

Chapter 2 BASIC EQUATIONS OF NONLINEAR CONSTITUTIVE MODELS

TYPES OF NONLINEAR CONSTITUTIVE MODELS

Geomaterials are characterized by nonlinear stress-strain behavior and, often, by time-dependent deformations. The nonlinear behavior may be simulated using several approaches, including nonlinear elastic models and elastoplastic models. Time- and rate-dependent behavior can be described by adding viscous effects to the model equations, as is done in the case of elastoviscoplasticity. This chapter contains a description of the types of constitutive models that are available for geomaterials, and a description of the basic principles of elastoplastic constitutive relations.

Nonlinear Elastic Models

Nonlinear elastic models attempt to simulate the nonlinear stress-strain behavior of geomaterials by making the elastic constants (and consequently the stiffness) depend on stress state and/or accumulated strain. Several nonlinear elastic models have been used to simulate the behavior of geomaterials. Two of the more common nonlinear elastic models are the K - G model or Barron-Sandler model (Naylor et al., 1981) and the hyperbolic model (Kondner, 1963) which was modified for Mohr-Coulomb strength parameters and applied to finite element analysis by Duncan and Chang (1970). Both models have simple hypoelastic formulations in that the elastic moduli (bulk modulus K and shear modulus G for the K - G model, and tangent elastic modulus E_t and tangent Poisson's ratio ν_t for the Duncan-Chang model) are functions of only the stress state and model parameters. The coefficients used to determine the elastic moduli are functions of only the failure criterion. The Duncan-Chang model generally can provide a better fit to laboratory data than the K - G model because the Duncan-Chang model allows the Poisson's ratio to vary throughout the analysis. This feature of the Duncan-Chang model allows critical values of deviator stress (*i.e.*, peak strength and failure) and volume change (*i.e.*, onset of dilatancy) to occur at different stages of the analysis, which is commonly observed in real geomaterials.

Nonlinear elastic models are plagued by several shortcomings as stated in the previous chapter: (1) stress history and path dependency cannot be taken into account during analysis. Therefore, incremental strains are a function of the incremental stresses, rather than of the stress state at which the incremental stress are applied; (2) dilatant response during incremental

compression loading (*e.g.*, shear) violates thermodynamic principles and cannot be simulated within the framework of elasticity; (3) unloading and other changes in loading direction cannot be properly simulated (except by ad hoc model modification) because the stiffness moduli are dependent only on stress and/or strain state.

Elastoplastic Models

Classical elastoplastic models are formulated using the concepts and principles described in this chapter: yield function, strain additivity, incremental elasticity, plastic flow rule, and plastic hardening rule. Elastoplastic models have their basis in the following ideas: (1) the set of allowable stress states in a material is limited to some finite set, which is defined by the yield surface; (2) the behavior of a material is elastic when its stress state is beneath the yield surface, and is elastoplastic when its stress state is on the yield surface; (3) the plastic or irrecoverable response of a material depends not on its incrementally applied load, but on its stress state; and (4) the yield function and set of allowable stress states may change as loading progresses. This evolution is a function of some plastic phenomenon, usually plastic strain or plastic work.

Elastoplastic models range from simple to complex. An example of a simple model is the nonhardening Mohr-Coulomb model with constant plastic flow direction and linear elasticity; this model includes a single yield surface that is also a failure surface. A more complex elastoplastic model is the Cam-Clay model. This model also has only a single yield surface, but is complicated by incorporating yield surface evolution due to isotropic hardening and softening, stress state-dependent plastic flow, and stress-dependent elasticity. Additional complications to elastoplastic models can arise when kinematic hardening, multiple nested yield surfaces, or bounding surfaces are introduced into the formulation. One limitation of elastoplastic models is that time and rate of loading effects do not appear in the constitutive equations; however, these effects can be incorporated into these models by reformulating the model as an elastoviscoplastic model. It is notable that most constitutive models that have been proposed for chalk are elastoplastic models.

Elastoviscoplastic Models

Elastoviscoplastic models are very similar to elastoplastic models except that effects of loading rate are included in the formulation. The concepts and principles on which elastoviscoplastic

models are based are the same as those for elastoplastic models except that for certain formulations, irrecoverable or inelastic phenomena may occur for reasons other than mechanical loading. Also, for some elastoviscoplastic formulations, the idea that the set of allowable stress states in a material is limited to some finite set, may be violated temporarily.

The original elastoviscoplastic approach (Perzyna, 1963) represents only a slight modification to the classical elastoplastic model in that ideas 2, 3, and 4 of classical elastoplasticity apply. However, stress states are allowed to lie outside the yield surface temporarily due to viscous properties of the material. As time passes, plastic straining and plastic hardening occurs and the stress state returns to the yield surface. In newer elastoviscoplastic formulations, the concept of purely elastic behavior disappears. Irreversible strains accumulate continuously due to aging of the material. The equations incorporated into elastoviscoplastic formulations are presented later in this chapter.

