Chapter Three: Turbulence Model Review

Introduction

The governing equations of Newtonian fluid flow are the Navier-Stokes equations, given here in their full, time-accurate form and assuming Stokes' hypothesis:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_i)}{\partial x_i} = 0 \tag{3.1a}$$

$$\frac{\partial (\rho U_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_i U_j - \tau_{ij}\right) = -\frac{\partial P}{\partial x_i} \tag{3.1b}$$

$$\frac{\partial}{\partial t}\left[\rho\left(e + \tfrac{1}{2} U_i U_i\right)\right] + \frac{\partial}{\partial x_j}\left[\rho U_j\left(h + \tfrac{1}{2} U_i U_i\right)\right] = \frac{\partial}{\partial x_j}\left(U_i \tau_{ij}\right) - \frac{\partial q_j}{\partial x_j} \tag{3.1c}$$

where

$$\tau_{ij} = \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \tfrac{2}{3}\mu \frac{\partial U_k}{\partial x_k}\delta_{ij} \tag{3.2}$$

$$q_j = -k\frac{\partial T}{\partial x_j} \tag{3.3}$$

In multi-species, nonreacting flows the first equation of the above set, the continuity equation, is replaced by the following set[40]:

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial (\rho_i U_j)}{\partial x_j} = -\frac{\partial \left(\rho_i \hat{V}_{ij}\right)}{\partial x_j} \tag{3.4}$$

where ρ_i is the density of species i and $\hat{V}_{ij}$ is the mass diffusion velocity of the species i in the j direction. The mass diffusion vector is usually modeled as

$$\rho_i \overline{\hat{V}}_i = -\rho \mathcal{D}_{l_i} \nabla c_i \tag{3.5}$$

The diffusion coefficient $\mathcal{D}_{l_i}$, sometimes written as $\mathcal{D}_{ij}$ (where the second index indicates the fluid into which species i diffuses), is in theory a complex function that depends not only upon the species i but also upon the composition of the surrounding medium; however, in practice it is often simplified and related to the laminar viscosity via

$$\rho \mathcal{D}_{l_i} = \frac{\mu}{Sc_1} \tag{3.6}$$

Sc_l is the laminar Schmidt number. Computational simplicity often requires the Schmidt number be considered constant for a given flow.

Most numerical simulations do not solve the full, time-accurate Navier-Stokes equations, due to limitations on machine memory and processor time. Instead, flow variables are separated into turbulent-fluctuation and mean-flow components, and the Navier-Stokes equations rewritten accordingly. The basic separation technique is called Reynolds-averaging, and for a given flow variable A,

$$A = \overline{A} + A' \tag{3.7}$$

where

$$\overline{A} \equiv \frac{1}{T_o} \int_{t}^{t+T_o} A(t)dt \tag{3.8}$$

T_o is a time period long with respect to turbulent fluctuations but short with respect to laminar flow evolution, and A' represents the fluctuating part of A. Note that $\overline{A'} = 0$.

In compressible or reacting flows velocity components and thermodynamic variables are averaged somewhat differently, to take into account the changing density. This density-weighted (or Favre) averaging technique uses

$$A = \tilde{A} + A'' \tag{3.9}$$

Here

$$\tilde{A} = \frac{\overline{\rho A}}{\overline{\rho}} \tag{3.10}$$

and

$$\overline{\rho A''} = 0 \tag{3.11}$$

When the density and pressure variables are decomposed in Reynolds form and the velocity components and thermodynamic variables are decomposed in Favre form, the Navier-Stokes equations can themselves be time-averaged. Fluctuating components occuring singly or multiplied by non-fluctuating variables average to zero, as in Equation 3.11 above, but products of fluctuating components do not. The result is as follows:

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho}\tilde{U}_i\right) = 0 \tag{3.12a}$$

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{U}_i\right)+\frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{U}_i\tilde{U}_j-\bar{\tau}_{ij}\right)=-\frac{\partial \bar{P}}{\partial x_i}-\frac{\partial}{\partial x_j}\left(\overline{\rho U''_i U''_j}\right) \qquad (3.12b)$$

$$\frac{\partial}{\partial t}\left(\bar{\rho}\left(\tilde{e}+\tfrac{1}{2}\tilde{U}_i\tilde{U}_i\right)+\tfrac{1}{2}\overline{\rho U''_i U''_i}\right)+\frac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{U}_j\left(\tilde{h}+\tfrac{1}{2}\tilde{U}_i\tilde{U}_i\right)+\tfrac{1}{2}\tilde{U}_j\overline{\rho U''_i U''_i}\right)$$

$$=\frac{\partial}{\partial x_j}\left(\tilde{U}_i\left(\bar{\tau}_{ij}-\overline{\rho U''_i U''_j}\right)\right)-\frac{\partial q_j}{\partial x_j}+\frac{\partial}{\partial x_j}\left(-\overline{\rho h'' U''_j}+\overline{\tau_{ij}U''_i}-\tfrac{1}{2}\overline{\rho U''_j U''_i U''_i}\right) \qquad (3.12c)$$

The species continuity equation can also be averaged, with the following result:

$$\frac{\partial \bar{\rho}_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\bar{\rho}_i\tilde{U}_j\right)=\frac{\partial}{\partial x_j}\left[\mathcal{D}_{l_i}\left(\bar{\rho}\nabla\tilde{c}_i+\overline{\rho\nabla c''_i}\right)\right]-\frac{\partial}{\partial x_j}\left(\overline{\rho_i U''_j}\right) \qquad (3.13)$$

For a complete derivation of the Reynolds-averaged Navier-Stokes equations and a thorough discussion of Reynolds- and Favre-averaging, see References 6 and 41. For information on multi-species and chemically reacting flows, see Reference 40.

