A Three-dimensional Model of Poroviscous Aquifer Deformation

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Abstract

A mathematical model is developed for quantification of aquifer deformation due to ground-water withdrawal and, with some modifications, is potentially applicable to petroleum reservoirs. A porous medium saturated with water is conceptually treated in the model as a nonlinearly viscous fluid continuum. The model employs a new three-dimensional extension, made in this thesis, of Helm's poroviscosity as a constitutive law governing the stress-strain relation of material deformation and Gersevanov's generalization of Darcy's law for fluid flow in porous media. Relative to the classical linear poroelasticity, the proposed model provides a more realistic tool, yet with greater simplicity, in modeling and prediction of aquifer movement.

Based on laboratory consolidation tests conducted on clastic sedimentary materials, three phases of skeletal compaction are recognized. They are referred to as "instantaneous compression", "primary consolidation" and "secondary compression" according to Terzaghi and Biot's theory of poroelasticity. Among the three modes of consolidation, material behavior during the secondary compression phase has a nonlinear stress-strain relationship and is strongly time-dependent, exhibiting a phenomenon often known as "creep". In poroelasticity, the primary and secondary compressions have been conceptually considered as two separate physical processes that require two sets of material parameters to be evaluated. In contrast, the proposed poroviscosity model is a unified theory of time-dependent skeletal compression that realistically describes the physical phenomena of sediment compression as one single transient process.

As a general model, two sets of governing equations are formulated for Cartesian and cylindrical coordinates, respectively, and allow for mechanical anisotropy and the assumption of principal hydraulic directions. Further simplifications of the governing equations are formulated by assuming mechanical isotropy, irrotational deformation and mechanical axisymmetry, which are more suitable for field applications. Incremental forms of the governing equations are also provided.

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List of Symbols

Symbol		Units[†]
Δ	Prefix of an incremental variable	
^{T}	(Superscript) Indicates transpose of a matrix	
∇	Gradient operator	1/L
$\nabla\cdot$	Divergence operator	1/L
d	Total derivative operator	
$\frac{d}{dt}$	Total derivative with respect to time t	1/t
$\frac{\partial}{\partial t}$	Partial derivative with respect to time t	1/t
$\mathbf{0}$	Vector or second-rank tensor with value 0 in each entry	
$\mathbf{I}$	Identity matrix	
$\mathbf{f}$	Body force; $\mathbf{f} = [\, f_x \;\; f_y \;\; f_z \,]^T$ in Cartesian coordinates x y z and $\mathbf{f} = [\, f_r \;\; f_\theta \;\; f_z \,]^T$ in cylindrical coordinates r θ z	M/L^2t^2
$\mathbf{k}$	Unit vector in direction of vertical z axis with sea level as the datum in both Cartesian and cylindrical coordinates	
$\mathbf{q}$	Darcy flux or specific flux of water	L/t
$\mathbf{q}_b$	Bulk Flux; $\mathbf{q}_b = [\, q_{bx} \;\; q_{by} \;\; q_{bz} \,]^T$ in Cartesian coordinates x y z and $\mathbf{q}_b = [\, q_{br} \;\; q_{b\theta} \;\; q_{bz} \,]^T$ in cylindrical coordinates r θ z	L/t
$\mathbf{q}_{bp}$	Bulk Flux at time $t = 0^-$	L/t
$\mathbf{u}$	Displacement vector of a material in general; $\mathbf{u} = [\, u_x \;\; u_y \;\; u_z \,]^T$ in Cartesian coordinates x y z and $\mathbf{u} = [\, u_r \;\; u_\theta \;\; u_z \,]^T$ in cylindrical coordinates r θ z (Chapter 2)	L
$\mathbf{u}_0$	Displacement $\mathbf{u}$ at time t = 0	L
$\mathbf{u}_s$	Displacement vector of solid skeletal frame; $\mathbf{u}_s = [\, u_x \;\; u_y \;\; u_z \,]^T$ in Cartesian coordinates x y z and $\mathbf{u}_s = [\, u_r \;\; u_\theta \;\; u_z \,]^T$ in cylindrical coordinates r θ z (Chapters 4~6)	L

[†] M: Mass, L: Length, t: Time; ML/t^2: equivalent to units of force; M/Lt^2: equivalent to units of pressure or force per unit area.

List of Symbols (continued)

Symbol		Units†
$\mathbf{u}_{sp}$	Displacement of solids at time $t = 0^-$; $\mathbf{u}_{sp} = [\, u_{xp} \;\; u_{yp} \;\; u_{zp} \,]^T$ in Cartesian coordinates x y z and $\mathbf{u}_{sp} = [\, u_{rp} \;\; u_{\theta p} \;\; u_{zp} \,]^T$ in cylindrical coordinates r θ z	L
$\mathbf{u}_w$	Displacement vector of water	L
$\mathbf{u}_{wp}$	Displacement of water at time $t = 0^-$	L
$\mathbf{u}_P$	Displacement vector of a point P of interest	L
$\mathbf{u}_{Pp}$	Displacement of a point P of interest at time $t = 0^-$	L
$\mathbf{v}$	Velocity vector; $\mathbf{v} = [\, v_x \;\; v_y \;\; v_z \,]^T$ in Cartesian coordinates x y z and $\mathbf{v} = [\, v_r \;\; v_\theta \;\; v_z \,]^T$ in cylindrical coordinates r θ z	L/t
$\mathbf{v}_0$	Velocity $\mathbf{v}$ at time $t = 0$	L/t
$\mathbf{v}_s$	Velocity of solids	L/t
$\mathbf{v}_{sp}$	Velocity of solids at time $t = 0^-$	L/t
$\mathbf{v}_w$	Velocity of water	L/t
$\mathbf{v}_{wp}$	Velocity of water at time $t = 0^-$	L/t
$\mathbf{v}_P$	Velocity of a point P of interest	L/t
$\mathbf{v}_{Pp}$	Velocity of a point P of interest at time $t = 0^-$	L/t
$\mathbf{v}_\alpha$	Velocity of constituent α of a porous medium	L/t
$\mathbf{x}$	Position vector of a material in general (Chapter 2) or solid skeletal frame (Chapters 4~6); $\mathbf{x} = [x \;\; y \;\; z]^T$ in Cartesian coordinates and $\mathbf{x} = [r \;\; \theta \;\; z]^T$ in cylindrical coordinates (Chapter 2)	L
$\mathbf{x}_0$	Position vector $\mathbf{x}$ at time $t = 0$	L
$\mathbf{x}_s$	Position vector of solids; $\mathbf{x}_s = [r \;\; \theta \;\; z]^T$ in cylindrical coordinates (Chapters 4~6)	L
$\mathbf{x}_{sp}$	Position vector of solids at time $t = 0^-$; $\mathbf{x}_{sp} = [\, r_p \;\; \theta_p \;\; z_p \,]^T$ in cylindrical coordinates	L
$\mathbf{x}_w$	Position vector of water	L

List of Symbols (continued)