FORMULATION OF INCREMENTAL CONSTITUTIVE RELATIONS FOR ELASTOPLASTICITY

Stress-strain relations for geomaterials are nonlinear. This nonlinear behavior may be simulated using several approaches, including nonlinear elastic models and elastoplastic models. As described above, nonlinear elastic models have the advantage of being easier to formulate and implement, but have the disadvantage that elastic models fail to simulate several phenomena which occur during mechanical loading of geomaterials.

Elastoplastic behavior of geomaterials is path-dependent, because the material behavior depends on the stress history of the material. For this reason, constitutive relations cannot be expressed in integrated form but can only be expressed in incremental form. The incremental constitutive relations can be written using the properties of strain additivity, incremental elasticity, plastic flow rule, and plastic hardening rule. First, the invariants of stress and strain and the concept of plastic yielding will be introduced. The properties listed above, which contribute to the constitutive equations of classical rate-independent elastoplasticity, will also be described. Rate-independence means that behavior does not depend on time or rate of loading. A brief discussion is also included on more complex formulations of plasticity, including elastoviscoplasticity and bounding surface plasticity.

Stress and Strain Invariants

Certain stress- and strain-based quantities called invariants have been defined as described below. These quantities are useful because they do not depend on the choice of reference axes and so are invariant to the orientation of the reference axes. Because they are independent of reference axes, stress and strain invariants are useful in constitutive modeling. The various invariants which have been defined for 2-dimensional and 3-dimensional space are listed in the following table.

	2-D	3-D
Stress invariants	p, q	p, q, θ
Strain and strain increment invariants	$d\varepsilon_v, d\varepsilon_s$	$d\varepsilon_v, d\varepsilon_s, \theta_\varepsilon$

The 3-dimensional stress invariants which are used in this report (mean stress p , deviatoric stress q , and Lode angle θ) are defined as follows:

$$p = \frac{1}{3}\sigma_{ii} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \quad (2.1)$$

$$q = \sqrt{3J_2} = \sqrt{\frac{3}{2}s_{ij}s_{ij}} = \sqrt{\frac{1}{2}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\sigma_{12}^2 + 6\sigma_{13}^2 + 6\sigma_{23}^2]} \quad (2.2)$$

$$\theta = \frac{1}{3}\sin^{-1}\left(-\frac{3\sqrt{3}J_3}{2J_2^{1.5}}\right) \quad (2.3)$$

In these definitions, J_2 and J_3 are themselves stress invariants, and are defined as:

$$J_2 = \frac{1}{2}s_{ij}s_{ij} = \frac{1}{6}[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6\sigma_{12}^2 + 6\sigma_{13}^2 + 6\sigma_{23}^2] \quad (2.4)$$

$$J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki} = |s_{ij}| \quad (2.5)$$

and s_{ij} is the deviatoric stress tensor:

$$s_{ij} = \sigma_{ij} - \delta_{ij}p \quad (2.6)$$

The quantity δ_{ij} is the Kronecker delta, which takes the value $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. The summation convention over repeated indices is adopted in the notations.

These stress invariants p , q , and θ will be used in this proposal instead of another set of stress invariants because these quantities can be visualized graphically as shown in Figure 2.1, and because elastoplastic constitutive models for geomaterials are typically most efficiently described in terms of these invariants. The mean stress p is proportional to the distance from the origin of stress space along the hydrostatic axis to the π -plane in which the stress point lies (p has a constant value on the π -plane); the deviatoric stress q is proportional to the distance in the π -plane from the hydrostatic axis to the stress point; and the Lode angle θ shows the position of the stress point in the π -plane with respect to the three principal stresses, showing especially the effect of the intermediate principal stress σ_2 . For triaxial compression conditions ($\sigma_1 > \sigma_2 = \sigma_3$), the Lode angle $\theta = -30^\circ$; for triaxial extension conditions ($\sigma_1 = \sigma_2 > \sigma_3$), the angle $\theta = 30^\circ$; and for other triaxial stress conditions, the Lode angle lies somewhere between these two extremes.

The 3-dimensional strain increment invariants (incremental volumetric strain $d\varepsilon_v$, incremental generalized shear strain $d\varepsilon_s$, and Lode angle θ_ε) are defined as:

$$d\varepsilon_v = d\varepsilon_{ii} = d\varepsilon_{11} + d\varepsilon_{22} + d\varepsilon_{33} \quad (2.7)$$

$$d\varepsilon_s = \sqrt{\frac{2}{3} de_{ij} de_{ij}} = \sqrt{\frac{2}{9} [(d\varepsilon_{11} - d\varepsilon_{22})^2 + (d\varepsilon_{22} - d\varepsilon_{33})^2 + (d\varepsilon_{33} - d\varepsilon_{11})^2 + 6d\varepsilon_{12}^2 + 6d\varepsilon_{13}^2 + 6d\varepsilon_{23}^2]} \quad (2.8)$$

$$\theta_\varepsilon = \frac{1}{3} \sin^{-1} \left(-\frac{3\sqrt{3}J'_3}{2J_2^{1.5}} \right) \quad (2.9)$$

