Finite Generation of Ext-Algebras for Monomial Algebras

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Abstract

The use of graphs in algebraic studies is ubiquitous, whether the graphs be finite or infinite, directed or undirected. Green and Zacharia have characterized finite generation of the cohomology rings of monomial algebras, and thereafter G. Davis determined a finite criteria for such generation in the case of cycle algebras. Herein, we describe the construction of a finite directed graph upon which criteria can be established to determine finite generation of the cohomology ring of "in-spoked cycle" algebras, a class of algebras that includes cycle algebras. We then show the further usefulness of this constructed graph by studying other monomial algebras, including d-Koszul monomial algebras and a new class of monomial algebras which we term "left/right-symmetric" algebras.

This dissertation is dedicated to Dr. Ed Green, my advisor and mentor. His love for mathematics and his care for his students know no bounds.

I would also like to acknowledge the love and support of my wife, Jodi. She has taught me the real meanings of personal strength and integrity.

Finally, I wish to thank Henry Lee for allowing me to teach him something of elliptic curves, and to remember all that fell on April 16th, 2007. You are not forgotten. We carry on.

Contents

CHAPTER 1

Introduction

The use of graphs in algebraic studies is ubiquitous, whether the graphs be finite or infinite, directed or undirected. Path algebras, whose multiplication is determined by a finite directed graph, are an accessible and well-established class of algebras [**2**]. In studying the module categories of such algebras, Gabriel showed, in [**8**], that any indecomposable module over a path algebra of finite representation type naturally corresponds to a certain type of Dynkin diagram; diagrams which are, in turn, indicated by finite undirected trees. This work has been continued, refined, and made more accessible in [**5**]. Other instances of the use of graphs in the studies of algebras are readily found, an important example being that of [**16**], which is an application of graph theory to the study of growth in algebras.

On the other hand, monomial algebras and their homological properties have also been studied extensively in recent years, receiving attention on several fronts; see [**3**], [**9**], [**10**], [**13**], and [**15**] for examples. In [**14**], Green and Zacharia characterized finite generation of the cohomology rings for monomial algebras. Thereafter, in [**6**], G. Davis determined a finite criteria for such generation in the case of cycle algebras. In this dissertation, we first describe the construction of a finite directed graph upon which criteria can be established to determine finite generation of the cohomology ring of "in-spoked cycle" algebras, a class of algebras that includes cycle algebras. We then show the further usefulness of this constructed graph by studying other monomial algebras, including d-Koszul monomial algebras and a new class of monomial algebras which we term "left/right-symmetric" algebras.

At times, when presenting the results of scientific research, the original motivations, the driving force behind the research, are passed over. In light of this occasional oversight, we wish to be clear about our earliest interests. The mathematics community has long been in the practice of studying an object via secondary, or consequential, related objects. For example, we may say much about an $n \times n$ matrix if we know its determinant, but we cannot say everything. The determinant gives us information, secondarily, about some fundamental quality of a matrix. In a similar way, we will study the cohomology ring of an algebra, and in doing so, learn some fundamental qualities about the original algebra. To complete our analogy, we note that the cohomology ring of an algebra plays the rôle of determinant above, which we study as a secondary object in order to better understand the original object, an algebra.

Such homological endeavors are relatively recent, in the scope of mathematical time, and we know only a little. For example, we may ask: "Is it easy to determine if the cohomology ring of an algebra Λ is finitely generated?" The answer to this question is known definitively for very few classes of algebras. Some preliminary answers have been provided, however, such as the results in [**7**] concerning group rings. At the outset of our studies, our original wish was to extend this answer to a wider range of algebras. Furthermore, our designs included explorations to find some way of encapsulating data about the generators of the cohomology rings of algebras, perhaps within the confines of a combinatorial object or expression. We found ourselves able to do so, at least in relatively simple cases, through the employment of a finite directed graph. Our first attempts were along the lines of those found in Chapter 7, about LR-symmetric algebras. After some further work, we generalized our approach sufficiently to treat a more general class of algebras, specifically monomial algebras, and that is with what this dissertation is primarily concerned.

With regards to the style of this dissertation, we note that even though the basic motivations and ideas behind the constructions are natural, the proofs of some of the supporting lemmas become quite technical. We therefore have relegated some of the more technical proofs to the appendices, so as to prevent too much distraction from the flow of the main work. Also, we have incorporated examples throughout, in hopes of aiding the reader's understanding via illustration. Before we launch into the main body of the dissertation, we here provide a short, but descriptive, outline of the subject and the manner in which this document will proceed.

In order to facilitate the treatment of the graph theoretic nature of many of our results, *Chapter 2* immediately starts us off in establishing important vocabulary for finite directed graphs, quivers, paths, and the like. Furthermore, the definition of *monomial algebra* is herein established.

Chapter 3 provides the mathematical foundation for the rest of the work at hand. The notion of *left-admissible sequence* is rigorously defined, and the sets Γ_n are constructed. The *Ext-algebra*, or cohomology ring, of a monomial algebra is then described, and its basis is put into correspondence with the elements of the sets Γ_n. This chapter concludes with a new result on the uniqueness of the minimal generating set for the cohomology ring of a monomial algebra, when that generating set is derived from a k-basis, as in [**14**].

Chapter 4 gives formal construction of the finite directed graph, or Ψ-graph, we associate to a monomial algebra.

Chapter 5 describes general results concerning Ψ-graphs for arbitrary monomial algebras.

Chapter 6 contains the main results for the present dissertation. The definition of cycle algebras and in-spoked cycle algebras are given here, and thereafter many properties of the Ψ-graph for cycle algebras are discussed. From there, formal proofs are provided for a theorem giving finite criteria on the Ψ-graph of an arbitrary cycle algebra Λ to determine finite generation of the Ext-algebra of Λ. The main theorem follows, which extends the results to the more general case of in-spoked cycle algebras.

Chapter 7 demonstrates further utility of the Ψ-graph to other classes of algebras, including a new algebra we term as left/right-symmetric. To finish the dissertation, and the chapter, some possibilities for further research are developed via a series of questions.

Finally, there is an index and some appendices. We hope the index is an helpful tool when vocabulary questions arise. The first two appendices are rather lengthy technical proofs to two of the results in Chapter 6. Appendix 3 contains algorithms to determine some necessary properties of Ψ-graphs, at least necessary in some cases, and the algorithms themselves aptly show the computational nature of much of theory that unfolds below.

CHAPTER 2

Graph Theory and Path Algebras

In this dissertation, finite directed graphs are referenced in several contexts, including in the creation of path algebras. In this chapter, we establish much of the vocabulary surrounding graphs, quivers, and path algebras. The definitions and vocabulary provided here will be used throughout the remainder of this document.

The primary object determining multiplication in a path algebra is the underlying finite directed graph, which we refer to as a *quiver*. It is important to note that in a quiver, and in our usage of the term *finite directed graph*, loops and multiple edges between vertices are permitted. At times, we will use the terms *finite directed graph* and *quiver* interchangeably, with the emphasis that a quiver is the finite directed graph determining multiplication for a path algebra.

1. Some Graph Theory

For a quiver Γ, we denote its set of vertices by Γ_0 and its set of arrows by Γ_1. To ease the burden in referring to the beginning and ending vertices for a particular arrow, we define two functions. For a quiver Γ, let $a \in \Gamma_1$, such that $a : v_i \to v_j$ for $v_i, v_j \in \Gamma_0$. We define the *origin function* σ to be:

$$\sigma : \Gamma_1 \to \Gamma_0, \text{ where } \sigma(a) = v_i.$$

Similarly, the *terminus function* τ we define as:

$$\tau : \Gamma_1 \to \Gamma_0, \text{ where } \tau(a) = v_j.$$

For convenience, we extend these functions slightly so that they are both also defined on vertices: for $v \in \Gamma_0$ we define $\sigma(v) = \tau(v) = v$.

In a quiver Γ, we define a *path* p to be a sequence of alternating vertices and arrows $p = v_1, a_1, v_2, ..., a_n, v_{n+1}$; where $v_i \in \Gamma_0$ are vertices, and $a_i \in \Gamma_1$ are arrows, for $1 \leq i \leq n$. Furthermore, we require $\sigma(a_i) = v_i$ and $\tau(a_i) = v_{i+1}$ for $1 \leq i \leq n$, and $\tau(a_i) = \sigma(a_{i+1})$ for each $1 \leq i \leq n-1$. A path with no repeated vertices we will refer to as a *simple path*. At times, it will be convenient to drop the vertices from the notation for a path, and simply write the sequence of arrows, and without the comma separation. That is, for a path $p = v_1, a_1, v_2, ..., a_n, v_{n+1}$ we will alternatively write $p = a_1 a_2 ... a_n$. The *length* of a path we define to be the number of arrows in its sequence representation. We will often write this functionally; and so for a path p of length n, we write $l(p) = n$. It should be noted that we consider a single vertex to be a path, and its length to be 0.

We may further extend our origin and terminus functions to the set of all paths in Γ. We denote the set of all possible paths in Γ by $\mathcal{B}_\Gamma$. Let p be an arbitrary element in $\mathcal{B}_\Gamma$, where $p = a_1a_2...a_n$, for $n \geq 1$. Our function extensions are then:

$$\sigma : \mathcal{B}_\Gamma \to \Gamma_0, \text{ where } \sigma(p) = \sigma(a_1),$$

and:

$$\tau : \mathcal{B}_\Gamma \to \Gamma_0, \text{ where } \tau(p) = \tau(a_n).$$

Such extensions will allow us even more convenience and simplicity in notation. For example, let $p, q \in \mathcal{B}_\Gamma$, where $p = a_1a_2...a_n$ and $q = b_1b_2...b_m$, for $a_i, b_j \in \Gamma_1$ with $1 \leq i \leq n$ and $1 \leq j \leq m$. If $\tau(p) = \sigma(q)$, then for the path $a_1...a_nb_1...b_m$ we will oftentimes simply write pq.

A *subpath* q of $p = v_1, a_1, v_2, ..., a_n, v_{n+1}$ is a path given by the alternating sequence of vertices and arrows $q = u_1, b_1, u_2, b_2, ..., b_m, u_{m+1}$; where u_j are vertices in Γ, and b_j are arrows in Γ, for $1 \leq j \leq m$. Additionally, we require that for q to be a subpath of p: there exists some k with $1 \leq k \leq m$ such that $m \leq n - k$; for $1 \leq j \leq m - 1$ we have $u_{j+1} = v_{k+j}$ and $b_{j+1} = a_{k+j}$; and finally that $u_{m+1} = v_{k+m}$. That is to say, q is a connected subsequence of elements from the sequence for p, starting and ending at a vertex, and occurring in precisely the same order as in p. We refer to q as a *prefix* of p if $p = qy$ for some $y \in \mathcal{B}_\Gamma$. Similarly, we refer to q as a *suffix* of p if $p = xq$ for some $x \in \mathcal{B}_\Gamma$. Finally, we say a subpath q lies *within* a path p if $p = xqy$ for $x, y \in \mathcal{B}_\Gamma$, and q lies *strictly within* p if $l(x), l(y) \geq 1$.

If v is a vertex in a quiver Γ, we define the *indegree* of v to be the number of arrows $a \in \Gamma_1$ with $\tau(a) = v$, and the *outdegree* of v to be the number of arrows $a \in \Gamma_1$ with $\sigma(a) = v$. In words, the outdegree of a vertex $v \in \Gamma$ is the number of arrows exiting v, and the indegree of v is the number of arrows coming into v. A vertex will be referred to as a *source* if its indegree is 0. Similarly, a vertex with an outdegree of 0 will be referred to as a *sink*. If a vertex v has outdegree ≥ 2, we say Γ has a *fork* at v, or Γ *forks* at v.

Certain types of structures within finite directed graphs hold particular interest for us; for example, the notion of a cycle. We define a *cycle* to be a path $p = v_1, a_1, ..., a_n, v_1$ of length ≥ 1 that starts and ends at the same vertex, and where $v_i \neq v_j$ for $i \neq j$, $1 \leq i, j \leq n$. Now, let $\mathcal{C}, \mathcal{D}$ be two cycles in a finite directed graph Γ, and let $\mathcal{C} = v_1, a_1, v_2, a_2, ..., a_n, v_1$ denote the alternating sequence of vertices and arrows determining the path for $\mathcal{C}$ in Γ. If $\mathcal{D}$ cannot be represented as a cyclic permutation of vertices and arrows of $\mathcal{C}$, we say $\mathcal{C}$ and $\mathcal{D}$ are *distinct* cycles. Since the sequence representation for a cycle $\mathcal{C}$ of length n may take on n different forms, via a cyclic permutation of the vertices and arrows, we establish notation for a cycle that specifies a particular sequence representation. For a vertex v lying on the cycle $\mathcal{C}$ we write $\mathcal{C}_v$ to indicate the path for $\mathcal{C}$ that starts and ends at the vertex v.

If a path p can be written as $p = qrs$ where r is a subpath of a cycle $\mathcal{C}$, we say p *intersects* $\mathcal{C}$. In furthering our efforts in simplifying notation, we also allow cycles in our representations for paths. For example, suppose p and q are paths in Γ and $\mathcal{C}$ a cycle in Γ. If $v = \tau(p) = \sigma(q)$ lies on $\mathcal{C}$, we may write $p\mathcal{C}q$ to represent the path $p\mathcal{C}_v q$. For $k \geq 1$, we may write $p\mathcal{C}^k q$ to represent the path $p\mathcal{C}_v ... \mathcal{C}_v q$, where the subpath $\mathcal{C}_v$ occurs k times. We say a path p *lies across* the cycle $\mathcal{C}$ if $p = r\mathcal{C}^k s$ for some simple path rs, and for some $k \geq 1$.

Let $\mathcal{C}$ be a cycle in a quiver Γ. Suppose that p is a simple path in Γ of length 1 or greater with the following properties: $\tau(p)$ lies on $\mathcal{C}$, all other vertices of p do not lie on $\mathcal{C}$, and p intersects no other cycles in Γ. We refer to such a path p as an *in-spoke* of $\mathcal{C}$. Similarly, if we change only the property from "$\tau(p)$ lies on $\mathcal{C}$" to "$\sigma(p)$ lies on $\mathcal{C}$", we have the description of an *out-spoke* of $\mathcal{C}$. The collection of all in-spokes and out-spokes for a cycle $\mathcal{C}$ are simply called *spokes* . A cycle $\mathcal{C}$ and all its spokes in a quiver Γ is said to be a *spoked cycle* . If Γ is a quiver consisting of only a single cycle and in-spokes, we refer to Γ as a *in-spoked cycle quiver* or a *in-spoked cycle graph.*

2. Monomial Algebras

For a field k, and a quiver Γ, we denote by $k\Gamma$ the k-vector space with paths in $\mathcal{B}_\Gamma$ as basis. In a manner similar to [**2**], we further establish $k\Gamma$ as a k-algebra. For paths $p, q \in \mathcal{B}_\Gamma$, we use the functions τ and σ, given above, to define the multiplication of p and q in $k\Gamma$ as follows:

$$p \cdot q = \begin{cases} pq & \text{if } \tau(p) = \sigma(q), \\ p & \text{if } \tau(p) = v \text{ and } q = v, \\ q & \text{if } \sigma(q) = w \text{ and } p = w, \\ 0 & \text{otherwise.} \end{cases}$$

From this point forward, we will suppress the dot $\cdot$ when expressing such multiplication, and simply write pq to represent the product of p and q. With this multiplication, we may now refer to $k\Gamma$ as the *path algebra* of Γ over k, where arbitrary elements in $k\Gamma$ are finite k-linear combinations of elements from $\mathcal{B}_\Gamma$. Since it is possible, by our definition, that $pq = 0$, we will refer to $\mathcal{B}_\Gamma$ as the *multiplicative basis with zero* of $k\Gamma$. Also, within the context of path algebras, it is common to refer to an element in $k\Gamma$ determined by a path as a *monomial*, and a finite k-linear combination of paths in $k\Gamma$ as a (noncommutative) *polynomial.* We will use these latter two terms freely throughout this document.

Now, let Γ be a quiver, k a field, and $k\Gamma$ the resulting path algebra. Furthermore, let $\rho = \{r_i\}_{i \in \mathfrak{I}}$ be a set of monomial generators for an ideal in $k\Gamma$, where $\mathfrak{I}$ is some index set, and for which each r_i has length 2 or greater. We say ρ is *reduced* if for each $r_i \in \rho$, r_i is not a subpath of r_j for all $j \in \mathfrak{I}$, $j \neq i$. We refer to the quotient $\Lambda = k\Gamma/\langle\rho\rangle$ as a *monomial algebra*, if ρ is a reduced set of monomial generators for the ideal $\langle\rho\rangle$ in $k\Gamma$. For the duration of this dissertation we will assume ρ is a reduced set, unless otherwise explicitly stated.

CHAPTER 3

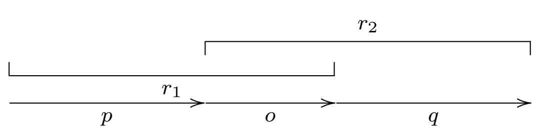
Results of Green and Zacharia

E. Green and D. Zacharia show in [**14**] that via a constructive, but not necessarily finite, process we are able to characterize finite generation of the cohomology ring for a monomial algebra Λ. As it turns out, the cohomology ring for Λ will have the additional structure of being a k-algebra when we have the setup as next described. For the rest of this chapter, let Γ be a quiver, k a field, and ρ a reduced monomial generating set for an ideal in $k\Gamma$. Also, set $\Lambda = k\Gamma/\langle\rho\rangle$, to denote the associated monomial algebra. When viewed as a k-algebra, the cohomology ring $E(\Lambda)$ is often referred to in the literature as an Ext-algebra. Because of this, we will use the terms *Ext-algebra* and *cohomology ring* interchangeably.

The point of this chapter is to describe fully the construction of $E(\Lambda)$ and its relation to certain sequences. These so-called "left-admissible" sequences, and the related sets Γ_n, are interesting and well-understood mathematical constructions, see [**1**] and [**10**]. In the first section of this chapter, we define left-admissible sequences of elements in ρ, and the construction of the sets Γ_n. In the second section, we describe the construction of $E(\Lambda)$, the Ext-algebra for Λ, and its relation to the sets Γ_n. In the final section of this chapter we indicate a new result on the uniqueness of the minimal generating set of $E(\Lambda)$, when it is derived from the multiplicative basis for $E(\Lambda)$.

1. Left-Admissible Sequences and the Γ_n's

As a reminder to the reader, we recall here that $\mathcal{B}_\Gamma$ is the set of all paths in the quiver Γ, where $\Lambda = k\Gamma/\langle\rho\rangle$. If $r_1, r_2 \in \rho$, we say r_2 *overlaps* r_1 *with intersection word* o if there exist $o, p, q \in \mathcal{B}_\Gamma$ such that $poq = r_1 q = pr_2$, with $1 \leq l(q) < l(r_2)$. These ideas are represented in the following overlap diagram:



We say that r_2 overlaps r_1 (with intersection word o) *maximally with respect to* ρ if no proper subword of poq is in ρ, other than r_1 or r_2. When r_2 overlaps r_1 with intersection word o, we refer to the subword q of the underlying path poq as the *tail* of the overlap. Also, when r_2 overlaps r_1 with intersection word o, we will sometimes use the notation $\overrightarrow{r_1 r_2}$ to refer to the underlying path poq.

Definition 3.1. Let ρ be a reduced set of monomial generators for the ideal $\mathcal{I}$ in $k\Gamma$. In the following, we will define recursively a *left-admissible sequence of length* n, and denote such a sequence by $\langle r_1, ..., r_n \rangle$.

A left-admissible sequence of length 1 is given by any single element $r_1 \in \rho$, and denoted by $\langle r_1 \rangle$. Let $r_1, r_2 \in \rho$. If r_2 overlaps r_1 maximally with respect to ρ and has underlying path $\overrightarrow{r_1 r_2} = poq$, then $\langle r_1, r_2 \rangle$ is a left-admissible sequence of length 2 with tail q. Note, in this latter case, since ρ is reduced we have $l(p) > 0$. For $r_i \in \rho$, $1 \leq i \leq n$, we say $\langle r_1, ..., r_n \rangle$ is a left-admissible sequence of length n with tail t if the sequence $\langle r_1, ..., r_{(n-1)} \rangle$ is left-admissible with tail t' and r_n overlaps t' maximally with respect to ρ. In this case, t is the tail of $\overrightarrow{t' r_n}$.

A left-admissible sequence $\langle r_1, ..., r_n \rangle$ has an underlying path in Γ, which will be referred to as the *overlap sequence word* for $\langle r_1, ..., r_n \rangle$, and denoted $\overrightarrow{r_1, ..., r_n}$. It will be natural, and convenient, when referring to the underlying path of a single relation $r_1 \in \rho$, to denote it by r_1 instead of $\overrightarrow{r_1}$, although we will use both notations at times when it is expressive and appropriate. We refer to $\langle s_1, ..., s_m \rangle$ as a *subsequence* of a left admissible sequence $\langle r_1, ..., r_n \rangle$ if $m \leq n$, and $s_j = r_{k+j}$ for some $0 \leq k \leq m-1$ and where $1 \leq j \leq m-k$. There is more: we require that a subsequence of a left-admissible sequence not only be a continuous subsequence of the original sequence of the same elements in the same order, but also that the overlaps and intersection words within the original left-admissible sequence be preserved.

If in Definition 3.1 we relax the requirement that each of the stated overlaps is maximal with respect ρ, and only require they be overlaps, then we may refer to such a sequence as a *left-overlap sequence*. We denote a left-overlap sequence by $(r_1, ..., r_n)$, where the r_i's are the relations in ρ. It is important to note that such a sequence $(r_1, ..., r_n)$ also has an underlying path in Γ which we analogously denote by $\overrightarrow{r_1, ..., r_n}$.

Let $\langle r_1, ..., r_n \rangle$ be a left-admissible sequence. We refer to the intersection word o given by the overlap of r_k by r_{k+1} in this admissible sequence as the *k-th intersection word* of the given sequence. So, a left-admissible sequence of length n has $n-1$ intersection words. Setting $\gamma = \overrightarrow{r_1, ..., r_n}$, we define the *k-th intersection word for* γ to be the k-th intersection word for $\langle r_1, ..., r_n \rangle$. We will denote the k-th intersection word o of $\langle r_i \rangle_{i=1}^n$ by $o \vdash_k \langle r_i \rangle_{i=1}^n$, or, if the position is not relevant by $o \vdash \langle r_i \rangle_{i=1}^n$. We will treat intersection words in γ similarly, and write $o \vdash_k \gamma$ or $o \vdash \gamma$.

We define and construct the sets Γ_n for $n \in \mathbb{Z}_{n \geq 0}$. Let ρ be a reduced set of monomial generators for the ideal $\mathcal{I}$ in $k\Gamma$. We define Γ_0 to be the set of vertices of the quiver Γ, and define Γ_1 to be the set of arrows of the quiver Γ. Set $\Gamma_2 = \rho$. For $n \geq 3$, we define Γ_n by:

$$\Gamma_n = \{\overrightarrow{r_1, ..., r_{n-1}} \mid \langle r_1, ..., r_{n-1} \rangle; r_i \in \rho\}.$$

That is, Γ_n is the set of underlying paths of all left-admissible sequences of length $n-1$ of elements $r_i \in \rho$.

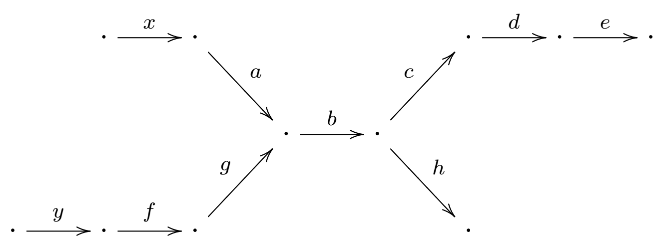
When we later construct a finite directed graph based on left-admissible sequences, it will be useful to have a way to refer to the relations that correspond to arrows in the graph. Let

$\langle r_1, ..., r_n\rangle$ be a left-admissible sequence, where $n \geq 3$. The *first sequence arrow word* is given by the subword po, where $\overrightarrow{r_1 r_2} = poq$. Similarly, the *n-th sequence arrow word* is given by subword oq, where $\overrightarrow{r_{n-1} r_n} = poq$. Then for each $1 < k < n$, $r_k = a_k x_k b_k$, where $r_{k-1} = p_k a_k$ and $r_{k+1} = b_k q_k$ for some $p_k, q_k \in \mathcal{B}_\Gamma$, we refer to the subword x_k of r_k as the *k-th sequence arrow word* of $\langle r_1, ..., r_n\rangle$. If $\gamma = \overrightarrow{r_1, ..., r_n}$, we also refer to x_k as the *k-th sequence arrow word for* γ.

