EFFICIENCY AND ACCURACY OF ALTERNATIVE IMPLEMENTATIONS OF NO-ARBITRAGE TERM STRUCTURE MODELS OF THE HEATH-JARROW-MORTON CLASS

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Abstract

Models of the term structure of interest rates play a central role in the modern theory of pricing bonds and other interest rate claims. Term structure models based on the principle of no-arbitrage, especially those of the Heath-Jarrow-Morton (1992) class, have become very popular recently, both with academics and practitioners. Surprisingly however, although the implied volatility function plays a crucial role in these no-arbitrage term structure models, there is little systematic evidence to guide optimal model specification within this broad class.

We study the implied volatility in the Heath-Jarrow-Morton framework using Eurodollar futures options data. We estimate a daily time series of forward rates within the HJM framework such that, by construction, the predicted futures prices from our model exactly match the observed futures prices. Next, we estimate a daily time series of volatility parameters such that the sum of squared errors between futures options prices predicted by the model and observed futures options prices is minimized. We use the six different volatility specifications suggested by Amin and Morton (1994) within the HJM class of models to price interest rate claims. Since the volatilities are the only unobservables, we use these models to infer the volatilities from the market prices of Eurodollar futures options over the 1987-1998 periods. The minimized sum of squared errors in the option prices is used as the measure of accuracy of each specific model. Each model differs from the others in its ability to match the market option prices and the time required for the computation. We compare the performances of the six volatility specifications in the accuracy-versus-computation time tradeoff. We document the

systematic biases between the model and market prices as a function of option type, maturity, and moneyness.

We also examine alternative numerical implementations of HJM models using the six volatility specifications. In particular, we analyze the impact on accuracy and computation time of using different numbers of time-steps. We also examine the effect of using time-steps of varying lengths within the same estimation procedure, and of ordering the time-steps in different ways.

Dedication

To the Memory of My Mother

Sook Ja Lee 1930-1999

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Chapter 1. Introduction

1.1. Volatility Estimation in a Heath-Jarrow-Morton Framework

The Heath-Jarrow-Morton class of models for pricing interest rate claims is based on the no-arbitrage principle. These models require only the specification of the form of the volatility function of forward interest rates along with the initial term structure of interest rates as inputs. They can then be used to infer the interest rate volatilities from the market prices of a given set of interest rate claims. The inferred volatilities can then be used for pricing other interest rate claims.

While there is a large body of literature which studies the implied volatility in equity options prices and its ability to forecast future volatility, there have been fewer studies on the implied volatility in fixed income markets. Part of the reason is the fact that HJM models are harder to implement. In general, HJM models are path-dependent. This implies that the whole forward rate term structure cannot be represented as a function of a few state variables whose evolution is governed by a Markov diffusion process. Therefore, the computational load expands exponentially with the number of steps.

There are two alternative approaches for implementing HJM models. The first is to use Monte Carlo simulation to simulate the evolution of the whole forward rate curve. The simulated values of all cash flows under each evolution are then used to calculate the present values of these cash flows by discounting at the realized spot rate. Unfortunately, this approach is slow and is hard to apply to American claims.

The other approach is to use a tree structure. Since many fixed income claims have American, i.e., early exercise features, we can use the tree method to correctly price in the early exercise. When the process is non-Markov, the tree becomes bushy and grows exponentially in size. However, it can be achieved within a reasonable time given the computing power and numerical techniques available today. Using a simple volatility specification, Heath, Jarrow, Morton and Spindel (1992) show that beyond five steps the error is always within 0.5% when benchmarked to the 12 time step price. They show that we can compute the value of options using five time-step models in much less time than 12 time-step models by sacrificing a relatively small amount of accuracy.

This paper follows the approach suggested by Amin and Morton (1994) for studying the implied volatility in Eurodollar futures options. We embed six different interest rate models within the HJM (1992) framework to price interest rate claims. Since the volatilities are the only unobservables, we can use these models to infer the volatilities from the market prices of Eurodollar futures options.

First, we specify the form of the volatility function of the forward rates, which then determines how our forward rate model evolves in time. In step one, the initial forward rate term structure is obtained that exactly fits the futures prices. We perform this procedure, which is described in section 4.2, for each of six different volatility functions.

In step two, given the term structure of initial forward rates, we infer the volatility function from market option prices. To do that, we estimate parameter values of the volatility function such that the sum of squared errors between model option prices and observed futures options prices is minimized. We compare the scale of forecast errors in futures options prices for each volatility model. We also compare the computation time required and the additional complexities involved in the process for each model. The details are described in section 5.1, 5.2, and 5.3.

We find that average computation time for two-parameter models is almost twice as large as that for one-parameter models while average absolute errors for two-parameter models are, in general, smaller than those for one-parameter models. We find that among the two-parameter models, the exponential models perform better than linear absolute model and linear proportional model in terms of accuracy and computation time. In terms of time steps we take in the model, 8b (3-2-2-1) time-step model results in the smallest average absolute error among the 8 time-step models. We also find that the standard deviation of average errors increase as the time-step decreases. In particular, the 90^{th} percentile of average absolute error for the exponential model stays at the lowest level in most cases across the options of different descriptions. We also find that there exist certain patterns among the biases in option prices. The biases are influenced by the different volatility specifications rather than by the time-steps we use in our implementations.

One of the arguments against the HJM model is that it is too slow to do the calculations for pricing American-style interest rate claims. We examine whether any numerical implementations seem clearly preferable over the others in the accuracy versus computation time tradeoff and find some patterns in which some models dominate others. We try to draw inferences about the effects of using time-steps of varying lengths within the same estimation procedure, and of ordering the time-steps in different ways. By using three time-steps for a single period, we can detect the early exercise of the option better than using one or two time-steps for the same period. As with equity options, there are significant biases as a function of strike price and maturity for all the models. Using an average of 36 options each day, we categorize and analyze the options each day by their type, maturity, and moneyness and examine those biases.

The dissertation is organized as follows. We survey the literature in Chapter 2 with special emphasis on Amin and Morton (1994). In Chapter 3, we review the general HJM model, the evolution of the discrete forward rate term structure, the six alternative volatility functions, and explain how to infer the volatility in HJM models. In Chapter 4, we describe the data and the empirical procedures used to estimate the models. In Chapter 5, we report and analyze the empirical results. Chapter 6 concludes. Appendices A and B present a flowchart and a detailed numerical example, respectively, to explain the estimation procedure. All tables and figures are presented at the end.

1.2. Market For Interest Rate Derivatives

The market for interest rate derivatives is both large and diverse. This section explains how the products in this market are categorized, and presents some summary evidence on how large the exchange-traded derivatives market and the OTC-traded interest rate derivatives market are. As reported below, these markets have experienced explosive growth in recent years, which highlights the importance of efficient and accurate models for pricing intetest rate claims.

The products in this market can be categorized as follows:

- Interest rate futures
- Interest rate futures options
- Interest rate options (caps, floors, collars)
- Interest rate forwards (Forward Rate Agreements, or FRAs)
- Interest rate swaps (swaps)
- Swaptions (options on interest rate swaps)
- Forward swaps (forward contracts on swaps)
- Embedded bond options (callable, puttable)
- Mortgage-backed securities (passthroughs, CMOs, stripped MBS)
- Accrual swaps
- Spread options

These products are used to manage interest rate risk. Traders can use these products to hedge against adverse interest rate moves. While interest rate futures and interest rate futures options are exchange-traded securities, other securities are traded on the over-the-counter market mostly among large corporations, financial institutions, and governments. The most popular exchange-traded interest rate options are those on Treasury bond futures, Treasury note futures, and Eurodollar futures.

The main advantages of exchange-traded products come from the liquidity benefits of standardization and the institutional features put in place to ensure that all parties perform their duty, i.e., to make a payment or to deliver the asset as specified in the contract. The disadvantage of exchange-traded products is that there is only a small set of contracts from which to choose. Interest rate futures are useful but since the eighties, OTC-interest rate derivatives have become more popular since they are customizable.

The interest rate derivatives market is huge and rapidly growing. Table 1.1 shows the summary of exchange-trade derivative market data. The numbers in each year represent the number of contracts, in millions, transacted during the year. For example, the number of interest rate derivative contracts in 2000 reached 997.8 million, accounting for more than 60% of the total number of contracts in the exchange-traded derivative market.

Table 1.2 shows the summary of OTC-interest rate derivative market data. The numbers in each year represent the notional amount outstanding in trillions of US dollars. For example, the combined total of outstanding interest rate swaps, interest rate forwards and interest rate options stood at $64.668 trillion in notional principal at December 31, 2000.

The growing interest rate derivatives market underlines the importance of efficient and accurate implementations of the interest rate term structure models to value those products.

Chapter 2. Literature Survey

Term structure modeling can be broadly categorized into two different approaches: equilibrium models and no-arbitrage models. Equilibrium approaches derive the interest rate term structure from underlying economic assumptions. In particular, they start with assumptions about economic variables, derive a stochastic process fro the short-term rate in a risk-neutral world, and examine what the process implies for interest rate claims. Thus, in an equilibrium model, the initial term structure of interest rates is an output from the model. The disadvantage of the equilibrium models is that they do not automatically fit the current term structure. On the other hand, the no-arbitrage models use the current term structure as an input, and choose parameters to model the evolution of interest rates in the future. By construction therefore, no-arbitrage models are designed to be exactly consistent with the current term structure.

2.1. Equilibrium Models

Vasicek (1977) presents one of the earliest models of the term structure of interest rates. It is a one-factor equilibrium model based on the short rate[1]. Under certain assumptions, it is shown that the expected rate of return on any bond in excess of the spot rate is proportional to its standard deviation. This property is then used to derive a PDE for bond prices. The diffusion process proposed by Vasicek is a mean-reverting normal process (Ornstein-Uhlenbeck process). The risk-neutral process for short rate r is

$$dr = \alpha (\gamma - r)\, dt + \sigma\, dz \qquad (2\text{-}1)$$

[1] The short rate at time t is the rate that applies to an infinitesimally short period of time at time t. It is sometimes also referred to as the instantaneous short rate.

where α is a speed-of-adjustment parameter, γ is the equilibrium level of interest rates, σ is volatility parameter, and dz is a Wiener process. Vasicek's objective is to choose a simplified tractable version of stochastic differential equation which adequately fits the actual stochastic behavior of the short rate. The solution of (2-1) is a Markov process with continuous sample paths and Gaussian increments. Vasicek makes three assumptions:

i) The spot rate follows a continuous Markov process,

ii) The price P(t, T) of a discount bond depends only on the spot process over the term of the bond, i.e., P(t, T) = P(t, T, r (t))[2], and

iii) The market is efficient.

Then, the basic equation for pricing of discount bonds in a market characterized by above assumptions is derived. The price of a discount bond solves the following PDE:

$$\partial P/\partial t + (f + \sigma q)\, \partial P/\partial r + (1/2)\sigma^2\, \partial^2 P/\partial r^2 - rP = 0 \qquad (2\text{-}2)$$

where P(t, T) is the price at time t of a discount bond maturing at time T, with unit maturity value P(T, T)=1, f is the instantaneous drift of the process r, and $q = (\mu - r) / \sigma$ is usually called the market price of risk. Once r and q are specified, the bond prices are obtained by solving (2-2) subject to the boundary condition P(T, T) = 1. Solutions of PDE, such as (2-2), can be represented in an integral form in terms of an underlying stochastic process. In the special case when the expected instantaneous rates of return on bonds of all maturities are the same, the bond price is given by

$$P(t, T) = E_t \exp\left(- \int_t^T r(\tau)\, d\tau\right)$$

One problem with Vasicek's model is that it allows for negative values of interest rates. It has other disadvantages as well:

i) it does not fit the initial term structure exactly,

ii) the single factor cannot capture complex term structure shifts, and

[2] The value of the spot rate is the only state variable for the whole term structure. Since there exists only one state variable, the instantaneous returns on bonds of different maturities are perfectly correlated. This means that the short bond and just one other bond completely spans the whole of the term structure.

iii) all spot and forward rates have the same volatility.

The Cox-Ingersoll-Ross (1985) model leads to the following form, known as the square-root process:

$$dr = \alpha (\gamma - r)\, dt + \sigma\sqrt{r}\, dz \tag{2-3}$$

where r is the short rate. It is a modification of the mean-reverting diffusion of Vasicek (1977). It differs from Vasicek in that it takes only positive values of interest rate due to the presence of the square root in the diffusion coefficient.

Brennan and Schwartz (1979) developed a two-factor term structure model with the short rate and the long rate as the two factors. They posit that the yield-curve behavior can be described in terms of the dynamics of two possibly unobservable state variables, u1 and u2. In particular, if the short and the long rates can be expressed as twice differentiable functions of the state variables, then they can be taken as their observable proxies. They show that the model can then be interpreted as being driven by level and slope of the yield curve. The drift and volatilities of the proxy variables, r and L, are taken to refer to the real world. The expected return from a security will, in general, be given by the sum of the riskless rate plus the risk premium connected with the variability in both the short and the long rate. Two market prices of risk in the BS model seems problematic but the BS model can offer the dynamic behavior of a two-factor model with the estimation complexity of a one-factor model once one of the market prices of risk can be made to disappear. Thus, if carefully implemented and parameterized, it can offer a valid alternative for the pricing of complex interest rate options.

Longstaff and Schwartz (1992) developed another two-factor model of interest rates. The two-factors, the short rate and the volatility of the short rate, allow interest rate derivative prices to reflect the current level of the short rate and its volatility. This view is derived from Dybvig (1990) who argued that the level and the slope might well explain 95% of the variability across rates of different maturity, but it might very well be that an additive second factor has a negligible effect on option values. If the second factor were instead taken to be the variance of the first factor, it could have small effect on bond prices, but perhaps a significant effect on bond option pricing. This would imply that a joint dynamics of the two factors of the form:

$$dr = \alpha_r \, dt + V \, dz_1 \tag{2-4}$$

$$dV = \alpha V \, dt + \sigma_v \, dz_2 \tag{2-5}$$

could efficiently account for observed option pricing. This is the view implicitly taken by Longstaff and Schwartz in the implementation of their two-factor model. Its disadvantage is that the joint process of two-factors was chosen for its analytical tractability rather than empirical realism.

2.2. No-Arbitrage Models

The no-arbitrage approach is an exact-fit approach that takes the observed market yield curve as given and specifies the stochastic evolution of the term structure so that there are no arbitrage possibilities.

Ho and Lee (1986) is a one-factor no-arbitrage short-rate model. It was the first no-arbitrage model of the term structure of interest rates. The continuous time limit of the model is:

$$dr = \theta(t) \, dt + \sigma \, dz \tag{2-6}$$

where r is the short rate, σ is a constant, and $\theta(t)$ is a function of time chosen to ensure that the model fits the initial term structure.

$$\theta(t) = F_t \, (0, t) + \sigma^2 \, t$$

where F_t (0, t) is the instantaneous forward rate at t=0.

Discount bonds and European options on bonds can be valued analytically.

$$P(t, T) = A(t, T)\, e^{-r(T-t)}$$

where

$$\ln A(t,T) = \ln \left[\frac{P(0, T)}{P(0, t)} \right] - (T-t) \left[\frac{\partial \ln P(0, t)}{\partial t} \right] - (1/2)\, \sigma^2\, t\, (T-t)^2$$

It has the advantages in that it is an analytically tractable Markov model and provides an exact fit to the current term structure of interest rates. However, it also has disadvantages in that all spot and forward rates have the same volatility and it contains no mean reversion.

Heath-Jarrow-Morton (1992) constructs a family of continuous-time models of the term structure consistent with the initial term structure data. However, the interest rate models that result from the HJM approach are usually non-Markov, so the distribution of interest rate in the next period depends not only on the current rate but also on the rates in the earlier periods. Moreover, it is difficult to obtain closed form solutions for the values of bond and interest rate derivatives.

Hull and White (1990, 1993) introduced a class of models that both incorporate deterministically mean-reverting features and allow perfect matching of an arbitrary yield curve. They extend the equilibrium models by letting parameters be time-varying. These generalized term structure models have more flexibility to fit a given yield curve and the term structure of volatility. However, except for the extended Vasicek model, their approach provides no closed form solution and has to rely on numerical methods. The extended Vasicek model has its own shortcomings. One stems from the possibility for interest rates to attain negative values. The other problem can arise if one attempts exact calibration of a normal one-factor model to cap[3] prices. The most general HW model will be written as

$$dr = [\, \theta(t) + \alpha\,(\gamma - r)\,]\, dt + \sigma\, r^{\beta}\, dz \qquad (2\text{-}7)$$

The Black-Derman-Toy (1990) model is a one-factor model algorithmically constructed in such a way as to price exactly any set of market discount bonds without requiring the explicit specification of investors' risk preferences[4]. As a result, swap rates, which can be expressed as linear combinations of discount bonds, can be priced exactly for any volatility input. While these features are shared by the Ho and Lee (1986) model, the BDT approach further assumes a lognormal process for the short rate. Besides preventing negative rates, this assumption allows the volatility input to be specified as a percentage volatility, thereby following market conventions and making model calibration to cap prices much easier. They eliminate some of the problems of Ho and Lee (1986), but create a new one: for a certain specification of the volatility function, the short rate can be mean-fleeting rather than mean-reverting.

To summarize, we explain the advantages and disadvantages of each approach in pricing interest rate derivatives. The advantage of the equilibrium models is that all interest rate derivatives are valued on a common basis. However, the equilibrium models have some disadvantages. First, they do not correctly price actual bonds and derivatives. Second, they have not incorporated sufficient empirical realism, i.e., model term structure does not fit the initial term structure and because of these reasons, they may admit arbitrage. On the other hand, the no-arbitrage models have an advantage that model term structure can fit the initial term structure. The disadvantages of the no-arbitrage models are as

[3] Caps are option derivatives that are designed to place a limit on a payout. Interest rate caps provide insurance against the rate of interest on a floating rate loan rising above a certain level.

[4] In equilibrium, there should be no possibility of forming a portfolio of bonds with a return that is free of risk. The ratio of excess expected return on any bond relative to its standard deviation must be the same regardless of the maturity of the bond. Equilibrium models, in general, begin by making assumptions regarding investors' beliefs and tastes. Then, the required risk premium for bonds at each maturity are obtained.

follows: First, there is no guarantee that the estimated function for the term structure of interest rate will be consistent with the previously estimated function. Second, it is difficult to obtain closed-form solutions for the value of bond and interest rate derivatives.

2.3 Empirical Studies

Amin and Morton (1994) is the first systematic empirical study which implements and tests a broad class of path-dependent HJM models. They test various volatility specifications of HJM class models using Eurodollar futures and options data from 1987-1992. They use the futures prices to determine the term structure of forward rates and fit the volatility parameters to all traded options. They compare different volatility models based on the stability of parameter values, fit between model and market prices, and the ability of the model to earn profits when it trades on perceived mispricing. They focus on Eurodollar futures options, which have the highest trading volume among interest rate options where arbitrage should ensure the informational efficiency. Besides, the Eurodollar series allows them to infer a complete initial term structure of forward interest rates that is contemporaneous with option prices.

To price Eurodollar futures option, we need a daily term structure of forward rates and an arbitrary volatility function. Once the term structure is determined, we only need to estimate the parameters in the volatility function. Amin and Morton (1994) carry out this procedure in two stages. In the first stage, they determine the forward rate term structure using market futures prices and a volatility from the previous day. In the second stage, given the term structure, they estimate the volatility parameters using Eurodollar futures option prices.

Let $\theta = (\theta_1, \theta_2)$ denote a vector of two parameters of the volatility function. The parameters θ_1 and θ_2 depend on the form of volatility function used. For example, if the absolute volatility model is used, only one parameter is needed and θ_2 becomes zero. If the exponential volatility model is used, we need both parameters.

Let Φ_t = (f(t, T_1), …., f(t, T_4)) be a vector of four forward rates whose maturity dates are approximately three months apart. Of course, the dimension of this vector can be extended beyond one year. In stage one, forward rate term structure Φ_t is estimated such that the sum of squared errors between model and market futures prices is set to zero. Amin and Morton (1994) indicate that forward bond prices to obtain an approximation of the entire term structure cannot be readily computed from futures prices due to two factors. First, futures prices are not equal to forward prices when interest rates are stochastic. Second, forward price does not exist which corresponds to the Eurodollar futures prices. This is because the terminal futures price is based on a three-month yield and is not a linear function of the price of a traded bond. Thus, they estimate forward interest rates for each futures maturity date and the forward rates are linearly interpolated between four maturity dates to obtain the forward rates for other maturities—e.g. for each month. The forward rates up to the first futures maturity date are assumed to be flat.

In stage two, given the term structure, only a volatility function is needed to price and hedge options. If the models are accurate at all for pricing options, the volatility function must be accurate in the Eurodollar futures options market. Then the volatility function can be inferred from market option prices by parameterizing the function and then estimating parameter values such that the sum of squared errors between market options prices and model options prices is minimized. The details are explained in section 4.2.
Based on those estimates, they test for model fit and parameter stationarity and set up simple trading strategies designed to exploit mispriced options. They document systematic strike-price and time-to-maturity biases for all models. They find that the implied volatility series is stationary and mean reverting irrespective of the model used. They also find that the one-parameter models fit slightly less well than two-parameter models, but their implied parameter estimates are more stable over time and they are able

to earn significantly larger and more consistent abnormal profits from the mispricings they detect.