J'_2 and J'_3 are themselves strain increment invariants, and are defined as:

$$J'_2 = \frac{1}{2} de_{ij} de_{ij} = \frac{1}{6} [(d\varepsilon_{11} - d\varepsilon_{22})^2 + (d\varepsilon_{22} - d\varepsilon_{33})^2 + (d\varepsilon_{33} - d\varepsilon_{11})^2 + 6d\varepsilon_{12}^2 + 6d\varepsilon_{13}^2 + 6d\varepsilon_{23}^2] \quad (2.10)$$

$$J'_3 = \frac{1}{3} de_{ij} de_{jk} de_{ki} = |de_{ij}| \quad (2.11)$$

and de_{ij} is the deviatoric incremental strain tensor, which is analogous to the deviatoric stress tensor:

$$de_{ij} = d\varepsilon_{ij} - \frac{1}{3} \delta_{ij} d\varepsilon_v \quad (2.12)$$

The angle θ_ε is analogous to the Lode angle θ in stress space.

The 2-dimensional stress and strain invariants which are used in this report are similar to those for 3-D:

$$p = \frac{1}{2}\sigma_{ii} = \frac{1}{2}(\sigma_{11} + \sigma_{22}) \quad (2.13)$$

$$q = \sqrt{3J_2} = \sqrt{\frac{1}{2}s_{ij}s_{ij}} = \sqrt{\frac{1}{4}[(\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2]} \quad (2.14)$$

$$d\varepsilon_v = d\varepsilon_{ii} = d\varepsilon_{11} + d\varepsilon_{22} \quad (2.15)$$

$$d\varepsilon_s = \sqrt{2de_{ij}de_{ij}} = \sqrt{(d\varepsilon_{11} - d\varepsilon_{22})^2 + 4d\varepsilon_{12}^2} \quad (2.16)$$

except that the 2-dimensional deviatoric incremental strain tensor de_{ij} is defined differently:

$$de_{ij} = d\varepsilon_{ij} - \frac{1}{2}\delta_{ij}d\varepsilon_v \quad (2.17)$$

The 2-dimensional stress invariants represent a special case of the 3-dimensional invariants, in which the Lode angle θ has a constant value corresponding to triaxial compression conditions.

Yield Surface and Yield Function

In stress space, it is common to divide stress states into those which are stable, unstable (*i.e.*, at yield or failure), and impossible (*i.e.*, in illegal stress space). The boundary between stable stress states and impossible stress states is called the yield surface. Stress states which lie inside the yield surface are stable and are represented by elastic material behavior. Stress states which lie outside the yield surface are impossible and cannot be attained in the context of rate-independent elastoplasticity (exceptions to this rule occur in the context of elastoviscoplasticity and will be discussed later). Stress states which lie on the yield surface are at yield, and are characterized by a combination of elastic and plastic behavior. See Figure 2.2 for an example of a yield surface in stress space. Depending on the particular constitutive model, the yield surface may be fixed in stress space, or the yield surface may change size, position, or shape in stress space as plastic deformation accumulates. The parameters which control the size, position, and shape of the yield surface are the plastic hardening parameters q_α .

The yield surface is usually described by an equation which describes the shape of the yield surface in stress space in terms of plastic hardening parameters and individual stress

components or stress invariants; stress invariants are preferred so that the model is not dependent on the reference axes. It is possible to then write the equation which describes a specific yield surface in stress space in such a form that the equation returns a value f for any combination of stress components or invariants and plastic hardening parameters in which the equation is expressed. Such an equation is called a yield function:

$$f = f(\sigma_{ij}, q_\alpha) \quad (2.18)$$

A yield function is usually expressed with the convention that $f=0$ if a stress point lies on the yield surface, $f<0$ if a stress point lies inside the yield surface, and $f>0$ if a stress point lies outside the yield surface, in impossible stress space.

Strain Additivity

The time-independent elastic and plastic strain increments act independently of each other. Elastic strains accompany deformation in which the energy of deformation is stored, such that the elastic deformation and energy of elastic deformation are fully reversible and may be recovered during removal of the deformation-causing loads. Plastic strains accompany deformation in which the energy of deformation is dissipated, so the plastic deformation and energy of plastic deformation are not recovered when the loads are removed. A total strain increment $d\varepsilon_{ij}$ may be additively decomposed into its recoverable or elastic ($d\varepsilon_{ij}^e$), and inelastic or irrecoverable ($d\varepsilon_{ij}^i$), components. For classical elastoplasticity, the irrecoverable strain increments are equal to the plastic strain increments ($d\varepsilon_{ij}^p$):

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p \quad (2.19)$$

The role of time-dependent deformations and strains may be incorporated into the additivity postulate using several different formulations as discussed later.