Definition 3.2. Let $\langle r_1, ..., r_n\rangle$ be a left-admissible sequence, where $n \geq 3$, and x_k the k-th arrow word in the sequence. If x_k is a vertex, then we say the sequence has an *adjacency at position k*.

To help illustrate some of the concepts and constructions above, we provide the following example.

Example 3.3. Let Γ be the quiver given by:



For k a field, we set $\Lambda = k\Gamma/\langle\rho\rangle$ where $\rho = \{xab, bcd, yfgb, bh, gbc, cde\}$.

From this definition of Λ, we note that Γ_0 is the set of (unlabeled) vertices of Γ, $\Gamma_1 = \{a, b, c, d, e, f, g, h, x, y\}$, and $\Gamma_2 = \rho$. The left-admissible sequences of length 2, of elements in ρ, are shown in the following configurations:

x a b h

x a b c d

y f g b c

y f g b h

b c d e

g b c d

These sequences indicate that $\Gamma_3 = \{xabh, xabcd, yfgbc, yfgbh, bcde, gbcd\}$. Continuing along these lines, we have the following two sequences as the only sequences of length 3:

$$\mathrm{x\,a\,b\,c\,d\,e}$$

$$\mathrm{y\,f\,g\,b\,c\,d\,e}$$

This give us: $\Gamma_4 = \{xabcde, yfgbcde\}$. Since there are no left-admissible sequences of length 4 or greater, $\Gamma_i = \emptyset$ for $i \geq 5$. We label the last of the sequences above by setting $Rel = \langle yfgb, gbc, cde\rangle$. Then the intersection words of the sequence are denoted by $gb \vdash_1 Rel$ and $c \vdash_2 Rel$.

2. The Ext-Algebra for Λ

This section is initially quite general in theory and application. Within, we first describe the construction of the Ext-algebra $E(\Lambda)$ for a monomial algebra Λ, and later describe the relation of the sets Γ_n to elements in $E(\Lambda)$. For the monomial algebra $\Lambda = k\Gamma/\langle\rho\rangle$, let Γ_1 denote the set of arrows in Γ, and $\langle\Gamma_1\rangle$ denote the ideal in $k\Gamma$ generated by Γ_1. By the ring isomorphism theorems, $\underline{\mathbf{r}} = \langle\Gamma_1\rangle/\langle\rho\rangle$ is an ideal in Λ, and so we let $\overline{\Lambda} = \Lambda/\underline{\mathbf{r}}$.

We start with description of the *Ext*-groups needed for the construction of $E(\Lambda)$, which are denoted by $\mathrm{Ext}^i_\Lambda(\overline{\Lambda}, \overline{\Lambda})$, for each $i \geq 0$. For our purposes, constructing the *Ext*-groups will involve the use of minimal projective resolutions. In order to give description of the minimality of such resolutions, we recall the following definition.

Definition 3.4. Let M be a Λ-module for some monomial algebra $\Lambda = k\Gamma/\langle\rho\rangle$, and $\underline{\mathbf{r}}$ as described above. A projective cover for M is a short exact sequence of Λ modules:

$$0 \hookrightarrow N \to P \to M \twoheadrightarrow 0$$

where P is a projective Λ-module, and where $P/\underline{\mathbf{r}}P \cong M/\underline{\mathbf{r}}M$.

Consider a projective resolution of the Λ-module $\overline{\Lambda}$:

$$(\mathcal{P}) \qquad \cdots \to P_i \xrightarrow{\delta_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\pi} \overline{\Lambda} \to 0$$

wherein each P_i is a projective Λ-module, $i \geq 0$, and the sequence is exact. We take the resolution $(\mathcal{P})$ to be minimal in the sense that, for each $i \geq 0$, the sequence

$$0 \hookrightarrow \ker\delta_i \to P_i \to \mathrm{im}\delta_i \twoheadrightarrow 0$$

is a projective cover for $\mathrm{im}\delta_i$.

We next apply the (contravariant) functor $\mathrm{Hom}_\Lambda(\cdot, \overline{\Lambda})$ to the deleted complex of $(\mathcal{P})$; the deleted complex being $(\mathcal{P})$ without the module $\overline{\Lambda}$ as part of the exact sequence:

$$(\mathcal{P}_{del}) \qquad \cdots \to P_i \xrightarrow{\delta_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} 0$$

Through this application of $\mathrm{Hom}_\Lambda(\cdot, \overline{\Lambda})$ to $(\mathcal{P}_{del})$, we obtain the following new complex:

$$0 \to \mathrm{Hom}_\Lambda(P_0, \overline{\Lambda}) \xrightarrow{\delta_1^*} \mathrm{Hom}_\Lambda(P_1, \overline{\Lambda}) \xrightarrow{\delta_2^*} \mathrm{Hom}_\Lambda(P_2, \overline{\Lambda}) \xrightarrow{\delta_3^*} \ldots$$

where δ_i^* is pre-composition. That is, for $\phi \in \mathrm{Hom}_\Lambda(P_{i-1}, \overline{\Lambda})$, $\delta_i^*(\phi) = \phi \circ \delta_i$, where $i \geq 0$. Note: we need only to apply the Hom functor to the simpler deleted complex, since the module $\overline{\Lambda}$ may be recovered as the cokernel of δ_1. From this new complex, we define our *Ext*-groups as follows:

$$\mathrm{Ext}^i_\Lambda(\overline{\Lambda}, \overline{\Lambda}) = \frac{\mathrm{ker}\delta^*_{i+1}}{\mathrm{im}\delta^*_i},$$

for $i \geq 0$. Since (P) is minimal, we have δ_i^* is the zero map for each $i \geq 0$. Therefore $\mathrm{ker}\delta^*_{i+1} = \mathrm{Hom}_\Lambda(P_i, \overline{\Lambda})$ and $\mathrm{im}\delta^*_i = 0$ for each $i \geq 0$, and so $\mathrm{Ext}^i_\Lambda(\overline{\Lambda}, \overline{\Lambda}) = \mathrm{Hom}_\Lambda(P_i, \overline{\Lambda})$, for each $i \geq 0$.

We may now define the Ext-algebra $E(\Lambda)$, for Λ, as:

$$E(\Lambda) = \coprod_{i=0}^{\infty} \mathrm{Ext}^i_\Lambda(\overline{\Lambda}, \overline{\Lambda}).$$

with the multiplicative structure given by the Yoneda product. This multiplication is defined in the following way. Let $\xi_i \in \mathrm{Hom}_\Lambda(P_i, \overline{\Lambda})$, and $\xi_j \in \mathrm{Hom}_\Lambda(P_j, \overline{\Lambda})$. We consider the following commutative diagram, where the P_i's are the projective Λ-modules from $(\mathcal{P})$:

$$\begin{tikzcd}
P_{i+j} \arrow[r] \arrow[d, "l_{i+j}"] & P_{i+j-1} \arrow[r] \arrow[d, "l_{i+j-1}"] & \cdots \arrow[r] & P_{i+1} \arrow[r, "\delta_{i+1}"] \arrow[d, "l_{i+1}"] & P_i \arrow[d, "l_i"] \arrow[dr, "\xi_i"] & & \\
P_j \arrow[r] \arrow[dr, "\xi_j"] & P_{j-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r, "\delta_1"] & P_0 \arrow[r] & \overline{\Lambda} \arrow[r] & 0 \\
& \overline{\Lambda} & & & & &
\end{tikzcd}$$

Since each of P_{i+k} is projective, we obtain each lifting l_{i+k}, for $1 \leq k \leq j$, as indicated. From this construction, we define the multiplication of ξ_j by ξ_i via the following composition of maps: $\xi_j \xi_i = \xi_j \circ l_{i+j}$. We now have the ring multiplicative structure defined for $E(\Lambda)$. But there is more: since each of the maps in $(\mathcal{P})$ and our commutative diagram are k-linear, we have by extension that $E(\Lambda)$ is a k-algebra. Furthermore, $E(\Lambda)$ is a graded k-algebra, under its preceding definition of ring multiplication. We will have use of the graded nature of $E(\Lambda)$ at the end of this chapter when we address the issue of obtaining a minimal generating set from the multiplicative basis for $E(\Lambda)$. However, we first pause to gain understanding of the relationship between the elements of the sets Γ_n and elements of $E(\Lambda)$.

Let now $(\mathcal{P})$ be a minimal projective projective resolution of $\overline{\Lambda}$ over Λ, given by:

$$(\mathcal{P}) \qquad \cdots \to P_i \xrightarrow{\delta_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} \overline{\Lambda} \to 0$$

Then by [**9**], we have $P_i = \coprod_{p \in \Gamma_i} e_p \Lambda$, for each $i \geq 0$, and where $e_p = \tau(p)$. That is, e_p is the idempotent corresponding to the endpoint of the path $p \in \mathcal{B}_\Gamma$. We need to be more precise about the nature of the elements e_p: if $p, q \in \mathcal{B}_\Gamma$ are distinct paths, then we choose e_p and

e_q also to be distinct, even if $e_p\Lambda$ and $e_q\Lambda$ are isomorphic. We may then identify $\mathrm{Ext}^i_\Lambda(\overline{\Lambda},\overline{\Lambda})$ with $\coprod_{p\in\Gamma_i}\overline{\Lambda}e_p$ for all $i\geq 0$.

Let $p\in\Gamma_i$ and $q\in\Gamma_j$. By the identifications $\mathrm{Ext}^i_\Lambda(\overline{\Lambda},\overline{\Lambda})=\coprod_{p\in\Gamma_i}\overline{\Lambda}e_p=\mathrm{Hom}(P_i,\overline{\Lambda})$, we have that e_p is represented by an element $f_p\in\mathrm{Hom}(P_i,\overline{\Lambda})$ such that:

$$f_p(e_q\lambda)=\begin{cases}0 & \text{if } q\neq p \text{ in } k\Gamma,\\ e_p\overline{\lambda} & \text{if } q=p \text{ in } k\Gamma\end{cases}$$

where $\overline{\lambda}$ indicates the element λ under the canonical surjection of Λ onto $\overline{\Lambda}$. Hence, each set Γ_i is identified with a k-basis of $\mathrm{Ext}^i_\Lambda(\overline{\Lambda},\overline{\Lambda})$ by taking $p\in\Gamma_i$ to $f_p\in\mathrm{Hom}(P_i,\overline{\Lambda})$, and hence also to the corresponding basis element of $\mathrm{Ext}^i_\Lambda(\overline{\Lambda},\overline{\Lambda})$. In [**9**], it is proved that these notions may be further extended, so that there is a one-to-one correspondence between the elements of $\bigcup_{i\geq 0}\Gamma_i$ and a k-basis $\mathcal{B}=\{e_p\}_{p\in\Gamma_0\cup\Gamma_1\cup\ldots}$ for $E(\Lambda)$. From $\mathcal{B}$, a minimal generating set for $E(\Lambda)$ may be determined. Our first result shows that a minimal generating set determined by this k-basis $\mathcal{B}$ is unique.

Lemma 3.5. *Let $\Lambda=k\Gamma/\mathcal{I}$ be a monomial algebra, and $E(\Lambda)$ its Ext-algebra. If $\mathcal{B}=\{e_p\}_{p\in\Gamma_0\cup\Gamma_1\cup\ldots}$ is a multiplicative k-basis for $E(\Lambda)$, as in* [**14**]*, then the minimal generating set for $E(\Lambda)$ as a k-algebra determined by this basis is unique.*

Proof. Let $\mathcal{E}$ be a minimal generating set for $E(\Lambda)$ as a k-algebra, determined by the k-basis $\mathcal{B}$. Set $\mathcal{F}$ to be another minimal generating set for $E(\Lambda)$, also determined by $\mathcal{B}$. $\mathcal{E}$ and $\mathcal{F}$ are $\mathbb{Z}_{\geq 0}$-graded since $\mathcal{B}$ is $\mathbb{Z}_{\geq 0}$-graded. Suppose there exists an element $f\in\mathcal{F}$ and $f\notin\mathcal{E}$. Let $deg(f)$ denote the degree of f, which is ≥ 2 since $\mathcal{E}$ and $\mathcal{F}$ must be the same in degrees 0 and 1. Since f is an element in the minimal generating set $\mathcal{F}$, f cannot be written as a finite sum of products of elements in $\mathcal{F}$ of degree less than $deg(f)$.

Since $f\notin\mathcal{E}$, but $\mathcal{E}$ is a minimal generating set for $E(\Lambda)$, we may write $f=\sum_{i=1}^n\alpha_i x_i$, where $\alpha_i\in k$ and $x_i=e_{p_{i,1}}...e_{p_{i,m_i}}$ for $e_{p_{i,j}}\in\mathcal{E}$, with $1\leq i\leq n$ and $1\leq j\leq m_i$. Now, since $f\in\mathcal{B}$ we have by linear independence that $f=e_{p_{k,1}}...e_{p_{k,m_k}}$, for some $1\leq k\leq n$, where $deg(f)=\sum_{i=1}^{m_k}deg(e_{p_{k,i}})$. If $m_k=1$ we are done, since then $f=e_{p_{k,1}}\in\mathcal{E}$ which contradicts our assumption that $f\notin\mathcal{E}$. Suppose now $m_k>1$. Then $deg(e_{p_{k,i}})<deg(f)$ for each $1\leq i\leq m_k$. Since $\mathcal{F}$ is a minimal generating set for $E(\Lambda)$, each $e_{p_{k,i}}$ can be written as a finite sum of products of elements in $\mathcal{F}$ with degree less than $deg(f)$. This contradicts the assumption that f is minimal in $\mathcal{F}$ and the result follows. □

CHAPTER 4

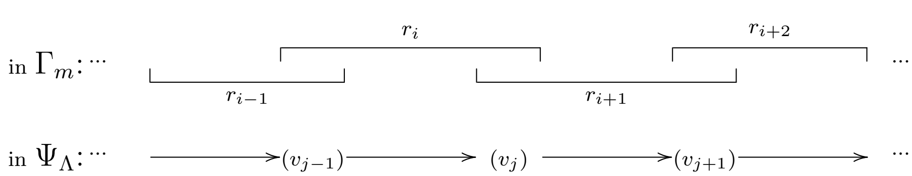
The Ψ Graph

In this section we describe the construction of a finite directed graph Ψ_Λ, whose nature determines a bound on how much work is necessary to determine finite generation of the cohomology ring of certain classes of monomial algebras. Although much of the construction is elementary, and somewhat natural, the notation and set descriptions become cumbersome at times. Therefore, in order to facilitate understanding of the construction, we will carry a concrete example throughout. The set up for the example will occur after some general description of our motivation.

To understand the nature of generation, finite or otherwise, of the cohomology ring of a monomial algebra, we analyze the elements of the sets Γ_n. The left-admissible sequences associated to the elements of the sets Γ_n determine whether or not generators for $E(\Lambda)$ exist in arbitrarily large degree.

Initially, we provide a general construction of the finite directed graph Ψ_Λ for a monomial algebra $\Lambda = k\Gamma/\mathcal{I}$, with ρ a finite reduced set of generators for $\mathcal{I}$. Essentially, the idea in the construction is to encode information about each left-admissible sequence into the graph by treating the intersection words of each sequence as vertices and the relations connecting those vertices as arrows. We will include more information than only such vertices and arrows, but this is our basic motivation.

For example, suppose ρ is a finite set of monomial generators for an ideal in the path algebra $k\Gamma$ for some quiver Γ. Further, suppose $\langle r_1, ..., r_{m+1}\rangle$ is a left-admissble sequence. Then we get some idea as to how a path in Ψ_Λ is determined by this sequence in the following diagram:



The (v_j) indicate the vertices in Ψ_Λ, and are determined by the intersection words of the left-admissible sequence. The arrows in Ψ_Λ are given by, for the most part, decompositions of relations r_i into their intersection words as determined by their occurrence the left-admissible sequence.

Assume $\rho = \{r_i\}_{i=1}^n$ is a finite set of reduced monomial generators for an ideal $\mathcal{I}$ in $k\Gamma$, k a field. Set $\Lambda = k\Gamma/\mathcal{I}$. The graph in this construction will be denoted by Ψ_Λ. Let $\Psi_\Lambda =$

$\{\Psi_0, \Psi_1\}$, where Ψ_0 are the vertices and Ψ_1 the arrows. We first describe the construction of Ψ_0, then Ψ_1.

Our primary example of illustration throughout this chapter is found in a continuance of the study of Example 3.3 from our last chapter. We repeat its definition here for the convenience of the reader:

EXAMPLE 4.1. Let Γ be the quiver given by:

$$\begin{tikzcd}
& \cdot \arrow[r, "x"] & \cdot \arrow[dr, "a"] & & & \cdot \arrow[r, "d"] & \cdot \arrow[r, "e"] & \cdot \\
& & & \cdot \arrow[r, "b"] & \cdot \arrow[ur, "c"] \arrow[dr, "h"] & & & \\
\cdot \arrow[r, "y"] & \cdot \arrow[r, "f"] & \cdot \arrow[ur, "g"] & & & \cdot & &
\end{tikzcd}$$

For k a field, set $\Lambda = k\Gamma/\langle\rho\rangle$ where $\rho = \{xab, bcd, yfgb, bh, gbc, cde\}$.

0.1. Vertices of Ψ_Λ. The set of vertices Ψ_0 is the disjoint union of the following three sets:

$$\Psi_0 = B \cup T \cup V.$$

The set B is a distinguished set of vertices in one-to-one correspondence with the elements of ρ, defined as:

$$B = \{\mathfrak{b}(r) \mid \text{for each } r \in \rho\}.$$

This set should be thought of as the set of all possible *beginning relations* in left-admissible sequences of elements from ρ, each of which will be indicated in the graph Ψ_Λ by a box and labeled with the respective relation.

The set T is a distinguished set of vertices in one-to-one correspondence with the elements of ρ; that is:

$$T = \{\mathfrak{t}(r) \mid \text{for each } r \in \rho\}.$$

This set indicates all possible *terminal relations* in left-admissible sequences of elements from ρ, and for which each such vertex will be indicated by an oval and labeled with the respective relation.

In practice, when we write the sets B and T, we simply identify them with the set ρ.

The set V is in one-to-one correspondence with the set of all possible intersection words in left-admissible sequences formed from elements in ρ. We define this set as:

$$V = \{\mathfrak{v}(o) \mid o \vdash \gamma; \gamma \in \Gamma_n, n \geq 3\}.$$

Each element $\mathfrak{v}(o)$ of V is indicated in Ψ_Λ by (o), and in practice, the elements in V are written as (o) rather than $\mathfrak{v}(o)$.

Hence, it is clear that when ρ is a finite set, so is Ψ_0.

For our Example 3.3, we have the following sets:

$$B = T = \rho,$$

$$V = \{(b), (gb), (bc), (c), (cd)\}$$

We graphically record the vertices for Ψ_Λ, for our example, as follows:

$$\begin{array}{cccc}
\boxed{xab} & (b) & & xab \\
\boxed{bcd} & (cd) & & bcd \\
\boxed{yfgb} & (gb) & (c) & yfgb \\
\boxed{cde} & & & cde \\
\boxed{bh} & & & bh \\
\boxed{gbc} & (bc) & & gbc
\end{array}$$

0.2. Arrows of Ψ_Λ. Next we define the set of arrows Ψ_1 as the following disjoint union:

$$\Psi_1 = A_\rho \cup A_B \cup A_T \cup A_\Theta \cup A_C.$$

To each arrow $\hat{a} \in \Psi_1 \backslash A_C$, we will assign a label $L(\hat{a})$ which will occur in the form of a triple (u, v, w), for some appropriate $u, v, w \in \mathcal{B}_\Gamma$. To each arrow in A_C we will also assign such a triple, as well as a relation $r \in \rho$. This will allow us, in future discussions of Ψ_Λ, to determine the **underlying paths** in Γ for paths in Ψ_Λ about which we are most interested; that is, paths which correspond to left-admissible sequences, and paths which correspond to "products". This is precisely why we will refer to this label as the *underlying path label*, or *up-label*, in much of what follows. It should be said that this secondary up-label is not necessarily a unique identifier for an arrow in Ψ_Λ, but rather instead a mechanism by which we record some additional, important information about underlying paths. Algorithms to determine the underlying paths in Γ for paths in Ψ_Λ, via this labeling scheme, are provided in Appendix C. Discussion about products and paths in Ψ_Λ, and their corresponding left-admissible sequences, will occur in the next chapter.

The first subset of arrows in Ψ_1 are in one-to-one correspondence with the relations in ρ. For each $r \in \rho$, there is an arrow connecting the vertex $\mathfrak{b}(r)$ to the vertex $\mathfrak{t}(r)$; that is, we define:

$$A_\rho = \{\mathfrak{r}(r) : \mathfrak{b}(r) \to \mathfrak{t}(r) \mid \text{ for each } r \in \rho\}$$

These arrows represent left-admissible sequences of length 1. When drawing Ψ_Λ, we label each arrow $\mathfrak{r}(r)$ from A_ρ with the unusual up-label $L(\hat{a}) = (r, r, r)$. It is unusual in the fact that there is no real decomposition of $\hat{a}$ into intersection words or sequence arrow words. Our choice is therefore to keep some consistency in the labeling of arrows of Ψ_Λ so that algorithms, such as those found in Appendix C, are easier to write. In practice, we will identify the set A_ρ with the set ρ, and we will often dispense with the drawing of these arrows in our graphical representations of Ψ_Λ, as to not clutter those diagrams. Nevertheless, they are an important feature of Ψ-graphs, and so, at the very least, we mustn't exclude them from our *interpretations* of the data encoded within Ψ_Λ.

The set A_B of *beginning sequence arrows* we define as:

$$A_B = \{\mathfrak{y}(\gamma) : \mathfrak{b}(r) \to \mathfrak{v}(o) \mid \gamma \in \Gamma_3; \gamma = \overrightarrow{rr'} = poq; r, r' \in \Gamma_2; o \vdash \gamma\}.$$

This is the set of arrows which connect vertices from B to vertices in V and which correspond to the beginning relations in left-admissible sequences of length ≥ 2. The elements in A_B will be represented in Ψ_Λ by hook-tailed arrows: $\hookrightarrow$. By definition, to each arrow $\hat{a} = \mathfrak{y}(\gamma)$ in A_B there corresponds a relation $r \in \rho$ with decomposition $r = po$, such that $o \vdash \gamma$ and $p, q \in \mathcal{B}_\Gamma$. When drawing an arrow $\hat{a}$ in Ψ_Λ, we label it (somewhat) respective of this decomposition with $L(\hat{a}) = (\sigma(r), p, o)$. There is an important subset of A_B consisting of those arrows $\mathfrak{y}(\gamma)$ where for $\gamma = poq$, the path p is an arrow in Γ_1. We denote this subset by:

$$A_B^* = \{\mathfrak{y}(\gamma) \in A_B \mid \text{ where for } \gamma \in \Gamma_3; \gamma = poq; p \in \Gamma_1\}.$$

Elements in A_B^* will be indicated in Ψ_Λ by hook-tailed arrows with an asterisk: $\overset{*}{\hookrightarrow}$, and labeled via the prescription given above for all elements in A_B.

The set A_T of *terminal sequence arrows* we define as:

$$A_T = \{\mathfrak{z}(\gamma) : \mathfrak{v}(o) \to \mathfrak{t}(r) \mid \gamma \in \Gamma_n; \gamma = \overrightarrow{r_1 \cdots r_{n-1}}; r = r_{n-1}; o \vdash_{n-1} \gamma\}.$$

This is the set of arrows which connect vertices from V to vertices in T and which correspond to the ending relations in left-admissible sequences of length $n \geq 3$. Elements in the set A_T will be indicated in Ψ_Λ by double-tipped arrows: $\twoheadrightarrow$. We wish to label these arrows in Ψ_Λ, and notice: to each arrow $\hat{a} = \mathfrak{z}(\gamma)$ in A_T, there corresponds a relation $r \in \rho$ with the decomposition $r = oq$, for $o \vdash_{n-1} \gamma$ and $q \in \mathcal{B}_\Gamma$. We label each arrow $\hat{a}$ in Ψ_Λ with $L(\hat{a}) = (o, \tau(o), q)$, respective of this decomposition.