Using a subset of Amin and Morton (1994) data, Amin and Ng (1997) study the implied volatility from several volatility specifications of the HJM (1992) models, and compare the results with those of popular historical volatility models in the Eurodollar options market. In contrast to earlier studies on the implied volatility from equity options, they find that the implied volatility from the HJM models explains much of the variation of realized interest rate volatility over both daily and monthly horizons. They also find that the implied volatility dominates the GARCH terms, the Glosten, Jagannathan, and Runkle (1993) type asymmetric volatility terms, and the interest rate level. Among the different volatility models, they find that the exponential and linear proportional volatility models perform better than the other implied volatility models.

The focus of Amin and Morton (1994) and Amin and Ng (1997) studies, however, was more theoretical than implementational in nature, i.e., they examined whether any of the different volatility specifications resulted in large and systematic pricing errors, and whether they gave rise to profitable trading strategies. Our study extends these studies by examining whether any volatility specification seems preferable over the others in the accuracy versus computation time tradeoff, and also by examining alternative numerical implementations of HJM models. In particular, we analyze the impact on accuracy and computation time of using different number of time steps. We also examine the effect of using time steps of varying lengths within the same estimation procedure. Moreover, our results are based on more comprehensive data – our sample period runs from 1987 to 1998, and we use an average of 36 options each day in testing model performance.[5]

Amin and Bodurtha (1995) develop an arbitrage-free discrete time model to price American-style claims. It converges weakly in the limit to the continuous time models in Amin and Jarrow (1991, 1992). Their model is an approach to valuing a broad range of

[5] Amin and Morton (1994) use data from 1987 to 1992, and use an average of 18 options per day.

instruments with exchange rate risk and interest rate risk along with two different term structures. It can be implemented in practice for up to five-year maturity options. Their general model is path-dependent and permits arbitrary volatility and covariance functions for both domestic and foreign interest rates and for the exchange rate. It may also be used to value regular derivatives or other path-dependent options with some modifications.

Amin (1991) develops a class of discrete, path-independent models as continuous time approximations to compute prices of American options within the Black-Scholes (1973) framework and analyze their convergence properties. This class includes models in which state variables have time-varying volatility functions and models with multiple state variables.

Heath, Jarrow, Morton and Spindel (1992) explains how to use the HJM models to price and hedge large interest rate derivative books and show that the HJM models for pricing interest rate options are not so slow for practical purposes. They measure running time on a 486 PC required to price a five-year cap and a 1 x 5 year European swaption under a path-dependent one-factor model. They show that running time roughly doubles for each additional time step due to the path-dependence. However, it is shown that beyond five steps the error is always within 0.5% when the price at each step is compared to the 12-step price.

Chapter 3. Alternative Volatility Specifications In HJM Models

3.1. The Heath-Jarrow-Morton Model

To model the uncertainty in interest rates, it is necessary to model the manner in which the term structure changes through time. A good term structure model would permit arbitrage-free pricing of bonds, derivatives on bonds, and interest rate derivatives. The HJM framework provides a general setting for arbitrage free models in which prices and hedge ratios are generated in tandem.

Let f(t, T) be the forward interest rate at date *t* for instantaneous and riskless borrowing or lending at date T . The spot interest rate at date t is given by r(t) = f(t, t). At each trading date t, HJM specify the simultaneous evolution of forward interest rates of every maturity T to follow the stochastic differential equation:

$$df(t, T) = a(t, T, f(t, T))\, dt + \sigma(t, T, f(t, T))\, dz(t). \tag{3-1}$$

where a(t, T, f(t, T)) and σ(t, T, f(t, T)) are the drift and dispersion coefficients for the forward interest rate of maturity T*,* and z(t) is a one-dimensional standard Brownian motion. Under complete markets, HJM (1992) show the existence of a "risk-neutral" pricing measure under which the price of every security discounted by the spot interest rate is a martingale. To illustrate, fix some final horizon date, S. Let P(t, T) be the price of a discount bond at date t that matures at date T and P(T, T) = 1. Define

$$Z(t, T) = P(t, T) \exp\{ - \textstyle\int_0^t r(u)\, du \} \quad \text{for } t \le T \le S. \tag{3-2}$$

For every discount bond of maturity T ≤ S, Z(t, T) is a martingale under the risk-neutral measure. Under this measure we can price every security as if investors are risk neutral. Therefore, the security's price equals the expectation of its value at any future date discounted back to today using the spot interest rate. Applying the martingale condition to discount bond prices yields the drift coefficient for each maturity, T :

$$a(t, T, f(t, T)) = \sigma(t, T, f(t, T)) \int_t^T \sigma(t, u, f(t, u))\, du \tag{3-3}$$

The function $\sigma(t, T, f(t, T))$ which is the instantaneous standard deviation of the forward interest rate of maturity T at date t, constitutes the parameter input(s) to the model. Specifying $\sigma(.)$ completely specifies the model since a(.) is then determined from Equation (3-3). If $\sigma(.)$ is a constant (as a function of both time and maturity), Equation (3-1) reduces to the Ho and Lee (1986) model. Furthermore, by appropriately specifying $\sigma(.)$ we can treat many spot rate models as special cases. For example, if the volatility of forward interest rates is an exponential function of time to maturity, the process for the spot interest rate is the same as assumed by Vasicek (1977). If $\sigma(.) = \sigma\, f(t, T)^{1/2}$ then the spot interest rate process is virtually identical to that in CIR(1985).

Although futures prices do not equal forward prices if interest rates are stochastic, futures contracts can be valued like any other contingent claim. The initial investment in a futures contract is zero. Therefore the expected change in the futures price under a risk-neutral measure must be zero.

Let $E_t[.]$ denote the expectation with respect to the risk-neutral measure conditional on the information set at date *t* . If the futures price at date *t* for maturity T is F(t, T), and F(T, T) is the terminal futures (and spot) price at maturity,

$$0 = E_t[\, F(T, T) - F(t, T)\,]$$

In other words,

$$F(t, T) = E_t[\, F(T, T)\,] \tag{3-4}$$

Consequently, the futures price for a continuously marked-to-market futures contract is a martingale under the risk-neutral measure. Given the terminal spot price distribution under the risk-neutral measure, we can use Equation (3-4) to determine any prior futures price. (See section section 4.2 for details) In practice, we can compute futures options prices using a path-dependent binomial-style model by discretizing Equations (3-1) and (3-3) under the risk-neutral measure.

A European option with a payoff C(T) at date T can be valued using the equation

$$C(t) = E_t \left[\exp \left\{ - \int_t^T r(u)\, du \right\} C(T) \right] \qquad (3\text{-}5)$$

Finally, American-style claims can be valued using a modified version of Equation (3-5), which accounts for early exercise.

3.2. The Evolution Of The Forward Rate Curve

The HJM model requires the specification of the volatility processes of all instantaneous forward rates at all future times. It then calculates the drift of each instantaneous forward rate and describes the evolution of the term structure of forward rates. As an approximation to the continuous time models, the discrete time model provides a useful computational tool. This is especially true when computing values for path-dependent interest rate derivatives like American Eurodollar futures options.

Figure 3.1 gives the one-factor forward rate curve evolution from time 0 to time T. The first column vector (with T+1 elements) represents the initial forward rate term structure at time zero. In the next time step, the forward rate term structure moves either up or down. At time 1, in either state, we have a new forward rate curve that consists of T forward rates (since f(0, 0) does not exist anymore). Similarly, at every step, the forward rate curve evolves up or down, up to the time T.

Jarrow (1996) provides a discrete time approximation to the single-factor version of HJM model in terms of its continuous-time limit[6]. Consider a one-factor economy in which the forward rate process can be expressed as follows:

$f_\Delta(t+\Delta, T; s_{t+\Delta}) = \alpha_\Delta(t, T; s_t)\, f_\Delta(t, T, s_t)$ if $s_{t+\Delta} = s_t\, u$ with probability $q_t^\Delta > 0$

$\beta_\Delta(t, T; s_t)\, f_\Delta(t, T; s_t)$ if $s_{t+\Delta} = s_t\, d$ with probability $1 - q_t^\Delta > 0$

where u denotes up state, d denotes down state, and $\tau\Delta - \Delta > T > t + \Delta$ (3-6)

Equation (3-6) can be reparameterized in terms of three stochastic processes μ, σ, and ϕ as follows:

$$\alpha_\Delta(t, T; s_t) = \exp\{\mu(t, T; s_t)\,\Delta - \sigma(t, T; s_t)\sqrt{\Delta}\}$$

$$\beta_\Delta(t, T; s_t) = \exp\{\mu(t, T; s_t)\,\Delta + \sigma(t, T; s_t)\sqrt{\Delta}\}$$

$$q_t^\Delta(s_t) = 0.5 + 0.5\,\phi(t; s_t)\sqrt{\Delta} \qquad (3\text{-}7)$$

By substituting these into (3-6), we get

$$\begin{aligned} f_\Delta(t+\Delta, T; s_{t+\Delta}) &= f_\Delta(t, T; s_t)\exp\{\mu(t, T; s_t)\,\Delta - \sigma(t, T; s_t)\sqrt{\Delta}\} \\ &\quad \text{with probability } 0.5 + 0.5\,\phi(t; s_t)\sqrt{\Delta} > 0 \text{ if } s_{t+\Delta} = s_t u \\ &= f_\Delta(t, T; s_t)\exp\{\mu(t, T; s_t)\,\Delta + \sigma(t, T; s_t)\sqrt{\Delta}\} \\ &\quad \text{with probability } 0.5 - 0.5\,\phi(t; s_t)\sqrt{\Delta} > 0 \text{ if } s_{t+\Delta} = s_t d \qquad (3\text{-}8) \end{aligned}$$

By taking natural logarithm of both sides of (3-8), we get

$$\begin{aligned} &\log f_\Delta(t+\Delta, T; s_{t+\Delta}) - \log f_\Delta(t, T; s_t) \\ &\quad = \{\mu(t, T; s_t)\,\Delta - \sigma(t, T; s_t)\sqrt{\Delta}\} \text{ with probability } 0.5 + 0.5\,\phi(t; s_t)\sqrt{\Delta} \\ &\quad = \{\mu(t, T; s_t)\,\Delta + \sigma(t, T; s_t)\sqrt{\Delta}\} \text{ with probability } 0.5 - 0.5\,\phi(t; s_t)\sqrt{\Delta} \qquad (3\text{-}9) \end{aligned}$$

The mean and variance of the above terms are

$$E_t(\log f_\Delta(t+\Delta, T; s_{t+\Delta}) - \log f_\Delta(t, T; s_t)) = [\mu(t, T) - \phi(t)\,\sigma(t, T)]\,\Delta$$
$$Var_t(\log f_\Delta(t+\Delta, T; s_{t+\Delta}) - \log f_\Delta(t, T; s_t)) = \sigma(t, T)^2\,\Delta - \phi(t)^2\,\sigma(t, T)^2\,\Delta^2 \qquad (3\text{-}10)$$

Dividing by Δ and taking limits of these moments as $\Delta \rightarrow 0$ give

$$\frac{\lim_{\Delta \to 0} E_t(\log f_\Delta(t+\Delta, T; s_{t+\Delta}) - \log f_\Delta(t, T; s_t))}{\Delta} = \mu(t, T) - \phi(t)\,\sigma(t, T)$$
$$\frac{\lim_{\Delta \to 0} Var_t(\log f_\Delta(t+\Delta, T; s_{t+\Delta}) - \log f_\Delta(t, T; s_t))}{\Delta} = \sigma(t, T)^2 \qquad (3\text{-}11)$$

This system is arbitrage free iff there exists an unique pseudo probability $\Pi_\Delta(t; s_t)$ [7]such that

$$\Pi_\Delta(t; s_t) = \frac{[r_\Delta(t; s_t) - d_\Delta(t, T; s_t)]}{[u_\Delta(t, T; s_t) - d_\Delta(t, T; s_t)]} \qquad (3\text{-}12)$$

We can relate the forward rate's rate of change parameters ($\alpha(.)$ and $\beta(.)$) to the zero-coupon bond price process's rate of return parameters u(.) and d(.) in the up and down states, respectively

$$u(t, T; s_t) = \frac{r(t; s_t)}{\Pi_{j=t+1}^{T-1}\,\alpha(t, j; s_t)}$$
$$d(t, T; s_t) = \frac{r(t; s_t)}{\Pi_{j=t+1}^{T-1}\,\beta(t, j; s_t)} \qquad (3\text{-}13)$$

Using (3-7) in (3-13) we get,

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[6] The analysis is based on Heath, Jarrow, and Morton (1991)

$$u_\Delta(t, T; s_t) = r_\Delta(t; s_t) \exp\{- \Sigma_{j=t+\Delta}^{T-\Delta} \mu(.) \Delta + \Sigma_{j=t+\Delta}^{T-\Delta} \sigma(.) \sqrt{\Delta}\}$$

$$d_\Delta(t, T; s_t) = r_\Delta(t; s_t) \exp\{- \Sigma_{j=t+\Delta}^{T-\Delta} \mu(.) \Delta - \Sigma_{j=t+\Delta}^{T-\Delta} \sigma(.) \sqrt{\Delta}\} \qquad (3\text{-}14)$$

By substituting these into (3-12) we get

$$\Pi_\Delta(t; s_t) = \frac{[\, 1 - \exp\{-\Sigma_{j=t+\Delta}^{T-\Delta} \mu(.) \Delta - \Sigma_{j=t+\Delta}^{T-\Delta} \sigma(.) \sqrt{\Delta}\} \,]}{[\exp\{-\Sigma_{j=t+\Delta}^{T-\Delta}\mu(.)\Delta - \Sigma_{j=t+\Delta}^{T-\Delta}\sigma(.)\sqrt{\Delta}\} - \exp\{-\Sigma_{j=t+\Delta}^{T-\Delta}\mu(.)\Delta - \Sigma_{j=t+\Delta}^{T-\Delta}\sigma(.)\sqrt{\Delta}\}]} \qquad (3\text{-}15)$$

From the relationship between the empirical and pseudo economies[8], we get the added restriction that

$$\Pi_\Delta(t; s_t) = 0.5 + 0\,(\sqrt{\Delta}) \qquad (3\text{-}16)$$

Equation (3-15) and (3-16) give the no-arbitrage restrictions such that the construction of approximating economies holds.

For convenience, we set the pseudo probabilities to $\Pi = 0.5$. Then, we get from (3-15) that

$$\begin{aligned} \exp\{\Sigma_{j=t+\Delta}^{T-\Delta} \mu(.) \Delta\} &= 0.5\, [\exp\{\Sigma_{j=t+\Delta}^{T-\Delta} \sigma(.) \sqrt{\Delta}\} + \exp\{-\Sigma_{j=t+\Delta}^{T-\Delta} \sigma(.) \sqrt{\Delta}\}] \\ &= \cosh(\Sigma_{j=t+\Delta}^{T-\Delta} \sigma(.) \sqrt{\Delta}) \qquad (3\text{-}17) \end{aligned}$$

Equation (3-17) can be rewritten as

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[5] A unique, strictly positive number less than one. It can be interpreted as a probability of the up outcomes occurring at time t.

[8] The evolution of observed zero-coupon bond prices and forward rates are generated by a continous empirical economy with parameters i) $\mu^*(t, T)$, the expected change in the forward rates per unit time, and ii) $\sigma(t, T)$, the standard deviation of changes in the forward rates per unit time. The assumption of no arbitrage gives the existence of unique pseudo probabilities π (t;st). Then, the continuous pseudo economy has parameters parameters i) $\mu(t, T)$, the expected change in the forward rates per unit time, and ii) $\sigma(t, T)$, the standard deviation of changes in the forward rates per unit time. Note that the expected changes of forward rates are not identical across the two economies. The details are explained in Jarrow (1996).

$$\exp\{\mu(t, t+\Delta; s_{t.})\, \Delta\} = \cosh(\sigma(.)\, \sqrt{\Delta})$$

$$\exp\{\mu(t, T; s_t)\, \Delta\} = \frac{\cosh(\Sigma_{j=t+\Delta}^{T}\, \sigma(.)\, \sqrt{\Delta})}{\cosh(\Sigma_{j=t+\Delta}^{T-\Delta}\, \sigma(.)\, \sqrt{\Delta})} \tag{3-18}$$

By substituting these into (3-8) we get

$$\begin{aligned} f_\Delta(t+\Delta, T; s_{t+\Delta}) &= f_\Delta(t, T; s_t)\ \frac{\cosh(\Sigma_{j=t+\Delta}^{T}\, \sigma(.)\, \sqrt{\Delta})}{\cosh(\Sigma_{j=t+\Delta}^{T-\Delta}\, \sigma(.)\, \sqrt{\Delta})}\ \exp(-\sigma(.)\sqrt{\Delta}) \quad \text{if } s_t+\Delta = s_t u \\ &= f_\Delta(t, T; s_t)\ )\ \frac{\cosh(\Sigma_{j=t+\Delta}^{T}\, \sigma(.)\, \sqrt{\Delta})}{\cosh(\Sigma_{j=t+\Delta}^{T-\Delta}\, \sigma(.)\, \sqrt{\Delta})}\ \exp(\sigma(.)\sqrt{\Delta}) \quad \text{if } s_t+\Delta = s_t d \end{aligned} \tag{3-19}$$

where we define

$$\cosh(\Sigma_{j=t+\Delta}^{t}\, \sigma(t, j: s_t)\, \sqrt{\Delta}\ ) \equiv 1$$

Equation (3-19) describes the evolution of the discrete-time forward rate curve. Under the pseudo probabilities, a vector of the volatility structure of forward rates

$$\begin{pmatrix} \sigma(t, t+\Delta: s_t) \\ \sigma(t, t+2\Delta: s_t) \\ \dots \\ \sigma(t, \tau\Delta-\Delta: s_t) \end{pmatrix}$$

is sufficient to determine the evolution of the forward rate term structure.

Next we explain how the no-arbitrage condition leads to a connection between the drift parameters and the volatility parameters in the HJM model. We may start with the observed yield curve, as described either by the collection of discount bonds given at time 0, P(0, T), or by the instantaneous forward rates, f(0, T), linked by

$$P(0, T) = \exp[\ - \textstyle\int^{T}_{0} f(0, s)\, ds\ ] \tag{3-20}$$

$$f(0, T) = -\frac{\partial \ln P(0, T)}{\partial T} \qquad (3\text{-}21)$$

It recovers by construction any given market yield curve. If one uses as numeraire the money market account, all assets instantaneously grow at the riskless rate. This must be true for discount bonds, P(t, T), for which one can therefore write:

$$dP(t, T) = r(t)\, P(t, T)\, dt + u(t, T, P(t, T))\, dz. \qquad (3\text{-}22)$$

where u(t, T) is the volatility of a bond of maturity T at time t, P(t, T) its price at time t, and r(t) the short rate at time t. In Equation (3-22), the drift component is specified by the no-arbitrage condition, but the maximum generality has been allowed for the price volatility of the discount bond, which can depend on calendar time, on maturity time, and on the discount bond price at time t.

Since the application of the HJM approach to a specific model is equivalent to the specification of the functional form for the volatility function, we have to distinguish what is implied by the HJM approach in general from what is a specific consequence of a particular implementation. Equation (3-22) is therefore taken as the most general starting point.

Let us define v(t, T, P) = u(t, T, P) / P(t, T). By applying Ito's lemma, we can write

$$d \ln P(t, T) = [r(t) - v(t, T, P)^2 / 2\,]\, dt + v(t, T)\, dz \qquad (3\text{-}23)$$

Therefore

$$d[\ln P(t,T_2) - \ln P(t,T_1)] = \tfrac{1}{2}[v(t,T_1,P)^2 - v(t,T_2,P)^2]dt + [v(t,T_2,P) - v(t,T_1,P)]dt \qquad (3\text{-}24)$$

The continuously compounded time-t forward rate is given by:

$$f(t, T_1, T_2) = -\frac{\ln [P(t, T_2) - \ln P(t, T_1)]}{(T_2 - T_1)} \tag{3-25}$$

From Equation (3-24) and Equation (3-25) we get

$$d[f(t, T_1, T_2)] = \frac{[\, v(t, T_2, P(t, T_2))^2 - v(t, T_1, P(t,T_1))^2]}{2(T_2 - T_1)} dt + \frac{[\, v(t, T_1, P(t, T_1)) - v(t, T_2, P(t, T_2))\,]}{(T_2 - T_1)} dz \tag{3-26}$$

Equation (3-26) shows that the risk-neutral process for f depends on the v(.)'s. It depends on r and the P(.)'s only to the extent that the v(.)'s themselves depend on these variables. As the limit approaches from T_2 to T_1 the discrete forward rate tends to the instantaneous forward rate f (t, T):

$$f(t, T) = -\frac{\partial \ \ln P(t, T)}{\partial T} \tag{3-27}$$

Then it follows that

$$df(t, T) = v(t, T, P)\, (\partial\, v / \partial T)\, dt + (\partial\, v / \partial T)\, dz \tag{3-28}$$

Once the function v(.) has been specified, the risk-neutral processes for the f(t, T)'s are known. Equation (3-28) therefore establishes the link that must exist in a risk-neutral world between the volatility of discount bonds and the drifts of forward rates. Integrating v(t, T, P) between t and T, we get

$$v(t, T, P) - v(t, t, P) = \int_t^T (\partial\, v(t, \tau, P) / \partial \tau)\, d\tau$$

We know that $v(t, t, P) = 0$ and if $a(t, T, f(t, T))$ and $\sigma(t, T, f(t, T))$ are the instantaneous drift and standard deviation of $f(t, T)$ so that

$$df(t, T) = a(t, T, f(t, T))\, dt + \sigma(t, T, f(t, T))\, dz$$

it follows from equation (3-28) that

$$a(t, T, f(t, T)) = \sigma(t, T, f(t, T)) \int_t^T \sigma(t, u, f(t, u))\, du$$

as stated in equation (3-3).

