An Embedded Toeplitz Problem

Bartleby Ordonez-Delgado

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Peter E. Haskell, Chair
Joseph A. Ball
William Floyd
John F. Rossi

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(ABSTRACT)

In this work we investigate multi-variable Toeplitz operators and their relationship with KK-theory in order to apply this relationship to define and analyze embedded Toeplitz problems. In particular, we study the embedded Toeplitz problem of the unit disk into the unit ball in $\mathbb{C}^2$. The embedding of Toeplitz problems suggests a way to define Toeplitz operators over singular spaces.

Dedication

Dedicated to my parents: Raul and Susana, and to my brothers. Special dedication to my younger brother Zubin who has been my motivation in the last year.

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Contents

Chapter 1

Introduction

The classifications of self-adjoint, normal and essentially normal operators have been largely studied and done completely. Brown-Douglas-Fillmore (BDF) introduced odd-degree K-homology, which they called Ext, in their classification of essentially normal operators. $Ext(X)$ is a group of equivalence classes of (noncommutative) C^*-algebra extensions of continuous functions on X by an algebra of compact operators. In the classification of essentially normal operators, the relevant X is the essential spectrum of an operator. BDF showed that to characterize essentially normal operators, the essential spectrum was not enough, as the essential spectrum was for the study of self-adjoint operators. The classification also requires the Fredholm index. To be more precise, BDF found that two essentially normal operators A and B are unitarily equivalent modulo compact operators if and only if they have the same essential spectrum and for all λ not in the essential spectrum $Index(A-\lambda I) = Index(B-\lambda I)$.

BDF brought a problem of Operator Theory into the framework of K-theory and Index Theory represented by the Atiyah-Singer Index Theorem. In [AT1], Atiyah knew that K-theory applied to noncommutative C^*-algebras, but he did not explain in detail where such algebras could be significant. Currently, noncommutative C^*-algebras have become one of the central foci of Index Theory, leading to references to noncommutative topology and geometry.

An early example of an essentially normal operator is the operator A defined by multiplication by $\bar{z}$ compressed to the Hardy space of boundary values of holomorphic functions on the unit disk (or the Bergman space of holomorphic functions). A is known as a Toeplitz operator, and the function $\bar{z}$ to be compressed is called the multiplier or symbol of the Toeplitz operator. For that operator, zero is not in the essential spectrum and the index of $A - 0I$ is 1. Note that the Toeplitz operator B, defined by multiplication by z, is also an essentially normal operator with the same essential spectrum but with Fredholm index equal to -1. Therefore, the operators A and B are not unitarily equivalent modulo compact operators. The BDF classification relies on a generalization of the indices of Toeplitz operators.

More general Toeplitz operators, defined over strictly pseudoconvex domains, have analogous connections with K-homology, index theory, and not necessarily commutative C^*-algebras. These operators are compressions of (matrix-valued) multiplication operators to analogues of Hardy or Bergman spaces. The operators are essentially normal if the multipliers take normal values on the boundary, and they are Fredholm if the multipliers are invertible on the boundary. Moreover, the C^*-algebra generated by these Toeplitz operators is a C^*-algebra extension of the algebra of (matrix-valued) continuous functions over the boundary of the underlying domain by the algebra of compact operators. Therefore, we could regard this extension as an element of the group $Ext(X)$, that is as an element in K-homology.

The index theory of Toeplitz operators is an interesting blend of topology, or geometry, and analysis. It is known that when the underlying domain is the unit disk, the index is restriction to the boundary of a Toeplitz operator, when this exists, is the negative of the winding number of the symbol. Moreover, for strictly pseudoconvex domains, Boutet de Monvel's index formula shows that the index depends only on the symbol and the underlying domain (see for example [BDM], [GHI] or Theorem 4.1.1 in this paper). These results can be recovered as special cases of the Atiyah-Singer index theorem, which in manifolds more complicated than strictly pseudoconvex domains in $\mathbb{C}^n$ carries additional information about the topology of the manifold.

The index of a Toeplitz operator can be viewed as the result of the pairing of a K-cohomology class, represented by the Toeplitz multiplier or symbol, with a K-homology class, which represents the subspace onto which the multiplier is compressed. The K-homology class is usually represented by a differential or pseudodifferential operator on the underlying manifold. For example, compression to a Bergman space is represented by a Dolbeault operator, of which the Bergman space is the kernel. In most of the interesting cases, this class is the K-homology fundamental class of the manifold. The K-theoretic formulation suggests a definition of Toeplitz operators on singular spaces having K-homology fundamental classes represented by differential operators on manifolds in which the singular spaces are embedded. The relationship between Toeplitz index theory and K-theory pairings motivates the use of Kasparov's bivariant K-theory also known as KK-theory. the intersection product of KK-theory (KK-product) is a flexible tool for calculating K-theoretic products involving differential operators.

A long-term goal is the definition and study of Toeplitz operators on singular spaces. The K-theoretic formulation of Toeplitz index theory suggests a definition of Toeplitz operators on singular spaces such as the intersection of algebraic varieties with the unit ball in $\mathbb{C}^n$. Results from [BFM] and [FOH] suggest that these singular spaces have K-homology fundamental classes represented by perturbed Dolbeault operators on the ball in which they are embedded. This paper takes the first steps in using embedding to define Toeplitz operators by checking that the approach can be implemented and recovers known results when applied to a much-studied manifold. This paper constructs, on the unit ball in a complex two-dimensional vector space, the K-homology class representing the one dimensional disk. This paper gives evidence that the index theory of the Toeplitz operators associated with this class matches the index theory of Toeplitz operators defined directly on the disk.

Now, we will proceed to describe the content of each chapter.

In the second chapter we go over some background material and some standard results about Toeplitz operators. The first section is a review of Fredholm operators and their indices. Fredholm operators are very important in KK-theory because elements in KK-theory can be thought as generalizations of Fredholm operators. In this chapter, we also go over some basic C^*-algebra theory and give some examples that will be used in the next chapters. The last section of this chapter is dedicated to some basic KK-theory such as the main definitions, some important theorems, and the Kasparov product.

In chapter three we study Toeplitz operators over strictly pseudoconvex domains and their representations as Kasparov products. This chapter is based on the paper [GHI]. In [GHI] it is shown that a Toeplitz operator T_F has the same index as an operator D_F that looks like a sharp product. We prove that the latter operator is the Kasparov product of the symbol F and the Dolbeault operator of the underlying domain.

In chapter four we start by calculating some indices of Toeplitz operators using Boutet de Monvel's formula for the unit ball in $\mathbb{C}^2$. These calculations will help us to build some evidence that the "perturbed" Dolbeault operator considered in chapter five is a good candidate for the K-homology fundamental class of the embedded Toeplitz problem of the disk into the unit ball in $\mathbb{C}^2$. Using sharp products, we show some evidence that the pairing of a multiplier and this perturbed Dolbeault operator could be realized by a KK-product. The rest of this chapter motivates our approach to the more general case of singular spaces.

In the last chapter, chapter five, we construct on the unit ball in $\mathbb{C}^2$ a candidate to represent the fundamental class of the embedded unit disk. We show that the operator we construct represents a class in the appropriate K-homology group. The kernel of the operator is isomorphic to the Bergman space in the z_2 unit disk. We show that the compressions of multipliers (or symbols) to this kernel have indices equal to those of the corresponding Toeplitz operators in the usual construction on the unit disk.

In the future we plan to generalize these constructions to singular spaces and to verify that the pairings of multipliers with perturbed Dolbeault operators can be realized by Kasparov products. This formulation of the pairings will lead to an index formula of Toeplitz operators on singular spaces.

Chapter 2

Background

In this chapter we shall state some definitions and mention some standard results that we will need throughout this paper.

2.1 Some Operator Theory

In this section we will mention some important properties of Fredholm operators. Most of the content in this section was taken from [Sch].

Definition 2.1.1. Let X and Y be Banach spaces. Then a linear operator $A : X \to Y$ is Fredholm if satisfies the following conditions:

1. $D(A)$ is dense in X.
2. A is closed.
3. $Ker(A)$ has finite dimension.
4. $Ran(A)$ is closed in Y.
5. $Coker(A)$ has finite dimension.

Remark 2.1.1. If $A \in B(X,Y)$, the space of bounded linear operators from X to Y, then A is Fredholm if

1. $Ker(A)$ has finite dimension.
2. $Ran(A)$ is closed in Y.
3. $Coker(A)$ has finite dimension.

Moreover, the closed range condition can be dropped because it is a consequence of the first and third conditions.

Notation: Let X and Y be Hilbert spaces. $K(X,Y)$ denotes the space of compact operators from X to Y. In particular, $K(X)$ denotes the space of compact operators on X.

Theorem 2.1.2. *Let X, Y be Hilbert spaces. If $A : X \to Y$ is a Fredholm operator, then there is an $S \in B(Y,X)$ such that*

1. $Ker(S) = Ran(A)^{\perp}$,
2. $Ran(S) = Ker(A)^{\perp} \cap D(A)$,
3. $SA = I$ *on* $Ker(A)^{\perp} \cap D(A)$,
4. $AS = I$ *on* $Ran(A)$.

   Moreover, there are finite rank operators $K_1 \in K(X)$, $K_2 \in K(Y)$ such that
5. $SA = I - K_1$ *on* $D(A)$,
6. $AS = I - K_2$ *on* Y,
7. $Ran(K_1) = Ker(A)$, $Ker(K_1) = Ker(A)^{\perp}$,
8. $Ran(K_2) = Ran(A)^{\perp}$, $Ker(K_2) = Ran(A)$.

The following remark will be very useful in next chapters.

Remark 2.1.2. If A is a Fredholm operator, then the third assertion in the Theorem 2.1.2 implies that A is bounded below on $Ker(A)^{\perp} \cap D(A)$. Moreover, using the last block of consequences we have that $SA = Proj_{Ker(A)^{\perp}}$ and $AS = Proj_{Ran(A)}$, where $Proj$ means orthogonal projection.

The following theorem will give us a characterization of Fredholm operators.

Theorem 2.1.3. *Let $A : X \to Y$ be a densely defined closed linear operator. Suppose there are operators $A_1, A_2 \in B(X,Y)$, $K_1 \in K(X)$, $K_2 \in K(Y)$ such that*

1. $A_1A = I - K_1$ *on* $D(A)$ *and*
2. $AA_2 = I - K_2$ *on* Y.

Then A is Fredholm.

Remark 2.1.3. The bounded operator version of the above theorem is:
A is Fredholm if and only if A is invertible modulo compact operators.

Remark 2.1.4. The space of bounded Fredholm operators, denoted by $\Phi(X)$, is the set of operators whose equivalent classes are invertible elements of the Calkin algebra $B(X)/K(X)$.

Next, we will talk about the index of a Fredholm operator.

Definition 2.1.4. Let T be a Fredholm operator. The number $Ind(T) = dim(Ker(T)) - dim(Coker(T))$ is called the index of T.

Theorem 2.1.5 (Atkinson's Theorem)**.** *The space of Fredholm operators is closed under composition, the adjoint operation and addition of compact operators.*

Corollary 2.1.6. *Let A, B be Fredholm operators and K be a compact operator, then*

1. $Ind(A^*) = -Ind(A)$
2. $Ind(AB) = Ind(A) + Ind(B)$
3. $Ind(A + K) = Ind(A)$

Theorem 2.1.7 (Dieudonne's Theorem)**.** *Let $\Phi(X)$ be the space of Fredholm operators. The index is constant on the connected components of $\Phi(X)$.*

2.2 C^*-algebras

In this section we will review some standard properties about C^*-algebras that will be used in this paper.

Definition 2.2.1. A C^*-algebra A is an algebra over $\mathbb{C}$ with a norm $a \mapsto \|a\|$ and an involution $a \mapsto a^*$, $a \in A$, such that A is complete with respect to the norm, and such that $\|ab\| \leq \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ for every $a, b \in A$.

Now, we give some examples of C^*-algebra that we will use in the following chapters.

Examples

1. The complex numbers $\mathbb{C}$.
2. The space of bounded operators over a Hilbert space $B(H)$.
3. The space of compact operators $K(H)$.
4. The space of bounded continuous functions $C(M)$ and the space of continuous functions vanishing at infinity $C_0(M)$.

Remark 2.2.1. Let $C^*(T)$ be the C^*-algebra generated by the bounded operator T. If T is a normal operator, the C^*-algebra $C^*(T)$ is commutative. If T is not normal, the C^*-algebra $C^*(T)$ is not commutative. In the next section, we will see that C^*-algebras of Toeplitz operators are interesting examples of noncommutative C^*-algebras.

2.3 Toeplitz Operators

In this section, we shall outline some standard results in the theory of Toeplitz operators. Most of the results mentioned in this section are taken from [Ord], which is based on [Upm].

2.3.1 Definition and basic properties

Let B_n be the unit ball in $\mathbb{C}^n$. Consider $L^2(B_n)$, the Lebesgue space of square integrable functions on B_n with Lebesgue measure $dV(z)$ and inner product

$$\langle f, g\rangle = \int_{B_n} \overline{f(z)} g(z) dV(z)$$

Definition 2.3.1. The space of L^2 holomorphic functions $H^2(B_n) := L^2(B_n) \cap \mathcal{O}(B_n)$ is called the Bergman Space where $\mathcal{O}(B_n)$ is the space of holomorphic functions on B_n.

Definition 2.3.2. The orthogonal projection $P : L^2(B_n) \to H^2(B_n)$ is called the Bergman projection.

Definition 2.3.3. For f a continuous function on $\bar{B_n}$ the Bergman-Toeplitz operator with symbol f, $T_f : H^2(B_n) \to H^2(B_n)$, is defined by $T_f(g) := P \circ m_f(g)$ where m_f is the multiplication operator by f.

Definition 2.3.4. The Bergman-Toeplitz C*-algebra over B_n is defined as the unital C*-algebra $\mathcal{T}(B_n) := C^*\langle T_f : f \in C(\bar{B_n})\rangle$ generated by all Toeplitz operators with continuous symbols.

Remark 2.3.1. In the one-dimensional case, the unit disk $\mathbb{D}$ case, it can be shown that $\mathcal{T}(\mathbb{D}) = C^*(T_z)$. But T_z is a non-normal operator, so $\mathcal{T}(\mathbb{D})$ is a noncommutative algebra. We will see later that $\mathcal{T}(\mathbb{D})$ has more important properties.