Next, we describe the set of arrows A_Θ that correspond to relations in left-admissible sequences of length ≥ 3, but which do not correspond to beginning or terminal relations. To simplify notation, we first introduce the set Θ which characterizes the occurrences of relations from ρ in left-admissible sequences:

$$\Theta = \{(o, p, o') \mid opo' = r \text{ for some } r \in \rho, \text{ with } v(o), v(o') \in V, \text{ and } p \in \mathcal{B}_\Gamma\}.$$

It is worth noting that a relation $r \in \rho$ maybe decomposed into a triple (o, p, o') in more than one way, where such decompositions are given by their occurrences in left-admissible sequences. More precisely, if $\theta_1 = (o, p, o') \in \Theta$ with $r = opo'$, then there may exist $\theta_2 =$

(o'', p', o''') with $r = o''p'o'''$ where $(o,p,o') \neq (o'',p',o''')$. With this understanding of indexing by Θ, we create the set of arrows A_Θ in one-to-one correspondence with Θ:

$$A_\Theta = \{\mathfrak{a}(\theta) : \mathfrak{v}(o) \to \mathfrak{v}(o') \mid \text{where } \theta \in \Theta, \text{ with } \theta = (o,p,o')\}.$$

Elements in A_Θ will be indicated in Ψ_Λ by regular arrows: $\to$. We label each arrow $\hat{a} = \mathfrak{a}(\theta)$ in Ψ_Λ with its respective decomposition: $L(\hat{a}) = (o,p,o')$. We will have cause to refer to the subset $A^*_\Theta \subseteq A_\Theta$, consisting of those arrows $\mathfrak{a}(\theta)$ which correspond to elements of Θ with the middle factor p a path of length zero in $\mathcal{B}_\Gamma$. We index this set with the following subset of Θ:

$$\Theta^* = \{\theta \in \Theta \mid p \in \Gamma_0 \text{ for } \theta = (o,p,o')\}.$$

We then have:

$$A^*_\Theta = \{\mathfrak{a}(\theta) : \mathfrak{v}(o) \to \mathfrak{v}(o') \mid \theta = (o,p,o') \in \Theta^*\}.$$

The set of arrows A^*_Θ correspond to possible adjacencies in left-admissible sequences, and will be represented in Ψ_Λ by regular arrows with an asterisk: $\xrightarrow{*}$.

Finally, we create the set A_C of *continuation arrows* which play an important role in determining the existence of left-admissible sequences of arbitrary length that correspond to generators in $E(\Lambda)$. We define A_C as follows:

$$A_C = \{\mathfrak{c}(a,r) : \mathfrak{v}(o) \to \mathfrak{y}(r) \mid a = \mathfrak{a}(\theta) \in A^*_\Theta; \theta = (o,p,o'); r = o'q; r \in \rho; q \in \mathcal{B}_\Gamma\}.$$

Again, we are careful in understanding the indexing of this set. A_C is indexed over the set of tuples of the form (a,r), where $a = \mathfrak{a}(\theta) \in A^*_\Theta$ and $r \in \rho$ with $\theta = (o,p,o')$ such that $r = o'q \in \rho$ for some $q \in \mathcal{B}_\Gamma$ (with $l(q) \geq 1$). Thus, to each arrow $\mathfrak{c}(a,r)$ in A_C there is associated an adjacency arrow $a = \mathfrak{a}(\theta) \in A^*_\Theta$ and a relation $r \in \rho$. Elements in A_C will be represented in Ψ_Λ by dashed arrows: $\dashrightarrow$. To each arrow $\mathfrak{c}(a,r)$ in A_C we assign the label $((o,p,o'),r)$, where $a = \mathfrak{a}(\theta)$ for some $\theta = (o,p,o')$, and $r \in \rho$ for some $r = o'q$, with $q \in \mathcal{B}_\Gamma$. This set is particularly interesting since the elements in A_C will indicate possible product paths in Ψ_Λ; the term "product path" will be defined in the next chapter.

For Example 3.3, we have the following sets:

$$A_B = \{ \boxed{\text{xab}} \hookrightarrow (b), \boxed{\text{bcd}} \hookrightarrow (cd), \boxed{\text{yfgb}} \hookrightarrow (gb), \boxed{\text{yfgb}} \hookrightarrow (b), \boxed{\text{gbc}} \hookrightarrow (bc) \};$$

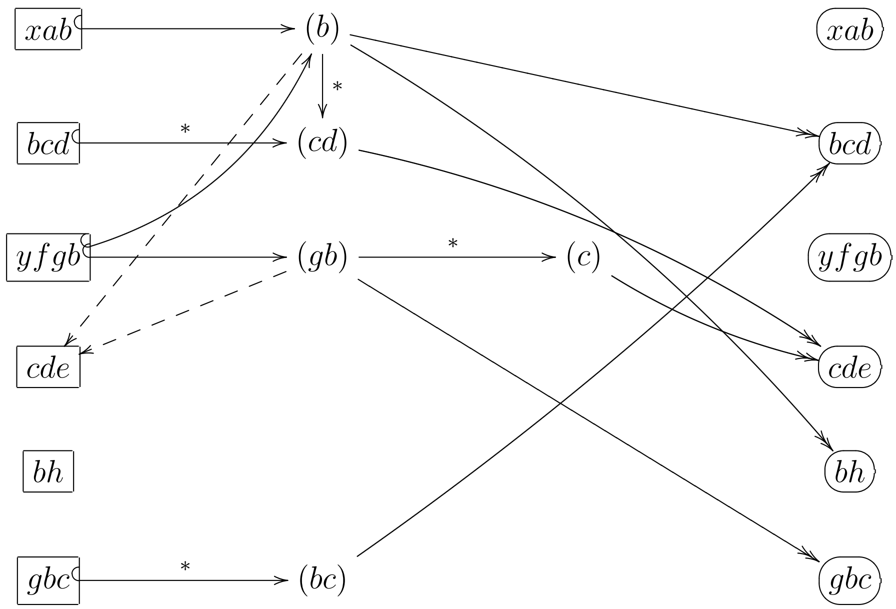
$$A_T = \{ (b) \twoheadrightarrow \boxed{\text{bcd}}, (b) \twoheadrightarrow \boxed{\text{bh}}, (cd) \twoheadrightarrow \boxed{\text{cde}}, (gb) \twoheadrightarrow \boxed{\text{gbc}}, (cd) \twoheadrightarrow \boxed{\text{cde}}, (bc) \twoheadrightarrow \boxed{\text{bcd}} \};$$

$$A_\Theta = A^*_\Theta = \{ (b) \xrightarrow{*} (cd), (gb) \xrightarrow{*} (c) \};$$

and:

$$A_C = \{ (gb) \dashrightarrow \boxed{\text{cde}}, (b) \dashrightarrow \boxed{\text{cde}} \}.$$

Including these (unlabeled) arrows in the Ψ-graph for our example yields:



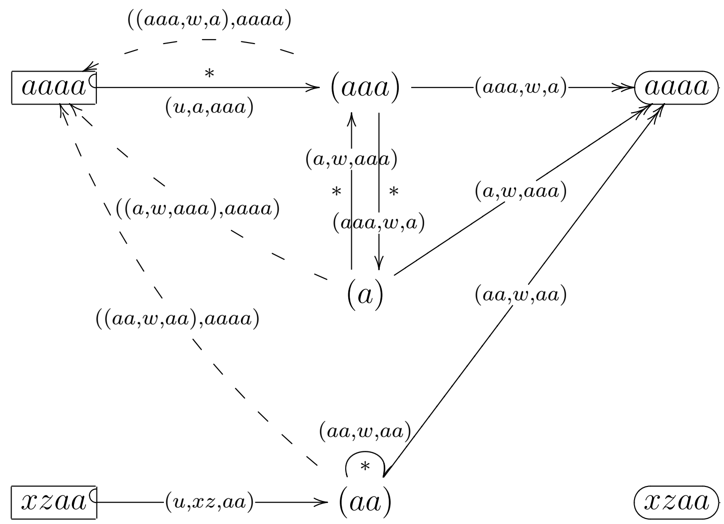
We are careful to remind ourselves that the arrows from A_ρ are not included in this figure. We will continue to follow this practice, when possible, throughout this document in order to keep the figures less cluttered.

As a final remark to this chapter, we point out that arrow labels were not shown on our main example. In some cases, this is entirely permissible, and even desirable, since it reduces the number of objects needed to indicate Ψ_Λ accurately. There are cases, however, when it is necessary to have all arrows labeled, so that the underlying paths in Γ maybe be easily retrieved; for example, when using the algorithms provided in Appendix C. In Chapter 6, we will have more to say about when we may dispense with the up-labels. Nevertheless, an example wherein arrow labels are included appears below.

Example 4.2. Let G be given by the quiver:

$$\begin{tikzcd} \cdot \arrow[r, "x"] & \cdot \arrow[r, "z"] & \cdot \arrow[loop right, "a"] \end{tikzcd}$$

and let $\rho = \{aaaa, xzaa\}$. We then set $\Lambda = k\Gamma/\langle\rho\rangle$, for a field k, and construct Ψ_Λ which is given as follows:



We will return to this example in Chapter 6, where will show that the inclusion of up-labels are entirely necessary here, when analyzing $E(\Lambda)$ based on this Ψ_Λ.

CHAPTER 5

General Properties of Ψ_Λ

When interpreting the information encoded in the Ψ-graph Ψ_Λ for a monomial algebra Λ, it is important to understand what particular paths represent. In the most practical sense, a left-admissible sequence of elements in ρ corresponds to a path $\hat{p} \in \Psi_\Lambda$ that starts at a box $\boxed{\text{a}}$ and ends at a rounded box (oval) $\boxed{\text{b}}$. So, looking from left to right in Ψ_Λ, we may read off paths that correspond to left-admissible sequences, which in turn, correspond to elements in the sets Γ_n. The elements of Γ_n are of primary interest to us, since they help us to determine if $E(\Lambda)$ is finitely generated, as indicated in Chapter 3. However, we must remain cautious in interpreting the paths in Ψ_Λ, since it is possible that a path in Ψ_Λ is somehow superfluous to our studies regarding $E(\Lambda)$. That is, a path that starts at a box and ends at an oval need not correspond to a left-admissible sequences of elements in ρ. In this chapter we will show that such superfluous paths can occur in Ψ_Λ, and we show precisely when this happens. This allows us to correctly interpret Ψ_Λ as a vehicle for encoding information about the generators of $E(\Lambda)$.

Throughout this chapter we will use $\mathcal{B}_{\Psi_\Lambda}$ to denote the set of all paths in the graph Ψ_Λ, where $\Lambda = k\Gamma/\langle\rho\rangle$ is the monomial algebra of interest. Much of the vocabulary and definitions included in this chapter will enable us to easily refer to the paths in $\mathcal{B}_{\Psi_\Lambda}$ which have corresponding left-admissible sequences of elements in ρ, and those which do not. We will have cause to continually refer to a particular subset of $\mathcal{B}_{\Psi_\Lambda}$, so we begin with its description: the set of all paths in $\mathcal{B}_{\Psi_\Lambda}$ having starting vertices in B and ending vertices in T we denote by $\mathcal{P}_{\Psi_\Lambda}$. Furthermore, let $\mathcal{P}seudo_{\Psi_\Lambda}$ be the subset of $\mathcal{P}_{\Psi_\Lambda}$ of all paths containing no continuation arrows. We refer to the set $\mathcal{P}seudo_{\Psi_\Lambda}$ as the set of all *pseudo-proper paths* in $\mathcal{B}_{\Psi_\Lambda}$.

Since we are most interested in those elements of $\mathcal{P}seudo_{\Psi_\Lambda}$ that correspond to left-admissible sequences of elements in ρ, we pause here to make clear the relationship between paths in $\mathcal{P}seudo_{\Psi_\Lambda}$ and those in $\mathcal{B}_\Gamma$. First, we wish to define the notion of an *underlying path* for an element $\hat{p} \in \mathcal{P}seudo_{\Psi_\Lambda}$. The intention here is to refer to underlying path $p \in \mathcal{B}_\Gamma$ that corresponds to the path $\hat{p} \in \mathcal{P}seudo_{\Psi_\Lambda}$.

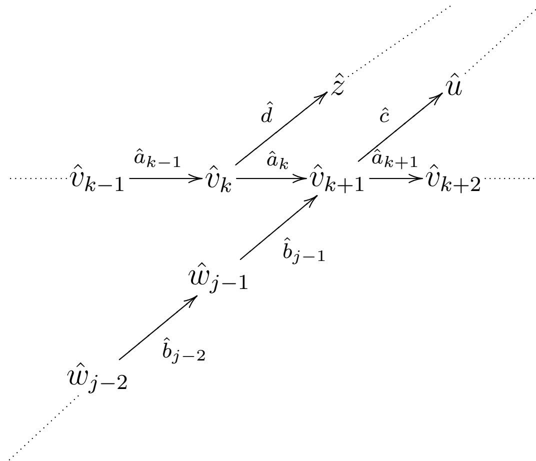
Let $\hat{p} \in \mathcal{P}seudo_{\Psi_\Lambda}$. If the length of $\hat{p}$ is one, then $\hat{p}$ is an element of the set A_ρ; say $\hat{p} = \mathfrak{r}(r)$ for some relation $r \in \rho$. This corresponds to a left-admissible sequence $\langle r\rangle$ of length 1, and has underlying path $\overrightarrow{r} \in \mathcal{B}_\Gamma$. We denote this underlying path by p. If $l(\hat{p}) > 1$, then there exists some left-overlap sequence $(r_1, ...r_{l(\hat{p})})$ that corresponds to the path $\hat{p}$. This left-overlap sequence has an underlying path $\overrightarrow{r_1, ...r_{l(\hat{p})}}$ which we also denote by p. We now set $\mathcal{P}roper_{\Psi_\Lambda}$ be the subset of $\mathcal{P}seudo_{\Psi_\Lambda}$ consisting of those paths $\hat{p} \in \mathcal{P}seudo_{\Psi_\Lambda}$ for which there exists a left-admissible sequence $\langle r_1, ..., r_{l(\hat{p})}\rangle$ such that, for the underlying path p in $\mathcal{B}_\Gamma$, we have $p = \overrightarrow{r_1, ..., r_{l(\hat{p})}}$, for some $r_i \in \rho$. We refer to $\mathcal{P}roper_{\Psi_\Lambda}$ as the set of *proper paths* in $\mathcal{B}_{\Psi_\Lambda}$.

For some classes of algebras, as we will see, the sets $\mathcal{P}seudo_{\Psi_\Lambda}$ and $\mathcal{P}roper_{\Psi_\Lambda}$ are identical. Our first goal is to characterize exactly when elements of $\mathcal{P}seudo_{\Psi_\Lambda}$ that are not members of $\mathcal{P}roper_{\Psi_\Lambda}$. We need only consider paths of length 2 or greater in $\mathcal{P}seudo_{\Psi_\Lambda}$, since each path of length 1 is contained in both $\mathcal{P}seudo_{\Psi_\Lambda}$ and $\mathcal{P}roper_{\Psi_\Lambda}$. In this discussion it is useful to recall a graph theoretic convention of writing a path in a directed graph as an alternating sequence of vertices and arrows. For example, if $\hat{p} \in \mathcal{P}seudo_{\Psi_\Lambda}$, we may write $\hat{p} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_n, \hat{v}_{n+1}$, where $\hat{v}_i \in \Psi_0$ and $\hat{a}_i \in \Psi_1$ for $1 \le i \le n$.

Now, let $\hat{p} \in \mathcal{P}seudo_{\Psi_\Lambda}$, $\hat{p} \notin \mathcal{P}roper_{\Psi_\Lambda}$, with $\hat{p} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_n, \hat{v}_{n+1}$, for $\hat{v}_i \in \Psi_0$ and $\hat{a}_i \in \Psi_1$ for $1 \le i \le n$, and $n \ge 2$. By the construction of Ψ_Λ, there must be some maximal k where the subpath $\hat{q} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_k, \hat{v}_{k+1}$ corresponds to a left-admissible sequence of elements in ρ along the underlying path q, but the subpath $\hat{r} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_{k+1}, \hat{v}_{k+2}$ does not correspond to a left-admissible sequence along the underlying path r. This must occur for some $k+1$ where $1 \le k < n-1$, since $\hat{p} \notin \mathcal{P}roper_{\Psi_\Lambda}$. Furthermore, this implies $\hat{v}_{k+1} \in V$; namely, $\hat{v}_{k+1} \notin B$ and $\hat{v}_{k+1} \notin T$.

Since $\hat{q}$ corresponds to a left-admissible sequence along the underlying path q, but $\hat{r}$ does not correspond to a left-admissible sequence along the underlying path r, there must exist some $\hat{u} \in \Psi_0$ and some $\hat{c} \in \Psi_1$ such that the path $\hat{s} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_k, \hat{v}_{k+1}, \hat{c}, \hat{u}$ corresponds to a left-admissible sequence of elements in ρ, where $s \ne r$. Additionally, since the arrow $\hat{a}_{k+1}$ corresponds to a relation in a left-admissible sequence, there must exist a path $\hat{t} = \hat{w}_1, \hat{b}_1, \hat{w}_2, ..., \hat{w}_j, \hat{a}_{k+1}, \hat{v}_{k+2}$ corresponding to a left-admissible sequence along t, for which $t \ne p$, and having $\hat{w}_1 \in B$ as well as $\hat{w}_i \in \Psi_0$, $\hat{b}_i \in \Psi_1$ for $1 \le i \le j$. Furthermore, $\hat{w}_j = \hat{v}_{k+1}$ in the path $\hat{t}$, and $j \ge 1$ since $\hat{v}_{k+1} \in V$.

Finally, since $\hat{p}$ does not correspond to an admissible sequence along the underlying path p, there must exist some $\hat{z} \in \Psi_0$ and some $\hat{d} \in \Psi_1$ such that the path $\hat{e} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_{k-1}, \hat{v}_k, \hat{d}, \hat{z}$ corresponds to a left-admissible sequence along p. Therefore, if Ψ_Λ has a path $\hat{p}$ that is in $\mathcal{P}seudo_{\Psi_\Lambda}$ but not in $\mathcal{P}roper_{\Psi_\Lambda}$, it must contain the following structure, where in terms of underlying paths, $s \ne r$ and $t \ne p$:



This discussion provides proof of the following technical lemma.

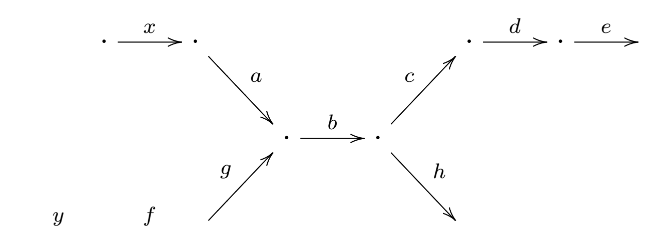
Lemma 5.1. *Let $\Lambda = k\Gamma/\mathcal{I}$ be a monomial algebra, where $\mathcal{I}$ has a finite reduced monomial generating set, and let Ψ_Λ be its associated Ψ-graph. If Ψ_Λ has a path $\hat{p}$ that is in $\mathcal{P}seudo_{\Psi_\Lambda}$ but not in $\mathcal{P}roper_{\Psi_\Lambda}$, it must contain the structure given in the figure above. Furthermore, in keeping with the notation preceding the lemma, we have that, in terms of underlying paths, $s \neq r$, $t \neq p$, and e is a subpath of p that starts at the same vertex as p.*

A partial converse of this lemma is also true, by the definition of proper path.

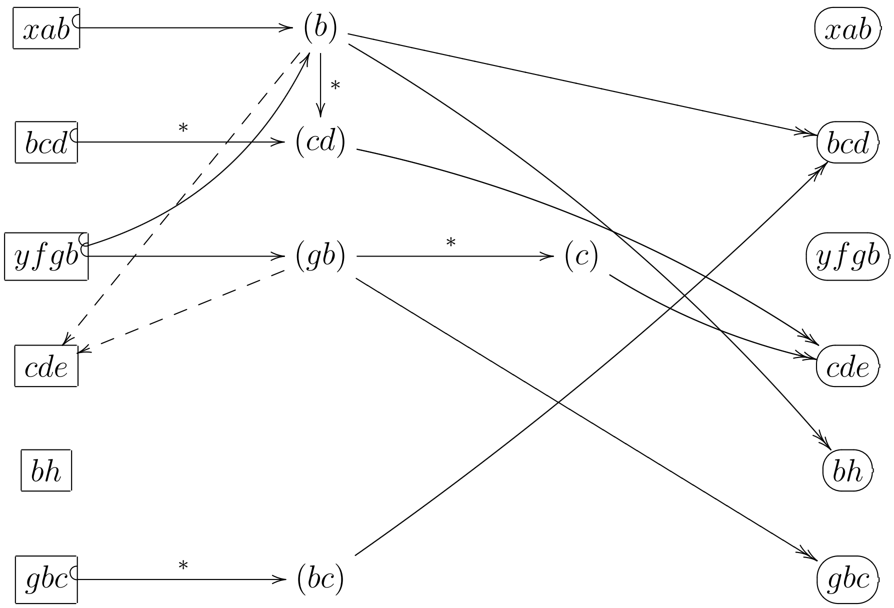
Lemma 5.2. *Let $\Lambda = k\Gamma/\mathcal{I}$ be a monomial algebra, where $\mathcal{I}$ has a finite reduced monomial generating set, and let Ψ_Λ be its associated Ψ-graph. Suppose $\hat{p} \in \mathcal{P}seudo_{\Psi_\Lambda}$, with $\hat{p} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_n, \hat{v}_{n+1}$, for $\hat{v}_i \in \Psi_0$ and $\hat{a}_i \in \Psi_1$ for $1 \leq i \leq n$, and $n \geq 2$. Also, suppose for some k with $1 < k < n$ the subpath $\hat{q}_k = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_{k-1}, \hat{v}_k$ does not correspond to an admissible sequence along the underlying path q_k, but $\hat{q}_{k-1} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_{k-2}, \hat{v}_{k-1}$ corresponds to an admissible sequence along the underlying path q_{k-1}. Then $\hat{p} \notin \mathcal{P}roper_{\Psi_\Lambda}$.*

In order to see these ideas at work, we return to Example 3.3 from Chapter 4. We repeat those diagrams here for the ease of the reader.

Example 5.3. The quiver Γ for $\Lambda = k\Gamma/\langle\rho\rangle$ is given to be:



with $\rho = \{xab, bcd, yfgb, bh, gbc, cde\}$. The resulting Ψ-graph is then:



The set $\mathcal{P}roper_{\Psi_\Lambda}$ consists of those paths that start at a box on the left-hand side of the page and end at oval on the right-hand side. To simplify the presentation of Ψ_Λ, the paths of length 1 given in the set A_ρ are not shown, but they are contained in $\mathcal{P}roper_{\Psi_\Lambda}$. The following list of paths, that correspond to left-admissible sequences of elements in ρ of length 2 or greater, complete the set $\mathcal{P}roper_{\Psi_\Lambda}$, as the reader may verify:

$$\boxed{\text{xab}} \hookrightarrow (b) \twoheadrightarrow \boxed{\text{bcd}}$$
$$\boxed{\text{xab}} \hookrightarrow (b) \longrightarrow (cd) \twoheadrightarrow \boxed{\text{cde}}$$
$$\boxed{\text{xab}} \hookrightarrow (b) \twoheadrightarrow \boxed{\text{bh}}$$
$$\boxed{\text{bcd}} \hookrightarrow (cd) \twoheadrightarrow \boxed{\text{cde}}$$
$$\boxed{\text{yfgb}} \hookrightarrow (gb) \twoheadrightarrow \boxed{\text{gbc}}$$
$$\boxed{\text{yfgb}} \hookrightarrow (gb) \twoheadrightarrow (c) \longrightarrow \boxed{\text{cde}}$$
$$\boxed{\text{yfgb}} \hookrightarrow (b) \twoheadrightarrow \boxed{\text{bh}}$$
$$\boxed{\text{gbc}} \hookrightarrow (c) \twoheadrightarrow \boxed{\text{cde}}$$

However, we notice that the following path also exists in Ψ_Λ:

$$\boxed{\text{yfgb}} \hookrightarrow (b) \twoheadrightarrow \boxed{\text{bcd}}.$$

This is clearly a path in $\mathcal{P}seudo_{\Psi_\Lambda}$, but not in $\mathcal{P}roper_{\Psi_\Lambda}$, since along the underlying path $yfgbcd$ in Γ, the relation gbc is maximal with respect to the relation $yfgb$. We may also identify in Ψ_Λ the necessary structure indicated in Lemma 5.1, if we take $\hat{v}_k$ to be $\boxed{\text{yfgb}}$. Then the vertex (b) corresponds to $\hat{v}_{k+1}$, the path $\boxed{\text{yfgb}} \hookrightarrow (b) \twoheadrightarrow \boxed{\text{bh}}$ corresponds to $\hat{s}$, $\boxed{\text{xab}} \hookrightarrow (b) \twoheadrightarrow \boxed{\text{bcd}}$ corresponds to $\hat{t}$, and $\boxed{\text{yfgb}} \hookrightarrow (gb) \twoheadrightarrow \boxed{\text{gbc}}$ to $\hat{e}$.

