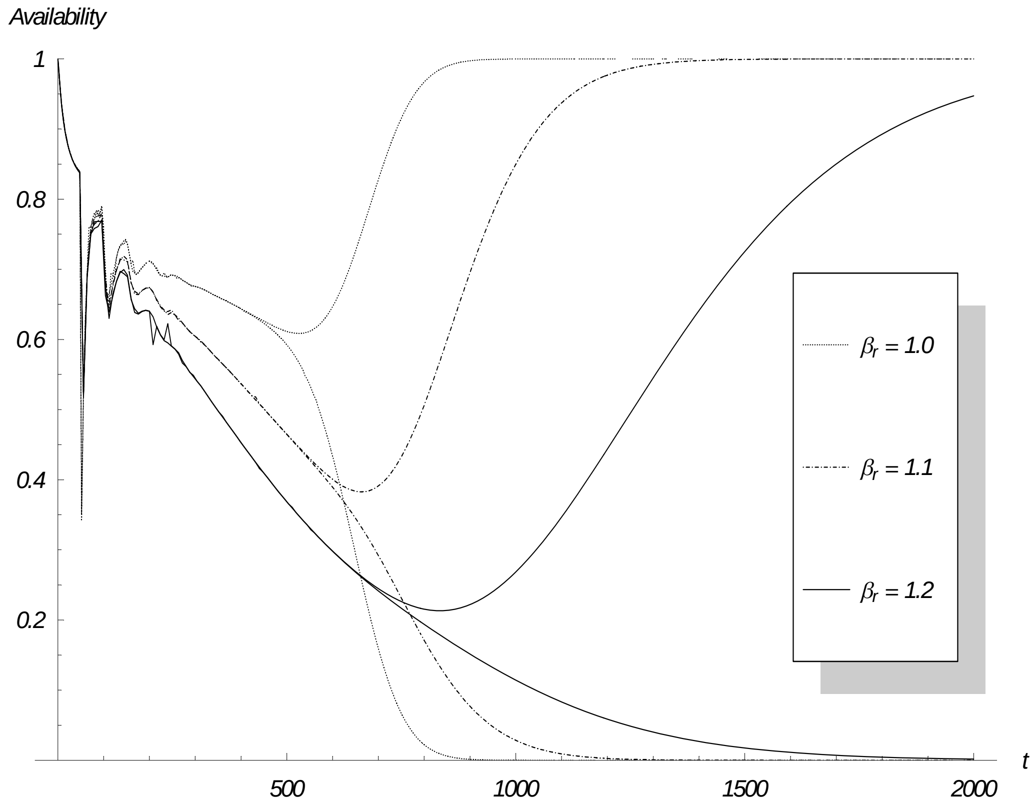

Figure 14. The plot of availability functions approximated using 14 terms for the exponential model with parameters λ = 0.01, ρ = 0.05, γ = 0.1, T_a = 50, α =0.9, β_p = 1, and β_r =1.0, 1.1 ,1.2.

Figure 15 depicts the effect of the various values of the deceleration rate on the repair time intervals β_r on the availability. When the deceleration rate is high (β_r is high), the repair intervals grow in length quickly, resulting in lower availability than other instances with lower deceleration rate. However, the accuracy of the availability for the highest β_r value stays the longest because the growing repair intervals cause the cycle lengths to grow quicker than others and hence the effect on numerical accuracy of truncation after c terms is less pronounced than other instances.

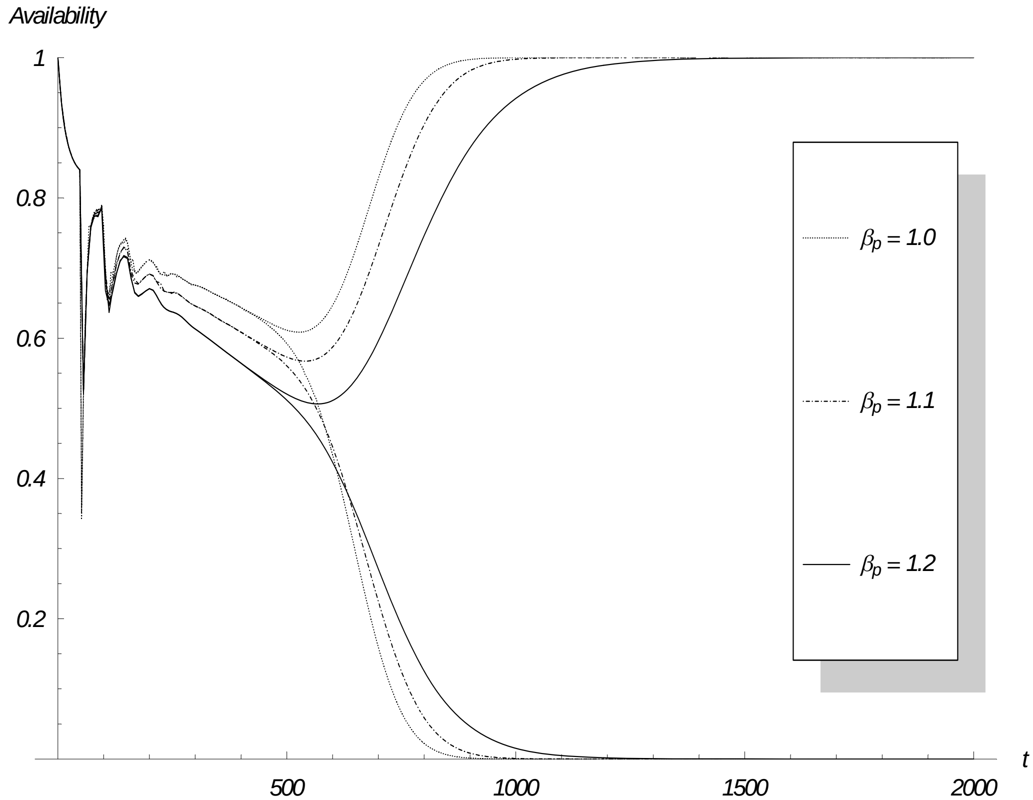

Figure 15. The plot of availability functions approximated using 14 terms for the exponential model with parameters λ = 0.01, ρ = 0.05, γ = 0.1, T_a = 50, α=0.9, β_r = 1, and β_p=1.0, 1.1 ,1.2.

Figure 15 depicts the effect of the various values of the deceleration rate on the repair time intervals β_p on the availability. The effect of values of β_p is similar to that of the values of β_r but less prominent. This confirms the fact that as t grows larger, a PM interval is less likely to occur and therefore β_p has less effect than β_r .

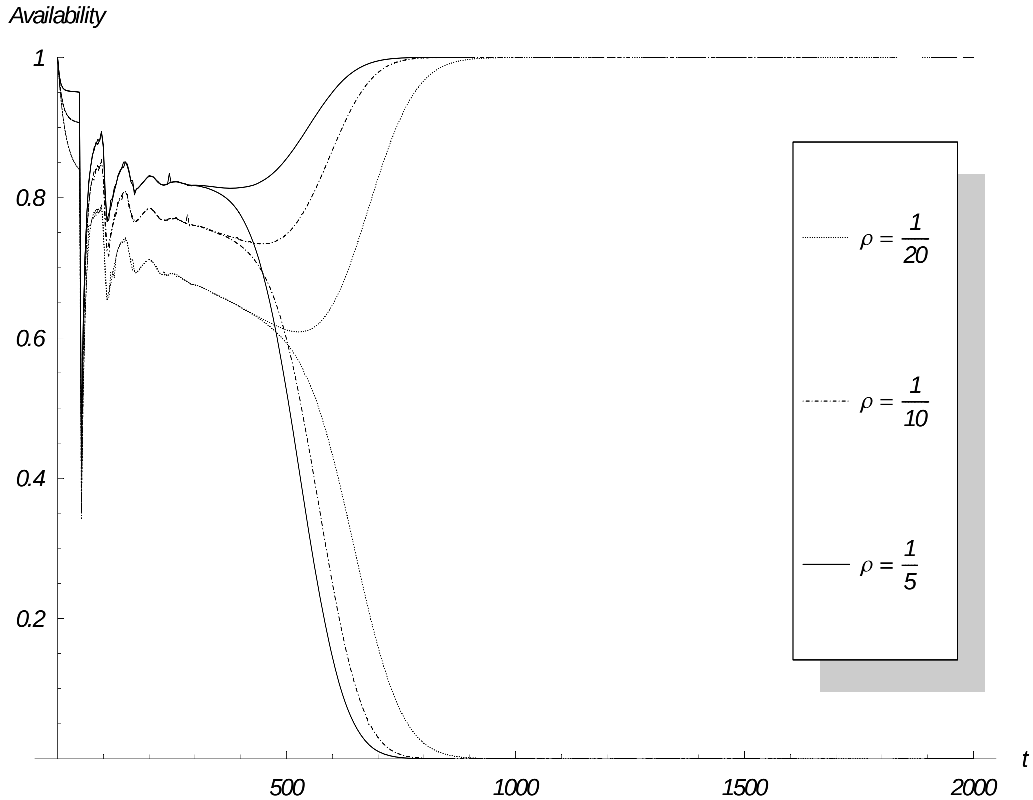

Figure 16. The plot of availability functions approximated using 14 terms for the exponential model with parameters λ = 0.01, γ = 0.1, α = 0.9, β_r = 1, β_p = 1, T_a = 50, and $\rho = \frac{1}{20}, \frac{1}{10}, \frac{1}{5}$.

Figure 16 displays the plot of the availability function for our exponential model using different mean repair times $\frac{1}{\rho}$ =20, 10, and 5. As expected, the model with the longest mean repair time yields a lowest availability plot while the shortest mean repair time yields the highest availability. However, the model with smallest mean repair time maintains accuracy for the shortest period because each cycle is shorter than the other two models, and hence the effect on numerical accuracy of truncation after c terms is the most pronounced.

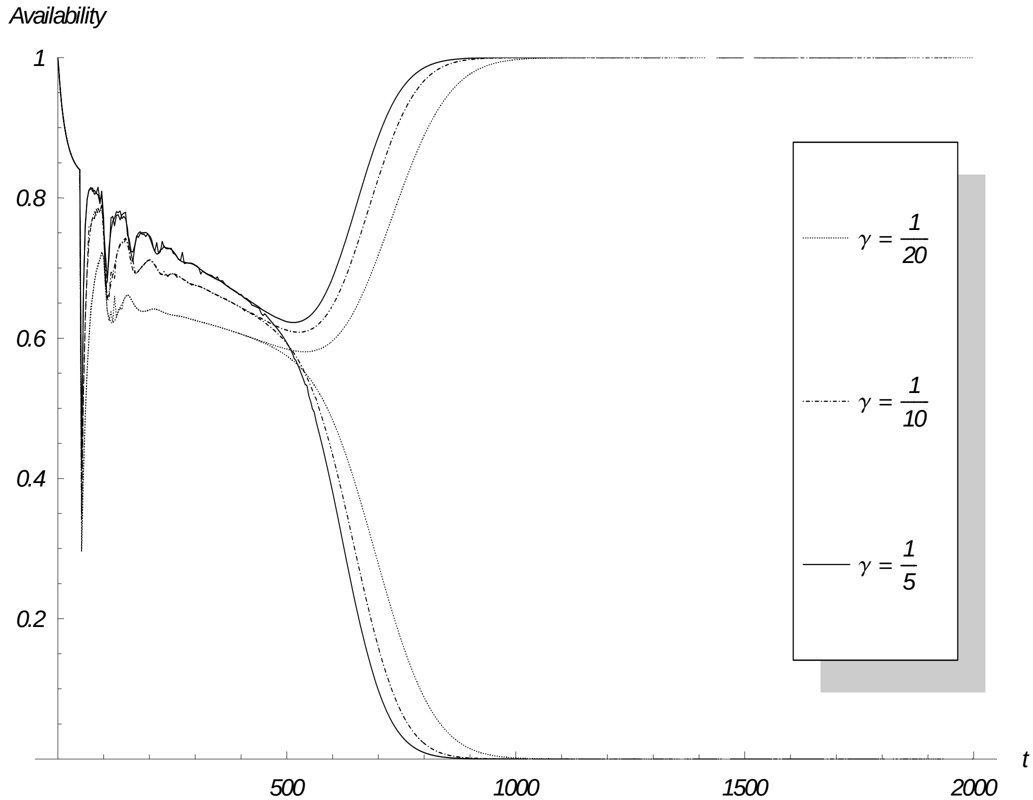

Figure 17. The plot of availability functions approximated using 14 terms for the exponential model with parameters λ = 0.01, ρ = 0.05, α = 0.9, β_r = 1, β_p = 1, T_a = 50, and $\gamma = \frac{1}{20}, \frac{1}{10}, \frac{1}{5}$.

Figure 17 displays the plot of the availability function for our base model using different mean PM times $\frac{1}{\gamma}$ =20, 10, and 5. Whenever the time equals a multiple value of T_a, a dent occurs. The width of the dents is consistent with the corresponding mean PM time as a larger mean PM time creates a wider dent. Moreover, the model with smaller mean PM time creates shallower dents and has higher overall availability than those with higher mean PM time. This is consistent with the fact that a shorter mean PM time will improve the availability.

When comparing Figure 16 and Figure 17, we can see that for a large value of t while the availability is still accurate (e.g. $t = 350$), the difference in the mean repair times creates bigger difference in the availability than that of the mean PM times. This confirms the fact that the PM cycles are less likely to occurs as t grows and therefore have lesser effect on the availability for a large value of t.

4.5 Gamma Distribution Model

4.5.1 Gamma Repair and PM Intervals

In our gamma quasi-renewal model with gamma repair and PM intervals, we assume the first operating interval is distributed according to $\text{Gamma}(a,\lambda)$ while the first repair time and the first PM time are distributed according to $\text{Gamma}(b,\rho)$ and $\text{Gamma}(c,\gamma)$ respectively. For any gamma distribution $\text{Gamma}(x,y)$ where x is the shape parameter and y is the scale parameter, the mean is given by $\frac{x}{y}$. In our base numerical example, we set the mean operating time for the first interval to be 100 ($a = 2, \lambda = \frac{2}{100}$), the mean for the first repair time to be 20 ($b = 2, \rho = \frac{2}{20}$), the mean for the first PM time to be 10 ($c = 2, \gamma = \frac{2}{10}$), T_a = 50, α =0.90, β_r = 1, and β_p = 1. The base example is used to compare against all other examples.

4.5.1.1 Quasi-renewal Function

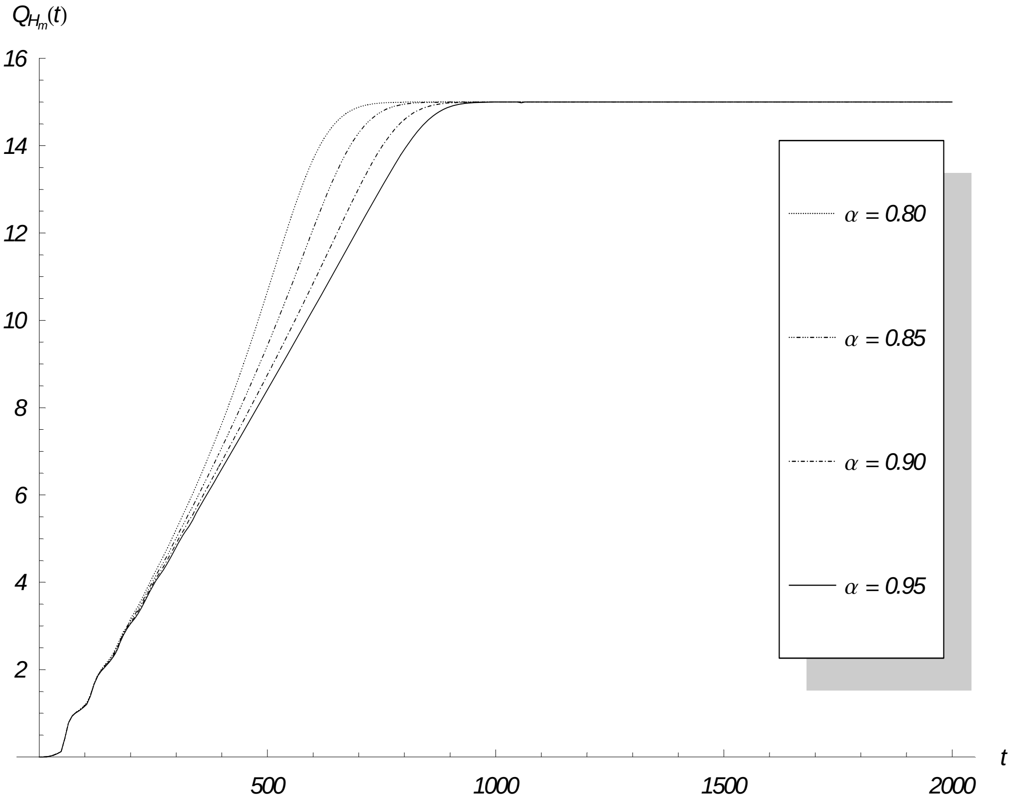

Figure 18. The plot of $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with gamma operating, repair, and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, b = 2, $\rho = \frac{2}{20}$, c = 2, $\gamma = \frac{2}{10}$, T_a = 50, β_r =1, β_p =1 , and α = 0.8, 0.85, 0.9, 0.95.

Figure 18 displays the quasi-renewal function plot from our gamma operating times and repair/PM times model with different deterioration rates α for the operating times. The effect of the difference in deterioration rates α in Figure 18 is similar to Figure 8, which shows the effect of the effect of α values on the quasi-renewal function under exponential model. That is, the graph of $\tilde{Q}_{H_m}(t)$ with lower α value will reach the value c sooner than others with higher α values because the operating times are stochastically shorten quicker. However, as t gets larger the effect of α values in the gamma model is more conspicuous than the exponential model. This is due to the difference in the two distributions. Although each operating interval from both models has the same mean, the probability for exponential distribution peaks around $t = 0$ while the probability for the gamma distribution peaks around $t > 0$. (See Figure 22.) Therefore, the probability $F_{T_k}(T_a)$ for the exponential distribution is greater than $F_{T_k}(T_a)$ for the gamma distribution, resulting in a shorter average operating interval length than that of the gamma distribution and hence $\tilde{Q}_{H_m}(t)$ reaches the value c sooner in the exponential model. The length difference is amplified as the deterioration rate α gets larger. That is why the effect of α is more noticeable in Figure 18 than in Figure 8.

In addition, the policy age $T_a = 50$ has similar effect on the quasi-renewal function around $t = 50$ and $t = 100$ as in the exponential distribution model (Figure 8), but the effect is more prominent in the gamma distribution model (Figure 18). This creates unique pattern for each model. This is because, as mentioned above, the probability $F_{T_k}(T_a)$ for the exponential distribution is greater than $F_{T_k}(T_a)$ for the gamma distribution. This results in more weight on the probability of PM cycle ($\bar{F}_{T_k}(T_a)$) in the expression $\tilde{Q}_{H_m}(t)$ under the gamma model and hence the expected number of restarts increase more rapidly, making the trend of plot more noticeably steeper after $t = 50$ and $t = 100$.

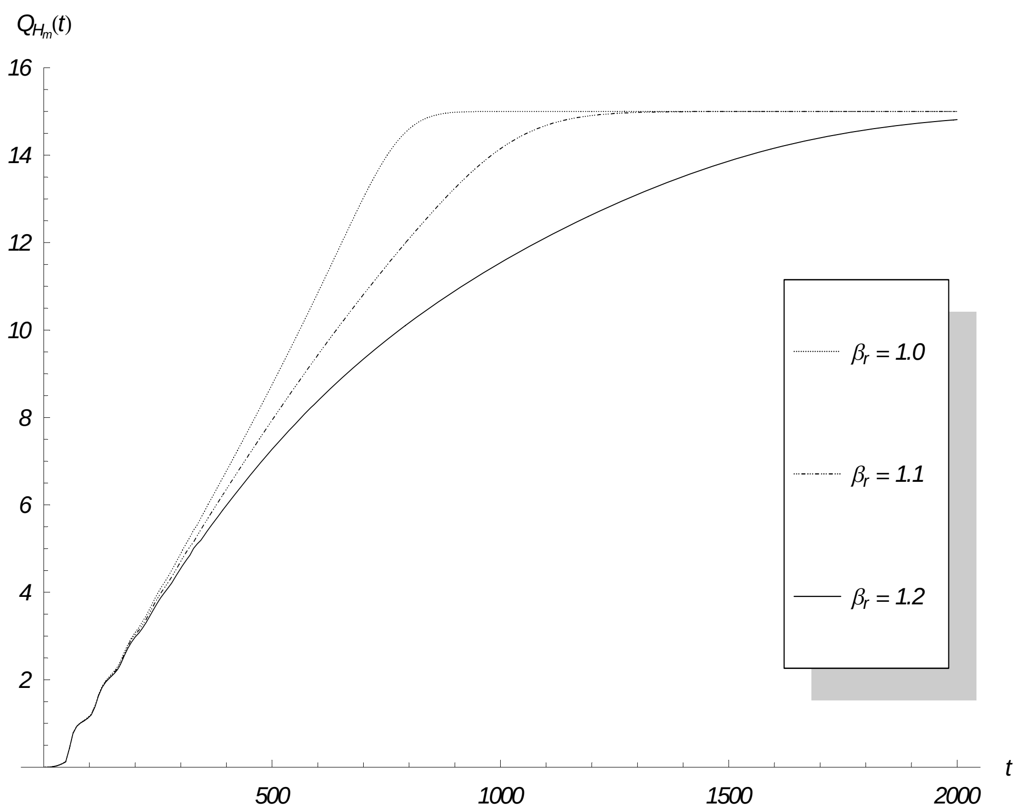

Figure 19. The plot of $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with gamma operating, repair, and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, b = 2, $\rho = \frac{2}{20}$, c = 2, $\gamma = \frac{2}{10}$, T_a = 50, α = 0.9, β_p =1 , and β_r =1.0, 1.1, 1.2.

Figure 19 displays the quasi-renewal function for the gamma model with different deceleration rates for the repair intervals β_r. The effect of different β_r values in Figure 19 is similar to the exponential model (Figure 9) and hence can be explained in a similar manner. That is, the higher β_r value is, the more quickly the repair times grow, resulting in longer time to reach c =15 cycles. However, the curve pattern around $t = 50$ and $t = 100$ remains unique to the distribution.

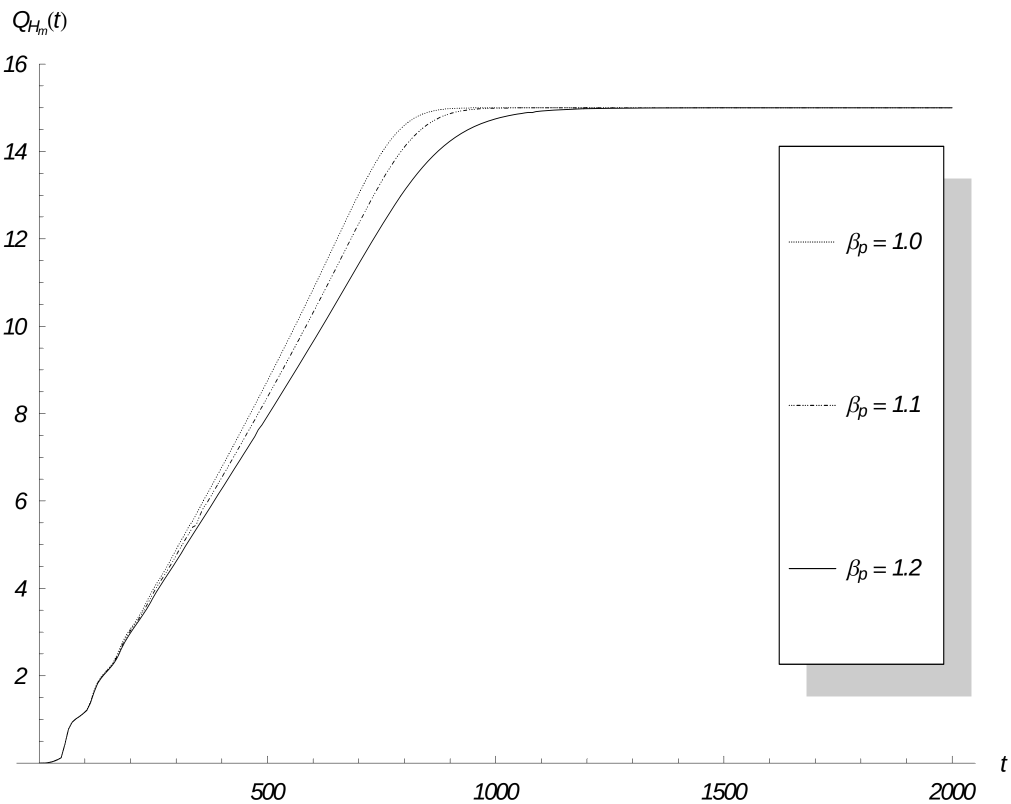

Figure 20. The plot of $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with gamma operating, repair, and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, b = 2, $\rho = \frac{2}{20}$, c = 2, $\gamma = \frac{2}{10}$, T_a = 50, α = 0.9, β_r =1 , and β_p =1.0, 1.1, 1.2.

Figure 20 depicts the effect of different deceleration rates for the PM intervals β_p on the quasi-renewal function under the gamma model. The effect is similar to that of the exponential model (Figure 10) and hence can be explained in the same manner. That is, the higher β_p is, the faster the PM time grow, resulting in longer time to reach the value c. The curve pattern around $t = 50$ and $t = 100$ remains unique to the distribution.