Similarly, we have a parallel theory on the boundary of the unit ball. Consider $L^2(\partial B_n)$ the Lebesgue space of square integrable functions on ∂B_n with surface measure $d\sigma(z)$, and inner product,

$$\langle f, g\rangle = \int_{\partial B_n} \overline{f(z)} g(z) d\sigma(z)$$

Definition 2.3.5. $H^2(\partial B_n)$, defined as the $L^2(\partial B_n)$ closure of $\{f|_{\partial B_n} : f \in C(\bar{B_n}) \cap \mathcal{O}(B_n)\}$, is called the Hardy Space over B_n.

Definition 2.3.6. The orthogonal projection $P : L^2(\partial B_n) \to H^2(\partial B_n)$ is called the Cauchy-Szegő projection.

Definition 2.3.7. For f a continuous function on ∂B_n the Hardy-Toeplitz operator with symbol f, $T_f : H^2(\partial B_n) \to H^2(\partial B_n)$, is defined by $T_f(g) := P \circ m_f(g)$.

Remark 2.3.2. Let T_f be a Bergman or Hardy Toeplitz operator. Then the function f is called symbol or multiplier.

Definition 2.3.8. The Hardy-Toeplitz C*-algebra over ∂B_n is defined as the unital C*-algebra $\mathcal{T}(\partial B_n) := C^*\langle T_f : f \in C(\partial B_n)\rangle$ generated by all Toeplitz operators with continuous symbols.

Remark 2.3.3. In the one-dimensional case, the unit circle S^1 case, it can be shown that $\mathcal{T}(S^1) = C^*(T_z)$. But T_z is a non-normal operator, so $\mathcal{T}(S^1)$ is a noncommutative C^*-algebra.

The Bergman and the Hardy Toeplitz problems meet in the following two theorems:

Theorem 2.3.9. *The Bergman-Toeplitz C*-algebra $\mathcal{T}(B_n)$ has the commutator ideal $\mathcal{K}(H^2(B_n))$ (compact operators), and there exists a C*-isomorphism*

$$\nu : C(\partial B_n) \to \mathcal{T}(B_n)/\mathcal{K}(H^2(B_n))$$

$$\nu(f) = T_{\tilde{f}} + \mathcal{K}(H^2(B_n)$$

where $\tilde{f}$ is a continuous extension of f to the unit ball B_n.

Theorem 2.3.10. *The Hardy-Toeplitz C*-algebra $\mathcal{T}(\partial B_n)$ has the commutator ideal $\mathcal{K}(H^2(\partial B_n))$ (compact operators), and there exists a C*-isomorphism*

$$\rho : C(\partial B_n) \to \mathcal{T}(\partial B_n)/\mathcal{K}(H^2(\partial B_n))$$

$$\rho(f) := T_f + \mathcal{K}(H^2(\partial B_n))$$

Conclusion: Since we are interested in Fredholm operators (we will explain this later), it turns out that (in any of the cases: Bergman or Hardy Toeplitz operator) a Toeplitz operator is Fredholm if the symbol never vanishes on the boundary ∂B_n. Moreover, if the symbol of a Toeplitz operator vanishes on the boundary, then the Toeplitz operator is compact. This is also true if we replace the unit ball by any strictly pseudoconvex domain (see [Upm]).

2.4 KK-Theory

In this section, we will make the reader aware of the basics of KK-theory. The definitions in this section are taken from [HIN].

One of the reasons for introducing KK-theory in this paper is because in [GHI] it is shown that a Fredholm Toeplitz operator is a compact perturbation of an operator that looks like a Kasparov product. In index theory, these operators are equivalent because they have the same index. Another reason to work with KK-theory is because we want to study Toeplitz operators over singular spaces, and there is good evidence that KK-theory is a powerful tool for study operators over these kind of spaces.

First, we need to generalize our idea of Hilbert space to study KK-theory.

Definition 2.4.1. Let A be a C^*-algebra. A Hilbert A-module $\mathcal{E}$ is a right A-module, with an A-valued form

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to A$$

satisfying:

1. $\langle \eta, \xi_1 + \xi_2 \rangle = \langle \eta, \xi_1 \rangle + \langle \eta, \xi_2 \rangle$;
2. $\langle \eta, \xi a \rangle = \langle \eta, \xi \rangle a$;
3. $\langle \eta, \xi \rangle^* = \langle \xi, \eta \rangle$;
4. $\langle \eta, \eta \rangle \geq 0$;
5. $\langle \eta, \eta \rangle = 0 \Leftrightarrow \eta = 0$; and
6. $\mathcal{E}$ is complete with respect to the norm $\|\eta\| = \|\langle \eta, \eta \rangle\|^{1/2}$.

where $\eta, \xi, \xi_1, \xi_2 \in \mathcal{E}$ and $a \in A$.

Examples
The two very basic examples are:

1. Every Hilbert space is a Hilbert $\mathbb{C}$-module.
2. Every C^*-algebra A with inner product $\langle a, b \rangle = a^* b$ is a Hilbert A-module.

Definition 2.4.2. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert A-modules. A function $T : \mathcal{E}_1 \to \mathcal{E}_2$ for which there is an adjoint $T^* : \mathcal{E}_2 \to \mathcal{E}_1$ such that

$$\langle \eta, T\xi \rangle = \langle T^* \eta, \xi \rangle$$

for all $\eta \in \mathcal{E}_1$ and all $\xi \in \mathcal{E}_2$ is called a Hilbert A-module operator.

Remark 2.4.1. The set of all Hilbert module operators from $\mathcal{E}_1$ to $\mathcal{E}_2$ is denoted by $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$. The space of operators from a Hilbert module $\mathcal{E}$ to itself is denoted by $\mathcal{L}(\mathcal{E})$. It is true that $\mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ is a C^*-algebra.

Definition 2.4.3. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert A-modules. A Hilbert A-module operator $T : \mathcal{E}_1 \to \mathcal{E}_2$ defined as

$$T(\xi) = \sum_{i=1}^{k} w_k \langle \eta_k, \xi \rangle$$

for some fixed $w_k \in \mathcal{E}_2$ and $\eta_k \in \mathcal{E}_1$, is called finite rank Hilbert A-module operator.

Definition 2.4.4. We define the space of generalized compact operators, denoted by $\mathcal{K}(\mathcal{E}_1, \mathcal{E}_2)$, as the operator norm closure of the space of finite rank Hilbert module operators.

Now, we will extend our definition of Fredholm operator.

Definition 2.4.5. A Hilbert A-module operator $T : \mathcal{E}_1 \to \mathcal{E}_2$ is called a generalized Fredholm operator if there is a Hilbert module operator $S : \mathcal{E}_2 \to \mathcal{E}_1$ (a parametrix) such that $1 - SF$ and $1 - FS$ are generalized compact.

Remark 2.4.2. If T is a generalized Fredholm operator then the Hilbert modules $Ker(T)$ and $Ker(T^*)$ are finitely generated.

Examples:

1. If $A = C_0(M)$ and T is a Hilbert A-module operator from A to A, then T is a generalized compact operator if and only if $T = m_f$ for some function $f \in A = C_0(M)$.

2. If $A = C_0(M)$ then an operator T is Fredholm if and only if $T = m_f$ where f is bounded away from zero outside a compact subset.

Definition 2.4.6. Let $\mathcal{E}_1$ be a Hilbert A-module and $\mathcal{E}_2$ a Hilbert B-module with a $*$-representation of A on $\mathcal{L}(\mathcal{E}_2)$. Consider the algebraic tensor product over $\mathbb{C}$ denoted by $\mathcal{E}_1 \odot_{\mathbb{C}} \mathcal{E}_2$ and define the bilinear form:

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \langle \xi_1, \langle \eta_1, \eta_2 \rangle \xi_2 \rangle$$

Define $\mathcal{E}_1 \otimes_A \mathcal{E}_2$ to be the completion of the quotient of $\mathcal{E}_1 \odot_{\mathbb{C}} \mathcal{E}_2$ by the submodule $\{\alpha : \langle \alpha, \alpha \rangle = 0\}$.

Remark 2.4.3. The tensor over A notation is suggested by the fact that elements of the form $\eta a \otimes \xi - \eta \otimes a\xi$ are in $\{\alpha : \langle \alpha, \alpha \rangle = 0\}$.

Example: Let M be a compact manifold and E a Hermitian bundle over M. Recall that E is a Hermitian bundle over M if E is a complex vector bundle over M with a smoothly varying positive-definite Hermitian form on each fiber. Then

$$C(M; E) \otimes_{C(M)} L^2(M) \cong L^2(M; E)$$

Definition 2.4.7. Let $\mathcal{E}_1$ be a Hilbert A-module and $\mathcal{E}_2$ a Hilbert B-module. Define $\mathcal{E}_1 \boxtimes_{\mathbb{C}} \mathcal{E}_2$ to be the completion of $\mathcal{E}_1 \odot_{\mathbb{C}} \mathcal{E}_2$ with respect to the norm induced by the inner product

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \langle \eta_1, \eta_2 \rangle \otimes \langle \xi_1, \xi_2 \rangle$$

Definition 2.4.8. A grading on a Hilbert module $\mathcal{E}$ is an orthogonal decomposition direct sum $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$. The subindices correspond to the grading zero and grading one of the Hilbert module.

Definition 2.4.9. Let $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ be a graded Hilbert module. Denote by $\partial\eta = 0$ if $\eta \in \mathcal{E}_0$ and $\partial\eta = 1$ if $\eta \in \mathcal{E}_1$. We define the graded commutator as

$$[\eta, \alpha]_g = \eta\alpha - (-1)^{\partial\eta\partial\alpha}\alpha\eta$$

Definition 2.4.10. Define the graded tensor product $\mathcal{E}\hat{\boxtimes}\mathcal{E}'$ as the tensor product $\mathcal{E} \boxtimes \mathcal{E}'$ equipped with the grading

$$(\mathcal{E}_0 \boxtimes \mathcal{E}'_0 \oplus \mathcal{E}_1 \boxtimes \mathcal{E}'_1) \oplus (\mathcal{E}_1 \boxtimes \mathcal{E}'_0 \oplus \mathcal{E}_0 \boxtimes \mathcal{E}'_1)$$

Definition 2.4.11. A generalized elliptic operator over A with coefficients in B consists of the following data:

(i) A Hilbert B-module operator $F : \mathcal{E}_0 \to \mathcal{E}_1$; and

(ii) *-representation of A as operators on $\mathcal{E}_0$ and $\mathcal{E}_1$ such that for every $a \in A$ the operators $aF - Fa$, $a(F^*F - 1)$ and $a(FF^* - 1)$ are generalized compact operators.

Definition 2.4.12. Let $T : \mathcal{E}_0 \to \mathcal{E}_1$ be a generalized elliptic operator. We say that $\mathcal{T} : \mathcal{E} \otimes_A \mathcal{E}_0 \to \mathcal{E} \otimes_A \mathcal{E}_1$ is a T-connection if for any $\eta \in \mathcal{E}$ we have that the graded commutator

$$\left[\begin{pmatrix} \mathcal{T} & 0 \\ 0 & T \end{pmatrix}, \begin{pmatrix} 0 & T_\eta \\ T_\eta{}^* & 0 \end{pmatrix}\right]_g$$

is compact where $T_\eta : \mathcal{E}_0 \to \mathcal{E} \otimes_A \mathcal{E}_0$ is defined as $T_\eta(\xi) := \eta \otimes \xi$.

Now we have all the elements to define Kasparov cycles and Kasparov products.

Definition 2.4.13. A Kasparov (A, B)-cycle is a triple $(F, \mathcal{E}, \phi)$ where

1. $\mathcal{E}$ is a graded Hilbert B-module.

2. ϕ is a grading degree zero $*$-representation of A on $\mathcal{L}(\mathcal{E})$.

3. $F : \mathcal{E} \to \mathcal{E}$ is a generalized Fredholm operator of grading degree one such that for every $a \in A$ the operators $aF - Fa$, $a(F^2 - 1)$ and $a(F - F^*)$ are generalized compact operators.

Definition 2.4.14. Let $(G, \mathcal{E}')$ be a Kasparov (A, B)-cycle and $(F, \mathcal{E})$ a (B, C)-cycle. A Kasparov product of these two cycles is any (A, C)-cycle $(G\#F, \mathcal{E}'\hat{\otimes}_B\mathcal{E})$ satisfying

1. $G\#F$ is an F-connection.

2. $$a^*\{(G\hat{\otimes}1)(G\#F) + (G\#F)(G\hat{\otimes}1)\}a$$

   is a positive operator modulo generalized compacts for every $a \in A$.

Chapter 3

Toeplitz Operators over Strictly Pseudoconvex Domains and KK-theory

The point of view in this chapter has been influenced by [GHI] and [HRo]. In this chapter we shall show that the Toeplitz operator T_F, can be represented as the Kasparov product of the class of the multiplier F and the class of the Dolbeault operator D.

3.1 Preliminaries

Let M be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and let $C_0(M)$ be the set of continuous functions vanish at infinity. Next, we shall prove two basic results about the C^*-algebra $C_0(M)$.

Definition 3.1.1. An approximate unit for a C^*-algebra A is an increasing net $\{u_\lambda\}_{\lambda\in\Lambda}$ of positive elements in the closed unit ball of A such that $a = \lim_\lambda au_\lambda$ for all $a \in A$ or equivalently $a = \lim_\lambda u_\lambda a$ for all $a \in A$.

Proposition 3.1.2. *$C_0(M)$ has an approximate unit.*

Proof. An approximate unit of $C_0(M)$ can be constructed as follows: Consider an open cover $\{U_n\}$ of M satisfying $\overline{U_n}$ is a compact set with $\overline{U_n} \subset U_{n+1}$ for each n. The approximate unit can be formed by taking continuous functions $\varphi_n : M \to [0, 1]$ with $supp\varphi_n \subset U_{n+1}$ and $\varphi \equiv 1$ on $\overline{U_n}$. $\square$

Proposition 3.1.3. *$\mathcal{K}(C_0(M)) \cong C_0(M)$ as C^*-algebras under the map $m_{fg^*} \mapsto fg^*$.*

Proof. To prove this isomorphism, we just need to show that if $f \in C_0(M)$ then $m_f \in \mathcal{K}(C_0(M))$. Let $\{\varphi_n\}$ be an approximate unit for $C_0(M)$. Then $\|f\varphi_n{}^* - f\| \to 0$. Therefore, $\|m_{f\varphi_n{}^*} - m_f\|_{op} \to 0$. Hence, $m_f \in \mathcal{K}(C_0(M)) \cong C_0(M)$. $\square$

Now, we will state three important results that will be very useful when working with first order elliptic operators. The following propositions and their proofs can be found in [HRo].