Paths in Ψ_Λ that contain no continuation arrows hold particular interest for us, so it will be useful and convenient to have the following definition:

Definition 5.4. Let $\hat{p}$ be a path in Ψ_Λ containing no continuation arrows. We refer to such a path as a *continuation-free path* in Ψ_Λ, or alternatively as a *cf-path*. If $\hat{p}$ is a cycle in Ψ_Λ, we refer to $\hat{p}$ as a *continuation-free cycle*, or *cf-cycle*.

We now show that if $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$, then for any continuation-free path $\hat{q}$ in $\mathcal{B}_{\Psi_\Lambda}$ there exists a proper path $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ for which $\hat{q}$ is a subpath.

Proposition 5.5. *Let Λ be a monomial algebra, whose ideal is determined by a finite reduced monomial generating set ρ, and let Ψ_Λ be its associated Ψ-graph. If $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$ then for any cf-path $\hat{q} \in \mathcal{B}_{\Psi_\Lambda}$, there exists a path $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ such that $\hat{q}$ is a subpath of $\hat{p}$.*

Proof. Suppose $\hat{q} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_m, \hat{v}_{m+1}$ is a cf-free path in $\mathcal{B}_{\Psi_\Lambda}$. If $\hat{v}_{m+1}$ is in T, we will set $\hat{x} = \hat{q}$ in the following argument. Otherwise, if the vertex $\hat{v}_{m+1}$ is not in T, there corresponds to $\hat{v}_{m+1}$ the intersection of two relations $r_1, r_2 \in \rho$ in a left-admissible sequence. By the construction of Ψ_Λ, there exists an arrow $\hat{b} \in A_T$ connecting $\hat{v}_{m+1}$ to the vertex $\sigma(r_2) \in T$. In this case we set $\hat{x} = \hat{q}\hat{b}$ in the following argument.

If $\hat{v}_1 \in B$, we set $\hat{p} = \hat{x}$, and note $\hat{p} \in \mathcal{P}seudo_{\Psi_\Lambda}$. By hypothesis $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$, and so $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ with $\hat{q}$ a subpath of $\hat{p}$.

If $\hat{v}_1 \in V$, there exists some cf-path $\hat{s}$ starting at a vertex in B and ending at the vertex $\hat{v}_1$ such that the path $\hat{p} = \hat{s}\hat{x}$ corresponds to an admissible sequence along the underlying path p. Otherwise, we contradict Lemma 5.2. So $\hat{p} \in \mathcal{P}seudo_{\Psi_\Lambda}$, and therefore $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ by hypothesis. Thus $\hat{q}$ is a subpath of $\hat{p}$ for $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$.

If $\hat{v}_1 \in T$, then $\hat{x} = \hat{v}_1$. By definition, $\hat{v}_1$ corresponds to a relation $r \in \rho$, and so there exists an arrow $\hat{p} \in A_\rho$ that corresponds to r. Since $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$, and $\hat{q}$ is a subpath of $\hat{p}$, we are done. $\square$

As we will see in the next chapter, there are well-known cases of monomial algebras where $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$. Among others, cycle algebras and in-spoked cycle algebras will have this property. In these cases, the fact that $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$ simplifies the interpretation of Ψ_Λ, since all paths in $\mathcal{P}_{\Psi_\Lambda}$ will correspond to left-admissible sequences, and, in turn, to generators or products of generators in $E(\Lambda)$.

CHAPTER 6

Results for Cycle and In-Spoked Cycle Algebras

We specialize now our treatment of Ψ_Λ, and investigate two classes of algebras: cycle algebras and in-spoked cycle algebras. In the beginning of this chapter we will take great pains in understanding the basic structure of Ψ_Λ for a cycle algebra. But first, we provide the definition of a cycle algebra. Let k be a field, and Γ a quiver consisting of only a single cycle. Let ρ be a finite reduced set of monomial generators for an ideal $\mathcal{I} \subset k\Gamma$. We refer to $\Lambda = k\Gamma/\mathcal{I}$ as a *cycle algebra*.

1. Structure of Ψ_Λ

There is an easy corollary to Lemma 5.1 concerning the Ψ-graph for a cycle algebra. The point of it is to notice that in the structure of a Ψ-graph for a cycle algebra, the sets $\mathcal{P}roper_{\Psi_\Lambda}$ and $\mathcal{P}seudo_{\Psi_\Lambda}$ are identical.

COROLLARY 6.1. *Let $\Lambda = k\Gamma/\mathcal{I}$ be a cycle algebra, and Ψ_Λ its associated Ψ-graph. Then $\mathcal{P}roper_{\Psi_\Lambda} = \mathcal{P}seudo_{\Psi_\Lambda}$.*

PROOF. Since Λ is a cycle algebra, there can be no such path s different from r in Γ for the paths $\hat{s}$ and $\hat{r}$ as in Lemma 5.1. $\square$

This seemingly innocuous corollary has an important implication when interpreting the information encoded in Ψ_Λ for a cycle algebra: any cf-path $\hat{p}$ traced from a box to an oval in Ψ_Λ will correspond either to a generator in $E(\Lambda)$, or a product of generators in $E(\Lambda)$. This is a consequence of the fact that such a path $\hat{p}$ necessarily corresponds to a left-admissible sequence of elements in ρ. Nonetheless, we move forward in our understanding of Ψ_Λ for a cycle algebra, and define what it means for two distinct cycles in Ψ_Λ to be *disjoint*.

DEFINITION 6.2. Let $\mathcal{C}_1, \mathcal{C}_2$ be two distinct cf-cycles in Ψ_Λ, and A_Θ the set of arrows associated to Ψ_Λ as described in Chapter 4. If there does not exist a path joining $\mathcal{C}_1$ and $\mathcal{C}_2$, comprised solely of arrows from A_Θ, and $\mathcal{C}_1, \mathcal{C}_2$ share no vertices, then we say $\mathcal{C}_1$ is *disjoint* from $\mathcal{C}_2$.

It is worth mentioning that for a cf-cycle $\mathcal{C}_1 = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_n, \hat{v}_1$ in Ψ_Λ, all vertices on this cycle are contained in the set V, since the indegree and the outdegree of v_i is ≥ 1 for all $1 \leq i \leq n$. Our next result continues to refine our understanding of the Ψ-graph of a cycle algebra in showing that distinct cf-cycles are disjoint. We recall that in our definition of cycle, we require its path length to be ≥ 1.

Lemma 6.3. *Let $\Lambda = k\Gamma/\mathcal{I}$ be a cycle algebra, and Ψ_Λ its associated Ψ-graph. Any two distinct cf-cycles $\mathcal{C}_1, \mathcal{C}_2$ in Ψ_Λ are disjoint.*

Proof. Let $\mathcal{C}_1, \mathcal{C}_2$ be distinct cf-cycles in Ψ_Λ, and let $\mathcal{C}_1 = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_n, \hat{v}_1$, $\mathcal{C}_2 = \hat{u}_1, \hat{b}_1, \hat{u}_2, \hat{b}_2, ..., \hat{b}_m, \hat{u}_1$ denote the alternating sequences of vertices and arrows determining the respective paths for $\mathcal{C}_1, \mathcal{C}_2$ in Γ; for some $m, n \geq 1$. We have two cases to consider.

First, if we assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ share some common vertex $\hat{w}$, then $\hat{w}$ lies on both $\mathcal{C}_1$ and $\mathcal{C}_2$. For the sake of simplicity of argument, we re-index the cycles so that $\hat{w} = \hat{v}_1 = \hat{u}_1$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are distinct, there exists some minimal k such that the subpath $\hat{q}_k = \hat{v}_1, \hat{a}_1, ..., \hat{a}_{k-1}, \hat{v}_k$ lies on both $\mathcal{C}_1$ and $\mathcal{C}_2$, but the subpath $\hat{q}_{k+1} = \hat{v}_1, \hat{a}_1, ..., \hat{a}_k, \hat{v}_{k+1}$ does not lie on $\mathcal{C}_2$, with $1 \leq k < min(m, n)$. Since $\mathcal{C}_2$ is a cycle, there exist some arrow $\hat{c} \in A_\Theta$ and some vertex $\hat{y} \in V$ such that $\hat{c} \neq \hat{a}_k$, and such that the subpath $\hat{v}_1, \hat{a}_1, ..., \hat{a}_{k-1}, \hat{v}_k, \hat{c}, \hat{y}$ lies on $\mathcal{C}_2$. By Proposition 5.5 there exists a path $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ for which $\hat{q}_k$ is a subpath. So, there exists some cf-path $\hat{s} \in \mathcal{B}_{\Psi_\Lambda}$ that starts at a vertex in B and ends at the vertex $\hat{w}$ and there exists some cf-path $\hat{t} \in \mathcal{B}_{\Psi_\Lambda}$ that starts at the vertex $\hat{v}_k$ and ends at some vertex in T so that $\hat{p} = \hat{s}\hat{q}_k\hat{t}$.

Suppose the path $\hat{x} = \hat{s}\hat{q}_k\hat{a}_k$ corresponds to an admissible sequence along the underlying path x. Then since Γ is a cycle, $\hat{c}$ must be equal to $\hat{a}_k$. Since this must be true for each k, $\mathcal{C}_1 = \mathcal{C}_2$. Suppose now that $\hat{x}$ does not correspond to an admissible sequence along the underlying path x. The beginning vertex for the path $\hat{x}$ lies in B. If the last vertex in the path $\hat{x}$ lies in T then $\hat{x} \in \mathcal{P}seudo_{\Psi_\Lambda}$ and $\hat{x} \notin \mathcal{P}roper_{\Psi_\Lambda}$. This contradicts the fact that $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$, given by Corollary 6.1 for the cycle algebra Λ. If the last vertex $\hat{v}_{k+1}$ in the path $\hat{x}$ does not lie in T, then there corresponds to $\hat{v}_{k+1}$ the intersection of two relations $r_1, r_2 \in \rho$ in a left-admissible sequence. By the construction of Ψ_Λ, there exists an arrow $\hat{d} \in A_T$ connecting $\hat{v}_{k+1}$ to the vertex $\sigma(r_2) \in T$. But then the path $\hat{x}\hat{d}$ is in $\mathcal{P}seudo_{\Psi_\Lambda}$ but not in $\mathcal{P}roper_{\Psi_\Lambda}$. And so $\hat{x}$ must correspond to an admissible sequence along x and we arrive at our previous case.

We now consider the case when $\mathcal{C}_1$ and $\mathcal{C}_2$ share no common vertices, but Ψ_Λ contains a path joining the two cycles. In this case, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are not disjoint, there must be some path $\hat{q}$ of length 1 or greater where $\hat{q} = \hat{w}_1, \hat{c}_1, ..., \hat{c}_{k-1}, \hat{w}_k$. Additionally, $\hat{q}$ starts at some vertex $\hat{w}_1$ on one cycle and ends at some vertex $\hat{w}_k$ on the other cycle, and all vertices $\hat{w}_i$ lie on neither $\mathcal{C}_1$ nor $\mathcal{C}_2$ for $1 < i < k$. Without loss of generality, we set $\hat{w}_1 = \hat{v}_2$ on $\mathcal{C}_1$ and $\hat{w}_k = \hat{u}_m$ on $\mathcal{C}_2$.

The argument proceeds similarly to that above, where by Proposition 5.5 there exists a path $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ for which $\hat{a}_1$ is a subpath. Then there exists some path $\hat{s}$ that starts at a vertex in B and ends at the vertex $\hat{v}_1$ and there corresponds to the path $\hat{s}\hat{a}_1$ a left-admissible sequence along the path sa_1. But now in considering the left-admissible sequences that correspond to $\hat{s}\hat{a}_1\hat{a}_2$ and to $\hat{s}\hat{a}_1\hat{c}_1$ it is easy to see that they must be the same sequence, else contradict Γ being a cycle or contradict $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$. □

In referring to paths in $\mathcal{P}_{\Psi_\Lambda}$, it is useful to establish some notation regarding continuation arrows. Let $\hat{p}$ be a path in $\mathcal{B}_{\Psi_\Lambda}$ having its starting vertex in B, its ending vertex in V, and containing no continuation arrows. We refer to $\hat{p}$ as a *pseudo-proper factor*, and denote by $\mathcal{F}s_{\Psi_\Lambda}$ the set of all pseudo-proper factors in $\mathcal{B}_{\Psi_\Lambda}$. We are most interested in the situation where our paths in $\mathcal{B}_{\Psi_\Lambda}$ correspond to left-admissible sequences, which again in this context we will refer to as "proper". Let $\hat{p}$ be a path in $\mathcal{F}s_{\Psi_\Lambda}$. We say the path $\hat{p}$ is a *proper factor* if there exists a left-admissible sequence $\langle r_1, ..., r_n\rangle$, for $r_i \in \rho$ and $n = l(\hat{p})$, such that, for the underlying path p in the quiver for Λ, we have $p = \overrightarrow{r_1, ..., r_n}$. We denote by $\mathcal{F}_{\Psi_\Lambda}$ the set of all proper factors in $\mathcal{B}_{\Psi_\Lambda}$.

We factor a path $\hat{p} \in \mathcal{P}_{\Psi_\Lambda}$ at the continuation arrows and write $\hat{p} = \hat{p}_1 \cdot \hat{p}_2 \cdot ... \cdot \hat{p}_n$, where there is a continuation arrow connecting the subpaths $\hat{p}_i$ to $\hat{p}_{i+1}$ for $1 \leq i \leq n-1$. Furthermore, each $\hat{p}_i$ is a pseudo-proper factor, for $1 \leq i \leq n-1$, and $\hat{p}_n$ is a pseudo-proper path. Within this context, our primary interest lies in those paths $\hat{p}$ in $\mathcal{P}_{\Psi_\Lambda}$ which are either proper paths, or those whose factorization $\hat{p} = \hat{p}_1 \cdot \hat{p}_2 \cdot ... \cdot \hat{p}_n$ consists only of proper factors for $\hat{p}_i$, for $1 \leq i \leq n-1$, and where $\hat{p}_n$ is a proper path.

Let $\hat{p}$ be a path in $\mathcal{P}_{\Psi_\Lambda}$ of length 2 or greater. There are two cases in which we consider $\hat{p}$ to be a *product path*. First, if the first arrow in the path $\hat{p}$ is contained in the set A_B^*, we say $\hat{p}$ is a product path. Second, if $\hat{p}$ is a path in $\mathcal{P}_{\Psi_\Lambda}$ containing at least one continuation arrow, we refer to $\hat{p}$ as a product path if, in the factorization $\hat{p} = \hat{p}_1 \cdot \hat{p}_2 \cdot ... \cdot \hat{p}_n$, each $\hat{p}_i$ is a proper factor, for $1 \leq i \leq n-1$, and where $\hat{p}_n$ is a proper path. We denote by $\mathcal{P}rod_{\Psi_\Lambda}$ the set of all product paths in $\mathcal{B}_{\Psi_\Lambda}$. If we need access to the continuation arrows directly, we may occasionally write such a factorization as $\hat{p} = \hat{p}_1\hat{c}_1\hat{p}_2\hat{c}_2...\hat{c}_{n-1}\hat{p}_n$, where $\hat{c}_i \in A_C$ and $\hat{p}_i$ is a proper factor, for each $1 \leq i < n$, and where $\hat{p}_n$ is a proper path.

There is a natural extension of the idea of the underlying path of a proper factor to that of the underlying path of a product path. Let $\hat{p}$ be a path in $\mathcal{P}rod_{\Psi_\Lambda}$, and $\hat{p} = \hat{p}_1 \cdot \hat{p}_2 \cdot ... \cdot \hat{p}_n$ be its factorization, where each $\hat{p}_i$ is a proper factor for $1 \leq i \leq n-1$, and $\hat{p}_n$ is a proper path. Since there is a well-defined multiplication on elements of $\mathcal{B}_\Gamma$, we define the underlying path for $\hat{p}$ to be $p = p_1...p_n$. That is, p is the multiplication of the underlying paths of the factors in $\hat{p}$.

For a vertex $\hat{v}$ in Ψ_Λ, we say Ψ_Λ has a *proper fork* at $\hat{v}$ if there exist arrows $\hat{a}_1, \hat{a}_2 \in \Psi_1 \backslash A_C$ such that $\sigma(\hat{a}_1) = \sigma(\hat{a}_2) = \hat{v}$, and $a_1 \neq a_2$ in the underlying paths. In general, Ψ_Λ cannot have a fork at a vertex in T, since each vertex in T has outdegree 0. Even more can be said of cycle algebras: in Ψ_Λ for a cycle algebra, there are no proper forks.

LEMMA 6.4. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a cycle algebra, and Ψ_Λ its associated Ψ-graph. Ψ_Λ contains no proper forks.*

PROOF. For each vertex $\hat{b} \in B$, the outdegree is at least 1 since there exists an arrow $\hat{a}_\rho \in A_\rho$ connecting $\hat{b}$ to its respective vertex in T. The outdegree of $\hat{b}$ is at most 2, since in a cycle algebra $\hat{b}$ can correspond to the initial relation of at most one left-admissible sequence of length 2; the initial relation being indicated in Ψ_Λ by an arrow $\hat{a} \in A_B$ connecting $\hat{b}$ to some vertex in V. Since the underlying paths of the two arrows are the same, that is, $a_\rho = a$, the graph Ψ_Λ cannot have a proper fork at $\hat{b}$.

Now, let $\hat{v} \in V$. Suppose $\hat{a}_1, \hat{a}_2 \in \Psi_1 \backslash A_C$ such that $\sigma(\hat{a}_1) = \sigma(\hat{a}_2) = \hat{v}$. Let r_1 and r_2 be the respective relations in ρ whose particular decompositions determine the arrows $\hat{a}_1$ and $\hat{a}_2$. Set $\hat{s}$ to be any cf-path that starts at a vertex in B and ends at the vertex $\hat{v}$. Since $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$ for cycle algebras, there exist left-admissible sequences corresponding to $\hat{s}\hat{a}_1$ and $\hat{s}\hat{a}_2$, along sa_1 and sa_2 and ending in the relations r_1 and r_2 respectively. Since Λ is a cycle algebra, both admissible sequences must lie along the same path. Hence $r_1 = r_2$. In turn, $a_1 = a_2$, and therefore Ψ_Λ does not properly fork at $\hat{v}$. □

Lemma 6.5. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a cycle algebra, and Ψ_Λ its associated Ψ-graph. For each $\hat{v} \in V$ there exists at most one continuation arrow $\hat{a} \in A_C$ such that $\sigma(\hat{a}) = \hat{v}$.*

Proof. Let $\hat{v}$ be a vertex in V, and suppose there exists at least one continuation arrow $\hat{a} \in A_C$ such that $\sigma(\hat{a}) = \hat{v}$. Then there is a relation $r \in \rho$ and a specific decomposition $\theta = (o, p, o')$ such that $\hat{v} = \mathfrak{v}(o)$, where $r = opo'$ and $l(p) = 0$. Since ρ is a reduced set and Γ is a cycle, there exists at most one relation $s \in \rho$ such that $s = o'p$ for some $p \in \mathcal{B}_\Gamma$. Hence, there exists at most one relation $s \in \rho$ such that $\hat{a} = \mathfrak{c}(r, s)$. □

Now we turn to say more about the nature of proper paths in Ψ_Λ for a cycle algebra Λ. Since V is a finite set, Ψ_Λ has a finite number of disjoint cf-cycles. Furthermore, every proper path in Ψ_Λ is either a simple path, or lies across a single cf-cycle; we see this in the following short argument. Let $n = |V| + 2$. Then any proper path $\hat{p}$ of length greater than n must have a repeated vertex. Since the cf-cycles in Ψ_Λ are disjoint for a cycle algebra, $\hat{p}$ lies across a single cf-cycle.

2. Products in Ψ_Λ

We may now begin to characterize finite generation of $E(\Lambda)$ based on the structure of Ψ_Λ. In [**14**], Green and Zacharia give the following characterization of finite generation of $E(\Lambda)$ for a monomial algebra Λ:

Proposition 6.6. *The algebra $E(\Lambda)$ is finitely generated if and only if there exists $N > 0$ such that for every $p \in \Gamma_n$ where $n \geq N$, we can write $p = q_1...q_t$ where each path $q_i \in \Gamma_{i_j}$ for some i_j and $i_1 + ... + i_t = n$.*

In order to make our correspondence between paths in Ψ_Λ and elements in Γ_n more direct, we define a weight function on the set of all proper paths and then extend this function to the set of product paths. For a path $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$, we define $w(\hat{p}) = l(\hat{p}) + 1$. Similarly, for a proper factor $\hat{q}_i$ in the product path $\hat{q} = \hat{q}_1 \cdot \hat{q}_2 \cdot ... \cdot \hat{q}_m$, we define $w(\hat{q}_i) = l(\hat{q}_i) + 1$. We then extend this function to the set $\mathcal{P}rod_{\Psi_\Lambda}$ by setting $w : \mathcal{P}rod_{\Psi_\Lambda} \to \mathbb{Z}_{\geq 0}$, where for $\hat{q} = \hat{q}_1 \cdot \hat{q}_2 \cdot ... \cdot \hat{q}_m$ we have $w(\hat{q}) = \sum_{i=1}^{m} w(\hat{q}_i)$. We next define two of the more crucial pieces of our vocabulary.

Definition 6.7. If, for $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$, there exists $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$ such that $p = q$ and $w(\hat{p}) = w(\hat{q})$, we say $\hat{p}$ is a *product*. If, for a particular $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$, we have $p = q$ and $w(\hat{p}) = w(\hat{q})$, we say $\hat{p}$ is a *product relative to* $\hat{q}$.

With this definition, we next analyze paths in Ψ_Λ to determine which we consider to be products. Suppose $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ and $\hat{q} = \hat{q}_1 \cdot ... \cdot \hat{q}_m \in \mathcal{P}rod_{\Psi_\Lambda}$ such that $p = q$ and $l(\hat{q}) > l(\hat{p})$, where $m \geq 2$. Then the factorization for $\hat{q}$ indicates that it contains exactly $m-1$ continuation arrows, and so $l(\hat{q}) = \sum_{i=1}^{m} l(\hat{q}_i) + (m-1)$. This yields $l(\hat{q}) + 1 = \sum_{i=1}^{m}(l(\hat{q}_i) + 1) = \sum_{i=1}^{m} w(\hat{q}_i) = w(\hat{q})$. Since $l(\hat{q}) > l(\hat{p})$, by assumption, we have $w(\hat{q}) = l(\hat{q}) + 1 > l(\hat{p}) + 1 = w(\hat{p})$. Through a completely analogous argument, if $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ and $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$ such that $p = q$ and $l(\hat{p}) > l(\hat{q})$, then $w(\hat{p}) > w(\hat{q})$. Summarizing this analysis, we have the following result:

Lemma 6.8. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a cycle algebra, and Ψ_Λ its associated Ψ-graph. Suppose $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ and $\hat{q} = \hat{q}_1 \cdot ... \cdot \hat{q}_m \in \mathcal{P}rod_{\Psi_\Lambda}$, $m \geq 2$, such that $p = q$ and $l(\hat{q}) \neq l(\hat{p})$. Then $w(\hat{q}) \neq w(\hat{p})$.*

This indicates that a product will occur in Ψ_Λ, for Λ a cycle algebra, only when the length of the proper path and an associated product path are the same.