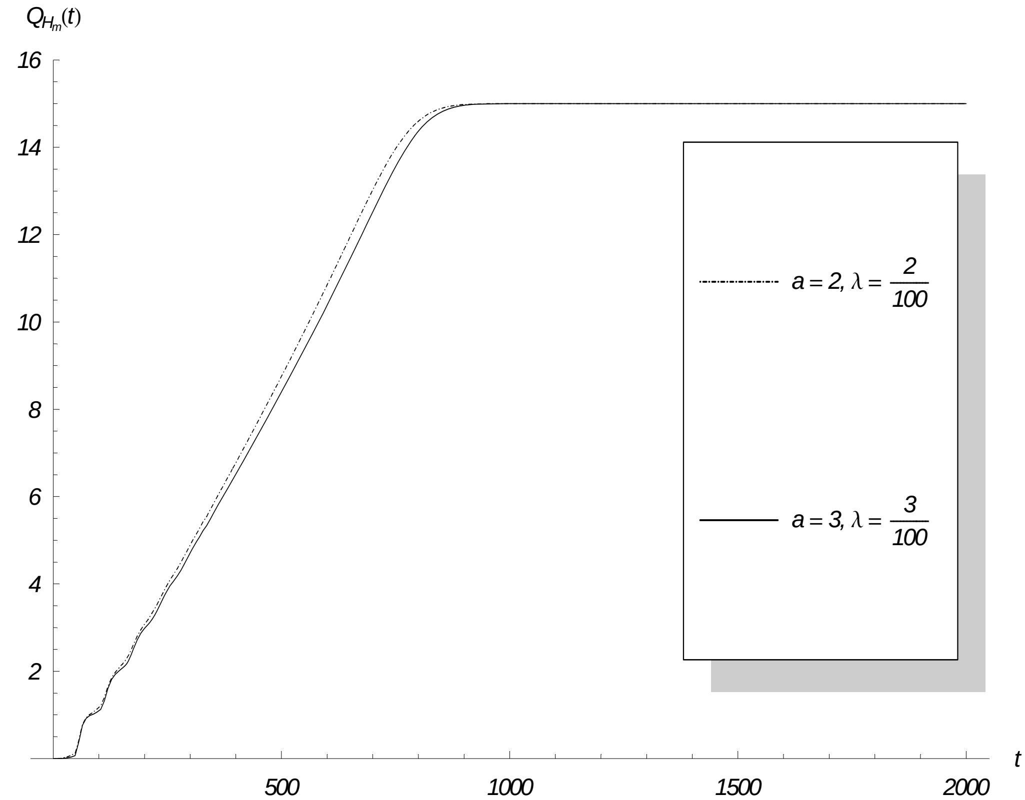

Figure 21. The plot of $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with gamma operating, repair, and PM intervals using parameters b = 2, $\rho = \frac{2}{20}$, c = 2, $\gamma = \frac{2}{10}$, T_a = 50, α = 0.9, β_p =1 , β_r =1, {a = 2, $\lambda = \frac{2}{100}$ } and {a = 3, $\lambda = \frac{3}{100}$}

Figure 21 displays the quasi-renewal function under two gamma models whose only difference is the distribution for the operating time of the first interval. The first model uses $\text{Gamma}\left(2, \frac{2}{100}\right)$ distribution while the second one uses $\text{Gamma}\left(3, \frac{3}{100}\right)$ distribution. Both distributions have equal mean which is 100 but the probability density function for $\text{Gamma}\left(3, \frac{3}{100}\right)$ distribution peaks at a higher t value. (See Figure 22.) This results in more weight on the probability of PM cycle ($\bar{F}_{T_k}(T_a)$) in the expression $\tilde{Q}_{H_m}(t)$ under the $\text{Gamma}\left(3, \frac{3}{100}\right)$ model than the $\text{Gamma}\left(2, \frac{2}{100}\right)$ model, making the average operating interval length longer and consequently the graph $\tilde{Q}_{H_m}(t)$ under the $\text{Gamma}\left(3, \frac{3}{100}\right)$ model takes longer to reach the truncation value c .

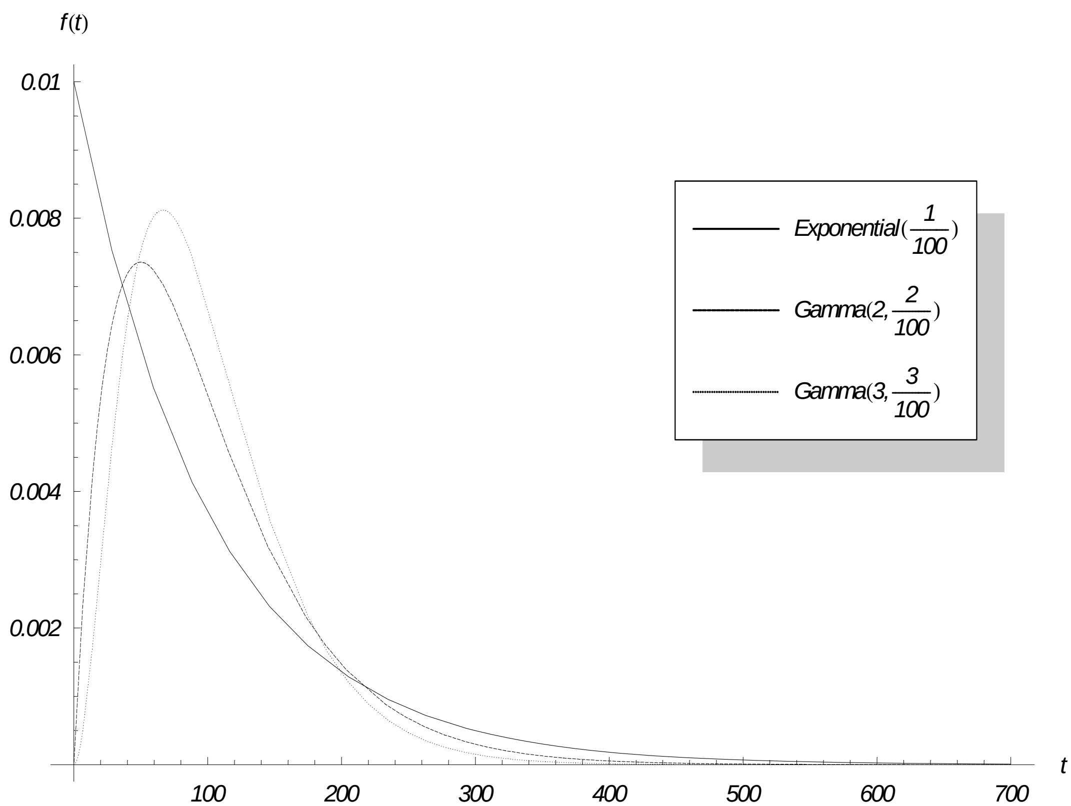

Figure 22. The plot comparing the probability density function for $\text{Exponential}\left(\frac{1}{100}\right)$, $\text{Gamma}\left(2,\frac{2}{100}\right)$**, and** $\text{Gamma}\left(3,\frac{3}{100}\right)$

4.5.1.2 Availability Function

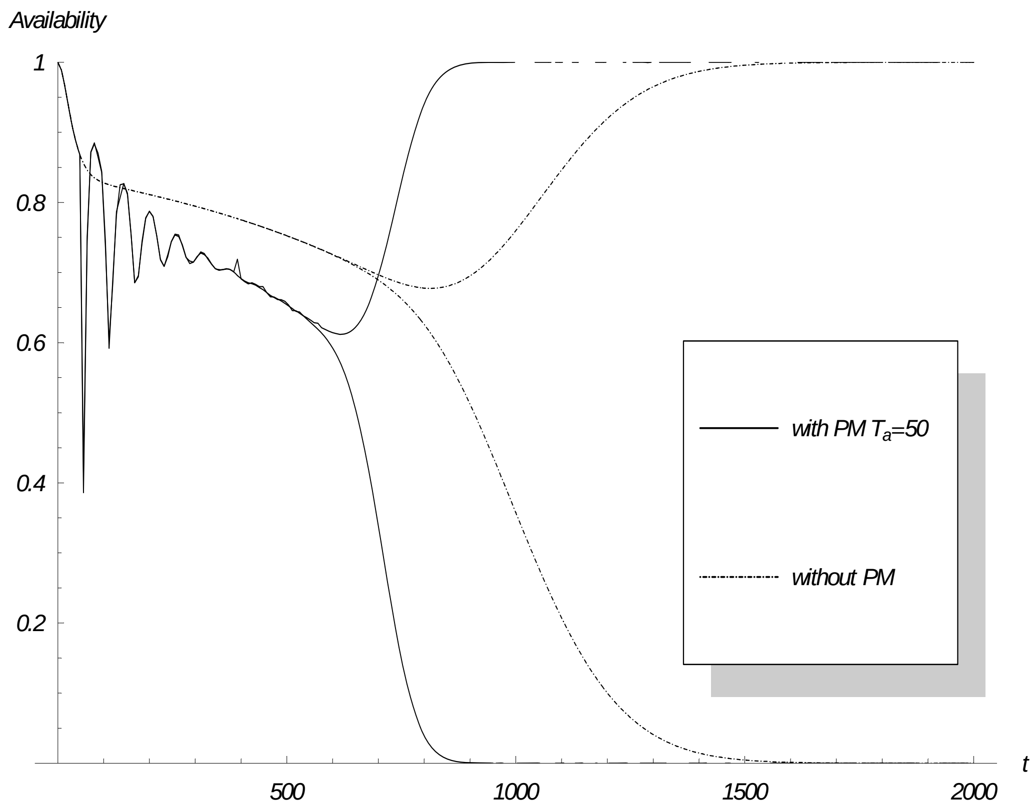

Figure 23. The plot comparing the availability from the base example and the no-PM example under the quasi-renewal model with gamma operating, repair, and PM intervals.

Figure 23 shows the availability plot comparing two gamma models: one with a PM policy and another without. (The one without PM actually matches Rehmert's plot [2].) The model with PM displays a series of dents whenever t equals a multiple value of $T_a = 50$ and the dents gets shallower as t grows because the probability of PM action decreases over time. This phenomenon is similar to that of the exponential case (Figure 11). However the plot for the gamma model has more visible dents than the exponential case. This is because, as mentioned before, the gamma distribution put more weight on the probability of PM cycle ($\bar{F}_{T_k}(T_a)$) in the availability expression and thus the effect of PM is more visible under the gamma model.

Interestingly, when comparing the PM and no-PM plots, we can see some improvement in availability right after the first and the second dents as the result of PM actions. This can be explained as follows. We assume the mean PM time is 10 while the mean repair time is 20. As the gamma distribution put more weight on the probability of PM cycle ($\bar{F}_{T_k}(T_a)$) in the availability expression, the shorter maintenance time (PM time) has more influence on the availability function than the repair time and thus causes the overall availability to rise up above the availability of the repair-cycle-only model. (In fact, we will see later in Figure 29 that the improvement gradually disappears as the mean PM time increases.) We do not see this incident in the exponential case because the weight on the probability of PM cycle ($\bar{F}_{T_k}(T_a)$) for the exponential case is not enough to cause such behavior.

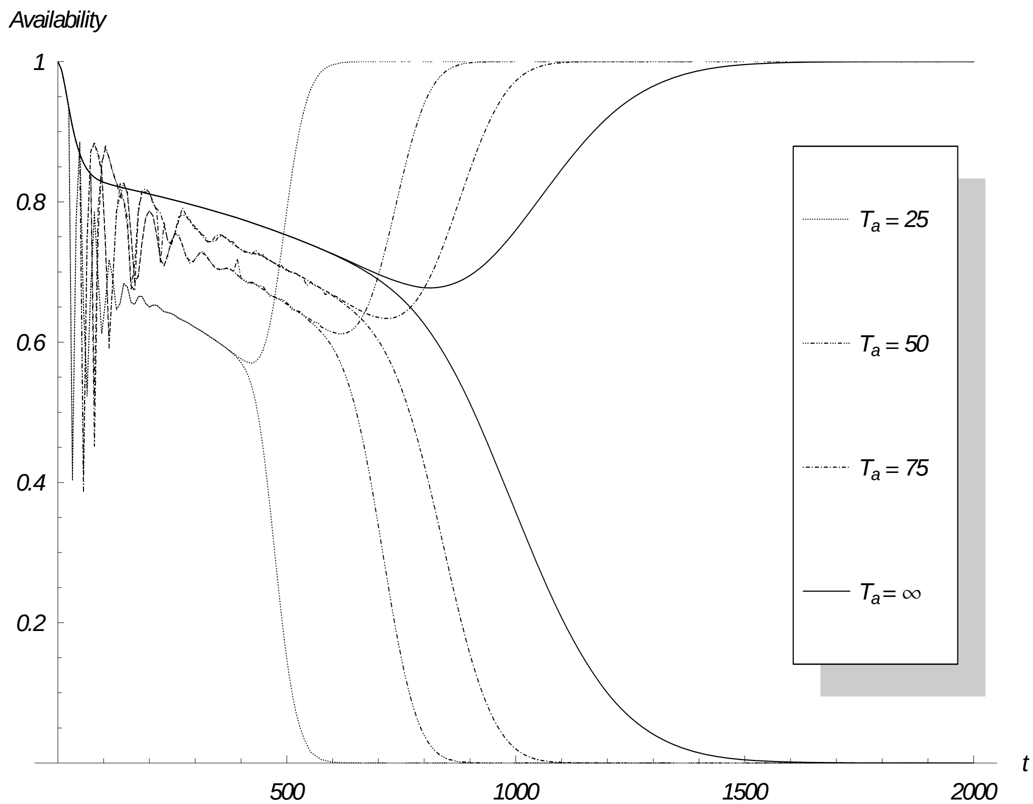

Figure 24. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating, repair, and PM intervals using parameters a **= 2,** $\lambda = \frac{2}{100}$**,** b **= 2,** $\rho = \frac{2}{20}$**,** c **= 2,** $\gamma = \frac{2}{10}$**,** α **= 0.9,** β_r **=1 ,** β_p **=1, and** T_a **= 25, 50, 75,** ∞**.**

Figure 24 depicts the plot of the uptime-based and downtime-based availability functions when the policy age T_a is set to 25, 50, 75, and ∞. The values of the policy age T_a have some effect on the availability comparable to the exponential case (Figure 12). That is the dents occurs on the multiple value of T_a and the dents get shallower as t grows. This can be explained in a similar manner as in the exponential case. Moreover, the overall availability of the instance with a lower value of T_a is lower than that of the higher value of T_a since a lower value of T_a causes more frequent interruptions.

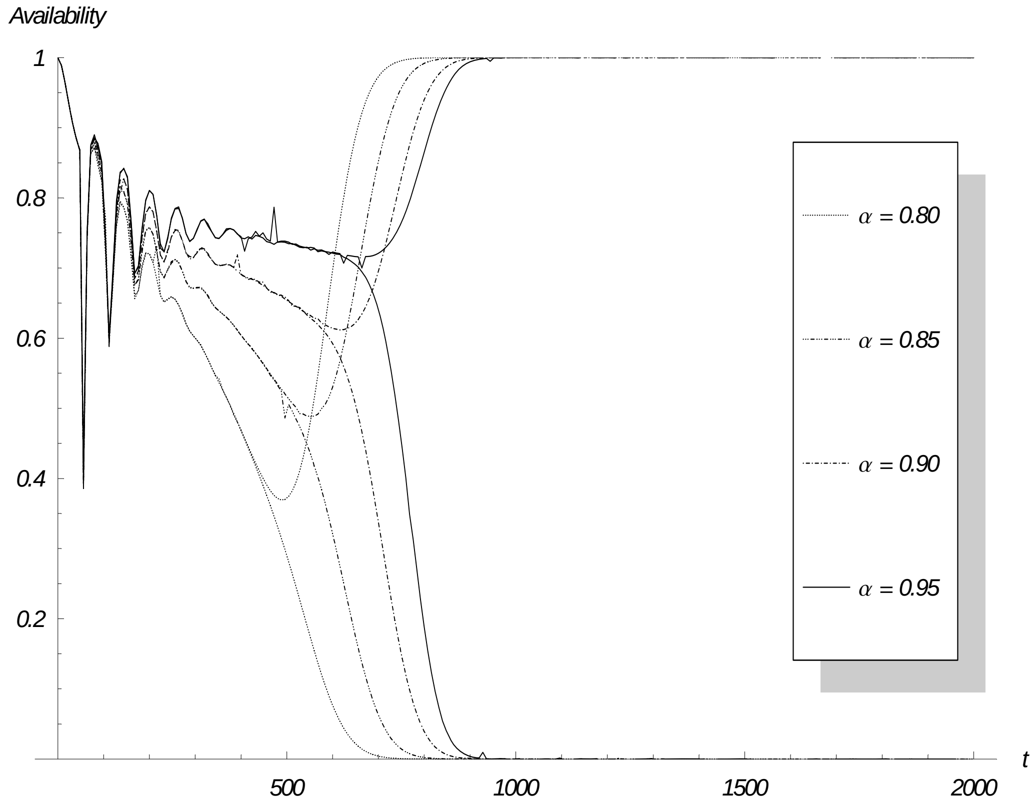

Figure 25. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating, repair, and PM intervals using parameters a **= 2,** $\lambda = \frac{2}{100}$**,** b **= 2,** $\rho = \frac{2}{20}$**,** c **= 2,** $\gamma = \frac{2}{10}$**,** T_a **= 50,** β_r **=1 ,** β_p **=1, and** α **=0.80, 0.85, 0.90, 0.95.**

Figure 25 displays the plot of the availability functions for the gamma model using different deterioration rates α =0.80, 0.85, 0.90, and 0.95. Comparing to the exponential case (Figure 13) , the effect of the deterioration rates α on the availability for the gamma case is similar. That is, the higher the deterioration rate is (α is low), the lower the overall availability will be. This is because the higher deterioration rate stochastically shortens the operating cycle more quickly and hence lowers the availability. In addition, the highest deterioration rate (lowest α) maintains the accuracy the shortest period of time because when α is low, each cycle is shortened quickly and hence reach the truncation value c sooner than other with higher α. Consequently, it loses its accuracy relatively quickly.

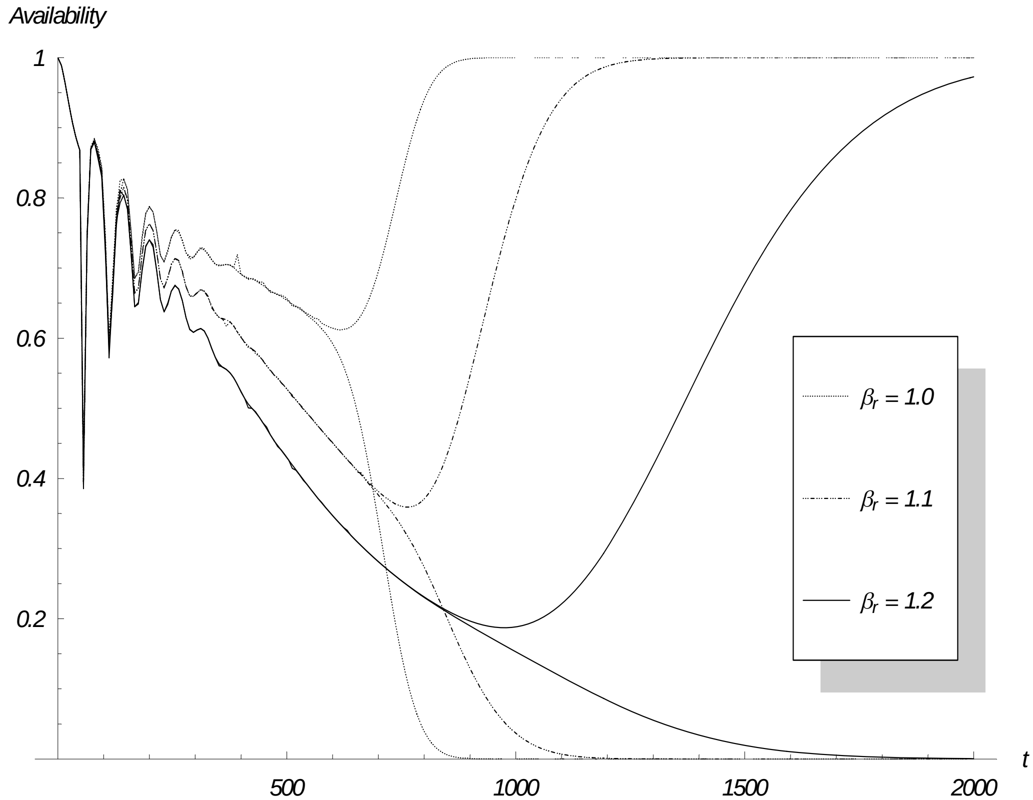

Figure 26. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating, repair, and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, b = 2, $\rho = \frac{2}{20}$, c = 2, $\gamma = \frac{2}{10}$, T_a = 50, α =0.9, β_p =1, and β_r =1.0, 1.1, 1.2.

Figure 26 shows the effect of different values of the deceleration rate on the repair time intervals β_r on the availability. The effect is similar to that of the exponential case (Figure 14) as a higher β_r lowers the overall availability. This is because a higher β_r stochastically creates longer repair intervals over time and hence decreases the availability. Consequently, each cycle is longer than other instances with lower β_r values and thus it takes longer time to reach the truncation value c. This is why we see the graph with highest β_r value maintains its accuracy the longest.

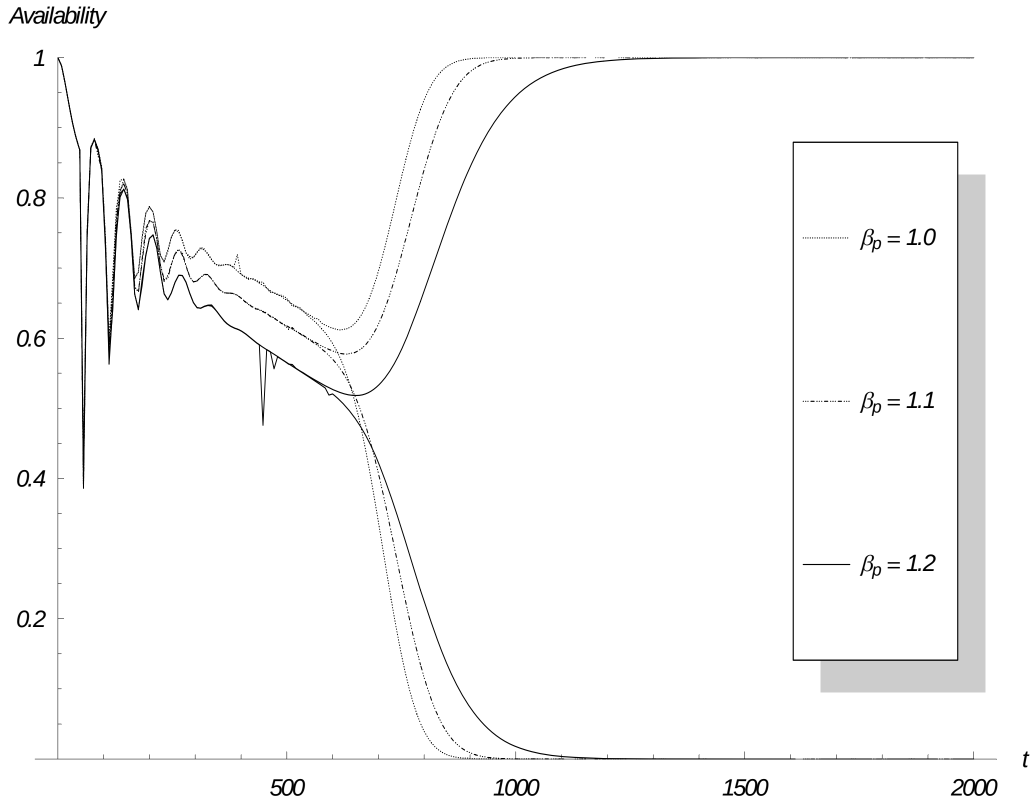

Figure 27. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating, repair, and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, b = 2, $\rho = \frac{2}{20}$, c = 2, $\gamma = \frac{2}{10}$, T_a = 50, α =0.9, β_r =1, and β_p =1.0, 1.1, 1.2.

Figure 27 depicts the effect of the various values of the deceleration rate on the repair time intervals β_p on the availability. The effect of values of β_p is similar to that of the values of β_r but less prominent. This is because as t grows larger, a PM interval is less likely to occur and therefore β_p has smaller effect than β_r. This behavior is similar to that of the exponential case (Figure 15).

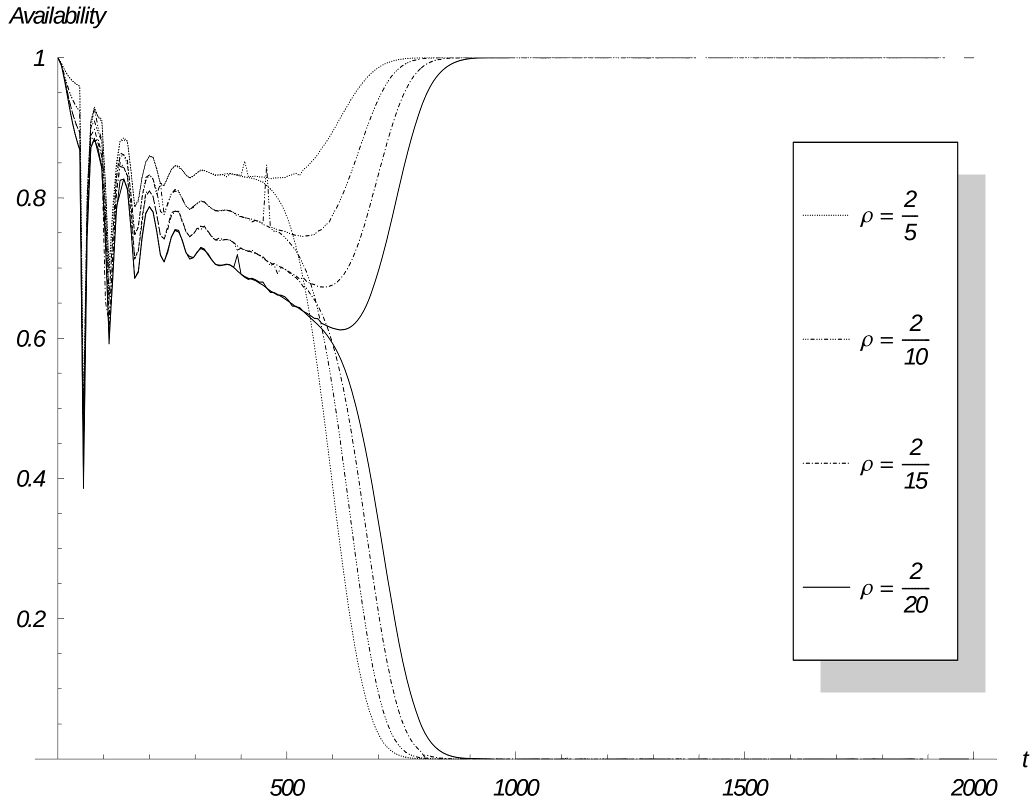

Figure 28. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating, repair, and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, c = 2, $\gamma = \frac{2}{10}$, T_a = 50, α =0.9, β_r =1, β_p =1, b = 2, and $\rho = \frac{2}{5}, \frac{2}{10}, \frac{2}{15}, \frac{2}{20}$.

Figure 28 displays the effect of different mean repair times $\frac{b}{\rho}$ = 5, 10, 15, 20 on the availability function. As expected, we see that the instance with lowest mean repair time ($\rho = \frac{2}{5}$) gives the highest overall availability. Moreover, it is also the one that loses its accurate the soonest since each cycle is stochastically shorter than that of other instances with higher mean repair times and thus the number of contribution terms reaches the truncation value c before others. This phenomenon is similar to that of the exponential case (Figure 16).