Symbol		Units†
$\mathbf{x}_{\mathrm{w}p}$	Position vector of water at time $t = 0^-$	L
$\mathbf{x}_{\mathrm{P}}$	Position vector of a point P of interest	L
$\mathbf{x}_{\mathrm{P}p}$	Position vector of a point P of interest at time $t = 0^-$; $\mathbf{x}_{\mathrm{P}p} = [\, r_{\mathrm{P}p} \;\; \theta_{\mathrm{P}p} \;\; z_{\mathrm{P}p} \,]^{\mathrm{T}}$ in cylindrical coordinates	L
$\mathbf{K}$	Hydraulic conductivity tensor	L/t
$\mathbf{K}_p$	Hydraulic conductivity tensor **K** at prestressed time $t = 0^-$	L/t
$\mathbf{R}$	Forcing function; $\mathbf{R} = [\, R_x \;\; R_y \;\; R_z \,]^{\mathrm{T}}$ in Cartesian coordinates x y z and $\mathbf{R} = [\, R_r \;\; R_\theta \;\; R_z \,]^{\mathrm{T}}$ in cylindrical coordinates r θ z	L/t
$\mathbf{R}_\Delta$	Incremental forcing function; $\mathbf{R}_\Delta = [\, R_{\Delta x} \;\; R_{\Delta y} \;\; R_{\Delta z} \,]^{\mathrm{T}}$ in Cartesian coordinates x y z and $\mathbf{R}_\Delta = [\, R_{\Delta r} \;\; R_{\Delta\theta} \;\; R_{\Delta z} \,]^{\mathrm{T}}$ in cylindrical coordinates r θ z	L/t
$\boldsymbol{\sigma}$	Total stress tensor	M/Lt^2
$\boldsymbol{\sigma}'$	Effective stress tensor	M/Lt^2
$\boldsymbol{\omega}$	Rotation vector of a material in general (Chapter 2) or solid skeletal frame (Chapters 4~6); $\boldsymbol{\omega} = [\, \omega_x \;\; \omega_y \;\; \omega_z \,]^{\mathrm{T}}$ in Cartesian coordinates x y z and $\boldsymbol{\omega} = [\, \omega_r \;\; \omega_\theta \;\; \omega_z \,]^{\mathrm{T}}$ in cylindrical coordinates r θ z	
a_{bx}, a_{by}, a_{bz}	Vector components of bulk volume acceleration in Cartesian coordinates x y z	L/t^2
f_{zp}	z component of body force at time $t = 0^-$	M/L^2t^2
g	Gravitational acceleration	L/t^2
h	Hydraulic head of a moving point in a porous medium saturated with water	L
h_p	Hydraulic head of a moving point in a porous medium saturated with water at time $t = 0^-$	L
h_{H}	Hydraulic head of a fixed point in a porous medium saturated with water	L

List of Symbols (continued)

Symbol		Units[†]
$\dot{m}_s$	Mass production (or loss) rate of solids within a bulk volume	M/L^3t
$\dot{m}_w$	Mass production (or loss) rate of water within a bulk volume	M/L^3t
n	Porosity	
n_p	Porosity at time $t = 0^-$	
p	Gauge water pressure of a moving point in a porous medium saturated with water	M/Lt^2
p_a	Absolute water pressure at an arbitrary elevation	M/Lt^2
p_p	Gauge water pressure of a moving point in a porous medium saturated with water at time $t = 0^-$	M/Lt^2
p_{sl}	Absolute water pressure at sea level	M/Lt^2
p_w	Gauge water pressure of a fixed point in a porous medium saturated with water	M/Lt^2
t	Time	t
t_c	The time when $\dot{\varepsilon} = \dot{\varepsilon}_i/2$ under a constant one-dimensional compressive stress since $t = 0$	t
t'_c	The time when $\sigma = 2\sigma_i$ under a one-dimensional compressive loading with constant strain rate since $t = 0$	t
t_t	The time when $\dot{\varepsilon} = 2\,\dot{\varepsilon}_i$ under a constant one-dimensional tensile stress since $t = 0$	t
t'_t	The time when $\sigma = \sigma_i/2$ under a one-dimensional tensile loading with constant strain rate since $t = 0$	t
u_{Pr}	Radial displacement of a point P of interest in cylindrical coordinates $r\,\theta\,z$	L
z_{sl}	Elevation of sea level	L
z_H	Elevation of a point fixed in space	L
z_P	Elevation of a point P of interest with sea level as the datum	L
z_{Pp}	Elevation of a point P of interest at time $t = 0^-$ with sea level as the datum	L

List of Symbols (continued)

Symbol		Units[†]
A	A poroviscous constitutive coefficient	
A_r, A_θ, A_z	Anisotropic poroviscous constitutive coefficients in cylindrical coordinates r θ z	
A_x, A_y, A_z	Anisotropic poroviscous constitutive coefficients in Cartesian coordinates x y z	
E	Young's modulus	M/Lt^2
G	Shear modulus	M/Lt^2
V	Bulk volume	L^3
I_1, I_2, I_3	First, second and third (total) stress invariants, respectively	M/Lt^2
$K_r, K_{\theta r}, K_{zr}$, $K_{r\theta}, K_\theta, K_{z\theta}$, $K_{rz}, K_{\theta z}, K_z$	Components of hydraulic conductivity tensor **K** in cylindrical coordinates r θ z	L/t
K_x, K_{yx}, K_{zx}, K_{xy}, K_y, K_{zy}, K_{xz}, K_{yz}, K_z	Components of hydraulic conductivity tensor **K** in Cartesian coordinates x y z	L/t
$\gamma_{r\theta}, \gamma_{\theta z}, \gamma_{zr}$, $\gamma_{\theta r}, \gamma_{z\theta}, \gamma_{rz}$	Shear strains in cylindrical coordinates r θ z (Chapter 2)	
$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$, $\gamma_{yx}, \gamma_{zy}, \gamma_{xz}$	Shear strains in Cartesian coordinates x y z (Chapter 2)	
ε	One-dimensional strain (Chapters 2 & 3)	
$\dot{\varepsilon}$	One-dimensional strain rate (Chapter 3)	1/t
ε_i	One-dimensional strain at time $t = 0^+$ (Chapter 3)	
$\dot{\varepsilon}_i$	One-dimensional strain rate at time $t = 0^+$ (Chapter 3)	1/t
ε_p	One-dimensional strain at time $t = 0^-$ (Chapter 3)	
$\varepsilon_r, \varepsilon_\theta, \varepsilon_z$	Normal strains of a material in general (Chapters 2 & 3) or solid skeletal frame (Chapters 4~6) in cylindrical coordinates r θ z	

List of Symbols (continued)

Symbol		Units[†]
$\dot{\varepsilon}_r, \dot{\varepsilon}_\theta, \dot{\varepsilon}_z$	Normal strain rates of solid skeletal frame in cylindrical coordinates r θ z	1/t
$\varepsilon_{rp}, \varepsilon_{\theta p}, \varepsilon_{zp}$	Normal strain of solid skeletal frame at time $t = 0^-$ in cylindrical coordinates r θ z	
ε_v	Volumetric strain of a material in general (Chapters 2 & 3) or solid skeletal frame (Chapters 4~6)	
$\varepsilon_x, \varepsilon_y, \varepsilon_z$	Normal strains of a material in general (Chapters 2 & 3) or solid skeletal frame (Chapters 4~6) in Cartesian coordinates x y z	
$\dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\varepsilon}_z$	Normal strain rates of a material in general (Chapter 3) or solid skeletal frame (Chapters 4~6) in Cartesian coordinates x y z	1/t
$\varepsilon_{xi}, \varepsilon_{yi}, \varepsilon_{zi}$	Normal strain of a material in general (Chapter 3) or solid skeletal frame (Chapters 4~6) at time $t = 0^+$ in Cartesian coordinates x y z	
$\dot{\varepsilon}_{xi}, \dot{\varepsilon}_{yi}, \dot{\varepsilon}_{zi}$	Normal strain rates of a material in general (Chapter 3) or solid skeletal frame (Chapters 4~6) at time $t = 0^+$ in Cartesian coordinates x y z	1/t
$\varepsilon_{xp}, \varepsilon_{yp}, \varepsilon_{zp}$	Normal strain of solid skeletal frame at time $t = 0^-$ in Cartesian coordinates x y z	
$\dot{\varepsilon}_{xp}, \dot{\varepsilon}_{yp}, \dot{\varepsilon}_{zp}$	Normal strain rates of solid skeletal frame at time $t = 0^-$ in Cartesian coordinates x y z	1/t
η	Dynamic viscosity	M/Lt
$\dot{\eta}$	Time rate of change of dynamic viscosity η in one dimension (Chapter 3)	M/Lt^2
η_i	Dynamic viscosity at time $t = 0^+$ (Chapter 3)	M/Lt
$\eta_r, \eta_\theta, \eta_z$	Anisotropic dynamic viscosities in cylindrical coordinates r θ z	M/Lt