Comparison of Equations 3.1 and 3.12 reveals that the continuity equation has not changed form, and that the momentum equations have changed only by the addition of the term $-\overline{\rho U''_i U''_j}$, which is given the symbol τ_{ij}^R. This term, called the Reynolds stress, contains all of the influence of turbulence on the momentum equations but cannot be computed without knowledge of the fluctuating velocity components, which are themselves unknown. Herein lies the center of turbulence modeling: the purpose of any turbulence model is to replace τ_{ij}^R with some known function of mean-flow variables that allows the Reynolds-averaged Navier-Stokes equations to simulate to the desired level of accuracy the properties and behavior of a given fluid-dynamic flow.

Most turbulence models follow the Boussinesq assumption that Reynolds stresses can be modeled by analogy with the shear stresses as

$$\tau_{ij}^R\equiv-\overline{\rho U''_i U''_j}=\mu_T\left(\frac{\partial\tilde{U}_i}{\partial x_j}+\frac{\partial\tilde{U}_j}{\partial x_i}-\tfrac{2}{3}\frac{\partial\tilde{U}_k}{\partial x_k}\delta_{ij}\right)-\tfrac{2}{3}\bar{\rho}\tilde{K}\delta_{ij} \qquad (3.14)$$

μ_T is a turbulent (eddy) viscosity, a model-dependent function which depends not only on the fluid but also on the flow. The last term in the above equation should be included to ensure consistency for i = j, since τ_{kk}^R is equal to twice the turbulence kinetic energy $\bar{\rho}\tilde{K} = \frac{1}{2}\overline{\rho U_k'' U_k''}$ See Reference 6.

The influence of turbulence upon the energy equation can be seen in a number of new terms. On the left-hand side of the equation the turbulence kinetic energy, defined above, appears in both the temporal and spatial derivatives. On the right-hand side, turbulence appears as three new terms. They are the turbulent heat flux vector, $-\overline{\rho h'' U_i''}$, the molecular diffusion vector, $\overline{\tau_{ij} U_i''}$, and the turbulent transport term, $\overline{\frac{1}{2}\rho U_j'' U_i'' U_i''}$.

The usual modeling of turbulent heat transfer term relates it to the mean-flow temperature gradient via

$$-\overline{\rho h'' U_i''} = -k_T \bar{\nabla} T = -\frac{\gamma \mu_T}{Pr_T} \bar{\nabla} \tilde{e} \qquad (3.15)$$

Pr_T is the turbulent Prandtl number. Like the laminar Schmidt number, computational simplicity often dictates that Pr_T be considered a constant for a given flow, especially in production-level CFD codes. However, there is little evidence to support such a simplification, and data which contradicts this simplification does exist[42].

The molecular diffusion and turbulent transport terms are modeled together as

$$\overline{\tau_{ij} U_i''} - \overline{\frac{1}{2}\rho U_j'' U_i'' U_i''} = \left(\mu + \frac{\mu_T}{\sigma_K}\right) \frac{\partial K}{\partial x_j} \qquad (3.16)$$

(See Reference 6) σ_K is another scaling factor like the turbulent Prandtl number, which is usually considered a constant set to 1.0.

The species continuity equation has two new terms,

$$\frac{\partial}{\partial x_j}\left[\mathcal{D}_{l_i}\left(\overline{\rho \nabla c_i''}\right) - \overline{\rho_i U_j''}\right] \qquad (3.17)$$

The first is usually neglected, and the second is modeled with a turbulent diffusion expression similar to the laminar diffusion:

$$\overline{\rho_i U_j''} = -\bar{\rho}\mathcal{D}_{t_i} \nabla \tilde{c}_i \qquad (3.18)$$

with

$$\overline{\rho}\mathcal{D}_{t_i} = \frac{\mu_t}{Sc_t} \tag{3.19}$$

Sc_t is the turbulent Schmidt number, analogous to the turbulent Prandtl number. Like the turbulent Prandtl number and laminar Schmidt number, it is often coded as a constant for a given flow, though there is little or no experimental justification.