There is another type of factorization of paths we wish to consider. Let $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ and assume $\hat{p}$ contains at least one adjacency arrow. We factor $\hat{p}$ at its adjacency arrows, and write $\hat{p} = \hat{p}_1 * \hat{p}_2 * ... * \hat{p}_n$, where each factor $\hat{p}_i$ is a subpath of $\hat{p}$ and contains no adjacency arrows, for $1 \leq i \leq n$ and $n \geq 2$. If $\hat{p} = \hat{p}_1 * \hat{p}_2 * ... * \hat{p}_n$ is a proper path and $\hat{q} = \hat{q}_1 \cdot \hat{q}_2 \cdot ... \cdot \hat{q}_m$ is a product path such that $p = q$, and for some $1 < j \leq m$ we have $\hat{p}_1 * ... * \hat{p}_i \neq \hat{q}_1 \cdot ... \cdot \hat{q}_j$ for all $1 \leq i \leq n$, we say that $\hat{q}$ has a *hidden product at position j, relative to* $\hat{p}$. The idea here is simple: we wish to identify if the factorization for $\hat{q}$ has a continuation arrow in some position j for which there does not correspond an adjacency arrow in $\hat{p}$. If $\hat{q}$ has no hidden products relative to $\hat{p}$, then we say the *adjacencies in $\hat{p}$ correspond to the product path $\hat{q}$*. As we will see, a proper path $\hat{p}$ is a product in the sense of Definition 6.7 when there exists a product path $\hat{q}$ such that all continuation arrows in $\hat{q}$ correspond to adjacencies in $\hat{p}$, where also $l(\hat{p}) = l(\hat{q})$ and $p = q$.

First, we consider a proper path $\hat{p}$ and an associated product path $\hat{q} = \hat{q}_1 \cdot \hat{q}_2 \cdot ... \cdot \hat{q}_m$ such that $p = q$. Since $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$, there is a corresponding left-admissible sequence $\langle r_i \rangle_{i=1}^{N}$, where $N \geq 1$. Let $\{o_i\}_{i=1}^{N-1}$ denote the set of intersection words for the left-admissible sequence $\langle r_i \rangle_{i=1}^{N}$, where $o_i \vdash_i \langle r_i \rangle_{i=1}^{N}$, $1 \leq i < N$. We remind the reader that the set $\{o_i\}_{i=1}^{N-1}$ corresponds to the vertices along the path $\hat{p}$. Now, for each $\hat{q}_j$, there is also a associated left-admissible sequence along q_j, since each $\hat{q}_j$ is a proper factor for $1 \leq j \leq m-1$, and $\hat{q}_m$ is a proper path. Respectively, we denote these associated left-admissible sequences by $Seq_j = \langle s_k \rangle_{k=1}^{n_j}$, where $n_j = l(\hat{p}_j)$ for $1 \leq j \leq m$. What is perhaps most interesting here is that each of the left-admissible sequences $\langle s_k \rangle_{k=1}^{n_j}$ must be related to a particular subsequence of $\langle r_i \rangle_{i=1}^{N}$, and in a particular way. It is these particulars that we concern ourselves with now.

Let $Seq_j = \langle s_k \rangle_{k=1}^{n_j}$ be the left-admissible sequence corresponding to $\hat{q}_j$, for $1 \leq j \leq m$. Since $p = q$, there exists some subsequence $Rel_j = \langle r_i \rangle_{i=K_j}^{N_j}$ of $\langle r_i \rangle_{i=1}^{N}$, of minimal length, such that in the underlying paths we have $\overrightarrow{s_1, ..., s_{n_j}}$ as a subpath of $\overrightarrow{r_{K_j}, ..., r_{N_j}}$, where $1 \le K_j \le N_j \le N$. For $\langle r_i \rangle_{i=K_j}^{N_j}$ to be of minimal length, we have that either $s_1 = r_{K_j}$, or s_1 starts at a vertex strictly within the intersection word o_{K_j-1}. Furthermore, for $\langle r_i \rangle_{i=K_j}^{N_j}$ to be of minimal length when $j < m$, we must also have that s_{n_j} ends within, but not necessarily strictly, the intersection word o_{N_j}; or when $j = m$ we have $s_{n_m} = r_{N_m}$. We simplify the overhead in keeping track of sequence and subsequence lengths by writing $|Seq_j| = n_j$, and $|Rel_j| = N_j - K_j + 1$. In terms of this notation, it can be show generally that $|Seq_1| = |Rel_1|$ $|Seq_j| \ge |Rel_j| - 1$ for $1 < j < m$, and $|Seq_m| = |Rel_m|$. The proof of these facts is rather lengthy and tedious, and so we refer the reader to Appendix A wherein a full proof is provided. We take these results, rework them into the vocabulary and notation of paths in Ψ_Λ, and use them to prove the following.

THEOREM 6.9. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a cycle algebra, and Ψ_Λ its associated Ψ-graph. Suppose $\hat{p} = \hat{p}_1 * \hat{p}_2 * ... * \hat{p}_n \in \mathcal{P}roper_{\Psi_\Lambda}$, $\hat{q} = \hat{q}_1 \cdot \hat{q}_2 \cdot ... \cdot \hat{q}_m \in \mathcal{P}rod_{\Psi_\Lambda}$ and $p = q$ in the underlying paths. Furthermore, assume $\hat{q}$ has a hidden product at position i relative to $\hat{p}$, for some $1 < i \le m$. Then $\hat{p}$ is not a product relative to $\hat{q}$.*

PROOF. Let $\hat{p}$ and $\hat{q}$ be as stated in the hypotheses. Then there must exist some maximal l such that $1 \le l < i \le m$ and $\hat{p}_1 * ... * \hat{p}_l = \hat{q}_1 \cdot ... \cdot \hat{q}_l$, but $\hat{p}_1 * ... * \hat{p}_i \ne \hat{q}_1 \cdot ... \cdot \hat{q}_i$. The goal is to show that $l(\hat{p}_{l+1} * ... * \hat{p}_n) < l(\hat{q}_{l+1} \cdot ... \hat{q}_m)$. Since $l(\hat{p}_1 * ... * \hat{p}_l) = l(\hat{q}_1 \cdot ... \hat{q}_l)$, we then have $l(\hat{p}) < l(\hat{q})$. By Lemma 6.8 we would therefore have our conclusion: if $\hat{q}$ has a hidden product relative to $\hat{p}$, then $w(\hat{p}) < w(\hat{q})$, and hence $\hat{p}$ is not a product relative to $\hat{q}$.

Let $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$, and so there corresponds to $\hat{p}$ a left-admissible sequence $\langle r_i \rangle_{i=1}^{N}$ of length N. Since $\hat{p}_1 * ... * \hat{p}_l = \hat{q}_1 \cdot ... \cdot \hat{q}_l$, we have $N = x + y + 1$, where $x \ge 1$ is the number of relations in $\langle r_i \rangle_{i=1}^{N}$ that occur in the subpath $\hat{p}_1 * ... * \hat{p}_l$, $y \ge 1$ is the number of relations in $\langle r_i \rangle_{i=1}^{N}$ that occur in the subpath $\hat{p}_{l+1} * ... * \hat{p}_n$, and the 1 in the sum corresponds to the adjacency arrow that connects the two subpaths. To each proper factor $\hat{q}_j$, with $l+1 \le j < m$, there corresponds a left-admissible sequence $Seq_j = \langle s_k \rangle_{k=1}^{n_j}$ as well as a subsequence $Rel_j = \langle r_t \rangle_{t=K_j}^{N_j}$ of $\langle r_i \rangle_{i=1}^{N}$. The same is true of the proper path $\hat{q}_m$: there corresponds a left-admissible sequence $Seq_m = \langle s_k \rangle_{k=1}^{n_m}$ as well as a subsequence $Rel_m = \langle r_t \rangle_{t=K_m}^{N_m}$ of $\langle r_i \rangle_{i=1}^{N}$. Furthermore, the path length of each $\hat{q}_j$ is equal to $|Seq_j|$ for $l+1 \le j \le m$. By the minimality of the length of each Rel_j, we have $\sum_{j=l+1}^{m} |Rel_j| = y$. This brings us to the length of the subpath $\hat{q}_{l+1} \cdot ... \cdot \hat{q}_m$, which is determined by the sum $\sum_{j=l+1}^{m} |Seq_j| + m - (l+1)$, where $m - (l+1)$ are the number of continuation arrows in the subpath.

By Lemma A.1, we have $|Seq_j| = |Rel_j|$ for $j = l+1$ and $j = m$, and $|Seq_j| \ge |Rel_j| - 1$ for $l+1 < j < m$, of which, in the latter case, there are $m - l - 2$ such sequences Seq_j. Hence $\sum_{j=l+1}^{m} |Seq_j| + m - (l+1) \ge \sum_{j=l+1}^{m} |Rel_j| - (m - l - 2) + m - (l+1)$, which reduces to $\sum_{j=l+1}^{m} |Seq_j| + m - (l+1) \ge y + 1$. This gives our result since $l(\hat{p}_{l+1} * ... * \hat{p}_n) < l(\hat{q}_{l+1} \cdot ... \cdot \hat{q}_m)$, and so $l(\hat{p}) < l(\hat{q})$. $\square$

One of the major consequences of Theorem 6.9 is that in Ψ_Λ, we need only concern ourselves with a limited number of product paths when trying to decide if a proper path is a product. We may make this limitation very precise when dealing with cycle algebras. A bit of vocabulary will be helpful here.

Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a cycle algebra, Ψ_Λ its associated Ψ-graph, and $\mathcal{C}$ a cf-cycle in Ψ_Λ. A cf-path $\hat{p} \in \Psi_\Lambda$ that is also a spoke for $\mathcal{C}$, we refer to as a *cf-spoke* of $\mathcal{C}$. An in-spoke $\hat{p}$ for a cf-cycle $\mathcal{C}$ that is also a proper factor, we will simply call a *cf-in-spoke* of $\mathcal{C}$. Now, if $\hat{x} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_n, \hat{v}_{n+1}$ is a path in Ψ_Λ such that $\hat{v}_1 \in V$, $\hat{a}_1 \in A_C$, and $l(\hat{x}) \geq 2$, we refer to $\hat{x}$ as a *continuation path from the vertex* $\hat{v}_1$ *to the vertex* $\hat{v}_{n+1}$. When $\hat{x}$ contains no other continuation arrows than that of $\hat{a}_1$, we say $\hat{x}$ is a *cf-continuation path*. When $\mathcal{C}$ is a cycle, $\hat{x}$ is a continuation path from a vertex on $\mathcal{C}$ and to a vertex on $\mathcal{C}$, and $\hat{x}$ contains no arrows from $\mathcal{C}$, we say $\hat{x}$ is a *return continuation path* to $\mathcal{C}$. Finally, if $\hat{x}$ is a return continuation path for $\mathcal{C}$, as well as a cf-continuation path, we call $\hat{x}$ a *cf-return continuation path*. In this vocabulary, it is clear that if $\hat{c}\hat{x}$ is a cf-return continuation path to $\mathcal{C}$, $\hat{c}$ being the initial (only) continuation arrow, then $\hat{x}$ is a cf-in-spoke of $\mathcal{C}$. We remind the reader that if $\hat{x}$ is a cf-in-spoke, then there corresponds to $\hat{x}$ a left-admissible sequence of elements in ρ.

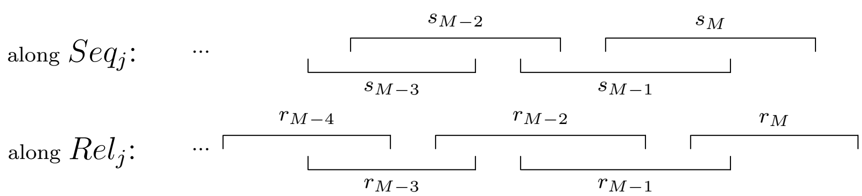
For a proper path $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$, we are primarily concerned with identifying only those paths $\hat{p} \in \mathcal{P}rod_{\Psi_\Lambda}$ such that $p = q$ and $w(p) = w(q)$. This rules out any path $\hat{q}$ that contains hidden products relative to $\hat{p}$. We remind ourselves that, in the case of a cycle algebra, for each adjacency arrow in Ψ_Λ there exists a single corresponding continuation arrow. Hence, Theorem 6.9 tells us we need only concern ourselves with product paths $\hat{q}$ whose continuation arrows correspond to the adjacency arrows of $\hat{p}$. In practical terms, this means when we are trying to determine if a proper path $\hat{p}$ is a product relative to a particular product path $\hat{q} = \hat{q}_1\hat{c}_1...\hat{c}_{m-1}\hat{q}_m$, we need only consider $\hat{q}$ if for each of $\hat{c}_i\hat{q}_{i+1}$, we have $\hat{c}_i\hat{q}_{i+1} = \hat{x}_{i+1}\hat{y}_{i+1}$ where $\hat{x}_{i+1}$ is a cf-continuation path and $\hat{y}_{i+1}$ is a cf-path in Ψ_Λ, for $1 \leq i < m$.

As a side note, we notice when ρ contains no elements that self-overlap, the following is true: if for $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ and $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$ we have $\sigma(\hat{p}) = \sigma(\hat{q})$, $\tau(\hat{p}) = \tau(\hat{q})$, and $l(\hat{p}) = l(\hat{q})$, then $p = q$. So, if ρ contains no self-overlaps, we may determine that $\hat{p}$ is a product relative to $\hat{q}$, simply by the knowledge that $\hat{p}$ and $\hat{q}$ start at the same vertex, end at the same vertex, and their path lengths are the same in Ψ_Λ. However, this is the case only when the continuation arrows of $\hat{q}$ each correspond to an adjacency arrow in $\hat{p}$. We next provide proof of this result, which allows us the convenience of simplifying the construction of Ψ_Λ, in some cases, by making unnecessary the arrow labels that indicate underlying paths.

Lemma 6.10. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a cycle algebra, and Ψ_Λ its associated Ψ-graph. Suppose for $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ and $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$, we have $\sigma(\hat{p}) = \sigma(\hat{q})$, $\tau(\hat{p}) = \tau(\hat{q})$, and $l(\hat{p}) = l(\hat{q})$. Further suppose that, for each continuation arrow in the factorization for $\hat{q}$, there corresponds an adjacency arrow in $\hat{p}$. If $p \neq q$, then ρ contains an element that overlaps itself.*

Proof. Let $\hat{q} = \hat{q}_1 \cdot \hat{q}_2 \cdot ... \cdot \hat{q}_m$ be the factorization for $\hat{q}$ at its continuations arrows, and $\hat{p} = \hat{p}_1 * \hat{p}_2 * ... * \hat{p}_n$ the factorization for $\hat{p}$ at its adjacency arrows. Since $l(\hat{p}) = l(\hat{q})$, and since the continuation arrows for $\hat{q}$ each correspond to an adjacency arrow in $\hat{p}$, there exists some $1 < i \leq n$ such that $l(\hat{p}_i * ... * \hat{p}_n) = l(\hat{q}_m)$. Let $M = l(\hat{q}_m)$. Now, since

$\sigma(\hat{p}) = \sigma(\hat{q})$, and Ψ_Λ contains no proper forks, we have $p_1 * ... * p_{i-1} = q_1 \cdot ... \cdot q_{m-1}$ in the underlying paths. By hypothesis $p \neq q$, which leaves us with $p_i * ... * p_n \neq q_m$. Let $\langle r_i \rangle_{i=1}^N$ be the left-admissible sequence corresponding to $\hat{p}$, $Rel = \langle r_j \rangle_{j=1}^M$ the subsequence of $\langle r_i \rangle_{i=1}^N$ corresponding to $\hat{p}_i * ... * \hat{p}_n$, and $Seq = \langle s_k \rangle_{k=1}^M$ the left-admissible sequence corresponding to the factor $\hat{q}_m$. Since Rel and Seq must start at the same relation, we have that $s_i = r_i$ for each odd $i \geq 1$. In the underlying paths we have $p_i * ... * p_n \neq q_m$, and so M cannot be odd. By hypothesis $\tau(\hat{p}) = \tau(\hat{q})$, so $s_M = r_M$ as elements in ρ. Finally, since $|Rel| = |Seq|$ and M is even, we have r_M overlaps itself, as in the following diagram:



□

Before we proceed with our main results, we pause a final time for brief mention of a technical fact that will make some calculations in the main results simpler and more precise. Consider a given cf-cycle $\mathcal{C} = \hat{v}_1, \hat{a}_1, \hat{v}_2, \hat{a}_2, ..., \hat{a}_n, \hat{v}_1$, of length n, in Ψ_Λ for a cycle algebra Λ. If Ψ_Λ contains another cf-cycle $\mathcal{D}$, distinct from $\mathcal{C}$ and of length greater than or equal to n, then the length of $\mathcal{D}$ is a multiple of n. Since the underlying path for $\mathcal{C}$ is the single cycle Γ, we are quite restricted in the placement of the vertices and arrows for the distinct second cf-cycle $\mathcal{D} = \hat{u}_1, \hat{b}_1, \hat{u}_2, \hat{b}_2, ..., \hat{b}_m, \hat{u}_1$, when $m \geq n$. Our task is then to make this restriction precise, which is done in Appendix B, and gives proof to the following:

LEMMA 6.11. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a cycle algebra, and Ψ_Λ its associated Ψ-graph. Suppose $\mathcal{C}$ is a cf-cycle in Ψ_Λ of length n, and $\mathcal{D}$ is another cf-cycle in Ψ_Λ distinct from $\mathcal{C}$, such that $l(\mathcal{D}) \geq n$. Then $l(\mathcal{D})$ is a positive integer multiple of n.*

3. Main Results

This brings us to the main subject at hand: determining when $E(\Lambda)$ is finitely generated for Λ a cycle algebra. We have essentially three cases to consider. The first is when Ψ_Λ contains no cf-cycles. In this case, there is clearly a maximum on the length of elements in $\mathcal{P}roper_{\Psi_\Lambda}$, which indicates $E(\Lambda)$ must be finitely generated, and for that matter, finite dimensional. The second case is where every path in $\mathcal{P}roper_{\Psi_\Lambda}$ starts with an arrow in A_B^*. Then each path in $\mathcal{P}roper_{\Psi_\Lambda}$ corresponds to a left multiplication by an element in Ext^1, and so $E(\Lambda)$ is finitely generated in this case also. The final case is the most interesting one, which we study next: where Ψ_Λ contains cf-cycles and not all paths in $\mathcal{P}roper_{\Psi_\Lambda}$ start with an arrow from A_B^*.

THEOREM 6.12. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a cycle algebra, $E(\Lambda)$ its Ext-algebra, and Ψ_Λ its associated Ψ-graph. Furthermore, let the following integer values be determined by: $M = max\{\ l(\mathcal{C}) \mid \mathcal{C}$ a cf-cycle in $\Psi_\Lambda\}$, $S = max\{\ l(\hat{s}) \mid \hat{s}$ an in-spoke for a cf-cycle $\mathcal{C} \in \Psi_\Lambda\}$, and set $N = 2M + 2S + 1$. Set $\mathcal{Q}$ to be the subset of all paths in $\mathcal{P}roper_{\Psi_\Lambda}$ that do not start with an arrow from A_B^*. $E(\Lambda)$ is finitely generated, if and only if, for each path $\hat{p} \in \mathcal{Q}$, of length N or greater, $\hat{p}$ contains an adjacency arrow for which the corresponding continuation arrow forms a cf-return continuation path $\hat{x}$ back to $\hat{p}$, where $\hat{x}$ is a subpath of some $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$ with $p = q$.*

PROOF. We begin with the reverse implication, and assume that Ψ_Λ contains cf-cycles, otherwise we are done by the remarks above. Since Λ is a cycle algebra, the set of all cf-cycles of Ψ_Λ is pairwise-disjoint, by Lemma 6.3. Furthermore, any path $\hat{p} \in \mathcal{Q}$ is a simple path, or lies across a single cf-cycle. By hypothesis, any path $\hat{p}$ of length $K = M + S + 1$ or greater must contain at least one adjacency arrow, since K is the longest length possible for a proper path without repeating an arrow. Also by hypothesis, this adjacency has a corresponding cf-return continuation path $\hat{x}$ back to $\hat{p}$, where $\hat{x}$ is a subpath of some $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$ with $p = q$. We claim for each path $\hat{p} \in \mathcal{Q}$ with length greater than or equal to N, the product path $\hat{q}$ with subpath $\hat{x}$, as indicated in our hypotheses, determines $\hat{p}$ as a product relative to $\hat{q}$. Therefore by taking this same value of N in Proposition 6.6, we conclude that $E(\Lambda)$ is finitely generated.

Let $\hat{p} \in \mathcal{Q}$, with length greater than or equal to N, and $\hat{p} = \hat{p}_1 * ... * \hat{p}_n$ denote its factorization at adjacency arrows. The path $\hat{p}$ must then lie across some cf-cycle $\mathcal{C}$ in Ψ_Λ. By the above remarks, there must exist some minimal $1 \leq i < n$ where the adjacency arrow connecting $\hat{p}_i$ to $\hat{p}_{i+1}$ corresponds to a cf-return continuation path to $\hat{p}$, of length at most S, and for which $l(\hat{p}_1 * ... * \hat{p}_i) \leq K$. Let $\hat{r} = \hat{p}_{i+1} \cdot ... \cdot \hat{p}_n$, and set $\hat{x}$ to be the cf-return continuation path corresponding to the adjacency at position i in $\hat{p}$. We now have two cases to consider: when $\hat{x}$ returns to $\hat{p}$ at $\tau(\hat{p})$, and when it returns to $\hat{p}$ at some vertex prior to $\tau(\hat{p})$. We dispense with the latter case first.

If $\tau(\hat{x}) \neq \tau(\hat{p})$, then $l(\hat{x}) \leq S$ since $\tau(\hat{x})$ must lie on $\mathcal{C}$. Now $l(\hat{x}) \leq S \leq l(\hat{r})$, and so we have that $\sigma(\hat{x})$ lies on $\hat{r}$, and x is a prefix of r. Furthermore, the left-admissible sequence Rel for $\hat{r}$ and the left-admissible sequence Seq for $\hat{x}$ start with the same relation, and since $\hat{x}$ ends at a vertex on $\hat{r}$, there is an underlying subsequence Rel_x of Rel that corresponds to Seq. By Lemma A.1 we have that $|Seq| = |Rel_x|$. Now, let $\hat{z}$ be the prefix of $\hat{r}$ that corresponds to Rel_x. Then we may write $\hat{r} = \hat{z}\hat{y}$ for some suffix $\hat{y}$ of $\hat{p}$, where $l(\hat{z}) = l(\hat{x})$. Finally, we have that $\hat{q} = \hat{q}_1 \cdot \hat{q}_2$ where $\hat{q}_1 = \hat{p}_1 * ... * \hat{p}_i$, $\hat{q}_2 = \hat{x}\hat{y}$, and where there is a continuation arrow between the paths $\hat{q}_1$ and $\hat{q}_2$. By hypothesis $p = q$, and since $l(\hat{p}) = l(\hat{q})$, we have $w(\hat{p}) = w(\hat{q})$. Hence $\hat{p}$ is a product relative to $\hat{q}$.

Now, if $\tau(\hat{x}) = \tau(\hat{p})$, then $l(\hat{x}) = l(\hat{p}_n)$ by Lemma A.1. Therefore $l(\hat{q}) = l(\hat{p})$, for $\hat{q} = \hat{q}_1\hat{x}$ and $\hat{q}_1 = \hat{p}_1 * ... * \hat{p}_{n-1}$. The major difference in this case, in contrast to the latter, is that $\hat{x}$ and $\hat{p}_n$ need not be restricted in length. That is, $\hat{x}$ and $\hat{p}_n$ lie across their respective disjoint cf-cycles $\mathcal{C}_{\hat{x}}$ and $\mathcal{C}_{\hat{p}}$, and give rise to arbitrarily long products. For $\hat{p}$ a product relative to $\hat{q}$ with $l(\hat{p}) \geq N$, we show there exist related paths $\hat{p}_{min}$ and $\hat{q}_{min}$, $\hat{p}$ and $\hat{q}$ respectively, such that $l(\hat{p}_{min}), l(\hat{q}_{min}) < N$. This will allow us to conclude that if $\hat{p}_{min}$ is a product relative to $\hat{q}_{min}$, then $\hat{p}$ a product relative to $\hat{q}$.

