Appendix B-1

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A NEW SOLUTION ALGORITHM FOR STRAIN-HARDENING ELASTOPLASTIC SOILS

UN NOUVEL ALGORITHME MODELISANT LE DURCISSEMENT DES SOLS ELASTOPLASTIQUES SOUMIS A DES DEFORMATIONS

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ABSTRACT. A new algorithm is proposed for integrating strain-hardening elastoplastic soil models. The algorithm determines the plastic multiplier λ directly by satisfying the yield function $f(\lambda)=0$ and solves the resulting non-linear equation using an iterative secant procedure. Implementation of the new algorithm on the constitutive level and finite element level is discussed, and a numerical example is presented.

RESUME. Cet article propose un nouvel algorithme intégrant les models mathématiques décrivant des sols elastoplastiques de durcissant soumis a des déformations. L'algorithme détermine le facteur de plasticité (λ) en résolvant directement l'équation non-lineaire $f(\lambda)=0$ par un système d'itérations et d'interpolations. L'utilisation de cet algorithme pour des models constitutifs (contraintes-deformations) et des éléments finis est discutée, et un exemple est présente.

1. Introduction

Numerical modeling of elastoplastic response in soils is necessary to correctly simulate certain material responses that are not explained by elasticity, including irrecoverable deformations and dilatancy on application of shear stresses. Many elastoplastic models have been developed to simulate soil behavior, including the Mohr-Coulomb and Cam-Clay models. Integration of these models on the constitutive and finite element levels often leads to problems with accuracy, stability, uniqueness and robustness of solution. These problems arise from the fact that the soil behavior is path dependent. Parameters which affect the solution of elastoplastic problem (e.g., the plastic multiplier λ, and gradients to the yield surface $\partial f/\partial\sigma_{ij}$ and the plastic potential surface $\partial g/\partial\sigma_{ij}$) may be evaluated at many different points along the stress paths, but evaluation of these parameters at different points in stress space yield different solutions.

Integration algorithms have been developed using two general procedures; these include incremental methods and iterative methods. For incremental methods, the initial load is subdivided into several smaller loads which are applied in succession. The stiffness matrix is modified at the start of each load step to account for material nonlinearity, but there is no guarantee that the final stress point will lie on the correct stress-strain curve. Incremental procedures are inaccurate due to "drift" from the true behavior.

For iterative methods, the total load is initially applied in full or in larger load increments than used in incremental procedures. To insure that the stress point always lies on the current yield surface, fractions of the initial load must be re-applied in subsequent iterations to account for material nonlinearity. The fraction of the initial load that must be applied for each iteration is a function of the residual stress between the initially predicted stress and the "true" stress. The residual stress varies with the magnitude of the plastic multiplier λ and with the gradients to the plastic potential surface. The residual stress is commonly determined using either a one-step return algorithm approach or a substepping approach. An elastic predictor-plastic corrector approach is commonly used as the return mapping algorithm, in which the trial stress point is calculated assuming purely elastic behavior and the final stress point is returned from the trial stress point to the final yield surface in the direction of plastic flow.

Return mapping algorithms are characterized by the stress points at which the functions of interest are evaluated. For explicit procedures, the functions of interest are evaluated at the starting stress point where the trial stress path intersects the original yield surface; for midpoint procedures, the functions of interest are evaluated at a point other than the initial stress point or the final stress point; for implicit procedures, the functions of interest are evaluated at the final stress point on the final yield surface. Various return algorithms are described by Ortiz and Simo (1986), and Borja and Lee (1990), among others.

Implementation of this type procedure into finite element calculations may be achieved by converting the residual stresses to residual forces and iteratively minimizing the residual forces, or by modifying the continuum stiffness matrix into a consistent stiffness matrix. Development of the consistent stiffness matrix for various models is described by Simo and Taylor (1985), and Jeremic and Sture (1997), among others. For the substepping approach, the residual stress is calculated by subdividing the strains from each iteration into equal "substeps" and finding the stress changes for each substep using one of the solution procedures (i.e., explicit, midpoint, implicit) described above. The substepping approach is described by Sloan (1987), and Potts and Ganendra (1994), among others.