Proposition 3.1.4. ***Rellich Lemma*** *Let S be a smooth vector bundle over M and let K be a compact subset of M. The inclusion of the first Sobolev space $W^1(K;S) \hookrightarrow L^2(M;S)$ is a compact operator.*

Proposition 3.1.5. ***Gårding's Inequality*** *Let D be a first order differential operator on M and let K be a compact subset of M. If D is elliptic over a neighborhood of K then there is a constant $c > 0$ such that*

$$\|u\| + \|Du\| \geq c\|u\|_1$$

for all $u \in W^1(K;S)$, where $\|\|$ denotes the norm in $L^2(M;S)$ and $\|\|_1$ denotes the norm in the Sobolev space $W^1(K;S)$.

Definition 3.1.6. An operator which has a unique self-adjoint extension is said to be essentially self-adjoint.

Proposition 3.1.7. *Let M be a manifold (not necessarily compact), and let D be an essentially selfadjoint differential operator on M. If D is elliptic over an open subset $U \subseteq M$, then for every $\phi \in C_0(\mathbb{R})$ and every $g \in C_0(U)$ the operator $g\phi(D) : L^2(M;S) \to L^2(M;S)$ is compact.*

The rest of this section is dedicated to describing our work environment. Our setting is the paper [GHI]. In the following lines we shall state some definitions, notations, and results stated and proven in [GHI].

Remark 3.1.1. **Collection of definitions, notations and results from [GHI]**

1. Let M be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and let f be a smooth function on $\overline{M}$ that is never zero on the boundary of M.

2. Give M the Hermitian metric defined by the formula

$$\sum_{i,j} h_{ij} dz_i \otimes d\bar{z}_j = -\sum_{i,j} \frac{\partial^2 log(r)}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j$$

where r is an appropriately chosen defining function on M.

3. Let $A^{p,q}$ be the space of smooth compactly supported differential forms of type (p,q) on M, with inner product induced by the Hermitian metric defined on M, and let ${L^2}_{p,q}$ be the Hilbert space completion of $A^{p,q}$.

4. Define the twisted Dolbeault operators:
$$D_+ = \bar{\partial} + \bar{\partial}^* : \bigoplus_{q \text{ even}} {L^2}_{n,q} \to \bigoplus_{q \text{ odd}} {L^2}_{n,q}$$
$$D_- = \bar{\partial} + \bar{\partial}^* : \bigoplus_{q \text{ odd}} {L^2}_{n,q} \to \bigoplus_{q \text{ even}} {L^2}_{n,q}$$
as unbounded operators and denote in the same way the closure of these operators in the sense of unbounded operator theory.

5. $KerD_+ = \{$ holomorphic square-integrable forms of type $(n,0)\}$

6. We also have a unitary isomorphism from the Hilbert space of functions square integrable with respect to the Lebesgue measure to the Hilbert space ${L^2}_{n,0}$
$$\begin{array}{rcl} L^2(M) & \to & {L^2}_{n,0} \\ f(z) & \mapsto & 2^{n/2} f(z) dz_1 dz_2 ... dz_n \end{array}$$

7. The Bergman space $H^2(M)$ is mapped isometrically onto $KerD_+$, denoted by ${L^2}_{hol}$.

8. The Toeplitz operator T_f on $H^2(M)$ is unitarily equivalent to the compression to $KerD_+$ of the multiplication operator m_f on $\bigoplus_{q \text{ even}} {L^2}_{n,q}$.

9. Let f act by the multiplication operator m_f, and define
$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} : \begin{array}{c} \bigoplus_{q \text{ even}} {L^2}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} {L^2}_{n,q} \end{array} \to \begin{array}{c} \bigoplus_{q \text{ even}} {L^2}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} {L^2}_{n,q} \end{array}$$
and
$$D_f = \begin{pmatrix} f & D_- \\ D_+ & -\bar{f} \end{pmatrix} : \begin{array}{c} \bigoplus_{q \text{ even}} {L^2}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} {L^2}_{n,q} \end{array} \to \begin{array}{c} \bigoplus_{q \text{ even}} {L^2}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} {L^2}_{n,q} \end{array}$$

10. It is known that D is self-adjoint i.e. ${D_+}^* = D_-$ and elliptic. Also, it can be proven that the operator D_f is Fredholm.

11. The operator D_- is bounded below. Let
$$E : \bigoplus_{q \text{ even}} {L^2}_{n,q} \rightarrow \bigoplus_{q \text{ odd}} {L^2}_{n,q}$$
be its generalized inverse i.e. $E = {D_-}^{-1} Proj_{RanD_-}$. Then D_-E is a bounded operator and $ED_- = Id_{DomD_-}$.

12. Let
$$P : \bigoplus_{q=1}^{n} {L^2}_{n,q} \rightarrow \bigoplus_{q=1}^{n} {L^2}_{n,q}$$
be the orthogonal projection onto $KerD_+$. Then $P = I - D_-E - ED_-$.

The following proposition is crucial for the next chapter. We will outline the proof which is done in [GHI].

Proposition 3.1.8. *Let f be a smooth function on $\bar{M}$ and $Q = I - P = D_-E - ED_-$. Then the operators PfQ and QfP are compact.*

Proof. Because f preserves the degree of a form, to analyze PfQ, it suffices to consider $Q's$ restriction D_E to forms of type (n, q) with q even. Because D_E is the identity on the range of D_-,
$$[f, D_-]E - D_-E[f, D_-]E = fD_-E - D_-fE - D_-EfD_-E + D_-ED_-fE$$
$$= fD_-E - D_-EfD_-E = (I - D_-E)fD_-E = PfQ$$
On the other hand
$$[f, D_-]\omega = \bar{\partial} f \wedge \omega + \bar{\partial}\bar{f} \lrcorner \omega$$
Then $\bar{\partial} f$ vanishes at infinity (see Lemma 2 in Section 1 in [GHI]) and so does $[f, D_-]$. Hence, the operator $[f, D_-]E$ is compact (see Lemma 2 in Section 3 in [GHI]). Because D_-E is bounded the operator $D_-E[f, D_-]E$ is also compact. Thus, PfQ is a compact operator. Taking adjoints shows that QfP is a compact operator, as well. □

In [GHI], it is shown that the Toeplitz operator T_f is Fredholm. Moreover, the operator T_f has the same index as D_f. Furthermore, the index of D_f is calculated by applying the Atiyah-Singer index theorem.

Our goal in this chapter is to show that the KK-class of the operator D_f can be realized as the Kasparov product of the class of f and the class of D. Therefore, our first task is to construct the representative elements of KK-classes for f and D. Using appropriate notation, we want to show

$$\begin{array}{ccc} KK(\mathbb{C}, C_0(M)) \times KK(C_0(M), \mathbb{C}) & \to & KK(\mathbb{C}, \mathbb{C}) \\ ([f], [D]) & \mapsto & [D_f] = [f] \otimes_{C_0(M)} [D] \end{array}$$

3.2 Construction of the KK-class of f in $KK(\mathbb{C}, C_0(M))$

In this section we will construct a K-theory class representative for the multiplier f. To have a KK-class in $KK(\mathbb{C}, C_0(M))$, we just need a representative element i.e. a $(\mathbb{C}, C_0(M))$-cycle. The usual way to construct a cycle is by generalized elliptic operators. The construction is done by doubling the generalized elliptic operator to make it self-adjoint of odd degree. This means, if F is a generalized elliptic operator then

$$\mathcal{F} = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$$

is a KK-cycle.

Now, recall from Definition 2.4.11 that a generalized elliptic operator over $\mathbb{C}$ with coefficients in $C_0(M)$ consists of the following data:
(i) A Hilbert $C_0(M)$-module operator $F : C_0(M) \to C_0(M)$; and
(ii) *-representation of $\mathbb{C}$ as operators on $C_0(M)$ such that for every $z \in \mathbb{C}$ the operators $zF - Fz$, $z(F^*F - 1)$ and $z(FF^* - 1)$ are generalized compact operators, denoted by $\mathcal{K}(C_0(M))$.

The natural representative for the class of f would be the multiplication operator

$$f : C_0(M) \to C_0(M)$$

This multiplication operator satisfies all the conditions to be a generalized elliptic operator except that $ff^* - 1$ is not necessarily in $\mathcal{K}(C_0(M)) \cong C_0(M)$. In fact, consider $M = \mathbb{D}$ the unit disk in $\mathbb{C}$ and $f(z) = 2z$. Then

$$ff^* - 1 = 2|z|^2 - 1 \text{ is not in } C_0(M)$$

Therefore, f does not define a generalized elliptic operator.

Since f is never zero on the boundary of M, there is a compact set K in M such that $f(z) \neq 0$ outside of K and there is a bump function φ such that

$$0 \leq \varphi \leq 1 \text{ on M with } \varphi \equiv 0 \text{ outside of K and } \varphi \equiv 1 \text{ on the zero set of } f$$

Now, define $F = \frac{f}{\varphi+|f|}$. Then,

$$FF^* - 1 = \frac{|f|^2}{(\varphi+|f|)^2} - 1 = 0 \text{ outside of K}$$

Thus $FF^* - 1 \in C_0(M)$ and F defines a generalized elliptic operator.

Remark 3.2.1. It is known that for strictly psedoconvex domains, the index of a Toeplitz operator depends only on the value of its symbol at the boundary points (see [Upm] or [Ord]). Therefore, we can consider any continuous function with the same boundary values of f to be a class representative. In particular, we can even consider this representative to be smooth if f is smooth on the boundary.

As we mentioned before, we can get a KK-cycle by defining

$$\mathcal{F} = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix} : \begin{matrix} C_0(M) \\ \oplus \\ C_0(M) \end{matrix} \longrightarrow \begin{matrix} C_0(M) \\ \oplus \\ C_0(M) \end{matrix}$$

Thus $[\mathcal{F}] \in KK(\mathbb{C}, C_0(M))$.

3.3 Construction of the KK-class of D in $KK(C_0(M), \mathbb{C})$

In this section, we will show that the Dolbeault operator

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} : \begin{matrix} \bigoplus_{q \text{ even}} {L^2}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} {L^2}_{n,q} \end{matrix} \longrightarrow \begin{matrix} \bigoplus_{q \text{ even}} {L^2}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} {L^2}_{n,q} \end{matrix}$$

where

$$D_+ = \bar{\partial} + \bar{\partial}^* : \bigoplus_{q \text{ even}} {L^2}_{n,q} \to \bigoplus_{q \text{ odd}} {L^2}_{n,q}$$

$$D_- = \bar{\partial} + \bar{\partial}^* : \bigoplus_{q \text{ odd}} {L^2}_{n,q} \to \bigoplus_{q \text{ even}} {L^2}_{n,q}$$

is an unbounded Kasparov module, an hence that $\mathcal{D} = D(1+D^2)^{-1/2}$ is a bounded Kasparov module.

D is an unbounded $(C_0(M), \mathbb{C})$-module if it satisfies the following conditions:
(i) D is a regular operator i.e. $1 + D^2$ has dense range.
(ii) $(1 + D^2)^{-1}\varphi$ is a compact operator for every $\varphi \in C_0(M)$.
(iii) $[D, \varphi]$ is a bounded operator for $\varphi \in {C_c}^\infty(M)$.

Proof. (i) Since D is self-adjoint, $(Ran(1+D^2))^{\perp} = Ker(1+D^2) = 0$. Therefore, $1+D^2$ has dense range. $\square$

Proof. (ii) Since D is self-adjoint, we will prove the equivalent condition $\varphi(1+D^2)^{-1}$ is a compact operator for every $\varphi \in C_0(M)$. Since $1+D^2 = (i+D)(i-D)$ and $(i-D)^{-1}$ is a bounded operator, it is enough to show that $\varphi(i+D)^{-1}$ is a compact operator for every $\varphi \in C_0(M)$. Also, we will assume that $\varphi \in {C_c}^{\infty}(M)$ because the general case can be done by an approximation argument.

Let $u \in {L_{p,q}}^2(M)$. Then $(i+D)^{-1}u = v$ for some $v \in Dom(D)$ i.e. in the Sobolev space $W^1(M)$. Therefore, by Gårding's inequality there is a constant c such that

$$\|(i+D)^{-1}u\|_1 = \|v\|_1 \leq (1/c)(\|v\| + \|Dv\|) = (1/c)\|(i+D)v\| = (1/c)\|u\|$$

This shows that $(i+D)^{-1} : {L_{p,q}}^2(M) \to W^1(M)$ is a bounded operator. Using Rellich's lemma, we have that the operator

$$\varphi(i+D)^{-1} : {L_{p,q}}^2(M) \to W^1(M) \to W^1(supp(\varphi)) \hookrightarrow {L_{p,q}}^2(M)$$

is compact. $\square$

Proof. (iii) Since $\varphi \in {C_c}^{\infty}(M)$ and $[\varphi, D_-]\omega = \bar{\partial}\varphi \wedge \omega + \bar{\partial}\bar{\varphi} \lrcorner \omega$, $[\varphi, D_-]$ is bounded by a multiple of the sup-norm of $grad(\varphi)$. Then the operator $[\varphi, D_-]$ is bounded. Similarly, $[\varphi, D_+]$ is a bounded operator. Therefore $[\varphi, D]$ is a bounded operator. $\square$

3.4 Construction of the KK-class of D_f in $KK(\mathbb{C}, \mathbb{C})$

The problem with the operator D_f is that ${D_f}^* D_f - 1$ is not compact and the domain is not the expected $(C_0(M) \oplus C_0(M)) \hat{\otimes} ({L_{p,q}}^2(M) \oplus {L_{p,q}}^2(M))$. We need to double and normalize this operator. Then we define

$$\mathcal{D}_f = \begin{pmatrix} 0 & 0 & \bar{f} & D_- \\ 0 & 0 & -D_+ & f \\ f & -D_- & 0 & 0 \\ D_+ & \bar{f} & 0 & 0 \end{pmatrix}$$

The lower left block of $\mathcal{D}_f$ is just

$$D_f \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Recall that

$$D_f = \begin{pmatrix} f & D_- \\ D_+ & -\bar{f} \end{pmatrix}$$

Since D_f is Fredholm, then the lower left block of $\mathcal{D}_f$ is also Fredholm and has the same index as D_f. Hence $\mathcal{D}_f$ is also Fredholm.