So it is that most every unknown term in the conventional modeling of the Favre-averaged Navier-Stokes equations is expressed as a function of the eddy viscosity, and the central problem of turbulence modeling becomes the development of an expression for μ_T. (The problem of modeling turbulence kinetic energy, K, will be addressed shortly.) Turbulence models vary greatly in their level of complexity and generality. Some are quite simple mathematically but rely heavily on experimental data and, as a consequence, are useful only in flows similar to those upon which they are based. Other turbulence models are more general or more accurate, but involve the repeated, expensive solution of complex equations. The best turbulence model for a given simulation is often a compromise between accuracy and cost and depends heavily on the degree of accuracy required for the particular simulation.

The problem of turbulence modeling becomes somewhat more tractable with the introduction of the Morkovin Hypothesis[43], at least at transonic and low supersonic speeds. According to the hypothesis, the relevant speed when considering compressibility effects in turbulence is not the speed of the flow itself, but the speed of the large eddies which (according to the theory) control the rate of turbulent dissipation of energy. So long as the eddies themselves move with subsonic velocities, it is the local mean density that is important to the determination of the turbulent structure, not the fluctuating component of density, which is believed to be relatively small. Since the speed of the large eddies is believed to be subsonic below a freestream Mach number of four or five, the effects of compressibility on turbulence should not be expected in slower flows

and incompressible turbulence models can be used without error if the local mean value of density is used. This hypothesis has been rather widely accepted and will be accepted for this investigation, due in large part to the unavailability of compressible versions of the new RNG turbulence models.

Algebraic Models

The simplest approach to the problem of eddy viscosity closure is the use of mean flow data in an algebraic equation. The eddy viscosity can be modeled directly, or a mixing length formulation can be used. The mixing length formulation developed by Prandtl is mathematically and physically independent of the eddy viscosity concept, but the final expressions can easily be converted from one form to the other.

The mixing length formulation is based upon an analogy between the motion of eddies in a turbulent fluid and the motion of molecules in a gas, as explained by the kinetic theory of gases[6, 44], and in this formulation turbulent eddies are considered to behave like molecules in some hypothetical gas. The mixing length ℓ_m represents an effective interaction distance between eddies, similar to a mean free path of molecules. Consider a simple, two-dimensional boundary layer flow with streamwise mean velocity U. The product $\ell_m \frac{\partial U}{\partial y}$ is equal to the magnitude of the change in U at a vertical location y that results when an eddy moves through a vertical distance ℓ_m, carrying with it the U-velocity of its previous location. This change in U is taken to approximate U'. By continuity (in two dimensions) $V' \approx -U'$, so $-\overline{\rho U'V'} = \rho \ell_m^2 \left|\frac{\partial U}{\partial y}\right|^2$. Clearly, a mixing length model can be converted to an eddy viscosity model by use of the equation

$$\mu_T = \rho \ell_m^2 \left|\frac{\partial U}{\partial y}\right| \tag{3.20}$$

This conversion equation can be extended to more complex flows as well.

Notice that this algebraic formulation is indeed an eddy viscosity model, in that it contains no reference to or expression for turbulence kinetic energy. Indeed, most or all algebraic models assume the turbulence kinetic energy contributions to the Reynolds stress and the various K-dependent terms in the Favre-averaged energy equation are negligible. These assumptions are quite valid for most subsonic flows (for which most algebraic models were derived) but become dubious at higher Mach numbers.

The formulation and historical development of algebraic turbulence models based on both the mixing length and the eddy viscosity concepts are well known and thoroughly documented[6, 44], and will not be repeated here. Most such models, if not all, use at least two equations: an equation for the inner region where the presence of the wall clearly effects the nature of turbulence, and a separate equation for the outer layer beyond the influence of the wall. All models must account for the laminar sublayer, either by adding a third equation or by providing near-wall damping for the wall-region equation.

Despite these multiple-region treatments, algebraic models retain a number of well-known weaknesses, including the lack of a sound basis in fundamental physics. They are semi-empirical models with a number of empirically derived constants, and care certainly should be taken when applying them to classes of flows far different than those used to tune the constants. Nonetheless some, such as the Baldwin-Lomax model, are widely used, in part for their simplicity, robustness, and low computational cost, and in part because more complex and expensive turbulence models are not always more successful in predicting flow behavior, especially for relatively simple wall-bounded flows with small pressure gradients. Because of their widespread use, no turbulence review would be complete without a mention of algebraic models.