The objective of the study described herein is to present a new method to integrate elastoplastic constitutive models, and to illustrate the efficiency of the new solution algorithm. A detailed description of the use of the new algorithm on the constitutive level and implementation of the algorithm into finite element calculations is given in the paper, with results and comparison to procedures commonly used at present.

2. Fundamental Equations of Elastoplasticity

The response of most elastoplastic materials can be characterized by the following constitutive equations:

$$d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p \tag{1}$$

$$d\sigma_{ij} = D_{ijkl} d\varepsilon_{kl}^e \tag{2}$$

$$d\varepsilon_{ij}^p = \lambda \frac{\partial g}{\partial \sigma_{ij}}(\sigma_{ij}, q_\alpha) \tag{3}$$

$$dq_\alpha = \lambda h_\alpha(\sigma_{ij}, q_\alpha) \tag{4}$$

In these equations, $d\varepsilon_{ij}$, $d\varepsilon_{ij}^e$, and $d\varepsilon_{ij}^p$ are increments of the total, elastic, and plastic stress tensors, $d\sigma_{ij}$ is the increment of the Cauchy stress tensor, D_{ijkl} is the elasticity tensor, λ is a plastic multiplier, $\partial g / \partial \sigma_{ij}$ is the plastic flow direction, q_α is a set of plastic variables, and h_α is the plastic hardening function. Eqs. (1)-(4) represent the properties of additivity, elasticity, flow rule, and hardening rule.

The plastic multiplier λ is related to the magnitude of plastic strain as indicated in the following equations (expressed in Karush-Kuhn-Tucker form), all of which must be satisfied simultaneously:

$$f(\sigma_{ij}, q_\alpha) \le 0 \tag{5}$$

$$\lambda \ge 0 \tag{6}$$

$$f\lambda = 0 \tag{7}$$

In Eqs. (5)-(7), $f(\sigma_{ij}, q_\alpha)$ represents the yield function of the material. The consequence of simultaneously satisfying these equations is that material must deform elastically when the current stress point is not on the current yield surface ($f < 0$), and the stress point must rest on the yield surface ($f = 0$) when plastic deformation occurs during loading. This last stipulation is represented by Prager's consistency condition:

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial q_\alpha} dq_\alpha \tag{8}$$

3. Description of New Algorithm

The core of the proposed procedure is the rejection of Prager's consistency condition as a basis for determining the plastic multiplier λ and for establishing the elastoplastic constitutive tensor. Instead, the proposed procedure will determine λ directly by satisfying the yield function $f(\sigma_{ij}, q_\alpha) = 0$ at all stages of plastic deformation. Given the current stresses σ_{ij}^o and plastic hardening variables q_α^o, the yield function should also be satisfied during a load increment causing a change in stress, $d\sigma_{ij}$, and change in plastic variables, dq_α:

$$f(\sigma_{ij}^o + d\sigma_{ij}, q_\alpha^o + dq_\alpha) = 0 \tag{9}$$

Satisfying the yield function is a much stronger requirement than satisfying Prager's consistency condition that the gradient to the yield function should be equal to zero. The reason for this is that Prager's consistency condition is only a first-order approximation of the yielding condition. This can be shown by taking the Taylor-series expansion of $f(\sigma_{ij}^o + d\sigma_{ij}, q_\alpha^o + dq_\alpha)$:

$$f(\sigma_{ij}^o + d\sigma_{ij}, q_\alpha^o + dq_\alpha) = f\left(\sigma_{ij}^o\right) + f'\left(\sigma_{ij}^o\right) d\sigma_{ij} + f'\left(h_\alpha^o\right) dh_\alpha + \frac{f''\left(\sigma_{ij}^o\right)}{2!} d\sigma_{ij}^2 + \frac{f''\left(h_\alpha^o\right)}{2!} dh_\alpha^2 + \ldots \tag{10}$$