Incremental Elasticity

The generalized Hooke's Law relates stress increment $d\sigma_{ij}$ (of the Cauchy stress tensor) to elastic strain increment $d\varepsilon_{ij}^e$ using the elastic constitutive tensor D_{ijkl}^e or the elastic compliance tensor C_{ijkl}^e :

$$d\sigma_{ij} = D_{ijkl}^e d\varepsilon_{kl}^e \quad (2.20a)$$

$$d\varepsilon_{ij}^e = C_{ijkl}^e d\sigma_{kl} \quad (2.20b)$$

These tensors are inverses of each other: $D_{ijkl}^e = (C_{ijkl}^e)^{-1}$.

Plastic Flow Rule

The direction of plastic flow, which determines the relative magnitudes of the plastic strain components during elastoplastic loading, is represented for each yield mechanism by a plastic potential surface g . Plastic flow occurs in the direction normal to the plastic potential surface. The magnitude of plastic strain is proportional to the plastic multiplier φ . Therefore, the plastic flow rule may be represented as:

$$d\varepsilon_{ij}^p = \varphi \frac{\partial g}{\partial \sigma_{ij}} \quad (2.21)$$

If the plastic potential surface is coincident with the yield surface at all points ($f = g$), plastic flow is said to be associated because the direction of plastic flow is associated with the yield surface. If the plastic potential surface is not coincident with the yield surface, flow is said to be non-associated. Figure 2.3 illustrates the difference between associated and non-associated flow.

Plastic Hardening Rule

A yield surface may be either hardening or non-hardening. A non-hardening yield surface remains in a fixed position in stress space at all stages of loading, and thus for rate-independent plasticity represents a set of limiting stress states which can never be exceeded. A hardening yield surface may undergo some type of change in position or shape in stress space when the material is subjected to certain loading conditions, such that the allowable stress states in the material change as material loading progresses. Hardening yield surfaces move outward, in the loading direction, as loading continues. Generally, since only the energy related to the elastic component of elastoplastic deformation is recoverable, hardening of a yield surface is related to some aspect of the plastic component of elastoplastic deformation (*e.g.*, plastic strain or plastic work).

The size, position, and/or shape of the yield surface in stress space may change as elastoplastic loading continues. The parameters which define the size, position, and/or shape of the yield surface are called the plastic hardening parameters q_α . The plastic hardening function h_α describes the evolution of the plastic hardening parameter dq_α as a function of either plastic strain or plastic work:

$$dq_\alpha = \phi h_\alpha \quad (2.22)$$

The plastic hardening function h_α and plastic hardening parameter dq_α may take various forms depending on the nature of the hardening function, so α may replace a different number of indices. For the case where the hardening function affects the size of the yield surface (*i.e.*, isotropic hardening), h_α and dq_α are scalar quantities; for the case where the hardening function affects the position of the yield surface in stress space (*i.e.*, translational and rotational kinematic hardening), these are second-order tensors; for the case where the hardening function affects the shape of the yield surface (*i.e.*, distortional kinematic hardening), these are fourth-order tensors.

Elastoviscoplasticity

It is possible to incorporate rate or time effects into the framework of elastoplasticity by modifying the rate-independent constitutive equations. The simplest and most common framework to incorporate rate-dependence is elastoviscoplasticity. In contrast to rate-independent elastoplasticity, elastoviscoplastic formulations are commonly expressed in terms of the constitutive rate equations. In elastoviscoplasticity, additivity postulate is modified such that the strain rate is additively decomposed. The decomposition of strain rates is modified such that the irreversible component becomes the viscoplastic component ($\dot{\epsilon}_{ij}^{vp}$):

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^{vp} \quad (2.23)$$

The superior dot indicates a time derivative. Elastoviscoplastic models are typically formulated using either the “overstress” approach or the “rate-type” approach. Perzyna (1963) formulated the viscoplastic strain rate using the overstress approach with the following flow rule:

$$\dot{\epsilon}_{ij}^{vp} = \gamma \langle \Phi(f) \rangle \frac{\partial g}{\partial \sigma_{ij}} \quad (2.24)$$

where γ is the fluidity parameter and Φ is a function of the yield function f ; these take the place of the plastic multiplier ϕ used in rate-independent elastoplasticity. The Macauley brackets indicate that the bracketed quantity equals zero if the quantity inside the brackets is negative, and equals the quantity inside the brackets if that value is nonnegative. According to the original viscoplastic formulation, the viscoplastic strain rate is zero if the stress point is located in the elastic zone below the yield surface, and is proportional to a function of the yield function if the stress point is located outside the yield surface; excursions outside the yield surface into “illegal stress space” are temporarily allowed in this formulation of elastoviscoplasticity.

In the original overstress approach of Perzyna (1963), viscoplastic behavior only occurs for stress states which lie outside the yield surface. As time passes, plastic straining and plastic hardening occurs and the stress state returns to the yield surface. Overstress models have been modified such that the overstress is measured relative to a reference surface which may not be the same as the yield surface. In this way, overstress models allow for time-dependent deformations to occur at stress states below the yield surface. The overstress approach is illustrated in Figure 2.4.