We write $\hat{x} = \hat{y}_{\hat{x}} \mathcal{C}_{\hat{x}}^{N_{\hat{x}}} \hat{a}_{\hat{x}}$ and $\hat{p}_n = \hat{y}_{\hat{p}} \mathcal{C}_{\hat{p}}^{N_{\hat{p}}} \hat{a}_{\hat{p}}$ for some simple paths $\hat{y}_{\hat{x}}, \hat{y}_{\hat{p}} \in \Psi_\Lambda$, terminal arrows $\hat{a}_{\hat{x}}, \hat{a}_{\hat{p}} \in A_T$, and some positive integers $N_{\hat{x}}, N_{\hat{p}}$. By Proposition B.1, and without loss of generality, we take $l(\mathcal{C}_{\hat{x}})$ as a positive integer multiple of $l(\mathcal{C}_{\hat{p}})$. Since $l(\hat{x}) = l(\hat{p}_n)$, we then have $l(\hat{y}_{\hat{x}}) + N_{\hat{x}} L \ l(\mathcal{C}_{\hat{p}}) = l(\hat{y}_{\hat{p}}) + M_{\hat{p}} l(\mathcal{C}_{\hat{p}}) + (N_{\hat{p}} - M_{\hat{p}}) l(\mathcal{C}_{\hat{p}})$, for some positive integer L, and some integer $0 \leq M_{\hat{p}} < L$. We remove $N_{\hat{x}}$ copies of the subpath $\mathcal{C}_{\hat{x}}$ from $\hat{x}$ to form the path $\hat{x}_{min}$. This directly corresponds to removing $(N_{\hat{p}} - M_{\hat{p}})L$ copies of the subpath $\mathcal{C}_{\hat{p}}$ from $\hat{p}_n$ to form $\hat{p}_{n,min}$. This leaves us with the arithmetic expression: $l(\hat{y}_{\hat{x}}) = l(\hat{y}_{\hat{p}}) + (M_{\hat{p}} L) \ l(\mathcal{C}_{\hat{p}})$, wherein we also make use of the fact that $l(\hat{y}_{\hat{p}}) \leq l(\hat{y}_{\hat{x}})$. Since $0 \leq M_{\hat{p}} < L$, we have $l(\hat{y}_{\hat{x}})$ is less than M, the maximum length of a cf-cycle in Ψ_Λ. This leaves us with $l(\hat{x}_{min}), l(\hat{p}_{n,min}) < M + 1$. We set $\hat{p}_{min} = \hat{p}_1 * ... * \hat{p}_{n-1} \hat{p}_{min}$ and $\hat{q} = \hat{q}_1 \hat{x}_{min}$. Since $l(\hat{p}_1 * ... * \hat{p}_{n-1}) \leq K$, we have $l(\hat{p}_{min}), l(\hat{q}_{min}) \leq S + 2M + 1 < N$, and the proof of the reverse implication is complete.

We now assume $E(\Lambda)$ is finitely generated, and proceed with the forward implication. By Proposition 6.6 there exists some N for which every path $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$, with $l(\hat{p}) \geq N$, there is a $\hat{q} = \hat{q}_1 \cdot ... \cdot \hat{q}_m \in \mathcal{P}rod_{\Psi_\Lambda}$ such that $p = q$, $w(\hat{p}) = w(\hat{q})$, and $m \geq 2$. Let $\hat{p}$ be such a path of length $\geq N$, and $\hat{q}$ as indicated. Without loss of generality, we assume $\hat{p}$ lies across a single cf-cycle $\mathcal{C}$ in Ψ_Λ. By Theorem 6.9, each continuation arrow in the factorization of $\hat{q}$ must correspond to an adjacency in $\hat{p}$, else $w(\hat{p}) \neq w(\hat{q})$. Furthermore, each such continuation arrow must be a prefix of a cf-return continuation path $\hat{x}$ to $\hat{p}$. That $\hat{x}$ must be a prefix for a return continuation path is clear. By Lemma 6.5, continuation arrows in Ψ_Λ are uniquely determined. So, if $\hat{x}$ were a return continuation path containing a continuation arrow associated to a vertex on a cf-cycle disjoint from $\mathcal{C}$, then $\hat{q}$ would have a hidden product relative to $\hat{p}$ at a position along $\hat{x}$. Again, by Theorem 6.9 we would then have $w(\hat{p}) \neq w(\hat{q})$. Hence, each continuation arrow in factorization of $\hat{q}$ must correspond to an adjacency in $\hat{p}$, and is the prefix of a cf-return continuation path back to $\hat{p}$, which gives us our result. □

There is a natural and easy extension of Theorem 6.12 to a more general class of algebras, which we describe next. Let Γ be an in-spoked cycle quiver. Let k be a field, and ρ a finite set of reduced monomial generators for an ideal in $k\Gamma$. We refer to $\Lambda = k\Gamma/\langle\rho\rangle$ as an *in-spoked cycle algebra*. There is an immediate corollary to Lemma 5.1 for in-spoked cycle algebras since there can be no path s different from r as stated there. Hence, $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$ in Ψ_Λ, for in-spoked cycle algebras. Furthermore, there is an almost identical argument to that of Lemma 6.4, yielding the fact that Ψ_Λ contains no proper forks for an in-spoked cycle algebra, and hence all cf-cycles in Ψ_Λ are disjoint. Lemmas 6.5 and 6.8 each have clear analogues for in-spoked cycle algebras. Theorem 6.9 and Lemma A.1 are both quite general. Through an easy recycling of notation and arguments from Theorem 6.12, wherein the numbers K, M, N, and S must be calculated for a given in-spoked cycle algebra Λ, we have the following:

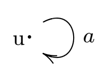
THEOREM 6.13. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be an in-spoked cycle algebra, $E(\Lambda)$ its Ext-algebra, and Ψ_Λ its associated Ψ-graph. Furthermore, let the following integer values be determined by: $M = max\{\ l(\mathcal{C}) \mid \mathcal{C}$ a cf-cycle in $\Psi_\Lambda\}$, $S = max\{\ l(\hat{s}) \mid \hat{s}$ an in-spoke for a cf-cycle $\mathcal{C} \in \Psi_\Lambda\}$, and set $N = 2M+2S+1$. Set $\mathcal{Q}$ to be the subset of all paths in $\mathcal{P}roper_{\Psi_\Lambda}$ that do not start with an arrow from A_B^*. $E(\Lambda)$ is finitely generated, if and only if, for each path $\hat{p} \in \mathcal{Q}$, of length N or greater, $\hat{p}$ contains an adjacency arrow for which the corresponding continuation arrow forms a cf-return continuation path $\hat{x}$ back to $\hat{p}$, where $\hat{x}$ is a subpath of some $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$ with $p = q$.*

We should take a moment here to say a word about the finite nature of the conditions given in Theorems 6.12 and 6.13. Let Λ be an in-spoked cycle algebra, Ψ_Λ its Ψ-graph, and N the bound whose calculation is described above. Since there are a finite number of proper paths in Ψ_Λ of length N (or less), we may determine whether or not each proper path is a product in a finite amount of time. That is, we may determine if $E(\Lambda)$ is finitely generated by examining the set of all proper paths in Ψ_Λ of length N or less, and determining if each element therein is a product relative to some $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$.

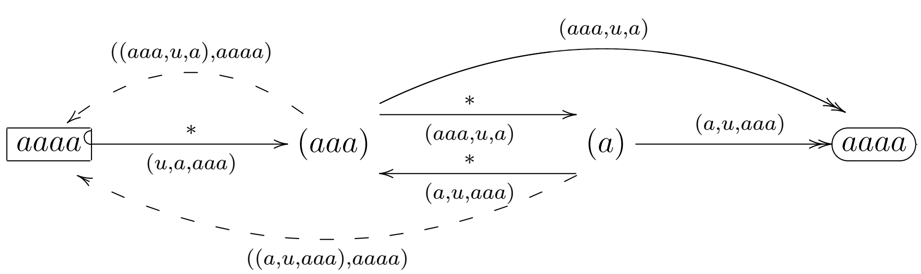
4. Examples

We now examine some examples of cycle algebras, in-spoked cycles algebras, and their corresponding Ψ-graphs. We begin with two very simple examples, then move to two more complicated ones.

EXAMPLE 6.14. First we examine $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is:



and $\rho = \{aaaa\}$. The corresponding Ψ_Λ graph is then:

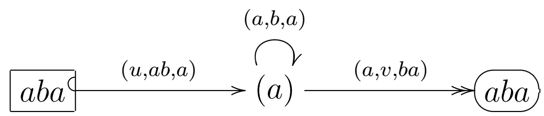


Although the full Ψ-graph for Λ is indicated above, including the up-labels, the interpretation of Ψ_Λ for this example is simple: every proper path of length 2 or greater must start with an arrow from A_B^*. Hence, by our work above, $E(\Lambda)$ is finitely generated.

EXAMPLE 6.15. Let $\Lambda = k\Gamma/\langle\rho\rangle$, where Γ is:

$$\begin{tikzcd} \cdot u \arrow[r, shift left, "a"] & \cdot v \arrow[l, shift left, "b"] \end{tikzcd}$$

and $\rho = \{aba\}$. We then have Ψ_Λ for this Λ as:



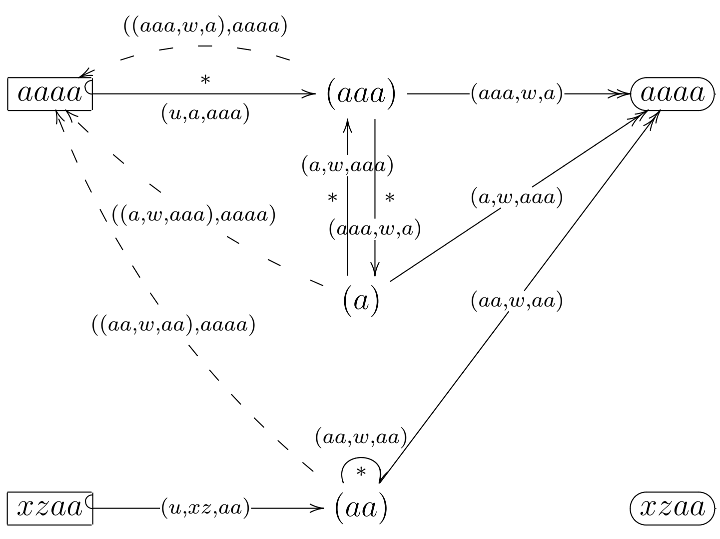
Again, our interpretation of Ψ_Λ is simple: any proper path in Ψ_Λ contains no adjacency arrows; thus, by Theorem 6.12, $E(\Lambda)$ is infinitely generated.

We finish this chapter with two examples that are a bit more subtle, but nevertheless covered by Theorems 6.12 and 6.13. The first of these examples is an apparently simple in-spoked algebra, within which we need to take care when examining the underlying paths of product paths.

EXAMPLE 6.16. We return an earlier example: Example 4.2 from Chapter 4. Let G be given by the quiver:

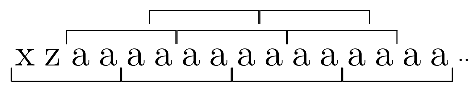
$$\begin{tikzcd} \cdot \arrow[r, "x"] & \cdot \arrow[r, "z"] & \cdot \arrow[loop right, "a"] \end{tikzcd}$$

and let $\rho = \{aaaa, xzaa\}$. We set $\Lambda = k\Gamma/\langle\rho\rangle$, for a field k, and construct Ψ_Λ, which is given as follows:



From this diagram, it is easy to see that for any proper path $\hat{p}$ of length 3 or greater, there exists a product path of the same length that starts and ends at the same vertices as $\hat{p}$.

However, this fact in itself does not guarantee each proper path is a necessarily a product, and the subtleties of the underlying paths begin to become apparent if we begin to consider some examples of particular proper paths. Let $\hat{p} \in \Psi_\Lambda$ of length 4, starting at the vertex $\boxed{\text{xzaa}}$, and whose corresponding left-admissible sequence is given by the initial relations in:



If we write out the proper path $\hat{p}$, we see it has the form:

$$\boxed{\text{xzaa}} \hookrightarrow (aa) \xrightarrow{*} (aa) \xrightarrow{*} (aa) \twoheadrightarrow \boxed{\text{aaaa}}$$

and has the underlying path $xzaaaaaaaa \in \mathcal{B}_\Gamma$. We further notice that there exists a path $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$ of length 4 which, like $\hat{p}$, starts at the vertex $\boxed{\text{xzaa}}$ and ends at vertex $\boxed{\text{aaaa}}$:

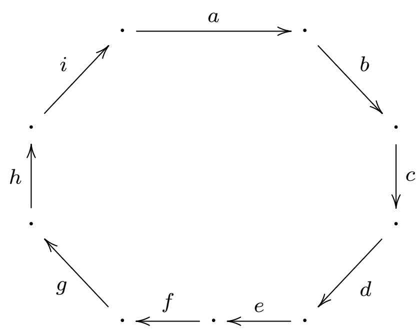
$$\boxed{\text{xzaa}} \hookrightarrow (aa) \dashrightarrow \boxed{\text{aaaa}} \overset{*}{\hookrightarrow} (aaa) \twoheadrightarrow \boxed{\text{aaaa}}$$

However, we see that the underlying path of $\hat{q}$ is $xzaaaaaaa$. It is a straightforward exercise to verify these underlying paths by examining the sequence diagram above, or compute them via the algorithms provided in the appendices. The consequence of these discussions is the following: since $p \neq q$, $\hat{p}$ is not a product relative to $\hat{q}$. In fact, we may further show $\hat{p}$ is not a product relative to any product path.

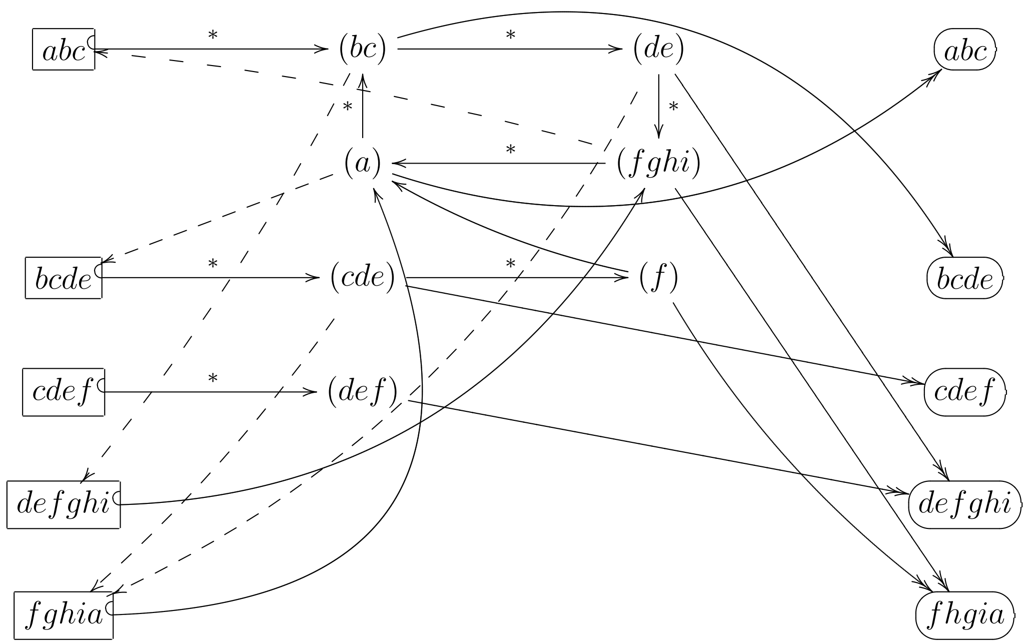
However, all is not lost, and Theorem 6.13 requires we need look only so far out, in terms of product path lengths, to determine if $E(\Lambda)$ is finitely generated. We compute the number N indicated in the theorem, and find that for this example, $N = 7$. If we now consider the proper path of length 7, starting at $\boxed{\text{xzaa}}$ and ending at vertex $\boxed{\text{aaaa}}$, we see by the above sequence diagram that it is indeed a product at the first adjacency. An even better example is found by examining the path of length 6, also starting at the vertex $\boxed{\text{xzaa}}$ and ending at vertex $\boxed{\text{aaaa}}$. Although this path has length one less than our bound $N = 7$, we see it provides additional insight since, upon examination, it is a product at its second adjacency, but not at its first.

In conclusion, we may verify that every proper path $\hat{p}$ of length 7 or greater contains an adjacency arrow for which the corresponding continuation arrow forms a cf-return continuation path $\hat{x}$ back to $\hat{p}$, where $\hat{x}$ is a subpath of some $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$ with $p = q$. This indicates every proper path of length 7 or greater is a product and, by Theorem 6.13, $E(\Lambda)$ is finitely generated.

Example 6.17. The final example for this chapter comes from page 567 of [**6**], the quiver of which is:



and $\rho = \{abc, bcde, cdef, defghi, fghia\}$. Since there are no self-overlapping relations in ρ, we exclude the up-labels for arrows in Ψ_Λ:



Since any proper path that starts with relations abc, $bcde$, or $cdef$ starts with an adjacency arrow, and is therefore a product, we are really only interested in proper paths that begin with $defghi$ or $fghia$. However, every proper path of length 2 or greater that starts with one of these latter two relations also lies across the only cf-cycle in Ψ_Λ. This cf-cycle has an adjacency arrow for every arrow in its path. From this, it is easy to confirm that every path of length $N = 2S + 2M + 1 = 2(3) + 2(4) + 1$, or greater, is a product.

CHAPTER 7

Further Results and Conclusions

The utility of the Ψ-graph may be extended to other classes of algebras. In this chapter we examine finite criteria on Ψ_Λ for d-Koszul monomial algebras Λ, and for a heretofore undefined class of algebras we term "LR-symmetric algebras" Λ, where "LR" refers to a kind of left-right symmetry on admissible sequences. Earlier in this document we hinted at this notion of symmetry by referring to these algebras, in advance of their definition, as "left/right-symmetric" algebras. The latter class of algebras is related to the work of Bardzell in [**3**]. In the former case, the authors of [**11**] determine that $E(\Lambda)$ is finitely generated for a d-Koszul monomial algebra Λ, when $d \geq 3$, and we simply confirm this result using our methods.

For both d-Koszul and LR-symmetric algebras, the finite criteria given on Ψ_Λ is distinctly different from that in our discussions of Ψ_Λ for cycle and in-spoked cycle algebras. The most obvious fact is that, in general, for arbitrary monomial algebras we no longer have the feature that $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$. As it turns out, we do have this property for the LR-symmetric algebra case, but do not need it in determining finite generation of $E(\Lambda)$ for d-Koszul monomial algebras.

1. d-Koszul Monomial Algebras

When $\Lambda = k\Gamma/\langle\rho\rangle$ is a d-Koszul monomial algebra, with $d \geq 3$, the authors of [**11**] show that Λ may be classified as an algebra such that $E(\Lambda)$ is generated in degrees 0, 1, and 2. We wish to exploit a unique property of $\langle\rho\rangle$ found in the work of [**11**], as well as [**12**], in our considerations of finite criteria on Ψ_Λ. In doing so, we confirm the result in [**11**] concerning the degrees in which $E(\Lambda)$ is generated, when Λ is a d-Koszul algebra with $d \geq 3$.

The unique property of $\langle\rho\rangle$ is this: if the relations $r, s \in \rho$ are such that s overlaps r, in any non-trivial way, then there exists a relation $t \in \rho$ such that $at = rb$ for arrows $a, b \in \mathcal{B}_\Gamma$. This has direct implications for Ψ_Λ, specifically: $A_B = A_B^*$. Since Γ can be quite general, we do not have the luxury of assuming $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$. However, we observe that the unique property of $\langle\rho\rangle$ implies every product path in Ψ_Λ, of length 2 or greater, begins with an initial sequence arrow and is therefore a product. Hence, $E(\Lambda)$ is generated in degrees 0,1, and 2. Since Γ and ρ are finite, we have $E(\Lambda)$ is finitely generated, and in the degrees listed.

2. LR-Symmetric Algebras

In Chapter 3, we defined the term *left-admissible sequence*. In addition, we described how we could, for a reduced monomial generating set ρ, form the sets Γ_n based on the left-admissible sequences of the elements of ρ. In a completely analogous fashion, we may define the concept of *right-admissible* sequence, and similarly determine a sequence of set H_n determined by such right-admissible sequences. In [**3**], Bardzell proved that $\Gamma_n = H_n$, for each $n \geq 0$. In this section, we follow similar notational conventions to that of Chapter 3, and express a right-admissible sequence of length $n \geq 2$ as $[s_j]_{j=1}^{n}$. In terms of the underlying path corresponding to $[s_j]_{j=1}^{n}$, we denote the path by $\overleftarrow{s_n, ..., s_1}$, and remark that, by Bardzell's results, $\overleftarrow{s_n, ..., s_1} \in \Gamma_{n+1}$.

Along these lines of consideration, we are motivated to describe conditions on the generators ρ of a monomial ideal, based on the equivalence of the underlying admissible sequences, both left and right, for **each element** in the sets Γ_i, for $i \geq 2$. We make this notion precise in the following definition.

Definition 7.1. A reduced set of monomial generators ρ is called *LR-symmetric* if the following is true about the sets Γ_n, with $n \geq 2$, constructed from ρ: for each element $\gamma \in \Gamma_n$, the left-admissible sequence for γ is the same as the right-admissible sequence for γ, in the reverse order. That is, for every Γ_n with $n \geq 2$, we have for each $\gamma \in \Gamma_n$: if $\gamma = \overrightarrow{r_1, ..., r_{n-1}}$, with left-admissible sequence $\langle r_1, ..., r_{n-1}\rangle$, then the right-admissible sequence for γ is $[r_j]_{j=n-i}^{1}$, for $1 \leq i < n$, and $r_i \in \rho$.

To reduce the notational overhead when discussing the equivalence of left- and right-admissible sequences, we adopt the following convention. Let $\gamma \in \Gamma_n$, for some $n \geq 2$. Then we denote its left-admissible sequence by $\overrightarrow{Rel}_{\gamma}^{n-1} = \langle r_i\rangle_{i=1}^{n-1}$ and its right-admissible sequence by $\overleftarrow{Rel}_{\gamma}^{n-1} = [s_j]_{j=1}^{n-1}$, where each $r_i, s_j \in \rho$. So, we may now more simply state Definition 7.1 through the employment of this new notation: a reduced set of monomial generators is called *LR-symmetric* if for every Γ_n, with $n \geq 2$, $\overrightarrow{Rel}_{\gamma}^{n} = \overleftarrow{Rel}_{\gamma}^{n}$ for each $\gamma \in \Gamma_n$. In keeping a parallel course with the definitions of the past few paragraphs, we say a monomial algebra is an *LR-symmetric algebra* if its monomial ideal has a monomial generating set that is LR-symmetric. As it turns out, we need only specify the LR-symmetric criteria in the case of Γ_4; all Γ_n with $n \geq 5$ inherit this property by induction.

Proposition 7.2. *Let $k\Gamma$ be a path algebra, and ρ a set of reduced monomial generators such that for each $\gamma \in \Gamma_4$ we have that $\overrightarrow{Rel}_{\gamma}^{3} = \overleftarrow{Rel}_{\gamma}^{3}$. Then for every Γ_n, with $n \geq 0$, $\overrightarrow{Rel}_{\gamma}^{n} = \overleftarrow{Rel}_{\gamma}^{n}$ for each $\gamma \in \Gamma_n$.*

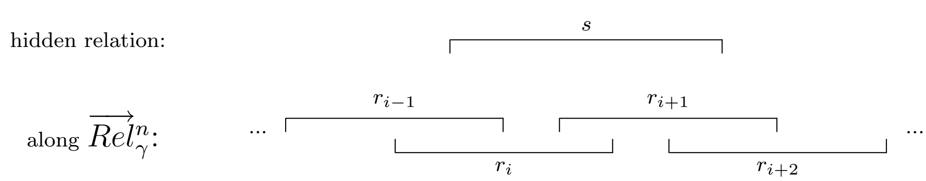
Proof. We proceed by induction on n, the index on the sets Γ_n, for $n \geq 0$. The cases when $n = 0, 1, 2, 3$ are obvious, from the definition of those sets, and the case $n = 4$ is given by hypothesis. We move to the initial case on our induction: when $n = 5$. Let $\gamma \in \Gamma_5$, and $\overrightarrow{Rel}_{\gamma}^{4} = \langle r_i\rangle_{i=1}^{4}$ its corresponding left-admissible sequence. Then the subsequence $\langle r_2, r_3, r_4\rangle$ must correspond to an element of Γ_4, else $\langle r_1, r_2, r_3\rangle$ must not, which contradicts our hypothesis on elements in Γ_4. Since $\overrightarrow{r_2, r_3, r_4} \in \Gamma_4$, we have its right-admissible sequence given by $[r_2, r_3, r_4]$. This implies $\overleftarrow{Rel}_\gamma^4 = [r_j]_{j=4}^1$, else $\overrightarrow{Rel}_\gamma^4$ is not left-admissible.