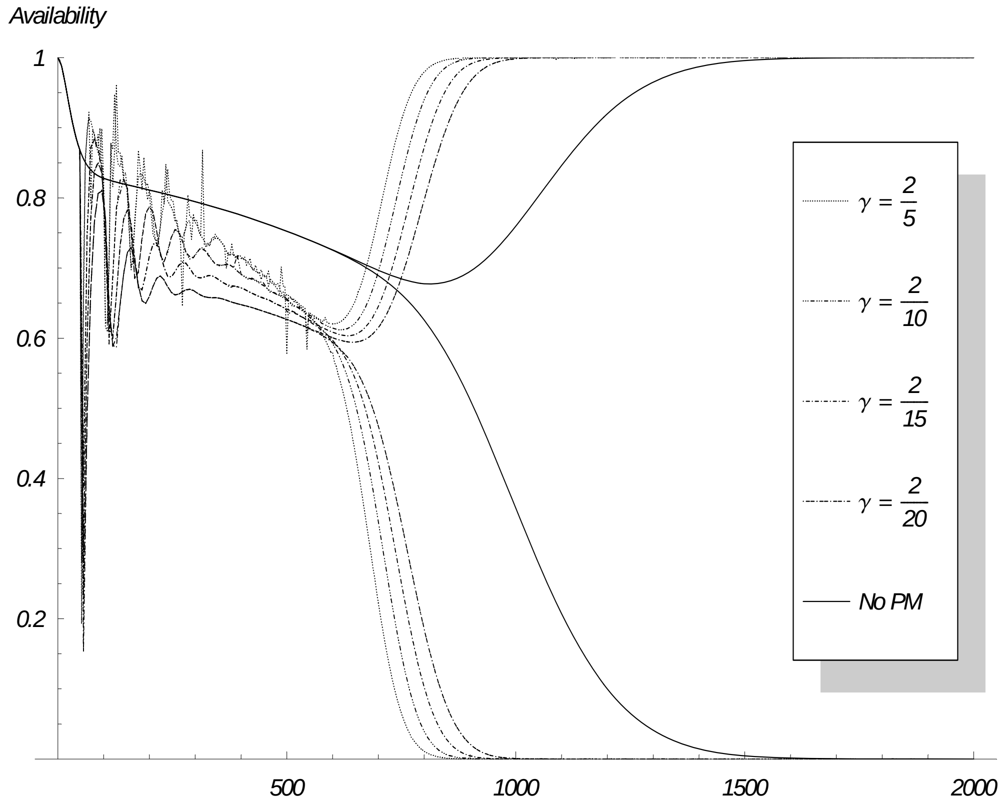

Figure 29. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating, repair, and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, b = 2, $\rho = \frac{2}{20}$, T_a = 50, α=0.9, β_r=1, β_p=1, c = 2, and $\gamma = \frac{2}{5}, \frac{2}{10}, \frac{2}{15}, \frac{2}{20}$.

Figure 29 displays the effect of various values of the mean PM time ($\frac{c}{\gamma}$ = 5, 10, 15, 20) on the availability function. As expected, the instance with the lowest mean PM time gives the highest overall availability. However, as t gets larger, the difference in availability among all instances becomes diminishing since the probability of the PM cycle is decreasing as t grows. A similar behavior can be seen in the exponential case (Figure 17).

Moreover, as the mean PM time gets larger, the improvement of availability as seen in Figure 23 fades away. This behavior does not happen elsewhere when we vary other parameters. This confirms that the improvement is the result of the PM times having shorter mean than the repair times.

4.5.2 Exponential Repair and PM Intervals

In our gamma quasi-renewal model with exponential repair and PM intervals, we assume the first operating interval is distributed according to $\text{Gamma}(a,\lambda)$ while the first repair time and the first PM time are distributed according to $\text{Exponential}(\rho)$ and $\text{Exponential}(\gamma)$ respectively. In our base example, we set the model parameters so that the mean operating time for the first interval is 100 ($a = 2, \lambda = \frac{2}{100}$), the mean for the first repair time is 20 ($\rho = \frac{1}{20}$), the mean for the first PM time is 10 ($\gamma = \frac{1}{10}$), T_a = 50, α =0.90, β_r = 1, and β_p = 1. The base example is used to compare against all other examples.

4.5.2.1 Quasi-renewal Function

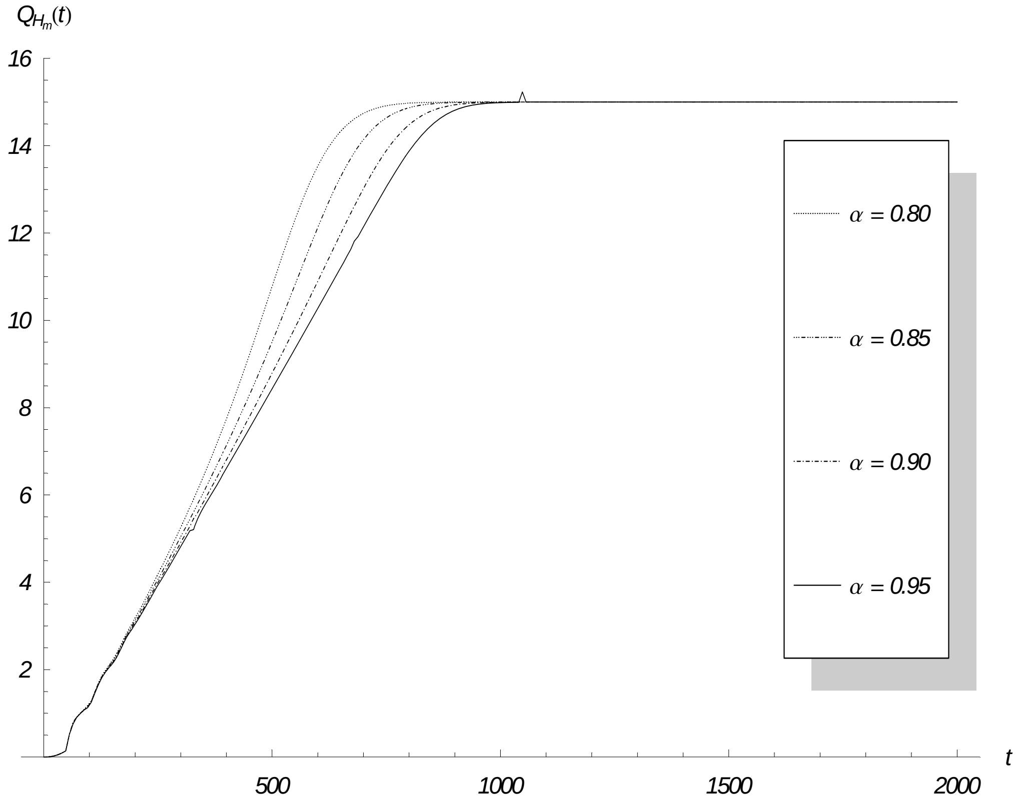

Figure 30. The plot of $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with gamma operating intervals and exponential repair and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, $\rho = \frac{1}{20}$, $\gamma = \frac{1}{10}$, T_a = 50, β_r =1, β_p =1 , and α = 0.8, 0.85, 0.9, 0.95.

Figure 30 depicts the quasi-renewal function truncated to 15 terms for the gamma model with exponential repair and PM intervals with different deterioration rates α for the operating times. The effect of various values of α is similar to other cases we have seen so far and can be explained in a similar way. That is, for the instance with the highest deterioration rate (α is the lowest), the operating times are stochastically shortened the quickest, resulting in shorter cycles than other instances and thus the expected number of restarts reaches the truncation value c before others.

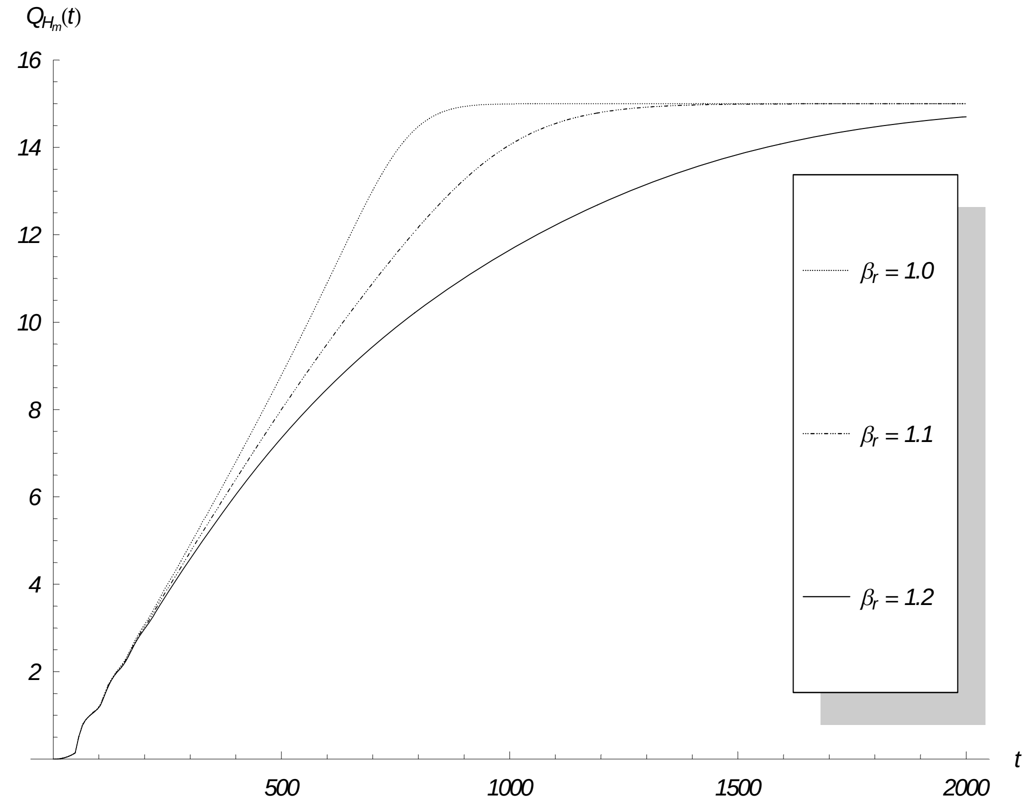

Figure 31. The plot of $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with gamma operating intervals and exponential repair and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, $\rho = \frac{1}{20}$, $\gamma = \frac{1}{10}$, T_a = 50, α = 0.9, β_p =1 , and β_r =1.0, 1.1, 1.2.

Figure 31 displays the effect of various values of the deceleration rate for the repair times β_r on the expected number of restarts. The effect is similar to that of other models and can be explained in a similar way. That is, for the instance with the highest deceleration rate (β_r is the highest), the repair times are stochastically expanded the quickest and hence it takes longer time than others to reach the truncation value c.

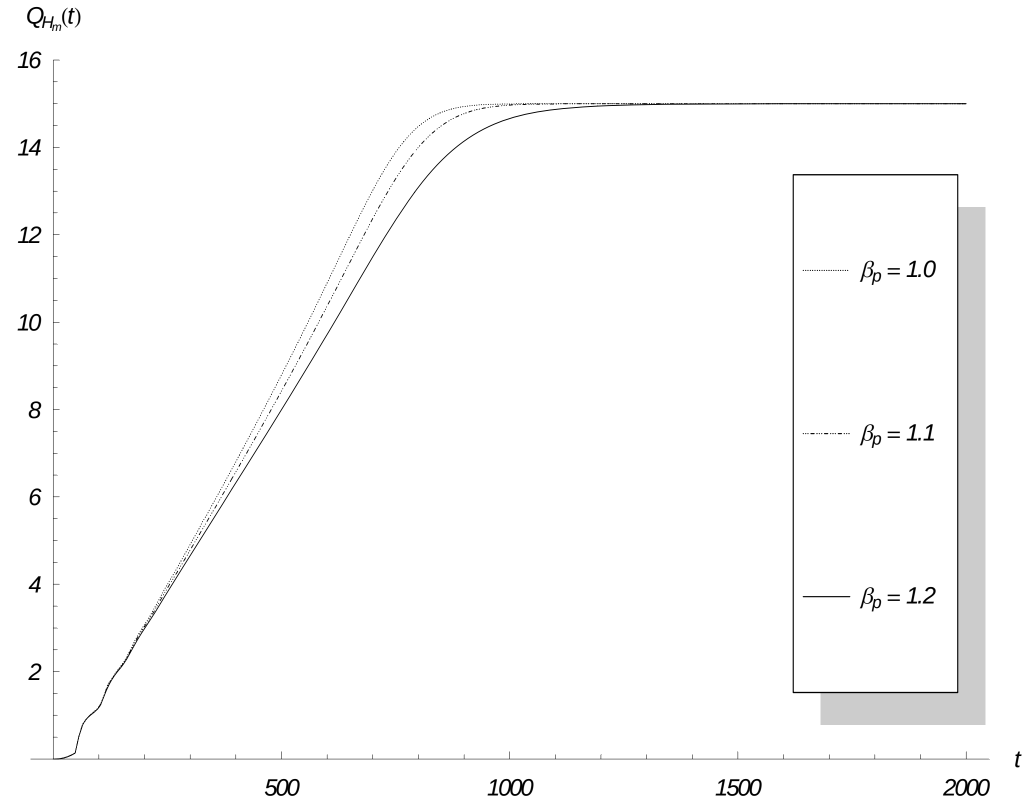

Figure 32. The plot of $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with gamma operating intervals and exponential repair and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, $\rho = \frac{1}{20}$, $\gamma = \frac{1}{10}$, T_a = 50, α = 0.9, β_r =1 , and β_p =1.0, 1.1, 1.2.

Figure 32 displays the effect of various values of the deceleration rate for the PM times β_p on the expected number of restarts. The effect is similar to that of other models where the higher the value β_p is, the longer it takes to reach the truncation value c. This is because the higher β_p values results in longer PM times and hence longer cycles.

4.5.2.2 Availability Function

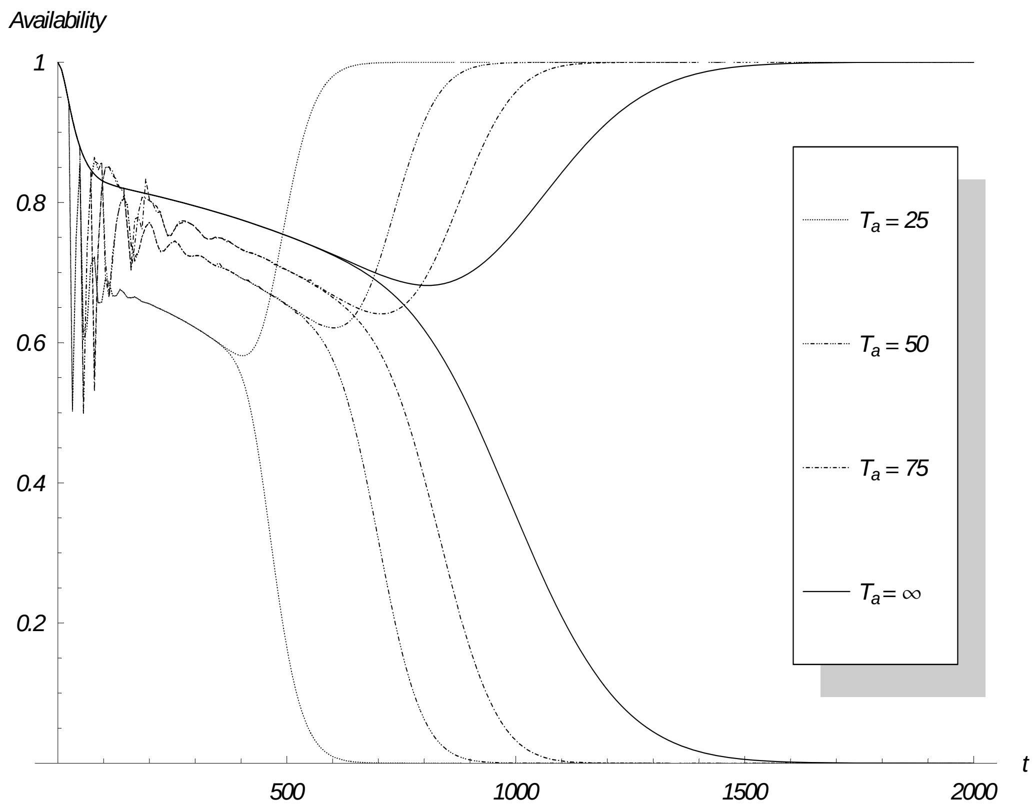

Figure 33. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating intervals and exponential repair and PM intervals using parameters a **= 2,** $\lambda = \frac{2}{100}$, $\rho = \frac{1}{20}$, $\gamma = \frac{1}{10}$, α **= 0.9,** β_r **=1 ,** β_p **=1, and** T_a **= 25, 50, 75,** ∞ **.**

Figure 33 depicts the effect of various values of the policy age on the availability. The plot is similar to that of the gamma model with gamma repair/PM intervals (Figure 24) and can be explained in a similar manner. The dents occur at a multiple value of T_a and the dents get shallower as t grows. The plot for a smaller T_a value loses the accuracy sooner and its dents disappear quicker than other plot with higher T_a values. Moreover, the overall availability for a smaller T_a value instance is lower than others with higher T_a value as a smaller T_a value causes more frequent interruptions. Moreover, we also see some improvement in the availability (as seen in Figure 23) around $t = T_a$ for T_a =50 and 75 in this gamma-exponential model.

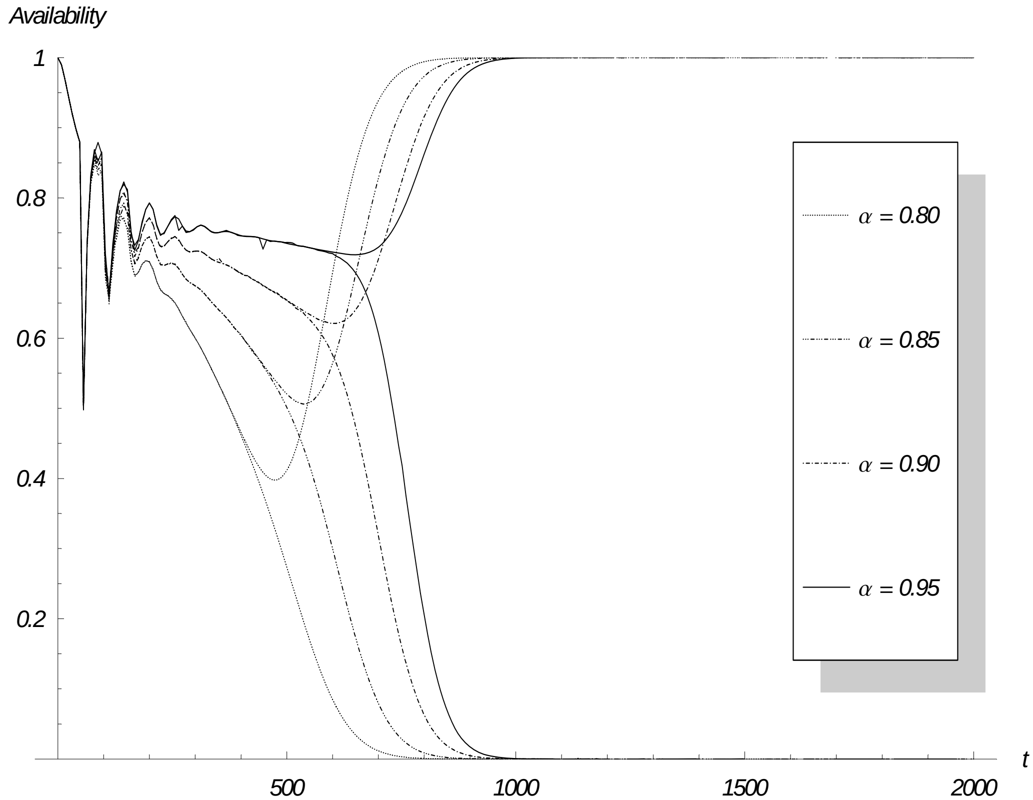

Figure 34. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating intervals and exponential repair and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, $\rho = \frac{1}{20}$, $\gamma = \frac{1}{10}$, T_a = 50, β_r =1 , β_p =1, and α =0.80, 0.85, 0.90, 0.95.

Figure 34 shows the effect of various deterioration rates α on the availability under the gamma operating intervals model with exponential repair/PM intervals. The plot shows the similar behavior to those from the other models we have seen and hence can be explained with the same reasoning. That is, an instance with high deterioration rate (α is low) has lower availability and loses its accuracy sooner than others with lower deterioration rates. This is because the operating cycles are stochastically shortened quickly.

Comparing with Figure 25 from the previous model (gamma operating, repair, and PM intervals), we can see that Figure 34 displays less visible dents on the availability plot given the same deterioration rate. This is because the exponential repair/PM intervals tend to have more probability mass towards zero than the gamma counterparts. (See Figure 22.) Thus their repair/PM intervals tend to be shorter and hence the gamma-exponential model availability plot shows less visible dents.

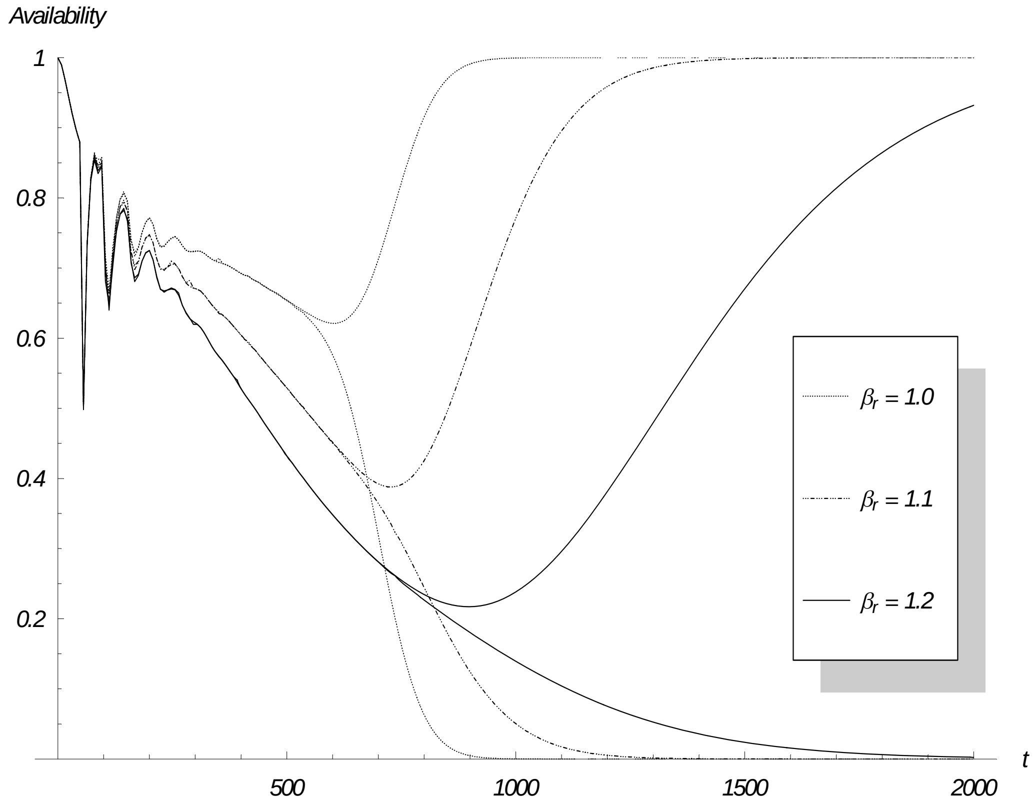

Figure 35. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating intervals and exponential repair and PM intervals using parameters a **= 2,** $\lambda = \frac{2}{100}$**,** $\rho = \frac{1}{20}$**,** $\gamma = \frac{1}{10}$**,** T_a **= 50,** α **=0.9,** β_p **=1, and** β_r **=1.0, 1.1, 1.2.**

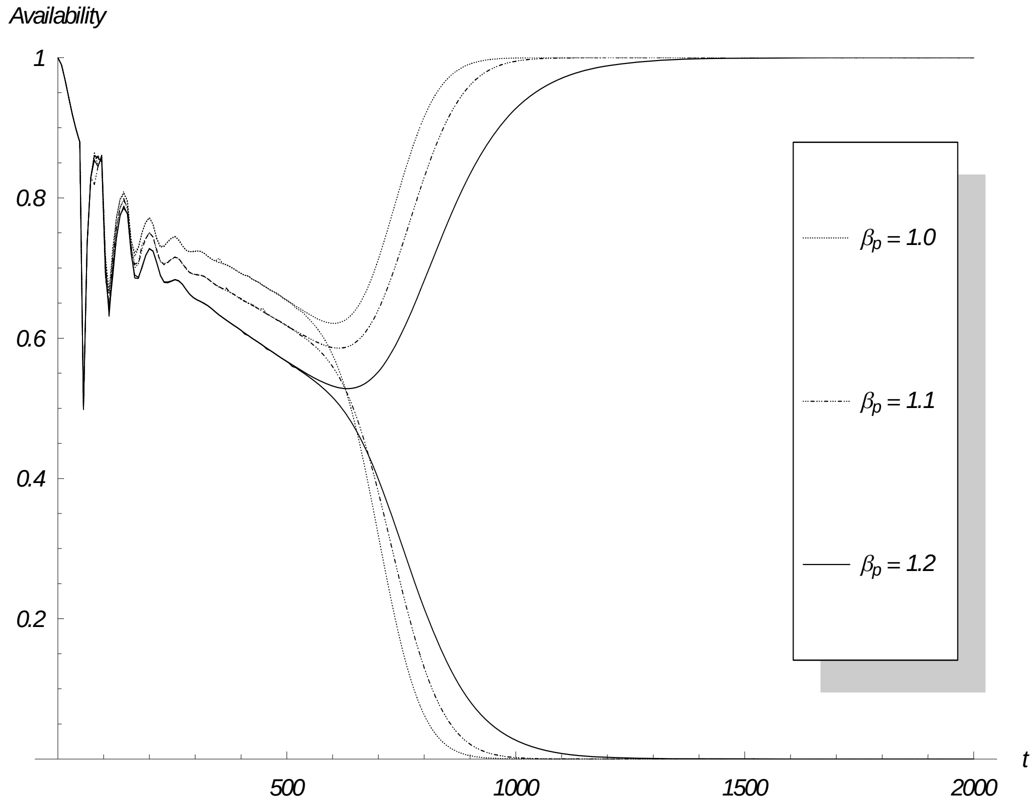

Figure 36. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating intervals and exponential repair and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, $\rho = \frac{1}{20}$, $\gamma = \frac{1}{10}$, T_a = 50, α =0.9, β_r =1, and β_p =1.0, 1.1, 1.2.

Figure 35 and Figure 36 show the effect of the deceleration rate for the repair intervals β_r and the deceleration rate β_p for the PM intervals on the availability, respectively. Both β_r and β_p have similar effect. That is, an instance with a larger deceleration rate value has lower availability than other instances with lower deceleration rates as it stochastically expands the length of maintenance intervals quicker. However, β_r has more prominent effect on the availability than β_p because the probability of PM cycles diminishes over time. We have seen a similar result from other models.