3.3. The Alternative Volatility Specifications

In HJM models, the volatility function determines the stochastic evolution of the entire term structure curve. We focus on models possessing the time invariance property that $\sigma(.)$ depends on t and T only through T-t. Still, a broad class of volatility structures is available. Since we can use arbitrary volatility functions, we choose the following six forms of volatility functions, first suggested by Amin and Morton (1994), of which the first three functions contain one parameter, and the others contain two parameters each. Some of these volatility specifications lead to models developed earlier. For example, the absolute volatility model yields the Ho-Lee (1986) model and the square root volatility model yields the CIR (1985) model.

Absolute: $\sigma(.) = \sigma_0$

Square Root: $\sigma(.) = \sigma_0\, f(t, T)^{1/2}$

Proportional: $\sigma(.) = \sigma_0\, f(t, T)$

Linear Absolute: $\sigma(.) = \sigma_0 + \sigma_1 \cdot (T - t)$

Exponential: $\sigma(.) = \sigma_0 \exp(-\lambda \cdot (T - t))$

Linear Proportional: $\sigma(.) = (\sigma_0 + \sigma_1 \cdot (T - t))\, f(t, T)$

These volatility functions are subsumed by the general function form:

$$\sigma(.) = (\sigma_0 + \sigma_1(T - t)) \exp(-\lambda \cdot (T - t)) f(t, T)^{\gamma}$$

Each of our six specific functional forms for the volatility function corresponds to particular choices of the parameters λ and γ.

First, let us look at the functional form:

$$\sigma(.) = \sigma_0 f(t, T)^{\gamma}$$

The structure described by this form permits the volatility to depend on the level of the forward rate. If $\gamma = 0$, the above equation states that the volatility is constant, as in the Absolute model, and the drift term determined by (3-3) is equal to $\sigma_0^2 (T - t)$. Interest rate claims can then be priced under the local expectations hypothesis, using the arbitrage-free process for forward rates given by

$$d f(t, T) = \sigma_0^2 (T - t) dt + \sigma dz(t)$$

If $\gamma = 0.5$, the volatility is given by the Square root model. If $\gamma = 1$, the volatility is proportional to the level of the forward interest rate, decreasing to zero when interest rates tend to zero, as in the Square root model. While we cannot avoid negative interest rates in Absolute model, we can avoid negative rates in Square root and Proportional models.

The Linear absolute model and Exponential model permit the volatility to depend on the forward rate maturity. The Exponential model has the exponentially dampened volatility structure. It exploits the fact that near-term forward rates are more volatile than distant forward rates. However, as in the Absolute volatility structure, the exact magnitude of volatility is independent of the level of the forward rate. The structure of the Linear proportional model permits the volatility to vary according to the level of forward interest rate and forward rate maturity.

Since the Absolute model assumes constant volatility, which is not always the case in the real world, and we cannot avoid negative interest rate in the Absolute model, we may expect that the Square Root model and the Proportional model perform better than the Absolute model in terms of minimizing option pricing errors. On the other hand, since the Exponential model exploits the fact that near-term forward rates are more volatile than distant forward rates with exponentially dampened volatility structure, as we observe frequently in the real world, we may expect that the Exponential model performs better than the Linear Absolute model, which has a linear relationship between volatility and the term to maturity.

3.4. The Implied HJM Volatility

We compute the implied volatility of the forward rate as follows. First, we compute implied parameter values from each model as explained in section 4.2. Second, using the functional form of each volatility specification, we compute the daily implied volatility of the forward rate, $\sigma(t, T, f(t, T))$ for each model. For example, for the square root, or CIR, specification, the implied volatility of the forward rate is

$$\sigma(t, T, f(t, T)) = \sigma_0 f(t, T)^{1/2}$$

where we substitute the implied parameter estimate for σ_0 in the above equation.

Chapter 4. Data and Empirical Procedures for Estimation

4.1. Data

We use 3-month Eurodollar futures prices and futures options prices with maximum maturity of one year to estimate the implied volatility of the forward rates. For our analysis, we use daily data taken from the Futures Industry Institute database containing settlement prices of Eurodollar futures and Eurodollar futures options for January months from January 1, 1987 to December 31, 1998. We have four futures, F(t, T), T = 3, 6, 9, and 12, everyday since we eliminate all futures with maturity of over one year, F(t, T), T > 12. On average, we have 36 options everyday. We eliminate all options, C(t, T, X) or P(t, T, X), for which the maturity is over one year, T > 12. We also check the trading volume of options contracts traded per day per contract and if it is less than fifty, we eliminate that particular option price data from our sample.

While the Eurodollar futures contract is traded on two exchanges – the Chicago Mercantile Exchange (CME) and the London International Financial Futures Exchange (LIFFE), we use the daily data for the Eurodollar futures contract traded on the Chicago Mercantile Exchange, as reported in the FII database. The Eurodollar interest rate is the rate of interest earned on 3-month Eurodollar deposits made by one bank with another bank and is also known as the 3-month London Interbank Offer Rate (LIBOR).

Eurodollar futures trade with maturities up to 10 years. Eurodollar futures contracts expire in March, June, September, and December. Contracts expire on the second business day before the third Wednesday of the month of expiration.

Options on Eurodollar futures contracts also trade on the Chicago Mercantile Exchange, and are included in the Futures Industry Institute database. The underlying asset for these options is the 3-month Eurodollar futures contract described above. Upon exercise, the cash flow to the holder of a call option[9] equals the difference between the current futures price and the exercise price. The owner of a put option receives the difference between the exercise price and the current futures price. The options are American, and have the same maturity dates as the underlying futures contracts. Most traded options have a maturity of less than 1 year. We eliminate options data if the maturity of the option is over one year, $T > 12$. We also check the number of contracts traded per day per contract and if it is less than fifty, we exclude that particular option price data from our sample.

4.2. Empirical Procedure For Estimation

The volatility estimation procedure consists of two-step estimation. In step one, we choose forward rates to exactly match current futures prices. In step two, we choose volatility parameters to minimize option pricing errors. Appendix A presents a flow diagram and Appendix B presents a numerical example illustrating the 2-step estimation process.

Since we are studying continuous compounding as the limit of the discretely compounded rates, the continuously compounded forward rate, $f^*(t, T)$, as described by equation (3-1), is that rate such that for small time intervals Δ, the following condition holds:

$$f_\Delta(t, T) \approx \exp(f^*(t, T)\,\Delta)$$

$f_\Delta(t, T)$ is the forward rate over $(T, T+\Delta)$ at time t. This forward rate is a one plus a percentage, while $f^*(t, T)$ is a percentage, expressed as a number between 0 and 1. This equation is only an approximation. The formal version of the equation is as follows:

[9] An option on a futures contract allows the holder to enter into a position with a futures price equal to the exercise price of the option. Upon exercise, a futures call option holder acquires a long futures position with a futures price equal to the exercise price. At the close of the day's trading the futures contract is marked to the market. At this time, the option holder is free to withdraw in cash the difference between the futures price and the exercise price.

$$f_{\Delta}(t, T) = \exp\left(\int_T^{T+\Delta} f^*(t, s)\, ds\right)$$

Now we explain how we choose time zero forward rates, $f(0, T)$, $T = 3, 6, 9$, and 12, to exactly match current futures prices, $F(0, T)$, $T = 3, 6, 9$, and 12. For each day, the data sample consists of four futures prices, $F(t, T)$, term to next maturity, $\Delta_{T2\text{-}T1}$, and a certain volatility parameter, σ_0 for the one-parameter models and σ_0 and σ_1 for the two-parameter models. We use the estimate in Amin and Morton (1994) as our initial value for volatility parameter and use the previous day's volatility parameter estimated from our model for the rest of the days. This is done by matching the market futures prices with the futures prices derived from the forward rate tree that was constructed with the initial volatility parameter estimate and the time zero forward rate term structure. Since the forward rates-to-futures rates mapping is model dependent, the estimated daily term structure is different for each model. (Refer to Appendices A & B)

We know that forward rate is not the same as the futures rate, but the starting value for the forward rate, $f(0, T)$, $T = 3, 6, 9$, and 12, is the futures rate that was computed from the current futures prices, $F(0, T)$, $T = 3, 6, 9$, and 12. Then, we construct the whole forward rate tree, $f(t, T; s_t)$, where t denotes the time steps we will be using in our analysis and s_t denotes the state being either u, up-state, or d, down-state, with certain volatility parameters, σ_0 and σ_1, using the Equation (3-19). (Refer to Figure 4.1)

Next, from the forward rate tree, $f(t, T; s_t)$, we compute the futures prices at time T. Figure 4.1 gives the one-factor futures price curve evolution from time 0 to time T. The futures prices at time T are computed from the forward rate tree, $f(t, T; s_t)$, since at time T, the forward rate, $f(T, T)$, equals the futures rate, $\mathit{f}(T, T)$. Since the futures price, $F(t, T)$, is a martingale, the futures price at time T-1, $F(T-1, T)$, equals the time T-1 expectation of futures price at time T, $E^Q_{T-1}\{F(T, T)\}$. Similarly, the time zero futures price, $F(0, T)$, becomes the time-0 expectation of the futures price at time T, $E^Q_0\{F(T, T)\}$.

We explain how the futures prices, $F(t, T)$, are actually computed from the rate quoted in the financial market. At maturity T, the futures settlement price $F(T, T)$ is

$$F(T, T) = 100 [1 - y(T)]$$

where $y(T)$ is the 3-month LIBOR interest rate at date T. $y(T)$ is the futures rate that applies to the deposit for the period from T to $T+ \Delta$, where Δ denotes 3 month in the Eurodollar futures contract. Our actual futures price data are computed by 1,000,000 x (1 – y(T) / 4) as the notional amount for each futures contract is \$1 million. Since each contract covers a 3-month rate (0.25 years), each basis point change in the interest rate corresponds to a \$25 (= 0.0001 x 0.25 x 1,000,000) change in the futures price.

We compute model futures price at time zero, $F(0, T)$, from the expectation of futures prices at time T, $E^{Q}_{0}[F(T, T)]$, which is computed from the forward rate curve as follows:

$$\begin{aligned} F(0, T) &= E^{Q}_{0} [F(T, T)] \\ &= E^{Q}_{0}[10^{6} (1 - f (T, T) \times 0.25)] \end{aligned}$$

If model futures prices are greater than observed futures prices, we reset $f(0, T)$ to a higher value. Again, we construct the forward rate curve, using the equation (3-19), from the new $f(0, T)$ value and compute the model futures price at time zero from the expectation of futures prices at time T. If model futures prices are less than observed futures prices, we reset $f(0, T)$ to a lower value. We then repeat the optimization process until model futures prices equals to the observed futures prices.

We use quadratic programming[10] as an algorithm for updating $f(0, T)$. It is faster than Levenberg-Marquardt suggested by Amin and Morton (1994). We then obtain $f(0, T)$ recursively for T = 3 months, 6 months, 9 months, and 12 months.

Next, we use linear interpolation to get the intermediate maturity forward rates, f(0, T), at monthly intervals. If we are to use 12 time-step model, four observed futures prices, F(0, T), T = 3, 6, 9, and 12, each day are insufficient to determine the number of forward rates desired. There are missing futures price observations, F(0, T), T = 1, 2, 4, 5, 7, 8, 10, and 11. We estimate forward interest rates for each futures maturity date and linearly interpolate between these dates to obtain the forward interest rates for other maturities.

$$f(0, T+j) = f(0, T) + \{f(0, T+\Delta) - f(0, T)\}/(j+1) \qquad T = 1, 2, \ldots, 12, \text{ and } j < \Delta$$

This approximates the forward rate curve with a step function. We assume that the instantaneous forward interest rate curve is constant to the end of the accrual period of the first futures contract. For example, if we are to use 12 time-step discrete time model for the evolution of forward interest rates, we linearly interpolate between 4 maturity dates to obtain 12 forward interest rates for each of 12 maturities with approximately one-month intervals. Thus, we assume that the forward interest rate curve is constant within each time step for our purpose of forward interest rate tree construction.

In step 2, after estimating the forward rates, f(0, T), T = 1, 2, … , 12, we use the interest rate model to estimate the volatility parameters, σ_0 and σ_1. Specifically, we minimize the sum of the squares of the errors between model and market prices of Eurodollar futures options, C(0, T, X) – C*(0, T, X) in case of call options. Model option prices are computed on a path-dependent tree with up to 12 time steps.

Figure 4.2 gives the one-factor futures options price evolution from time 0 to time T, C(t, T, X), $t \leq T \leq 12$. Given the futures prices, F(T, T), we can compute the call option prices at time T, C(T, T, X), which is the maximum of the futures price at time T, F(T, T), minus the exercise price, X, and zero. Since the option is American-style, the call option price at time T-1, C(T-1, T, X), is the maximum of option value when option is exercised and the option value when the option is not exercised. Similarly, the call option price at

[10] Both quadratic programming and Levenberg-Marquardt are in-built algorithms for numerically solving

time zero, C(0, T, X), is the expectation of time 1 option value, E^Q_0(C(1, T, X), discounted by the interest rate over the time period 0 to 1, f(0, 0). We calculate option prices for an average of 36 options every day, and compare the calculated prices to the observed prices. The quadratic programming algorithm is used again to iteratively identify that value of the volatility parameter(s) that minimize the sum of squared errors between predicted and observed option prices. Repeating this process for each model, for each day in the sample period, yields a daily time series of implied parameter estimates and implied volatilities.

Next, we examine alternative numerical implementations of HJM models using the six volatility specifications. In particular, we analyze the impact on accuracy and computation time of using different numbers of time steps. Since Eurodollar futures and options contracts mature approximately every three months, we have four maturity dates in a year that are approximately three months apart. Let us suppose we use 12 time steps for the 1-year maturity. Then we have three time-steps for the first 3-month period, another three time steps for the next 3-month period, and so on. We call this the 12 time-step model. In an 8 time-step model, we use two time steps instead of three time steps for each period. Similarly, in 4 time-step models, we use one time step for each 3-month period. The more time steps we use, the computation time gets longer as our bushy tree grows fast. On the other hand, it allows us to compute more accurate value of options.

We estimate the volatility parameters by minimizing the sum of the squares of the errors between model and market prices of Eurodollar futures options. Therefore, by using 12 time steps, we can detect the early exercise of the option better than using fewer time steps for the same period. However, due to the path-dependence, running time roughly doubles for each additional time step.

We perform the analysis for each group of call, put, in-the-money, at-the-money, out-of-the-money, 3-month, 6-month, 9-month, and 12-month options.

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nonlinear equations in Matlab.

The schedule is as follows in terms of time-steps for each model:

	First period	Second period	Third period	Fourth period
12 time steps	3	3	3	3
8 time steps	2	2	2	2
4 time steps	1	1	1	1

We also examine the effect of using time steps of varying lengths within the same estimation procedure, and of ordering the time steps in different ways. The time required for the calculation increases almost proportionately to each additional time step due to the path-dependence. Considering the accuracy and computation time trade-off, we choose 8 time-step models and use time-steps of varying lengths.

	First period	Second period	Third period	Fourth period
8 time steps	2	2	2	2
8a time steps	3	3	1	1
8b time steps	3	2	2	1

Chapter 5. Empirical Results

In this section, we examine alternative numerical implementations of HJM models using the six volatility specifications. In particular, we analyze the impact on accuracy and computation time using different numbers of time steps. We also examine the effect of using time steps of varying lengths within the same estimation procedure, and of ordering the time steps in different ways.

5.1. Estimated Implied Volatility Parameters

We compute a daily time series of the implied volatility parameters for the six different models for the January of each year from 1987 to 1998. Table 5.1 and Figure 5.1 show the implied volatility parameters of the absolute volatility model for 12, 8, 8a, 8b, and 4 time-step implementations. Signs for volatility parameters are consistent with Amin and Morton (1994) except for the linear proportional model. It shows that 8b (3-2-2-1) time-step model is the closest to the 12 time-step model. Table 5.2 and Figure 5.2 show the implied volatility parameters of the square root model. Table 5.3 and Figure 5.3 show the implied volatility parameters of the proportional model. Similarly, they show that 8b time-step model is the closest to the 12 time-step model.

5.2. Times Required For Computation

We report the time required for the computation using different volatility models and using various time-steps in Table 5.4 and Figure 5.4, where the y-axis represents time required in minutes and the x-axis represents six volatility specifications in each time-steps. The average computation time for two-parameter models is almost twice as long as that for one-parameter models.

On the other hand, the increase in average computation time appears to be a multiple of the increase in time step for more time steps. In fact, the increase in time is a function of the increase in time step and the distribution of the maturities of futures and futures options. In general, the total number of nodes we need to compute in constructing our binomial tree is the summation of 2^i from i=1 to N, where N is the number of time steps we take in the tree. Thus, if it is not a linear relationship, we could expect the increase in average computation time as we take more time steps and more long-dated maturity in futures and futures options data.

5.3. Error Measures

The error is equal to the difference between the market option price observed on that day and the model value we computed using the estimated forward rate term structure. We compute the average errors and average absolute errors for six volatility models. We also examine the effect of using time-steps of varying lengths within the same estimation procedure, and of ordering the time-steps in different ways. We regroup the options by different criteria, such as call/put, time-to-maturity, and moneyness. Then, we compute the errors for each group of options to find any pattern within that relationship.

In calculating the errors, we defined the option moneyness as follows: In-the-money option refers the option when futures price minus strike price (strike price minus futures price) is greater than 40 dollars if the option is a call (a put). At-the-money option refers the option when the absolute value of futures price minus strike price is less than 40 dollars. Out-of-the-money option refers the option when strike price minus futures price (futures price minus strike price) is greater than 40 dollars if the option is a call (a put).

As we increase the time-step in the computation, we see the trade-off between the scale of errors and computation time (Table 5.5 and Figure 5.5). For example, 12 time-step models have smaller errors than 8 time-step models, but 12 time-step models also require longer computation time. Similarly, 8 time-step models have smaller errors and require longer computation time than 4 time-step models. In terms of accuracy and computation time, the two-parameter models are grouped together and connected by the solid line. The one-parameter models are grouped together and connected by another solid line. There is a tradeoff between the two groups in terms of accuracy and computation time. Within the same time-step schemes, two-parameter models have smaller errors and require longer computation time than one-parameter models. However, two-parameter models with 8b time steps seem to be preferable to one-parameter models with 12 time steps in terms of accuracy versus computation time since the former require less computation time with the same level of errors.

In Table 5.6 and Figure 5.6, we report average absolute errors, where each column represents different volatility specifications and each row represents different time steps applied in the procedure. The average absolute errors range from 0.0142 to 0.0240 for 12 time-step models. The scale of error gets larger as the time steps decrease. The average absolute errors range from to 0.0946 to 0.1026 for 4 time-step models.

Note that within the 8 time-step models, the scale of error for 8b (3-2-2-1) time-step model drops almost in half compared to an 8 time-step model. It implies that we can predict market option prices faster and more accurately by using an 8b time-step model.

In Table 5.7 and Figure 5.7, we report the standard deviation of average errors for six volatility specifications and for each time-steps. The standard deviation of average errors increases as the time-step decreases. Also note that within the 8 time-step models, the standard error for the 8b (3-2-2-1) time-step model is much smaller than that of the 8 time-step model. However, it is not clear which volatility model has the smallest standard error.

Tables 5.8, 5.9, and Figure 5.8, 5.9 exhibit the percentiles for absolute errors as grouped by the type of options. The 90th percentile of errors for one-parameter models is much higher than that for two-parameter models. Specifically, the 90th percentile of errors for the linear absolute and exponential models stays at the lowest level for both type of options.

Tables 5.10, 5.11, 5.12, and Figures 5.10, 5.11, 5.12 exhibit the percentiles for absolute errors as grouped by the option moneyness. Here we find a very distinctive pattern for each group of options. While we do not find much difference in the 90th percentile of errors with in-the-money options across six volatility models but the 75th percentile of errors for two parameter models are lower than that of one-parameter models. The 90th percentile of errors for the linear absolute and exponential models stays at the lowest level with at-the-money options. In the out-of-the-money options, the 90th percentile of errors for the linear proportional model stays at the lowest level.

Table 5.13 and Figures 5.13 – 5.18 exhibit the pattern of errors as a function of time steps as grouped by the moneyness. Table 5.14 summarizes all the results in a table. For in-the-money call options, the models tend to overprice short-dated call options while underpricing medium- and long-dated call options. Also, all the models tend to underprice puts except long-dated options, which are overpriced by the two-parameter models. Note that two-parameter models are a better fit for most options.

For at-the-money options, short-dated options are overpriced while long-dated options are underpriced both for call options and put options. Note that two-parameter models are a better fit than one-parameter models for at-the-money options.

For out-of-the-money options, all the models tend to underprice options except the short-dated call options that are overpriced.