Suppose now $\overrightarrow{Rel}_\gamma^{n-1} = \overleftarrow{Rel}_\gamma^{n-1}$ for each $\gamma \in \Gamma_n$, for $n > 5$. Let $\gamma \in \Gamma_{n+1}$, and $\overrightarrow{Rel}_\gamma^n = \langle r_i \rangle_{i=1}^n$ its corresponding left-admissible sequence. Then the subsequence $\langle r_2, ..., r_n \rangle$ must correspond to an element of Γ_n, else $\langle r_1, ..., r_{n-1} \rangle$ must not, which contradicts our hypothesis on elements in Γ_n. Since $\overrightarrow{r_2, ..., r_n} \in \Gamma_n$, we have its right-admissible sequence given by $[r_2, ..., r_n]$. This implies $\overleftarrow{Rel}_\gamma^n = [s_j]_{j=n}^1$, else $\overrightarrow{Rel}_\gamma^n$ is not left-admissible, and our result follows. $\square$

One practical result of the preceding is that, in order to satisfy the LR-symmetric property on the generators of a monomial ideal, we need only verify the criteria on the set Γ_4. We next wish to tie this notion of LR-symmetry to that of the relations involved in constructing paths in Ψ_Λ. We begin this synthetic process of relating together the two notions with the following definition.

Definition 7.3. Let ρ be a reduced set of monomial generators for an ideal in a path algebra $k\Gamma$, and let γ be an element in the constructed set Γ_{n+1}, for $n \geq 1$. We set $\overrightarrow{Rel}_\gamma^n$ to be the left-admissible sequence associated to γ. If the relation $s \in \rho$ is a subword of γ but does not occur as a sequence element of $\overrightarrow{Rel}_\gamma^n$, at least not in the order in which the relations appear in $\overrightarrow{Rel}_\gamma^n$, then we say s is a *hidden relation of* γ. Similarly, we refer to s as a *hidden relation* of the left-admissible sequence $\overrightarrow{Rel}_\gamma^n$.

We see an example of a hidden relation $s \in \rho$ for $\gamma \in \Gamma_{n+1}$ in the following diagram, where $\overrightarrow{Rel}_\gamma^n = \langle r_i \rangle_{i=1}^n$ is the associated left-admissble sequence for γ, and each $r_i \in \rho$:



Now, let $\gamma \in \Gamma_{n+1}$ be such that it has no hidden relations. If $\overrightarrow{Rel}_\gamma^n$ is its associated left-admissible sequence, then $\overrightarrow{Rel}_\gamma^n = \overleftarrow{Rel}_\gamma^n$, else γ has a hidden relation. Conversely, if γ is such that $\overrightarrow{Rel}_\gamma^n = \overleftarrow{Rel}_\gamma^n$, then γ has no hidden relations, otherwise we have that either $\overrightarrow{Rel}_\gamma^n$ is not left-admissible, or $\overleftarrow{Rel}_\gamma^n$ is not right-admissible. From these ideas we have a proof of the following proposition.

Proposition 7.4. *Let ρ be a reduced set of monomial generators for an ideal in a path algebra $k\Gamma$, and let γ be an element in the constructed set Γ_{n+1}, for $n \geq 2$. The element γ has no hidden relations if and only if $\overrightarrow{Rel}_\gamma^n = \overleftarrow{Rel}_\gamma^n$.*

If we combine the preceding result with that of Proposition 7.2, we have the following corollary.

Corollary 7.5. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a LR-symmetric algebra, where ρ is a set of reduced monomial generators, and Γ_n are the sets constructed from elements in ρ. Then for every $n \geq 2$, each $\gamma \in \Gamma_n$ has no hidden relations.*

This corollary has implications regarding the sets $\mathcal{P}seudo_{\Psi_\Lambda}$ and $\mathcal{P}roper_{\Psi_\Lambda}$ for LR-symmetric algebras. In particular, if Λ is an LR-symmetric algebra, then for a path $\hat{p} \in \Psi_\Lambda$ to be an element of $\mathcal{P}seudo_{\Psi_\Lambda}$, but not an element of $\mathcal{P}roper_{\Psi_\Lambda}$, there needs to correspond to $\hat{p}$ an associated structure in Ψ_Λ given in Lemma 5.1. However, this structure may only occur when the underlying path $\hat{p}$ contains a hidden relation. Specifically, and in the context and notation of the Lemma, the arrow $\hat{a}_{k+1}$ corresponds to a hidden relation to an element γ with e as a prefix. This gives us the following corollary to Lemma 5.1:

Lemma 7.6. *Suppose Λ is an LR-symmetric algebra, and Ψ_Λ its corresponding Ψ-graph. Then $\mathcal{P}seudo_{\Psi_\Lambda} = \mathcal{P}roper_{\Psi_\Lambda}$.*

The LR-symmetric criteria on an algebra Λ also reduces the complications of determining when a path in Ψ_Λ corresponds to a product in $E(\Lambda)$. That is to say, if $\hat{p} = \hat{p}_1 * ... * \hat{p}_n$ is a proper path factored at the adjacency arrows, where $n \geq 2$, then because the underlying path $p \in \Gamma_{n+1}$ contains no hidden relations, each $\hat{p}_i$ corresponds to a factor $\hat{q}_i$ in a product path $\hat{q} = \hat{q}_1 \cdot ... \cdot \hat{q}_n$, where $l(q_i) = l(p_i)$ and, in the underlying paths, $q_i = p_i$, for each $1 \leq i \leq n$. Hence, $p = q$, and so $\hat{p}$ is a product relative to $\hat{q}$. In practical terms, for LR-symmetric algebras, the preceding argument allows us to state our criteria on Ψ_Λ for finite generation of $E(\Lambda)$ in terms of adjacency arrows only, and dispense with the inclusion of continuation arrows in Ψ_Λ altogether.

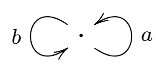
Theorem 7.7. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a LR-symmetric algebra, where ρ is a finite set of reduced monomial generators, and Ψ_Λ is the associated Ψ-graph without continuation arrows. $E(\Lambda)$ is finitely generated if and only if every cycle in Ψ_Λ contains an adjacency arrow.*

Proof. For both implications, we assume Ψ_Λ contains cycles; otherwise, the results are clear. These cycles are cf-cycles, in particular, since continuation arrows were not included in Ψ_Λ. We begin with the reverse implication and suppose every cycle in Ψ_Λ contains an adjacency arrow. We let $\mathrm{N} = \mid \Psi_0 \mid + 1$, the number of vertices in Ψ_Λ, plus 1. Since ρ is a finite set, N is a finite number. Let $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ be a path of length N or greater. Then $\hat{p}$ contains a repeated vertex, and therefore contains some cycle $\mathcal{C}$ as a subpath. This implies that $\hat{p}$ contains an adjacency arrow and is therefore a product. Since $\hat{p}$ was an arbitrary path of length N or greater, we take the value N for that in Proposition 6.6 and the reverse implication is proved.

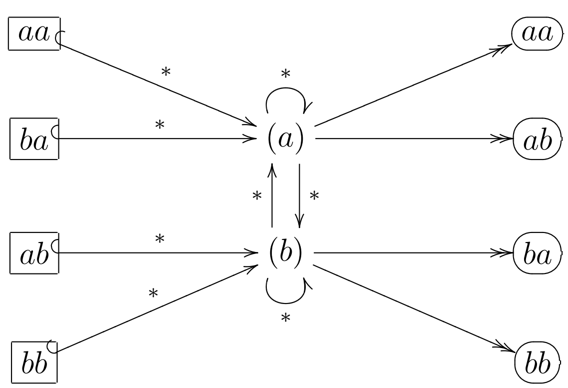
The forward is proved by contraposition. Assume there is some cycle $\mathcal{C}$ in Ψ_Λ that contains no adjacency arrows. Let $\hat{a}$ be an arrow in Ψ_Λ that lies on $\mathcal{C}$, and so $\hat{a}$ is not an adjacency arrow. We set $\hat{v} = \sigma(\hat{a})$, $\hat{w} = \tau(\hat{a})$, and write the cycle $\mathcal{C}$ in the particular cyclic permutation form $\mathcal{C}_{\hat{v}} = \hat{v}, \hat{a}, \hat{w}, ..., \hat{v}$. Now, to $\hat{a}$ there corresponds a relation $r_a \in \rho$ and, by construction, a (source) vertex $\mathfrak{b}(r_a)$. Furthermore, there exists a beginning sequence arrow $\mathfrak{y}(r_a) \in A_B \backslash A_B^*$ connecting $\mathfrak{b}(r_a)$ to $\hat{w}$. Otherwise, any left-admissible sequence of elements in ρ that contains an overlap sequence word corresponding to $\hat{v}$ must contain a hidden relation, which would contradict our hypothesis that ρ is LR-symmetric. Also, and this is simply by construction, there exists some vertex $\hat{t} \in T$ and some terminal sequence arrow $\hat{b}$ such that $\hat{b}$ connects the vertex $\hat{w}$ to $\hat{t}$. We now consider the set of paths $\hat{p}^i = \mathfrak{b}(r_a)\mathcal{C}_{\hat{v}}^i\hat{b}$, for $i \geq 0$. Since each corresponds to an admissible sequence of length $n_i = i \mid C \mid +2$, each also corresponds to an element in its respective set Γ_{n_i+1}. Since each $\hat{p}^i$ contains no adjacency arrows, each $\hat{p}^i$ is not a product. This implies there exists a non-product element $\hat{p}^i$ in the set Γ_{n_i+1}, where n_i becomes arbitrarily large as $i \to \infty$. Hence, by Proposition 6.6, $E(\Lambda)$ is infinitely generated, and thus the forward implication is proved. $\square$

In the examples that follow, we see that the up-labels have been excluded, in addition to the continuation arrows. This is with good reason, and is a consequence of the fact that proper paths themselves include information about whether they do or do not constitute a product, and thus implicitly contain information about underlying paths.

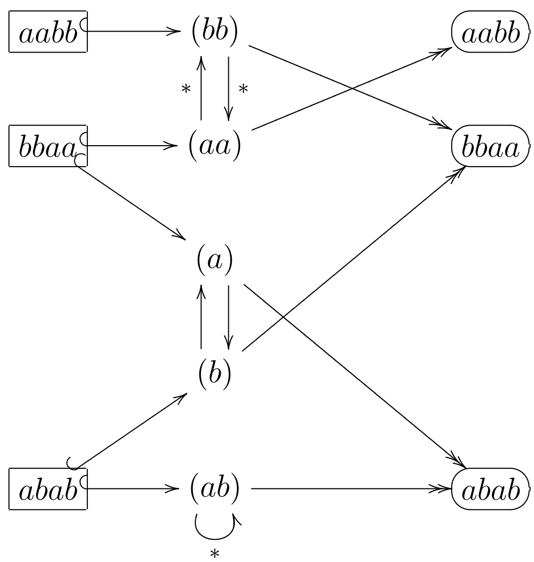
Example 7.8. Let Γ be the quiver given by:



For k a field, we set $\Lambda = k\Gamma/\langle\rho\rangle$ where $\rho = \{aa, ab, ba, bb\}$. There are several ways in which we may determine $E(\Lambda)$ is finitely generated. We observe that the criteria are met on Γ_4, and so Λ is an LR-symmetric algebra. Hence, Theorem 7.7 applies and we see $E(\Lambda)$ is finitely generated since every cycle in Ψ_Λ contains an adjacency arrow:



Example 7.9. Let Γ be as in Example 7.8, and k a field. We set $\Lambda = k\Gamma/\langle\rho\rangle$, where $\rho = \{aabb, bbaa, abab\}$. It is clear that Λ is an LR-symmetric algebra. Since Ψ_Λ has a cycle that contains no adjacency arrows, by Theorem 7.7 we have that $E(\Lambda)$ is infinitely generated:



3. Further Explorations

We wish to conclude this dissertation with a few words about possible future work based on the studies above. Some of these possibilities are expressed as questions. For example, in the preceding we showed that there exists finite criteria for determining finite generation for $E(\Lambda)$, when Λ is an in-spoked cycle algebra. This result generalized what was known in [**6**] about cycle algebras. Can we use our current formulation for Ψ_Λ in an even wider, or different, generalization of cycle algebras? Perhaps along the lines of "out-spoked" or "spoked" cycle algebras?

Continuing, when we undertake such considerations, would we need to modify the construction of Ψ_Λ to deal with forks? The answer to the latter is conjectured to be "yes". If this is the case, is Ψ_Λ the best combinatorial way of understanding finite generation of $E(\Lambda)$ for Λ monomial? In answering this last question, some thought has been given to what changes to Ψ_Λ might be necessary to deal with more general classes of monomial algebras. In conclusion, we end with a more direct set of questions: what can be said of Ψ_Λ when the square of the monomial ideal $\mathcal{I} = \langle \rho \rangle$ for Λ is zero? What about when $\mathcal{I}^n = 0$ for $n > 2$, and $\mathcal{I}^j \neq 0$ for each $1 < j < n$?

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Index

APPENDIX A

Proof of Product Lemma

We repeat some of the notation and vocabulary from Chapter 6 for ease of reading.

Let $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ and $\hat{q} = \hat{q}_1 \cdot \hat{q}_2 \cdot ... \cdot \hat{q}_m \in \mathcal{P}rod_{\Psi_\Lambda}$ such that $p = q$. Since $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$, there is a corresponding left-admissible sequence $\langle r_i \rangle_{i=1}^N$, where $N \geq 1$. Let $\{o_i\}_{i=1}^{N-1}$ denote the set of intersection words for the left-admissible sequence $\langle r_i \rangle_{i=1}^N$, where $o_i \vdash_i \langle r_i \rangle_{i=1}^N$, $1 \leq i < N$. Note: the set $\{o_i\}_{i=1}^{N-1}$ corresponds to the vertices along the path $\hat{p}$. Now, for each $\hat{q}_j$, there is also a associated left-admissible sequence along q_j, since each $\hat{q}_j$ is a proper factor for $1 \leq j \leq m-1$, and $\hat{q}_m$ is a proper path. Respectively, we denote these associated left-admissible sequences by $Seq_j = \langle s_k \rangle_{k=1}^{n_j}$, where $n_j = l(\hat{p}_j)$ for $1 \leq j \leq m$.

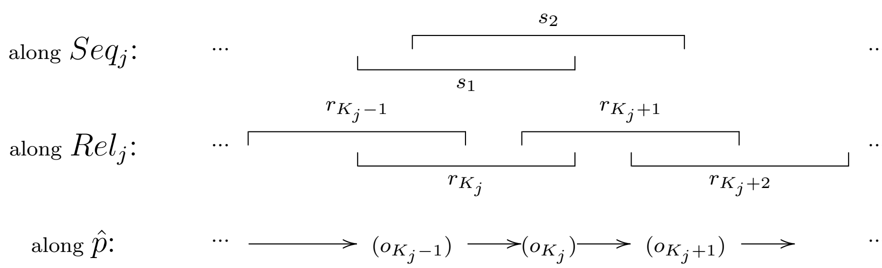
We wish to be careful about what we mean when we say a relation $s_k \in \rho$ "starts within" an intersection word $o_l \vdash \langle r_i \rangle_{i=1}^N$, or that it "ends within" an intersection word $o_{l+1} \vdash \langle r_i \rangle_{i=1}^N$. Due to the left-admissibility of $\langle r_i \rangle_{i=1}^N$, when we say a relation s_k *starts within* the intersection word o_l, we are stating that the initial vertex $\sigma(s_k)$ of s_k is within o_l, excepting the case when $\sigma(s_k) = \tau(o_l)$. Due to the fact that ρ is a reduced set, when we say s_k *ends within* the intersection word o_{l+1}, we mean that $\tau(s_k)$ is within o_{l+1}, excepting the case when $\tau(s_k) = \tau(o_{l+1})$. By saying s_k *starts strictly within* o_l, or *ends strictly within* o_{l+1}, we are stating $\sigma(s_k) \neq \sigma(o_l)$ and $\sigma(s_k) \neq \tau(o_l)$, or $\sigma(s_k) \neq \sigma(o_{l+1})$ and $\sigma(s_k) \neq \tau(o_{l+1})$ respectively.

Let $Seq_j = \langle s_k \rangle_{k=1}^{n_j}$ be the left-admissible sequence corresponding to $\hat{q}_j$, for $1 \leq j \leq m$. Since $p = q$, there exists some subsequence $Rel_j = \langle r_i \rangle_{i=K_j}^{N_j}$ of $\langle r_i \rangle_{i=1}^N$, of minimal length, such that in the underlying paths we have $\overrightarrow{s_1, ..., s_{n_j}}$ as a subpath of $\overrightarrow{r_{K_j}, ..., r_{N_j}}$, where $1 \leq K_j \leq N_j \leq N$. For $\langle r_i \rangle_{i=K_j}^{N_j}$ to be of minimal length, we have that either $s_1 = r_{K_j}$, or s_1 starts at a vertex strictly within the intersection word o_{K_j-1}. Furthermore, for $\langle r_i \rangle_{i=K_j}^{N_j}$ to be of minimal length when $j < m$, we must also have that s_{n_j} ends within, but not necessarily strictly, the intersection word o_{N_j}; or when $j = m$ we have $s_{n_m} = r_{N_m}$. We simplify the overhead in keeping track of sequence and subsequence lengths by writing $|Seq_j| = n_j$, and $|Rel_j| = N_j - K_j + 1$. The following is a consequence of "embedding" the subsequences Seq_j in the left-admissible sequence $\langle r_i \rangle_{i=1}^N$.

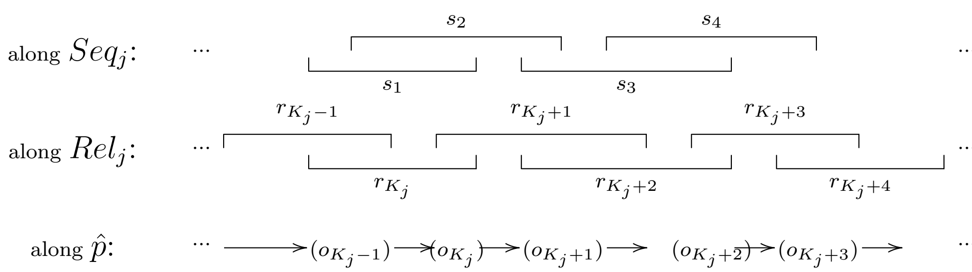
Lemma A.1. *$|Seq_1| = |Rel_1|$, $|Seq_j| \geq |Rel_j| - 1$ for $1 < j < m$, and $|Seq_m| = |Rel_m|$.*

We assume first that $j < m$, and that $Rel_j = \langle r_i \rangle_{i=K_j}^{N_j}$ is the minimal length subsequence of $\langle r_i \rangle_{i=1}^N$ that corresponds to $\hat{q}_j$. Then we also have that Rel_j corresponds to the left-admissible sequence $Seq_j = \langle s_k \rangle_{k=1}^{n_j}$. We further assume that $Seq_j \neq Rel_j$, otherwise it is immediate that $|Seq_j| = |Rel_j|$.

If first we suppose that the initial relations of Seq_j and Rel_j are the same, that is, $s_1 = r_{K_j}$, then s_2 must start strictly within the intersection word o_{K_j-1} and end within the intersection word o_{K_j+1}. If s_2 is the last relation in Seq_j, then $s_2 = s_{n_j}$ and $r_{N_j} = r_{K_j+1}$ and so $|Seq_j| = 2 = |Rel_j|$. We express these ideas in the overlap diagram:

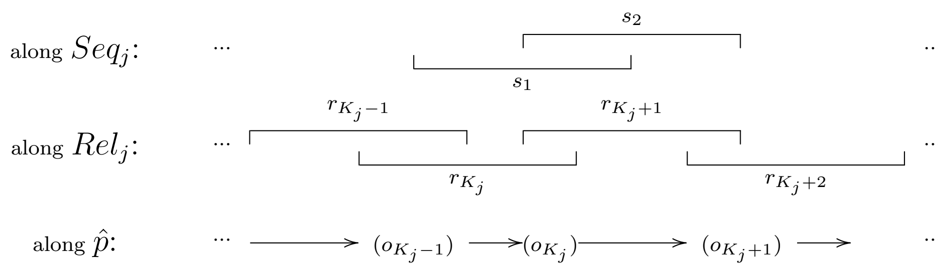


Note: although it is not indicated on the above diagram, the case can occur when $\tau(s_2) = \sigma(o_{K_j+1})$. Continuing, if instead s_3 is the last relation in Seq_j, then by the left-admissibility of Rel_j, s_3 must be equal to r_{K_j+2}, and so again $|Seq_j| = |Rel_j|$. If s_4 is the last relation in Seq_j then s_4 must start strictly within o_{K_j+1}, but after s_2, and end within o_{K_j+3}. We are served well by examining the following diagram:

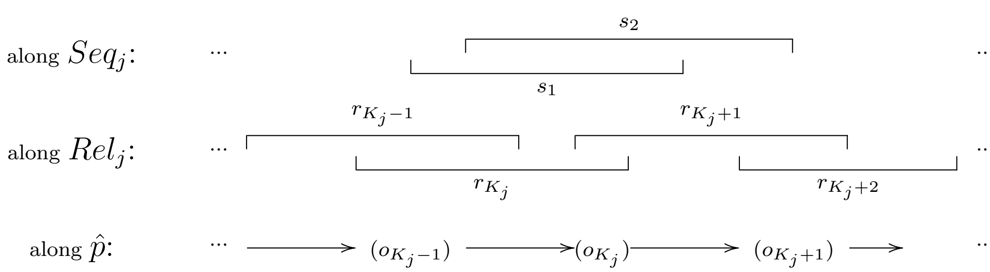


Following this same line of reasoning, we see for odd $k \geq 1$ that $s_k = r_{K_j+k-1}$, and so for n_j odd we have $s_{n_j} = r_{K_j+n_j-1}$ with $|Seq_j| = n_j = |Rel_j| = N_j - K_j + 1 = (K_j + n_j - K_j - 1) + 1$. Furthermore, for even $k > 1$, s_k must end strictly within o_{K_j+k-1}, and so, for $n_j > 1$ even, we have that s_{n_j} must end strictly within $o_{K_j+n_j-1}$. This again yields $|Seq_j| = n_j = |Rel_j|$ for $n_j > 1$ even. Hence, in all cases when the initial relations of Seq_j and Rel_j are the same, then $|Seq_j| = |Rel_j|$. Particularly, we have $|Seq_1| = |Rel_1|$.

Next we consider the case when the initial relations of Seq_j and Rel_j are not the same; that is, when $s_1 \neq r_{K_j}$. We must then have s_1 starting strictly within the intersection word o_{K_j-1} and: (a) ending either before r_{K_j+2} begins, or (b) ending strictly within the intersection word o_{K_j+1}. If, as in case (a), s_1 ends before r_{K_j+2} begins, then either: (a.i) $s_2 = r_{K_j+1}$, or (a.ii) s_2 starts strictly within o_{K_j-1} and ends within o_{K_j+1}. The case (a.i) is the easiest; by left-admissibility of $\langle r_i \rangle_{i=1}^{N}$ we have $s_k = r_{K_j+k-1}$ for each $k \geq 2$, and hence $|Seq_j| = |Rel_j|$ is always true of this case:

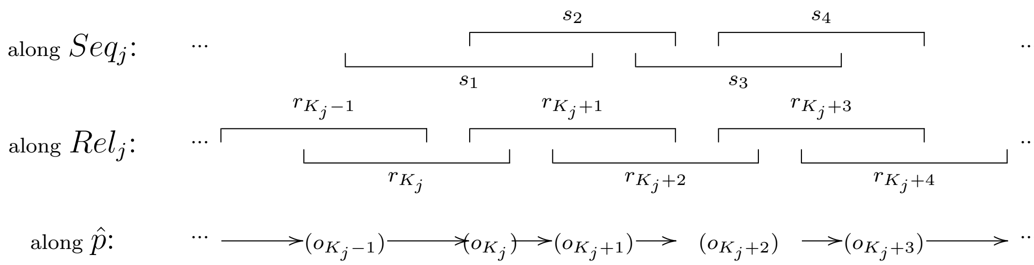


Now we turn to sub-case (a.ii), when s_2 starts strictly within o_{K_j-1} and ends within o_{K_j+1}. If $\tau(s_2) = \sigma(o_{K_j+1})$, then s_2 must be the final relation in Seq_j and so $|Seq_j| = |Rel_j| = 2$. On the other hand, if s_2 starts strictly within o_{K_j-1} and ends strictly within o_{K_j+1}, we arrive at the situation where $s_k = r_{K_j+k-1}$ for each $k \geq 3$. Hence, for $n_j \geq 3$, we have $s_{n_j} = r_{K_j+n_j-1}$, and so $|Seq_j| = n_j = |Rel_j| = N_j - K_j + 1 = (K_j + n_j - 1) + 1$ always in this case as well. The first few relations of this case are show in:

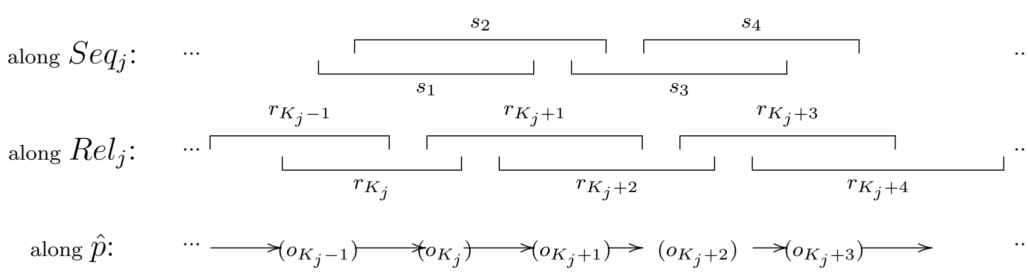


We next turn to case (b) from above, where s_1 starts strictly within the intersection word o_{K_j-1} and ends strictly within the intersection word o_{K_j+1}. If $|Seq_j| = 1$, then we are done, and see that $|Seq_j| \geq |Rel_j| - 1$, since $|Rel_j| = 2$. If not, we again have two sub-cases to consider where: (b.i) $s_2 = r_{K_j+1}$, or (b.ii) s_2 starts strictly within o_{K_j-1}, after the beginning of s_1, and s_2 ends strictly within o_{K_j+1}, after the end of s_1.