Two-Equation Models

A step up in complexity from algebraic models are the one-equation turbulence energy models, most of which solve an additional partial differential equation for turbulence kinetic energy, K, for use in the eddy viscosity as $\mu_T = \rho K^{0.5} l_m$, where l_m is some length scale. (An exception is the Bradshaw model, which assumes $\tau_{ij}^R \propto K$, instead of making the usual Boussinesq approximation. See Reference 6.) These models are somewhat more fundamental in their physical development because they attempt to calculate at least one property of the turbulence and because they allow the inclusion of the K-dependent Reynolds-stress and energy-equation terms neglected by algebraic equations, but in general they are little more accurate than algebraic models. One reason for the lack of improvement is that most one-equation models, like algebraic models, use a length scale based upon distance from the wall. (Once again the Bradshaw model is an exception.) The length scale is calculated via a universal function and is essentially independent of flow conditions. In order to overcome this deficiency many researchers have adopted one of several two-equation models, which solve an additional partial differential equation for some other turbulent variable. The second variable is then used in a dimensional analysis along with K to form a length scale that depends upon the turbulence of the flow, not only the distance from the wall. While a number of parameters have been used as the second variable, perhaps the best-known two-equation models are the $K-\varepsilon$ model, in which the second variable is the turbulence dissipation rate, and the $K-\omega$ model, in which the second variable is the specific dissipation rate, ε/K. For wall-bounded flows such as the ones considered in this investigation the $K-\omega$ model may be superior. However, the $K-\varepsilon$ model has a longer history and is more widely used. Its prevalence in production-level simulation codes warrants its inclusion in this investigation. The $K-\omega$ model is unfortunately not included, due to a lack of sufficient computer time for another set of calculations.

The $K-\varepsilon$ turbulence model has a forty-year history and has been thoroughly discussed by a number of authors. A good, recent review of the subject is found in Wilcox[6].

The derivation of both the K and ε equations rely upon mathematical manipulation of the Navier-Stokes equations. The equations can be derived either in incompressible (Reynolds-averaged) form or compressible (Favre-averaged) form, with the resulting equations being strongly similar. The Favre-averaged K equation contains two terms the Reynolds-averaged K equation does not, the pressure work and pressure dilation terms, but most practical treatments neglect both these terms because insufficient data is available for modeling. For similar reasons, most attempts at modeling the other terms in the Favre-averaged turbulence equations express the compressible terms in forms identical to their incompressible counterparts. The basic form of the Favre-averaged turbulence kinetic energy equation, then, is exactly like the Reynolds-averaged equation, with only the notation changed. (Though a good deal of modeling can and has been done to produce an equation more suited for compressible flows, as will be discussed shortly.) Fortunately, the Morkovin hypothesis[43] discussed above suggests that the error involved in such a simplification should be small below Mach numbers of about five. For these reasons, the Favre-averaged turbulence kinetic energy equation will not be derived here, but will be presented in their final form.

The Reynolds-averaged turbulence kinetic energy equation has the following form[6] before modeling:

$$\overline{\rho}\frac{\partial \overline{K}}{\partial t}+\overline{\rho}\overline{U}_j\frac{\partial \overline{K}}{\partial x_j}=\tau_{ij}^R\frac{\partial \overline{U}_i}{\partial x_j}-\overline{\rho}\overline{\varepsilon}+\frac{\partial}{\partial x_j}\left[\mu\frac{\partial \overline{K}}{\partial x_j}-\tfrac{1}{2}\overline{\rho U_i' U_i' U_j'}-\overline{P' U_j'}\right] \qquad (3.21)$$

where the dissipation per unit mass (or simply dissipation) is

$$\overline{\varepsilon}\equiv\frac{\mu}{\overline{\rho}}\overline{\frac{\partial U_i'}{\partial x_k}\frac{\partial U_i'}{\partial x_k}} \qquad (3.22)$$

The triple correlation in Equation 3.21 represents turbulent transport, and the last term in the same equation is the pressure diffusion term, also a form of turbulent transport. These two terms,

along with the Reynolds stress term, represent additional unknowns and must be modeled. The dissipation itself is determined from another transport equation, also derived by taking a moment of the Navier-Stokes equation, grouping terms, and modeling all additional unknown quantities that result. Before modeling, the Reynolds-averaged dissipation equation has the following form:

$$\begin{aligned}\bar{\rho}\frac{\partial\bar{\varepsilon}}{\partial t}+\bar{\rho}\bar{U}_j\frac{\partial\bar{\varepsilon}}{\partial x_j} &= -2\mu\left[\overline{U'_{i,k}U'_{j,k}}+\overline{U'_{k,i}U'_{k,j}}\right]\frac{\partial\bar{U}_i}{\partial x_j} \\ &\quad -2\mu\overline{U'_kU'_{i,j}}\frac{\partial^2\bar{U}_i}{\partial x_k\partial x_j}-2\mu\overline{U'_{i,k}U'_{i,m}U'_{k,m}}-2\frac{\mu^2}{\bar{\rho}}\overline{U'_{i,km}U'_{i,km}} \\ &\quad +\frac{\partial}{\partial x_j}\left[\mu\frac{\partial\bar{\varepsilon}}{\partial x_j}-\mu\overline{U'_jU'_{i,m}U'_{i,m}}-2\frac{\mu}{\bar{\rho}}\overline{P'_{j,m}U'_{j,m}}\right] \qquad (3.23)\end{aligned}$$

Unlike the turbulence kinetic energy equation, in which the modeling assumptions are at least to some degree based on data, the dissipation equation contains multiple additional double and triple correlations for which there is little if any reliable experimental data. Therefore, all modeling must be done without a significant physical basis. Indeed, the lack of experimental data for modeling renders all dissipation-equation models at least somewhat suspect. This suspicion is one of the more common criticisms of the $K-\varepsilon$ turbulence model.