List of Symbols (continued)

Symbol		Units[†]
$\eta_{rp}, \eta_{\theta p}, \eta_{zp}$	Anisotropic viscosities at time $t = 0^-$ in cylindrical coordinates r θ z	M/Lt
η_x, η_y, η_z	Anisotropic dynamic viscosities in Cartesian coordinates x y z	M/Lt
$\eta_{xi}, \eta_{yi}, \eta_{zi}$	Anisotropic viscosities at time $t = 0^+$ in Cartesian coordinates x y z	M/Lt
$\eta_{xp}, \eta_{yp}, \eta_{zp}$	Anisotropic viscosities at time $t = 0^-$ in Cartesian coordinates x y z	M/Lt
ν	Poisson's ratio	
ρ_b	Bulk density	M/L^3
ρ_s	Density of solid grains	M/L^3
ρ_w	Density of water	M/L^3
ρ^α	Partial density of constituent α of a porous medium	M/L^3
σ	One-dimensional stress (scalar) (Chapter 3)	M/Lt^2
σ_i	One-dimensional stress at time $t = 0^+$ (Chapter 3)	M/Lt^2
σ_m	Mean total stress	M/Lt^2
σ'_m	Mean effective stress	M/Lt^2
σ_{mp}	Mean total stress at time $t = 0^-$	M/Lt^2
σ'_{mp}	Mean effective stress at time $t = 0^-$	M/Lt^2
$\sigma_r, \sigma_\theta, \sigma_z$	Total normal stresses and normal components of total stress tensor $\boldsymbol{\sigma}$ in cylindrical coordinates r θ z	M/Lt^2
$\sigma'_r, \sigma'_\theta, \sigma'_z$	Effective stresses and normal components of effective stress tensor $\boldsymbol{\sigma}'$ in cylindrical coordinates r θ z	M/Lt^2
$\sigma'_{re}, \sigma'_{\theta e}, \sigma'_{ze}$	Equivalent effective stresses in cylindrical coordinates r θ z	M/Lt^2
$\sigma_x, \sigma_y, \sigma_z$	Total normal stresses and normal components of total stress tensor $\boldsymbol{\sigma}$ in Cartesian coordinates x y z	M/Lt^2
$\sigma'_x, \sigma'_y, \sigma'_z$	Effective stresses and normal components of effective stress tensor $\boldsymbol{\sigma}'$ in Cartesian coordinates x y z	M/Lt^2

List of Symbols (continued)

Symbol		Units[†]
$\sigma'_{xe}, \sigma'_{ye}, \sigma'_{ze}$	Equivalent effective stresses in Cartesian coordinates x y z	M/Lt^2
$\sigma_{xei}, \sigma_{yei}, \sigma_{zei}$	Equivalent normal stresses at time $t = 0^+$ in Cartesian coordinates x y z	M/Lt^2
$\sigma'_{xep}, \sigma'_{yep}, \sigma'_{zep}$	Equivalent effective stresses at time $t = 0^-$ in Cartesian coordinates x y z	M/Lt^2
$\sigma_{xi}, \sigma_{yi}, \sigma_{zi}$	Total normal stresses at time $t = 0^+$ in Cartesian coordinates x y z	M/Lt^2
$\sigma_{xp}, \sigma_{yp}, \sigma_{zp}$	Total normal stresses at time $t = 0^-$ in Cartesian coordinates x y z	M/Lt^2
$\sigma'_{xp}, \sigma'_{yp}, \sigma'_{zp}$	Effective stresses at time $t = 0^-$ in Cartesian coordinates x y z	M/Lt^2
$\tau_{r\theta}, \tau_{\theta z}, \tau_{zr}$, $\tau_{\theta r}, \tau_{z\theta}, \tau_{rz}$	Shear stresses and shear components of stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ in cylindrical coordinates r θ z	M/Lt^2
$\tau_{xy}, \tau_{yz}, \tau_{zx}$, $\tau_{yx}, \tau_{zy}, \tau_{xz}$	Shear stresses and shear components of stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ in Cartesian coordinates x y z	M/Lt^2
Φ	Hubbert potential (mechanical energy per unit mass of slowly moving water)	L^2/t^2

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Chapter 1 Introduction

Land subsidence and earth fissuring are two detrimental features known to be associated with ground-water and petroleum development and have caused tremendous property and infrastructure damages worldwide. In the United States alone, the affected area is estimated to be 80,000 km^2 (30,900 mi^2), of which 6,200 km^2 (2,400 mi^2) is in Los Banos-Kettleman city within the San Joaquin Valley, California, where a maximum total settlement of more than 9 meters (29 feet) occurred over a 49-year period (Johnson, 1998, pp. 51~55). Land subsidence, as the term implies, refers to only the vertical dimension of the land settlement while earth fissuring is another phenomenon of land movement occurring in the horizontal dimension. One of the well documented areas impacted by earth fissures is the Las Vegas Valley (Bell and Price, 1991; Bell *et al.*, 1992) located in the southwestern United States. Within 4,100 km^2 (1,600 mi^2) of the valley lowland, the total length of visible surface fissures recorded amounts to more than 18 km (11 mi) (Bell and Helm, 1998, p. 168). Property and infrastructure damage caused by subsidence and earth fissures collectively cost governments and individuals worldwide hundreds of millions of dollars every year both to investigate and to mitigate. The problems are ongoing and have been drawing attention from scientists and researchers to understand the causes and mechanism of land movement so that past mistakes can be avoided and effective remedial action can be taken.

Aquifer mechanics as a broad area of research focuses on the full three-dimensional aspects of land movement. One of the theories of aquifer mechanics that nearly all existing modeling and prediction tools are based upon is collectively classified as poroelasticity (Meinzer, 1923; Terzaghi, 1925a, 1925b; Jacob, 1940; Biot, 1941). Other more complicated theories of aquifer mechanics including viscoelasticity have also been developed (Goodman, 1980, p. 196; Jaeger and Cook, 1984, pp. 314~325).

Poroelasticity is founded on the classical theory of linear elasticity known as "Hooke's Law" and is best represented by consolidation theory of Terzaghi and Biot. (Please refer to Terzaghi (1960) for a collection of his important papers published and to Wang (2000) and Detournay and Cheng (1993) for an excellent summary of Biot's poro-elasticity and the state-of-the-art development.) Poroelasticity has gained wide popularity over more than fifty years and has been the predominant theory available for field applications of ground-water aquifer and petroleum reservoir compaction. Compared with other sophisticated theories, poroelasticity is relatively simple to apply.