In considering sub-case (b.i) where $s_2 = r_{K_j+1}$, if also $s_2 = s_{n_j}$, then $|Seq_j| = |Rel_j|$. If $s_2 \neq s_{n_j}$, then s_3 must be such that: $s_3 \neq r_{K_j+1}$, s_3 starts strictly within o_{K_j+1} after s_1 ends, and s_3 ends within o_{K_j+3}. If $s_3 = s_{n_j}$ then $N_j = K_j + 3$ and so $|Seq_j| = 3 = |Rel_j| - 1$. If $n_j = 4$, by the left-admissibility of $\langle r_i \rangle_{i=1}^{N}$, we have that $s_4 = r_{K_j+1}$. This implies $|Seq_j| = 4 = |Rel_j|$. Continuing in this fashion, we see that $s_k = r_{K_j+k-1}$ for even $k > 1$, and for odd $k > 1$ that s_k must start strictly within o_{K_j+k-2} after s_{k-2} ends, and end within o_{K_j+k}, for $k \geq 3$. Hence, for even $n_j > 1$, we have $N_j = K_j + n_j - 1$, and so $|Seq_j| = n_j = |Rel_j| = N_j - K_j + 1 = (K_j + n_j - 1) - K_j + 1$. For odd $n_j > 1$, since s_{n_j} must end within $o_{K_j+n_j}$, we have $|Seq_j| = n_j = |Rel_j| - 1 = (N_j - K_j + 1) - 1 = ((K_j + n_j) - K_j + 1) - 1$. Viewing this case in diagram form yields some intuition:



Next, we consider sub-case (b.ii), wherein s_2 starts strictly within o_{K_j-1}, after the beginning of s_1, and s_2 ends strictly within o_{K_j+1}, after the end of s_1. If $s_2 = s_{n_j}$, then $|Seq_j| = |Rel_j| = 2$. For $n_j > 2$, we see that $s_3 \neq r_{K_j+2}$ since s_1 ends strictly within o_{K_j+1}. Therefore s_3 must be such that it: starts strictly within o_{K_j+1}, starts after s_1 ends, and starts before s_2 ends. Furthermore, s_3 must end strictly within o_{K_j+3}. If $s_3 = s_{n_j}$ then $N_j = K_j + 3$ and so $|Seq_j| = 3 = |Rel_j| - 1$. If $n_j = 4$, then either $s_4 = r_{K_j+3}$ or s_4 is such that it: starts strictly within o_{K_j+1}, starts after s_2 ends, starts before s_3 ends, and ends strictly within o_{K_j+3} after s_3 ends. If $s_4 = r_{K_j+3}$, then it is clear that $|Seq_j| = 4 = |Rel_j|$. If not, then we still have $N_j = K_j + 3$, and so $|Seq_j| = 4 = |Rel_j|$. Continuing along these lines, we see for $k > 1$ odd, s_k must be such that it: starts strictly within o_{K_j+k-2}, starts after s_{k-2} ends, starts before s_{k-1} ends, and ends strictly within o_{K_j+k}. Therefore for odd $n_j > 1$ we have s_{n_j} ending strictly within $o_{K_j+n_j}$, and hence: $|Seq_j| = n_j = |Rel_j| - 1 = (N_j - K_j + 1) - 1 = ((K_j + n_j) - K_j + 1) - 1$. For even $n_j > 1$, the situation is much simpler since $s_{n_j} = r_{K_j+n_j-1}$, or s_{n_j} ends strictly within $o_{K_j+n_j-1}$. In either situation, even $n_j > 1$ implies $|Seq_j| = n_j = |Rel_j| = N_j - K_j + 1 = ((K_j + n_j - 1 - K_j) + 1)$. The diagram for this case is as follows:



Finally, we arrive at the final case for our analysis: when $j = m$. However, we have already provided argument above. For, when $s_{n_m} = r_{N_m}$ in the sequences Rel_m and Seq_m, the work above shows that the following equality is always true: $|Seq_m| = n_m = |Rel_m|$.

APPENDIX B

Proof of Cf-Cycle Lengths as Multiples

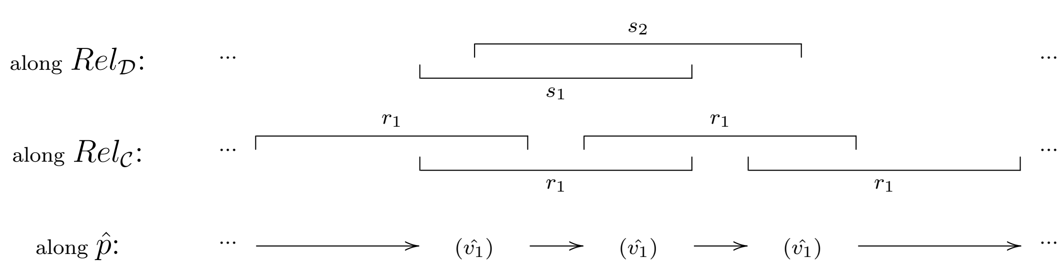
This appendix contains the full proof of the following lemma from Chapter 6:

Lemma B.1. *Let $\Lambda = k\Gamma/\langle\rho\rangle$ be a cycle algebra, and Ψ_Λ its associated Ψ-graph. Suppose $\mathcal{C}$ is a cf-cycle in Ψ_Λ of length n, and $\mathcal{D}$ is another cf-cycle in Ψ_Λ distinct from $\mathcal{C}$, such that $l(\mathcal{D}) \geq n$. Then $l(\mathcal{D})$ is a positive integer multiple of n.*

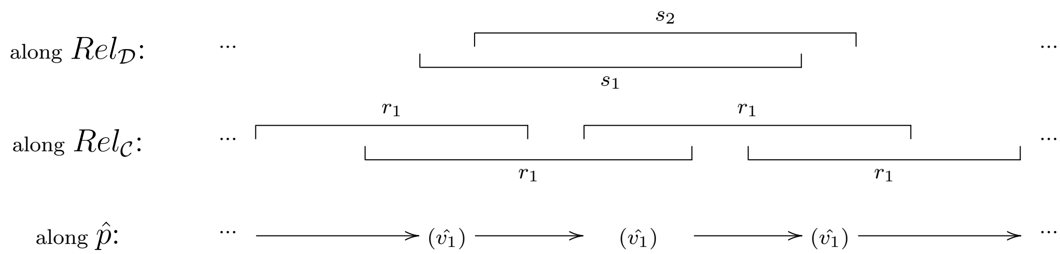
Proof. We find the subsequence of a left-admissible sequence that corresponds to the arrows (and vertices) of the cf-cycle $\mathcal{C}$ in the following way. First, we choose a vertex $\hat{v}_1$ on $\mathcal{C}$, and consider $\mathcal{C}_{\hat{v}_1}$ to be a simple path in Ψ_Λ starting and ending at $\hat{v}_1$. By Proposition 5.5 there exists a path $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$ such that $\mathcal{C}_{\hat{v}_1}$ is a subpath. Let $Rel_\mathcal{C} = \langle r_i \rangle_{i=1}^n$ be the subsequence of the left-admissible sequence for $\hat{p}$ that corresponds to $\mathcal{C}_{\hat{v}_1}$. The cycle $\mathcal{D}$ may be similarly treated, and we denote by $Rel_\mathcal{D} = \langle s_j \rangle_{j=1}^m$ its respective subsequence of a left-admissible sequence for a proper path containing $\mathcal{D}$ as a subpath. The subsequences $Rel_\mathcal{C}$ and $Rel_\mathcal{D}$ are clearly each unique, up to cyclic permutation.

For simplicity's sake, and without loss of generality, we reindex $Rel_\mathcal{D}$ so that one of two things is true: s_1 is equal to r_1, or s_1 starts strictly within v_1 and ends strictly within $v_{3(\text{mod } n)}$. It turns out, either case is satisfactory, and we re-index this way only so that we have a distinguished starting point for the path $\mathcal{D}$ relative to $\mathcal{C}$. We will treat each of the two cases independently, for each integer $n \geq 1$. It is clear that the restrictions on the arrows and vertices of $\mathcal{D}$ are simply the restrictions on the placement of the relations $s_j \in Rel_\mathcal{D}$ relative to $Rel_\mathcal{C}$. The proof proceeds by induction on $n = l(\mathcal{C})$; we cover the cases $n = 1$ and $n = 2$ first.

When $n = 1$, we the first of our two possibilities for the position of s_1 relative to v_1: when $s_1 = r_1$. Then we have that s_2 starts strictly within v_1 and ends strictly within v_1. Since s_1 is the next relation to be left-admissible along the cycle Γ, we have that $l(\mathcal{D}) = 2 = 2l(\mathcal{C})$. Diagrammatically, we see this in:



If $s_1 \neq r_1$, then we assume $s_2 \neq r_1$, else we devolve into the latter subcase for $n = 1$. Here, both s_1 and s_2 start strictly within v_1, and end strictly within v_1 in the following way:

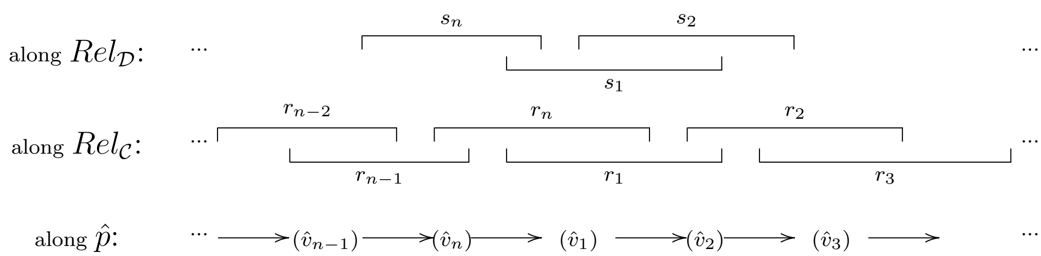


That is, s_1 must start strictly within v_1 and end strictly within v_1. Similarly, s_2 must start strictly within v_1 after s_1 starts, and end strictly within v_1 after s_1 ends. Furthermore, s_2 must overlap s_1, but not overlap s_2. That is, s_2 must end before s_2 begins, but after s_1 begins. These latter requirements are necessary, else $Rel_{\mathcal{D}}$ will not determine a cf-cycle in Ψ_Λ, or it will determine a cycle that is not disjoint from $\mathcal{C}$, yet it is distinct from $\mathcal{C}$. Again, from this we see $l(\mathcal{D}) = 2 = 2l(\mathcal{C})$.

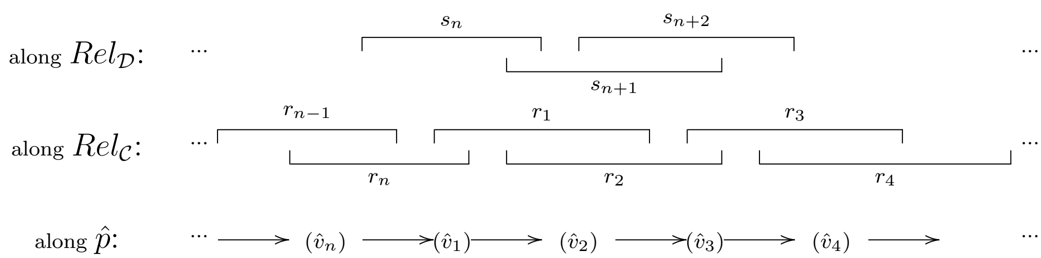
For the case $n = 2$, we have essentially the same arguments as those for $n = 1$, since v_1 is the only possible starting and ending vertex for relations not equal to s_1.

We have two major cases to consider, when $n \geq 3$, those cases being: when $s_1 = r_1$, and when $s_1 \neq r_1$. We dispatch with the first case here. If $s_1 = r_1$, then s_2 starts strictly within v_1 and ends strictly within v_3. For n even, we then have for $1 < j < n$: (i) if j is odd, then $s_j = r_j$; and (ii) if j is even, then s_j starts strictly within v_{j-1}, after s_{j-2} ends), and ends strictly within v_{j+1}. Furthermore, if n is even, then s_n must start strictly within v_{n-1}, after s_{n-2} ends, and end strictly within v_1. Finally, we must have s_n ending before s_2 starts, otherwise the sequence $Rel_{\mathcal{D}}$ will not determine a cf-cycle in Ψ_Λ, or it will determine a cycle that is not disjoint from $\mathcal{C}$, yet it is distinct from $\mathcal{C}$. If n is odd then we proceed as above, with the exception that s_n starts strictly within v_{n-1}, after s_{n-2} ends, and ends strictly within v_2. In continuing this line of thinking, we see for j such that $n < j < 2n$ we have: (i) if j is odd, then $s_j = r_{j-n}$, and (ii) if j is even, then s_j starts strictly within v_{j-n-1}, after s_{j-2} ends, and ends strictly within v_{j-n+1}. Finally, s_{2n} must start strictly within s_{2n-1}, after s_{2n-2} ends, and must end strictly within v_1, before s_2 begins. This latter fact is necessary, otherwise the sequence $Rel_{\mathcal{D}}$ will not determine a cf-cycle in Ψ_Λ, or it will determine a cycle that is not disjoint from $\mathcal{C}$, yet it is distinct from $\mathcal{C}$. Therefore, whether n is even, or n is odd, we have that when $s_1 = r_1$, $l(\mathcal{D})$ is a positive integer multiple of $l(\mathcal{C})$.

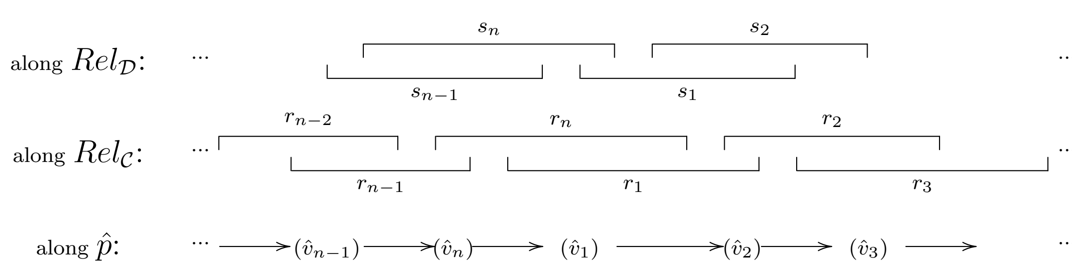
The following two diagrams will assist the reader in visualizing the principles outlined in the last paragraph. The first diagram corresponds to the case when $s_1 = r_1$, and n is an even number:



The second diagram illustrates the case when $s_1 = r_1$, n is an odd number, and when we are considering the relations halfway through the sequence $Rel_{\mathcal{D}}$. That is, how the relations s_n, s_{n+1}, and s_{n+2} fall relative to $Rel_{\mathcal{C}}$:



In our analysis of the relationship between $l(\mathcal{D})$ and $l(\mathcal{C})$, the other major case is when s_1 starts strictly within v_1 and ends strictly within v_3. If $s_2 = r_2$ then we simply devolve into the latter case, so we assume $s_2 \neq r_2$. We then have that s_2 starts strictly within v_1, after s_1 starts, and ends strictly within v_3, after s_1 ends. For n even, when j is such that $2 < j \leq n$, we have: (i) if j is odd, then s_j starts strictly within v_j after s_{j-2} ends, but before s_{j-1} ends, and s_j ends strictly within $v_{j+2(\text{mod } n)}$; and (ii) if j is even, then s_j starts strictly within v_{j-1} after s_{j-2} ends, and ends strictly within $v_{j+1(\text{mod } n)}$. Furthermore, we have that s_{n-1} must end before s_1 begins, and s_n must end before s_2 begins and s_n must overlap s_1, otherwise the sequence $Rel_{\mathcal{D}}$ does not determine a cf-cycle in Ψ_Λ, or it will determine a cycle that is not disjoint from $\mathcal{C}$, yet it is distinct from $\mathcal{C}$. The following diagram illustrates these requirements for $Rel_{\mathcal{D}}$ near v_1:



Finally, when n is odd, we proceed as earlier in this case when n is even, with the following exceptions. The first exception is that s_n must start strictly within v_n after s_{n-2} ends, but before s_{n-1} ends, and s_n ends strictly within v_2. The second exception is that s_{n+1} must start strictly within v_n after s_{n-1} ends, and s_{n+1} ends strictly within v_2. Continuing, when n is odd and j is such that $n-1 \leq j \leq 2n$, we have: (i) if j is odd, then s_j starts strictly within $v_{j(\text{mod } n)}$ after s_{j-2} ends, but before s_{j-1} ends, and s_j ends strictly within $v_{j+2(\text{mod } n)}$; and (ii) if j is even, then s_j starts strictly within $v_{j-1(\text{mod } n)}$ after s_{j-2} ends, and ends strictly within $v_{j+1(\text{mod } n)}$. Furthermore, we have that s_{2n-1} must end before s_1 begins, s_{2n} must end before s_2 begins, and s_{2n} must overlap s_1. The latter properties are required of s_{2n-1} and s_{2n}, otherwise the sequence $Rel_{\mathcal{D}}$ does not determine a cf-cycle in Ψ_Λ, or it will determine a cycle that is not disjoint from $\mathcal{C}$, yet it is distinct from $\mathcal{C}$. Therefore, whether n is even, or n is odd, we have also in this second major case that $l(\mathcal{D})$ is a positive integer multiple of $l(\mathcal{C})$

Our conclusion is to realize that, in all cases above, the sequence $Rel_{\mathcal{D}}$ must start and end at v_1. Hence, $l(\mathcal{D})$ is a positive integer multiple of $l(\mathcal{C})$, whenever $l(\mathcal{D}) \geq l(\mathcal{C})$. $\square$

APPENDIX C

Algorithms

Herein are given two, related, algorithms. The first, **UPPROPER**, calculates the underlying path $p \in \mathcal{B}_\Gamma$ for a proper path $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$. The second, **UPPRODUCT**, calculates the underlying path $q \in \mathcal{B}_\Gamma$ for a product path $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$. In both cases we use notation outlined in Chapter 4, so that every arrow $\hat{a} \in \Psi_1 \backslash A_C$ has a label in the form of a triple (u, v, w), for some appropriate decomposition of the underlying path a of $\hat{a}$. That is, $a = uvw$ for the subpaths $u, v, w \in \mathcal{B}_\Gamma$. These labels are referred to as *underlying path labels*, or, alternatively, as *up-labels*. There are exceptions to the labeling scheme just outlined, based on appropriate decomposition of the different types of arrows, and the reader is encouraged to refer back to Chapter 4 for a fuller description of the labeling for arrows in Ψ_1.

In this appendix, we will use the convention that the first element of the up-label triple for the arrow $\hat{a}$ may also be denoted by $L_0(\hat{a})$, and hence $L_0(\hat{a}) = u$ by the notes above. We will make the following similar denotations for the other components in the triple: $L_1(\hat{a}) = v$ and $L_2(\hat{a}) = w$. Finally, in the algorithms that follow, we use the symbol $\wedge$ to signify concatenation of paths. So, in terms of our continued example of $\hat{a}$ from above, we have $L_0(\hat{a}) \wedge L_1(\hat{a}) \wedge L_2(\hat{a}) = uvw$.

1. UPPROPER

The idea behind the algorithm provided here is quite simple. Because of the chosen labeling scheme for arrows in Ψ_Λ, we may determine the underlying path p of a proper path $\hat{p}$, of length ≥ 2, by first assigning the vertex $L_0(\hat{a}_1)$ to p, and then, for each arrow $\hat{a}_i$ of $\hat{p}$, we concatenate p and $L_1(\hat{a}_i) \wedge L_2(\hat{a}_i)$. Note: if $\hat{p}$ has length 1, then it receives special treatment by the algorithm, since the triple assigned in labeling such a $\hat{p}$ is one of the special cases in arrow labeling, wherein $L_0(\hat{a}_1) = p$. The algorithm itself appears on the next page in order to display it in its entirety.

INPUT: labeled proper path $\hat{p} \in \mathcal{P}roper_{\Psi_\Lambda}$
OUTPUT: underlying path $p \in \mathcal{B}_\Gamma$ of the given path $\hat{p}$
Assumptions: $\hat{p} = \hat{a}_1\hat{a}_2 \cdots \hat{a}_n$, for some $n \geq 1$.

Algorithm 1 UPPROPER

```
p ← L_0(â_1)
if n ≠ 1 then
  i ← 1
  while i ≤ n do
    p ← p ∧ L_1(â_i) ∧ L_2(â_i)
    i ← i + 1
  end while
end if
return p
```

2. UPPRODUCT

As in the previous case, the idea behind the present algorithm is again quite simple. For a product path $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$, we write $\hat{q} = \hat{q}_1 \cdot ... \cdot \hat{q}_m$, where, by definition, $m \geq 2$. Also by definition, $\hat{q}$ has been factored at its continuation arrows, so no factor $\hat{q}_i$ contains continuation arrows. It is worth pointing out that the variable q, which will be successively built up to be the underlying path for $\hat{q}$, is initialized with the starting vertex $\sigma(L_0(\hat{a}_{(1,1)}))$ of the underlying path of the first arrow $\hat{a}_{(1,1)}$ of the first factor $\hat{q}_1$. Finally, we treat each factor $\hat{q}_i$ in much the same way as that seen in **UPPROPER** above, going so far as to call it to compute the underlying path for the final factor $\hat{q}_m$. Again, the algorithm itself appears on the next page in order to display it in its entirety.

INPUT: labeled product path $\hat{q} \in \mathcal{P}rod_{\Psi_\Lambda}$
OUTPUT: underlying path $q \in \mathcal{B}_\Gamma$ of the given path $\hat{q}$
Assumptions: $\hat{q} = \hat{q}_1 \cdot ... \cdot \hat{q}_m$, for some $m \geq 2$,
$\hat{q}_i = \hat{a}_{(i,1)}\hat{a}_{(i,2)}...\hat{a}_{(i,n_i)}$, for $n_i \geq 1$ and $1 \leq i \leq m$.

Algorithm 2 UPPRODUCT

```
i ← 1
q ← σ(L_0(â_(1,1)))
while i < m do
  if n_i = 1 then
    q ← q ∧ L_0(â_(i,1))
  else
    j ← 1
    while j ≤ n_i do
      q ← q ∧ L_1(â_(i,j)) ∧ L_2(â_(i,j))
      j ← j + 1
    end while
  end if
  i ← i + 1
end while
return q ∧ UPPROPER(q̂_m)
```

$i \leftarrow 1$
$q \leftarrow \sigma(L_0(\hat{a}_{(1,1)}))$
while $i < m$ **do**
  if $n_i = 1$ **then**
    $q \leftarrow q \wedge L_0(\hat{a}_{(i,1)})$
  else
    $j \leftarrow 1$
    while $j \leq n_i$ **do**
      $q \leftarrow q \wedge L_1(\hat{a}_{(i,j)}) \wedge L_2(\hat{a}_{(i,j)})$
      $j \leftarrow j + 1$
    end while
  end if
  $i \leftarrow i + 1$
end while
return $q \wedge$ **UPPROPER**$(\hat{q}_m)$