For the case of $f(\sigma_{ij}^o) = 0$ and if second- and higher-order derivatives are neglected, Eq. (10) reduces to:

$$f'\left(\sigma_{ij}^o\right) d\sigma_{ij} + f'\left(h_\alpha^o\right) dh_\alpha = 0 \tag{11}$$

It can be seen from Eq. (11) that the consistency condition (Eq. 8) corresponds to the first-order term of the Taylor-series expansion, and thus, Prager's consistency condition is only a first-order approximation of the $f(\sigma_{ij}^o + d\sigma_{ij}, q_\alpha^o + dq_\alpha) = 0$ condition.

To insure a more robust integration of elastoplastic models, it is proposed to derive the plastic multiplier λ directly from the yield function $f(\sigma_{ij}^o + d\sigma_{ij}, q_\alpha^o + dq_\alpha) = 0$.

Combining Eqs. (1), (2) and (3) gives the stress increments $d\sigma_{ij}$ in terms of the given strain increments $d\varepsilon_{ij}$ and the plastic multiplier λ:

$$d\sigma_{ij} = D_{ijkl}\left(d\varepsilon_{kl} - \lambda \frac{\partial g}{\partial \sigma_{kl}} \right) \tag{12}$$

Substituting Eq. (12) together with Eq. (4) in Eq. (11) yields an equation in terms of λ:

$$f(\lambda) = f\left(\sigma_{ij}^o + D_{ijkl}\left(d\varepsilon_{kl} - \lambda \frac{\partial g}{\partial \sigma_{kl}} \right), q_\alpha^o + \lambda h_\alpha \right) = 0 \tag{13}$$

Solution of Eq. (13) yields an exact value of the plastic multiplier λ regardless of the magnitude of the given strain increments. The yield function is satisfied at all stages of plastic loading, and the value of λ determined from Eq. (13) ensures that the change in the plastic hardening variable is consistent with the stress change. Once the plastic multiplier λ has been determined, the stress increments $d\sigma_{ij}$ can be solved by substituting λ in Eq. (12).

As will be shown below, Eq. (13) is, in general, non-linear in λ except for certain types of perfectly plastic, non-hardening models with linear yield and plastic potential functions. Thus, iterative solution procedures have to be resorted to in solving λ in case the yield function $f(\sigma_{ij}, q_\alpha)$, the plastic potential $g(\sigma_{ij}, q_\alpha)$, or the hardening function $h_\alpha(\sigma_{ij}, q_\alpha)$ is non-linear. Newton-Raphson iterative methods are commonly used to solve Eq. (13), starting from the point $\lambda = 0$ which corresponds to purely elastic response. The application of the Newton-Raphson iteration is described by Ortiz and Simo (1986), and Borja and Lee (1990), among others.

A new algorithm proposed here has its basis in the premise that it is possible to calculate the value of the yield function $f(\lambda)$ at two points, which represent purely elastic behavior and perfectly plastic behavior. A strain-hardening material gives a response between purely elastic response and perfectly plastic response, as shown in Figure 1. Note that the values of λ can be calculated analytically for most elastoplastic models at both of these points, which are represented by $\lambda_{\min} = 0$ (purely elastic response) and $\lambda_{\max}$ (perfectly plastic response).