The rate-type approach has its roots in the concept of time-lines and equivalent time (Bjerrum, 1967) and represents a more significant conceptual change to the classical elastoplastic model. In this case, the concept of purely elastic behavior disappears because any strain increment has both “instant” and “delayed” components. Irreversible strains accumulate continuously due to aging of the material, and the rate of irreversible straining depends on its equivalent age t_e , relative to an equivalent newly deposited material of minimum age. Each time-line (Figure 2.5) represents a material age which corresponds to a steady-state viscoplastic strain rate. If a material is loaded at a constant rate, its stress-strain curve becomes tangent to a time-line.

Hypoplasticity and Bounding Surface Plasticity

Hypoplasticity is a more complex formulation than classical plasticity. In classical plasticity, the plastic strain rate direction depends only on the stress state. In hypoplasticity, the plastic strain rate direction depends on the stress state and on the stress rate direction (Dafalias, 1986). One way to account for hypoplasticity in constitutive modeling is by using anisotropic hardening models.

Anisotropic hardening constitutive models were introduced to modeling of geomaterials in an effort to more accurately simulate the response of geomaterials, especially to reversed and cyclic loading. A problem in anisotropic hardening models occurs in describing the evolution of the yield surface and determining the direction of plastic flow for a given loading step. Mroz (1967) introduced multi-surface plasticity to geomechanics, in which many nested yield surfaces are encountered and activated in succession during continued elastoplastic loading (Figure 2.6). Each yield surface has its own yield function; if yield surface i is active, $f_i = 0$, while $f_i < 0$ if it is inactive. These nested yield surfaces are formulated such that each yield surface can become tangent to the next surface, but may never cross it. Each active yield surface contributes to the plastic hardening modulus. In multi-surface plasticity formulations, the hardening functions for all active yield surfaces must be calculated and kept in memory to determine the instantaneous hardening modulus.

Bounding surface plasticity was first applied to the study of metal behavior by Krieg (1975) and later applied to geomechanics by Mroz et al. (1978). Bounding surface plasticity formulations have been commonly used in geomechanics in recent years (*e.g.*, Dafalias, 1986; Manzari and Nour, 1997; Borja et al., 2001). Bounding surface formulations are more efficient than general multi-surface formulations in that only two surfaces in stress space are required to describe the behavior of a material. These surfaces include the loading surface and bounding surface, shown in Figure 2.7. The loading surface is analogous to the yield surface in that a stress state inside the loading surface (if possible) is characterized by elastic behavior, while a stress state on the loading surface is characterized by elastoplastic behavior. The loading surface evolves during continued elastoplastic loading in the same way that the yield surface evolves for classical rate-independent elastoplastic formulations. The role of the bounding surface is to determine the value of the hardening modulus during elastoplastic loading. As for general multi-surface plasticity, the loading surface may become tangent to but never cross the bounding surface; often, the loading surface is similar to the bounding surface. The hardening modulus, and therefore the evolution of the loading surface, is determined by some function of the proximity of the conjugate stress points. The conjugate stress points include the stress point on the loading surface and the image stress point $\underline{\sigma}_{ij}$ on the bounding surface (Figure 2.7). The form of a typical hardening function is:

$$h_\alpha = h_\alpha \left[(\underline{\sigma}_{ij} - \sigma_{ij}) (\underline{\sigma}_{ij} - \sigma_{ij}) \right] \quad (2.25)$$

In addition to the evolution of the loading surface, the bounding surface itself may evolve as a function of some external variable such as a state variable.

Incremental Relations

The incremental relations for an elastoplastic model may be written using the properties of strain additivity, incremental elasticity, and plastic flow rule. Under stress-controlled loading conditions, the strain increment may be calculated as a function of the elastic compliance tensor, stress increment, direction of plastic flow, and plastic multiplier:

$$d\varepsilon_{ij} = C_{ijkl}^e d\sigma_{kl} + \varphi \frac{\partial g}{\partial \sigma_{ij}} \quad (2.26)$$