Chapter 6. Conclusion

Several models of the term structure of interest rates have been proposed in the literature. Among them, the Heath-Jarrow-Morton (1991,1992) class of models for pricing interest rate claims, based on the no-arbitrage principle, is one of the most widely used. Under complete markets, HJM show the existence of a risk-neutral pricing measure under which the price of every security discounted by the spot interest rate is a martingale. Their class of models requires only the specification of the form of the volatility function of forward interest rates along with the initial term structure as inputs. We study the implied volatility in the Heath-Jarrow-Morton framework using different volatility specifications and time step schemes.

While the HJM class of models has been widely adopted, there is very little academic evidence available on issues relating to their implementation. This is particularly important since we have to use a path-dependent binomial model to correctly price the early exercise feature. The fact that usually there are no closed–form solutions implies that we need accurate and stable numerical methods to approximate the claim values and assess their sensitivities with respect to the different input parameters. One of the arguments against the HJM model is that it is too slow for pricing American-style interest rate claims as the model needs numerical fitting. In our study, we examine the trade-off between accuracy and computation time in implementing models of the HJM class. Since we can use arbitrary volatility functions, we choose six different forms of volatility functions. Our results clearly show this tradeoff between accuracy and computation time.

Previous work in this area includes Amin and Morton (1994) and Amin and Ng (1997), who implement and test a broad class of path-dependent HJM models. The focus of those studies, however, was more theoretical than implementational in nature, i.e., they examined whether any of the different volatility specifications resulted in large and systematic pricing errors, and whether they gave rise to profitable trading strategies. Our study extends these studies by examining whether any volatility specification seems preferable over the others in the accuracy versus computation time tradeoff, and also by examining alternative numerical implementations of HJM models. In particular, we analyze the impact on accuracy and computation time of using different number of time steps. We also examine the effect of using time steps of varying lengths within the same estimation procedure. Moreover, our results are based on more comprehensive data – our sample period runs from 1987 to 1998, and we use an average of 36 options each day in testing model performance.

The square root, proportional, and linear proportional models have structures that permit the volatility to depend on the level of the forward rate. The linear absolute, exponential, and linear proportional models have structures that permit the volatility to depend on time to maturity. The absolute model has a structure in which the exact magnitude of volatility is independent of the level of the forward rate and time to maturity.

Each model differs from the others in its ability to match the market option prices and the time required for the computation. We find that the average computation time for two-parameter models is almost twice as large as that for one-parameter models while average absolute errors for two-parameter models are, in general, smaller than those for one-parameter models.

We find that among the two-parameter models, the exponential model performs better than the linear absolute model and the linear proportional model in terms of accuracy and computation time. We also find that the 90th percentile of the standard deviation of average errors for the exponential model stay at the lowest level for both types of options. It suggests that its exponentially dampened volatility structure leads to better prediction of interest rate claims prices.

In terms of the number of time steps, the 8b (3-2-2-1) time step model results in the smallest average absolute error among the eight time-step models, suggesting that the 8b time step model is to be preferred for practical purposes. For example, two-parameter models with 8b time steps seem to be preferable to one-parameter models with 12 time steps in terms of accuracy versus computation time since the former require less computation time with the same level of errors.

We also find certain patterns among the biases in option prices. The biases are influenced by the different volatility specifications rather than by the time steps used in the implementations. For in-the-money call options, the models tend to overprice short-dated call options while underpricing medium- and long-dated call options. Also, all the models tend to underprice puts except long-dated options which are overpriced among the two parameter models.

For at-the-money options, short-dated options are overpriced while long-dated options are underpriced both in call options and in put options. Note that two-parameter models are better fit that the one parameter models in the at-the-money options.

For out-of-the-money options, all the models tend to underprice options except the short-dated call options that are overpriced.

There are several important directions in which future research may extend our findings. One limitation of this study is that only one-factor models are examined. A single source of uncertainty can explain much of the variations in forward rates with short-term maturities. However, future studies could extend this paper by examining the effects of two factors in the HJM class of models. In doing so, they should be careful in disentangling the effects of two factors.

Another limitation is that this paper uses only short-term futures and futures options data. Future studies could use longer-term futures and futures options data. In dealing with the prices of the futures options with longer maturities, they should be careful in dealing with the data with smaller trading volumes, since futures market liquidity declines quickly with increasing maturity. Also, it may be useful to examine the data on futures expiring in months other than March, June, September, and December.

Finally, while this study finds that certain numerical implementations seem preferable over the others in the accuracy versus computation time tradeoff, future studies could examine how changes of volatility structure over time affect numerical implementations of HJM models.

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Table 1.1 Summary Statistics for the Exchange-Traded Derivatives Market

The numbers for each year represent the number (in millions) of exchange-traded contracts transacted during the year.

	TOTAL	INTEREST RATES	EQUITIES	CURRENCIES
1999	1,356.5	792.8	522.3	41.4
2000	1,612.3	997.8	677.4	47.0
% change	18.85 %	11.98 %	29.71 %	13.47 %

Source: *Futures Industry,* Futures Industry Association, February/March 2001

Table 1.2 Summary Statistics for the Over-The-Counter Interest Rate Derivatives Market

The numbers for each year represent the notional amount outstanding (in trillions of US dollars), of over-the-counter interest rate derivatives at the end of the year.

	TOTAL	INTEREST RATE OPTIONS	INTEREST RATE FORWARDS	INTEREST RATE SWAPS
1998.12	50.0	8.0	5.8	36.3
1999. 6	54.1	8.6	7.1	38.4
1999.12	60.1	9.4	6.8	43.9
2000.12	64.7	9.5	6.4	48.8

Source: Bank for International Settlements, Press release, May 16, 2001

Table 5.1 Implied Volatility Parameters in Absolute Model

This table provides a daily time series of implied volatility parameters using the Absolute volatility model for the 5 different number of time steps, as described in Section 4.2. Each column denotes the different time steps and each row denotes the trading days in January from 1987 to 1988.

	12	8	8b	8a	4
1987	0.011076	0.011043	0.010879	0.011233	0.010521
	0.009890	0.009750	0.009841	0.009794	0.009262
	0.010457	0.010288	0.010423	0.010357	0.010054
	0.010325	0.010212	0.010264	0.010188	0.009890
	0.009800	0.009763	0.009755	0.009721	0.009490
	0.009170	0.009112	0.009153	0.009206	0.008779
	0.008819	0.008783	0.008836	0.008937	0.008479
	0.008969	0.008994	0.008975	0.009060	0.008711
	0.008770	0.008801	0.008828	0.008926	0.008509
	0.009169	0.009137	0.009201	0.009080	0.008891
	0.008809	0.008719	0.008790	0.008871	0.008562
	0.008883	0.008794	0.008881	0.008910	0.008700
	0.008996	0.008851	0.008980	0.008860	0.008537
	0.008747	0.008672	0.008758	0.008659	0.008705
	0.008511	0.008226	0.008453	0.008239	0.008032
	0.008735	0.008620	0.008679	0.008674	0.008595
	0.008668	0.008540	0.008669	0.008635	0.008264
	0.008402	0.008314	0.008362	0.008426	0.008119
	0.008364	0.008120	0.008268	0.008218	0.008030
	0.008310	0.008148	0.008273	0.008494	0.008070
	0.008792	0.008643	0.008767	0.008759	0.008572
1988	0.019951	0.019892	0.019828	0.020234	0.019176
	0.020026	0.019743	0.019737	0.019831	0.018576
	0.020153	0.019827	0.019821	0.019960	0.019119
	0.019911	0.019693	0.019763	0.019647	0.019043
	0.019147	0.019075	0.019128	0.018993	0.018608
	0.019789	0.019664	0.019798	0.019867	0.018989
	0.018615	0.018539	0.018769	0.018863	0.018011
	0.019325	0.019379	0.019596	0.019522	0.019020
	0.018402	0.018467	0.018703	0.018729	0.018027
	0.018578	0.018513	0.018579	0.018398	0.017953
	0.017165	0.016990	0.017156	0.017286	0.016711
	0.016698	0.016531	0.016763	0.016750	0.016421
	0.016542	0.016275	0.016465	0.016292	0.015653
	0.016440	0.016299	0.016457	0.016275	0.016357
	0.016752	0.016191	0.016393	0.016217	0.015577
	0.015847	0.015638	0.015961	0.015735	0.015807
	0.015789	0.015556	0.016094	0.015729	0.015342
	0.015310	0.015150	0.015465	0.015353	0.015016
	0.015713	0.015255	0.015424	0.015439	0.014980
	0.015224	0.014927	0.015185	0.015562	0.014812

Table 5.2 Implied Volatility Parameters in Square Root Model

This table provides a daily time series of implied volatility parameters using the Square root volatility model for the 5 different number of time steps, as described in Section 4.2. Each column denotes the different time steps and each row denotes the trading days in January from 1987 to 1988.

	12	8	8b	8a	4
1987	0.045507	0.045372	0.044821	0.045207	0.042966
	0.040473	0.039925	0.040281	0.039459	0.037990
	0.042343	0.041811	0.042245	0.041074	0.040779
	0.041942	0.041544	0.041716	0.040615	0.040126
	0.039955	0.039853	0.039784	0.039432	0.038698
	0.037500	0.037323	0.037488	0.038694	0.035888
	0.035905	0.035751	0.035953	0.034764	0.034522
	0.036362	0.036482	0.036362	0.036345	0.035376
	0.035501	0.035640	0.035671	0.034718	0.034666
	0.037047	0.036866	0.037198	0.036995	0.035970
	0.035916	0.035683	0.035873	0.035200	0.034892
	0.036356	0.035990	0.036347	0.033889	0.035951
	0.036717	0.036246	0.036606	0.036130	0.034928
	0.035740	0.035491	0.035740	0.035327	0.035575
	0.034635	0.033706	0.034471	0.032864	0.032825
	0.035399	0.035016	0.035224	0.033666	0.034943
	0.035160	0.034619	0.035149	0.034209	0.033619
	0.033969	0.033584	0.033751	0.033224	0.033088
	0.033756	0.032844	0.033382	0.031788	0.032283
	0.033596	0.032959	0.033512	0.031370	0.032601
	0.035165	0.034523	0.035080	0.034455	0.034292
1988	0.070893	0.070683	0.070251	0.070427	0.067344
	0.070765	0.069807	0.069845	0.068993	0.065873
	0.071296	0.070400	0.070144	0.069159	0.067710
	0.070040	0.069375	0.069474	0.067824	0.066826
	0.067279	0.067107	0.067258	0.066398	0.065422
	0.069724	0.069395	0.069752	0.071944	0.066775
	0.066901	0.066614	0.067501	0.064775	0.064814
	0.068428	0.068654	0.069188	0.068396	0.067312
	0.065649	0.065906	0.066584	0.064201	0.064708
	0.065948	0.065626	0.065735	0.065855	0.063565
	0.061937	0.061535	0.062144	0.060701	0.060445
	0.059600	0.059000	0.059556	0.055555	0.058907
	0.058923	0.058167	0.058712	0.057981	0.056021
	0.058676	0.058267	0.059048	0.057997	0.058775
	0.059985	0.058376	0.059195	0.056917	0.056368
	0.057189	0.056570	0.057580	0.054389	0.057121
	0.056878	0.056003	0.057922	0.055340	0.055401
	0.055792	0.055160	0.056441	0.054569	0.055332
	0.057247	0.055700	0.056190	0.053909	0.054340
	0.056386	0.055317	0.056201	0.05265	0.054673

Table 5.3 Implied Volatility Parameters in Proportional Model

This table provides a daily time series of implied volatility parameters using the Proportional volatility model for the 5 different number of time steps, as described in Section 4.2. Each column denotes the different time steps and each row denotes the trading days in January from 1987 to 1988.

	12	8	8b	8a	4
1987	0.186522	0.185784	0.185587	0.184111	0.176164
	0.165454	0.163272	0.164604	0.161799	0.155489
	0.171063	0.169524	0.170705	0.166284	0.164911
	0.170025	0.168661	0.169235	0.164758	0.161653
	0.162706	0.162372	0.162055	0.160841	0.157429
	0.153157	0.152639	0.152473	0.159104	0.146505
	0.145847	0.144916	0.146035	0.140855	0.140410
	0.147197	0.147853	0.147136	0.147080	0.143463
	0.143474	0.143525	0.144083	0.140884	0.140428
	0.149401	0.148602	0.149981	0.149317	0.145233
	0.146092	0.145836	0.146172	0.144081	0.143064
	0.148650	0.147655	0.148506	0.139746	0.146692
	0.149405	0.148038	0.148920	0.148006	0.142772
	0.145969	0.145301	0.145740	0.144700	0.144931
	0.141250	0.137907	0.140657	0.134498	0.133410
	0.143232	0.141864	0.142684	0.136281	0.141876
	0.142499	0.140269	0.142269	0.139711	0.135992
	0.137149	0.135368	0.136126	0.134239	0.133847
	0.135942	0.132402	0.134544	0.129438	0.130326
	0.135591	0.133208	0.135278	0.127845	0.131655
	0.140604	0.138288	0.140300	0.137873	0.137107
1988	0.250063	0.249074	0.248013	0.246831	0.235420
	0.249088	0.245803	0.246366	0.243586	0.232723
	0.250906	0.248649	0.247534	0.243896	0.239132
	0.245520	0.243550	0.243697	0.237913	0.232779
	0.235828	0.235344	0.235842	0.233125	0.229110
	0.244454	0.243627	0.244912	0.253946	0.235326
	0.240521	0.238986	0.243111	0.232288	0.233747
	0.240524	0.241596	0.242585	0.240333	0.236529
	0.234761	0.234844	0.237020	0.230522	0.231007
	0.232877	0.231632	0.231754	0.232746	0.224417
	0.222870	0.222479	0.224654	0.219801	0.219877
	0.211351	0.209936	0.210837	0.198691	0.208262
	0.209383	0.207467	0.208965	0.207422	0.200338
	0.208838	0.207882	0.210799	0.207022	0.209629
	0.214961	0.209873	0.213145	0.204685	0.202163
	0.206659	0.204685	0.207267	0.196630	0.206093
	0.204644	0.201441	0.207468	0.200640	0.198314
	0.203518	0.200875	0.204956	0.199199	0.201525
	0.208087	0.202668	0.204024	0.198131	0.197628
	0.208423	0.204760	0.207924	0.196517	0.202355

Table 5.4 Computation Time

This table shows the computation time required (in minutes) to estimate forward rate term structures and volatility parameter(s) for 256 sets of daily observations with 4 futures contracts and an average of 36 futures options per day, for the 6 different volatility specifications and 4 different numbers of time steps. Under the 8b implementation scheme, 3 steps are used for the time-to-maturity of the first futures contract, 2 steps for the second and third futures contracts, and 1 step for the fourth futures contract.

	TIME STEPS			
VOLATILITY MODEL	12	8b	8	4
Absolute	152.37	12.65	11.13	1.72
Square Root	164.71	11.72	11.87	1.49
Proportional	169.76	16.66	11.48	1.65
Linear absolute	400.26	32.12	29.33	3.58
Exponential	395.33	26.28	27.18	3.28
Linear proportional	455.57	25.73	34.13	3.15

Table 5.5 Accuracy Versus Time Required

This table shows the trade-off between computation time and accuracy in estimating the forward rate term structures and volatility parameter(s) for the 6 different volatility specifications. Computation time is the time (in minutes) required for 256 daily sets of observations. Average absolute error indicates the mean absolute value (in dollars) of option pricing errors, i.e., the difference between observed and model-predicted option prices, for an average of 36 options per day. Under the 8b implementation scheme, 3 steps are used for the time-to-maturity of the first futures contract, 2 steps for the second and third futures contracts, and 1 step for the fourth futures contract.

TIME STEPS	MODEL	AVG ABS ERRORS	TIME IN MINUTES
12 step	Absolute	0.0240	152.4
	Square root	0.0219	164.7
	Proportional	0.0199	169.8
	Linear absolute	0.0143	400.3
	Exponential	0.0142	395.3
	Linear proportional	0.0153	455.6
8b step	Absolute	0.0289	12.6
	Square root	0.0281	11.7
	Proportional	0.0267	16.7
	Linear absolute	0.0213	32.1
	Exponential	0.0213	26.3
	Linear proportional	0.0220	25.7
8 step	Absolute	0.0514	11.1
	Square root	0.0501	11.9
	Proportional	0.0489	11.5
	Linear absolute	0.0431	29.3
	Exponential	0.0427	27.2
	Linear proportional	0.0437	34.1
4 step	Absolute	0.1026	1.7
	Square root	0.1023	1.5
	Proportional	0.1019	1.7
	Linear absolute	0.0955	3.6
	Exponential	0.0946	3.3
	Linear proportional	0.0961	3.1

Table 5.6 Average Absolute Errors

This table shows the means of absolute values (in dollars) of option pricing errors, i.e., the difference between observed and model-predicted option prices, for an average of 36 options per day, for 256 daily sets of observations. Under the 8b implementation scheme, 3 steps are used for the time-to-maturity of the first futures contract, 2 steps for the second and third futures contracts, and 1 step for the fourth futures contract.

TIME STEPS	VOLATILITY MODEL					
	ABSOLUTE	SQUARE ROOT	PROPORTIONAL	LINEAR ABSOLUTE	EXPONENTIAL	LINEAR PROPORTIONAL
12 step	0.024043	0.021901	0.019906	0.01429	0.014187	0.015326
8b step	0.028884	0.028143	0.026742	0.021278	0.021289	0.021995
8 step	0.051445	0.050107	0.048936	0.043106	0.042749	0.043665
4 step	0.102582	0.102298	0.101916	0.095476	0.094608	0.096114

Table 5.7 Standard Deviation of Average Errors

This table shows the standard deviations (in dollars) of option pricing errors, i.e., the difference between observed and model-predicted option prices, for an average of 36 options per day, for 256 daily sets of observations. Under the 8b implementation scheme, 3 steps are used for the time-to-maturity of the first futures contract, 2 steps for the second and third futures contracts, and 1 step for the fourth futures contract.

TIME STEPS	VOLATILITY MODEL					
	ABSOLUTE	SQUARE ROOT	PROPORTIONAL	LINEAR ABSOLUTE	EXPONENTIAL	LINEAR PROPORTIONAL
12 step	0.006538	0.006453	0.003590	0.003457	0.003558	0.003743
8b step	0.003745	0.003975	0.003885	0.003914	0.004029	0.004040
8 step	0.006395	0.006193	0.005974	0.005795	0.005916	0.005768
4 step	0.015760	0.016141	0.014866	0.018103	0.015223	0.028022

Table 5.8 Percentiles For Errors in Calls

This table provides a percentile break-down of absolute values (in dollars) of option pricing errors, i.e., the difference between observed and model-predicted option prices, for an average of 18.0 call options per day, for 256 daily sets of observations. Reported percentile values are for the 12-step implementation schemes.

	Volatility Model					
Percentiles	Absolute	Square root	Proportional	Linear Absolute	Exponential	Linear Proportional
10	0.001755	0.001976	0.001620	0.000808	0.000949	0.001173
25	0.008428	0.007222	0.006678	0.003994	0.004014	0.005232
50	0.017749	0.015570	0.014724	0.010000	0.010000	0.010399
75	0.033694	0.030678	0.029170	0.018586	0.019338	0.021486
90	0.052224	0.050264	0.048101	0.030194	0.030392	0.036693

Table 5.9 Percentiles For Errors in Puts

This table provides a percentile break-down of absolute values (in dollars) of option pricing errors, i.e., the difference between observed and model-predicted option prices, for an average of 15.5 put options per day, for 256 daily sets of observations. Reported percentile values are for the 12-step implementation schemes.

	VOLATILITY MODEL					
PERCENTILES	ABSOLUTE	SQUARE ROOT	PROPORTIONAL	LINEAR ABSOLUTE	EXPONENTIAL	LINEAR PROPORTIONAL
10	0.002885	0.002137	0.002191	0.001806	0.001759	0.001566
25	0.009871	0.007921	0.005990	0.006307	0.006270	0.005234
50	0.022132	0.017523	0.013921	0.012304	0.012594	0.011523
75	0.040302	0.034383	0.029526	0.020652	0.020432	0.021636
90	0.058094	0.053277	0.049946	0.031289	0.031250	0.035367

Table 5.10 Percentiles For Errors in In-the-Money Options

This table provides a percentile break-down of absolute values (in dollars) of option pricing errors, i.e., the difference between observed and model-predicted option prices, for an average of 4.1 in-the-money options per day, for 256 daily sets of observations. Reported percentile values are for the 12-step implementation schemes.

	Volatility Model					
Percentiles	Absolute	Square root	Proportional	Linear Absolute	Exponential	Linear Proportional
10	0.000530	0.000501	0.000326	0.000135	0.000221	0.000217
25	0.009418	0.008860	0.009171	0.006799	0.006140	0.005955
50	0.019754	0.019582	0.019462	0.013564	0.013641	0.016423
75	0.037742	0.037828	0.039190	0.030905	0.032307	0.030499
90	0.060490	0.060430	0.060452	0.059810	0.059765	0.059770

Table 5.11 Percentiles For Errors in At-the-Money Options

This table provides a percentile break-down of absolute values (in dollars) of option pricing errors, i.e., the difference between observed and model-predicted option prices, for an average of 16.1 at-the-money options per day, for 256 daily sets of observations. Reported percentile values are for the 12-step implementation schemes.