Let $\tilde{P}$ be the orthogonal projection onto $Ker\mathcal{D}_f$. Then $\tilde{P}$ is a finite rank operator. Now we define the operator

$$\mathcal{T} = \mathcal{D}_f(\tilde{P} + {\mathcal{D}_f}^2)^{-1/2} = \begin{cases} \frac{\mathcal{D}_f}{|\mathcal{D}_f|} & \text{on} \quad Ker{\mathcal{D}_f}^{\perp} \\ 0 & \text{on} \quad Ker\mathcal{D}_f. \end{cases}$$

Note that:
(i) $\mathcal{T}$ is self-adjoint and
(ii) $\mathcal{T}^2 - 1 = -\tilde{P}(\tilde{P} + {\mathcal{D}_f}^2)^{-1}$ is a compact operator.
Thus $[\mathcal{T}] \in KK(\mathbb{C}, \mathbb{C})$.

3.5 The Kasparov product

Our goal in this section is to show that $[\mathcal{F}] \otimes [\mathcal{D}] = [\mathcal{T}]$. The definition of Kasparov product requires that the following conditions be satisfied:
(i) $\mathcal{T}$ is a $\mathcal{D}$-connection i.e. for any $\eta \in C_0(M) \oplus C_0(M)$ we have that the graded commutator

$$\left[\begin{pmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{D} \end{pmatrix}, \begin{pmatrix} 0 & T_\eta \\ {T_\eta}^* & 0 \end{pmatrix} \right]_g$$

is compact. Here $T_\eta(\xi) := \eta \otimes \xi$.

(ii) $(\mathcal{F}\hat{\otimes}1)\mathcal{T} + \mathcal{T}(\mathcal{F}\hat{\otimes}1)$ is positive modulo compact operators.

Proposition 3.5.1. *$\mathcal{T}$ is a $\mathcal{D}$-connection.*

Proof. **Case 1:** $\partial\eta = 0$

This means that the graded commutator is just the commutator. Define

$$T = \begin{pmatrix} D_f & 0 \\ 0 & D \end{pmatrix}$$

By functional calculus, there is a continuous function g such that

$$g(T) = \begin{pmatrix} \tilde{P} + {D_f}^2 & 0 \\ 0 & 1 + D^* \end{pmatrix}^{-1/2}$$

Then

$$Tg(T) = \begin{pmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{D} \end{pmatrix}$$

Using this representation we have

$$\left[\begin{pmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{D} \end{pmatrix}, \begin{pmatrix} 0 & T_\eta \\ {T_\eta}^* & 0 \end{pmatrix}\right] = \left[Tg(T), \begin{pmatrix} 0 & T_\eta \\ {T_\eta}^* & 0 \end{pmatrix}\right]$$

$$= T\left[g(T), \begin{pmatrix} 0 & T_\eta \\ {T_\eta}^* & 0 \end{pmatrix}\right] + \left[T, \begin{pmatrix} 0 & T_\eta \\ {T_\eta}^* & 0 \end{pmatrix}\right] g(T) \tag{3.5.1}$$

We will split the proof of this case in two following lemmas:

Lemma 3.5.2. *The second term of the equation (3.5.1) is a compact operator.*

Proof.

$$\left[T, \begin{pmatrix} 0 & T_\eta \\ {T_\eta}^* & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & \mathcal{D}_f T_\eta - T_\eta D \\ D{T_\eta}^* - {T_\eta}^* \mathcal{D}_f & 0 \end{pmatrix}$$

But

$$\mathcal{D}_f T_\eta - T_\eta D = \begin{pmatrix} 0 & 0 & \bar{f} & D_- \\ 0 & 0 & -D_+ & f \\ f & -D_- & 0 & 0 \\ D_+ & \bar{f} & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \eta \end{pmatrix} - \begin{pmatrix} \eta & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

$$\mathcal{D}_f T_\eta - T_\eta D = \begin{pmatrix} 0 & [D_-, \eta] \\ 0 & f\eta \\ -f\eta & 0 \\ [D_+, \eta] & 0 \end{pmatrix}$$

Similarly

$$D{T_\eta}^* - {T_\eta}^* \mathcal{D}_f = \begin{pmatrix} 0 & 0 & -\bar{f}\bar{\eta} & -[D_-, \bar{\eta}] \\ -[D_+, \bar{\eta}] & \bar{f}\bar{\eta} & 0 & 0 \end{pmatrix}$$

Assume $\eta \in {C_c}^\infty(M)$. Using Gårding's inequality and Rellich lemma, we have $(\mathcal{D}_f T_\eta - T_\eta D)\,(1+D^2)^{-1/2}$ and $(D{T_\eta}^* - {T_\eta}^*)\,\mathcal{D}_f(P+{\mathcal{D}_f}^2)^{-1/2}$ are compact operators. Therefore

$$\left[T, \begin{pmatrix} 0 & T_\eta \\ {T_\eta}^* & 0 \end{pmatrix}\right] g(T)$$

is a compact operator.

The general case follows by a density argument. □

Lemma 3.5.3. *The first term of equation (3.5.1) is compact.*

Proof. We want to show that

$$T\left[g(T), \begin{pmatrix} 0 & T_\eta \\ {T_\eta}^* & 0 \end{pmatrix}\right]$$

is compact. We introduce some notation that will help to make the calculation less tedious.

$$\mathbf{T}_\eta = \begin{pmatrix} 0 & T_\eta \\ {T_\eta}^* & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \tilde{P} + {D_f}^2 & 0 \\ 0 & 1 + D^* \end{pmatrix} = T^2 + \begin{pmatrix} \tilde{P} & 0 \\ 0 & 1 \end{pmatrix}$$

With this new notation we have

$$T\left[g(T), \begin{pmatrix} 0 & T_\eta \\ {T_\eta}^* & 0 \end{pmatrix}\right] = T[A^{-1/2}, \mathbf{T}_\eta]$$

Since $\mathbf{T}_\eta$ is bounded and using Riemann sum approximation, we get

$$T[A^{-1/2}, \mathbf{T}_\eta] = T\left[\frac{2}{\pi}\int_0^\infty \lambda^{-1/2}(A+\lambda)^{-1}d\lambda, \mathbf{T}_\eta\right] = T\left(\frac{2}{\pi}\int_0^\infty \lambda^{-1/2}[(A+\lambda)^{-1}, \mathbf{T}_\eta]d\lambda\right)$$

Using a Riemann sum approximation and the fact that T is closed, we have

$$T[A^{-1/2}, \mathbf{T}_\eta] = \frac{2}{\pi}\int_0^\infty \lambda^{-1/2} T[(A+\lambda)^{-1}, \mathbf{T}_\eta]d\lambda \tag{3.5.2}$$

Now, we analyze the integrand of the equation (3.5.2). Notice that

1. $[(A+\lambda)^{-1}, \mathbf{T}_\eta] = (A+\lambda)^{-1}\left[\mathbf{T}_\eta, (A+\lambda)\right](A+\lambda)^{-1} = (A+\lambda)^{-1}\left[\mathbf{T}_\eta, A\right](A+\lambda)^{-1}$
2. $[\mathbf{T}_\eta, A] = [\mathbf{T}_\eta, T^2] + \left[\mathbf{T}_\eta, \begin{pmatrix} \tilde{P} & 0 \\ 0 & 1 \end{pmatrix}\right]$
3. $[\mathbf{T}_\eta, T^2] = [\mathbf{T}_\eta, T]\,T + T\,[\mathbf{T}_\eta, T]$
4. $\left[\mathbf{T}_\eta, \begin{pmatrix} \tilde{P} & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 0 & T_\eta - \tilde{P}T_\eta \\ {T_\eta}^*\tilde{P} - {T_\eta}^* & 0 \end{pmatrix}$
5. If we compose the operator $\left[\mathbf{T}_\eta, \begin{pmatrix} \tilde{P} & 0 \\ 0 & 1 \end{pmatrix}\right]$ with a bounded operator with range in the Sobolev space $W^1(M)$, we get a compact operator by the Rellich lemma.

Then, the operator in the integrand of the equation (3.5.2)

$$T[(A+\lambda)^{-1},\mathbf{T}_\eta] = T(A+\lambda)^{-1}\left([\mathbf{T}_\eta,T]\,T + T\,[\mathbf{T}_\eta,T] + \left[\mathbf{T}_\eta,\begin{pmatrix}\tilde{P} & 0\\ 0 & 1\end{pmatrix}\right]\right)(A+\lambda)^{-1} \quad (3.5.3)$$

$$= T(A+\lambda)^{-1}\,[\mathbf{T}_\eta,T]\,T(A+\lambda)^{-1} + T(A+\lambda)^{-1}T\,[\mathbf{T}_\eta,T]\,(A+\lambda)^{-1} \quad (3.5.4)$$
$$+T(A+\lambda)^{-1}\left[\mathbf{T}_\eta,\begin{pmatrix}\tilde{P} & 0\\ 0 & 1\end{pmatrix}\right]T(A+\lambda)^{-1}$$

The second term in equation (3.5.4) equals

$$\left\{T(A+\lambda)^{-1}\right\}\left\{T\,[\mathbf{T}_\eta,T]\,(A+\lambda)^{-1/4}\right\}\left\{(A+\lambda)^{-3/4}\right\}. \quad (3.5.5)$$

In equation (3.5.5), the first factor is a bounded operator, the second factor is compact and the third factor together with $\lambda^{-1/2}$ make a power λ^{-2} that will guarantee the convergence of the integral at ∞ in equation (3.5.2). Therefore, the second term in equation (3.5.4) is compact.

Following a similar argument, the first and third terms in equation (3.5.4) are compact operators, and convergence of the integral in equation (3.5.2) is guaranteed by the power of λ in the integrand. The integral in equation 3.5.2 is compact. This concludes the proof of the first case when $\partial\eta = 0$. □

Case 2: $\partial\eta = 1$

We want to show that the operator

$$[Tg(T),\mathbf{T}_\eta]_g = T\,[g(T),\mathbf{T}_\eta] + [T,\mathbf{T}_\eta]_g\,g(T) \quad (3.5.6)$$

is compact. We will split the proof of this case into the two following lemmas:

Lemma 3.5.4. *The second term on the right of equation (3.5.6) is compact.*

Proof. Note that

$$[T,\mathbf{T}_\eta]_g = \begin{pmatrix} 0 & \mathcal{D}_f T_\eta + T_\eta D\\ D{T_\eta}^* + {T_\eta}^*\mathcal{D}_f & 0\end{pmatrix}$$

But

$$\mathcal{D}_f T_\eta + T_\eta D = \begin{pmatrix} \bar{f}\eta & 0\\ -[D_+,\eta] & 0\\ 0 & -[D_-,\eta]\\ 0 & \bar{f}\eta\end{pmatrix}$$

Similarly

$$DT_\eta{}^* + T_\eta{}^*\mathcal{D}_f = \begin{pmatrix} f\bar\eta & [D_-,\bar\eta] & 0 & 0 \\ 0 & 0 & [D_+,\bar\eta] & f\bar\eta \end{pmatrix}$$

Assume $\eta \in C_c{}^\infty(M)$. Using Gårding's inequality and the Rellich lemma as in case 1, we have that $[T,\mathbf{T}_\eta]_g\, g(T)$ is a compact operator. The general case follows by a density argument. □

Lemma 3.5.5. *The first term on the right of equation (3.5.6) is compact.*

Proof. We want to prove that $T\,[g(T),\mathbf{T}_\eta]$ is compact. As in the first case, we have the integral representation

$$T\,[g(T),\mathbf{T}_\eta] = \frac{2}{\pi}\int_0^\infty \lambda^{-1/2} T[(A+\lambda)^{-1},\mathbf{T}_\eta]d\lambda$$

Using the same calculations as in the first case and considering that $[\mathbf{T}_\eta, T^2] = [\mathbf{T}_\eta, T]_g\, T - T\,[\mathbf{T}_\eta, T]_g$, we get

$$T[(A+\lambda)^{-1},\mathbf{T}_\eta] = T(A+\lambda)^{-1}\left([\mathbf{T}_\eta,T]_g\,T - T\,[\mathbf{T}_\eta,T]_g + \left[\mathbf{T}_\eta, \begin{pmatrix} \tilde{P} & 0 \\ 0 & 1 \end{pmatrix}\right]\right)(A+\lambda)^{-1}$$

Using the same argument as in the grading zero case, we conclude that the operator

$$T\,[g(T),\mathbf{T}_\eta]$$

is compact. □

Thus, this completes the proof that $\mathcal{T}$ is a $\mathcal{D}$-connection. □

Proposition 3.5.6. $(\mathcal{F}\hat\otimes 1)\mathcal{T} + \mathcal{T}(\mathcal{F}\hat\otimes 1)$ *is positive modulo compact operators.*

Before we start with the proof of this proposition, we will make some remarks.

Remark 3.5.1. We can assume that f is unitary on the boundary of M with small gradient on M. Then, $F = f$.

Remark 3.5.2.

$$\left(\mathcal{F}\hat\otimes 1\right) = \begin{pmatrix} 0 & 0 & \bar f & 0 \\ 0 & 0 & 0 & f \\ f & 0 & 0 & 0 \\ 0 & \bar f & 0 & 0 \end{pmatrix}$$

Remark 3.5.3.

$$(\mathcal{F}\hat\otimes 1)\mathcal{D}_f + \mathcal{D}_f(\mathcal{F}\hat\otimes 1) = \begin{pmatrix} 2\bar f f & [D_-,\bar f] & 0 & 0 \\ [f,D_+] & 2\bar f f & 0 & 0 \\ 0 & 0 & 2\bar f f & [f,D_-] \\ 0 & 0 & [D_+,\bar f] & 2\bar f f \end{pmatrix} = (2\bar f f)I + \mathfrak{M}$$

where

$$\mathfrak{M} = \begin{pmatrix} 0 & [D_-, \bar{f}] & 0 & 0 \\ [f, D_+] & 0 & 0 & 0 \\ 0 & 0 & 0 & [f, D_-] \\ 0 & 0 & [D_+, \bar{f}] & 0 \end{pmatrix}$$

Remark 3.5.4. Since the gradient of f vanishes at infinity, $\mathfrak{M}$ vanishes at infinity. Then $\mathfrak{M}(\tilde{P} + \mathcal{D}_f{}^2)$ is compact.