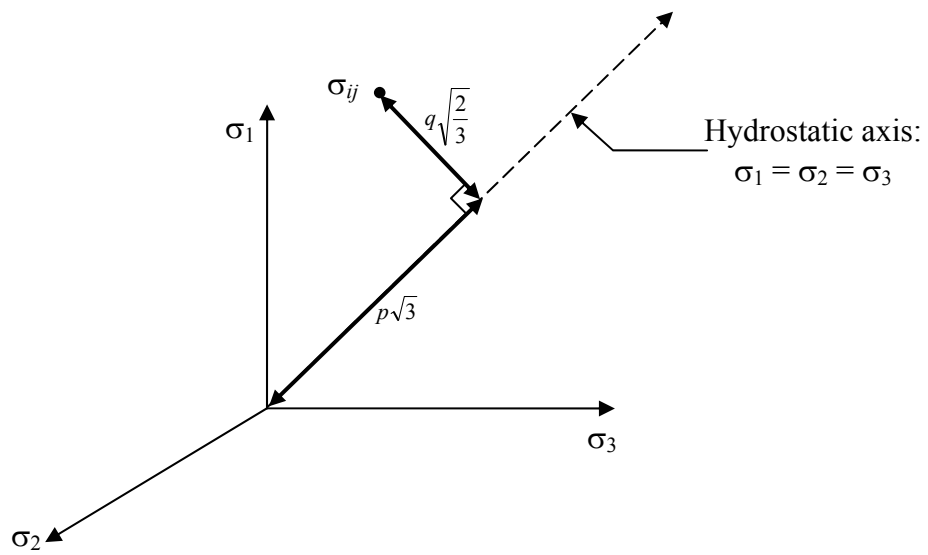
Under strain-controlled loading conditions, the stress increment may be calculated as a function of the elastic constitutive tensor, total strain increment, direction of plastic flow, and plastic multiplier:

$$d\sigma_{ij} = D_{ijkl}^e \left(d\varepsilon_{kl} - \varphi \frac{\partial g}{\partial \sigma_{kl}} \right) \quad (2.27)$$

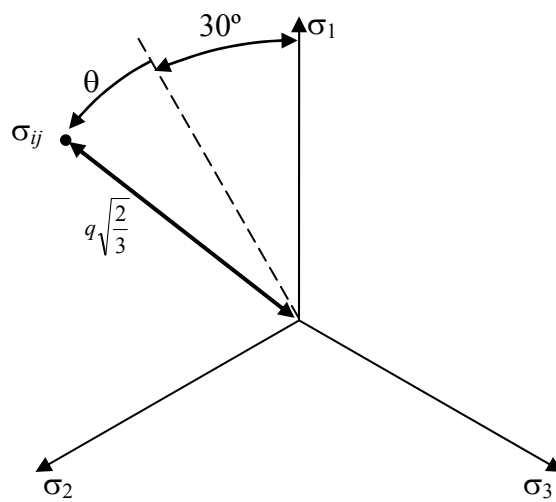
Integration of the incremental constitutive relations over a finite loading step is discussed in Chapter 7. For elastoviscoplastic formulations, the constitutive equations are formulated as rate equations, so the strain rate must be integrated with respect to time to obtain the strain increment.

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(a)



(b)

Figure 2.1. Illustration of the 3-dimensional stress invariants p , q , and θ : (a) in principal stress space; (b) projected on the π -plane.

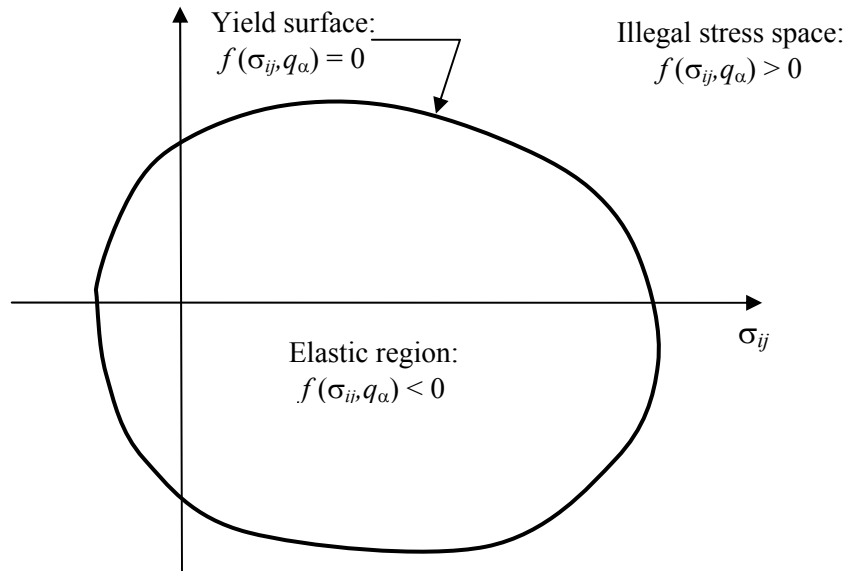


Figure 2.2. Example yield surface in stress space, showing the yield surface as the boundary between the elastic region and impossible stress space.