	VOLATILITY MODEL					
PERCENTILES	ABSOLUTE	SQUARE ROOT	PROPORTIONAL	LINEAR ABSOLUTE	EXPONENTIAL	LINEAR PROPORTIONAL
10	0.004819	0.003806	0.003033	0.001593	0.001781	0.002533
25	0.012343	0.010220	0.008012	0.004962	0.004791	0.006546
50	0.025709	0.022772	0.018553	0.010559	0.010331	0.013728
75	0.044824	0.042106	0.035750	0.019221	0.019699	0.025518
90	0.060393	0.059576	0.056906	0.029432	0.030191	0.038580

Table 5.12 Percentiles For Errors in Out-of-the-Money Options

This table provides a percentile break-down of absolute values (in dollars) of option pricing errors, i.e., the difference between observed and model-predicted option prices, for an average of 13.3 out-of-the-money options per day, for 256 daily sets of observations. Reported percentile values are for the 12-step implementation schemes.

	Volatility Model					
Percentiles	Absolute	Square root	Proportional	Linear Absolute	Exponential	Linear Proportional
10	0.001545	0.001427	0.001064	0.000891	0.000741	0.000785
25	0.005730	0.005344	0.004713	0.004246	0.004715	0.004106
50	0.013003	0.011051	0.010171	0.010000	0.010000	0.008929
75	0.026086	0.021431	0.018888	0.017116	0.017280	0.015000
90	0.040831	0.033807	0.030193	0.026154	0.024895	0.022435

Table 5.13 Average Errors in Option Prices

This table reports means of option pricing errors (in dollars), i.e., the difference between observed and model-predicted option prices, broken down by option type (call/put), maturity (short/medium/long), and moneyness (in-the-money/at-the-money/out-of-the-money). Short maturity options have up to 3 months to maturity, medium maturity options have more than 3 and up to 9 months to maturity, and long maturity options have more than 9 months to maturity. Average errors are reported for each of the 6 volatility specifications.

Moneyness	Option type	Maturity	Absolute	Square Root	Proportional	Linear Absolute	Exponential	Linear Proportional
ITM	Call	Short	-0.013299	-0.013345	-0.012432	-0.009646	-0.009454	-0.011128
ITM	Call	Medium	-0.013299	-0.013345	-0.012432	-0.009646	-0.009454	-0.011128
ITM	Call	Long	0.013937	0.011082	0.012181	-0.002373	-0.006234	0.001544
ITM	Put	Short	0.007306	0.008643	0.009824	0.009601	0.009244	0.009127
ITM	Put	Medium	0.010236	0.014548	0.018891	0.013224	0.013234	0.018439
ITM	Put	Long	0.011548	0.009202	0.007273	-0.008332	-0.007834	0.000525
ATM	Call	Short	-0.025428	-0.023025	-0.019330	-0.005730	-0.006539	-0.010558
ATM	Call	Medium	-0.013554	-0.012286	-0.008009	-0.002203	-0.001299	-0.003533
ATM	Call	Long	0.013990	0.011666	0.012301	-0.003585	-0.006194	0.004078
ATM	Put	Short	-0.022239	-0.020099	-0.015972	-0.004053	-0.005046	-0.010091
ATM	Put	Medium	-0.009099	-0.008587	-0.005226	0.003896	0.005786	-0.000788
ATM	Put	Long	0.024517	0.020497	0.018730	-0.000225	-0.002320	0.007679
OTM	Call	Short	-0.006529	-0.003732	-0.001276	0.001899	0.001886	0.002830
OTM	Call	Medium	-0.006791	-0.003149	0.001791	-0.000451	0.000010	0.004153
OTM	Call	Long	0.003003	0.004307	0.007041	-0.006257	-0.007590	0.004395
OTM	Put	Short	0.000839	0.001030	0.001626	0.003424	0.003000	0.001690
OTM	Put	Medium	0.004210	0.001455	0.000866	0.011120	0.012058	0.002839
OTM	Put	Long	0.028167	0.021308	0.016610	0.012474	0.010594	0.008829

Table 5.14 Biases In Option Prices by Option Type, Maturity, and Moneyness

This table reports the directional biases of option pricing errors, i.e., the difference between observed and model-predicted option prices, broken down by option type (call/put), maturity (short/medium/long), and moneyness (in-the-money/at-the-money/out-of-the-money). 1 refers to short maturity options with up to 3 months to maturity, 2 to medium maturity options with more than 3 and up to 9 months to maturity, and 3 to long maturity options with more than 9 months to maturity. Biases are reported for each of the 6 volatility specifications – ab: Absolute, sq: Square Root, pr: Proportional, la: Linear Absolute, ex: Exponential, and lp: Linear Proportional. A positive bias indicates that, on average, the model's predicted option price is greater than the observed option price, while a negative bias indicates the opposite.

		ITM						ATM						OTM					
		ab	sq	pr	la	ex	lp	ab	sq	pr	la	ex	lp	ab	sq	pr	la	ex	lp
Call	1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
	2	-	-	-	-	-	-	+	+	+	-	-	-	-	-	-	-	-	-
	3	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
Put	1	-	-	-	-	-	-	+	+	+	+	+	+	-	-	-	-	-	-
	2	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
	3	-	-	-	+	+	+	-	-	-	-	-	-	-	-	-	-	-	-

ab: Absolute model
sq: Square Root model
pr: Proportional model
la: Linear Absolute model
ex: Exponential model
lp: Linear Proportional model
1 : maturity <= 3 months
2: 3 months < maturity <= 9 months
3: maturity > 9 months
+ : model option price >= market option price
- : model option price < market option price

Figure 3.1 One-factor Forward Rate Curve Evolution From Time-0 To Time-T

This figure depicts the evolution of the forward rate term structure from time-0 (on the left) to time-T (on the right). At each step, the entire term structure either moves up or moves down, following the transition relation described by equations (3-19). At each step, the number of elements in the vector of forward rates decreases by one, as the shortest maturity forward rate matures and stops trading.

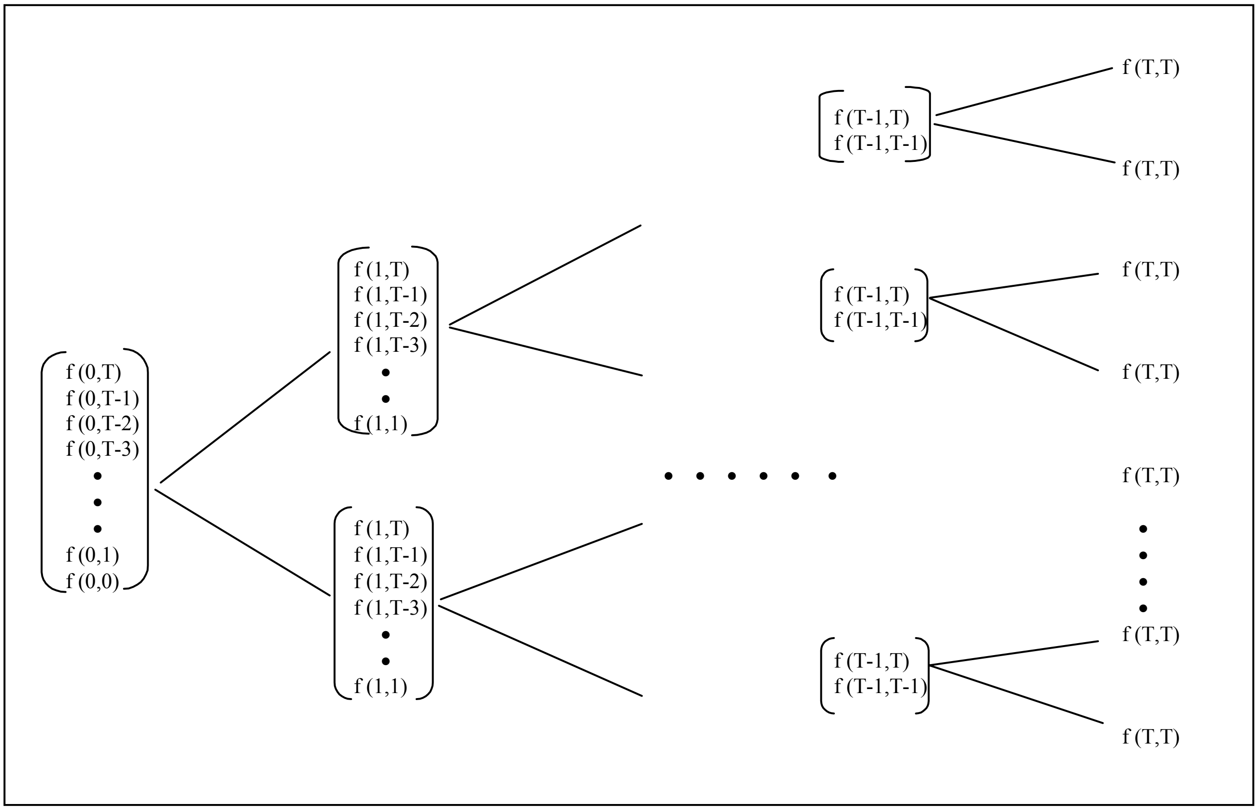


Figure 4.1 One-factor Futures Price Evolution From Time-0 To Time-T

This figure depicts the evolution of futures prices based on the forward rate tree shown in Figure 3.1. At time-T (on the right), the futures price F(T, T) can be calculated from the time-T forward rate f(T, T) using the relation $F(T, T) = 10^6 (1 - 0.25 \times f(T, T))$. Moving to time-0 (on the left), one step at a time, the futures price at time-t ($0 \le t < T$) is given by $F(t, T) = E^Q_t[F(T, T)]$, so that at time-0, $F(0, T) = E^Q_0[F(T, T)]$, where the expectations operator E^Q is with respect to the risk neutral probabilities of 0.5 each for up and down moves.

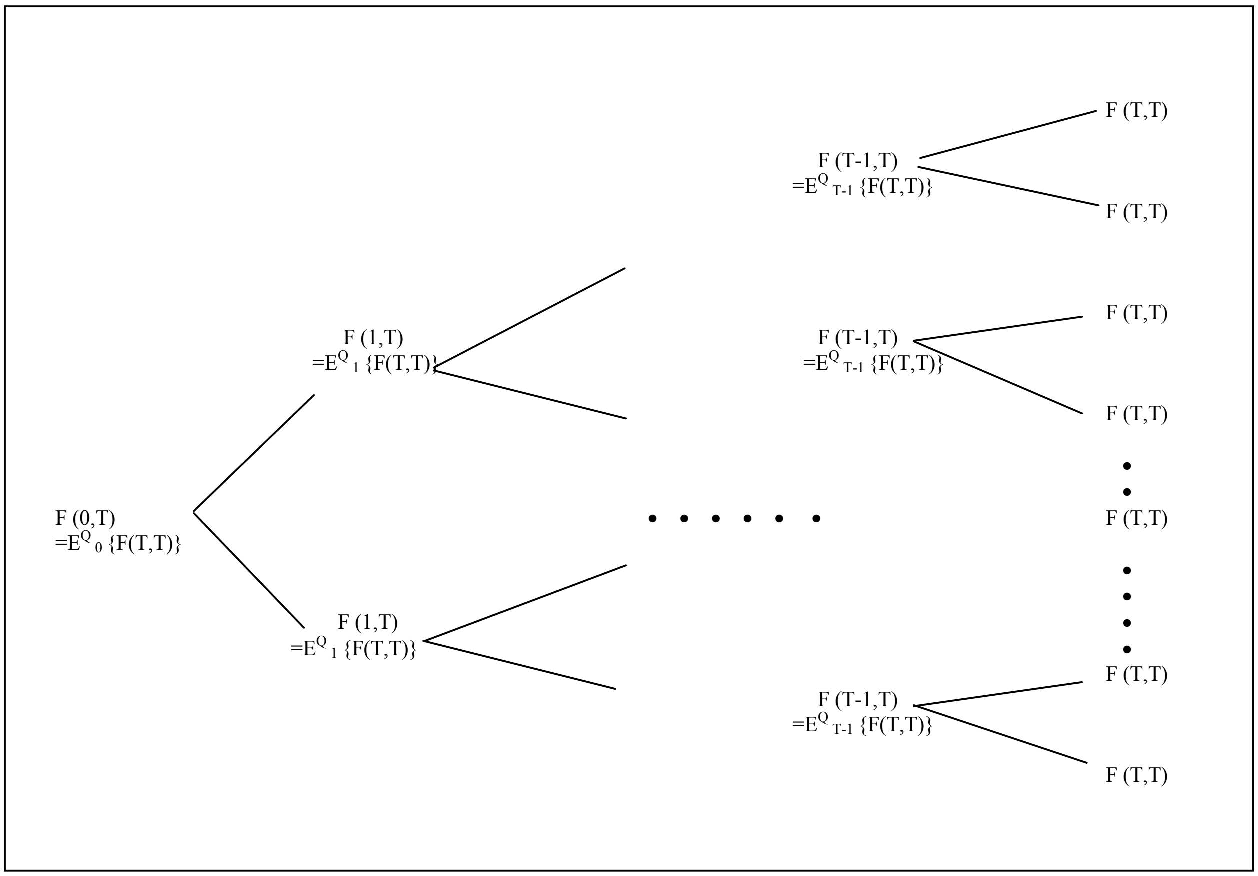


Figure 4.2 One-factor Futures Option Price Evolution From Time-0 To Time-T.

This figure depicts the evolution of a call option price based on the futures price tree shown in Figure 4.1. At time-T (on the right), the call price C(T) can be calculated from the time-T futures price F(T, T) as shown. Moving to time-0 (on the left), one step at a time, the call option price C(t) at each node is calculated as the maximum of the value of early exercise, F(t, T) - K, and the value of keeping the option $E^Q_t[C(t+1)] / f(t, t)$, i.e., the discounted expected value of the option in the next period, where the expectations operator E^Q is with respect to the risk neutral probabilities of 0.5 each for up and down moves.

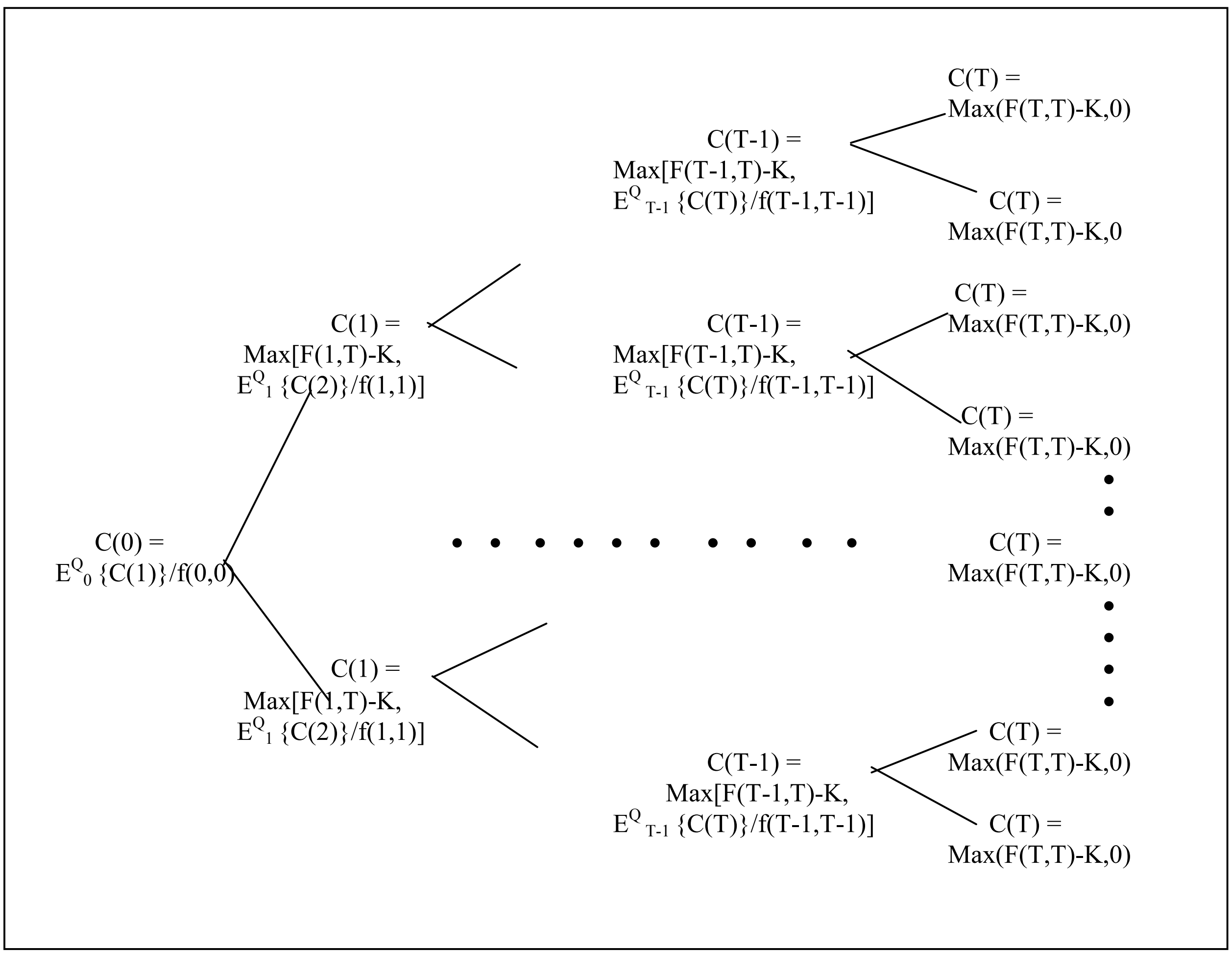


Figure 5.1 Implied Volatility Parameters in the Absolute Volatility Model

This figure provides a daily time series of implied volatility parameters using the Absolute volatility model. The Y axis denotes the estimated volatility parameter and the X axis denotes the total number of trading days in January from 1987 to 1998. The figure plots the estimated volatility parameter for implementation schemes using 12, 8, 8a, 8b, and 4 time steps, as described in Section 4.2.

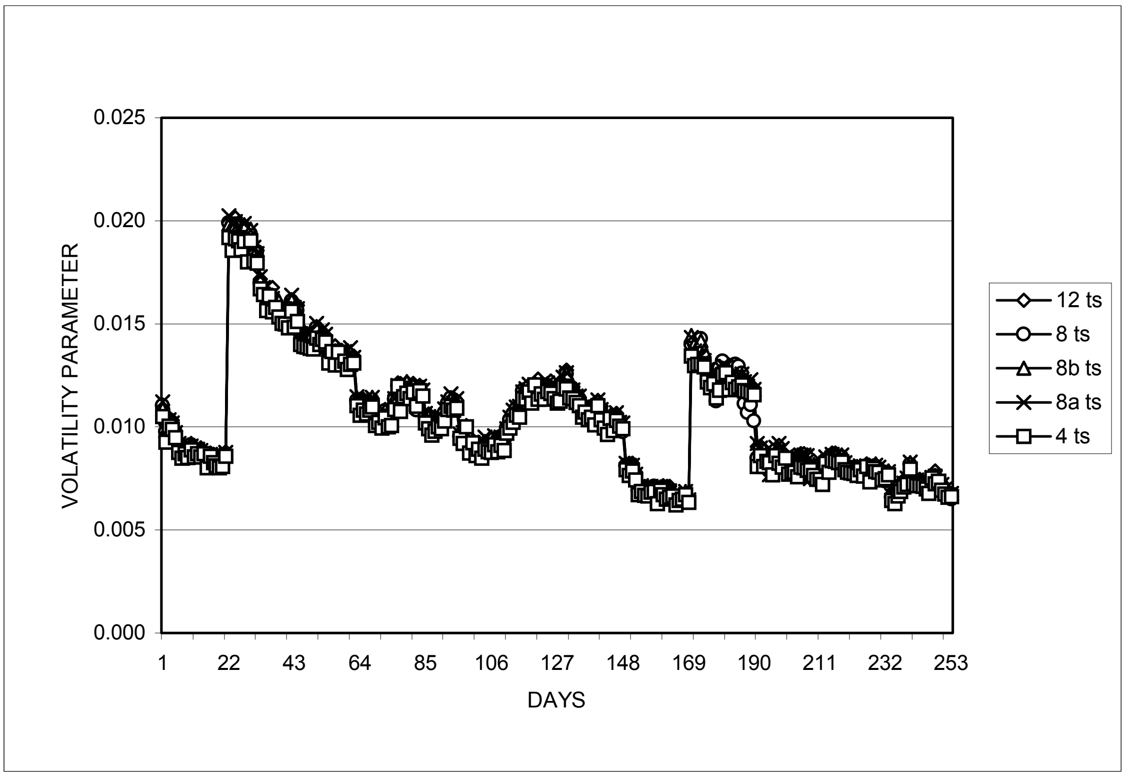


Figure 5.2 Implied Volatility Parameters in Square Root Model

This figure provides a daily time series of implied volatility parameters using the Absolute volatility model. The Y axis denotes the estimated volatility parameter and the X axis denotes the total number of trading days in January from 1987 to 1998. The figure plots the estimated volatility parameter for implementation schemes using 12, 8, 8a, 8b, and 4 time steps, as described in Section 4.2.