Despite the popularity of poroelasticity, the theory itself is not without short-comings and constraints which are addressed in Section 1.2.1. This belief is shared by scientists in general which is why new theoretical and numerical developments are necessary for accurate prediction.

Recently, a new poroviscosity theory was introduced by Helm (1998) which, for one-dimensional problems, offers even more simplicity than does poroelasticity. At the same time poroviscosity debuted, Helm (1998) introduced a pioneer work in its own right, namely, a new nonlinear material constitutive law that represents the backbone of the theory of poroviscosity. Although the term "poroviscosity" with prefix "poro" seems to imply that the theory is applicable only to porous media, Helm's constitutive law actually can be applied potentially to many other materials which exhibit time-dependent strain behavior. The subject of poroviscosity as a constitutive law is introduced and discussed in detail in Section 3.2 of this thesis.

In a one-dimensional consolidation testing in the laboratory, compression of a porous material has been observed to theoretically go through three continuous periods of different behavior, namely, the early-time, intermediate-time and late-time behavior (Helm, 1998, p. 396). These three periods are viewed in poroelasticity theory as three separate physical processes: instantaneous, primary and secondary compressions, respectively, (Lambe and Whitman, 1979, pp. 299, 411 & 419) (to be explained later in Section 1.2.1). The theory of poroviscosity (Helm, 1998), however, incorporates these three phases of strain behavior into a single unified theory of time-dependent skeletal compression. Also, from an intuitive and practical point of view, the capacity of a porous medium to transmit fluids, which is grossly represented by hydraulic conductivity, must be affected by the possible closure of the interconnected pore space as the medium is being compressed. The theory of poroviscosity (Helm, 1998) has taken this practical consideration into account and allows hydraulic conductivity to vary, which is otherwise traditionally neglected in poroelasticity. Recently, Kim and Parizek (1999) introduced a poroelasticity model that allows hydraulic conductivity to vary during time-dependent compression. For more features of poroviscosity and its advantages over poroelasticity, readers are encouraged to consult the original paper by Helm (1998).

The theory of poroviscosity has opened a new area of research in aquifer mechanics that will change the way scientists traditionally perceive porous media to behave under stressed conditions. Since the theory was introduced, there have been only a handful of related research papers reported on the topic (Li and Helm, 1995, 1998; Jackson *et al*., 2004). Sections 1.2.2 and 1.2.3 of this thesis clarifies some of the technical issues and addresses some of the problems of the existing poroviscosity theory (Helm, 1998).

The research contained in this thesis is built directly on the fundamental principles of Darcy-Gersevanov's empirical law (Darcy, 1856; Gersevanov, 1937) that governs the flow of fluids in porous media and Helm's (1998) poroviscosity constitutive law that dictates how materials deform in response to stress. As a result, a new three-dimensional poroviscosity model is proposed and can serve as a theory to quantitatively describe the stressed behavior of aquifers or oil reservoirs. The use of this theory will ultimately lead to improved prediction of the long-term aquifer and reservoir movement with greater simplicity and more realistic results.

1.1 Organization of Thesis

The objectives and scope of this dissertation are addressed in Section 1.3 following a problem statement in Section 1.2. Some terms, notation and conventions used and adopted in this thesis are defined in the last section of this chapter (Section 1.4).

Subsequent organization of the thesis consists of five chapters. All variables used in the thesis are introduced and defined in Chapter 2 followed by Chapter 3 which is dedicated to the background of poroviscosity as a constitutive law. Included in Chapter 3 are an introduction to the one-dimensional poroviscosity (Helm, 1998) (Section 3.2) and the proposed extension of the poroviscosity constitutive law (Section 3.3).

Chapter 4 is where the proposed poroviscosity model is assembled from some fundamental principles presented in Section 4.3 and by invoking the three-dimensional extension of the poroviscosity constitutive relation (Sections 3.3 and 4.2). The proposed model in the form of governing equations is reached and presented in Section 4.4 followed by the simplification of the model with necessary assumptions in Section 4.5.

The incremental forms of the governing equations that are commonly employed in field applications are formulated in Chapter 5. The resulting incremental governing equations in the most general forms, that is, with less assumptions, are presented in Section 5.5 whereas their much simplified forms along with the necessary assumptions are summarized in Section 5.6.

Finally, Chapter 6 summarizes the conclusions and findings of this research and addresses the areas that are needed for future studies.

1.2 Statement of Problem

1.2.1 Limitations of Poroelasticity

While some materials, such as artificially manufactured metals and alloys, may exhibit behavior of instantaneous deformation and stress-strain linearity which, for practical purposes, can be explained and modeled by classical theory of linear elasticity with acceptable accuracy, the movement and deformation of most natural earth materials under stress are neither completely instantaneous nor linearly proportional to the applied stress (Lambe and Whitman, 1979, p. 151). The nature of nonlinear deformation of porous materials was recognized (Biot, 1973) in the early period of development of the poroelasticity theory.

As mentioned previously, stressed behavior of a porous material is viewed by poroelasticity as three separate physical processes, namely, the instantaneous, primary and secondary compressions (Lambe and Whitman, 1979, pp. 299, 411 & 419). Figure 1.1 displays a typical one-dimensional compression curve that is often observed in a laboratory testing. Among the three processes, the instantaneous compression (or the initial strain), which is represented by a theoretical poroelasticity asymptote in Figure 1.1, is difficult to measure with confidence in the lab for it occurs instantaneously as loading is applied. From the outset of loading, material compression essentially proceeds smoothly from the primary consolidation to the secondary compression (Figure 1.1). The theory of poroelasticity, however, draws an artificial limit of 100 % consolidation to the primary compression which is represented by a second poroelasticity asymptote (Figure 1.1). Also,

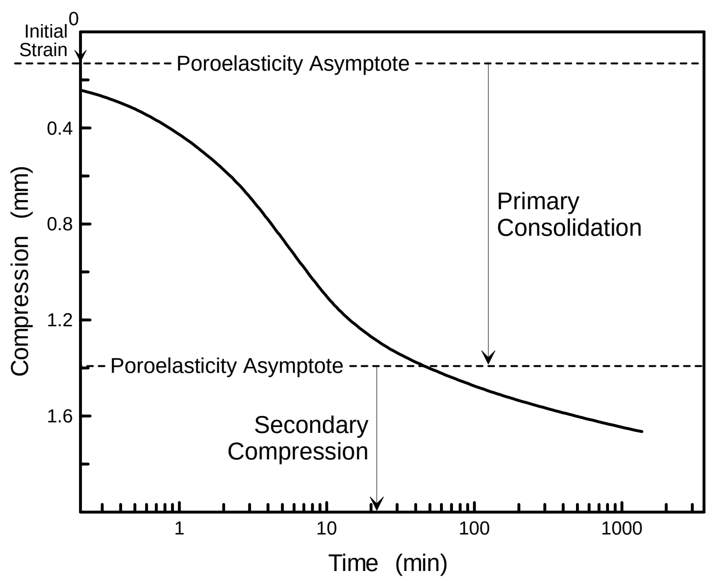

Figure 1.1 Theoretical one-dimensional consolidation curve and three independent physical processes: instantaneous (initial), primary and secondary compressions required by poroelasticity. (Modified from Helm (1998))

secondary compression is often a nonlinear stress-strain relationship and is strongly time-dependent which is a material behavior known as "creep" (Goodman, 1980, pp. 74, 193~195, 204 & 240~247; Jaeger and Cook, 1984, pp. 308~310). "The phenomenon of secondary compression greatly complicates prediction of the time history and final magnitude of settlement" (Lambe and Whitman, 1979, p. 420). In order to cope with the problem using poroelasticity, secondary compression must be treated separately from primary consolidation. As a result, two separate models are required to study the compression process and, at the same time, two sets of material parameters need to be estimated. Even so, poroelasticity still fails to account for the fact that materials do not stop deforming when the theoretical "hundred percent consolidation" occurs as Helm (1998, p. 396) points out.