The so-called "standard" $K-\varepsilon$ model, with the necessary modeling done and adjustable constants fixed, is based upon work reported in two papers, Jones and Launder[45] and Launder and Sharma[46]. As mentioned above, much work in modeling the equations has been done by many researchers, to develop a compressible $K-\varepsilon$ model and one that will give better results than the standard model in the near-wall region. One of the resulting models, the Chien $K-\varepsilon$ model[47, 48], was chosen for use in this investigation. When expressed in Favre-averaged form, the Chien $K-\varepsilon$ model calculates the eddy viscosity needed for the Navier-Stokes equations as $\mu_T=\bar{\rho}C_\mu\tilde{K}^2\left(1.0-\exp\left(-0.0115y^+\right)\right)/\tilde{\varepsilon}$, and the turbulence quantities are calculated from the following equations:

$$\frac{\partial\left(\bar{\rho}\tilde{K}\right)}{\partial t}+\frac{\partial\left(\bar{\rho}\tilde{K}\tilde{U}_j\right)}{\partial x_j}=\underline{P}-\bar{\rho}\tilde{\varepsilon}\left(1+\frac{2\mu\tilde{K}}{\bar{\rho}y^2\tilde{\varepsilon}}\right)\left(1+\frac{\tilde{K}}{a^2}\right)+\frac{\partial}{\partial x_j}\left[\left(\mu+\mu_T/\sigma_k\right)\frac{\partial\tilde{K}}{\partial x_j}\right] \tag{3.24}$$

$$\frac{\partial\left(\bar{\rho}\tilde{\varepsilon}\right)}{\partial t}+\frac{\partial\left(\bar{\rho}\tilde{U}_j\tilde{\varepsilon}\right)}{\partial x_j}=C_{\varepsilon 1}\frac{\tilde{\varepsilon}}{\tilde{K}}\underline{P}-\bar{\rho}\frac{\tilde{\varepsilon}^2}{\tilde{K}}\left(C_{\varepsilon 2}E+\frac{2\mu F\tilde{K}}{\rho y^2\tilde{\varepsilon}}\right)+\frac{\partial}{\partial x_j}\left[\left(\mu+\mu_T/\sigma_\varepsilon\right)\frac{\partial\tilde{\varepsilon}}{\partial x_j}\right] \tag{3.25}$$

where

$$\underline{P}=\frac{\partial\tilde{U}_i}{\partial x_j}\tau_{ij}^R \tag{3.26}$$

$$E=1-\tfrac{2}{9}\exp\left(-\bar{\rho}^2\tilde{K}^4/36\mu^2\tilde{\varepsilon}^2\right) \tag{3.27}$$

$$F=\exp\left(-0.5y^+\right) \tag{3.28}$$

$C_{\varepsilon 1}=1.35$ $\quad C_{\varepsilon 2}=1.8$ $\quad C_\mu=0.09$

$\sigma_k=1.0$ $\quad \sigma_\varepsilon=1.3$

The symbol τ_{ij}^R in the equations for K and ε is the same Reynolds stress encountered in the Navier-Stokes equations, but it is not necessarily modeled in the same way. The current treatment of Reynolds stress as it appears in the turbulence equations is the Boussinesq approximation (once again, using Favre variables),

$$\tau_{ij}^R=-\tfrac{2}{3}\delta_{ij}\bar{\rho}\tilde{K}+\mu_T\left(\frac{\partial\tilde{U}_i}{\partial x_j}+\frac{\partial\tilde{U}_j}{\partial x_i}-\tfrac{2}{3}\delta_{ij}\frac{\partial\tilde{U}_k}{\partial x_k}\right) \tag{3.29}$$

As mentioned above, $K-\varepsilon$ turbulence models suffer from the necessity of modeling a number of quantities for which there is little if any reliable experimental data. While this necessity is a fundamental weakness of the $K-\varepsilon$ approach to turbulence modeling, it is by no means the only one. A further and perhaps more fundamental uncertainty lies in the assumption that turbulence kinetic energy and its dissipation rate are necessary and sufficient turbulence variables in the mathematical simulation of turbulent flows. Nonetheless, the model is widely used and has been attributed with some significant simulation successes. No matter what its weaknesses or uncertainties, the $K-\varepsilon$ model will likely remain popular for quite some time.

Renormalized Group Theory Models

One of the common complaints leveled against $K-\varepsilon$ turbulence models is that they tend to be very *ad hoc*: the equations are derived from the Navier-Stokes equations with no new physics and little empirical data available to support the modeling of the numerous intractable terms. Furthermore, there are a number of empirical "constants" whose values are assigned differently by different researchers[6]. The Renormalized Group Theory (RNG) is a mathematical technique that has been used in recent years to attempt to overcome this deficiency by providing a sound mathematical and physical derivation of the $K-\varepsilon$ and other common turbulence models, including mathematically derived constant coefficients to replace emperical ones. The details of RNG are very complex; however a brief overview need not be intractable, and the resulting $K-\varepsilon$ turbulence model is not nearly as complicated as the theory used in its derivation.