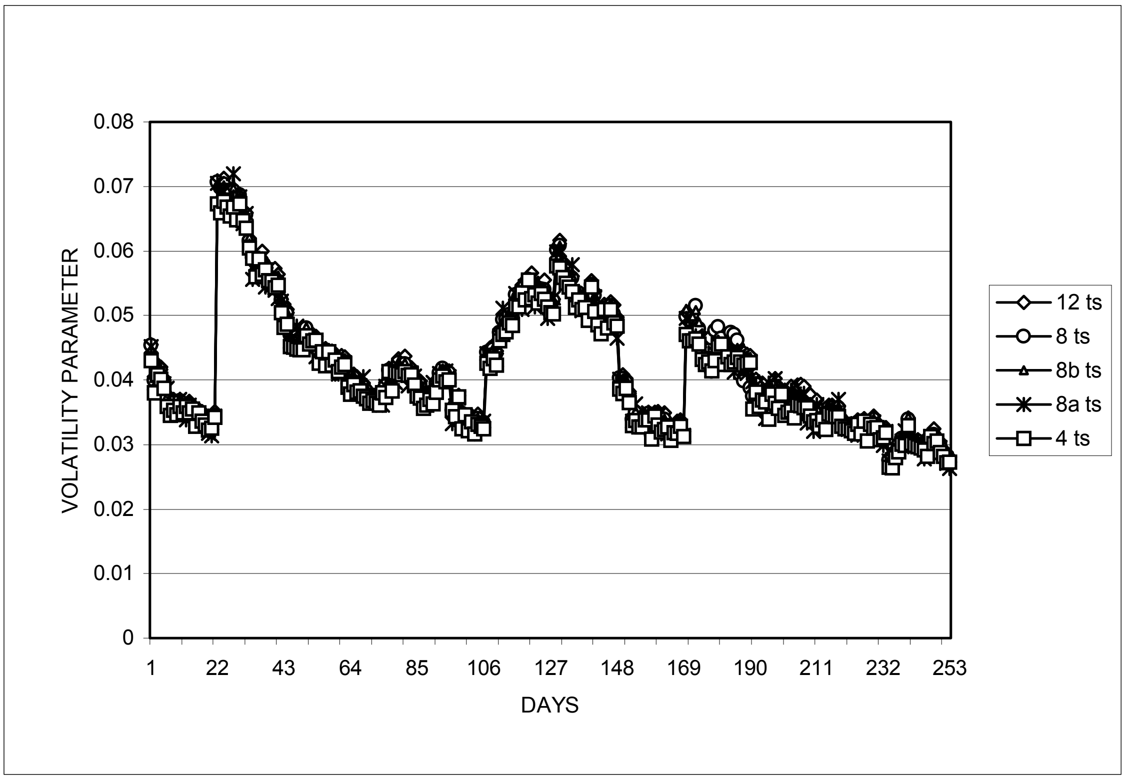


Figure 5.3 Implied Volatility Parameters in Proportional Model

This figure provides a daily time series of implied volatility parameters using the Absolute volatility model. The Y axis denotes the estimated volatility parameter and the X axis denotes the total number of trading days in January from 1987 to 1998. The figure plots the estimated volatility parameter for implementation schemes using 12, 8, 8a, 8b, and 4 time steps, as described in Section 4.2.

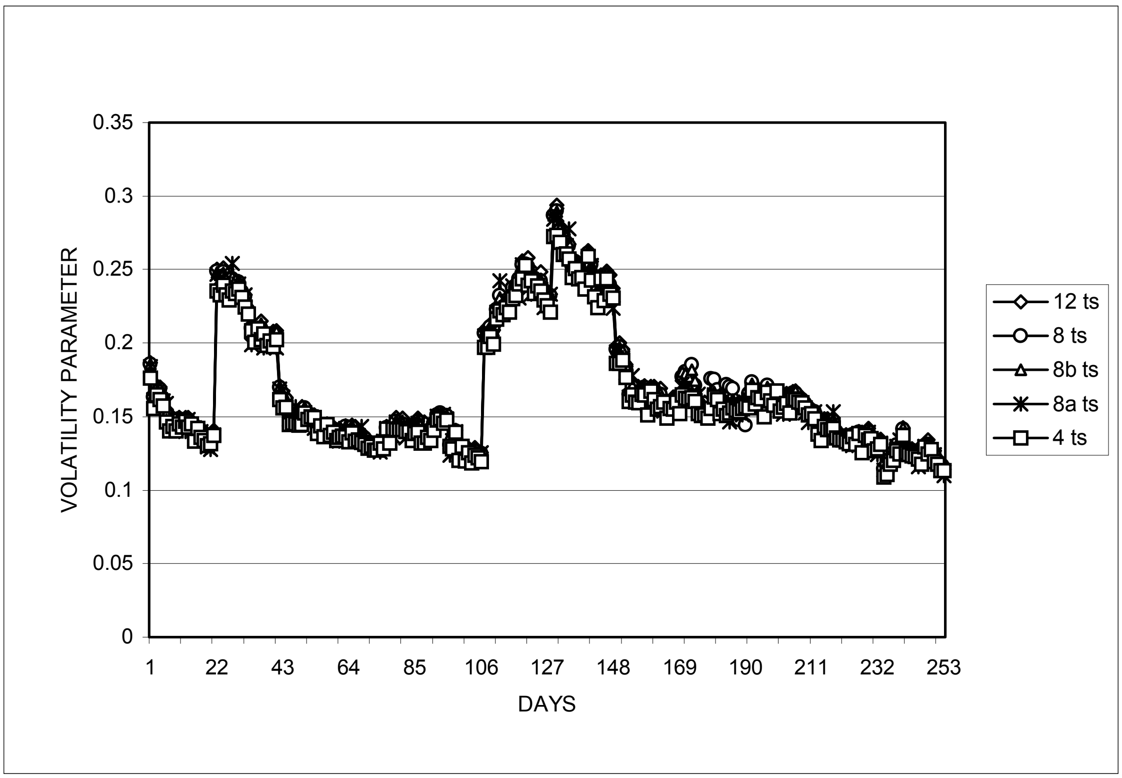


Figure 5.4 Computation Time

This figure shows the computation time required for each of six volatility models using different number of time steps. The Y axis denotes the computation time required to estimate the volatility parameters for each model, while the X axis denotes the six volatility models in each time steps.

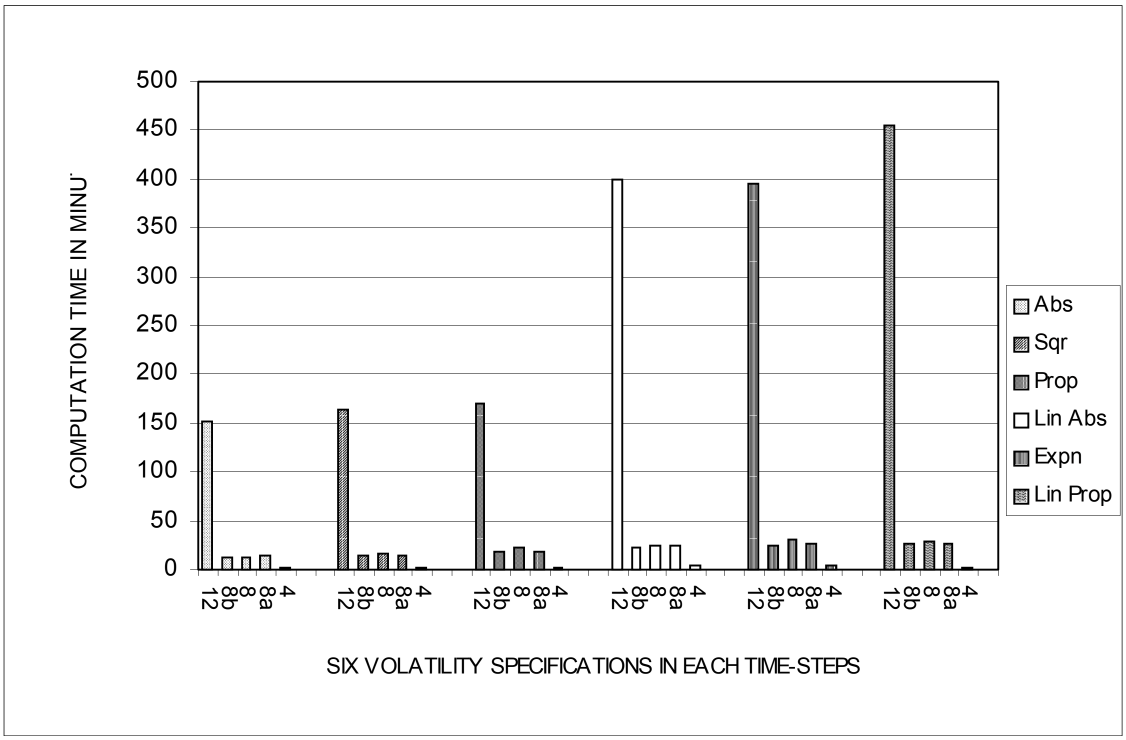


Figure 5.5 Accuracy Versus Time Required

This figure shows the tradeoff between accuracy and computation time required for each of six volatility models using different number of time steps. The Y axis denotes the computation time required to estimate the volatility parameters for each model, while the X axis denotes the average difference between model option price and market option price. 'lpro' in the label means linear proportional model, 'labs' for linear absolute, 'exp' for exponential, 'pro' for proportional, 'sqr' for square root, and 'abs' for absolute model.

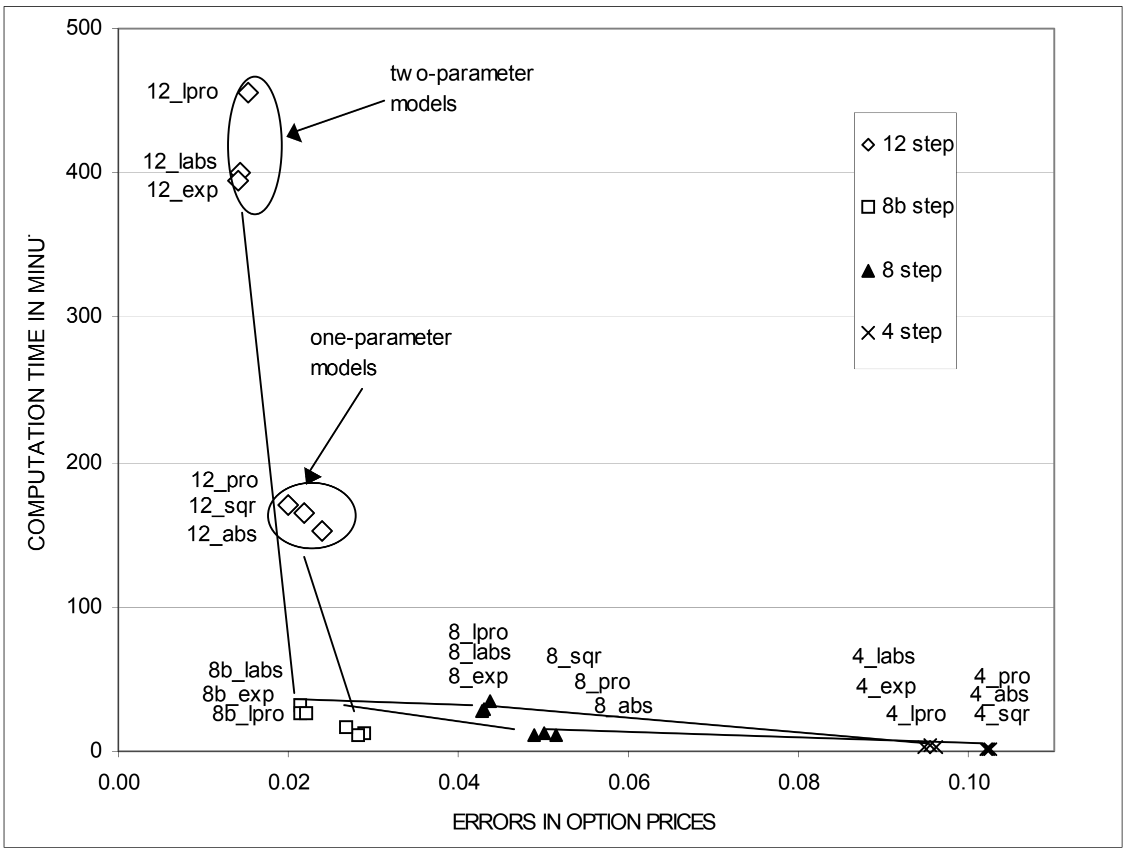


Figure 5.6 Average Absolute Errors

This figure shows the average absolute errors between the model option prices and the market option prices. The X axis represents six volatility models, the Y axis represents time steps taken for each model, and the Z axis represents the option price errors.

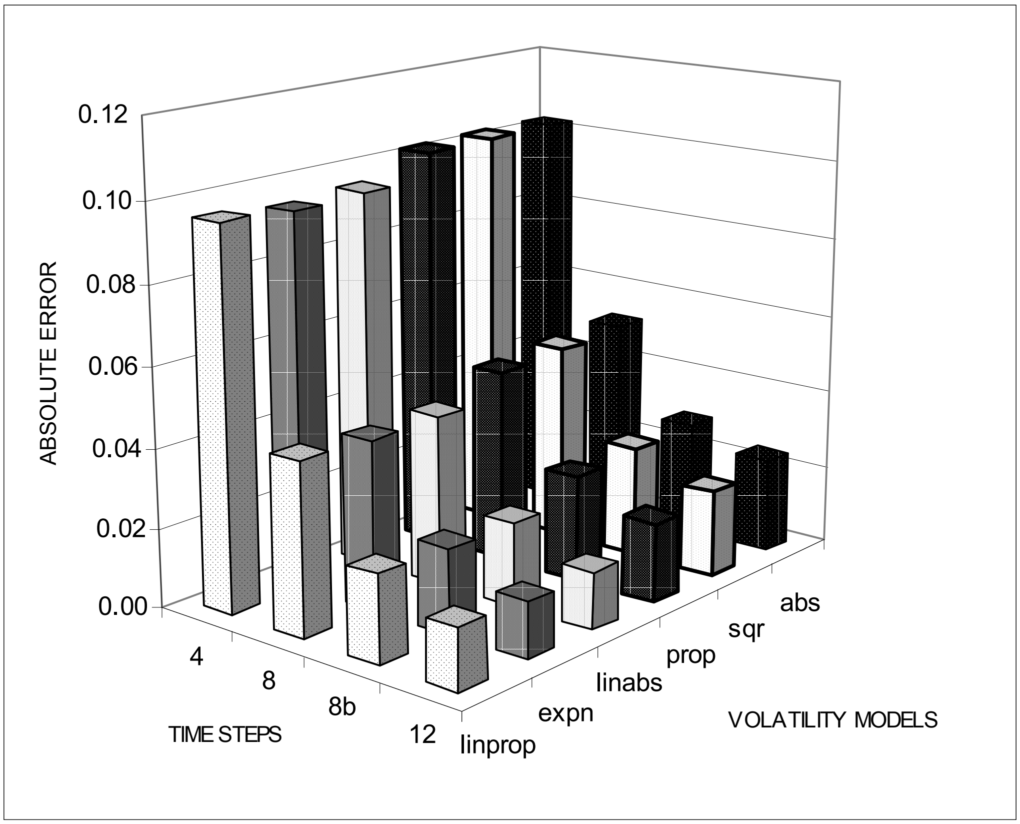


Figure 5.7 Standard Deviation of Average Errors

This figure shows the standard deviation of average errors between the model option prices and the market option prices. The X axis represents six volatility models, the Y axis represents time steps taken for each model, and the Z axis represents the standard deviation of option price errors.

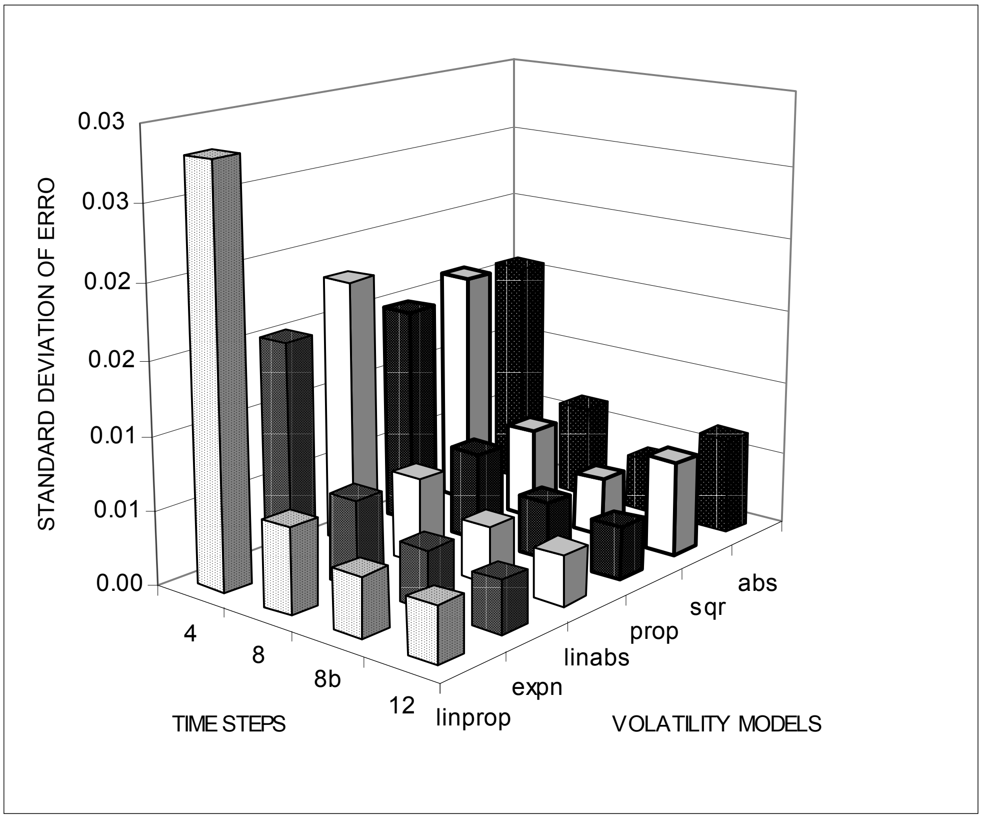


Figure 5.8 Percentiles For Errors in Calls

This figure shows the percentiles for average absolute errors in call option prices. The Y axis represents average absolute errors and the X axis represents six different volatility models. The legend denotes percentiles from the 10th percentile on the top to the 90th percentile on the bottom.

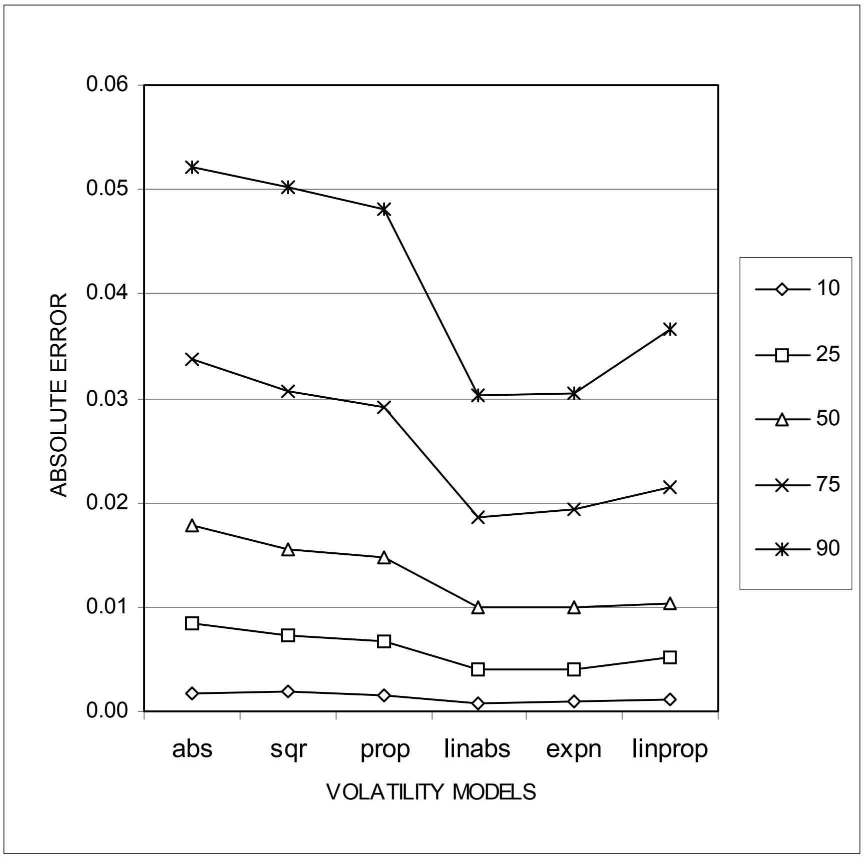


Figure 5.9 Percentiles For Errors in Puts

This figure shows the percentiles for average absolute errors in put option prices. The Y axis represents average absolute errors and the X axis represents six different volatility models. The legend denotes percentiles from the 10^{th} percentile on the top to the 90^{th} percentile on the bottom.