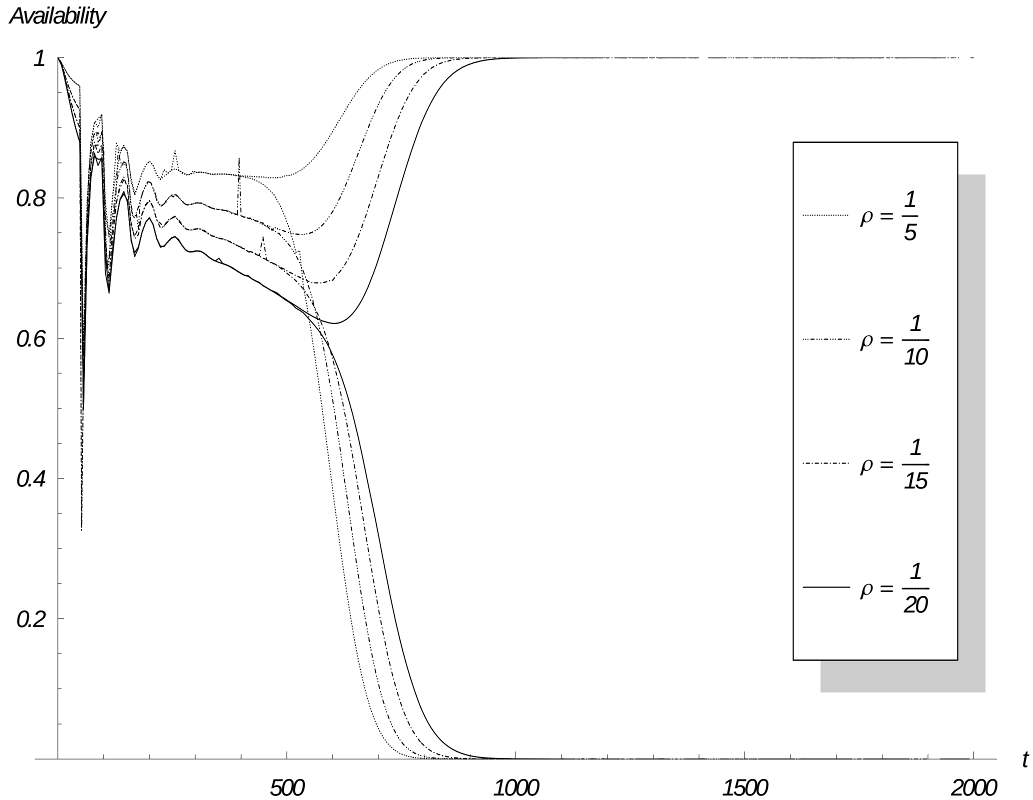

Figure 37. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating intervals and exponential repair and PM intervals using parameters a **= 2,** $\lambda = \frac{2}{100}$**,** $\gamma = \frac{1}{10}$**,** T_a **= 50,** α **=0.9,** β_r **=1,** β_p **=1, and** $\rho = \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}$**.**

Figure 37 depicts the effect of various values of the mean repair time ($\frac{1}{\rho}$ = 5, 10, 15, 20) on the availability function. As expected, the higher mean repair time is, the lower the availability becomes. This behavior is found in other models as well and can be explained using the same reasoning.

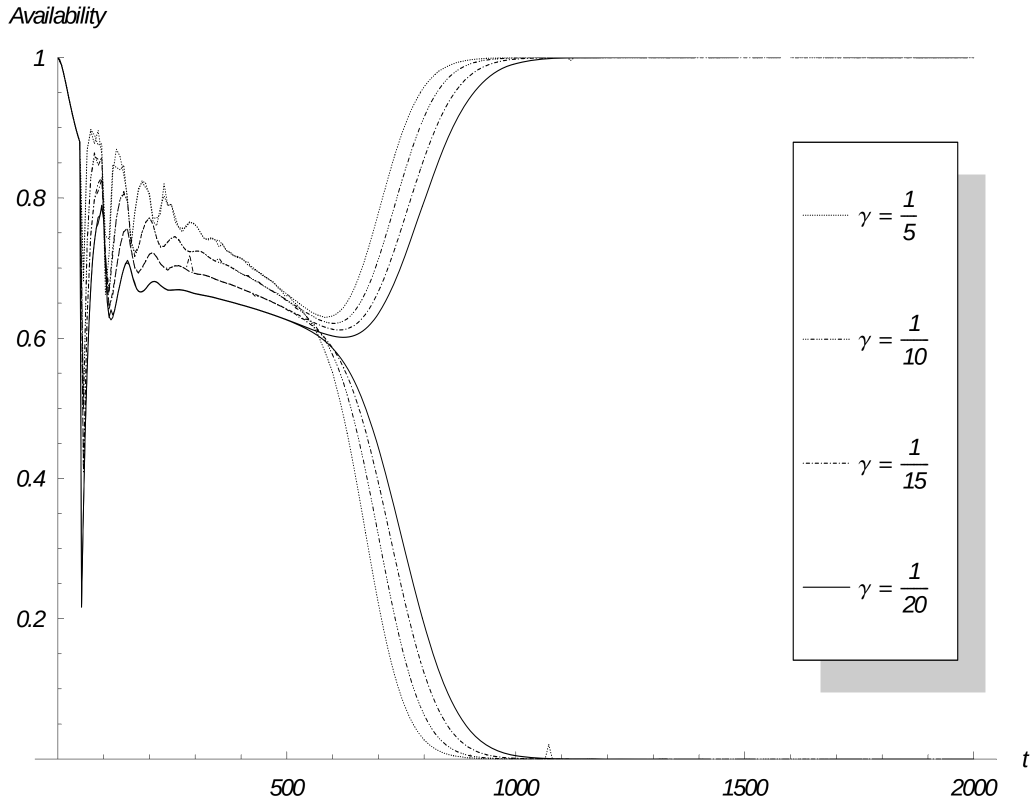

Figure 38. The plot of availability functions approximated using 14 terms under the quasi-renewal model with gamma operating intervals and exponential repair and PM intervals using parameters a = 2, $\lambda = \frac{2}{100}$, $\rho = \frac{1}{20}$, T_a = 50, α =0.9, β_r =1, β_p =1, and $\gamma = \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}$.

Figure 38 displays the effect of various values of the mean PM time ($\frac{1}{\gamma}$ = 5, 10, 15, 20) on the availability function. An instance with a higher mean PM time has lower overall availability than other instances with lower mean PM times. This behavior is seen in other model as well and can be explained in a similar manner.

4.6 Normal Distribution Model

4.6.1 Normal Repair and PM Intervals

In our normal quasi-renewal model with normal repair and PM intervals, we assume the first operating interval is distributed according to $\text{Normal}(\mu,\sigma)$ while the first repair time and the first PM time are distributed according to $\text{Normal}(\mu_r,\sigma_r)$ and $\text{Normal}(\mu_p,\sigma_p)$ respectively. In our base example, we set the model parameters so that the mean operating time for the first interval is 100 ($\mu = 100, \sigma = 25$), the mean for the first repair time is 10 ($\mu_r = 10, \sigma_r = 2.5$), the mean for the first PM time is 5 ($\mu_p = 5, \sigma_p = 1.5$), T_a = 50, α =0.90, β_r = 1, and β_p = 1. The standard deviation parameters are chosen so that the probability mass for $t < 0$ is negligible. The base example is used to compare against all other examples.

4.6.1.1 Quasi-renewal Function

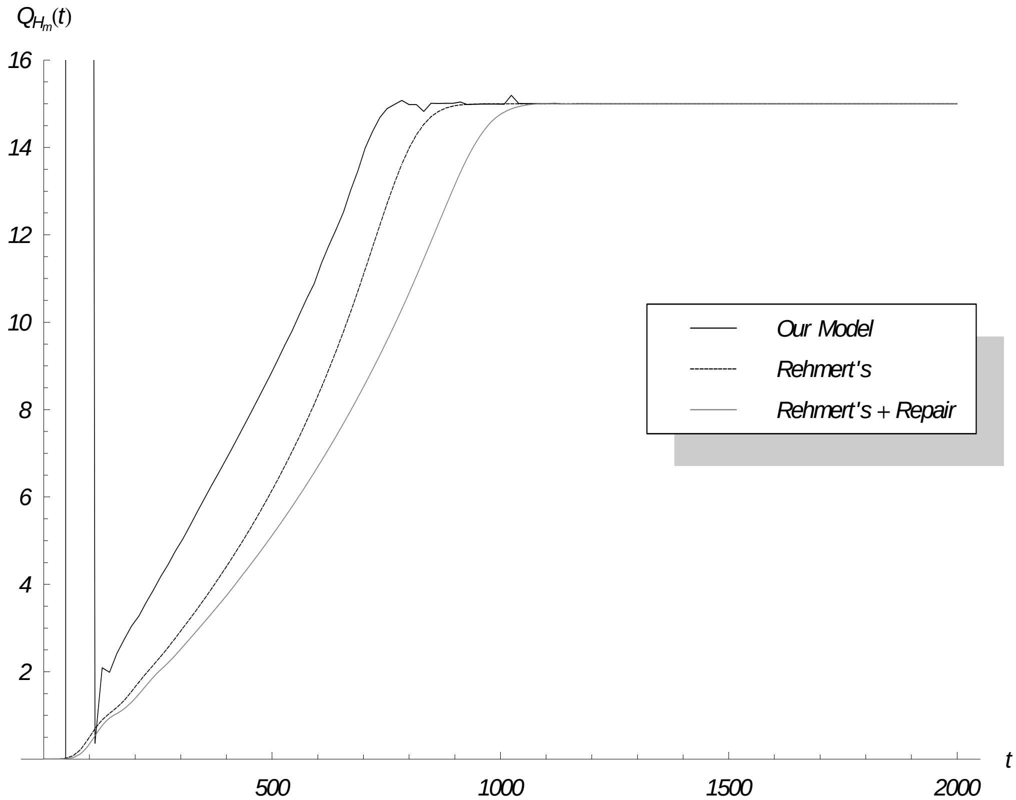

Figure 39 The plot of $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with normal operating intervals and normal repair and PM intervals using parameters μ = 100, $\sigma = 25$, μ_r = 10, $\sigma_r = 2.5$, μ_p = 5, $\sigma_p = 1.5$, α = 0.9, β_p =1 , β_r =1, and T_a = 50, comparing with Rehmert's plot.

Figure 39 displays the quasi-renewal function from our base example comparing with the plot from Rehmert's expression [2] for the normal operating and repair intervals (without PM). Rehmert's quasi-renewal function expression is created under the assumption that the repair times are negligible. His quasi-renewal function expression can be written using our notations as

$$Q_{H_m}(t) = \sum_{n=1}^{\infty} \Phi\left(\frac{t - \frac{\mu\left(1-\alpha^n\right)}{1-\alpha}}{\sqrt{\frac{\sigma^2\left(1-\alpha^{2n}\right)}{1-\alpha^2}}} \right) \quad \text{“Rehmert’s”} \tag{4.6.1}$$

However, our model takes repair times into consideration. Therefore, to compare our quasi-renewal function with Rehmert’s, it makes sense for us to add the repair terms into (4.6.1) (with β_r=1) and obtain the “Rehmert’s + Repair” expression:

$$Q_{H_m}(t) = \sum_{n=1}^{\infty} \Phi\left(\frac{t - \left(\frac{\mu\left(1-\alpha^n\right)}{1-\alpha} + n\mu_r \right)}{\sqrt{\frac{\sigma^2\left(1-\alpha^{2n}\right)}{1-\alpha^2} + n{\sigma_r}^2}} \right) \quad \text{“Rehmert’s + Repair”} \tag{4.6.2}$$

Figure 39 shows that the “Rehmert’s + Repair” plot reaches the truncation value later than the original Rehmert’s plot. This is because when the repair times are considered, each cycle length is longer than the original model and hence it takes longer time to reach the limit value.

Since the quasi-renewal function essentially represents the expected number of restarts, it must be a monotone nondecreasing function. However, Figure 39 exhibits a huge jump in the plot from our model approximately between $t = 50$ and $t = 100$, which should not happen. This problem is possibly caused by the numerical error or the transient effect. After the transient period where the noise occurs, the plot appears to be smooth and show no sign of any major anomaly. Despite the huge noise, the quasi-renewal function plot from our model seems to be above Rehmert’s plot throughout until they reach the truncation value 15. This is possible since PM actions from our model will produce more cycles than Rehmert’s model especially during the initial time period, resulting in increasing in the number of restarts.

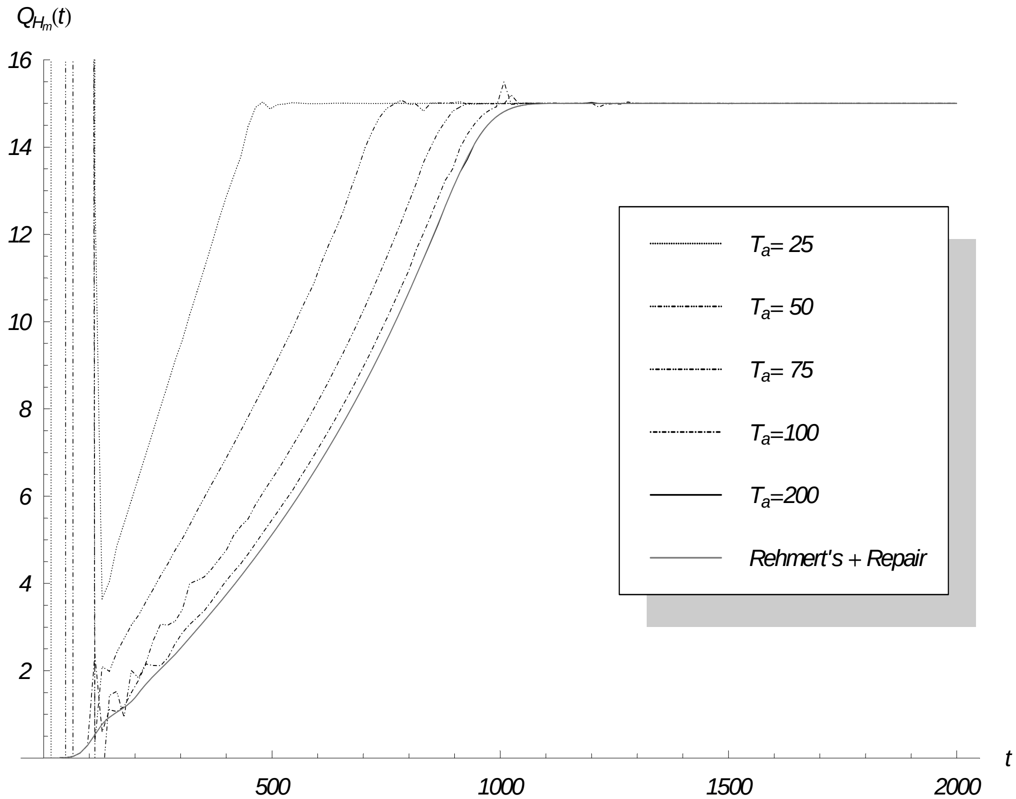

Figure 40 The plot of $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with normal operating intervals and normal repair and PM intervals using parameters μ = 100, $\sigma = 25$, μ_r = 10, $\sigma_r = 2.5$, μ_p = 5, $\sigma_p = 1.5$, α = 0.9, β_p =1 , β_r =1, and T_a = 25, 50,75, 100, 200, comparing with Rehmert's plot.

Figure 40 displays the effect of various policy age values T_a on the quasi-renewal function comparing with "Rehmert's+Repair" plot. As expected, when T_a gets smaller, the system is more likely to be interrupted more frequently by a PM action. This causes the number of restarts to go up. As T_a gets larger, a PM action is less likely to occur. In fact, when $T_a = 200$, the plot practically coincides with "Rehmert's+Repair" plot, indicating that a PM action never happens, probabilistically speaking. Table 2 shows that our model and "Rehmert's+Repair" model produce similar numerical results. Essentially, the probability of a PM action for the k^{th} cycle $\bar{F}_{T_k}(200) = 1 - \Phi\left(\frac{200 - 100\alpha^{k-1}}{25\alpha^{k-1}}\right)$ is almost zero for $k = 1, \ldots, 15$.

All plots (except for $T_a = 200$) indicates a huge jump in the neighborhood of $t = 100$, which is possibly caused by the numerical error or the transient effect. After this period, the plot seems to be smooth and no major irregularities are found. However, we are cautious about the correctness of the data especially for the base example, which we intend to use for comparing with other examples. Therefore, we provide no further investigation into the impact of other parameters on the quasi-renewal function for this model.

	t	**Our Model**	**Rehmert's + Repair**
$t < T_a$	32	0.000953023755946	0.000953023755946
	64	0.033567999052253	0.033567999052253
	96	0.289063070659287	0.289063070659287
	128	0.770803668061778	0.770803668061778
	160	1.046532053212876	1.046532053212876
	192	1.299628726845418	1.299628726845418
$t \geq T_a$	224	1.687128776795798	1.686287265712048
	256	2.040759315259337	2.042070513866752
	288	2.374297885858573	2.374468510064372
	320	2.757316100765062	2.756827507246999
	352	3.144436327493147	3.144430817239569
	384	3.537270279870152	3.537374889685295

Table 2 Some Quasi-Renewal Function Values from the case $T_a = 200$

4.6.1.2 Availability Function

Figure 41 - Figure 45 displays the availability plot for the normal quasi-renewal model where T_a=25, 50, 75, 100, and 200, respectively. In every case, the plot for $t < T_a$ fits nicely with Rehmert's plot, where PM is not considered. However, for the plots with a small value of T_a (T_a=25, 50, and 75), the numerical error is so severe for $t \geq T_a$ since the upper and lower bounds do not match and the plots appear to have random ups and downs. No useful behavior observation can be concluded from these plots. However, as T_a gets larger, the random noise starts to disappear and the plot becomes more and more similar to Rehmert's plot. In fact, when $T_a = 200$, the plot from our model virtually coincides with Rehmert's plot. Some numerical values from Table 3 and Table 4 confirm that our model and Rehmert's model produce similar results for the case $T_a = 200$. This is because when T_a is large, a PM action hardly occurs. In fact, as mentioned before, the probability of a PM action for the k^{th} cycle $\overline{F}_{T_k}(200)$ is almost zero for $k = 1,\ldots,15$. The only maintenance action performed is almost always corrective and our model correctly exhibits this behavior.

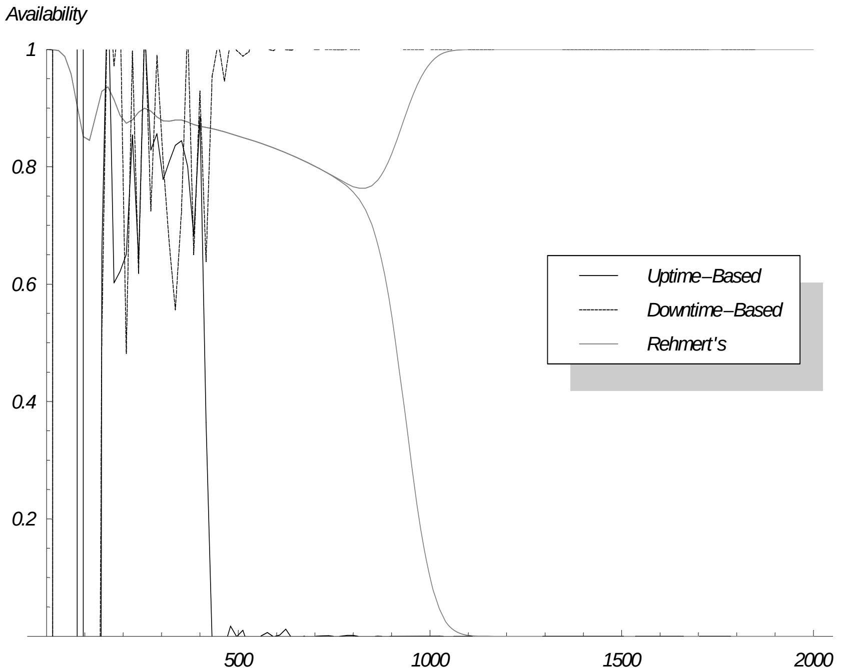

Figure 41 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating, repair, and PM intervals using parameters μ = 100, $\sigma = 25$, μ_r = 10, $\sigma_r = 2.5$, μ_p = 5, $\sigma_p = 1.5$, α = 0.9, β_p =1 , β_r =1, and T_a = 25.

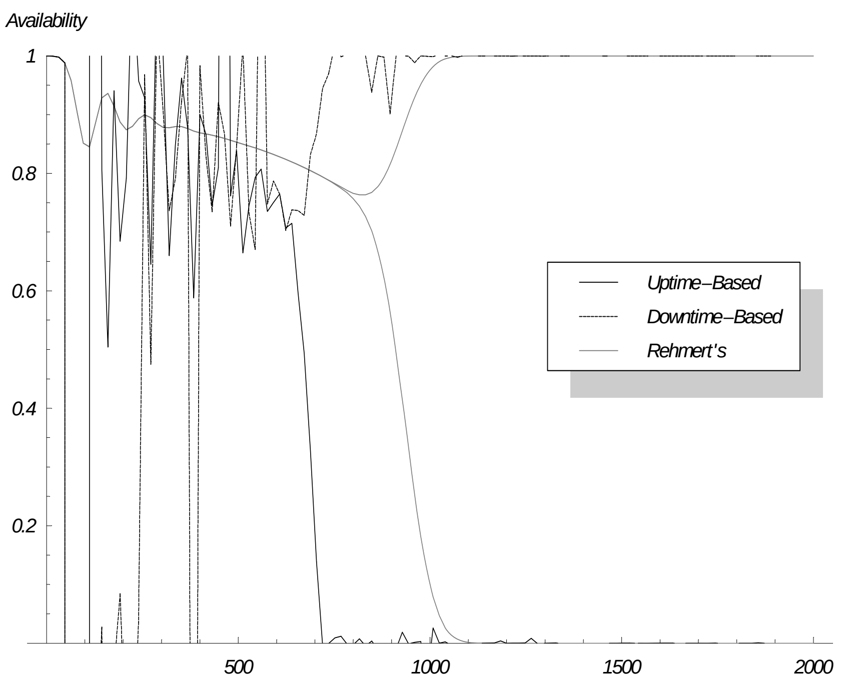

Figure 42 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating, repair, and PM intervals using parameters μ = 100, $\sigma = 25$, μ_r = 10, $\sigma_r = 2.5$, μ_p = 5, $\sigma_p = 1.5$, α = 0.9, β_p =1 , β_r =1, and T_a = 50.

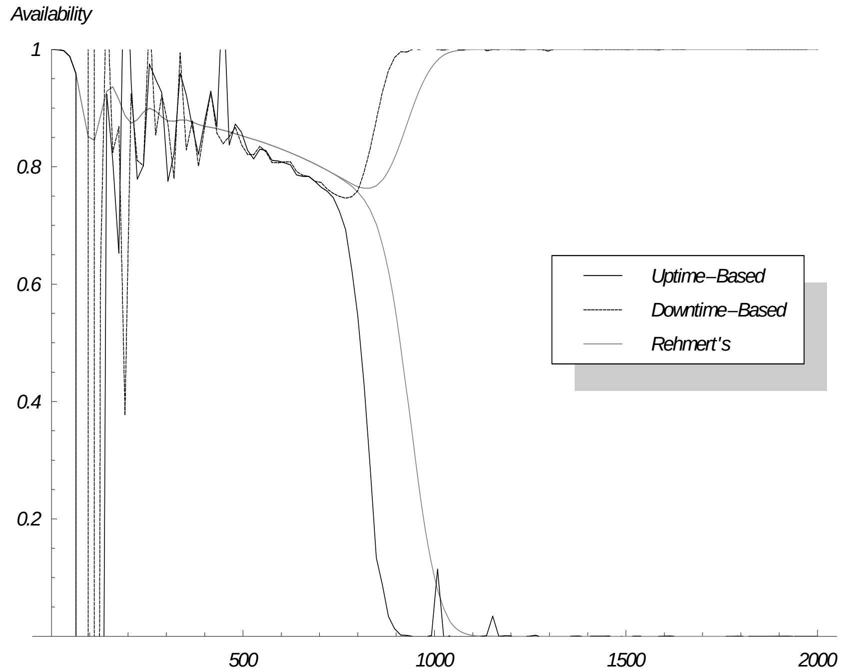

Figure 43 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating, repair, and PM intervals using parameters μ = 100, $\sigma = 25$, μ_r = 10, $\sigma_r = 2.5$, μ_p = 5, $\sigma_p = 1.5$, α = 0.9, β_p =1 , β_r =1, and T_a = 75.

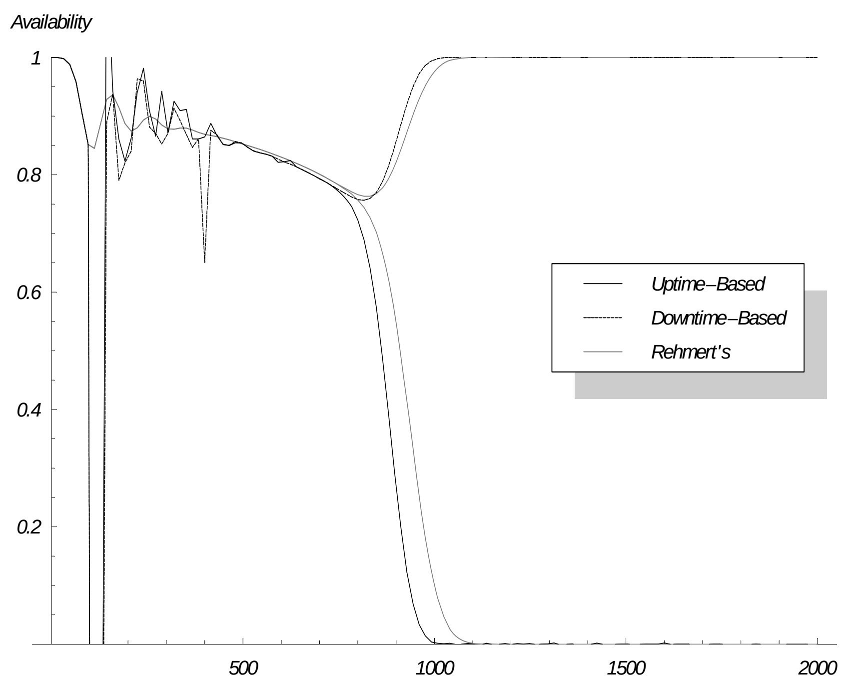

Figure 44 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating, repair, and PM intervals using parameters μ = 100, $\sigma = 25$, μ_r = 10, $\sigma_r = 2.5$, μ_p = 5, $\sigma_p = 1.5$, α = 0.9, β_p =1 , β_r =1, and T_a = 100.