RNG as applied to turbulence modeling can be summarized as follows: The process begins by performing a Fourier transform on the Navier-Stokes equations, so that the primary variable is the wavenumber of the flow structures. (Wavenumber is the reciprocal of wavelength. The wavenumber range extends from zero in the long-wavelength limit up through the smallest resolvable structures, and further through the turbulent spectrum to the high-wavenumber Kolmogorov microscale.) The velocity field is then decomposed by wavenumber into a number of bands, and an iterative process is used to express the flow effects of each wavenumber band as a function of the adjacent, smaller-wavenumber variables. In this way the high wavenumber bands (corresponding to flow features which are too small to be resolved) are systematically removed from the Fourier-space equations and replaced by expressions of lower wavenumber variables that reproduce the high-wavenumber effects. The process is continued until the only velocity bands remaining correspond to resolvable flow features, at which point the equations can be inverse Fourier-

transformed to return them to conventional form. To the accuracy of the mathematics involved at each step, the resulting equations contain all the effects of the subgrid-scale turbulence, expressed as functions of resolvable (meanflow) variables. Turbulence models from algebraic eddy viscosity to Reynolds stress formulations can be created in this manner, but we will consider only the RNG $K-\varepsilon$ models.

The first known attempt to apply RNG analysis to the Navier-Stokes equations to derive a $K-\varepsilon$ turbulence model was reported by Forster *et. al.* in 1977[49], which was followed by a good deal of work beginning in the mid nineteen-eighties[50-55]. Central to these derivations were the introduction of a forcing function into the momentum equation to generate a velocity field described by the Kolmogorov energy spectrum[50] and the introduction of a small parameter as an exponent in the forcing function. The theory was developed in the limit for the parameter much less than one, but recovery of the Kolmogorov spectrum required the parameter be set to four[56, 57]. A further limitation of this theory is that it can account only for nonlocal wavespace interactions. That is, modeling is made only of the interactions of subgrid (unresolvable) structures with those whose wavenumbers approach zero (far larger than the smallest grid-resolvable structures)[58], even though evidence has been offered suggesting that most of the energy leaving grid-resolvable structures is transferred between structures with wavenumbers that differ only slightly[59]. The result is a $K-\varepsilon$ turbulence model that is perhaps as good as or better than traditional, *ad hoc* $K-\varepsilon$ models for certain applications[60], but which fails to reach the original goal of providing the $K-\varepsilon$ turbulence model with a rational mathematical basis.

The practical differences between traditional $K-\varepsilon$ models and these RNG-based models are slightly different values for the (previously empirical) constants and a nonlinear addition to the Reynolds stress tensor used in the turbulence kinetic energy and

dissipation equations. This addition is a consequence of the nonlocal interactions and has the following form[61] (using Favre notation):

$$\underline{P} = \frac{\partial \tilde{U}_i}{\partial x_j}\left(\tau_{ij}^{R} + \tau_{ij}^{R^*}\right) \tag{3.30}$$

$$\frac{\tau_{ij}^{R^*}}{\bar{\rho}} = -\frac{\tilde{K}^3}{\tilde{\varepsilon}^2}\left[C_{\tau 1}\left(\frac{\partial \tilde{U}_i}{\partial x_\alpha}\frac{\partial \tilde{U}_j}{\partial x_\alpha}\right)^* + C_{\tau 2}\left(\frac{\partial \tilde{U}_i}{\partial x_\alpha}\frac{\partial \tilde{U}_\alpha}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_\alpha}\frac{\partial \tilde{U}_\alpha}{\partial x_i}\right)^* + C_{\tau 3}\left(\frac{\partial \tilde{U}_\alpha}{\partial x_i}\frac{\partial \tilde{U}_\alpha}{\partial x_j}\right)^*\right] \tag{3.31}$$

with

$$C_{\tau 1}=0.034 \qquad C_{\tau 2}=0.104 \qquad C_{\tau 3}=-0.014$$

A different approach to the renormalized group theory is the so-called recursion-RNG approach, introduced first by Rose in 1977[62] and further examined by Zhou *et. al.*[7, 59, 63-65]. Recursion-RNG is different in that it eliminates entirely the use of the forcing function and its associated small exponential parameter. An additional feature of recursion-RNG is its ability to model the effects of energy transfer between grid-resolvable structures and unresolved structures of similar but slightly higher wavenumbers, in addition to the nonlocal wavespace interactions[7]. These local wavespace interactions lead to the inclusion of two new, anisotropic, higher-order terms to the Reynolds stress equation, in addition to those due to nonlocal interactions given in Equation 3.31 above. The full equation for turbulence production is then given by Equation 3.30 with