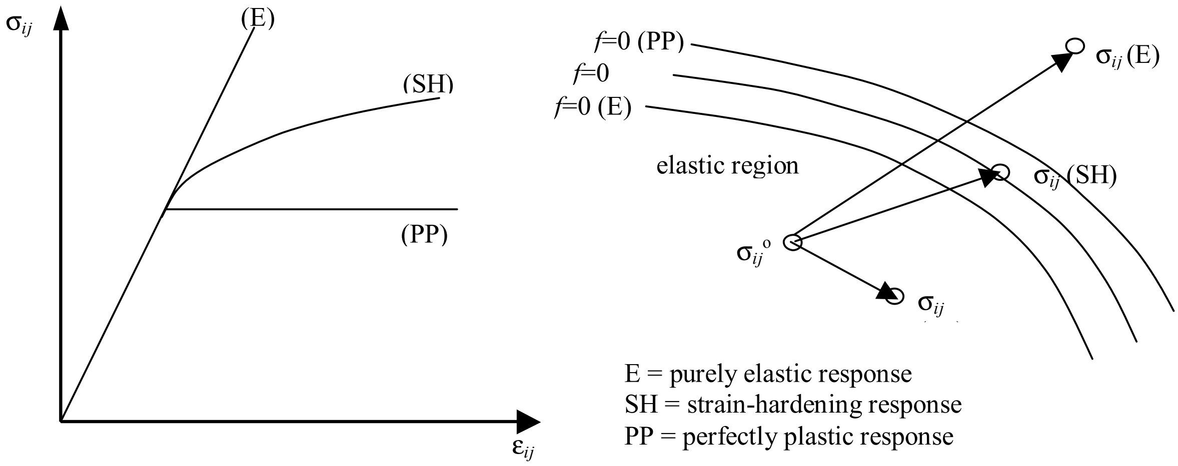

Figure 1. Graphical representation that strain-hardening response is bounded by purely elastic response and perfectly plastic response in both the stress-strain space (left) and the stress space (right).

Given the values of $\lambda_{\min}$ and $\lambda_{\max}$, Eq. (13) may be solved iteratively by interpolating between these points as shown in Figure 2 where the function $f(\lambda)$ is plotted against λ. A secant method is used assuming a local linear variation of $f(\lambda)$ between $\lambda_{\min}$ and $\lambda_{\max}$. From Figure 2 an improved estimate of the plastic multiplier λ_{new} can be obtained as:

$$\lambda_{new} = \frac{\lambda_{\max} f(\lambda_{\min}) - \lambda_{\min} f(\lambda_{\max})}{f(\lambda_{\min}) - f(\lambda_{\max})} \tag{14}$$

The value of $f(\lambda_{new})$ is then calculated. This procedure can be repeated iteratively by successively refining the endpoints $\lambda_{\min}$ and $\lambda_{\max}$ using the improved estimate λ_{new} until a specified accuracy is achieved (e.g., $|f(\lambda)| \le f_{TOL}$, where f_{TOL} is a small number). Note that in order to calculate $f(\lambda)$ it is necessary to make an assumption on where to evaluate the gradients to the plastic potential $\partial g / \partial \sigma_{ij}$. One alternative is to evaluate the gradients at the stress point interpolated from the stress points corresponding to $\lambda_{\min}$ and $\lambda_{\max}$. As mentioned in the introduction, the gradient can also be evaluated at other stress points. The iterative secant procedure is elaborated in the flowchart given in Table I.