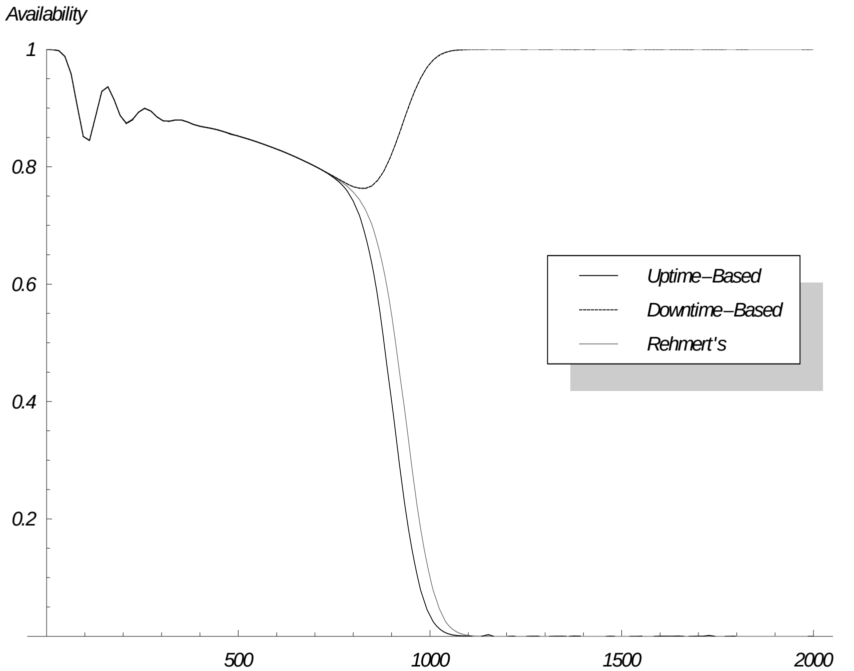

Figure 45 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating, repair, and PM intervals using parameters μ = 100, $\sigma = 25$, μ_r = 10, $\sigma_r = 2.5$, μ_p = 5, $\sigma_p = 1.5$, α = 0.9, β_p =1 , β_r =1, and T_a = 200.

	t	**Our Model**	**Rehmert's Model**
$t < T_a$	32	0.997688611951542	0.997688611951542
	64	0.958606689787854	0.958606689787854
	96	0.851599584897483	0.851599584897483
	128	0.885754047326895	0.885754047326895
	160	0.936473018349896	0.936473018349896
	192	0.887469257130496	0.887469257130496
$t \geq T_a$	224	0.880615051877672	0.879766475248994
	256	0.899760299965941	0.899816378249625
	288	0.884948176001539	0.884927491024350
	320	0.877600847116977	0.877588737825013
	352	0.879494746770126	0.879539645965725
	384	0.872009681407244	0.872012731203263

Table 3 Some Upper Bound Values from the case $T_a = 200$

	t	**Our Model**	**Rehmert's Model**
$t < T_a$	32	0.997688611951542	0.997552867468958
	64	0.958606689787854	0.957589216943250
	96	0.851599584897483	0.851257419413294
	128	0.885754047326895	0.886701252735578
	160	0.936473018349896	0.936090878075716
	192	0.887469257130496	0.887139404416978
$t \geq T_a$	224	0.879626231709656	0.880042240220956
	256	0.899181215895460	0.899749540182686
	288	0.884879108631178	0.884772128574151
	320	0.877563073320408	0.877632499903144
	352	0.879501727748991	0.879483016367508
	384	0.871985991484423	0.871944501190467

Table 4 Some Lower Bound Values from the case $T_a = 200$

Although a large value of T_a creates a well-behaved graph, the policy age is normally set to be smaller than the mean operating time in practice. We include the results from a large T_a value here to show that our model does correctly behave like a model without PM as expected. As for smaller value of T_a, the error from the numerical example is so severe we cannot make any reasonable conclusion from the results, especially for the base example. We are cautious about the correctness of the base example data and therefore we provide no further discussion on the impact of other parameters on the availability for this model.

4.6.2 Exponential Repair and PM Intervals

In our normal quasi-renewal model with exponential repair and PM intervals, we assume the first operating interval is distributed according to $\text{Normal}(\mu,\sigma)$ while the first repair time and the first PM time are distributed according to $\text{Exponential}(\rho)$ and $\text{Exponential}(\gamma)$ respectively. In our base example, we set the model parameters so that the mean operating time for the first interval is 100 ($\mu = 100, \sigma = 25$), the mean for the first repair time is 20 ($\rho = \frac{1}{20}$), the mean for the first PM time is 10 ($\gamma = \frac{1}{10}$), T_a = 50, α=0.90, β_r = 1, and β_p = 1. The standard deviation value σ is chosen so that the probability mass for $t < 0$ is negligible. The base example is used to compare against all other examples.

Rehmert [2] does not provide any data discussion for this normal-exponential model because the *Mathematica* built-in Laplace transform inversion routine is not able to invert the expressions from this model either numerically or symbolically. However, we use a different numerical Laplace transform inversion routine (i.e. GWR routine) and we are able to obtain some results although the results are impaired greatly by unexpected large noise.

4.6.2.1 Quasi-renewal Function

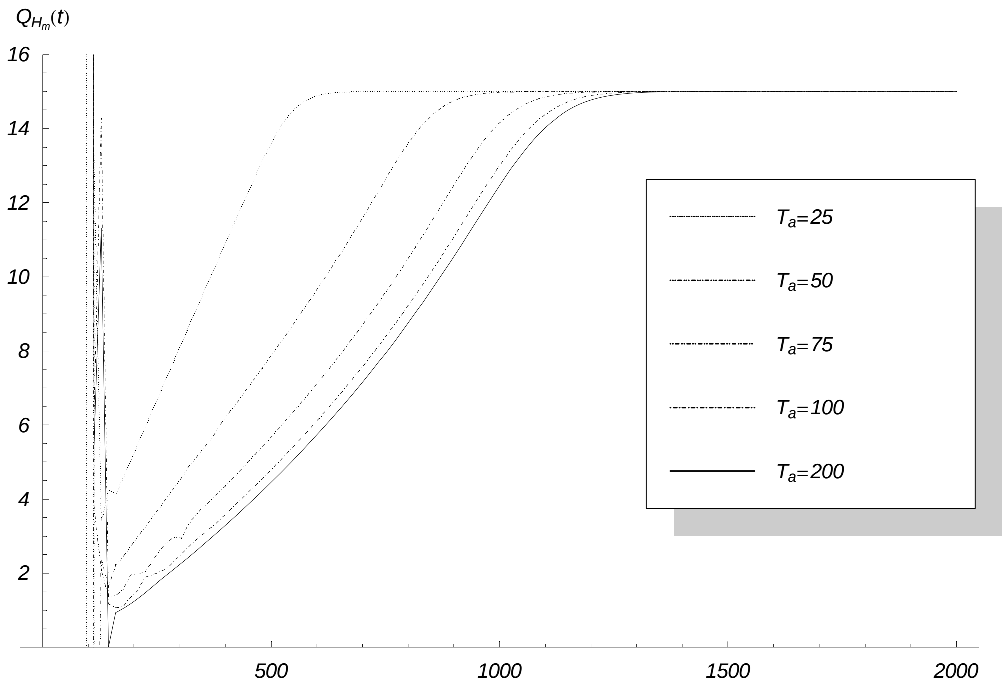

Figure 46 The plot of $\tilde{Q}_{H_m}(t)$ truncated to 15 terms for the quasi-renewal model with normal operating intervals and exponential repair and PM intervals using parameters μ = 100, $\sigma = 25$, ρ = 0.05, γ = 0.1, α = 0.9, β_p =1 , β_r =1, and T_a = 25, 50,75, 100, 200.

Figure 46 depicts the effect of different values of the policy age on the quasi-renewal function for the normal quasi-renewal model with exponential repair/PM times. Figure 46 also displays some huge jumps during the beginning time period similar to Figure 40 except that the jump appears for all values of T_a. In fact, several overflow warnings are received while the data are being generated from our model. This is possibly the result of the numerical error or the transient effect. After this troublesome period, the plot appears to be smooth without major noise and can be used for comparison.

Despite the noise, the effect of various policy age values is similar to that of the previous model (Figure 40), whose operating times and repair/PM times are normally distributed. That is a smaller T_a value causes the plot to reach the limiting value sooner than a larger T_a value because the number of restarts occurs more often.

Since there is significant noise presented in the data, we are cautious about the correctness of the data for our base example. Therefore, we provide no further investigation into the impact of other parameters on the quasi-renewal function for this model.

4.6.2.2 Availability Function

Figure 47 - Figure 52 display the availability plot for the normal quasi-renewal model with exponential repair/PM times where T_a =25, 50, 75, 100, 200, and ∞, respectively. Every figure exhibits some noise in the neighborhood of $t = 100$. Several overflow warnings are also received while the data are being generated. This causes unusable data for $t < 200$ in most cases.

Figure 53 combines the plots from Figure 47 to Figure 52 and displays a familiar behavior when the noisy transient period is disregarded. That is, a smaller T_a value creates more interruptions and therefore the overall availability is lower and loses its accuracy sooner than others with larger T_a value. Moreover, a smaller T_a value seems to create more visible dents on the availability plot than a larger T_a value. In addition, the plots from $T_a = 200$ and $T_a = \infty$ are almost identical since the probability of a PM action is almost zero when $T_a = 200$.

Since there is significant noise presented in the data, we are cautious about the correctness of the data especially for our base example. Therefore, we provide no further investigation into the impact of other parameters on availability for this model.

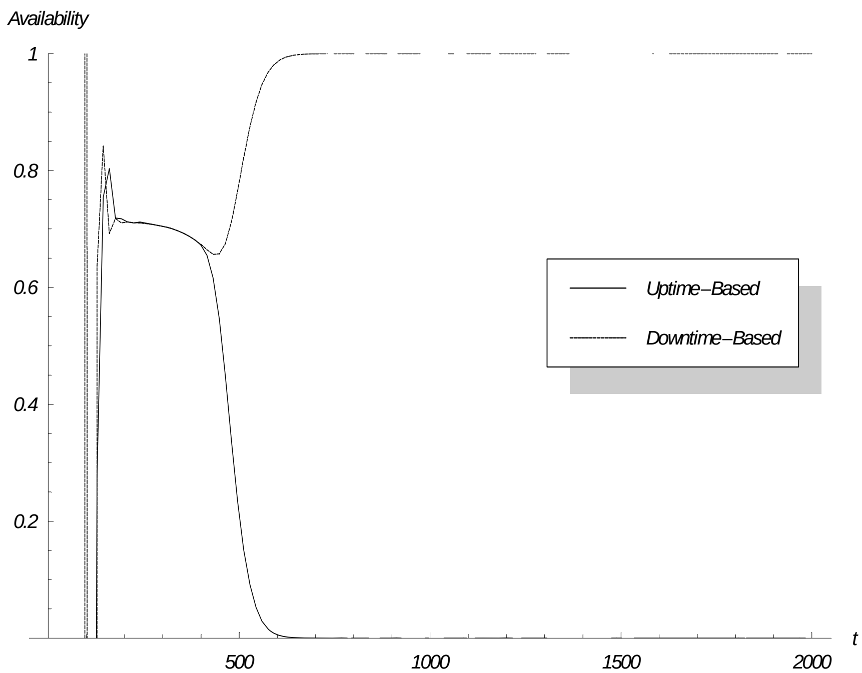

Figure 47 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating intervals and exponential repair and PM intervals using parameters μ = 100, $\sigma = 25$, ρ = 0.05, γ = 0.1, α = 0.9, β_p =1 , β_r =1, and T_a = 25.

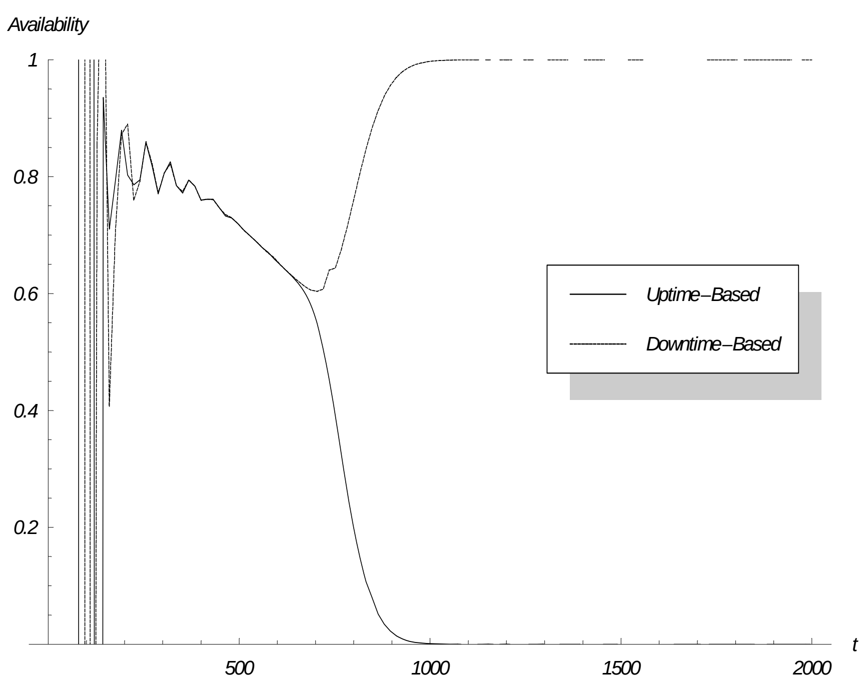

Figure 48 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating intervals and exponential repair and PM intervals using parameters μ = 100, $\sigma = 25$, ρ = 0.05, γ = 0.1, α = 0.9, β_p =1 , β_r =1, and T_a = 50.

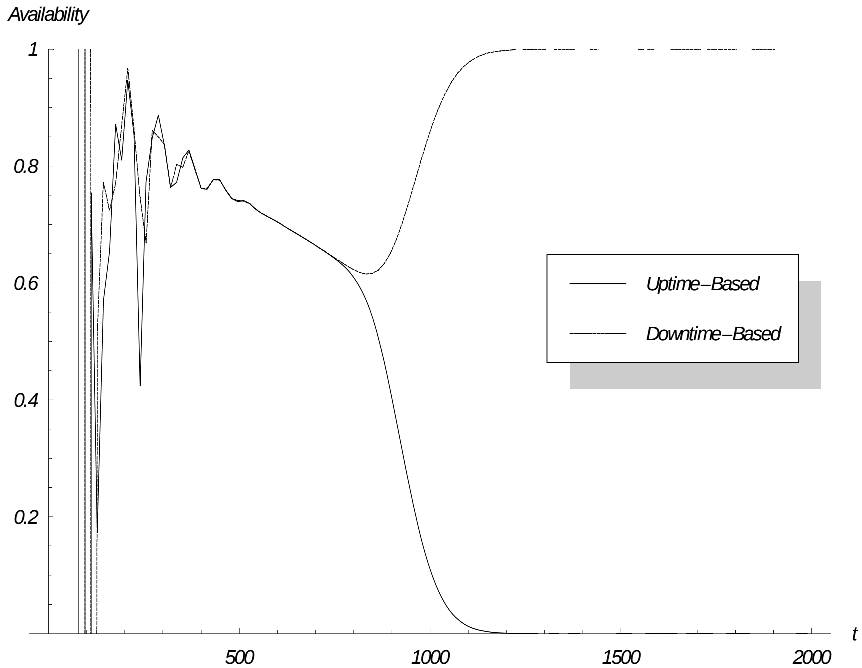

Figure 49 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating intervals and exponential repair and PM intervals using parameters μ = 100, $\sigma = 25$, ρ = 0.05, γ = 0.1, α = 0.9, β_p =1 , β_r =1, and T_a = 75.

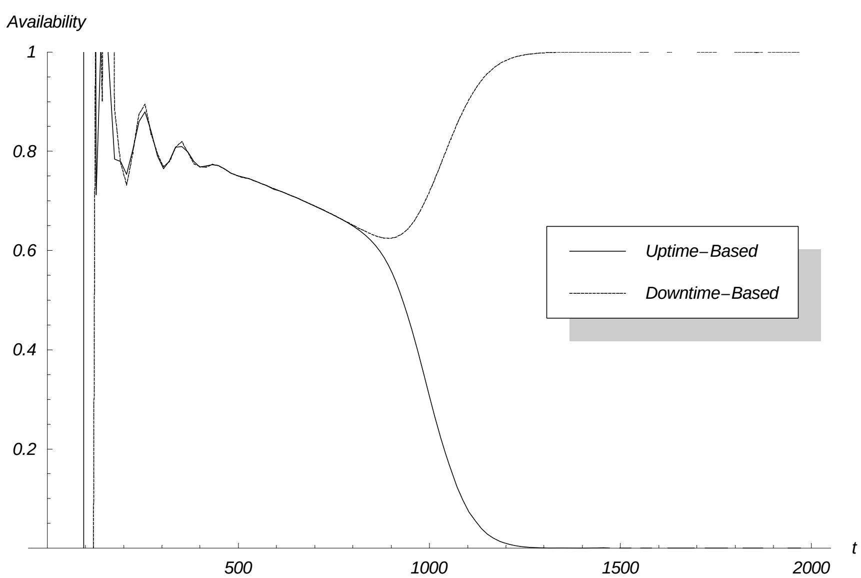

Figure 50 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating intervals and exponential repair and PM intervals using parameters μ = 100, $\sigma = 25$, ρ = 0.05, γ = 0.1, α = 0.9, β_p =1 , β_r =1, and T_a = 100.

Figure 51 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating intervals and exponential repair and PM intervals using parameters μ = 100, $\sigma = 25$, ρ = 0.05, γ = 0.1, α = 0.9, β_p =1 , β_r =1, and T_a = 200.

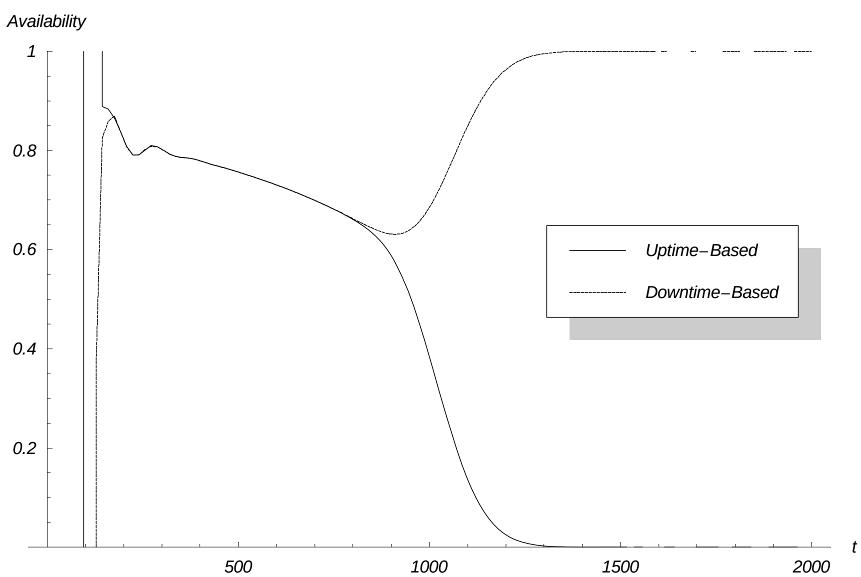

Figure 52 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating intervals and exponential repair and PM intervals using parameters μ = 100, $\sigma = 25$, ρ = 0.05, γ = 0.1, α = 0.9, β_p =1 , β_r =1, and $T_a = \infty$ (no PM).

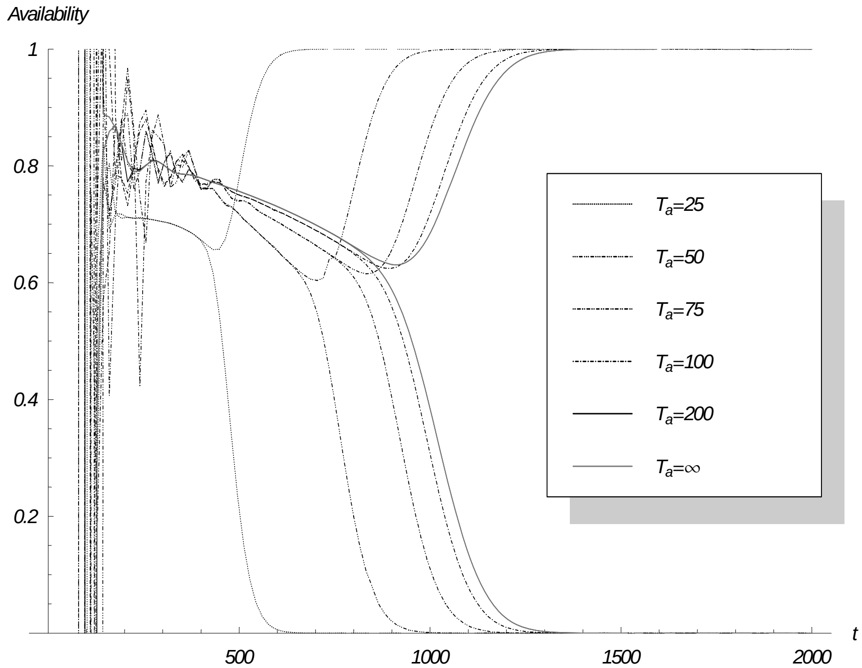

Figure 53 The plot of availability functions approximated using 14 terms under the quasi-renewal model with normal operating intervals and exponential repair and PM intervals using parameters μ = 100, $\sigma = 25$, ρ = 0.05, γ = 0.1, α = 0.9, β_p =1 , β_r =1, and T_a =25, 50, 75, 100, 200, ∞.

Chapter 5 Conclusions

This dissertation extends the work done by Murdock [1] and Rehmert [2] by providing the expression for the quasi-renewal function and the point availability functions for models described by a quasi-renewal process integrated with an age dependent preventive maintenance policy. A non-stationary process is more realistic and more suitable for modeling a system sample path in many real-world situations than the traditional renewal process because many systems do not return to a brand-new condition after repair. Although a small number of studies have defined their models using non-homogeneous processes, most of them describe the repair using the imperfect repair assumption, which does not reflect the deterioration process of the system. A quasi-renewal process is more realistic than the imperfect repair model in that regard. Our research study is one of a few studies that use the quasi-renewal process in the models. Even fewer studies whose models are described by the quasi-renewal process incorporate a preventive maintenance policy, which gains more and more interest in the reliability research arena. Our research is unique since we provide the point availability analysis for such models with an age dependent preventive maintenance policy. The point availability is the most informative measure of availability because it describes the actual behavior of the system, but it is often difficult to evaluate. This measure is important for any application where system availability for each point in time is crucial such as applications in health care or military field operations. Moreover, other availability measures can be easily derived from the point availability measure.

In this research, an expression for the quasi-renewal function and two point availability expressions are developed. The quasi-renewal function essentially represents the expected number of restarts for each point in time and is frequently used in a cost optimization problem. The two point availability expressions are equivalent and derived from two equivalent notions of availability. Some numerical examples are provided to illustrate the behavior of the models. No optimization problem is discussed in this research.

All expressions that we develop contain an infinite sum. To obtain numerical results, we must truncate the infinite sum to acquire an approximation. Interestingly, the two approximated availability expressions form upper and lower bounds on the true value of the point availability. The availability is accurate only when the bounds meet or get very close to each other and loses its accuracy when the bounds diverge. The analysis shows that the contribution from each term in the sum is localized and moving to the right as the term index gets higher. This suggests that if we are interested in the availability for a small enough period of time, only a certain number of terms in the sum are needed to obtain the point availability with a high degree of accuracy. One may arbitrarily increase the number of terms to expand the period where accuracy is needed. In fact, the truncation at the lower end as well as the upper end of the infinite sum can also be done to evaluate a segment of the point availability plot with a high degree of accuracy. This is because the contribution from each term is localized and hence only a certain number of terms are needed for such an evaluation. However, one must further investigate a method of identifying the needed terms given the desired segment of the point availability and the degree of accuracy.

Our quasi-renewal function and the point availability expressions are written as the sum of Laplace transform terms. To obtain numerical values from these expressions, the Laplace transform terms must be inverted. While some terms can be inverted symbolically, most of them are too complicated and numerical inversion is needed. We employ a numerical Laplace transform inversion routine called *GWR*, which is developed by Valkó and Abate [62], in our computation. However, numerical inversion of a complicated Laplace transform is computationally intensive and time consuming. To minimize the computation time as well as the error posed by the inversion algorithm, we find the inverse of these terms algebraically whenever possible and numerically otherwise.

The numerical examples are obtained for models where the operating, repair, and PM times are distributed according to various theoretical distributions. The impact of model parameters on the model behavior is analyzed and discussed. The parameters of interest are the deterioration rate for the operating time quasi-renewal process (α), the deceleration rate for the repair and PM time quasi-renewal process (β_r and β_p respectively), the policy age (T_a), and other parameters that determine the mean of the operating times, the repair times, and the PM times. In most cases, the effect of these parameters is predictable. For example, when the deterioration rate or the deceleration rate increases, the availability decreases. When the mean operating time decreases or when the mean repair or PM times increases, the availability also decreases. When the policy age decreases, the availability drops. When the policy age is set to infinity, the point availability plot matches Rehmert's [2].

Any computation must involve some type of numerical round-off errors because of the finite discrete nature of computer memory. Most of the time, these errors are unnoticeable but sometimes they are amplified and appear as noise in the results. For the exponential and gamma distribution models, the noise is small and can be ignored. However, this is not the case for the normal distribution model.

For the quasi-renewal model with normally distributed operating, repair, and PM times, we are able to find the closed form for the point availability expression when $t < T_a$. Our plot coincides with Rehmert's [2] plot and shows no sign of noise. However, the point availability expression when $t \geq T_a$ cannot be written in a closed form and numerical Laplace transform inversion is necessary. The algorithm produces the results in the base example that contain so much noise that no useful information can be concluded. Nevertheless, when the policy age increases, the noise appears less. In fact, when $T_a = 200$ (which is unrealistic), the noise seems to disappear and the plot practically coincides with Rehmert's [2] plot, which indicates that our model correctly behaves as if there was no PM policy.

For the quasi-renewal model with normally distributed operating and exponential repair/PM times, we are not able to find a closed form of the availability expression. Even Rehmert [2] does not present any numerical results for this model because the numerical inversion routine that he uses is incapable of handle such a complicated Laplace transform expression. Our inversion routine, however, is able to produce some results although the numerical errors are so severe during the starting time period that no useful observation can be made. After the initial major noise, the plot appears to be smooth and the noise is less noticeable. Regardless of the initial noise, the model exhibits a familiar behavior when we vary the value of the policy age. That is, the overall availability increases as the policy age increases.