$$\frac{\tau_{ij}^{R^*}}{\bar{\rho}} = -\frac{\tilde{K}^3}{\tilde{\varepsilon}^2}\left[C_{\tau 1}\left(\frac{\partial \tilde{U}_i}{\partial x_\alpha}\frac{\partial \tilde{U}_j}{\partial x_\alpha}\right)^* + C_{\tau 2}\left(\frac{\partial \tilde{U}_i}{\partial x_\alpha}\frac{\partial \tilde{U}_\alpha}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_\alpha}\frac{\partial \tilde{U}_\alpha}{\partial x_i}\right)^* + C_{\tau 3}\left(\frac{\partial \tilde{U}_\alpha}{\partial x_i}\frac{\partial \tilde{U}_\alpha}{\partial x_j}\right)^*\right] \tag{3.32}$$

$$+C_{R1}\frac{\tilde{K}^4}{\tilde{\varepsilon}^3}\left[\frac{\partial \tilde{U}_i}{\partial x_\alpha}\frac{\partial \tilde{U}_j}{\partial x_\beta}\frac{\partial \tilde{U}_\beta}{\partial x_\alpha} + i \leftrightarrow j\right] + C_{R2}\frac{\tilde{K}^7}{\tilde{\varepsilon}^5}\left[\frac{\partial \tilde{U}_i}{\partial x_\alpha}\frac{\partial}{\partial x_j}\left(\frac{\partial \tilde{U}_\beta}{\partial x_\gamma}\frac{\partial^2 \tilde{U}_\gamma}{\partial x_\alpha \partial x_\beta}\right) + i \leftrightarrow j\right]$$

(Once again using Favre notation. The symbol $i \leftrightarrow j$ implies a term similar to the preceding one but with the indices i and j interchanged.) The coefficients C_{R1} and C_{R2} are in theory complicated functions of the flow quantities, including turbulence kinetic energy, dissipation rate, strain rate, and rotation rate, but for the sake of simplicity and following Zhou *et. al.*[7] they are considered to be

constants with the values $25.0*10^{-3}$ and $-0.0352*10^{-3}$, respectively. The rest of the standard $K-\varepsilon$ turbulence model remains unchanged.

While RNG can be used to calculate in closed form all the constant coefficients needed for the $K-\varepsilon$ turbulence model, Zhou, *et. al.* in their recent work[7] have chosen to use the more common empirical values for all constants except those in the nonlinear terms of the Reynolds stress equation. The motivation, presumably, is to allow the specific effects of the nonlinear terms to be isolated from those of altered constants. Furthermore, the RNG $K-\varepsilon$ turbulence model of Zhou, *et. al.* has been developed for incompressible flow, as have been many others. No compressible RNG $K-\varepsilon$ turbulence model is available at this time, though in light of the Morkovin hypothesis discussed above the incompressible form of RNG $K-\varepsilon$ is presumed to be worthy of trial in non-hypersonic compressible flows.

A number of unanswered questions remain about the validity and usefulness of the renormalized group theory and its application to turbulence modeling. The theory itself is mathematically very complex and in general poorly understood. Its application to turbulence modeling is a work in progress rather than a completed project, and much time and effort will be necessary either to substantiate fully the theory or to dismiss it completely. While the recursion-RNG approach has avoided many of the mathematical contradictions present in earlier forms, it is still too early in the development of RNG turbulence theory to predict whether it will accomplish the rather broad objectives of improving the performance of $K-\varepsilon$ and other turbulence models while at the same time providing them with a sound mathematical basis. Detailed analysis of the mathematics of RNG-based turbulence models is beyond the scope of this work, but RNG $K-\varepsilon$, like any other turbulence model, is finally judged by the degree of success with which it predicts turbulent flows of practical aerodynamic interest. It is to this end that an RNG turbulence model is included in this study.

The RNG $K-\varepsilon$ turbulence model chosen for this study is that of Zhou *et. al.*[7] Three variations were implemented, all modifications of the Chien $K-\varepsilon$ model. The first, referred to hereafter as "full" RNG, differs from the "standard" Chien $K-\varepsilon$ model only in that Reynolds stresses are given now by Equations 3.30 and 3.32 instead of Equation 3.26 and 3.29. Note that the Navier-Stokes equations themselves are unchanged.

The second variation, hereafter called "fast" RNG, is exactly like the first except that in "fast" RNG the last term in Equation 3.32 (the C_{R2} term) is omitted. The omitted term is of a higher order and requires the calculation of new derivatives at significantly greater expense than the lower-order terms. "Fast" RNG was implemented to investigate the relative importance and cost of the omitted term.

The third variation of RNG is called "mixing" RNG and unlike the others involves modification of the meanflow continuity equations themselves. In particular, mixing RNG involves a change in the treatment of the turbulent diffusion and heat flux.

As explained above, the usual treatment of the turbulent diffusion term in the species continuity equation is by analogy with the laminar diffusion, thus replacing the rather complex expression with a ratio of the turbulent viscosity to a constant turbulent Schmidt number. (See Equations 3.13 and 3.16-3.18) This technique is most certainly too simple to be accurate. A proper RNG treatment of the turbulent diffusion term would involve the application of the renormalized group theory to the species continuity equation itself, a prodigious undertaking which has not yet been attempted and which is beyond the scope of this investigation.