Another problem associated with poroelasticity is the assumption of constant hydraulic conductivity. As mentioned previously, the ability of a porous medium in transmitting fluid may change as compaction may diminish the pore volume and break the connectivity of pore space. The assumption of constant hydraulic conductivity is an impractical constraint.

The issues stated above that are associated with poroelasticity can be solved with a model incorporating poroviscosity (Helm, 1998).

1.2.2 One-Dimensional Poroviscosity

The new poroviscosity constitutive relation (Helm, 1998) offers an improved alternative to poroelasticity in modeling land subsidence due to ground-water withdrawal. As a material constitutive law and as a theory of aquifer mechanics, poroviscosity in both development and application is still limited to one-dimensional problems. The performance of poroviscosity theory in three-dimensional field applications is yet unknown and can be assessed only if a three-dimensional poroviscosity model is available.

A so-called "incremental form" of any mathematical model in general is often preferred and desired for field applications and requires transformation of the original system of governing equations with some degree of simplifying assumptions for the reference state. For a nonlinear system of problem domain such as poroviscosity, this is especially true.

Therefore, transformation of a nonlinear poroviscosity model into its incremental form should not involve only a meager change of verbal meanings of variables without explicitly stating any assumptions. The author has found that the implication on any assumed reference state of the one-dimensional poroviscosity model posed by Helm (1998, p. 400) is obliviously vague, although validity of the model is unaffected. The problems associated with incremental formulation in general are addressed in the following section.

1.2.3 Posing Field Problems in Incremental Form

In seeking solutions to a field problem, a set of governing equations representing the physical and dynamic conditions of the problem domain under consideration are often solved numerically or, in rare occasions, analytically with appropriately specified prestressed and boundary conditions of the system. A certain state of equilibrium is usually assumed for the prestressed condition which, in practice, requires field monitoring and testing and sometimes rigorous research to estimate with confidence.

In reality, however, most research objectives and the pressing questions that need to be answered are often of predictive nature with the lack of interest on how a system behaved in the past. To serve such a purpose and also to simplify problems at hand, an incremental approach is commonly adopted which eliminates the need to know the preloading condition in most field problems. Any mathematically posed problem in incremental form is thereby expressed in terms of increments of variables from the respective states at preloading time. Also, by this definition, the increment of any variable at the preloading time must be zero-valued.

The idea of incremental variables is fairly intuitive but formulating a problem in incremental form is historically vague. To the author's knowledge, there are very few, if any, papers so far addressing the conditions and assumptions which are essential to the incremental form of a problem, especially one that is mathematically nonlinear. The ambiguities arise mainly from the lack of statements declaring the assumptions inherent to the preloading condition.

Without the knowledge of the reference state, the validity of an incremental model may not be jeopardized as stated previously regarding the one-dimensional poro-viscosity model (Helm, 1998, p. 400). Although the problem may be minor, potential assumptions associated with the incremental forms of the governing equations are investigated in Chapter 5.

1.3 Objectives and Scope

The objectives of the dissertation are (1) to extend the existing one-dimensional poroviscosity constitutive law (Helm, 1998) to three dimensions and ultimately (2) to develop a three-dimensional poroviscosity model for quantification of the movement and deformation of ground-water aquifers. The model is limited in scope to a set of governing equations using Cartesian and cylindrical coordinates that are aimed at potential field applications to aquifer systems. No attempt is made to solve for any analytical solutions to the proposed governing equations although some solutions may exist for problems with appropriately simple initial and boundary conditions.

1.4 Terms, Notation and Conventions

1.4.1 Representative Elementary Volume

In this dissertation, a conceptual model accounting for physical laws of fluid flow and material deformation is proposed in the form of mathematical equations which quantitatively describe the motion of a porous material as it deforms while fluid flows freely through interconnected pore space.

All discussions on and related to mathematical equations are made with respect to a material particle or point representing the centroid of a certain infinitesimal volume within a problem domain of interest, rather than the whole region, unless stated otherwise along with specified boundary conditions. An equation therefore should be considered a mathematical statement applicable to only an arbitrary single material point in general. Any properties, if not constant, or variables associated with the point are assumed continuous functions of space and time (or other properties and variables) and averaged over a volume that is often referred to as the "Representative Elementary Volume" or simply "REV".

An REV of a porous medium is conceptually a physical volume in a macroscopic point of view, which overlooks minute features such as molecular structures and texture or solid arrangement of the medium. An REV is characterized by a certain length scale within which the average of a physically meaningful and stable property or variable associated with the REV can be acquired at any time. The concept of Representative Elementary Volume was introduced by Hubbert (1956, pp. 34~36; 1969, pp. 275~277) and is also explained in detail in the literature (Bear, 1988, pp. 19~21; Bear and Verruijt, 1987, pp. 17~21). In particular, Bear and Verruijt (1987, pp. 17~21) introduced the term "Arbitrary Elementary Volume" (AEV) to refer to a conceptual volume at the microscopic level.

In this thesis, a porous medium saturated with water is collectively treated as a viscous continuum consisting of material points, in the sense described above, for which mathematical equations are written in terms of macroscopic state variables in accordance with the notion of a Representative Elementary Volume. A material point, therefore, may possess certain properties of both solid and fluid phases.

1.4.2 Prestressed Time and Initial Time

The expression "time $t = 0$" is used in this thesis to refer to the very beginning of an observation on a system with new hydraulic and/or mechanical stress conditions. The usage of the notation is usually sufficient for any interior point of a problem domain of interest. However, for any point that is located at a stress boundary, abrupt material motion may occur at so-called "time $t = 0$" which makes further distinction of $t = 0^-$ and $t = 0^+$ necessary. This can be illustrated by pumping water from aquifers. As a pump is turned on, a sudden surge of water flowing across the well screen takes place which also induces a spontaneous movement of solids toward the screen. Occurrence of this phenomenon is rapid and almost instantaneous. Without conceptually discerning the conditions at time t = 0, this physically unique material motion will be left unaccounted for. The term "prestressed" or "preloading" is employed to refer to the conditions or state of a variable at time $t = 0^{-}$ and any variable associated is denoted by an appended subscript "*p*" (in italic). Meanwhile, the term "initial" is used to refer to the time at $t = 0^{+}$ and a subscript "*i*" (in italic) in conjunction with the notation of a variable denotes the initial state of the variable.

For an internal point, especially one that is remote from stress boundaries, abrupt material motion is, theoretically speaking, unlikely and the states of any variable at $t = 0^{-}$ and $t = 0^{+}$ are indistinguishable and are assumed to be the same. Therefore, for interior points of a problem domain under consideration, time designation of simply "t = 0" suffices and the differentiation of prestressed time from initial time is unnecessary.