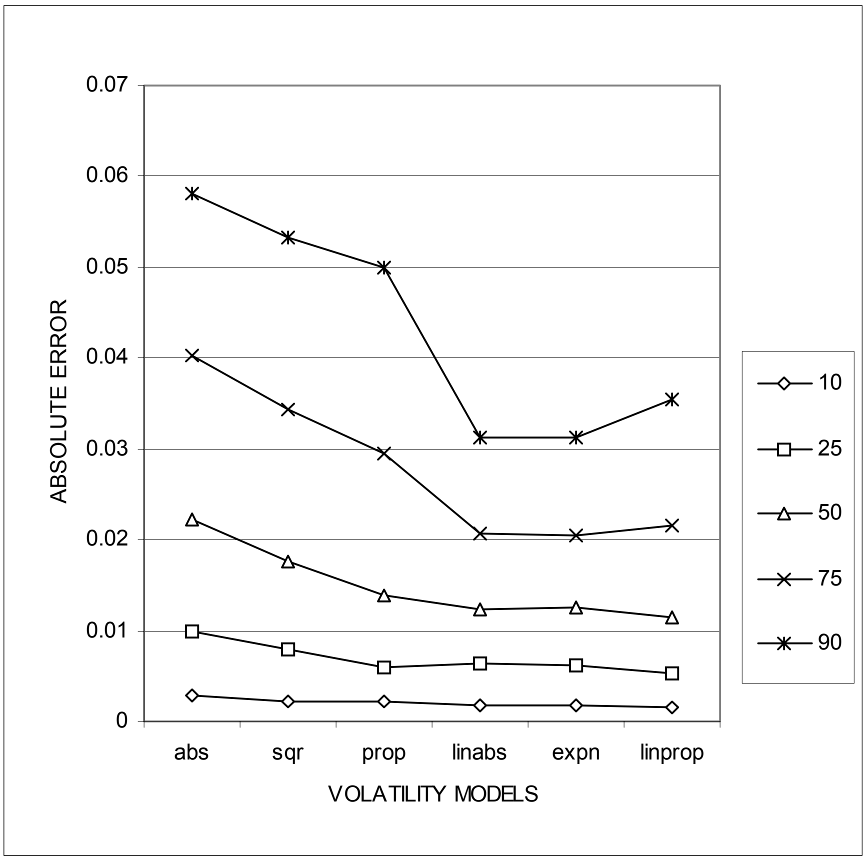


Figure 5.10 Percentiles For Errors in ITMs

This figure shows the percentiles for average absolute errors in ITM option prices. The Y axis represents average absolute errors and the X axis represents six different volatility models. The legend denotes percentiles from the 10th percentile on the top to the 90th percentile on the bottom.

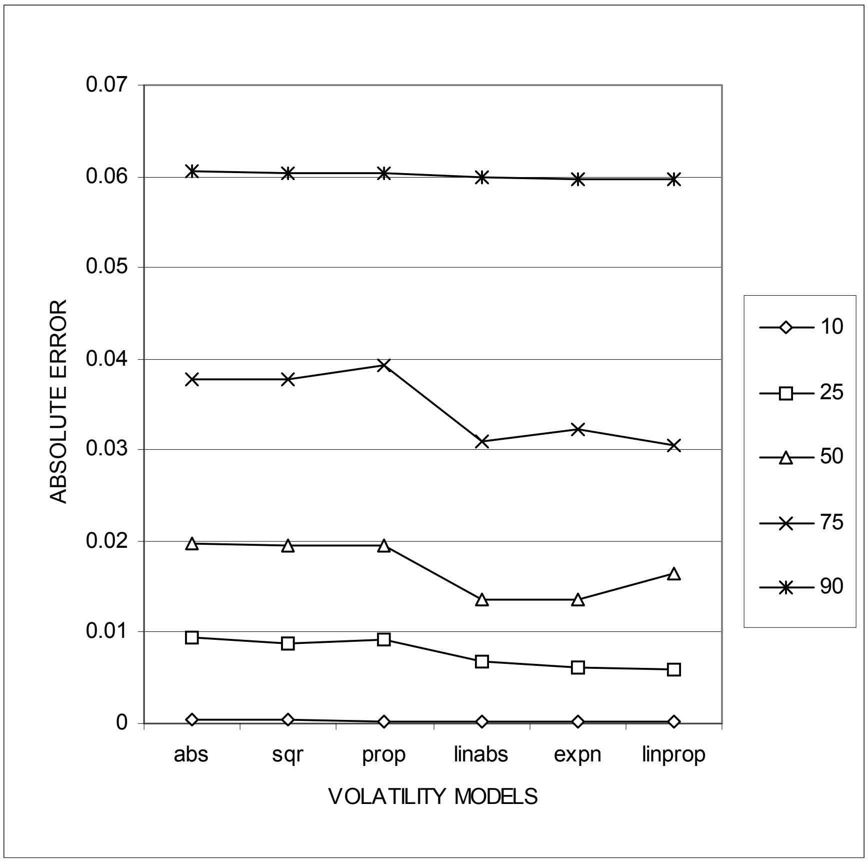


Figure 5.11 Percentiles For Errors in ATMs

This figure shows the percentiles for average absolute errors in ATM option prices. The Y axis represents average absolute errors and the X axis represents six different volatility models. The legend denotes percentiles from the 10th percentile on the top to the 90th percentile on the bottom.

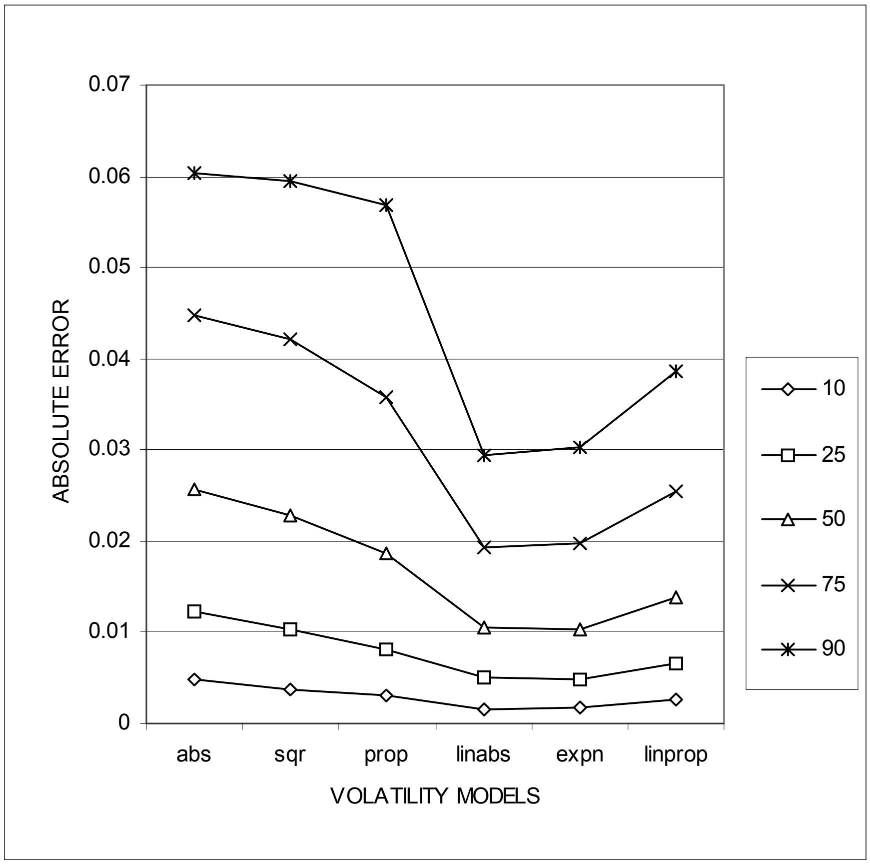


Figure 5.12 Percentiles For Errors in OTMs

This figure shows the percentiles for average absolute errors in OTM option prices. The Y axis represents average absolute errors and the X axis represents six different volatility models. The legend denotes percentiles from the 10th percentile on the top to the 90th percentile on the bottom.

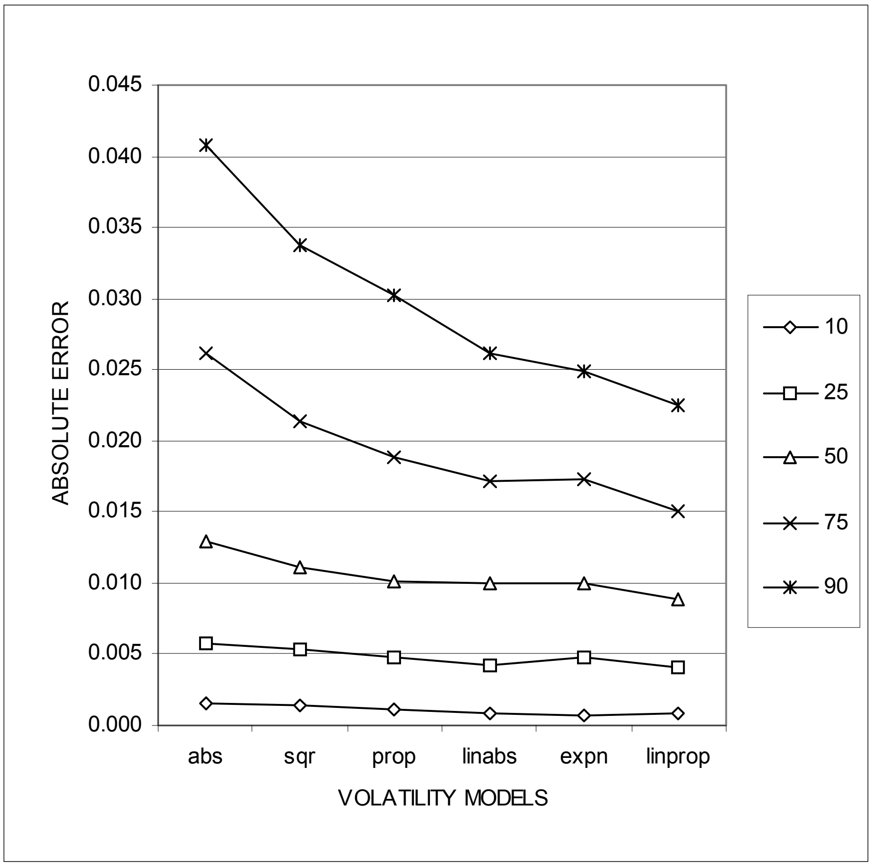


Figure 5.13 Average Errors in ITM Calls

This figure shows the average errors in option pricing for ITM call options. The Y axis denotes average errors in dollar, the X axis denotes six volatility models, and the Z axis denotes the term to maturity with S1 being short-dated and S3 being long-dated.

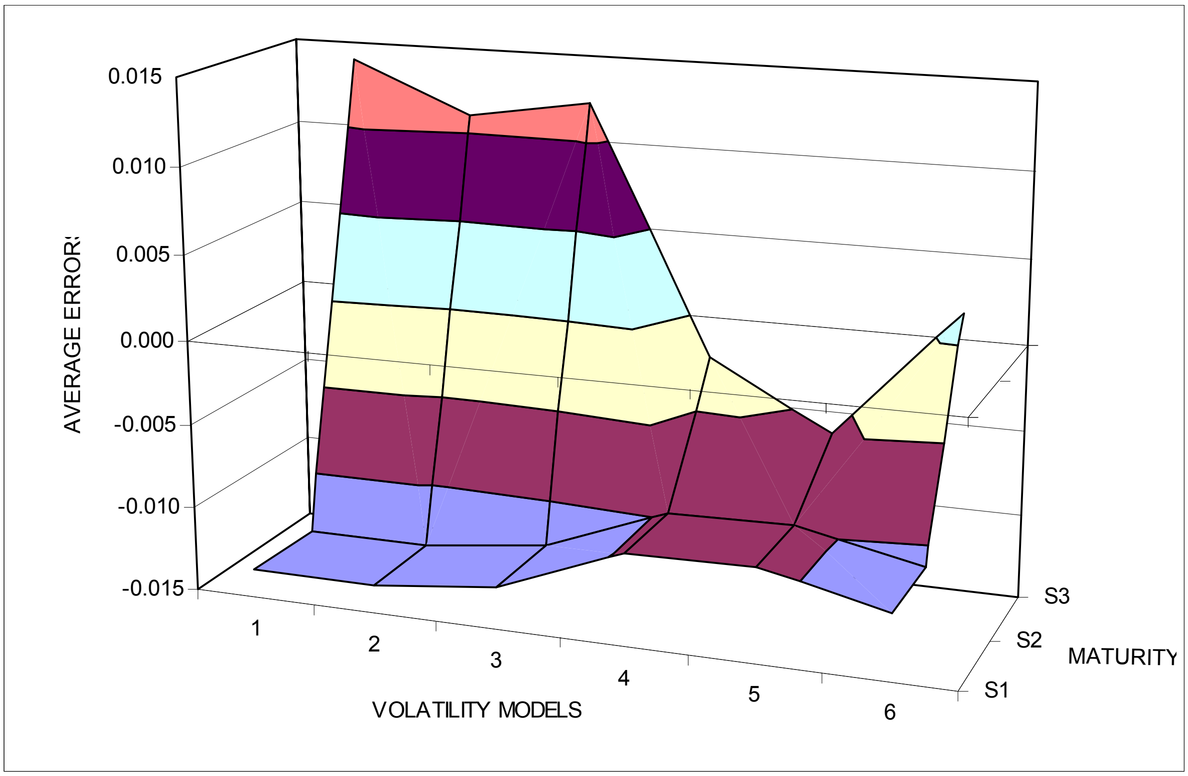


Figure 5.14 Average Errors in ITM Puts

This figure shows the average errors in option pricing for ITM put options. The Y axis denotes average errors in dollar, the X axis denotes six volatility models, and the Z axis denotes the term to maturity with S1 being short-dated and S3 being long-dated.

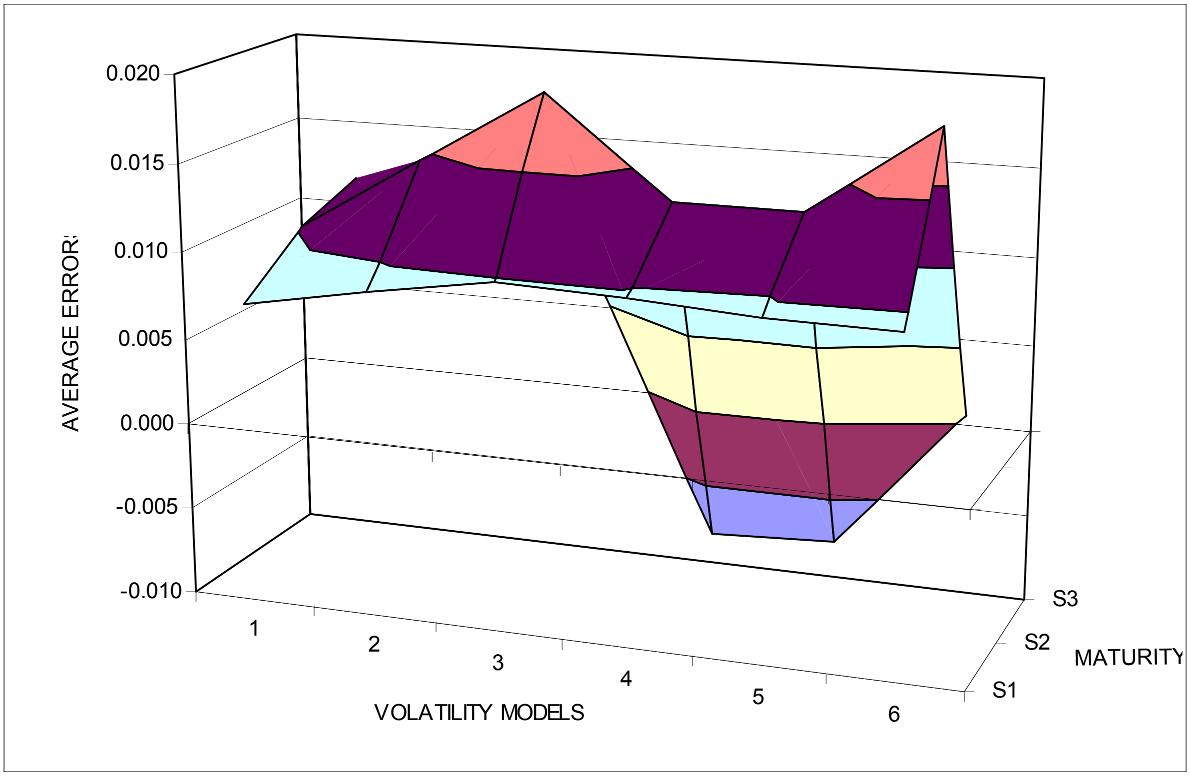


Figure 5.15 Average Errors in ATM Calls

This figure shows the average errors in option pricing for ATM call options. The Y axis denotes average errors in dollar, the X axis denotes six volatility models, and the Z axis denotes the term to maturity with S1 being short-dated and S3 being long-dated.

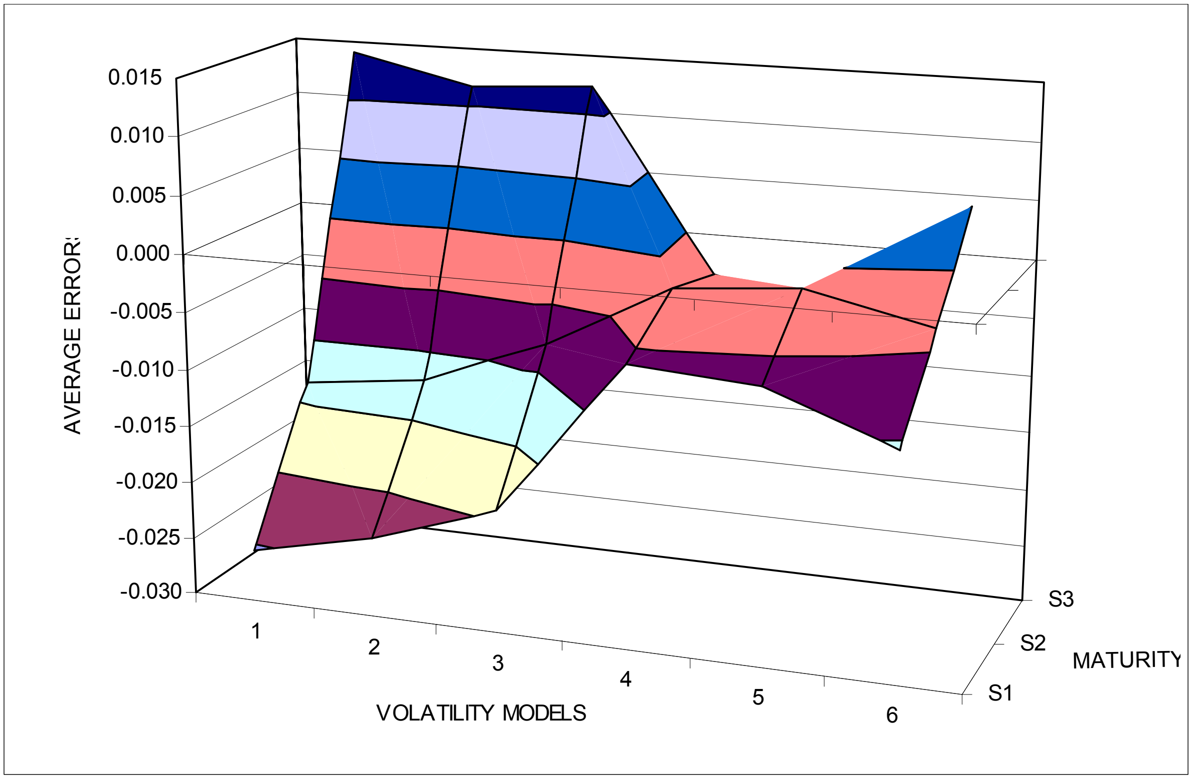


Figure 5.16 Average Errors in ATM Puts

This figure shows the average errors in option pricing for ATM put options. The Y axis denotes average errors in dollar, the X axis denotes six volatility models, and the Z axis denotes the term to maturity with S1 being short-dated and S3 being long-dated.

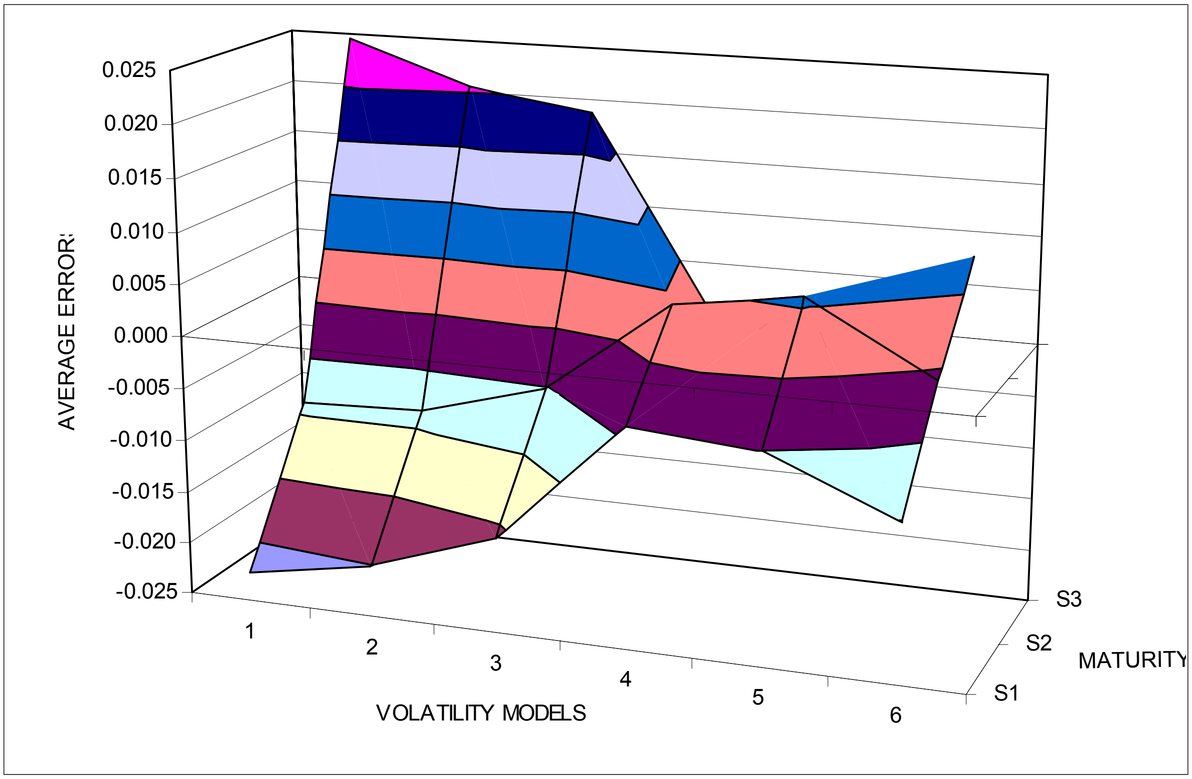


Figure 5.17 Average Errors in OTM Calls

This figure shows the average errors in option pricing for OTM call options. The Y axis denotes average errors in dollar, the X axis denotes six volatility models, and the Z axis denotes the term to maturity with S1 being short-dated and S3 being long-dated.