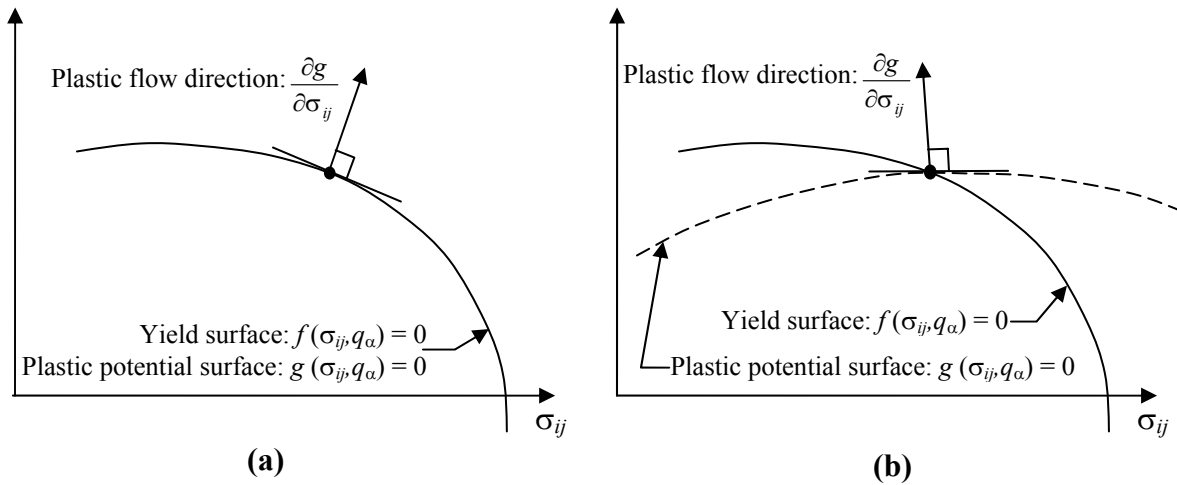


Figure 2.3. Plastic potential surface and plastic flow direction in stress space, for (a) associated flow; (b) non-associated flow.

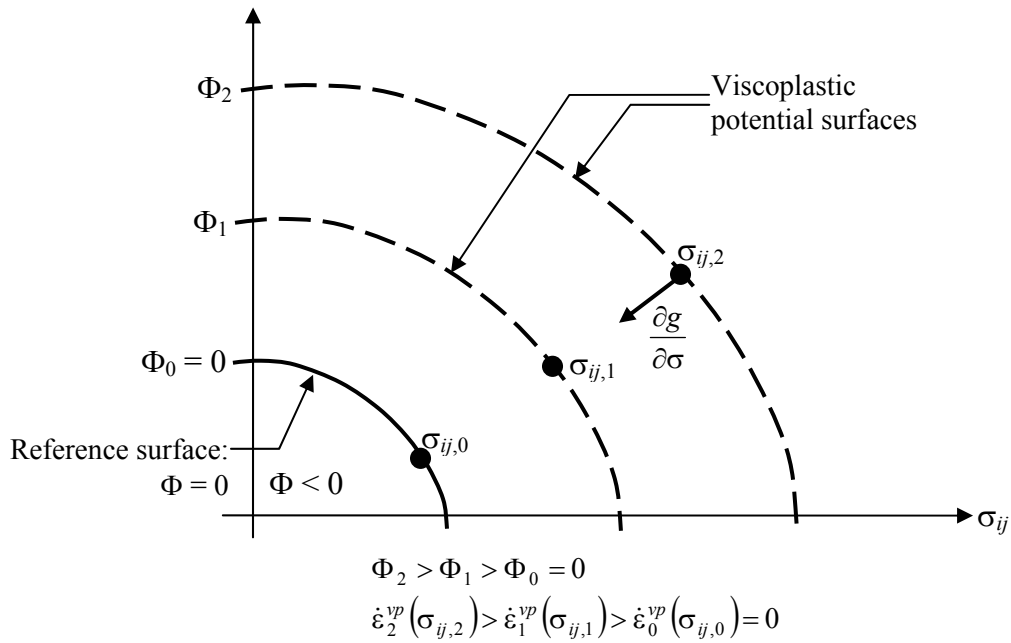


Figure 2.4. Main components of overstress elastoviscoplastic formulation. Viscoplastic strain rate depends on distance in stress space from reference surface, and viscoplastic flow direction is determined by gradient to viscoplastic potential surface.