With the introduction of the two terminologies "prestressed" and "initial", time reference of "t = 0" (without specifying "0^{-}" or "0^{+}") may appear, for brevity, in this thesis at times provided that clarity is maintained. Chapter 3 introduces the topics of poroviscosity as a constitutive law which applies to an ideal material in direct contact with some specified forces. Therefore, the mechanical behavior of materials described in Chapter 3 should be considered the material behavior under boundary conditions. The prevailing conditions pertaining to t = 0, and considered in Chapter 3, are the "initial" conditions, namely, when $t = 0^{+}$, unless otherwise indicated. In contrast, governing equations for field conditions are assembled in Chapters 4 and 5 where only the internal domain of a field problem is considered. (For boundary points, hydraulic and stress conditions are often accounted for through the specification of boundary conditions when solving the governing equations.) The brief reference of time "t = 0" in Chapters 4 and 5 for the same preloading and initial time is thus sufficient according to the explanation above. Regarding variable notations, however, any prestressed variable that appears in Chapters 4 and 5 is still denoted by the subscript "*p*" to emphasize the specific time reference to $t = 0^{-}$. Notably, in Chapter 5 where incremental forms of the governing equations are formulated for field conditions, all variables are expressed specifically as increments from the respective states of the variables at the prestressed time $t = 0^{-}$.

1.4.3 Sign Conventions

Introduced here are two sign conventions that are simultaneously used in this thesis and define the mechanical meanings of stress and strain variables in an opposite way.

Engineering Mechanics Sign Convention:

> A sign convention that considers a tensile stress and the associated tensile strain positive (> 0) quantities; and, a compressive stress and the associated compressional strain negative (< 0) quantities. Traditionally, mechanical engineers, material scientists and structural engineers, among others, use this convention.

Geomechanics Sign Convention:

> A sign convention that considers a compressive stress and the associated compressional strain positive (> 0) quantities; and, a tensile stress and the

associated tensile strain negative (< 0) quantities. Ground-water hydrologists, geological engineers and some, not all, civil engineers often use this sign convention. Among the civil engineers adapted to use this sign convention are the geotechnical engineers who frequently deal with foundation and slope stability problems involving soils and rocks.

These are the two known sign conventions in use that simultaneously give stress and strain variables the same quantitative sign (positive "+" or negative "–") reflecting the nature of force involved. In other words, with a specific convention consistently adopted in formulation of a set of governing equations representing a system under certain stress conditions, a positive stress and a positive strain will represent the same mechanical state: tensional or compressional depending on the sign convention adopted; and likewise for both stress and strain being negative.

There are, however, other sign conventions under which, for example, a compressive stress is considered to be a positive (> 0) quantity whereas the associated compressional strain is considered a negative (< 0) quantity; and, a tensile stress is considered negative (< 0) whereas the associated tensile strain is considered positive (> 0). Mixed sign conventions of this sort do exist but will not be used in this thesis.

Therefore, the dual notation "±" or "∓" is employed as an algebraic operator and an algebraic sign, as well, in this thesis to consolidate two sets of equations or expressions for both conventions mentioned above. Wherever "∓", for example, appears in this thesis, one who adopts the engineering mechanics convention should take the sign on top, or "–" in this case. Otherwise, if the geomechanics convention is preferred, the sign at the bottom ("+") should be followed. With this rule, the example expression

$$a = \mp b \pm c \tag{1.1}$$

as written essentially represents two alternative expressions, that is, for engineering mechanics convention,

$$a = -b + c \tag{1.1e}$$

and for geomechanics convention,

$$a = b - c \; . \tag{1.1g}$$

If an expression is applicable only for the engineering mechanics convention, its equation label will consist of a trailing letter "e" as in Equation (1.1e). Likewise, expression (1.1g), with label ended with a letter "g", is meaningful only when one adopts the geomechanics convention.

1.4.4 Incremental Notation

The term "incremental" as in "incremental variable" is commonly used to indicate the change of a variable from its reference state. The variable includes scalar, vector and tensor quantities. What exactly the reference state is depends on what kind of solution to a particular problem is being sought and how the solution is approached. There are at least two solution techniques calling for the use of incremental notation which expresses variables in incremental form.

First, in a problem solving process with no interest in the prestressed condition, taking an approach that eliminates the need to investigate the prestressed state of a problem domain is often preferred. In this case, the reference state is placed at the prestressed state of the problem domain under consideration. Using incremental notation, the problem is then posed in terms of incremental variables with reference to the prestressed condition.

Second, some nonlinear problems can be linearized and solved afterwards as linear problems. Incremental notation facilitates the linearization of some nonlinear problems which can then be treated and solved as linear ones. In studying the nonlinear creep behavior of deformable porous media using the theory of poroelasticity, this incremental approach is commonly employed. In this sense, an incremental variable represents the increment between two discrete times with the interval so selected such that it is appropriately small and the condition pertaining to the beginning of the time interval is the reference state.

The usage of incremental notation in this thesis is to serve the first purpose stated above. In this thesis, a symbol "$\mathit{\Delta}$" (delta) preceding a variable is used to indicate the incremental quantity of that variable from the reference state (or the prestressed condition when time $t = 0^{-}$). For example, if $\mathbf{K}$ is a hydraulic conductivity tensor and is allowed to vary in space and time from the prestressed state $\mathbf{K}_p$ when $t = 0^{-}$, then $\mathbf{K}$ is written in terms of the increment $\mathit{\Delta}\mathbf{K}$ from $\mathbf{K}_p$ as $\mathbf{K} = \mathbf{K}_p + \mathit{\Delta}\mathbf{K}$ and, therefore,

$$\mathit{\Delta}\mathbf{K} = \mathbf{K} - \mathbf{K}_p \tag{1.2}$$

of which $\mathit{\Delta}\mathbf{K}$ is also referred to in this thesis as the "incremental form of hydraulic conductivity $\mathbf{K}$" or simply "incremental hydraulic conductivity". For $t = 0^{-}$, $\mathbf{K} = \mathbf{K}_p$ and, therefore, from (1.2),

$$\mathit{\Delta}\mathbf{K} = \mathbf{0} \qquad (\text{for } t = 0^{-}) \tag{1.3}$$

where $\mathbf{0}$ is a tensor with all components equal to 0. Equation (1.3) also serves as a demonstration that the increment of any variable from the reference state for which time $t = 0^{-}$ is zero-valued by definition.

In this thesis, **0** also appears frequently as a vector. In this sense, all components of **0** should be regarded to be zero-valued.

Chapter 2 General Background

In dealing with deformable materials in the field of mechanics, displacement, velocity, strain and rotation are considered kinematic quantities and stress is a dynamic quantity. This chapter introduces the fundamental definitions of these variables. Definitions of displacement and velocity are introduced first followed by that of strain and rotation in Section 2.2. The description of the stress state of a material is covered in Section 2.3. Hydraulic conductivity is an intrinsic property of a porous medium and its tensor notation and components used in this thesis are briefly introduced in the end of the chapter in Section 2.4.

2.1 Displacement and Velocity

For a point or a material particle moving in space and time by following a certain velocity field, displacement is the difference between the position vectors of the point at two different times. If displacement is designated by **u** [L] and the positions of the point at times $t = t_1$ and $t = t_2$ $(\geq t_1)$ are indicated by the position vectors $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ [L], respectively, then $\mathbf{u} = \mathbf{x}(t_2) - \mathbf{x}(t_1)$ and the displacement **u** is dependent on both $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ under a certain coordinate system. And, the change of **u** in time is the velocity **v** [L/t]. Thus, magnitude of the displacement vector **u** is a linear distance traveled by the particle from times t_1 to t_2 $(\geq t_1)$. A graphical rendition of a particle P moving along a curve *S* in a three-dimensional space is displayed in Figure 2.1 in which Figure 2.1(a) depicts the relative positions of the particle at times t_1 and t_2 $(\geq t_1)$ in a Cartesian coordinate system x y z with origin *C* fixed in space.