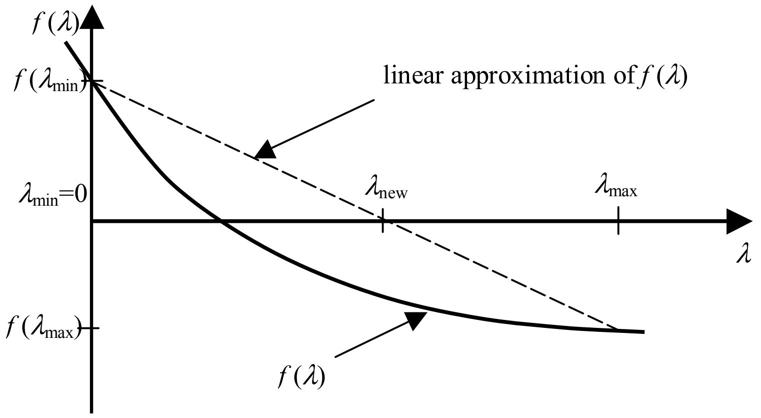

Figure 2. Illustration of secant method to iteratively solve for λ

Table I. Flowchart for new solution algorithm

1. Initialize $k = 0$.
2. Find $\lambda_{\min} = \lambda_{elastic}$ (=0) and $\lambda_{\max} = \lambda_{perfectly\ plastic}$ for the current stress point σ_{ij}^{o}.
3. Find σ_{ij}^{trial} and q_{α}^{trial}, as necessary, for $\lambda_{\min}$ or $\lambda_{\max}$, using gradient at σ_{ij}^{o}.
4. Solve for $f(\lambda_{\min})$ and $f(\lambda_{\max})$ using equation (13).
5. Linearly interpolate between $\lambda_{\min}$ or $\lambda_{\max}$ to find $\lambda_{new}^{(k+1)}$ corresponding to $f(\lambda_{new}^{(k+1)}) = 0$.
6. Find σ_{ij}^{trial} and q_{α}^{trial}, as necessary, for $\lambda_{new}^{(k+1)}$, using gradient at an appropriate stress point, which may include σ_{ij}^{o}, $\sigma_{ij}^{trial}(\lambda_{new}^{(k)})$, or a stress point between these.
7. Solve for $f(\lambda_{new}^{(k+1)})$ using equation (13).
8. If $|f| \leq f_{TOL}$, update σ_{ij} and q_{α}, and exit this algorithm loop; otherwise, reduce the interval on which we are interpolating by replacing $\lambda_{\min}$ or $\lambda_{\max}$, as appropriate, with $\lambda_{new}^{(k+1)}$, and go to step 4.

The response of a strain-hardening material is located between perfectly elastic response and perfectly plastic response, as shown in Figure 1. The final stress point for a strain-hardening material is therefore bounded by the purely elastic and the perfectly plastic response. Because the algorithm proposed here interpolates between these solutions (i.e., on a closed interval), the algorithm is guaranteed to produce a converged solution to Eq. (13). In contrast, there is no guarantee that Newton-Raphson iterative methods will converge to a solution because the Newton-Raphson procedure is a tangent method that iterates on an open interval.

There are several advantages of the new algorithm over Newton-Raphson procedure. As shown above, the solution obtained using the new algorithm is bounded and converges unconditionally. As will be shown later using a numerical example, fewer iterations are generally required to obtain a converged solution using the new algorithm. Another advantage is that less computation is required for each iteration. Newton-Raphson procedures use the tangent method to estimate trial values of λ, which requires that the derivatives $\partial f(\lambda) / \partial \sigma_{ij}$ be calculated at each trial value. Because the new algorithm uses the secant method, each trial value λ_{new} is calculated from values of $f(\lambda)$ at the interval endpoints, not from their derivatives $\partial f(\lambda)/\partial\sigma_{ij}$. Since there is no need to evaluate derivatives $\partial f/\partial\sigma_{ij}$ using the proposed algorithm, the dilemma of choosing appropriate stress points to evaluate these derivatives is avoided.

4. Implementation of New Algorithm on the Constitutive Level

The application of the proposed on the constitutive level is demonstrated in the following section using the strain hardening model of Poorooshasb and Pietruszczak (1985). This model for sands is expressed in terms of the following stress and strain invariants:

$$p = \tfrac{1}{2}\sigma_{kk}\ ,\ q = \sqrt{\tfrac{1}{2}\left(\sigma_{ij} - \delta_{ij}p\right)\left(\sigma_{ij} - \delta_{ij}p\right)} \tag{15}$$

$$d\varepsilon_v = d\varepsilon_{kk}\ , d\varepsilon_s = \sqrt{(d\varepsilon_{ij} - \delta_{ij}d\varepsilon_v)(d\varepsilon_{ij} - \delta_{ij}d\varepsilon_v)} \tag{16}$$