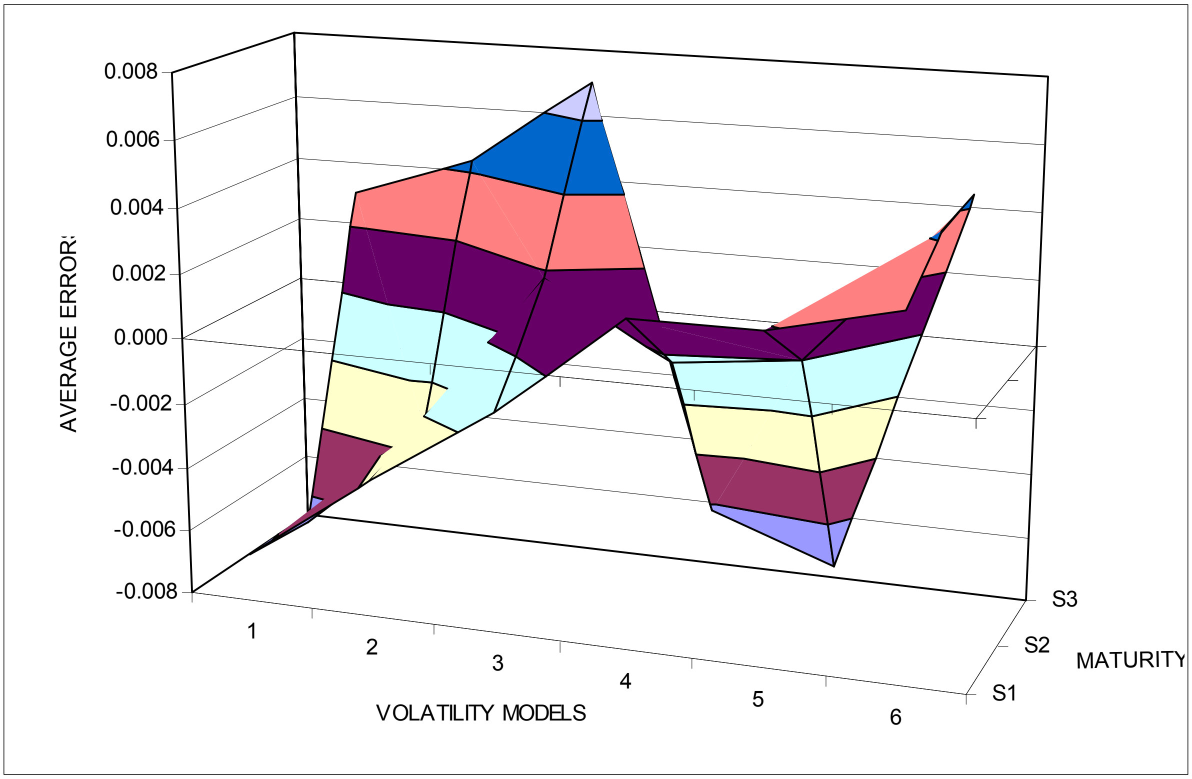
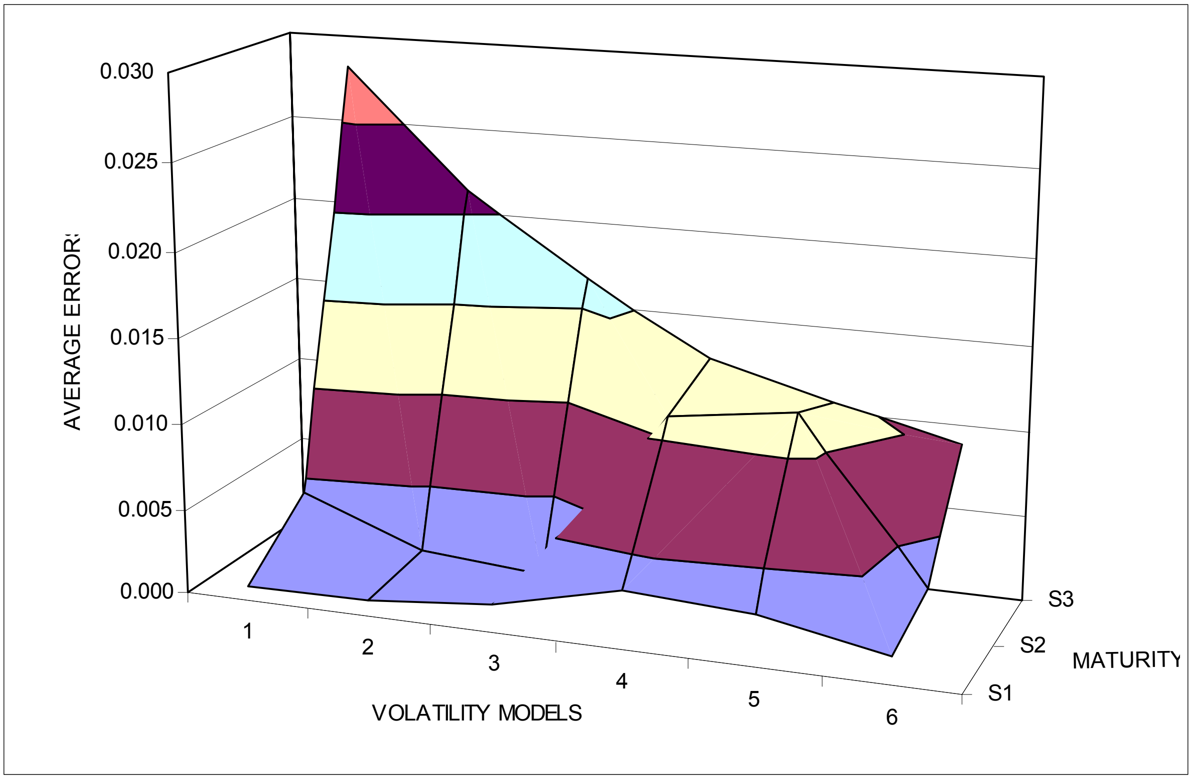
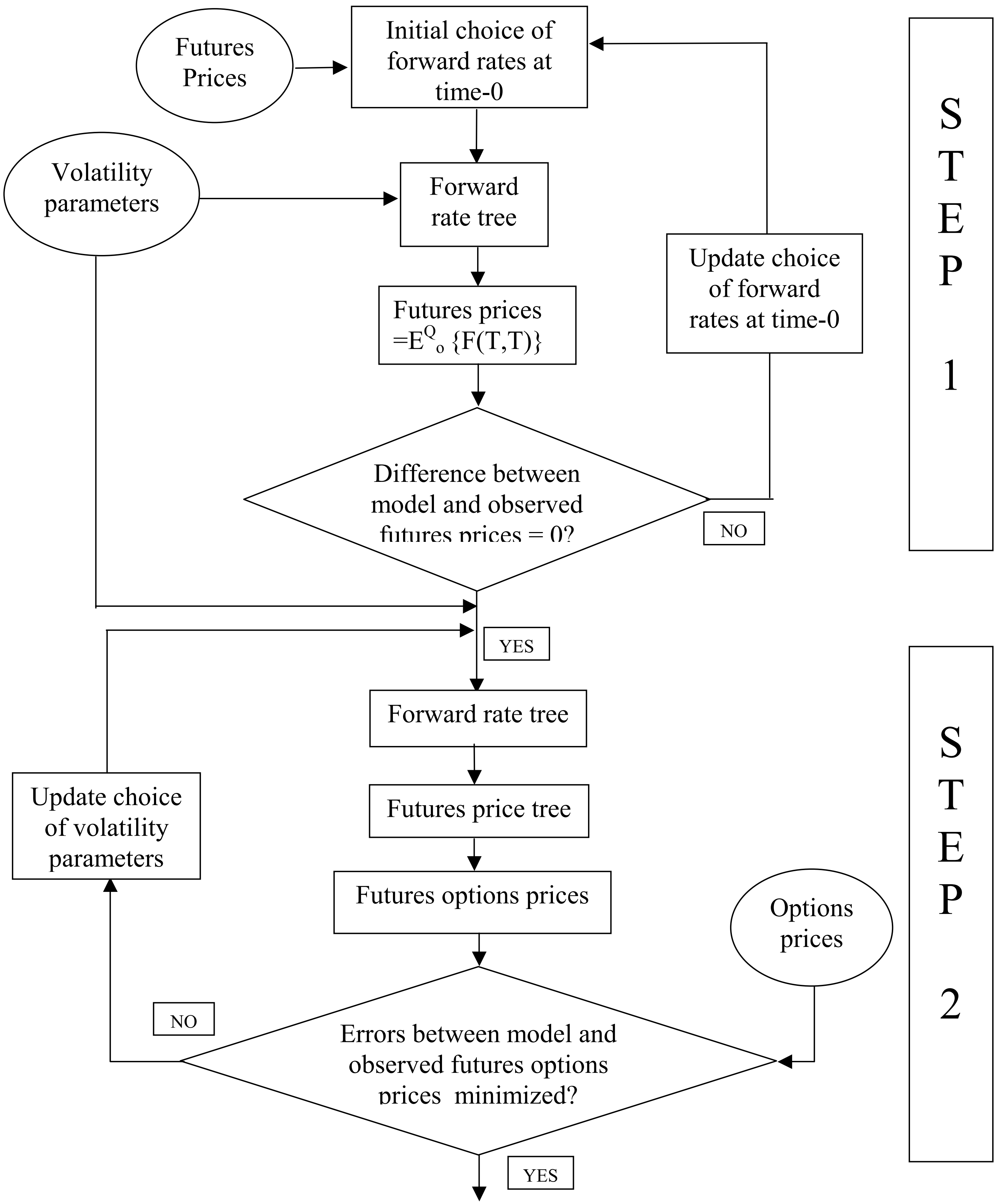


Figure 5.18 Average Errors in OTM Puts

This figure shows the average errors in option pricing for OTM put options. The Y axis denotes average errors in dollar, the X axis denotes six volatility models, and the Z axis denotes the term to maturity with S1 being short-dated and S3 being long-dated.



Appendix A. Flowchart for Empirical Estimation Procedure



Appendix B. Numerical Example

This example explains how to estimate forward rates and volatility parameters from a given set of Eurodollar futures prices and futures options prices using the Proportional volatility specification ($\sigma(.) = \sigma_0 \, f(t, T)$) within the framework of HJM model. For illustration purposes, we use only one futures price and three option prices out of the daily data set for January 12, 1995. The date was picked so that the remaining days up to the next maturity becomes exactly 60 days and we split the period into two sub-periods, with 30 days each, to construct the binomial tree. The input variables are as follows:

$\Delta_1 = (t_1 - t_0) / 365 = 30 / 365$

$\Delta_2 = (t_2 - t_1) / 365 = 30 / 365$

Spot rate $s = f(0, 0)$	: 0.0625
Initial volatility parameter σ_0	: 0.1820[11]
Three month futures price that matures in March F(0, 2)	: 982,875

	Option 1	Option 2	Option 3
Price	45.0	1.0	25.0
Type	call	call	call
Expiration	March	March	March
Strike	9,275	9,375	9,300

Step 1: Construction of Initial Forward Rate Curve

First, we assign the initial values to our forward rates.

$f(0, 0) = 1.0625$

$f(0, 2) = 1.0685 = 4 * (1 – F(0, 2) / 1{,}000{,}000)$

Note that we use the futures rate as our beginning value for forward rate f(0, 2) and use interpolation method to fill in for the intermediate maturity.

From equation (3-19), we have

$$f_\Delta(t+\Delta, T; s_{t+\Delta}) = f_\Delta(t, T; s_t) \frac{\cosh(\Sigma_{j=t+\Delta}^{T} \sigma(.)\sqrt{\Delta})}{\cosh(\Sigma_{j=t+\Delta}^{T-\Delta} \sigma(.)\sqrt{\Delta})} \exp(-\sigma(.)\sqrt{\Delta}) \text{ if } s_t+\Delta = s_t u$$
$$= f_\Delta(t, T; s_t)) \frac{\cosh(\Sigma_{j=t+\Delta}^{T} \sigma(.)\sqrt{\Delta})}{\cosh(\Sigma_{j=t+\Delta}^{T-\Delta} \sigma(.)\sqrt{\Delta})} \exp(\sigma(.)\sqrt{\Delta}) \text{ if } s_t+\Delta = s_t d \tag{3-19}$$

where we define

$\cosh(\Sigma_{j=t+\Delta}^{t} \sigma(t, j: s_t) \sqrt{\Delta}) \equiv 1$

First, we compute the value of the argument within the cosh function.

$\sigma(0, 1)) \sqrt{\Delta_1} = \sigma_0(f(0, 1) – 1))\sqrt{\Delta_1} = 0.182 * 0.0655 * 0.286691 = 0.003418$
$\sigma(0, 2)) \sqrt{\Delta_1} = \sigma_0(f(0, 2) – 1))\sqrt{\Delta_1} = 0.182 * 0.0685 * 0.286691 = 0.003574$

Now, we compute the forward rate for each node at time t_1.

$f(1, 2; u) = f(0, 2) * \cosh(\sigma(0, 1)\sqrt{\Delta_1} + \sigma(0, 2)\sqrt{\Delta_1}) / \cosh(\sigma(0, 1)\sqrt{\Delta_1}) * \exp(-\sigma(0, 2)\sqrt{\Delta_1})$
$= 1.0685 * 1.000024 / 1.000006 * 0.996432 = 1.064708$
$f(1, 2; d) = f(0, 2) * \cosh(\sigma(0, 1)\sqrt{\Delta_1} + \sigma(0, 2)\sqrt{\Delta_1}) / \cosh(\sigma(0, 1)\sqrt{\Delta_1}) * \exp(\sigma(0, 2)\sqrt{\Delta_1})$
$= 1.0685 * 1.000024 / 1.000006 * 1.003581 = 1.072346$

Next, we compute the forward rate for each node at time t_2.

[11] This number comes from the estimate in Amin and Morton (1994).

$f(2, 2; uu) = f(1, 2; u) * \cosh(\sigma(1, 2; u) \sqrt{\Delta_2}) * \exp(-\sigma(1, 2; u) \sqrt{\Delta_2})$

$= 1.064708 * 1.000006 * 0.996629 = 1.061125$

$f(2, 2; ud) = f(1, 2; u) * \cosh(\sigma(1, 2; u) \sqrt{\Delta_2}) * \exp(\sigma(1, 2; u) \sqrt{\Delta_2})$

$= 1.064708 * 1.000006 * 1.003382 = 1.068325$

$f(2, 2; du) = f(1, 2; d) * \cosh(\sigma(1, 2; d) \sqrt{\Delta_2}) * \exp(-\sigma(1, 2; d) \sqrt{\Delta_2})$

$= 1.072346 * 1.000007 * 0.996232 = 1.068313$

$f(2, 2; dd) = f(1, 2; d) * \cosh(\sigma(1, 2; d) \sqrt{\Delta_2}) * \exp(\sigma(1, 2; d) \sqrt{\Delta_2})$

$= 1.072346 * 1.000007 * 1.003782 = 1.076409$

From these forward rates, we calculate the futures price F(2, 2) at each node at time t_2:

$F(2, 2, uu) = 10^6 [1 - 0.25 \times 0.061125] = 984{,}719$

$F(2, 2, ud) = 10^6 [1 - 0.25 \times 0.068325] = 982{,}919$

$F(2, 2, du) = 10^6 [1 - 0.25 \times 0.068313] = 982{,}922$

$F(2, 2, dd) = 10^6 [1 - 0.25 \times 0.076409] = 980{,}898$

Since the futures price is a martingale under the risk-neutral measure,

$F(0, 2) = E^Q_t \{F (2, 2)\} = 982{,}865$

which is different from the market futures price. Using the solver function in Excel, we find the value of f(0, 2) that generates an exact match between the model and the market futures prices. This value is 1.068460. (In our estimation with the complete data set, instead of Excel, we use the fmincon function in Matlab, with the quadratic programming solution algorithm, for solving for the forward rates.)

Now, we have estimated the initial forward rate curve as follows:

$f(0, 0) = 1.062500$

$f(0, 1) = (f(0, 0) + f(0, 2)) / 2 = 1.065480$

$f(0, 2) = 1.068460$

From this curve, we can construct the evolution of two period forward rate curve.

$f(0, 2) = 1.068460$ $f(1, 1; u) = 1.061852$ $f(2, 2; uu) = 1.061089$

$f(1, 1; d) = 1.069133$ $f(2, 2; ud) = 1.068274$

$f(2, 2; du) = 1.068273$

$f(2, 2; dd) = 1.076364$

Step 2: Computation of Volatility Parameters

From the forward rate tree, we can compute futures price tree.

$F(0, 2) = 982{,}875$ $F(1, 1; u) = 983{,}833$ $F(2, 2; uu) = 984{,}728$

$F(1, 1; d) = 981{,}924$ $F(2, 2; ud) = 982{,}931$

$F(2, 2; du) = 982{,}932$

$F(2, 2; dd) = 980{,}909$

Given the futures prices, we can compute the call option prices at time $T=t_2$.

$C(2; uu) = Max(F(2, 2; uu) – K, 0)$

$= Max (984{,}728 – 10^6 * (1 – 0.25 * (1- 9{,}275 / 10{,}000)), 0)$

$= 2{,}853$

$C(2; ud) = Max(F(2, 2; ud) – K, 0)$

$= Max (982{,}931 – 10^6 * (1 – 0.25 * (1- 9{,}275 / 10{,}000)), 0)$

$= 1{,}056$

$C(2; du) = Max(F(2, 2; du) – K, 0)$

$= Max (982{,}932 – 10^6 * (1 – 0.25 * (1- 9{,}275 / 10{,}000)), 0)$

$= 1{,}057$

$C(2; dd) = Max(F(2, 2; dd) – K, 0)$

$= \text{Max} (980{,}909 – 10\text{^}6 * (1 – 0.25 * (1- 9{,}275 / 10{,}000)), 0)$
$= 0$

Now, the call option price at time T-1 is the maximum of option value when option is exercised and the option value when not exercised.

$C(1; u) = \text{Max}(983{,}833 – 10\text{^}6 * (1 – 0.25 * (1 – 9{,}275 / 10{,}000)), (C(2; uu) + C(2; ud)) / 2 * \exp(-f(1, 1; u) * \Delta_2))$
$= \text{Max}(1{,}958, (2{,}853 + 1{,}056) / 2 * \exp(-(1.061852 - 1) * 30 / 365))$
$= 1{,}957.60$
$C(1; d) = \text{Max}(981{,}924 – 10\text{^}6 * (1 – 0.25 * (1 – 9{,}275 / 10{,}000)), (C(2; du) + C(2; dd)) / 2 * \exp(-f(1, 1; d) * \Delta_2))$
$= \text{Max}(49, (1{,}057 + 0) / 2 * \exp(-(1.069133 - 1) * 30 / 365))$
$= 525.40$

Similarly,

$C(0) = \text{Max}(982{,}885 – 10\text{^}6 * (1 – 0.25 * (1 – 9{,}275 / 10{,}000)), (C(1; u) + C(1; d)) / 2 * \exp(-f(0, 0)* \Delta_1))$
$= \text{Max}(49, (1{,}957.60 + 525.40) / 2 * \exp(-(1.0625 - 1) * 30 / 365))$
$= 1{,}235.14$

Now call option value computed from our model give us 1,235.14 / 2,500 = 49.41, which is different from our market option price 45.0. We repeat the process for two other options and find the volatility parameter that minimize the sum of squared errors between our model option prices and market option prices, again using the solver function in Excel.

Thus, we get

$\sigma_0 = 0.139248$.

In principle, it is possible to go back to step 1 with this value of σ_0, re-estimate the forward rate term structure, and iterate. However, we find that the forward rate term structure estimate is not at all sensitive to this iteration.

Vita

Tae Young Park earned a doctorate in Finance from the Virginia Polytechnic Institute & State University in 2001. His research interest is in the fixed income and derivatives area. He began his career in 1986 at Hyundai Research Institute, before joining the Ph.D. in Finance program at the VPI & SU. Tae has an MS in Finance from Drexel University and an MBA from the University of Illinois at Urbana.

Tae was born in Seoul, South Korea and lives with his wife Jung Sook Nam and two sons, Oliver Sungjae and Edward Sungjin in Blacksburg, Virginia.