The effects of turbulence on the energy equation come about from two sources, the Reynolds stress term itself and the turbulent heat flux. (See Equations 3.12 and 3.15) The turbulent heat flux is like the turbulent diffusion term in that it, too, is a complex and unknown function usually represented by a simple expression dependent upon the turbulent viscosity and an adjustable constant. Once again the proper RNG treatment would require extensive

mathematical manipulation in wavespace of the entire equation, which has not been accomplished and will not be attempted as part of the investigation.

Until a technique such as RNG can be applied to derive a rigorous expression for turbulent diffusion and turbulent heat flux, a reasonable substitute would be to continue to use the traditional form of the expressions, as given in Equation 3.18 and 3.15 respectively, but to modify the value of the turbulent viscosity to reflect the effect of RNG analysis of the turbulence equations themselves. The turbulence model referred to as "mixing" RNG accomplishes that task as explained below.

As stated above, the effect of RNG upon the turbulence equations comes through the turbulence production term given in Equation 3.30. Standard $K-\varepsilon$ has the turbulence production term

$$\underline{P} = \frac{\partial \tilde{U}_i}{\partial x_j}\left(-\tfrac{2}{3}\overline{\rho}\tilde{K}\delta_{ij} + \mu_T\left(\frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_i} - \tfrac{2}{3}\delta_{ij}\frac{\partial \tilde{U}_k}{\partial x_k}\right)\right) \tag{3.33}$$

and RNG has a production term with the form

$$\underline{P} = \frac{\partial \tilde{U}_i}{\partial x_j}\left(-\tfrac{2}{3}\overline{\rho}\tilde{K}\delta_{ij} + \mu_T\left(\frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_i} - \tfrac{2}{3}\delta_{ij}\frac{\partial \tilde{U}_i}{\partial x_j}\right) + \tau_{ij}^{R^*}\right) \tag{3.34}$$

We can preserve the form of the standard $K-\varepsilon$ production term if we define a variable μ_T^* such that

$$\mu_T^* = \mu_T + \frac{\tau_{ij}^{R^*}\dfrac{\partial \tilde{U}_i}{\partial x_j}}{\left(\dfrac{\partial \tilde{U}_i}{\partial x_j} + \dfrac{\partial \tilde{U}_j}{\partial x_i} - \tfrac{2}{3}\delta_{ij}\dfrac{\partial \tilde{U}_i}{\partial x_j}\right)\dfrac{\partial \tilde{U}_i}{\partial x_j}} \tag{3.35}$$

Notice that each multiplicative term in the numerator and denominator of the fraction in Equation 3.35 is a three-by-three tensor, not a scaler, and the $\frac{\partial \tilde{U}_i}{\partial x_j}$ terms cannot be "canceled". Then

$$\underline{P} = \frac{\partial \tilde{U}_i}{\partial x_j}\left(-\tfrac{2}{3}\overline{\rho}\tilde{K}\delta_{ij} + \mu_T^*\left(\frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_i} - \tfrac{2}{3}\delta_{ij}\frac{\partial \tilde{U}_i}{\partial x_j}\right)\right) \tag{3.36}$$

It is clear, therefore, that we can consider the higher-order additions to the Reynolds stress term to be modifications to the

turbulent viscosity, and this modified turbulent viscosity can be used in Equations 3.18 and 3.15 to calculate the turbulent diffusion and turbulent heat flux. This formulation is not used to calculate turbulent viscosity for the momentum or turbulence equations nor is it used to calculate the Reynolds stress itself in the energy equation. In those the turbulent viscosity is calculated via the Chien relation

$$\mu_T = \frac{\overline{\rho} C_\mu \tilde{K}^2 \left(1.0 - \exp\left(-0.0115 y^+\right)\right)}{\tilde{\varepsilon}} \tag{3.37}$$

and the Reynolds stress is calculated via Equation 3.14. Certainly the more standard relations are the correct formulation for the momentum and turbulence equations and for the Reynolds stress term in the energy equation. RNG theory has provided a detailed derivation of the $K-\varepsilon$ equations based upon this expression. But in the absence of any kind of formal analysis, the substitution of μ_T^* for μ_T in the turbulent diffusion and turbulent heat flux expressions is a reasonable attempt to extend the benefits of RNG analysis to the areas of turbulent heat and mass transfer. If RNG does indeed provide a systematic and rigorous derivation of the $K-\varepsilon$ turbulence model, no *ad hoc* function, no matter how elaborate or what its origin, can replace a thorough mathematical and physical analysis of turbulent heat and mass transfer. Nonetheless one may presume that the higher-order form of the turbulent viscosity given in Equation 3.35 and called μ_T^* might be a reasonable trial function to replace the more usual μ_T given in Equation 3.37, and such a trial function might prove to be a useful tool until a more exact formulation based upon rigorous analysis is available. "Mixing" RNG, therefore, is identical to "fast" RNG except that Equation 3.35 is used for μ_T in calculating the turbulent binary diffusion coefficient and heat flux vector.