We start with the proof of the Proposition 3.5.6

Proof.

$$(\mathcal{F}\hat{\otimes}1)\mathcal{T} + \mathcal{T}(\mathcal{F}\hat{\otimes}1) = (\mathcal{F}\hat{\otimes}1)\mathcal{D}_f(\tilde{P} + D_f{}^2)^{-1/2} + \mathcal{D}_f(\tilde{P} + D_f{}^2)^{-1/2}(\mathcal{F}\hat{\otimes}1)$$

$$= \mathcal{D}_f \left[(\tilde{P} + D_f{}^2)^{-1/2}, \mathcal{F}\hat{\otimes}1\right] + \left((\mathcal{F}\hat{\otimes}1)\mathcal{D}_f + \mathcal{D}_f(\mathcal{F}\hat{\otimes}1)\right)(\tilde{P} + D_f{}^2)^{-1/2} \tag{3.5.7}$$

We will be done if we prove that the two terms in equation (3.5.7) are essentially positive. We split the proof in the following lemmas.

Lemma 3.5.7. *The second term in the equation (3.5.7) is essentially positive.*

Proof. Using equations (3.5.3) and (3.5.4) we get

$$\left((\mathcal{F}\hat{\otimes}1)\mathcal{D}_f + \mathcal{D}_f(\mathcal{F}\hat{\otimes}1)\right)(\tilde{P} + D_f{}^2)^{-1/2} = (2\bar{F}f + \mathfrak{M})(\tilde{P} + D_f{}^2)^{-1/2}$$

$$= 2\bar{F}f(\tilde{P} + D_f{}^2)^{-1/2} + \text{ compact}$$

Then,

$$2\bar{f}f(\tilde{P} + D_f{}^2)^{-1/2} = 2\bar{f}(\tilde{P} + D_f{}^2)^{-1/2}f + 2\bar{f}\left[f, (\tilde{P} + D_f{}^2)^{-1/2}\right]$$

The first term on the right of the previous equation is positive. In the second term on the right of the previous equation, we have

$$\left[(\tilde{P} + D_f{}^2)^{-1/2}, f\right] = \frac{2}{\pi}\int_0^\infty \lambda^{-1/2}\left[(\tilde{P} + D_f{}^2 + \lambda)^{-1}, f\right] d\lambda$$

Notice that

$$\left[(\tilde{P} + D_f{}^2 + \lambda)^{-1}, f\right] = (\tilde{P} + D_f{}^2 + \lambda)^{-1}\left[f, \tilde{P} + D_f{}^2\right](\tilde{P} + D_f{}^2 + \lambda)^{-1}$$

But

$$\left[f, \tilde{P} + D_f{}^2\right] = [f, \tilde{P}] + [f, D_f{}^2]$$

$$= [f, \tilde{P}] + [f, D_f]\, D_f + D_f\, [f, D_f] \tag{3.5.8}$$

The first term in the equation (3.5.8) is a finite rank operator because $\tilde{P}$ is finite rank. The second term and the third term when composing with $(\tilde{P} + {D_f}^2 + \lambda)^{-1/2}$ are compact by the Rellich lemma. Thus $\left[(\tilde{P} + {D_f}^2)^{-1/2}, f\right]$ is compact. □

Lemma 3.5.8. *The first term in the equation (3.5.7) is compact. This means that we want to show that* $\mathcal{D}_f \left[(\tilde{P} + {D_f}^2)^{-1/2}, \mathcal{F}\hat{\otimes}1\right]$ *is compact.*

Proof.

$$\left[(\tilde{P} + {D_f}^2)^{-1/2}, \mathcal{F}\hat{\otimes}1\right] = \frac{2}{\pi}\int_0^\infty \lambda^{-1/2}\left[(\tilde{P} + {D_f}^2 + \lambda)^{-1}, \mathcal{F}\hat{\otimes}1\right] d\lambda$$

Note that

$$\left[(\tilde{P} + {D_f}^2 + \lambda)^{-1}, \mathcal{F}\hat{\otimes}1\right] = (\tilde{P} + {D_f}^2 + \lambda)^{-1}\left[\mathcal{F}\hat{\otimes}1, \tilde{P} + {D_f}^2\right](\tilde{P} + {D_f}^2 + \lambda)^{-1} \tag{3.5.9}$$

But

$$\left[\mathcal{F}\hat{\otimes}1, \tilde{P} + {D_f}^2\right] = \left[\mathcal{F}\hat{\otimes}1, \tilde{P}\right] + \left[\mathcal{F}\hat{\otimes}1, {D_f}^2\right]$$

The first term on the right in the last equation is finite rank. The second term

$$\left[\mathcal{F}\hat{\otimes}1, {D_f}^2\right] = \left[\mathcal{F}\hat{\otimes}1, D_f\right]_g D_f - D_f\left[\mathcal{F}\hat{\otimes}1, D_f\right]_g$$

Now, using the equation (3.5.3)

$$\left[\mathcal{F}\hat{\otimes}1, {D_f}^2\right] = [2\bar{f}f, D_f] + \mathfrak{M}D_f - D_f\mathfrak{M}$$

Going back to what we want to prove

$$\mathcal{D}_f\left[(\tilde{P} + {D_f}^2)^{-1/2}, \mathcal{F}\hat{\otimes}1\right] =$$

$$\mathcal{D}_f(\tilde{P} + {D_f}^2)^{-1/2}(\tilde{P} + {D_f}^2)^{1/2}\left(\frac{2}{\pi}\int_0^\infty \lambda^{-1/2}\left[(\tilde{P} + {D_f}^2 + \lambda)^{-1}, \mathcal{F}\hat{\otimes}1\right] d\lambda\right)$$

Equation (3.5.9) shows that we have the operator $(\tilde{P}+{D_f}^2+\lambda)^{-1}$ on the right and left of the expression containing the operator $\mathfrak{M}$ in the integrand. However, the above equation shows that we have to multiply by $(\tilde{P} + {D_f}^2)^{1/2}$ on the left in the integrand. Therefore, writing $(\tilde{P}+{D_f}^2+\lambda)^{-1} = (\tilde{P}+{D_f}^2+\lambda)^{-1/2}(\tilde{P}+{D_f}^2+\lambda)^{-1/2}$ and recalling that $\mathfrak{M}(\tilde{P}+{D_f}^2+\lambda)^{-1/2}$ is compact, we conclude that $\mathcal{D}_f\left[(\tilde{P} + {D_f}^2)^{-1/2}, \mathcal{F}\hat{\otimes}1\right]$ is compact.

□

This concludes the proof that $(\mathcal{F}\hat{\otimes}1)\mathcal{T} + \mathcal{T}(\mathcal{F}\hat{\otimes}1)$ is positive modulo compact operators.

□

Chapter 4

The Embedded Toeplitz Problem

4.1 Index Calculation of Some Toeplitz Operators over the Unit Ball $\mathbb{B} \subset \mathbb{C}^2$

In the present section we focus our attention on the discussion of Toeplitz operators and Boutet de Monvel's index formula for strictly pseudoconvex domains. In particular, we will concentrate on calculating indices of Toeplitz operators for some particular symbols that will motivate our approach to the embedded Toeplitz problem. First, let us start by start by stating the Boutet de Monvel's index formula [BDM].

Theorem 4.1.1. *Let B be a strictly pseudoconvex domain in $\mathbb{C}^n$. Suppose that the restriction of F to the boundary ∂B is an invertible matrix-valued function. Then the Toeplitz operator T_F is Fredholm and the following index formula holds:*

$$Index(T_F) = \frac{-(n-1)!}{(2n-1)!(2\pi i)^n} \int_{\partial B} trace((F^{-1}dF)^{2n-1})$$

We will do the index calculation for a class of examples on the unit ball $\mathbb{B} \subset \mathbb{C}^2$

Proposition 4.1.2. *Let*

$$F = \begin{pmatrix} \bar{f}(z_2) & -z_1 \\ \bar{z}_1 & f(z_2) \end{pmatrix}$$

If $f(z_2) = {z_2}^n$ then $Index(T_F) = n$. And, if $f(z_2) = \bar{z}_2^n$ we have $Index(T_F) = -n$.

Proof. Consider the matrix-valued function

$$F = \begin{pmatrix} \bar{f}(z_2) & -z_1 \\ \bar{z}_1 & f(z_2) \end{pmatrix}$$

where $f(z_2)$ is a function which is never 0 on the boundary of the unit ball $\mathbb{B} \subset \mathbb{C}^2$. The determinant

$$det(F) = |f(z_2)|^2 + |z_1|^2 \neq 0 \text{ on } \partial\mathbb{B}$$

Then F is invertible on $\partial\mathbb{B}$. By Theorem 4.1.1, the operator T_F is Fredholm and its index can be computed using the Boutet de Monvel's formula.

First, let us compute $trace((F^{-1}dF)^3)$.

$$F^{-1} = \frac{1}{det(F)} \begin{pmatrix} f(z_2) & z_1 \\ -\bar{z}_1 & \bar{f}(z_2) \end{pmatrix}$$

$$dF = \begin{pmatrix} d\bar{f} & -dz_1 \\ d\bar{z}_1 & df \end{pmatrix}$$

Then

$$F^{-1}dF = \frac{1}{det(F)} \begin{pmatrix} fd\bar{f} + z_1 d\bar{z}_1 & -fdz_1 + z_1 df \\ -\bar{z}_1 d\bar{f} + \bar{f} d\bar{z}_1 & \bar{z}_1 dz_1 + \bar{f}df \end{pmatrix}$$

$$(F^{-1}dF)^3 = \frac{1}{(det(F))^3} \begin{pmatrix} fd\bar{f} + z_1 d\bar{z}_1 & -fdz_1 + z_1 df \\ -\bar{z}_1 d\bar{f} + \bar{f} d\bar{z}_1 & \bar{z}_1 dz_1 + \bar{f}df \end{pmatrix}^3$$

Then

$$Trace((F^{-1}dF)^3) = \frac{3}{(det(F))^2} \left(\bar{f}dz_1 d\bar{z}_1 df - fdz_1 d\bar{z}_1 d\bar{f} + \bar{z}_1 df d\bar{f} dz_1 - z_1 df d\bar{f} d\bar{z}_1 \right) \quad (4.1.1)$$

Now, we specialize to the case $f(z_2) = z_2{}^n$. Using polar coordinates, we get

$$z_1 = re^{i\theta_1} \ , \ z_2 = \sqrt{1-r^2}e^{i\theta_2}$$

where $0 \leq r \leq 1$ and $0 \leq \theta_1, \theta_2 \leq 2\pi$. Then

$$dz_1 = e^{i\theta_1}dr + ire^{i\theta_1}d\theta_1 \ , \ dz_2 = \frac{-r}{\sqrt{1-r^2}}e^{i\theta_2}dr + i\sqrt{1-r^2}e^{i\theta_2}d\theta_2$$

$$d\bar{z}_1 = e^{-i\theta_1}dr - ire^{-i\theta_1}d\theta_1 \ , \ d\bar{z}_2 = \frac{-r}{\sqrt{1-r^2}}e^{-i\theta_2}dr - i\sqrt{1-r^2}e^{-i\theta_2}d\theta_2$$

$$dz_1 d\bar{z}_1 = -2irdrd\theta_1 \ , \ dz_2 d\bar{z}_2 = 2irdrd\theta_2$$

Consequently

$$dz_1 d\bar{z}_1 dz_2 = 2r\sqrt{1-r^2}e^{i\theta_2}drd\theta_1 d\theta_2$$

$$dz_1 d\bar{z}_1 d\bar{z}_2 = -2r\sqrt{1-r^2}e^{-i\theta_2}drd\theta_1 d\theta_2$$

$$dz_2 d\bar{z}_2 dz_1 = 2r^2 re^{i\theta_1}drd\theta_1 d\theta_2$$

$$dz_2 d\bar{z}_2 d\bar{z}_1 = -2r^2 r e^{-i\theta_1} dr d\theta_1 d\theta_2$$

Using these identities and making the following substitutions $f(z_2) = {z_2}^n$, $df = n{z_2}^{n-1}dz_2$ and $det(F) = r^2 + (1-r^2)^n$ in the equation (4.1.1), we obtain

$$Trace((F^{-1}dF)^3) = \frac{12nr(1-r^2)^{(n-1)}(1+(n-1)r^2)}{(r^2+(1-r^2)^n)^2}$$

Now, we use the index formula

$$Index(T_F) = \frac{-1}{3!(2\pi i)^2}\int_{\partial B} trace((F^{-1}dF)^3)$$

$$= \frac{1}{6(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^1 \frac{12nr(1-r^2)^{(n-1)}(1+(n-1)r^2)}{(r^2+(1-r^2)^n)^2} dr d\theta_1 d\theta_2$$

$$= \int_0^1 \frac{2nr(1-r^2)^{(n-1)}(1+(n-1)r^2)}{(r^2+(1-r^2)^n)^2} dr = \int_0^1 \frac{n(1-x)^{(n-1)}(1+(n-1)x)}{(x+(1-x)^n)^2} dx$$

$$= -\frac{n(1-x)}{x^n+(1-x)}|_0^1$$

Thus $Index(T_F) = n$. Similarly, if we use $f(z_2) = \bar{z}_2^n$ we have $Index(T_F) = -n$. □

4.2 Motivation of the Embedded Toeplitz Problem

In this section we shall explain the embedded Toeplitz problem of the unit disc $\mathbb{D} \subset \mathbb{C}$ into the unit ball $\mathbb{B} \subset \mathbb{C}^2$. Note that the operator

$$F\#D_+ = \begin{pmatrix} F & -D_- \\ D+ & \bar{F} \end{pmatrix}$$

has the same index as the operator $D_F = \begin{pmatrix} F & D_- \\ D+ & -\bar{F} \end{pmatrix}$ because $F\#D_+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = D_F$. Since they represent the same class, I will use the abuse of notation $F\#D_+ = D_F$. Here we paired the multiplier F with the Dolbeault operator D_+ whose kernel is the space of L^2 holomorphic functions. The idea of the embedded problem is to recover the Toeplitz problem in the unit disk by pairing a multiplier invertible on the boundary of unit disk with a perturbed Dolbeault operator on the unit ball in $\mathbb{C}^2$. Our previous calculations will motivate the choice of a candidate for this perturbed Dolbeault operator.