Because a large number of significant errors are presented in the data for the normal distribution model, we are cautious about the correctness of the rest of the data, especially for the base example. Therefore we provide no further investigation on the impact of other parameters on the quasi-renewal function and availability function.

We assume the PM action does not improve the system condition. However, the mean PM times are half of the mean repair times. Any improvement in the availability from our model over the non-PM model is merely the result from the fact that the PM times are probabilistically shorter than the repair times. The improvement can be seen in the gamma distribution models but not in the exponential distribution models because gamma distribution puts more weight on the probability of PM cycles. In most cases, the improvement does not occur. In fact, the overall availability from a model with an age-dependent PM policy is lower than a model without PM policy. However, we should also consider the fact that usually a PM action costs significantly less than a repair action and can be performed at convenient times (which usually are the times where the damaging cost of shutting down the system is minimal). The trade-off between cost and availability must be made to find an optimal PM policy. Moreover, the assumption that the PM action does not improve the system working condition may not be realistic in many situations. We should see more availability improvement when this assumption is changed in a more optimistic direction.

Chapter 6 Future Research

A non-homogeneous process provides a more realistic model for repairable system behavior than traditional models where a repair is assumed to be either perfect or minimal. In this dissertation, we construct a quasi-renewal process model with an age-dependent policy and provide the expressions for the quasi-renewal function and the point availability function. There are other non-stationary processes such as Kijima's virtual age models [21] that can be used instead of the quasi-renewal process in the availability model, which would make an interesting future research study.

Our model assumes that the PM action does not improve system working condition. One may extend this model by assuming that the PM action does improve the system. Any age improvement after PM concept such as Malik [38] and Nakagawa [40] would be a good candidate. It may be more appropriate to use an age improvement PM concept in conjunction with a virtual age model. Instead of an age-dependent PM policy, other types of preventive maintenance policy can also be used such as a periodic PM policy and other policies presented in Section 2.2.3.

Our research provides mathematical expressions that can be used in an optimization problem. For example, the quasi-renewal function, which represents the number of expected restarts, can be used to construct a cost model. Moreover, we have seen that there is a trade-off between cost and availability in maintenance planning. Constructing an optimization problem to determine an optimal PM policy for our model would make an interesting research project. Usually, when both cost and availability are considered in an optimization problem, we either minimize the cost while keeping the availability above a certain requirement or maximize the availability while keeping the cost under a certain budget. In any case, determining the optimal PM policy for our model implies that the policy age T_a is the decision variable. To determine the optimal policy age, one should limit the search range to the time period between 0 and the mean system operating time to help speed up the search.

Using our model in a real-world application requires an appropriate means to find the suitable parameters. Moreover, obtaining numerical results is computationally intensive and time consuming, which may not be practical in the real-world application. One may investigate a way to make a practical application using the results from our research. One way to reduce the computing time is to reduce the number of terms in the approximated sum by truncating both the lower and the upper ends of the infinite sum as described in the third paragraph of page 131. One still needs to construct an algorithm for identifying the needed terms given the desired segment of the availability plot and degree of accuracy. Another way is to use a curve fitting technique with the data points from the availability plot to approximate the point availability function with a (piecewise) polynomial. The method is suitable for applications that require repeated availability data retrieval of a previously known model with known parameters. The data retrieval avoids the Laplace transform inversion altogether and should be relatively fast. Such an application includes Response Surface Methodology (RSM). A well-known textbook on RSM is written by Myers and Montgomery [68].

Additionally, in some real-world situations, there may be different types of repair and consequently the distribution on the repair time of each type is different. To incorporate this type of situation into our model, we may construct a mixture of the distribution functions consisting of the sum of all repair distribution functions where each term is weighted by the probability of the corresponding type of repair. The mixture distribution then can be used as the repair distribution function in our model. However, the availability expressions will be more complicated and harder to evaluate than that of the model with just one type of repair.

The numerical results presented in this dissertation are obtained from exponential, gamma, and normal distribution models. However, Weibull distribution is the most used distribution in reliability, but its Laplace transform is difficult to obtain. Rehmert [2] suggests using Lomnicki's [69] method for approximating the Weibull distribution.

Rehmert [2]'s normal distribution model availability function can be written in a closed form and does not require numerical Laplace transform inversion. Since the normal distribution may be used to approximate other distribution under certain conditions, the ease of use of the normal availability expression may be exploited. However, the numerical results from our normal distribution model show too many major numerical errors in the data. If this issue can be resolved, our normal availability expression may be exploited in the same way.

Laplace transform inversion is an important part of this research. Unfortunately, the inversion process is difficult for a complicated Laplace transform expression. Numerical inversion for such a complex expression requires a sizable amount of computing time and computer power. Moreover, our current inversion routine cannot handle the normal distribution model very well because too many major errors occur in the data. A mainframe computer may help reduce the time and obtain a better result. In addition, a better inversion routine that reduces the effect of machine round-off error and increases the computation speed will help improve the results from our model.

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Appendix

A. 1. The construction of expression (3.6.6)

From (3.6.5),

$$f_{cc,T_k}(t) = \left(1 - u_{T_a}(t)\right)\frac{1}{\alpha^{k-1}\sigma\sqrt{2\pi}}e^{-\frac{\left(t-\alpha^{k-1}\mu\right)^2}{2\left(\alpha^{k-1}\sigma\right)^2}}.$$

For simplicity, let

$$\mu_k := \alpha^{k-1}\mu \text{ and } \sigma_k := \alpha^{k-1}\sigma. \qquad \text{(A.1.1)}$$

Hence, (3.6.5) becomes

$$f_{cc,T_k}(t) = \left(1 - u_{T_a}(t)\right)\frac{1}{\sigma_k\sqrt{2\pi}}e^{-\frac{\left(t-\mu_k\right)^2}{2\sigma_k{}^2}}. \qquad \text{(A.1.2)}$$

Since the mean and variance parameters are chosen so that the probability mass for $t < 0$ is negligible, the Laplace transform of $f_{cc,T_k}(t)$ can be expressed as

$$\begin{aligned}
f^*_{cc,T_k}(s) &= \int_{-\infty}^{\infty} e^{-st} f_{cc,T_k}(t)dt \\
&= \int_{-\infty}^{\infty} e^{-st}\left(1 - u_{T_a}(t)\right) f_{T_k}(t)dt \\
&= \int_{-\infty}^{T_a} e^{-st} f_{T_k}(t)dt \\
&= \int_{-\infty}^{\infty} e^{-st} f_{T_k}(t)dt - \int_{T_a}^{\infty} e^{-st} f_{T_k}(t)dt
\end{aligned}$$

Therefore,

$$f^*_{cc,T_k}(s) = e^{-s\mu_k + \frac{1}{2}s^2\sigma_k{}^2} - \frac{1}{\sigma_k\sqrt{2\pi}}\int_{T_a}^{\infty} e^{-\left(\frac{\left(t-\mu_k\right)^2}{2\sigma_k{}^2}+st\right)}dt \qquad \text{(A.1.3)}$$

Consider the term $\frac{\left(t-\mu_k\right)^2}{2\sigma_k{}^2}+st$ in (A.1.3):

$$\begin{aligned}
\frac{\left(t-\mu_k\right)^2}{2\sigma_k{}^2}+st &= \frac{t^2-2\mu_k t+\mu_k{}^2+2\sigma_k{}^2 st}{2\sigma_k{}^2} \\
&= \frac{1}{2\sigma_k{}^2}\left(t^2+2\left(-\mu_k+\sigma_k{}^2 s\right)t+\mu_k{}^2\right) \\
&= \frac{1}{2\sigma_k{}^2}\left(t^2+2\left(-\mu_k+\sigma_k{}^2 s\right)t+\left(-\mu_k+\sigma_k{}^2 s\right)^2+\mu_k{}^2-\left(-\mu_k+\sigma_k{}^2 s\right)^2\right) \\
&= \frac{1}{2\sigma_k{}^2}\left(\left(t-\mu_k+\sigma_k{}^2 s\right)^2+\mu_k{}^2-\left(\mu_k{}^2-2\mu_k\sigma_k{}^2 s+\sigma_k{}^4 s^2\right)\right) \\
&= \frac{\left(t-\mu_k+\sigma_k{}^2 s\right)^2}{2\sigma_k{}^2}+s\mu_k-\frac{1}{2}s^2\sigma_k{}^2
\end{aligned}$$

Hence (A.1.3) can be written as

$$\begin{aligned}
f^*_{cc,T_k}(s) &= e^{-s\mu_k+\frac{1}{2}s^2\sigma_k{}^2}-\frac{1}{\sigma_k\sqrt{2\pi}}\int_{T_a}^{\infty}e^{-\left(\frac{\left(t-\mu_k+\sigma_k{}^2 s\right)^2}{2\sigma_k{}^2}+s\mu_k-\frac{1}{2}s^2\sigma_k{}^2\right)}dt \\
&= e^{-s\mu_k+\frac{1}{2}s^2\sigma_k{}^2}-\frac{e^{-s\mu_k+\frac{1}{2}s^2\sigma_k{}^2}}{\sigma_k\sqrt{2\pi}}\int_{T_a}^{\infty}e^{-\frac{\left(t-\mu_k+\sigma_k{}^2 s\right)^2}{2\sigma_k{}^2}}dt
\end{aligned}$$

Therefore,

$$f^*_{cc,T_k}(s)=e^{-s\mu_k+\frac{1}{2}s^2\sigma_k{}^2}\left(1-\frac{1}{\sigma_k\sqrt{2\pi}}\int_{T_a}^{\infty}e^{-\frac{\left(t-\mu_k+\sigma_k{}^2 s\right)^2}{2\sigma_k{}^2}}dt\right) \tag{A.1.4}$$

Let $x=\frac{t-\mu_k+\sigma_k{}^2 s}{\sqrt{2}\sigma_k}$. Hence $dx=\frac{dt}{\sqrt{2}\sigma_k}$ or $\sqrt{2}\sigma_k dx=dt$. $t=\infty$ implies $x=\infty$ and $t=T_a$ implies $x=\frac{T_a-\mu_k+\sigma_k{}^2 s}{\sqrt{2}\sigma_k}$. Therefore, by the change of variable technique, (A.1.4) can be written as

$$
\begin{aligned}
f^*_{cc,T_k}(s) &= e^{-s\mu_k + \frac{1}{2}s^2\sigma_k{}^2}\left(1 - \frac{1}{\sqrt{\pi}}\int_{\frac{T_a - \mu_k + \sigma_k{}^2 s}{\sqrt{2}\sigma_k}}^{\infty} e^{-x^2}\,dx\right) \\
&= e^{-s\mu_k + \frac{1}{2}s^2\sigma_k{}^2}\left(1 - \frac{1}{2}\operatorname{erfc}\left(\frac{T_a - \mu_k + \sigma_k{}^2 s}{\sqrt{2}\sigma_k}\right)\right) \\
&= \frac{1}{2}e^{-s\mu_k + \frac{1}{2}s^2\sigma_k{}^2}\left(2 - \operatorname{erfc}\left(\frac{T_a - \mu_k + \sigma_k{}^2 s}{\sqrt{2}\sigma_k}\right)\right)
\end{aligned}
$$

Therefore,

$$
f^*_{cc,T_k}(s) = \frac{1}{2}e^{-s\mu_k + \frac{1}{2}s^2\sigma_k{}^2}\left(1 + \operatorname{erf}\left(\frac{T_a - \mu_k + \sigma_k{}^2 s}{\sqrt{2}\sigma_k}\right)\right) \tag{A.1.5}
$$

Moreover, since the standard normal cdf $\Phi(x)$ can be expressed as

$$
\Phi(x) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right), \tag{A.1.6}
$$

we can also express (A.1.5) as

$$
f^*_{cc,T_k}(s) = e^{-s\mu_k + \frac{1}{2}s^2\sigma_k{}^2}\,\Phi\left(\frac{T_a - \mu_k + \sigma_k{}^2 s}{\sigma_k}\right). \tag{A.1.7}
$$

After plugging (A.1.1) into (A.1.5) and (A.1.7), we obtain the expression (3.6.6).

A. 2. Mathematica Source Code for GWR Routine

```
(* :Name: User`NumericalLaplaceInversion` *)

(* :Title: Numerical Inversion of Laplace Transform with
Multiple Precision*)

(* :Author: Peter P. Valko and Joe Abate*)

(* :Summary:
This package provides only one function: GWR. The function
calculates the value of the inverse of a Laplace transform
at a specified time point. The Laplace transform should be
provided as a function ready for multiple-precision
evaluation. In other words, approximate numbers (with
decimal point) or Mathematica functions starting with the
letter 'N' are not allowed.
*)

(* :Context: User`NumericalLaplaceInversion` *)

(* :Package Version: 1.0 *)

(* :Copyright: Copyright 2002, Peter P. Valko and Joe
Abate*)

(* :History: Originally written by Peter P. Valko, Dec 01,
2002. *)

(* :Keywords:
Laplace transform, Numerical inversion, Multiple-precision,
Gaver functional, Wynn-Rho algorithm
*)

(* :Source:
Valkó, P. P. and Abate, J.:  Comparison of Sequence
Accelerators for the Gaver Method of Numerical Laplace
Transform Inversion,  Computers & Mathematics with
Applications, (2002), accepted for publication (CAM 5307)
*)

(* :Warnings: None. *)

(* :Mathematica Version: 4.1 *)

(* :Limitations:
The Laplace transform function should be given in a form,
that permits multiple precision evaluation.
*)

(* :Discussion:
```

```
The basic difference between this package and all other
Mathematica packages aimed to do the same job is the
systematic use of multiple precision arithmetics. The
algorithm is also new.
*)

BeginPackage["User`NumericalLaplaceInversion`"]

GWR::usage =

"GWR[F, t, M, precin ] gives the inverse of the Laplace
transform function named 'F' for a given time point 't'. The
method involves the calculation of 'M' terms of the Gaver-
functional. The obtained series is accelerated using the
Wynn-Rho convergence acceleration scheme. The precision of
internal calculations is set to 'precin'.
\n
\n
GWR[F, t, M ] does the same, but the precision of the
internal calculations is selected automatically:  precin =
2.1 M).
\n
\n
GWR[F, t] uses M = 32 terms and precin = 67 as defaults. It
should give reasonable results for many problems.
\n
\n
Important note: The Laplace transform should be degfined as
a function of one argument. It can involve anything from a
simple Mathematica expression to a sophisticated Module.
Since the Laplace transform will be evaluated with non-
standard (multiple) precision, approximate numbers (with
decimal point) or Mathematica functions starting with the
letter 'N' are not allowed in the function definition.
\n
\n
Example usage:
\n
\n
fun[s_]=(1/s) Exp[-Sqrt[ (s^2 + (37/100) s + 1)/(s^2 + s +
Pi)]]
\n
t0=100
\n
GWR[fun,t0]
"
Unprotect[GWR];

Begin["User`NumericalLaplaceTransformInversion`Private`"]
GWR[F , t , M :32, precin :0] := Module[
    {M1, G0, Gm, Gp, best, expr, \[Tau] = Log[2]/t, Fi,
broken, prec},
    If[precin <= 0, prec = 21 M/10, prec = precin];
    If[prec <= $MachinePrecision, prec = $MachinePrecision];
```

```
    broken = False;

If[Precision[\[Tau]]<prec,\[Tau]=SetPrecision[\[Tau],prec]];
    Do[Fi[i] = N[F[i \[Tau]], prec], {i, 1, 2 M}];
    M1 = M;
    Do[
      G0[n - 1] = \[Tau](2n)!/(n!(n - 1)!)Sum[
            Binomial[n, i](-1)^i  Fi[n + i], {i, 0, n}];
      If[Not[NumberQ[G0[n - 1]]], M1 = n - 1; G0[n - 1] =.;
Break[]];
      , {n, 1, M}];
    Do[Gm[n] = 0, {n, 0, M1}];
    best = G0[M1 - 1];

    Do[
      Do[
        expr = G0[n + 1] - G0[n];
        If[Or[Not[NumberQ[expr]], expr == 0], broken = True;
Break[]];
        expr = Gm[n + 1] + (k + 1)/expr;
        Gp[n] = expr;
        If[OddQ[k],
          If[n == M1 - 2 - k, best = expr]
          ];
        , {n, M1 - 2 - k, 0, -1}];
      If[broken, Break[]];
      Do[Gm[n] = G0[n]; G0[n] = Gp[n], {n, 0, M1 - k}];
      , {k, 0, M1 - 2}];
    best
    ]

End[]  (* User`NumericalLaplaceTransformInversion` *)

Protect[GWR];

EndPackage[] (* User`NumericalLaplaceInversion` *)
```

A. 3. An Example of Mathematica Source Code for the Exponential Distribution Model

```
Off[General::spell1];
<<NumericalMath`ComputerArithmetic`
SetArithmetic[10,10, RoundingRule→RoundToInfinity,
ExponentRange→{-1000,1000}, MixedMode→False,
IdealDivide→False]
(* Call numerical inversion of Laplace transform package.
The package provides only one function called GWR*)
thisDir=ToFileName[

("FileName"/.NotebookInformation[EvaluationNotebook[]])
        [[1]]];
SetDirectory[thisDir];
<<NumericalLaplaceInversion.m
<<Graphics`MultipleListPlot`
 Ta =50;
 α = 9/10;
 βr=1;
 βp=1;
 (* parameter for exponential operating time *)
 λ = 1/100;
 (* parameter for exponential  preventive time *)
 γ = 1/10;
 (* parameter for exponential  repair time *)
 ρ = 1/20;
 (* number of terms *)
K =2;
numTerms = 15;
 Date[]
 (* Density and Distribution Functions *)
 F[t_, 1] := 1 - 1/ⅇ^(λ t);
 F[t_, k_] := F[t/α^(k-1), 1];
 Fbar[t_, k_] := 1 - F[t, k];
 f[t_, 1] := λ/ⅇ^(λ t);
 f[t_, k_] := f[t/α^(k-1), 1]/α^(k-1);
```

$$\texttt{fcc[t\_, k\_] := (1 - UnitStep[t - Ta]) f[t, k];}$$

$$\texttt{fc[t\_, k\_] :=} \frac{\texttt{fcc[t, k]}}{\texttt{F[Ta, k]}};$$

$$\texttt{Fc[t\_, k\_] := UnitStep[t - Ta] + (1 - UnitStep[t - Ta])} \frac{\texttt{F[t, k]}}{\texttt{F[Ta, k]}};$$

$$\texttt{Fcbar[t\_, k\_] := 1 - Fc[t, k];}$$

$$\texttt{Fccbar[t\_, k\_] := (1 - UnitStep[t - Ta]) Fbar[t, k];}$$

$$\texttt{fcStar[s\_, 1] :=} \frac{\lambda\ (1 - e^{-(s+\lambda)\ Ta})}{(1 - e^{-\lambda\, Ta})\ (s + \lambda)};$$

$$\texttt{fcStar[s\_, k\_] :=} \frac{\lambda\ \left(1 - e^{-\left(s+\alpha^{-k+1}\lambda\right)\ Ta}\right)}{\left(1 - e^{-\alpha^{-k+1}\lambda\, Ta}\right)\ (\alpha^{k-1}\, s + \lambda)};$$

$$\texttt{fccStar[s\_, 1] :=} \frac{\lambda\ (1 - e^{-(s+\lambda)\ Ta})}{(s + \lambda)};$$

$$\texttt{fccStar[s\_, k\_] :=} \frac{\lambda\ \left(1 - e^{-\left(s+\alpha^{-k+1}\lambda\right)\ Ta}\right)}{(\alpha^{k-1}\, s + \lambda)};$$

$$\texttt{FcbarStar[s\_, k\_] :=} \frac{1}{s}\ \texttt{(1 - fcStar[s, k]);}$$

$$\texttt{FccbarStar[s\_, 1] :=} \frac{1 - e^{-(s+\lambda)\ Ta}}{s + \lambda};$$

$$\texttt{FccbarStar[s\_, k\_] :=} \frac{1 - e^{-\left(s + \frac{\lambda}{\alpha^{k-1}}\right)\ Ta}}{s + \frac{\lambda}{\alpha^{k-1}}};$$

$$\texttt{FbarStar[s\_, k\_] :=} \frac{1}{s + \alpha^{-k+1}\, \lambda};$$

$$\texttt{grStar[s\_, k\_] :=} \frac{\beta r^{-k+1}\, \rho}{s + \beta r^{-k+1}\, \rho};$$

$$\texttt{oneMinusgrStar[s\_, k\_] :=} \frac{s}{s + \beta r^{-k+1}\, \rho};$$

$$\texttt{gpStar[s\_, k\_] :=} \frac{\beta p^{-k+1}\, \gamma}{s + \beta p^{-k+1}\, \gamma};$$

$$\texttt{oneMinusgpStar[s\_, k\_] :=} \frac{s}{s + \beta p^{-k+1}\, \gamma};$$

$$\text{hmvkStar}[s\_,\ k\_] := \frac{\beta r^{-k+1}\,\lambda\,\rho\,\left(1 - e^{-\left(s+\alpha^{-k+1}\lambda\right)\,Ta}\right)}{\left(\alpha^{k-1}\,s+\lambda\right)\,\left(s+\beta r^{-k+1}\,\rho\right)} + \frac{\beta p^{-k+1}\,\gamma\,e^{-\left(s+\alpha^{-k+1}\lambda\right)\,Ta}}{s+\beta p^{-k+1}\,\gamma};$$

$$\text{hm}\theta\text{nStar}[s\_,\ n\_] := \prod_{k=1}^{n} \text{hmvkStar}[s,\ k];$$

$$\text{hrvkStar}[s\_,\ 1] := \frac{\lambda\,\rho}{(s+\lambda)\,(s+\rho)};$$

$$\text{hrvkStar}[s\_,\ k\_] := \frac{\lambda\,\rho}{\left(\alpha^{k-1}\,s+\lambda\right)\,\left(\beta r^{k-1}\,s+\rho\right)};$$

$$\text{hmUnStar}[s\_,\ n\_] := \prod_{k=1}^{n} \text{hrvkStar}[s,\ k];$$

$$\text{quasiRenFnStar1}[s\_] := \frac{1}{s}\sum_{k=1}^{\text{numTerms}} \text{hmUnStar}[s,\ k];$$

$$\text{quasiRenFnStar2}[s\_] := \frac{1}{s}\sum_{k=1}^{\text{numTerms}} \text{hm}\theta\text{nStar}[s,\ k];$$

```
(* Preparing Plot *)
points = Table[8 x, {x, 0, 250}];
```

$$\text{points}[[1]] = \frac{1}{100};$$

```
(* number of terms*)
terms = {2, 4, 6, 8, 10, 12, 14};
numPoints = Length[points];
(*********** Calculating uptime-based availability
***********)
 (* Create Table of seven empty entries *)
AUpTime = Table[0, {7}];
AUpTimeTerms = Table[0, {14}];
(* Populate the data for ploting *)
startTime = TimeUsed[];

(* Evaluate the first term *)
myTable= Table[0,{numPoints}];
Do[
    t=points[[j]];
    If[t < Ta,
      myTable[[j]] = Fbar[t,1];,
      myTable[[j]] = 0;
      ]
    , {j, numPoints}
    ];
AUpTimeTerms[[1]] = myTable;

(* Evaluate the remaining terms *)
Do[
    myTable= Table[0,{numPoints}];
    sumTermsInAStar1[s_] :=FbarStar[s,n]  hmUnStar[s, n-1] ;
```

```
    sumTermsInAStar2[s_] :=FccbarStar[s,n]  hmθnStar[s, n-1]
;
    Do[
      t=points[[j]];
      If[t < Ta,
        myTable[[j]] = GWR[sumTermsInAStar1,t];,
        myTable[[j]] = GWR[sumTermsInAStar2,t];
        ]
      , {j, numPoints}
      ];
    AUpTimeTerms[[n]] = myTable;
    ,{n, 2,14}
    ];

(* Evaluate the approximation with two terms *)
myTable1 = AUpTimeTerms[[1]];
myTable2 = AUpTimeTerms[[2 ]];
myTable3 = Table[0,{numPoints}];
Do[
    myTable3[[j]] =  myTable1[[j]]+ myTable2[[j]];
    , {j, numPoints}
    ];
AUpTime[[1]] = myTable3;


(* Evaluate the approximation with more than two terms *)
Do [
    myTable = AUpTime[[n - 1]];  (* cumulative sum upto 2
x(n-1) terms *)
    myTable1 = AUpTimeTerms[[2 n - 1]];
    myTable2 = AUpTimeTerms[[2 n ]];
    myTable3 = Table[0,{numPoints}];
    Do[
      myTable3[[j]] = myTable[[j]]+ myTable1[[j]]+
myTable2[[j]];
      , {j, numPoints}
      ];
    AUpTime[[n]] = myTable3;
    , {n,2,7}
    ];

endTime = TimeUsed[];
(* Display the time usage in seconds*)
Print["CPU Time Usage: ",SetPrecision[endTime-startTime,5],
" seconds"];
 (*********** Calculating downtime-based availability
***********)
 (* Create Table of seven empty entries *)
ADownTime = Table[0, {7}];
ADownTimeTerms = Table[0, {14}];
startTime = TimeUsed[];
```

```
(* Evaluate the first term *)
myTable = Table[0, {numPoints}];
(* For t < Ta *)
If[λ ≠ ρ,
 firstTerm1[t_] := - λ (ⅇ^(-ρ t) - ⅇ^(-λ t)) / (ρ - λ) ;,
 firstTerm1[t_] := λ t ⅇ^(-λ t);
]
(* For t ≥ Ta *)
If[λ ≠ ρ,
 firstTerm2[t_] :=
   - λ (ⅇ^(-ρ t) - ⅇ^(-λ t)) / (ρ - λ)  +  ( λ ⅇ^(- λ Ta) (ⅇ^(-ρ (t - Ta)) - ⅇ^(-λ (t-Ta))) / (ρ - λ) + ⅇ^(-γ (t - Ta) - λ Ta) );,
 firstTerm2[t_] := λ Ta ⅇ^(-λ t) + ⅇ^(-γ (t - Ta) - λ Ta);
]
Do[
   t=points[[j]];
   If[t < Ta,
     myTable[[j]] = firstTerm1[t];,
     myTable[[j]] = firstTerm2[t] ;
     ];
   , {j, numPoints}
   ];
ADownTimeTerms[[1]] = myTable;
```

```
 (* Evaluate the remaining terms *)
 Do[
   myTable = Table[0, {numPoints}];
   sumTermsInAStar1[s_] := βr^(n-1) ρ^(n-1) λ^n ∏_{i=1}^{n} 1/((α^(i-1) s + λ) (βr^(i-1) s + ρ));
   sumTermsInAStar2[s_] :=
    1/s hmθnStar[s, n - 1]
     (fccStar[s, n] oneMinusgrStar[s, n] +
       Fbar[Ta, n] e^(-Ta s) oneMinusgpStar[s, n]);
   Do[
    t = points[[j]];
    If[t < Ta,
     myTable[[j]] = GWR[sumTermsInAStar1, t];,
     myTable[[j]] = GWR[sumTermsInAStar2, t];
    ];
    , {j, numPoints}
   ];
   ADownTimeTerms[[n]] = myTable;
   , {n, 2, 14}
  ];
 (* Evaluate the approximation with first two terms *)
myTable1 = ADownTimeTerms[[1]];
myTable2 = ADownTimeTerms[[2]];
myTable3 = Table[0,{numPoints}];
Do[
    myTable3[[j]] = 1-( myTable1[[j]]+ myTable2[[j]]);
    , {j, numPoints}
    ];
ADownTime[[1]] = myTable3;


(* Evaluate the approximation with more than two terms *)
Do [
    myTable = ADownTime[[ n -1]]; (* 1 - cumulative sum upto
2(n-1) terms *)
    myTable1 = ADownTimeTerms[[2 n - 1]];
    myTable2 = ADownTimeTerms[[2 n ]];
    myTable3 = Table[0,{numPoints}];
    Do[
      myTable3[[j]] = myTable[[j]]-( myTable1[[j]]+
myTable2[[j]]);
      , {j, numPoints}
      ];
    ADownTime[[n]] = myTable3;
    , {n,2,7}
```

```
    ];

endTime = TimeUsed[];

(* Display the time usage in seconds*)
Print["CPU Time Usage:",SetPrecision[endTime-startTime,5], "
seconds"];
 (*********** Calculating quasi-renewal function
***********)
QTable = Table[0, {numPoints}];
startTime = TimeUsed[];
Do[
    t=points[[m]];
    If[t < Ta,
      QTable [[m]] =  GWR[quasiRenFnStar1,t];,
      QTable [[m]] =  GWR[quasiRenFnStar2,t];
      ];
    , {m, numPoints}
    ];
endTime = TimeUsed[];

(* Display the time usage in seconds*)
Print["CPU Time Usage:",SetPrecision[endTime-startTime,5], "
seconds"];
 Print[SetPrecision[SessionTime[],5]];
Date[]
```