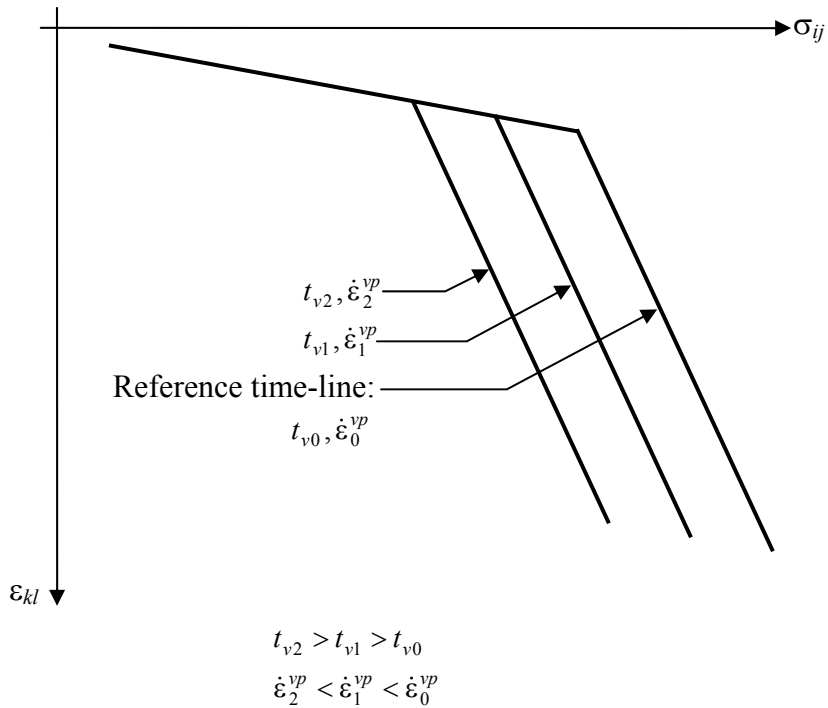


Figure 2.5. Main components of rate-type elastoviscoplastic formulation. Viscoplastic strain rate depends on age of material compared to reference age.

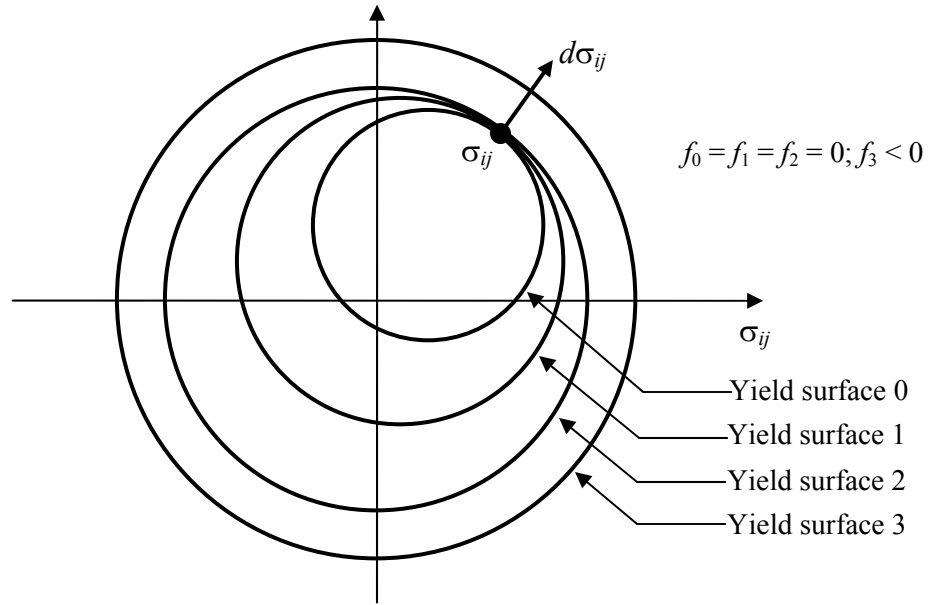


Figure 2.6. Multiple nested yield surfaces in stress space for a multi-surface formulation. Surfaces 0, 1, and 2 are active; surface 3 is not.

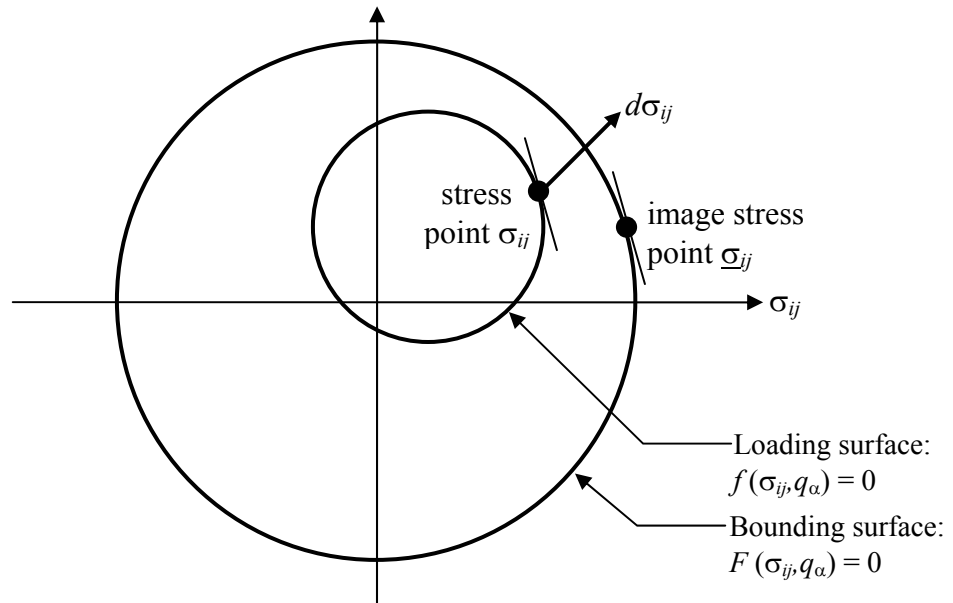


Figure 2.7. Main components of bounding surface formulation. Stress point and image stress point are located at similar points on loading surface and bounding surface.