In the paper [GHI] it is shown that $Index(D_F) = Index(T_F)$. Then we could say that the operator D_F represents the Toeplitz operator with symbol F. In the previous section, we showed that for the symbol

$$F = \begin{pmatrix} \bar{z}_2 & -z_1 \\ \bar{z}_1 & z_2 \end{pmatrix}$$

we have $Index(T_F) = Index(D_F) = 1$. For this particular function F, we have $\bar{z}_2 \# \bar{z}_1 = F$. Then, for appropriate identification of spaces

$$(\bar{z}_2 \# \bar{z}_1) \# D_+ = D_F$$

Hence

$$\bar{z}_2 \# (\bar{z}_1 \# D_+) = D_F$$

We claim that the operator $D_{\bar{z}_1} := \bar{z}_1 \# D_+$ is a good candidate to be the perturbed Dolbeault operator on the unit ball for the embedded Toeplitz problem for the following reasons:

1. Observe that in the z_2-unit disk, i.e. when $z_1 = 0$, the operator $D_{\bar{z}_1}$ is the Dolbeault operator D.

2. We will show later that a compact perturbation of the operator $D_{\bar{z}_1}$ has as kernel the L^2 holomorphic functions in the variable z_2.

3. If the operator $D_{\bar{z}_1}$ is the right perturbed Dolbeault operator, then the pairing $\bar{z}_2 \# D_{\bar{z}_1}$ should represent the Toeplitz operator $T_{\bar{z}_2}$ in the z_2-unit disk. Recall that we showed that the operator
$$(\bar{z}_2 \# \bar{z}_1) \# D_+ = \bar{z}_2 \# D_{\bar{z}_1}$$
has index 1.

4. Furthermore, if we let $f(z_2)$ be invertible on the boundary of the z_2-unit disk with winding number $-n$. Then
$$Index(T_f) = Index(T_{\bar{z}_2^n}) = n = Index(T_F) = Index(\bar{z}_2^n \# D_{\bar{z}_1})$$
where
$$F = \begin{pmatrix} \bar{z}_2^n & -z_1 \\ \bar{z}_1 & {z_2}^n \end{pmatrix}$$

Hence, the above remarks tell us that the Toeplitz index problem in the z_2-unit disk is equivalent to the index problem of the pairing $\bar{z}_2^n \# D_{\bar{z}_1}$. Thus, the operator $D_{\bar{z}_1}$ is a good candidate to be a perturbed Dolbeault operator that could define a K-homology element defined on the two-dimensional ball and representing the disk.

Being more ambitious, if all these ideas work as expected we could go further and have the following program: Consider the zero set of a "nice" function φ (such as $\bar{z}_1$ which was the defining function of the z_2-unit disk). In particular, we are interested in the case when the zero set of the function φ defines a singular space. This zero set of φ can be embedded into a strictly pseudoconvex domain M. In this ambient space M, we have a Dolbeault operator D_+. Then we could define the perturbed Dolbeault operator D_φ and this would be a good candidate to be a K-homology element. Therefore, we could extend our definition of Toeplitz operators to zero sets of holomorphic function in the following way: The Toeplitz operator with symbol f could be defined as the Kasparov product of the multiplier f and the perturbed Dolbeault operator D_φ i.e.

$$T_f := [f] \otimes [D_\varphi]$$

A nice example of a singular space is

$$N = \left(\mathbb{C}^2 \times \{0\} \cup \{0\} \times \mathbb{C}^2\right) \cap \mathbb{B}^3 \subset \mathbb{C}^3$$

N can be seen as the zero set of $\varphi(z) = z_1 z_3$. The boundary of N can be seen as two copies of S^3 intersecting transversely in a circle S^1. More precisely, the boundary of N is $S^3 \times \{0\} \cup \{0\} \times S^3$.

Chapter 5

The K-homology of the Embedded Toeplitz Problem

The purpose of this chapter is to find an adequate K-homology element in $KK(C_0(\widetilde{M}), \mathbb{C})$ that plays the role of the Dolbeault operator in the case of strictly pseudoconvex domains. Here $\widetilde{M}$ is the closed unit ball $\mathbb{B}$ in $\mathbb{C}^2$ except the points in the circle $\{z \in \bar{\mathbb{B}} : z_1 = 0, |z_2| = 1\}$. A strong candidate is the operator $D_{\bar{z}_1}$ as noted in the last comments in the previous chapter where

$$D_{\bar{z}_1} = \begin{pmatrix} \bar{z}_1 & D_- \\ D_+ & -z_1 \end{pmatrix} : \begin{matrix} \bigoplus_{q \text{ even}} {L^2}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} {L^2}_{n,q} \end{matrix} \to \begin{matrix} \bigoplus_{q \text{ even}} {L^2}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} {L^2}_{n,q} \end{matrix}$$

It turns out that the right representative for this K-homology class is a compact perturbation of the operator $D_{\bar{z}_1}$.

In the next two sections, we build some machinery we will need later and study some properties of the operator $D_{\bar{z}_1}$.

5.1 Definitions and basic properties

Definition 5.1.1. For any $x \in [0,1]$ and $Re(a)$, $Re(b) \geq 0$, we define

$$B(x; a, b) := \int_0^x (1-t)^{b-1} t^{a-1} dt$$

This function is called the generalized beta function. The particular case $B(a,b) := B(1; a, b)$ is called the beta function.

Definition 5.1.2. The regularized beta function is defined as follows:

$$I_x(a,b) := \frac{B(x; a, b)}{B(a,b)}$$

for any $x \in [0,1]$. The integral representation of the regularized beta function is

$$I_x(a,b) = \frac{\int_0^x (1-t)^{b-1} t^{a-1} dt}{B(a,b)}$$

For definitions of the different types of beta functions see for example [PaR].

Remark 5.1.1. The regularized beta function satisfies the following recurrence relations:

$$I_x(a+1,b) = I_x(a,b) - \frac{x^a(1-x)^b}{aB(a,b)}$$

$$I_x(a,b+1) = I_x(a,b) + \frac{x^a(1-x)^b}{bB(a,b)}$$

This means that the regularized beta function is decreasing with respect to the first entry and it is increasing with respect to the second entry.

Proof. Use the integral representation of the regularized beta function and then take derivatives with respect to x on both sides of the equations applying the fundamental theorem of calculus to verify that the functions on the left and the functions on the right have the same derivative. Therefore, the functions on the right and the functions on the left differ by constants. But all these functions vanish at $x = 0$. Thus, the equalities hold. □

Remark 5.1.2.

$$I_{1-p}(n-k, k+1) \leq \exp\left(-2\frac{(np-k)^2}{n}\right)$$

holds for $k \leq np$.

Proof. The proof is based on representing the regularized beta function in terms of the cumulative distribution function $I_{1-p}(n-k, k+1) = F(k; n, p)$ and then using Hoeffding's inequality for $k \leq np$ from probability theory. For details of the proof see for example [AlS]. □

Lemma 5.1.3. *An orthonormal basis for the Bergman space $H^2(\mathbb{B})$ is $\{\phi_\alpha\}_{\alpha \geq 0}$ where $\phi_\alpha(z) = c_\alpha z^\alpha = c_\alpha {z_1}^{\alpha_1} {z_2}^{\alpha_2}$ for $\alpha_1, \alpha_2 \geq 0$ and $c_\alpha = \frac{1}{\pi}\sqrt{\frac{(\alpha_1+\alpha_2+2)!}{\alpha_1!\alpha_2!}}$.*

Proof.

$$\int_{\mathbb{B}} |z_1|^{2\alpha_1} |z_2|^{2\alpha_2} dV = \int_{|z_2|<1} \left(\int_{|z_1| \leq \sqrt{1-|z_2|^2}} {r_1}^{2\alpha_1} |z_2|^{2\alpha_2} r_1 d\theta_1 dr_1 \right) dV(z_2)$$

$$= 2\pi \int_{|z_2|<1} \frac{(1-|z_2|^2)^{\alpha_1+1}}{2\alpha_1+2} |z_2|^{2\alpha_2} dV(z_2) = \frac{(2\pi)^2}{2\alpha_1+2} \int_0^1 (1-{r_2}^2)^{\alpha_1+1} {r_2}^{2\alpha_2} r_2 dr_2$$

$$= \frac{(2\pi)^2}{4\alpha_1+4}\int_0^1 (1-u)^{\alpha_1+1}u^{\alpha_2}du = \frac{\pi^2}{\alpha_1+1}B(\alpha_2+1,\alpha_1+2)$$

Recall that the gamma function $\Gamma(n) = (n-1)!$ and the beta function $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(n+m)}$ for n and m nonnegative integers. Therefore

$$\int_{\mathbb{B}} |z_1|^{2\alpha_1}|z_2|^{2\alpha_2}dV = \frac{\pi^2}{\alpha_1+1}\frac{\Gamma(\alpha_2+1)\Gamma(\alpha_1+2)}{\Gamma(\alpha_1+\alpha_2+3)}$$

$$= \frac{\pi^2}{\alpha_1+1}\frac{\alpha_2!(\alpha_1+1)!}{(\alpha_1+\alpha_2+2)!} = \pi^2\frac{\alpha_1!\alpha_2!}{(\alpha_1+\alpha_2+2)!}$$

Because orthonormality requires that

$$c_\alpha{}^2 = \left[\int_{\mathbb{B}} |z_1|^{2\alpha_1}|z_2|^{2\alpha_2}dV\right]^{-1}$$

the preceding calculation shows that $c_\alpha = \frac{1}{\pi}\sqrt{\frac{(\alpha_1+\alpha_2+2)!}{\alpha_1!\alpha_2!}}$. ☐

Lemma 5.1.4. *For $R < 1$, let $K_R = \{(z_1,z_2) \in \bar{\mathbb{B}} : |z_2| \le R\}$ and $\|\phi_\alpha\|_{K_R}$ be the norm of ϕ_α restricted to the compact set K_R. Then $\|\phi_\alpha\|_{K_R} = \sqrt{I_{R^2}(\alpha_2+1,\alpha_1+2)}$.*

Proof.

$$\|\phi_\alpha\|^2{}_{K_R} = \int_{K_R} \phi_\alpha\bar{\phi}_\alpha dV$$

$$= c_\alpha{}^2\int_{K_R}|z_1|^{2\alpha_1}|z_2|^{2\alpha_2}dV = c_\alpha{}^2\int_{|z_2|\le R}\left(\int_{|z_1|\le\sqrt{1-|z_2|^2}} r_1{}^{2\alpha_1}|z_2|^{2\alpha_2}r_1d\theta_1dr_1\right)dV(z_2)$$

$$= c_\alpha{}^2 2\pi\int_{K_R}\frac{(1-|z_2|^2)^{\alpha_1+1}}{2\alpha_1+2}|z_2|^{2\alpha_2}dV(z_2) = c_\alpha{}^2\frac{(2\pi)^2}{2\alpha_1+2}\int_0^R(1-r_2{}^2)^{\alpha_1+1}r_2{}^{2\alpha_2}r_2dr_2$$

Making the change of variable $u = r_2{}^2$

$$\|\phi_\alpha\|^2{}_{K_R} = c_\alpha{}^2\frac{(2\pi)^2}{4\alpha_1+4}\int_0^{R^2}(1-u)^{\alpha_1+1}u^{\alpha_2}du = \frac{\int_0^{R^2}(1-u)^{\alpha_1+1}u^{\alpha_2}du}{\int_0^1(1-u)^{\alpha_1+1}u^{\alpha_2}du}$$

Thus, using the integral representation of the regularized beta function we have

$$\|\phi_\alpha\|^2{}_{K_R} = I_{R^2}(\alpha_2+1,\alpha_1+2)$$

. ☐

5.2 The Operator $D_{\bar{z}_1}$

In this section we shall study the operator

$$D_{\bar{z}_1} = \begin{pmatrix} \bar{z}_1 & D_- \\ D_+ & -z_1 \end{pmatrix} : \begin{matrix} \bigoplus_{q \text{ even}} L^2{}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} L^2{}_{n,q} \end{matrix} \quad \rightarrow \quad \begin{matrix} \bigoplus_{q \text{ even}} L^2{}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} L^2{}_{n,q} \end{matrix}$$

which will play an important role in the construction of the K-homology element for the embedded Toeplitz problem in the unit ball.

Recall that P is the orthogonal projection from the $\bigoplus_q L^2{}_{n,q}$ onto the kernel of D, which is the space of L^2 holomorphic $(n, 0)$-forms, denoted by L^2_{hol}.

Lemma 5.2.1. *Let f be a smooth function on the closed unit ball in $\mathbb{C}^2$. Then*

$$PD_fP = PfP$$

Proof. Recall that

$$D_f = \begin{pmatrix} f & D_- \\ D_+ & -\bar{f} \end{pmatrix} : \begin{matrix} \bigoplus_{q \text{ even}} L^2{}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} L^2{}_{n,q} \end{matrix} \quad \rightarrow \quad \begin{matrix} \bigoplus_{q \text{ even}} L^2{}_{n,q} \\ \bigoplus \\ \bigoplus_{q \text{ odd}} L^2{}_{n,q} \end{matrix}$$

and that

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

vanishes on L^2_{hol} which is the image of P. We can rewrite D_f as follows:

$$D_f = D + \mathfrak{F}$$

where

$$\mathfrak{F} = \begin{pmatrix} f & 0 \\ 0 & -\bar{f} \end{pmatrix}$$

Note that only the upper left corner of $\mathfrak{F}$ acts on L^2_{hol}. Then

$$PD_fP = PDP + P\mathfrak{F}P = PfP$$

□

Remark 5.2.1. Observe that the operator $PD_fP = PfP$ is Toeplitz and it is bounded. Also, note that PD_fP and $PD_{\bar{f}}P$ are adjoints of each other.