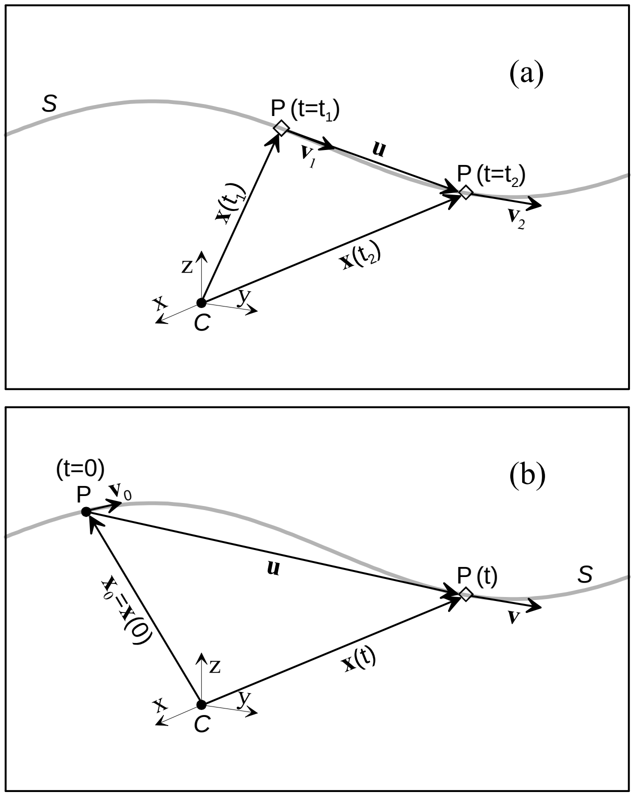

Figure 2.1 Displacement of a particle P moving along a curve *S* in Cartesian coordinates with fixed origin *C* and x, y and z axes for (a) two arbitrary times t_1 and t_2 ($\geq t_1$) and (b) any time $t \geq 0$. (•: a point fixed in space, ◇: a moving particle)

For a practical purpose to relax the dependency of displacement on two position variables ($\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$) at the same time, t_1 is usually set to zero, for convenience, to mark the time at the very beginning of an observation or a new stress scenario and t_2 is assigned as an arbitrary later time as illustrated in Figure 2.1(b). Accordingly, displacement is defined in this thesis as

$$\mathbf{u} = \mathbf{x} - \mathbf{x}_0 = \int_0^t \mathbf{v}\,dt \qquad (\text{for } t \geq 0) \tag{2.1}$$

in which $\mathbf{x}_0 = \mathbf{x}(0)$ (independent of time t) is the position vector at time t = 0 and $\mathbf{x} = \mathbf{x}(t)$

(a function of time t) is the position vector at an arbitrary time t (≥ 0). Thus, displacement and velocity so defined are functions of space and time, namely,

$$\begin{cases} \mathbf{u} = \mathbf{u}(\mathbf{x}, t) \\ \mathbf{v} = \mathbf{v}(\mathbf{x}, t) \end{cases} \tag{2.2}$$

and, in particular,

$$\begin{cases} \mathbf{u}_0 = \mathbf{u}(\mathbf{x}, 0) = \mathbf{0} \\ \mathbf{v}_0 = \mathbf{v}(\mathbf{x}, 0) \end{cases} \quad \text{(for t = 0)} \tag{2.3}$$

where $\mathbf{v}_0 = \mathbf{v}_0(\mathbf{x})$ becomes a function of space $\mathbf{x}$ only. Also, it follows immediately from (2.1) that

$$\mathbf{v} = \frac{d\mathbf{u}}{dt} \ . \tag{2.4}$$

The coordinates denoted by the position vector $\mathbf{x}_0$ in (2.1)~(2.3) above is referred to as the "material coordinates" and $\mathbf{x}$ as the "spatial coordinates" of the particle initially at $\mathbf{x}_0$ (Malvern, 1969, p. 139; Bear, 1988, pp. 70~71). Displacement and velocity in (2.1), as well as (2.4), are defined, therefore, in terms of the spatial coordinates $\mathbf{x}$ as declared in (2.2). The reference description or coordinate reference frame employed in this thesis is therefore the Eulerian description by definition (Malvern, 1969, pp. 138~145; Lliboutry, 1987, pp. 41~42; Bear, 1988, pp. 70~71).

In a Cartesian coordinate system x y z, position vector $\mathbf{x}$, displacement $\mathbf{u}$ and velocity $\mathbf{v}$ are denoted in vector notation as

$$\begin{cases} \mathbf{x} = [x \quad y \quad z]^T \\ \mathbf{u} = [u_x \quad u_y \quad u_z]^T \\ \mathbf{v} = [v_x \quad v_y \quad v_z]^T \end{cases} \tag{2.5}$$

where the superscript "T" indicates the transpose of a vector (so that a vector is a column vector) and vector components are also some functions of time t and/or spatial coordinates x, y and z. Similarly, in cylindrical coordinates r θ z,

$$\begin{cases} \mathbf{x} = [\mathrm{r} \;\; \theta \;\; \mathrm{z}]^{\mathrm{T}} \\ \mathbf{u} = [\mathrm{u}_{\mathrm{r}} \;\; \mathrm{u}_{\theta} \;\; \mathrm{u}_{\mathrm{z}}]^{\mathrm{T}} \\ \mathbf{v} = [\mathrm{v}_{\mathrm{r}} \;\; \mathrm{v}_{\theta} \;\; \mathrm{v}_{\mathrm{z}}]^{\mathrm{T}} \end{cases} . \tag{2.6}$$

In Equations (2.4), $\mathrm{d}\mathbf{u}/\mathrm{dt}$ is the total or material derivative of the displacement vector $\mathbf{u}$ and is equivalent to a vector with the total (time) derivative operator d/dt applied to each vector component of **u**, (2.5) or (2.6). In other words, if written out for Cartesian coordinates,

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} = \left[\frac{\mathrm{du}_{\mathrm{x}}}{\mathrm{dt}} \;\; \frac{\mathrm{du}_{\mathrm{y}}}{\mathrm{dt}} \;\; \frac{\mathrm{du}_{\mathrm{z}}}{\mathrm{dt}} \right]^{\mathrm{T}}$$