A. 4. An Example of Mathematica Source Code for the Gamma Distribution Model with Gamma Repair and PM Intervals

```
Off[General::spell1];
<<NumericalMath`ComputerArithmetic`
SetArithmetic[10,10, RoundingRule→RoundToInfinity,
ExponentRange→{-1000,1000}, MixedMode→False,
IdealDivide→False]
(* Call numerical inversion of Laplace transform package.
The package provides only one function called GWR*)
thisDir=ToFileName[

("FileName"/.NotebookInformation[EvaluationNotebook[]])
        [[1]]];
SetDirectory[thisDir];
<<NumericalLaplaceInversion.m
<<Graphics`MultipleListPlot`
 Ta = 50;
 α = 9/10;
 βr = 1;
 βp = 1;
 (* gamma parameters for operating time *)
 λ = 2/100;
 a = 2;
 (*  gamma parameters for repair time *)
 ρ = 2/20;
 b = 2;
 (* gamma parameters for preventive time *)
 γ = 2/10;
 c = 2;
 (* number of terms *)
K =2;
numTerms = 15;
 Date[]
```

(* Density and Distribution Functions *)

$$\text{Fbar[t\_, k\_]} := e^{-\alpha^{-k+1}\lambda t}\sum_{j=0}^{a-1}\frac{(\alpha^{-k+1}\lambda t)^{j}}{j!};$$

$$\text{FccbarStar[s\_, 1]} := \sum_{j=0}^{a-1}\left(\frac{\lambda^{j}}{(s+\lambda)^{j+1}}\left(1-e^{-(s+\lambda)\,Ta}\sum_{i=0}^{j}\frac{((s+\lambda)\,Ta)^{i}}{i!}\right)\right);$$

$$\text{FccbarStar[s\_, k\_]} := \sum_{j=0}^{a-1}\left(\frac{\alpha^{k-1}\lambda^{j}}{(\alpha^{k-1}s+\lambda)^{j+1}}\left(1-e^{-(s+\alpha^{-k+1}\lambda)\,Ta}\sum_{i=0}^{j}\frac{((s+\alpha^{-k+1}\lambda)\,Ta)^{i}}{i!}\right)\right);$$

$$\text{fccStar[s\_, 1]} := \left(\frac{\lambda}{s+\lambda}\right)^{a}\left(1-e^{-(s+\lambda)\,Ta}\sum_{i=0}^{a-1}\frac{((s+\lambda)\,Ta)^{i}}{i!}\right);$$

$$\text{fccStar[s\_, k\_]} := \left(\frac{\lambda}{\alpha^{k-1}s+\lambda}\right)^{a}\left(1-e^{-(s+\alpha^{-k+1}\lambda)\,Ta}\sum_{i=0}^{a-1}\frac{((s+\alpha^{-k+1}\lambda)\,Ta)^{i}}{i!}\right);$$

$$\text{grStar[s\_, k\_]} := \left(\frac{\beta r^{-k+1}\rho}{s+\beta r^{-k+1}\rho}\right)^{b};$$

oneMinusgrStar[s_, k_] := 1 - grStar[s, k];

$$\text{gpStar[s\_, k\_]} := \left(\frac{\beta p^{-k+1}\gamma}{s+\beta p^{-k+1}\gamma}\right)^{c};$$

oneMinusgpStar[s_, k_] := 1 - gpStar[s, k];

$$\text{hmvkStar[s\_, k\_]} := \text{fccStar[s, k] grStar[s, k]} + e^{-Ta\,s}\,\text{Fbar[Ta, k] gpStar[s, k]};$$

$$\text{hm}\theta\text{nStar[s\_, n\_]} := \prod_{k=1}^{n}\text{hmvkStar[s, k]};$$

$$\text{hmUnStar[s\_, n\_]} := \prod_{k=1}^{n}\text{grStar[s, k]}\left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^{a};$$

$$\text{quasiRenFnStar1[s\_]} := \frac{1}{s}\sum_{k=1}^{\text{numTerms}}\text{hmUnStar[s, k]};$$

$$\text{quasiRenFnStar2[s\_]} := \frac{1}{s}\sum_{k=1}^{\text{numTerms}}\text{hm}\theta\text{nStar[s, k]};$$

(* Preparing Plot *)

points = Table[8 x, {x, 0, 250}];

$$\text{points[[1]]} = \frac{1}{100};$$

(* number of terms*)

```
terms = {2,  4, 6, 8, 10, 12, 14};
numPoints = Length[points];
```

```
 (*********** Calculating uptime-based availability
***********)
 (* Create Table of seven empty entries *)
AUpTime = Table[0, {7}];
AUpTimeTerms = Table[0, {14}];
(* Populate the data for ploting *)
startTime = TimeUsed[];
(* Evaluate the first term *)
myTable= Table[0,{numPoints}];
 Do[
   t = points[[j]];
   If[t < Ta,
    myTable[[j]] = e^(-λ t) Sum[(λ t)^j / j!, {j, 0, a-1}];,
    myTable[[j]] = 0;
   ]
   , {j, numPoints}
  ];
 AUpTimeTerms[[1]] = myTable;
 (* Evaluate the remaining terms *)
 Do[
   myTable = Table[0, {numPoints}];
   sumTermsInAStar1[s_] :=
    1/s (1 - (λ/(α^(n-1) s + λ))^a) (Product[grStar[s, i] (λ/(α^(i-1) s + λ))^a, {i, 1, n-1}]);
   sumTermsInAStar2[s_] := FccbarStar[s, n] hmθnStar[s, n - 1] ;
   Do[
    t = points[[j]];
    If[t < Ta,
     myTable[[j]] = GWR[sumTermsInAStar1, t];,
     myTable[[j]] = GWR[sumTermsInAStar2, t];
    ]
    , {j, numPoints}
   ];
   AUpTimeTerms[[n]] = myTable;
   , {n, 2, 14}
  ];
 (* Evaluate the approximation with two terms *)
myTable1 = AUpTimeTerms[[1]];
myTable2 = AUpTimeTerms[[2 ]];
myTable3 = Table[0,{numPoints}];
Do[
```

```
    myTable3[[j]] =  myTable1[[j]]+ myTable2[[j]];
    , {j, numPoints}
    ];
AUpTime[[1]] = myTable3;
 (* Evaluate the approximation with more than two terms *)
Do [
    myTable = AUpTime[[n - 1]];  (* cumulative sum upto 2
x(n-1) terms *)
    myTable1 = AUpTimeTerms[[2 n - 1]];
    myTable2 = AUpTimeTerms[[2 n ]];
    myTable3 = Table[0,{numPoints}];
    Do[
      myTable3[[j]] = myTable[[j]]+ myTable1[[j]]+
myTable2[[j]];
      , {j, numPoints}
      ];
    AUpTime[[n]] = myTable3;
    , {n,2,7}
    ];

endTime = TimeUsed[];
(* Display the time usage in seconds*)
Print["CPU Time Usage: ",SetPrecision[endTime-startTime,5],
" seconds"];
 (*********** Calculating downtime-based availability
***********)
 (* Create Table of seven empty entries *)
ADownTime = Table[0, {7}];
ADownTimeTerms = Table[0, {14}];
startTime = TimeUsed[];
(* Evaluate the first term *)
myTable= Table[0,{numPoints}];
 (* For t < Ta; assuming a = b = 2 *)
```

firstTerm1Star[s_] := $\frac{1}{s}\left(\frac{\lambda}{\lambda+s}\right)^{a}$ oneMinusgrStar[s, 1];

(* For t ≥ Ta *)

firstTerm2Star[s_] :=

$\frac{1}{s}$ (fccStar[s, 1] oneMinusgrStar[s, 1] +

Fbar[Ta, 1] $e^{-Ta\, s}$ oneMinusgpStar[s, 1]);

```
 Do[
    t=points[[j]];
    If[t < Ta,
      myTable[[j]] = GWR[firstTerm1Star, t];,
      myTable[[j]] = GWR[firstTerm2Star, t] ;
      ];
    , {j, numPoints}
    ];
ADownTimeTerms[[1]] = myTable;
```

```
(* Evaluate the remaining terms *)
Do[
  myTable = Table[0, {numPoints}];
  sumTermsInAStar1[s_] :=
```

$$\frac{1}{s}\left(\left(\frac{\beta r^{n-1}\,s+\rho}{\rho}\right)^{b}-1\right)\left(\lambda^{a}\rho^{b}\right)^{n}\prod_{i=1}^{n}\left(\frac{1}{\alpha^{i-1}\,s+\lambda}\right)^{a}\left(\frac{1}{\beta r^{i-1}\,s+\rho}\right)^{b};$$

```
  sumTermsInAStar2[s_] :=
```

$$\frac{1}{s}\,\text{hm}\theta\text{nStar}[s,\ n-1]\,\big(\text{fccStar}[s,n]\ \text{oneMinusgrStar}[s,n]+\text{Fbar}[Ta,n]\ e^{-Ta\,s}\ \text{oneMinusgpStar}[s,n]\big);$$

```
  Do[
   t = points[[j]];
   If[t < Ta,
    myTable[[j]] = GWR[sumTermsInAStar1, t];,
    myTable[[j]] = GWR[sumTermsInAStar2, t];
   ];
   , {j, numPoints}
  ];
  ADownTimeTerms[[n]] = myTable;
  , {n, 2, 14}
 ];
 (* Evaluate the approximation with first two terms *)
myTable1 = ADownTimeTerms[[1]];
myTable2 = ADownTimeTerms[[2]];
myTable3 = Table[0,{numPoints}];
Do[
    myTable3[[j]] = 1-( myTable1[[j]]+ myTable2[[j]]);
    , {j, numPoints}
    ];
ADownTime[[1]] = myTable3;

(* Evaluate the approximation with more than two terms *)
Do [
    myTable = ADownTime[[ n -1]]; (* 1 - cumulative sum upto
2(n-1) terms *)
    myTable1 = ADownTimeTerms[[2 n - 1]];
    myTable2 = ADownTimeTerms[[2 n ]];
    myTable3 = Table[0,{numPoints}];
    Do[
      myTable3[[j]] = myTable[[j]]-( myTable1[[j]]+
myTable2[[j]]);
      , {j, numPoints}
      ];
    ADownTime[[n]] = myTable3;
    , {n,2,7}
```

```
    ];
endTime = TimeUsed[];
(* Display the time usage in seconds*)
Print["CPU Time Usage:",SetPrecision[endTime-startTime,5], "
seconds"];
 (*********** Calculating quasi-renewal function
***********)

QTable = Table[0, {numPoints}];
startTime = TimeUsed[];
Do[
    t=points[[m]];
    If[t < Ta,
      QTable [[m]] =  GWR[quasiRenFnStar1,t];,
      QTable [[m]] =  GWR[quasiRenFnStar2,t];
      ];
    , {m, numPoints}
    ];
endTime = TimeUsed[];

(* Display the time usage in seconds*)
Print["CPU Time Usage:",SetPrecision[endTime-startTime,5], "
seconds"];
 Print["Session Time in Seconds: ",
SetPrecision[SessionTime[],5]];
Print["Session Time in Minutes: ",
SetPrecision[SessionTime[]/60,5]];
Print["Session Time in Hours: ",
SetPrecision[SessionTime[]/3600,5]];
Date[]
```

A. 5. An Example of Mathematica Source Code for the Gamma Distribution Model with Exponential Repair and PM Intervals

```
Off[General::spell1];
<<NumericalMath`ComputerArithmetic`
SetArithmetic[10,10, RoundingRule→RoundToInfinity,
ExponentRange→{-1000,1000}, MixedMode→False,
IdealDivide→False]
(* Call numerical inversion of Laplace transform package.
The package provides only one function called GWR*)
thisDir=ToFileName[

("FileName"/.NotebookInformation[EvaluationNotebook[]])
        [[1]]];
SetDirectory[thisDir];
<<NumericalLaplaceInversion.m
<<Graphics`MultipleListPlot`
Ta =50;
 α = 9/10;
 βr=1;
βp=1;
 (* gamma parameters for operating time *)
 λ = 2/100;
 a = 2;
 (*  gamma parameters for repair time *)
 ρ = 1/20;
 b = 1;
 (* gamma parameters for preventive time *)
 γ = 1/10;
 c = 1;
 (* number of terms *)
K =2;
numTerms = 15;
 Date[]
 (* Density and Distribution Functions *)
```

$$\text{Fbar[t\_, k\_] := } e^{-\alpha^{-k+1}\lambda t}\sum_{j=0}^{a-1}\frac{(\alpha^{-k+1}\lambda t)^j}{j!};$$

FccbarStar[s_, 1] :=

$$\sum_{j=0}^{a-1}\left(\frac{\lambda^j}{(s+\lambda)^{j+1}}\left(1-e^{-(s+\lambda)\,Ta}\sum_{i=0}^{j}\frac{((s+\lambda)\,Ta)^i}{i!}\right)\right);$$

FccbarStar[s_, k_] :=

$$\sum_{j=0}^{a-1}\left(\frac{\alpha^{k-1}\lambda^j}{(\alpha^{k-1}s+\lambda)^{j+1}}\left(1-e^{-(s+\alpha^{-k+1}\lambda)\,Ta}\sum_{i=0}^{j}\frac{((s+\alpha^{-k+1}\lambda)\,Ta)^i}{i!}\right)\right);$$

$$\text{fccStar[s\_, 1] := }\left(\frac{\lambda}{s+\lambda}\right)^a\left(1-e^{-(s+\lambda)\,Ta}\sum_{i=0}^{a-1}\frac{((s+\lambda)\,Ta)^i}{i!}\right);$$

fccStar[s_, k_] :=

$$\left(\frac{\lambda}{\alpha^{k-1}s+\lambda}\right)^a\left(1-e^{-(s+\alpha^{-k+1}\lambda)\,Ta}\sum_{i=0}^{a-1}\frac{((s+\alpha^{-k+1}\lambda)\,Ta)^i}{i!}\right);$$

$$\text{grStar[s\_, k\_] := }\left(\frac{\beta r^{-k+1}\rho}{s+\beta r^{-k+1}\rho}\right);$$

oneMinusgrStar[s_, k_] := 1 - grStar[s, k];

$$\text{gpStar[s\_, k\_] := }\left(\frac{\beta p^{-k+1}\gamma}{s+\beta p^{-k+1}\gamma}\right);$$

oneMinusgpStar[s_, k_] := 1 - gpStar[s, k];

hmvkStar[s_, k_] := fccStar[s, k] grStar[s, k] +
$e^{-Ta\,s}$ Fbar[Ta, k] gpStar[s, k];

$$\text{hm}\theta\text{nStar[s\_, n\_] := }\prod_{k=1}^{n}\text{hmvkStar[s, k]};$$

$$\text{hmUnStar[s\_, n\_] := }\prod_{k=1}^{n}\text{grStar[s, k]}\left(\frac{\lambda}{\lambda+\alpha^{k-1}s}\right)^a;$$

$$\text{quasiRenFnStar1[s\_] := }\frac{1}{s}\sum_{k=1}^{\text{numTerms}}\text{hmUnStar[s, k]};$$

$$\text{quasiRenFnStar2[s\_] := }\frac{1}{s}\sum_{k=1}^{\text{numTerms}}\text{hm}\theta\text{nStar[s, k]};$$

```
(* Preparing Plot *)
points = Table[8 x, {x, 0, 250}];
```

$$\text{points[[1]] = }\frac{1}{100};$$

```
(* number of terms*)
terms = {2,  4, 6, 8, 10, 12, 14};
numPoints = Length[points];
```

```
(*********** Calculating uptime-based availability
***********)
(* Create Table of seven empty entries *)
AUpTime = Table[0, {7}];
AUpTimeTerms = Table[0, {14}];
(* Populate the data for ploting *)
startTime = TimeUsed[];
(* Evaluate the first term *)
myTable= Table[0,{numPoints}];
Do[
  t = points[[j]];
  If[t < Ta,
```

$$\texttt{myTable[[j]]} = e^{-\lambda t} \sum_{j=0}^{a-1} \frac{(\lambda t)^j}{j!} \texttt{;,}$$

```
   myTable[[j]] = 0;
  ]
  , {j, numPoints}
 ];
AUpTimeTerms[[1]] = myTable;
(* Evaluate the remaining terms *)
Do[
  myTable = Table[0, {numPoints}];
  sumTermsInAStar1[s_] :=
```

$$\frac{1}{s} \left(1 - \left(\frac{\lambda}{\alpha^{(n-1)}\ s + \lambda}\right)^a\right) \left(\prod_{i=1}^{n-1} \texttt{grStar[s, i]} \left(\frac{\lambda}{\alpha^{i-1}\, s + \lambda}\right)^a\right);$$

```
  sumTermsInAStar2[s_] := FccbarStar[s, n] hmθnStar[s, n - 1] ;
  Do[
   t = points[[j]];
   If[t < Ta,
    myTable[[j]] = GWR[sumTermsInAStar1, t];,
    myTable[[j]] = GWR[sumTermsInAStar2, t];
   ]
   , {j, numPoints}
  ];
  AUpTimeTerms[[n]] = myTable;
  , {n, 2, 14}
 ];
(* Evaluate the approximation with two terms *)
myTable1 = AUpTimeTerms[[1]];
myTable2 = AUpTimeTerms[[2 ]];
myTable3 = Table[0,{numPoints}];
Do[
```

```
    myTable3[[j]] =  myTable1[[j]]+ myTable2[[j]];
    , {j, numPoints}
    ];
AUpTime[[1]] = myTable3;
(* Evaluate the approximation with more than two terms *)
Do [
    myTable = AUpTime[[n - 1]];  (* cumulative sum upto 2
x(n-1) terms *)
    myTable1 = AUpTimeTerms[[2 n - 1]];
    myTable2 = AUpTimeTerms[[2 n ]];
    myTable3 = Table[0,{numPoints}];
    Do[
      myTable3[[j]] = myTable[[j]]+ myTable1[[j]]+
myTable2[[j]];
      , {j, numPoints}
      ];
    AUpTime[[n]] = myTable3;
    , {n,2,7}
    ];

endTime = TimeUsed[];
(* Display the time usage in seconds*)
Print["CPU Time Usage: ",SetPrecision[endTime-startTime,5],
" seconds"];
 (*********** Calculating downtime-based availability
***********)
 (* Create Table of seven empty entries *)
ADownTime = Table[0, {7}];
ADownTimeTerms = Table[0, {14}];
startTime = TimeUsed[];
 (* Evaluate the first term *)
myTable= Table[0,{numPoints}];
 (* For t < Ta; assuming a = b = 2 *)
 If[λ ≠ ρ,
```

$$\texttt{firstTerm1[t\_]} := e^{-\rho t}\left(\frac{\lambda}{\lambda-\rho}\right)^{a} - e^{-\lambda t}\left(\frac{\lambda}{\lambda-\rho}\right)^{a}\sum_{j=0}^{a-1}\frac{((\lambda-\rho)\ t)^{j}}{j!};,$$

$$\texttt{firstTerm1[t\_]} := e^{-\lambda t}\frac{(\lambda\ t)^{a}}{a!};$$

```
 ]
 (* For t ≥ Ta *)
 If[λ ≠ ρ,
```

$$\texttt{firstTerm2[t\_]} := e^{(\gamma-\lambda)\ Ta-\gamma\ t}\,(1+\lambda\,Ta) +$$

$$e^{-\rho t}\left(\frac{\lambda}{\lambda-\rho}\right)^{2}\left(1-e^{(\rho-\lambda)\ Ta}\,(1+(\lambda-\rho)\ Ta)\right);,$$

$$\texttt{firstTerm2[t\_]} := \frac{\lambda^{2}}{2}\,Ta^{2}\,e^{-\lambda t} + e^{-\gamma\ (t-Ta)-\lambda\,Ta}\,(1+\lambda\,Ta);$$

```
 ]
 Do[
```

```
     t=points[[j]];
     If[t < Ta,
       myTable[[j]] = firstTerm1[t];,
       myTable[[j]] = firstTerm2[t] ;
       ];
     , {j, numPoints}
     ];
ADownTimeTerms[[1]] = myTable;
  (* Evaluate the remaining terms *)
  Do[
    myTable = Table[0, {numPoints}];
    sumTermsInAStar1[s_] :=
```

$$\lambda^{a\,n}\,(\beta r\,\rho)^{n-1}\prod_{i=1}^{n}\frac{1}{(\alpha^{i-1}\,s+\lambda)^{a}\,(\beta r^{i-1}\,s+\rho)};$$

```
    sumTermsInAStar2[s_] :=
```

$$\frac{1}{s}\,\text{hm}\theta\text{nStar}[s,\,n-1]\,\big(\text{fccStar}[s,\,n]\ \text{oneMinusgrStar}[s,\,n]+\text{Fbar}[\text{Ta},\,n]\,e^{-\text{Ta}\,s}\,\text{oneMinusgpStar}[s,\,n]\big);$$

```
    Do[
     t = points[[j]];
     If[t < Ta,
      myTable[[j]] = GWR[sumTermsInAStar1, t];,
      myTable[[j]] = GWR[sumTermsInAStar2, t];
     ];
     , {j, numPoints}
    ];
    ADownTimeTerms[[n]] = myTable;
    , {n, 2, 14}
   ];
  (* Evaluate the approximation with first two terms *)
myTable1 = ADownTimeTerms[[1]];
myTable2 = ADownTimeTerms[[2]];
myTable3 = Table[0,{numPoints}];
Do[
    myTable3[[j]] = 1-( myTable1[[j]]+ myTable2[[j]]);
     , {j, numPoints}
     ];
ADownTime[[1]] = myTable3;

(* Evaluate the approximation with more than two terms *)
Do [
    myTable = ADownTime[[ n -1]]; (* 1 - cumulative sum upto
2(n-1) terms *)
    myTable1 = ADownTimeTerms[[2 n - 1]];
    myTable2 = ADownTimeTerms[[2 n ]];
```

```
    myTable3 = Table[0,{numPoints}];
    Do[
      myTable3[[j]] = myTable[[j]]-( myTable1[[j]]+
myTable2[[j]]);
      , {j, numPoints}
      ];
    ADownTime[[n]] = myTable3;
    , {n,2,7}
    ];

endTime = TimeUsed[];

(* Display the time usage in seconds*)
Print["CPU Time Usage:",SetPrecision[endTime-startTime,5], "
seconds"];
 (*********** Calculating quasi-renewal function
***********)
QTable = Table[0, {numPoints}];
startTime = TimeUsed[];
Do[
    t=points[[m]];
    If[t < Ta,
      QTable [[m]] =  GWR[quasiRenFnStar1,t];,
      QTable [[m]] =  GWR[quasiRenFnStar2,t];
      ];
    , {m, numPoints}
    ];
endTime = TimeUsed[];

(* Display the time usage in seconds*)
Print["CPU Time Usage:",SetPrecision[endTime-startTime,5], "
seconds"];
 Print[SetPrecision[SessionTime[],5]];
Date[]
```