The yield function and plastic potential function are given as:

$$f = q - \eta p \tag{17}$$

$$g = q + \eta_{cr} p \ln\left(\frac{p}{p_o}\right) \tag{18}$$

where $\eta = (q/p)$ is the mobilized stress ratio, η_{cr} is the critical stress ratio which separates contractive behavior from dilatant behavior, and p_o is a factor which varies with the stress point to satisfy $g = 0$. A hyperbolic hardening function that is a function of plastic shear strain was used:

$$\eta = (\eta_{peak} - \eta_{initial})\frac{\varepsilon_s^p}{A + \varepsilon_s^p} + \eta_{initial} \tag{19}$$

where A is a hardening parameter, η_{peak} is the peak shear stress ratio, and $\eta_{initial}$ is the initial size of the elastic region in the $p-q$ space.

Due to the non-linear nature of the yield, plastic potential and hardening functions, the plastic multiplier λ is solved iteratively following the procedure described above. A numerical example is shown to illustrate the performance of the new algorithm. The example was performed in $p-q$ space with strain-controlled loading. Values of all model parameters used in the simulation are listed in Table II. As an illustration, an undrained loading is simulated where shear strains are applied monotonically but no volumetric strain is allowed.

Table II. Values of Parameters Used for Example

Bulk modulus, K (MPa)	6667
Shear modulus, G (MPa)	4000
Initial shear stress ratio, $\eta_{initial}$	0.15
Critical shear stress ratio, $\eta_{critical}$	0.70
Peak shear stress ratio, η_{peak}	0.85
Hardening parameter, A	0.02
Initial mean stress, p (MPa)	200
Initial shear stress, q (MPa)	0
Tolerance for convergence, f_{TOL} (MPa)	0.001

The stress path and stress-strain curve for the example are shown in Figure 3. For comparison, the same problem was solved using a standard Newton-Raphson procedure. Figure 4 shows the number of iterations required to achieve convergence for the new algorithm and for the Newton-Raphson procedure. While identical results are obtained, it is apparent that the new algorithm requires fewer iterations to converge to the solution than the Newton-Raphson procedure.

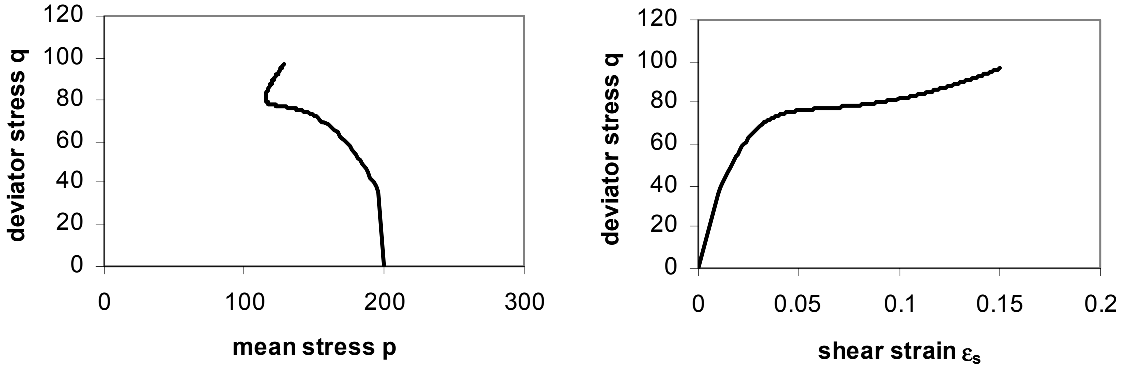

Figure 3. Stress path and stress-strain curve for example calculation.