Lemma 5.2.2. *Let f be a smooth function on the closed unit ball in $\mathbb{C}^2$. Then, the operators QD_fP and PD_fQ are compact.*

Proof. Using the fact that D vanishes on the image of P and that only the upper left corner of $\mathfrak{F}$ acts on L^2_{hol}, we have

$$QD_fP = QDP + Q\mathfrak{F}P = QfP$$

By Proposition 3.1.8, $QD_fP = QfP$ is a compact operator. Taking adjoints, it turns out that PD_fQ is a compact operator, as well. $\square$

5.2.1 The Operator $PD_{\bar{z}_1}P$

By Lemma 5.2.1, we have

$$PD_{\bar{z}_1}P = P\bar{z}_1P : L^2_{hol} \to L^2_{hol}$$

We will analyze this operator on the orthonormal basis elements of L^2_{hol}. Since the Bergman space $H^2(\mathbb{B})$ is unitarily isomorphic to L^2_{hol} (see Remark 3.1.1), the orthonormal basis $\{\phi_\alpha\}_{\alpha\geq 0}$ of $H^2(\mathbb{B})$ is mapped onto an orthonormal basis of L^2_{hol}. We will use the same notation for the latter orthonormal basis.

$$\begin{aligned} P\bar{z}_1P(\phi_\alpha) = P(\bar{z}_1\phi_\alpha) &= \sum_\beta \langle \bar{z}_1\phi_\alpha, \phi_\beta\rangle\phi_\beta \\ &= \langle \bar{z}_1\phi_\alpha, \phi_{(\alpha_1-1,\alpha_2)}\rangle\phi_{(\alpha_1-1,\alpha_2)} \end{aligned}$$

But

$$\langle \bar{z}_1\phi_\alpha, \phi_{(\alpha_1-1,\alpha_2)}\rangle = \int_{\mathbb{B}} c_\alpha c_{(\alpha_1-1,\alpha_2)}|z_1|^{2\alpha_1}|z_2|^{2\alpha_2}dV = c_\alpha c_{(\alpha_1-1,\alpha_2)}\frac{1}{{c_\alpha}^2} = \frac{c_{(\alpha_1-1,\alpha_2)}}{c_\alpha}$$

By Lemma 5.1.3

$$\langle \bar{z}_1\phi_\alpha, \phi_{(\alpha_1-1,\alpha_2)}\rangle = \frac{c_{(\alpha_1-1,\alpha_2)}}{c_\alpha} = \sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2+2}}$$

Then, the operator

$$PD_{\bar{z}_1}P(\phi_\alpha) = P\bar{z}_1P(\phi_\alpha) = \sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2+2}}\phi_{(\alpha_1-1,\alpha_2)}$$

is a weighted left shift in z_1.

Remark 5.2.2. $Ker(PD_{\bar{z}_1}P) = L^2_{hol}(z_2) = L^2_{hol}$ with $\alpha_1 = 0$

Remark 5.2.3. $PD_{\bar{z}_1}P$ is not bounded below on the orthogonal complement of its kernel because if we fix α_1 and let $\alpha_2 \to \infty$ we have that

$$\sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2+2}} \to 0$$

5.2.2 The Operator $PD_{z_1}P$

By Lemma 5.2.1, we have

$$PD_{z_1}P = Pz_1P : L^2_{hol} \to L^2_{hol}$$

Then

$$Pz_1P(\phi_\alpha) = z_1\phi_\alpha = c_\alpha {z_1}^{\alpha_1+1}{z_2}^{\alpha_2}$$

$$= \frac{c_\alpha c_{(\alpha_1+1,\alpha_2)}}{c_{(\alpha_1+1,\alpha_2)}} {z_1}^{\alpha_1+1}{z_2}^{\alpha_2} =$$

$$= \frac{c_\alpha}{c_{(\alpha_1+1,\alpha_2)}} \phi_{(\alpha_1+1,\alpha_2)}$$

By Lemma 5.1.3

$$\frac{c_\alpha}{c_{(\alpha_1+1,\alpha_2)}} = \sqrt{\frac{\alpha_1+1}{\alpha_1+\alpha_2+3}}$$

Then, the operator

$$PD_{z_1}P(\phi_\alpha) = Pz_1P(\phi_\alpha) = \sqrt{\frac{\alpha_1+1}{\alpha_1+\alpha_2+3}}\phi_{(\alpha_1+1,\alpha_2)}$$

is a weighted right shift in z_1.

Remark 5.2.4. $Ker(PD_{z_1}P) = \{0\}$

Remark 5.2.5. $PD_{z_1}P$ is not bounded below on the orthogonal complement of its kernel because if we fix α_1 and let $\alpha_2 \to \infty$ we have that

$$\sqrt{\frac{\alpha_1+1}{\alpha_1+\alpha_2+3}} \to 0$$

5.2.3 More Properties of $D_{\bar{z}_1}$

Notice that

$$(PD_{z_1}P)(PD_{\bar{z}_1}P)(\phi_\alpha) = \frac{\alpha_1}{\alpha_1+\alpha_2+2}\phi_\alpha$$

and

$$(PD_{\bar{z}_1}P)(PD_{z_1}P)(\phi_\alpha) = \frac{\alpha_1+1}{\alpha_1+\alpha_2+3}\phi_\alpha$$

Then, the eigenvalues of these operators are in the set $\left\{\frac{\alpha_1}{\alpha_1+\alpha_2+2} : \alpha_1,\alpha_2 \in \mathbb{Z}_0^+\right\}$. By [GHI], there is a smooth function ζ which agrees with $\bar{z}_1$ on the boundary of the unit ball $\mathbb{B}$ such that the operator $QD_\zeta Q$ is bounded below where $Q = I - P$.

Definition 5.2.3. Define

$$T_{\bar{z_1}} = PD_{\bar{z_1}}P + QD_{\zeta}Q$$

Remark 5.2.6.

$$T_{\bar{z_1}} = D_{\zeta}$$

modulo compact operators.

Proof. Observe that

$$T_{\bar{z_1}} = PD_{\bar{z_1}}P - PD_{\zeta}P + D_{\zeta} - PD_{\zeta}Q - QD_{\zeta}P$$

By Lemma 5.2.1 and Lemma 5.2.2, we have

$$T_{\bar{z_1}} = P(\bar{z_1} - \zeta)P + D_{\zeta}$$

modulo compact operators. But the operator $P(\bar{z_1}-\zeta)P$ is Toeplitz with a symbol vanishing on the boundary. Then it is a compact operator (see the conclusion of Section 2.3). Thus

$$T_{\bar{z_1}} = D_{\zeta}$$

modulo compact operators. □

Definition 5.2.4. Define

$$T = \begin{pmatrix} 0 & {T_{\bar{z_1}}}^* \\ T_{\bar{z_1}} & 0 \end{pmatrix}$$

Notice that by taking adjoints, we have that ${T_{\bar{z_1}}}^* = PD_{z_1}P + QD_{\bar{\zeta}}Q$. The operator

$$T^2 = \begin{pmatrix} {T_{\bar{z_1}}}^* T_{\bar{z_1}} & 0 \\ 0 & T_{\bar{z_1}}{T_{\bar{z_1}}}^* \end{pmatrix}$$

has eigenvalues in the set $\left\{\frac{\alpha_1}{\alpha_1+\alpha_2+2} : \alpha_1, \alpha_2 \in \mathbb{Z}_0^+\right\}$ on $Im(P)$ and eigenvalues bounded away from zero on $Im(Q)$. Then, the operator $|T|$ has eigenvalues in the set $\left\{\sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2+2}} : \alpha_1, \alpha_2 \in \mathbb{Z}_0^+\right\}$ on $Im(P)$ and eigenvalues bounded away from zero on $Im(Q)$.

5.3 The K-Homology Element

A defining function for the embedding of the unit disk into the unit ball is $\bar{z_1}$. We would like to find an operator that does the job of the Dolbeault operator in the KK-product for strictly pseudoconvex domains. This operator should have the same crucial properties as the Dolbeault operator. One of those important properties is that the kernel of this operator should be the space of L^2 holomorphic functions in z_2. Also, we should expect this operator to be related to the defining function $\bar{z}_1$. One natural candidate could be the operator $D_{\bar{z}_1}$. The problem with this operator is that its kernel is hard to identify. Therefore, we will consider a compact perturbation of this operator that has the properties we want, the operator T defined in the previous section.

We shall show that the operator $\frac{T}{|T|}$ satisfies the following conditions:

(i) $m_f\left(\left(\frac{T}{|T|}\right)^2 - 1\right)$ is a compact operator for any $f \in C_0(\widetilde{M})$.

(ii) $\left[m_f, \frac{T}{|T|}\right]$ is a compact operator for any $f \in C_0(\widetilde{M})$ and thus, it defines a K-homology element in $KK(C_0(\widetilde{M}), \mathbb{C})$.

Remark 5.3.1. In most of the following proofs if $f \in C_0(\widetilde{M})$, it is enough to assume that f is supported in K_R for R close to 1 where $K_R = \{(z_1, z_2) \in \bar{\mathbb{B}} : |z_2| \leq R\}$. Because, for any $f \in C_0(\widetilde{M})$ and $\epsilon > 0$ there are $R < 1$ and f_ϵ with support in K_R such that $\|f - f_\epsilon\|_\infty < \epsilon$ and hence with $\|m_f - m_{f_\epsilon}\|_{op} < \epsilon$. Most of the following proofs will use this approximation argument.

Theorem 5.3.1. $m_f\left(\left(\frac{T}{|T|}\right)^2 - 1\right)$ *is a compact operator for any* $f \in C_0(\widetilde{M})$.

Proof. Assume that f is supported in K_R for R close to 1 where $K_R = \{(z_1, z_2) \in \bar{\mathbb{B}} : |z_2| \leq R\}$. Note that

$$\left(\frac{T}{|T|}\right)^2 - 1 = P_{Ker(T)} = P_{L^2_{hol}(z_2)}$$

The problem simplifies to proving that the multiplication operator m_f acting on $L^2_{hol}(z_2)$ is compact.

Claim: The operator

$$P_{Ker(T)} m_{\bar{f}} : \bigoplus_{q=0}^{n} L^2_{n,q} \to L^2_{hol}(z_2)$$

is compact.

Proof. Define

$$\mathfrak{z}_j : \bigoplus_{q=0}^{n} L^2_{n,q} \to L^2_{hol}(z_2)$$

$$\mathfrak{z}_j(h) := \sum_{\alpha_2 \geq j} \langle \bar{f} h, \phi_{\alpha_2} \rangle \phi_{\alpha_2} = \sum_{\alpha_2 \geq j} \langle h, f\phi_{\alpha_2} \rangle \phi_{\alpha_2}$$

Assume that $h \in \bigoplus_{q=0}^{n} L^2_{n,q}$ has norm 1. Then

$$\|\mathfrak{Z}_j(h)\|^2 = \sum_{\alpha_2 \geq j} |\langle h, f\phi_{\alpha_2}\rangle|^2$$

Using the Schwarz inequality

$$|\langle h, f\phi_{\alpha_2}\rangle| \leq \|h\|\|f\phi_{\alpha_2}\| = \|f\phi_{\alpha_2}\|$$

Therefore

$$\|\mathfrak{Z}_j(h)\|^2 \leq \sum_{\alpha_2 \geq j} \|f\phi_{\alpha_2}\|^2_{K_R} \leq \|f\|^2_\infty \sum_{\alpha_2 \geq j} |c_{\alpha_2}|^2 \int_{K_R} |z_2|^{2\alpha_2} dV$$

$$\leq \|f\|^2_\infty \sum_{\alpha_2 \geq j} |c_{\alpha_2}|^2 R^{2\alpha_2} Vol(\mathbb{B}) \leq \|f\|^2_\infty \frac{1}{\pi^2} \sum_{\alpha_2 \geq j} (\alpha_2+2)(\alpha_2+1) R^{2\alpha_2} Vol(\mathbb{B})$$

$$\leq \|f\|^2_\infty \frac{1}{\pi^2} \sum_{\alpha_2 \geq j} (\alpha_2+2)^2 R^{2\alpha_2} Vol(\mathbb{B})$$

For some large j, we have $(\alpha_2+2)^4 R^{2\alpha_2} < 1$ for every $\alpha_2 \geq j$. Then

$$\|\mathfrak{Z}_j(h)\|^2 < \|f\|^2_\infty \frac{1}{\pi^2} Vol(\mathbb{B}) \sum_{\alpha_2 \geq j} \frac{1}{(\alpha_2+2)^2}$$

$$\to 0 \text{ as } j \to \infty$$

because the series

$$\sum_{\alpha_2=0}^{\infty} \frac{1}{(\alpha_2+2)^2}$$

converges. Hence, $\|\mathfrak{Z}_j\|_{op} \to 0$ as $j \to \infty$.

Thus $P_{Ker(T)} m_{\bar{f}}$ can be approximated by finite rank operators. □

Using the claim and taking the adjoint of $P_{Ker(T)} m_{\bar{f}}$, the multiplication operator m_f acting on $L^2_{hol}(z_2)$ is a compact operator.

□

Theorem 5.3.2. *The operator* $\left[m_f, \frac{T}{|T|}\right]$ *is compact for any* $f \in C_0(\widetilde{M})$.

Before doing the proof of the above theorem we shall prove some lemmas to make the exposition more organized.