A. 6. An Example of Mathematica Source Code for the Normal Distribution Model with Normal Repair and PM Intervals

```
Off[General::spell1];
<<NumericalMath`ComputerArithmetic`
SetArithmetic[10,10, RoundingRule→RoundToInfinity,
ExponentRange→{-1000,1000}, MixedMode→False,
IdealDivide→False]
(* Call numerical inversion of Laplace transform package.
The package provides only one function called GWR*)
thisDir=ToFileName[

("FileName"/.NotebookInformation[EvaluationNotebook[]])
        [[1]]];
SetDirectory[thisDir];
<<NumericalLaplaceInversion.m
<<Graphics`MultipleListPlot`
<<Statistics`NormalDistribution`
Ta =50;
 α = 9/10;
 (* mean and standard deviation for operating time *)
μ=100;
σ=25;
 (* mean and standard deviation for repair time *)
 μr = 10;
 σr = 5/2;
 (* mean and standard deviation for preventive time *)
 μp = 5;
 σp = 3/2;
 (* number of terms *)
K =2;
numTerms = 15;
 Date[]
 (* Density and Distribution Functions *)
 stdndist = NormalDistribution[0, 1];
 Φ[t_] := CDF[stdndist, t];
 F[t_, 1] := CDF[NormalDistribution[μ, σ], t];
 F[t_, k_] := CDF[NormalDistribution[α^(k-1) μ, α^(k-1) σ], t];
 Fbar[t_, k_] := 1 - F[t, k];
 f[t_, 1] :=  PDF[NormalDistribution[μ, σ], t];
 f[t_, k_] := f[t/α^(k-1), 1] / α^(k-1);
```

$$\texttt{fcc[t\_, k\_] := (1 - UnitStep[t - Ta]) f[t, k];}$$

$$\texttt{fc[t\_, k\_] :=} \frac{\texttt{fcc[t, k]}}{\texttt{F[Ta, k]}};$$

$$\texttt{Fc[t\_, k\_] := UnitStep[t - Ta] +} \quad \texttt{(1 - UnitStep[t - Ta])} \frac{\texttt{F[t, k]}}{\texttt{F[Ta, k]}};$$

$$\texttt{Fcbar[t\_, k\_] := 1 - Fc[t, k];}$$

$$\texttt{Fccbar[t\_, k\_] := (1 - UnitStep[t - Ta]) Fbar[t, k];}$$

$$\texttt{fcStar[s\_, 1] :=} \frac{e^{-s\mu+\frac{1}{2}s^2\sigma^2}}{\Phi\left[\frac{Ta-\mu}{\sigma}\right]}\Phi\left[\frac{Ta-\mu+s\,\sigma^2}{\sigma}\right];$$

$$\texttt{fcStar[s\_, k\_] :=} \frac{e^{-s\alpha^{k-1}\mu+\frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}}{\Phi\left[\frac{Ta-\alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right]}\Phi\left[\frac{Ta-\alpha^{k-1}\mu+s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right];$$

$$\texttt{fccStar[s\_, 1] :=} e^{-s\mu+\frac{1}{2}s^2\sigma^2}\Phi\left[\frac{Ta-\mu+s\,\sigma^2}{\sigma}\right];$$

$$\texttt{fccStar[s\_, k\_] :=} e^{-s\alpha^{k-1}\mu+\frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}\Phi\left[\frac{Ta-\alpha^{k-1}\mu+s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right];$$

$$\texttt{FcbarStar[s\_, k\_] :=} \frac{1}{s}\texttt{(1 - fcStar[s, k]);}$$

$$\texttt{FccbarStar[s\_, 1] :=} \frac{1-e^{-s\,Ta}}{s} + \frac{1}{s}\left(e^{-s\,Ta}\Phi\left[\frac{Ta-\mu}{\sigma}\right] - e^{-s\mu+\frac{1}{2}s^2\sigma^2}\Phi\left[\frac{Ta-\mu+s\,\sigma^2}{\sigma}\right]\right);$$

$$\texttt{FccbarStar[s\_, k\_] :=} \frac{1-e^{-s\,Ta}}{s} + \frac{1}{s}\left(e^{-s\,Ta}\Phi\left[\frac{Ta-\alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right] - e^{-s\alpha^{k-1}\mu+\frac{1}{2}s^2\left(\alpha^{k-1}\sigma\right)^2}\Phi\left[\frac{Ta-\alpha^{k-1}\mu+s\left(\alpha^{k-1}\sigma\right)^2}{\alpha^{k-1}\sigma}\right]\right);$$

```
gr[t_] :=PDF[NormalDistribution[μr,σr],t];
gp[t_] := PDF[NormalDistribution[μp,σp],t];
```

$$\texttt{grStar[s\_] :=} e^{-s\mu r+\frac{1}{2}s^2\sigma r^2};$$

$$\texttt{oneMinusgrStar[s\_] :=} 1-e^{-s\mu r+\frac{1}{2}s^2\sigma r^2};$$

$$\texttt{gpStar[s\_] :=} e^{-s\mu p+\frac{1}{2}s^2\sigma p^2};$$

$$\texttt{oneMinusgpStar[s\_] :=} 1-e^{-s\mu p+\frac{1}{2}s^2\sigma p^2};$$

```
hmvkStar[s_, 1] := ⅇ^(-s (μ + μr) + 1/2 s^2 (σ^2 + σr^2)) Φ[(Ta - μ + s σ^2)/σ] +
   (1 - Φ[(Ta - μ)/σ]) ⅇ^(-s (μp + Ta) + 1/2 s^2 σp^2);
hmvkStar[s_, k_] :=
  ⅇ^(-s (α^(k-1) μ + μr) + 1/2 s^2 ((α^(k-1) σ)^2 + σr^2)) Φ[(Ta - α^(k-1) μ + s (α^(k-1) σ)^2)/(α^(k-1) σ)] +
   (1 - Φ[(Ta - α^(k-1) μ)/(α^(k-1) σ)]) ⅇ^(-s (μp + Ta) + 1/2 s^2 σp^2);
hmθnStar[s_, n_] := ∏_{k=1}^{n} hmvkStar[s, k];
quasiRenFn1[t_] := ∑_{k=1}^{numTerms} Φ[(t - (μ (1-α^k)/(1-α) + k μr)) / Sqrt[σ^2 (1-α^(2 k))/(1-α^2) + k σr^2]];;

quasiRenFnStar2[s_] := 1/s ∑_{k=1}^{numTerms} hmθnStar[s, k];
(* Preparing Plot *)
points = Table[16 x, {x, 0, 125}];
points[[1]] = 1/100;
(* number of terms*)
terms = {2, 4, 6, 8, 10, 12, 14};
numPoints = Length[points];
(*********** Calculating uptime-based availability
***********)
(* Create Table of seven empty entries *)
AUpTime = Table[0, {7}];
    AUpTimeTerms = Table[0, {14}];
    (* Populate the data for ploting *)
    startTime = TimeUsed[];
    (* Evaluate the first term *)
    myTable= Table[0,{numPoints}];
Do[
   t = points[[j]];
   If[t < Ta,
    myTable[[j]] = 1 - Φ[(t - μ)/σ];,
    myTable[[j]] = 0;
   ]
   , {j, numPoints}
  ];
AUpTimeTerms[[1]] = myTable;
```

```
 (* Evaluate the remaining terms *)
 Do[
   myTable = Table[0, {numPoints}];
   sumTermsInAStar1[s_] := 0;
   sumTermsInAStar2[s_] :=
    FccbarStar[s, n] hmθnStar[s, n - 1] ;
   Do[
    t = points[[j]];
    If[t < Ta,
```

$$\text{myTable[[j]]} = \Phi\left[\frac{t - \left((n-1)\,\mu r + \frac{\mu\,(1-\alpha^{n-1})}{1-\alpha}\right)}{\sqrt{(n-1)\,\sigma r^2 + \frac{\sigma^2\,(1-\alpha^{2\,(n-1)})}{1-\alpha^2}}}\right] -$$

$$\Phi\left[\frac{t - \left(\alpha^{n-1}\mu + (n-1)\,\mu r + \frac{\mu\,(1-\alpha^{n-1})}{1-\alpha}\right)}{\sqrt{(\alpha^{n-1}\,\sigma)^2 + (n-1)\,\sigma r^2 + \frac{\sigma^2\,(1-\alpha^{2\,(n-1)})}{1-\alpha^2}}}\right];,$$

```
     myTable[[j]] = GWR[sumTermsInAStar2, t];
    ]
    , {j, numPoints}
   ];
   AUpTimeTerms[[n]] = myTable;
   , {n, 2, 14}
  ];
 (* Evaluate the approximation with two terms *)
myTable1 = AUpTimeTerms[[1]];
myTable2 = AUpTimeTerms[[2 ]];
myTable3 = Table[0,{numPoints}];
Do[
    myTable3[[j]] =  myTable1[[j]]+ myTable2[[j]];
    , {j, numPoints}
    ];
AUpTime[[1]] = myTable3;

(* Evaluate the approximation with more than two terms *)
Do [
    myTable = AUpTime[[n - 1]];  (* cumulative sum upto 2
x(n-1) terms *)
    myTable1 = AUpTimeTerms[[2 n - 1]];
    myTable2 = AUpTimeTerms[[2 n ]];
    myTable3 = Table[0,{numPoints}];
    Do[
      myTable3[[j]] = myTable[[j]]+ myTable1[[j]]+
myTable2[[j]];
```

```
      , {j, numPoints}
      ];
    AUpTime[[n]] = myTable3;
    , {n,2,7}
    ];

endTime = TimeUsed[];
(* Display the time usage in seconds*)
Print["CPU Time Usage: ",SetPrecision[endTime-startTime,5],
" seconds"];
 (*********** Calculating downtime-based availability
***********)
 (* Create Table of seven empty entries *)
ADownTime = Table[0, {7}];
ADownTimeTerms = Table[0, {14}];
startTime = TimeUsed[];
 (* Evaluate the first term *)
 myTable = Table[0, {numPoints}];
 firsttermStar[s_] =
```

$$\frac{1}{s}\,e^{-s\mu+\frac{1}{2}s^2\sigma^2}\,\Phi\left[\frac{Ta-\mu+s\,\sigma^2}{\sigma}\right]\left(1-e^{-s\,\mu r+\frac{1}{2}s^2\sigma r^2}\right);$$

```
 Do[
   t = points[[j]];
   If[t < Ta,
```

$$\text{myTable[[j]]} = \Phi\left[\frac{t-\mu}{\sigma}\right]-\Phi\left[\frac{t-(\mu r+\mu)}{\sqrt{\sigma^2+\sigma r^2}}\right];,$$

$$\text{myTable[[j]]} = \left(1-\Phi\left[\frac{Ta-\mu}{\sigma}\right]\right)\left(1-\Phi\left[\frac{t-Ta-\mu p}{\sigma p}\right]\right)+$$

```
      GWR[firsttermStar, t];
   ];
   , {j, numPoints}
  ];
 ADownTimeTerms[[1]] = myTable;
```

```
(* Evaluate the remaining terms *)
Do[
  myTable = Table[0, {numPoints}];
  sumTermsInAStar1[t_] :=
   Φ[(t - ((n - 1) μr + μ (1-α^n)/(1-α)))/Sqrt[(n - 1) σr^2 + σ^2 (1-α^2 n)/(1-α^2)]] - Φ[(t - (n μr + μ (1-α^n)/(1-α)))/Sqrt[n σr^2 + σ^2 (1-α^2 n)/(1-α^2)]];
  sumTermsInAStar2[s_] :=
   1/s hmθnStar[s, n - 1]
    (fccStar[s, n] oneMinusgrStar[s] +
      Fbar[Ta, n] e^(-Ta s) oneMinusgpStar[s]);
  Do[
   t = points[[j]];
   If[t < Ta,
    myTable[[j]] = sumTermsInAStar1[t],
    myTable[[j]] = GWR[sumTermsInAStar2, t];
   ];
   , {j, numPoints}
  ];
  ADownTimeTerms[[n]] = myTable;
  , {n, 2, 14}
 ];
 (* Evaluate the approximation with first two terms *)
myTable1 = ADownTimeTerms[[1]];
myTable2 = ADownTimeTerms[[2]];
myTable3 = Table[0,{numPoints}];
Do[
    myTable3[[j]] = 1-( myTable1[[j]]+ myTable2[[j]]);
    , {j, numPoints}
    ];
ADownTime[[1]] = myTable3;

(* Evaluate the approximation with more than two terms *)
Do [
    myTable = ADownTime[[ n -1]]; (* 1 - cumulative sum up
to 2(n-1) terms *)
    myTable1 = ADownTimeTerms[[2 n - 1]];
    myTable2 = ADownTimeTerms[[2 n ]];
    myTable3 = Table[0,{numPoints}];
    Do[
      myTable3[[j]] = myTable[[j]]-( myTable1[[j]]+
myTable2[[j]]);
      , {j, numPoints}
      ];
```

```
    ADownTime[[n]] = myTable3;
    , {n,2,7}
    ];
endTime = TimeUsed[];

(* Display the time usage in seconds*)
Print["CPU Time Usage:",SetPrecision[endTime-startTime,5], "
seconds"];
 (*********** Calculating quasi-renewal function
***********)

QTable = Table[0, {numPoints}];
startTime = TimeUsed[];
Do[
    t=points[[m]];
    If[t < Ta,
      QTable [[m]] =  quasiRenFn1[t];,
      QTable [[m]] =  GWR[quasiRenFnStar2,t];
      ];
    , {m, numPoints}
    ];
endTime = TimeUsed[];

(* Display the time usage in seconds*)
Print["CPU Time Usage:",SetPrecision[endTime-startTime,5], "
seconds"];

 Print[SetPrecision[SessionTime[],5]];
Date[]
```

A. 7. An Example of Mathematica Source Code for the Normal Distribution Model with Exponential Repair and PM Intervals

```
Off[General::spell1];
<<NumericalMath`ComputerArithmetic`
SetArithmetic[10,10, RoundingRule→RoundToInfinity,
ExponentRange→{-1000,1000}, MixedMode→False,
IdealDivide→False]
(* Call numerical inversion of Laplace transform package.
The package provides only one function called GWR*)
thisDir=ToFileName[

("FileName"/.NotebookInformation[EvaluationNotebook[]])
        [[1]]];
SetDirectory[thisDir];
<<NumericalLaplaceInversion.m
<<Graphics`MultipleListPlot`
<<Statistics`NormalDistribution`
Ta =50;
α = 9/10;
βr=1;
βp=1;
 (* mean and standard deviation for operating time *)
μ=100;
σ=25;
 (* parameter for exponential  repair time *)
 ρ = 1/20;
 (* parameter for exponential  preventive time *)
 γ = 1/10;
 (* number of terms *)
K =2;
numTerms = 15;
 Date[]
 (* Density and Distribution Functions *)
stdndist =NormalDistribution[0,1];
Φ[t_] :=CDF[stdndist, t];
F[t_,1] := CDF[NormalDistribution[μ,σ],t];
 F[t_, k_] := CDF[NormalDistribution[α^(k-1) μ, α^(k-1) σ], t];
 Fbar[t_, k_] := 1 - F[t, k];
 f[t_, 1] := PDF[NormalDistribution[μ, σ], t];
 f[t_, k_] := f[t/α^(k-1), 1] / α^(k-1);
```

```
fcc[t_, k_] := (1 - UnitStep[t - Ta]) f[t, k];
```

$$\texttt{fc[t\_, k\_]} := \frac{\texttt{fcc[t, k]}}{\texttt{F[Ta, k]}};$$

$$\texttt{Fc[t\_, k\_]} := \texttt{UnitStep[t - Ta]} + (1 - \texttt{UnitStep[t - Ta]})\,\frac{\texttt{F[t, k]}}{\texttt{F[Ta, k]}};$$

```
Fcbar[t_,k_] := 1- Fc[t,k];
Fccbar[t_,k_] := (1-UnitStep[t - Ta ])Fbar[t,k];
```

$$\texttt{fcStar[s\_, 1]} := \frac{e^{-s\mu+\frac{1}{2}s^2\sigma^2}}{\Phi\left[\frac{Ta-\mu}{\sigma}\right]}\,\Phi\left[\frac{Ta-\mu+s\,\sigma^2}{\sigma}\right];$$

$$\texttt{fcStar[s\_, k\_]} := \frac{e^{-s\alpha^{k-1}\mu+\frac{1}{2}s^2(\alpha^{k-1}\sigma)^2}}{\Phi\left[\frac{Ta-\alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right]}\,\Phi\left[\frac{Ta-\alpha^{k-1}\mu+s\,(\alpha^{k-1}\sigma)^2}{\alpha^{k-1}\sigma}\right];$$

$$\texttt{fccStar[s\_, 1]} := e^{-s\mu+\frac{1}{2}s^2\sigma^2}\,\Phi\left[\frac{Ta-\mu+s\,\sigma^2}{\sigma}\right];$$

$$\texttt{fccStar[s\_, k\_]} := e^{-s\alpha^{k-1}\mu+\frac{1}{2}s^2(\alpha^{k-1}\sigma)^2}\,\Phi\left[\frac{Ta-\alpha^{k-1}\mu+s\,(\alpha^{k-1}\sigma)^2}{\alpha^{k-1}\sigma}\right];$$

$$\texttt{FcbarStar[s\_, k\_]} := \frac{1}{s}\,(1-\texttt{fcStar[s, k]});$$

$$\texttt{FccbarStar[s\_, 1]} := \frac{1-e^{-s\,Ta}}{s}+\frac{1}{s}\left(e^{-s\,Ta}\,\Phi\left[\frac{Ta-\mu}{\sigma}\right]-e^{-s\mu+\frac{1}{2}s^2\sigma^2}\,\Phi\left[\frac{Ta-\mu+s\,\sigma^2}{\sigma}\right]\right);$$

$$\texttt{FccbarStar[s\_, k\_]} := \frac{1-e^{-s\,Ta}}{s}+\frac{1}{s}\left(e^{-s\,Ta}\,\Phi\left[\frac{Ta-\alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right]-e^{-s\alpha^{k-1}\mu+\frac{1}{2}s^2(\alpha^{k-1}\sigma)^2}\,\Phi\left[\frac{Ta-\alpha^{k-1}\mu+s\,(\alpha^{k-1}\sigma)^2}{\alpha^{k-1}\sigma}\right]\right);$$

$$\texttt{grStar[s\_, k\_]} := \frac{\beta r^{-k+1}\rho}{s+\beta r^{-k+1}\rho};$$

$$\texttt{oneMinusgrStar[s\_, k\_]} := \frac{s}{s+\beta r^{-k+1}\rho};$$

$$\texttt{gpStar[s\_, k\_]} := \frac{\beta p^{-k+1}\gamma}{s+\beta p^{-k+1}\gamma};$$

$$\texttt{oneMinusgpStar[s\_, k\_]} := \frac{s}{s+\beta p^{-k+1}\gamma};$$

$$\texttt{hmvkStar[s\_, k\_]} := e^{-s\alpha^{k-1}\mu+\frac{1}{2}s^2(\alpha^{k-1}\sigma)^2}\,\Phi\left[\frac{Ta-\alpha^{k-1}\mu+s\,(\alpha^{k-1}\sigma)^2}{\alpha^{k-1}\sigma}\right]\texttt{grStar[s, k]}+\left(1-\Phi\left[\frac{Ta-\alpha^{k-1}\mu}{\alpha^{k-1}\sigma}\right]\right)e^{-s\,Ta}\,\texttt{gpStar[s, k]};$$

```
hmθnStar[s_, n_] := ∏_{k=1}^{n} hmvkStar[s, k];
```

$$\text{hmUnStar[s\_, n\_] := } e^{-s\mu \frac{1-\alpha^{n}}{1-\alpha} + \frac{1}{2} s^{2}\sigma^{2} \frac{1-\alpha^{2n}}{1-\alpha^{2}}} \prod_{k=1}^{n} \frac{\beta r^{-k+1}\rho}{s + \beta r^{-k+1}\rho};$$

$$\text{quasiRenFnStar1[s\_] := } \frac{1}{s} \sum_{k=1}^{numTerms} \text{hmUnStar[s, k]};$$

$$\text{quasiRenFnStar2[s\_] := } \frac{1}{s} \sum_{k=1}^{numTerms} \text{hm}\theta\text{nStar[s, k]};$$

```
(* Preparing Plot *)
points = Table[16 x, {x, 0, 125}];
points[[1]] = 1/100;
(* number of terms*)
terms = {2, 4, 6, 8, 10, 12, 14};
numPoints = Length[points];
(*********** Calculating uptime-based availability
***********)
(* Create Table of seven empty entries *)
AUpTime = Table[0, {7}];
AUpTimeTerms = Table[0, {14}];
(* Populate the data for ploting *)
startTime = TimeUsed[];
(* Evaluate the first term *)
myTable= Table[0,{numPoints}];
Do[
  t = points[[j]];
  If[t < Ta,
   myTable[[j]] = 1 - Φ[(t - μ)/σ];,
   myTable[[j]] = 0;
  ]
  , {j, numPoints}
 ];
AUpTimeTerms[[1]] = myTable;
```

```
(* Evaluate the remaining terms *)
Do[
  myTable = Table[0, {numPoints}];
  sumTermsInAStar1[s_] :=
```

$$\frac{1}{s}\left(1-e^{-s\,\alpha^{n-1}\,\mu+\frac{1}{2}\,s^{2}\,\left(\alpha^{n-1}\,\sigma\right)^{2}}\right)\text{hmUnStar[s, n];}$$

```
  sumTermsInAStar2[s_] := FccbarStar[s, n] hmθnStar[s, n - 1] ;
  Do[
   t = points[[j]];
   If[t < Ta,
    myTable[[j]] = GWR[sumTermsInAStar1, t];,
    myTable[[j]] = GWR[sumTermsInAStar2, t];
   ]
   , {j, numPoints}
  ];
  AUpTimeTerms[[n]] = myTable;
  , {n, 2, 14}
 ];
 (* Evaluate the approximation with two terms *)
myTable1 = AUpTimeTerms[[1]];
myTable2 = AUpTimeTerms[[2 ]];
myTable3 = Table[0,{numPoints}];
Do[
    myTable3[[j]] =  myTable1[[j]]+ myTable2[[j]];
    , {j, numPoints}
    ];
AUpTime[[1]] = myTable3;
 (* Evaluate the approximation with more than two terms *)
Do [
    myTable = AUpTime[[n - 1]];  (* cumulative sum up to 2
x(n-1) terms *)
    myTable1 = AUpTimeTerms[[2 n - 1]];
    myTable2 = AUpTimeTerms[[2 n ]];
    myTable3 = Table[0,{numPoints}];
    Do[
      myTable3[[j]] = myTable[[j]]+ myTable1[[j]]+
myTable2[[j]];
      , {j, numPoints}
      ];
    AUpTime[[n]] = myTable3;
    , {n,2,7}
    ];
 endTime = TimeUsed[];
(* Display the time usage in seconds*)
Print["CPU Time Usage: ",SetPrecision[endTime-startTime,5],
" seconds"];
 (*********** Calculating downtime-based availability
***********)
```

```
 (* Create Table of seven empty entries *)
ADownTime = Table[0, {7}];
ADownTimeTerms = Table[0, {14}];
startTime = TimeUsed[];
 (* Evaluate the first term *)
myTable= Table[0,{numPoints}];
```

$$\text{firsttermStar1[s\_]} = \frac{e^{-s\mu+\frac{1}{2}s^2\sigma^2}}{s+\rho};$$

$$\text{firsttermStar2[s\_]} = \text{fccStar[s, 1] oneMinusgrStar[s, 1]} + \text{Fbar[Ta, 1]}\, e^{-\text{Ta}\, s}\, \text{oneMinusgpStar[s, 1]};$$

```
 Do[
    t=points[[j]];
    If[t < Ta,
      myTable[[j]] =GWR[firsttermStar1,t];,
      myTable[[j]] =GWR[firsttermStar2,t];
      ];
    , {j, numPoints}
    ];
ADownTimeTerms[[1]] = myTable;
  (* Evaluate the remaining terms *)
  Do[
    myTable = Table[0, {numPoints}];
    sumTermsInAStar1[s_] :=
```

$$\beta r^{n-1}\rho^{n-1}\, e^{-s\mu\frac{1-\alpha^n}{1-\alpha}+\frac{1}{2}s^2\sigma^2\frac{1-\alpha^{2n}}{1-\alpha^2}}\prod_{k=1}^{n}\frac{\beta r^{-k+1}}{s+\beta r^{-k+1}\rho};$$

```
    sumTermsInAStar2[s_] :=
```

$$\frac{1}{s}\,\text{hmoneStar[s, n - 1]}\,(\text{fccStar[s, n] oneMinusgrStar[s, n]} + \text{Fbar[Ta, n]}\, e^{-\text{Ta}\, s}\,\text{oneMinusgpStar[s, n]});$$

```
    Do[
     t = points[[j]];
     If[t < Ta,
      myTable[[j]] = GWR[sumTermsInAStar1, t];,
      myTable[[j]] = GWR[sumTermsInAStar2, t];
     ];
     , {j, numPoints}
    ];
    ADownTimeTerms[[n]] = myTable;
    , {n, 2, 14}
   ];

 (* Evaluate the approximation with first two terms *)
myTable1 = ADownTimeTerms[[1]];
```

```
myTable2 = ADownTimeTerms[[2]];
myTable3 = Table[0,{numPoints}];
Do[
    myTable3[[j]] = 1-( myTable1[[j]]+ myTable2[[j]]);
    , {j, numPoints}
    ];
ADownTime[[1]] = myTable3;


(* Evaluate the approximation with more than two terms *)
Do [
    myTable = ADownTime[[ n -1]]; (* 1 - cumulative sum upto
2(n-1) terms *)
    myTable1 = ADownTimeTerms[[2 n - 1]];
    myTable2 = ADownTimeTerms[[2 n ]];
    myTable3 = Table[0,{numPoints}];
    Do[
      myTable3[[j]] = myTable[[j]]-( myTable1[[j]]+
myTable2[[j]]);
      , {j, numPoints}
      ];
    ADownTime[[n]] = myTable3;
    , {n,2,7}
    ];

endTime = TimeUsed[];

(* Display the time usage in seconds*)
Print["CPU Time Usage:",SetPrecision[endTime-startTime,5], "
seconds"];
 (*********** Calculating quasi-renewal function
***********)

QTable = Table[0, {numPoints}];
startTime = TimeUsed[];
Do[
    t=points[[m]];
    If[t < Ta,
      QTable [[m]] =  GWR[quasiRenFnStar1,t],
      QTable [[m]] =  GWR[quasiRenFnStar2,t];
      ];
    , {m, numPoints}
    ];
endTime = TimeUsed[];

(* Display the time usage in seconds*)
Print["CPU Time Usage:",SetPrecision[endTime-startTime,5], "
seconds"];

Print[SetPrecision[SessionTime[],5]];
Date[]
```

Vita

Boonyarit Intiyot

Boonyarit Intiyot was born to the family of Am-on and Somboon Intiyot in Lampang, a northern province of Thailand. In 1990, he was admitted to the Development and Promotion of Science and Technology Talents Project (DPST), a Thai government project whose main objective is to produce quality scientists for the Thai science community. The project provided scholarship funding for Boonyarit's study from high school through the doctorate level. He arrived in the United States in 1994 to pursue his study in science and received a Bachelor of Science Cum Laude and Degree with Distinction in Mathematics in May 1998 from the University of Delaware, Newark, Delaware. During his final year, he worked on a senior thesis in the cryptography area. In the same year, he became a member of Pi Mu Epsilon Honorary National Mathematics Society.

In December 2001, Boonyarit received a Master of Science in Mathematics from the University of Arizona, Tucson, Arizona. His master thesis research area is in cryptography and statistics. During his stay at the University of Arizona, he was a member of the American Mathematical Society and wrote cryptography articles for Vcharkarn.com magazine, a monthly online magazine for Thai students.

In December 2007, Boonyarit earned his Doctor of Philosophy in Industrial and Systems Engineering with a concentration in Operations Research from Virginia Tech, Blacksburg, Virginia. His dissertation research area is in availability modeling and analysis. During his stay at Virginia Tech, Boonyarit worked on a number of simulation projects. He created a number of Java graphical user interface programs as a part of a tutorial/educational package for TRANSIMS, a transportation planning/simulation software. The application package was used in the TRANSIMS short course for the U.S. Department of Transportation. Later, he joined the Virginia Tech Consortium on Energy Restructuring research group as a graduate research assistant whose main task is to develop a simulation program for energy grid and energy market. The software was intended to be used with small-scale energy grids for educational purposes. He made a presentation on the projects for many occasions. He was also a web administrator for the website of the Virginia Tech Chapter of INFORMS organization during his Ph.D. program.

Boonyarit is currently working at the Department of Mathematics, Faculty of Science, Chulalongkorn University in Bangkok, Thailand. His research interests include mathematical modeling in reliability, simulation application, and optimization.