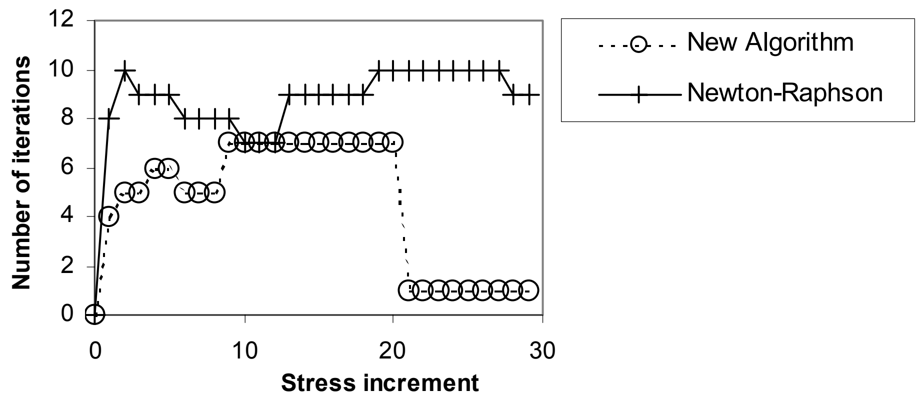

Figure 4. Comparison of results in terms of number of iterations required to achieve convergence for the new algorithm and the Newton-Raphson method.

5. Finite Element Implementation of the New Algorithm

As mentioned in the introduction, return mapping algorithms can be implemented by using the consistent stiffness matrix or by iteratively minimizing the force residuals due to the differences in the "true stresses" and the "predicted stresses." In the following, the finite element implementation of the new algorithm is presented using the force residual minimization approach. In the iterative force residual approach, the trial or predicted stress tensor σ_{ij}^{pred} is calculated assuming elastic response and assuming that the elasticity tensor D_{ijkl} is linear and constant. The predicted stresses are then substituted in the yield function keeping the plastic hardening variable constant to determine if plastic yielding should occur. If $f\left(\sigma_{ij}^{pred}, q_{\alpha}\right) < 0$ then the response is elastic and the new stresses are equal to the predicted stresses, that is, $\sigma_{ij}^{new} = \sigma_{ij}^{pred}$. Calculations can then be carried out for the next load increment. Otherwise, the response during the application of $d\varepsilon_{kl}$ is elastoplastic and the predicted stresses must be corrected to account for plastic response.

Following the procedure outlined above, the plastic multiplier λ is determined at the constitutive level. The plastic multiplier is then substituted in Eq. (12) to get the correct stress increments, giving the correct new stresses σ_{ij}^{new}. The differences between σ_{ij}^{new} and σ_{ij}^{pred} give the residual stresses σ_{ij}^{R} necessary to bring the predicted stresses back to the yield surface. It can be shown that the residual stresses are equal to:

$$\sigma_{ij}^{R} = \sigma_{ij}^{pred} - \sigma_{ij}^{new} = \lambda D_{ijkl} \frac{\partial g}{\partial \sigma_{kl}} \tag{20}$$

To obtain the correct stresses in the finite element calculations, the residual stresses are converted to residual forces. Following standard finite element procedure, the residual force vector $\{R\}$ is equal to:

$$\{R\} = \int_V [B]^T \{\sigma^R\} dV \tag{21}$$

where $[B]$ is the strain-displacement matrix, and V is the element volume. The global nodal residual force vector is obtained by summing the residual forces for all elements connected to each node. The force vector is minimized using an iterative solution producing the required plastic deformations.

6. Conclusions

Satisfaction of the yield criterion $f = 0$ is difficult in elastoplastic geotechnics because the equation which must be satisfied is, in general, a nonlinear equation. Use of Prager's consistency condition as a substitute for the yield criterion may introduce error because it is only an approximation of the yield criterion. The equation which represents satisfaction of the yield criterion may be solved numerically using iterative procedures. A new algorithm is proposed to arrive at a solution to $f = 0$ by using the secant method, and differs from the Newton-Raphson approach which uses the tangent method. The basis for the algorithm is that it is possible to find known bounds on the possible elastoplastic stress-strain response. Because the algorithm uses interpolation to arrive at a solution, convergence is guaranteed. The algorithm may be easily implemented on the constitutive and finite element levels. The proposed algorithm should be more computationally efficient and robust than standard methods used currently.

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