Lemma 5.3.3. *Let f be supported in K_R and let $\mathcal{L}_H$ be the line*

$$\mathcal{L}_H : \alpha_2 + 1 = m_H(\alpha_1 + 1)$$

where $m_H = \frac{3+R^2}{1-R^2}$. Then, for any $(\alpha_1 + 1, \alpha_2 + 1)$ above the line $\mathcal{L}_H$ we have

$$\|\phi_\alpha\|^2_{K_R} \leq \exp(-(1 - R^2)(\alpha_1 + 1)) \exp(-(1 - R^2)(\alpha_2 + 1))$$

Proof. By Lemma 5.1.4, we know that $\|\phi_\alpha\|^2_{K_R} = I_{R^2}(\alpha_2 + 1, \alpha_1 + 2)$.
Denote $k = \alpha_1 + 1$, $n = \alpha_1 + \alpha_2 + 2$ and $p = 1 - R^2$.
Notice that

$$k \leq \frac{np}{2} \Leftrightarrow \alpha_1 + 1 \leq \frac{(\alpha_1 + \alpha_2 + 2)}{2}(1 - R^2) \Leftrightarrow \frac{1 + R^2}{1 - R^2}(\alpha_1 + 1) \leq \alpha_2 + 1 \tag{5.3.1}$$

and

$$k \leq \frac{np}{4} \Leftrightarrow \alpha_1 + 1 \leq \frac{(\alpha_1 + \alpha_2 + 2)}{4}(1 - R^2) \Leftrightarrow \frac{3 + R^2}{1 - R^2}(\alpha_1 + 1) \leq \alpha_2 + 1 \tag{5.3.2}$$

Let $(\alpha_1 + 1, \alpha_2 + 1)$ be a point above the line $\mathcal{L}_H$. By equations (5.3.1) and (5.3.2), $(\alpha_1 + 1, \alpha_2 + 1)$ satisfies the inequality $k \leq \frac{np}{4}$. Then, using the last inequality and Remark 5.1.2 we get

$$\|\phi_\alpha\|^2_{K_R} = I_{R^2}(\alpha_2 + 1, \alpha_1 + 2) \leq \exp\left(-2\frac{(np - k)^2}{n}\right)$$

$$\exp\left(-2\frac{(n^2p^2 - 2npk + k^2)}{n}\right) \leq \exp\left(-2\frac{(n^2p^2 - 2npk)}{n}\right) = \exp\left(-2p(np - 2k)\right)$$

$$= \exp\left(-2p(\frac{np}{2} + \frac{np}{2} - 2k)\right) \leq \exp(-np^2) = \exp(-(1-R^2)^2(\alpha_1+1)) \exp(-(1-R^2)^2(\alpha_2+1))$$

□

Remark 5.3.2. Denote $\alpha \geq \mathcal{L}_H$ the points above the line $\mathcal{L}_H$. By the previous lemma,

$$\sum_{\alpha \geq \mathcal{L}_H} \|\phi_\alpha\|^2_{K_R} \leq \sum_{\alpha \geq \mathcal{L}_H} \exp(-(1 - R^2)^2(\alpha_1 + 1)) \exp(-(1 - R^2)^2(\alpha_2 + 1))$$

$$\leq \int_0^\infty \int_{m_H x}^\infty e^{-p^2 x} e^{-p^2 y} dx dy < \infty$$

Given $\epsilon > 0$, choose a natural number A such that

$$\int_A^\infty \int_{m_H x}^\infty e^{-p^2 x} e^{-p^2 y} dy dx < \frac{\epsilon^2}{2}$$

There are finitely many α_1's such that $\alpha_1 + 1 < A$. For each of these α_1 choose N_{α_1} such that

$$\int_{N_{\alpha_1}}^{\infty} e^{-p(\alpha_1)} e^{-py} dy < \frac{\epsilon^2}{2A}$$

Let $\widetilde{H_2}(\epsilon) = \overline{span\{\phi_\alpha : \alpha \geq \mathcal{L} \text{ except } (\alpha_1 < A \wedge \alpha_2 < N_{\alpha_1})\}}$. Note that this space is infinite dimensional. Thus

$$\sum_{\widetilde{H_2}(\epsilon)} \|\phi_\alpha\|^2_{K_R} < \epsilon^2$$

Proposition 5.3.4. *Let f have support in K_R. Then, there exists a continuous function φ bounded away from zero, such that $\left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)}\right) m_f$ is compact.*

Proof. Take a smooth function φ with the following properties:

1. $\varphi(x) = 1$ in a small neighborhood of 0.
2. $\delta \leq \varphi(x) \leq 1$ if $0 \leq x \leq \delta$
3. $\varphi(x) = x$ if $\delta \leq x$

where δ is to be chosen. We will choose δ positive satisfying the following conditions:

1. Since the spectrum of $Q|T|Q$ is bounded below, then we can choose δ to be a lower bound of the spectrum of $Q|T|Q$.
2. The eigenvalues coming from $P|T|P$ are of the form $\sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2+2}}$. Assume $\delta \neq 1$. Then

$$\sqrt{\frac{\alpha_1}{\alpha_1 + \alpha_2 + 2}} \geq \delta \Leftrightarrow \frac{\alpha_1}{\alpha_1 + \alpha_2 + 2} \geq \delta^2$$

$$\Leftrightarrow \left(\frac{1}{\delta^2} - 1\right)(\alpha_1 + 1) - \frac{1}{\delta^2} \geq (\alpha_2 + 1)$$

Denote by $\mathcal{L}_\delta$ to be the line $\left(\frac{1}{\delta^2} - 1\right)(\alpha_1 + 1) - \frac{1}{\delta^2} = (\alpha_2 + 1)$.

We choose δ so the slope $\left(\frac{1}{\delta^2} - 1\right) > m_H$ where m_H is the same as in Lemma 5.3.3.

Claim: Let $\epsilon_1 > 0$. Take $\epsilon = \frac{\epsilon_1}{\|f\|_\infty}$ and define $H_2 = \widetilde{H_2}(\epsilon)$. Then

$$\|\left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)}\right) P_{H_2} m_f\|_{op} < \epsilon_1$$

where P_{H_2} is the orthogonal projection onto H_2.

Proof. Let $v \in Im(P) \oplus Im(Q)$ be a vector with norm 1.

$$(P_{H_2}m_f)(v) = \sum_{H_2} \langle fv, \phi_\alpha \rangle \phi_\alpha$$

where the subindex H_2 in the sum means that $\phi_\alpha \in H_2$. Then

$$\|(P_{H_2}m_f)(v)\|^2 = \sum_{H_2} |\langle fv, \phi_\alpha \rangle|^2 = \sum_{H_2} |\langle v, \bar{f}\phi_\alpha \rangle|^2$$

$$\leq \sum_{H_2} \|v\|^2 \|\bar{f}\phi_\alpha\|^2 \leq \sum_{H_2} \|f\|_\infty^2 \|\phi_\alpha\|_{K_R}^2$$

Using Remark 5.3.2 we have

$$\| \left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)} \right) P_{H_2}m_f\|_{op} \leq \|P_{H_2}m_f\|_{op} < \epsilon_1$$

□

Now, we make a partition of the Hilbert space $Im(P)$ into the three following Hilbert spaces:
$H_3 :=$ the space generated by the ϕ_α such that α is in the region that is bounded above by $\mathcal{L}_\delta$ and $\mathcal{L}_H$.
$H_1 :=$ the orthogonal complement Hilbert space of H_2 and H_3 in ImP. The space H_1 is finite-dimensional.

Define P_{H_i} to be the orthogonal projection onto the space H_i for $i = 1, 2, 3$.
Remark 5.3.3. $\varphi(|T|) = |T|$ on H_3.

Now, we have all the ingredients to complete the proof of the proposition.

$$\left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)} \right) m_f = \left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)} \right) Pm_f + \left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)} \right) Qm_f$$

Notice that the second term on the right hand side in the above equation is 0 (this is because of the particular choice of δ). Then

$$\left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)} \right) m_f = \left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)} \right) P_{H_1}m_f + \left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)} \right) P_{H_2}m_f$$
$$+ \left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)} \right) P_{H_3}m_f$$

Note that the first term on the right hand side in the above equation is finite rank and the last term is 0.

Now, we are ready to finish the proof of the proposition.

$$\left(\frac{T}{|T|}-\frac{T}{\varphi(|T|)}\right)m_f=\left(\frac{T}{|T|}-\frac{T}{\varphi(|T|)}\right)P_{H_1}m_f+\left(\frac{T}{|T|}-\frac{T}{\varphi(|T|)}\right)P_{H_2}m_f$$

Then $\left(\frac{T}{|T|}-\frac{T}{\varphi(|T|)}\right)m_f$ equals a compact operator plus an operator with norm less than ϵ_1, where $\epsilon_1>0$ was arbitrary. Since the C^*-algebra of compact operators is closed under the operator norm, $\left(\frac{T}{|T|}-\frac{T}{\varphi(|T|)}\right)m_f$ is a compact operator. □

Lemma 5.3.5. *The operator $(\varphi(|T|)^2-T^2)m_f$ is compact.*

Proof. We will proceed as we did in Proposition 5.3.4.

$$(\varphi(|T|)^2-T^2)m_f=(\varphi(|T|)^2-T^2)Pm_f+(\varphi(|T|)^2-T^2)Qm_f=(\varphi(|T|)^2-T^2)Pm_f$$

Next, we decompose the projection P as follows:

$$P=P_{H_1}+P_{H_2}+P_{H_3}$$

and note that $\|(\varphi(|T|)^2-T^2)\|_{op}\leq 1$. The proof follows as in Proposition 5.3.4. Thus, the operator $(\varphi(|T|)^2-T^2)m_f$ is compact. □

Lemma 5.3.6. *The operator $[(\varphi(|T|)^2-T^2),m_f]$ is compact.*

Proof. By the previous lemma $(\varphi(|T|)^2-T^2)m_{\bar f}$ is compact. Then, taking adjoints we get that the operator

$$\left((\varphi(|T|)^2-T^2)m_{\bar f}\right)^*=m_f(\varphi(|T|)^2-T^2)$$

is compact. Hence, $[(\varphi(|T|)^2-T^2),m_f]$ is a compact operator. □

Lemma 5.3.7. *The equality $[T,m_f]=[D,m_f]$ holds modulo compacts.*

Proof. By Remark 5.2.6, $T_{\bar z_1}=D_\zeta$ modulo compacts. Then

$$[T_{\bar z_1},m_f]=[D_\zeta,m_f]=[D,m_f]$$

modulo compacts. Similarly,

$$[T_{z_1},m_f]=[D_{\bar\zeta},m_f]=[D,m_f]$$

Thus, $[T,m_f]=[D,m_f]$ modulo compacts. □

Proposition 5.3.8. *The operator $\left[m_f,\frac{T}{\varphi(|T|)}\right]$ is compact.*

Proof.

$$\frac{1}{\varphi(|T|)} = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(\varphi(|T|)^2+\lambda)^{-1}d\lambda$$

$$\left[m_f, \frac{T}{\varphi(|T|)}\right] = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\left[m_f, T(\varphi(|T|)^2+\lambda)^{-1}\right]d\lambda$$

Let us focus on the integrand

$$\left[m_f, T(\varphi(|T|)^2+\lambda)^{-1}\right] = T\left[m_f, (\varphi(|T|)^2+\lambda)^{-1}\right] + [m_f, T](\varphi(|T|)^2+\lambda)^{-1} \tag{5.3.3}$$

Lemma 5.3.9. *The operator* $[m_f, T](\varphi(|T|)^2+\lambda)^{-1}$ *is compact.*

Proof. By Lemma 5.3.7,

$$[m_f, T](\varphi(|T|)^2+\lambda)^{-1} = [m_f, D](\varphi(|T|)^2+\lambda)^{-1}$$

modulo compact operator. But $[m_f, D]$ vanishes at infinity. By Proposition 3.1.7,

$$[m_f, T](\varphi(|T|)^2+\lambda)^{-1}$$

is compact. □

Lemma 5.3.10. *The operator* $T\left[m_f, (\varphi(|T|)^2+\lambda)^{-1}\right]$ *is compact.*

Proof.

$$T\left[m_f, (\varphi(|T|)^2+\lambda)^{-1}\right] = T(\varphi(|T|)^2+\lambda)^{-1}[(\varphi(|T|)^2+\lambda), m_f](\varphi(|T|)^2+\lambda)^{-1}$$

$$= T(\varphi(|T|)^2+\lambda)^{-1}[(\varphi(|T|)^2), m_f](\varphi(|T|)^2+\lambda)^{-1}$$

$$= T(\varphi(|T|)^2+\lambda)^{-1}[(\varphi(|T|)^2-T^2), m_f](\varphi(|T|)^2+\lambda)^{-1} + T(\varphi(|T|)^2+\lambda)^{-1}[T^2, m_f](\varphi(|T|)^2+\lambda)^{-1}$$

The first term in the last equality is compact by Lemma 5.3.6. The second term

$$T(\varphi(|T|)^2+\lambda)^{-1}[T^2, m_f](\varphi(|T|)^2+\lambda)^{-1} =$$

$$T(\varphi(|T|)^2+\lambda)^{-1}T[T, m_f](\varphi(|T|)^2+\lambda)^{-1} + T(\varphi(|T|)^2+\lambda)^{-1}[T, m_f]T(\varphi(|T|)^2+\lambda)^{-1}$$

Again $[T, m_f] = [D, m_f]$ modulo compacts and $[D, m_f]$ vanishes at infinity. The result follows by Proposition 3.1.7. □

Since the operator in the equation (5.3.3) is compact, it follows that the operator $\left[m_f, \frac{T}{\varphi(|T|)}\right]$ is compact. □

Proposition 5.3.11. *The operator* $\left[m_f, \frac{T}{|T|}\right]$ *is compact for all* $f \in C_0(\widetilde{M})$.

Proof.

$$\left[m_f, \frac{T}{|T|}\right] = \left[m_f, \frac{T}{\varphi(|T|)}\right] + m_f\left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)}\right) - \left(\frac{T}{|T|} - \frac{T}{\varphi(|T|)}\right)m_f$$

By the last two propositions, the right hand side terms are compact operators so the left hand side is too. □

Conclusion: By Theorems 5.3.1 and 5.3.2, the operator $\frac{T}{|T|}$ satisfies the following conditions:
(i) $m_f\left(\left(\frac{T}{|T|}\right)^2 - 1\right)$ is a compact operator for any $f \in C_0(\widetilde{M})$;
(ii) $\left[m_f, \frac{T}{|T|}\right]$ is a compact operator for any $f \in C_0(\widetilde{M})$, and thus, it defines a K-homology element in $KK(C_0(\widetilde{M}), \mathbb{C})$.

Remark 5.3.4. The kernel of the operator $\frac{T}{|T|}$ is $L^2_{hol}(z_2)$ which can be identified with the Bergman space on the z_2-disk. Then, the compression of the multiplication operator $\bar{z}_2$, acting on $L^2_{hol}(z_2)$, onto $L^2_{hol}(z_2)$ is similar to the usual Toeplitz operator with symbol $\bar{z}_2$ on the z_2-disk. Therefore, the operator

$$\mathfrak{T}_{\bar{z}_2} = P_{L^2_{hol}(z_2)} m_{\bar{z}_2} : L^2_{hol}(z_2) \to L^2_{hol}(z_2)$$

has index 1.

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