with

$$\begin{cases} \dfrac{\mathrm{du}_{\mathrm{x}}}{\mathrm{dt}} = \dfrac{\partial \mathrm{u}_{\mathrm{x}}}{\partial \mathrm{t}} + \left(\mathrm{v}_{\mathrm{x}} \dfrac{\partial \mathrm{u}_{\mathrm{x}}}{\partial \mathrm{x}} + \mathrm{v}_{\mathrm{y}} \dfrac{\partial \mathrm{u}_{\mathrm{x}}}{\partial \mathrm{y}} + \mathrm{v}_{\mathrm{z}} \dfrac{\partial \mathrm{u}_{\mathrm{x}}}{\partial \mathrm{z}} \right) \\ \dfrac{\mathrm{du}_{\mathrm{y}}}{\mathrm{dt}} = \dfrac{\partial \mathrm{u}_{\mathrm{y}}}{\partial \mathrm{t}} + \left(\mathrm{v}_{\mathrm{x}} \dfrac{\partial \mathrm{u}_{\mathrm{y}}}{\partial \mathrm{x}} + \mathrm{v}_{\mathrm{y}} \dfrac{\partial \mathrm{u}_{\mathrm{y}}}{\partial \mathrm{y}} + \mathrm{v}_{\mathrm{z}} \dfrac{\partial \mathrm{u}_{\mathrm{y}}}{\partial \mathrm{z}} \right) \\ \dfrac{\mathrm{du}_{\mathrm{z}}}{\mathrm{dt}} = \dfrac{\partial \mathrm{u}_{\mathrm{z}}}{\partial \mathrm{t}} + \left(\mathrm{v}_{\mathrm{x}} \dfrac{\partial \mathrm{u}_{\mathrm{z}}}{\partial \mathrm{x}} + \mathrm{v}_{\mathrm{y}} \dfrac{\partial \mathrm{u}_{\mathrm{z}}}{\partial \mathrm{y}} + \mathrm{v}_{\mathrm{z}} \dfrac{\partial \mathrm{u}_{\mathrm{z}}}{\partial \mathrm{z}} \right) \end{cases} . \tag{2.7}$$

However, for brevity, (2.7) sometimes appears in a compact form as

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{dt}} = \frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \mathbf{v} \cdot \nabla \mathbf{u} \tag{2.8}$$

where the convective term $\mathbf{v} \cdot \nabla \mathbf{u}$ represents the dot product of **v** and $\nabla \mathbf{u}$ and is written as $\mathbf{v}^{\mathrm{T}} \nabla \mathbf{u}$ (the transpose of vector **v** multiplied by $\nabla \mathbf{u}$). Note that $\nabla \mathbf{u}$ ("gradient" of the displacement vector **u**) is not a conventional mathematical notation and should be defined, for Cartesian coordinates, as

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{bmatrix} \tag{2.9}$$

so that expressions (2.7) and (2.8) are mathematically compatible. By the same token, if (2.8) is written out for cylindrical coordinates, displacement **u** and velocity **v** consist of respective vector components given by (2.6) and it can be verified that

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{\partial u_\theta}{\partial r} & \frac{\partial u_z}{\partial r} \\ \frac{1}{r}\frac{\partial u_r}{\partial \theta} & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} \\ \frac{\partial u_r}{\partial z} & \frac{\partial u_\theta}{\partial z} & \frac{\partial u_z}{\partial z} \end{bmatrix}. \tag{2.10}$$

2.2 Strain and Rotation

For a one-dimensional bar material with uniform cross section and initial length L subjected to a pair of tensile forces F [ML/t^2] on both ends (Figure 2.2) normal strain is defined as the amount of elongation per unit bar length and is calculated as

$$\varepsilon = \frac{\delta L}{L} \tag{2.11}$$

where ε is the normal strain [dimensionless] and δL ($=\delta L_1 + \delta L_2$) [L] is the total increase

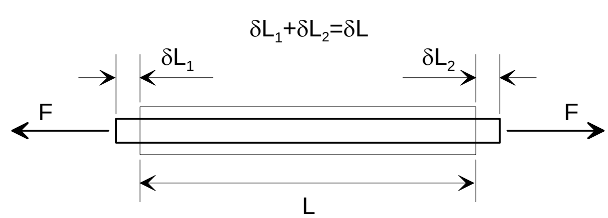

Figure 2.2 Extension of a bar under tensile forces F.

in bar length upon being stretched. Most material increases in length after being stretched. δL, in this case, is always measured and taken to have a positive quantity ($\delta L > 0$) as an observation.

If a certain "sign convention" is adopted so that the value of an extensional strain is considered positive (> 0) and that of a compressional strain is considered negative (< 0), then the normal strain ε computed according to (2.11) for the above bar material will be positive because $\delta L > 0$ and obviously the initial length of a bar is a positive material quantity ($L > 0$). Using also the same sign convention, normal strain of the bar, if subjected to a pair of compressional forces instead, will turn out to be a quantity equal in magnitude but opposite in algebraic sign, that is, a negative quantity, which indicates that straining within the bar is now compressional in nature. Traditionally, this sign convention has been widely adopted by researchers of mechanical engineering, material science and other disciplines. This particular sign convention is referred to, in this thesis, as "engineering mechanics (sign) convention". (See Section 1.4.3 for details on sign conventions.)

The above example of a deformed bar illustrates the significance of whether the numerical quantity inherent to a defined parameter, the normal strain ε in this case, is positive or negative if a specific sign convention is selected *a priori*. It reflects the state of material deformation, compressional or tensile, under the influence of applied forces.

There is, however, a different sign convention in use within the engineering community including notably the geotechnical engineering and geomechanics disciplines due to the fact that earth materials encountered in most field problems are, in large part, in a state of compression under most natural conditions. Under this convention, observation of the same physical phenomenon bears the same magnitude but are taken as a quantity opposite in sign: a normal strain is considered positive (> 0) if compressional and negative (< 0) if extensional. This set of convention is referred to as "geomechanics (sign) convention".

Without invoking any specific constitutive relation, this section revisits the definition of strain in terms of displacement by considering, in a Cartesian coordinate system x y z, the deformation of a cubic element with initial dimensions dx dy dz (Figure 2.3) in an infinitesimal displacement field $\mathbf{u} = [\,u_x \;\; u_y \;\; u_z\,]^T$ due to certain stress conditions, say, induced internally and/or at the remote boundary, for example. $u_x = u_x(x, y, x, t)$, $u_y = u_y(x, y, x, t)$ and $u_z = u_z(x, y, x, t)$ are the three components [L] of the displacement vector **u** in x, y and z directions, respectively, and are continuous scalar functions in space and time t. Their partial derivatives with respect to x, y and z, respectively, are assumed to exist, at least, up to the first order for the purpose of the following discussions.

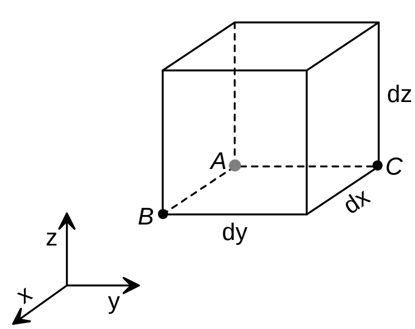

Figure 2.3 A deformable cube element in Cartesian coordinate system.

A complete description of the state of deformation of the cubic element depicted in Figure 2.3 consists of nine strain components, namely, three components of normal strain: ε_x , ε_y and ε_z parallel to x, y and z axes, respectively, and six components of shear strain: γ_{xy} , γ_{yx} (in xy plane), γ_{yz}, γ_{zy} (in yz plane), γ_{zx} and γ_{xz} (in zx plane). It will be shown in Section 2.2.2 that $\gamma_{xy} \equiv \gamma_{yx}$, $\gamma_{yz} \equiv \gamma_{zy}$ and $\gamma_{zx} \equiv \gamma_{xz}$ and, as a result, the number of independent shear strain components reduces to three. In the following two subsections, only the definitions of normal strain ε_x and shear strain γ_{xy} are discussed since the principles in reaching the expressions of the rest of the strain components are the same.

As the cubic element (Figure 2.3) deforms, rotation occurs as well and is characterized by a rotation vector $\boldsymbol{\omega}$ which is to be defined in Section 2.2.3.

2.2.1 Normal Strain

Consider the relocation of points *A* and *B* of the cubic element in Figure 2.3 under the infinitesimal displacement field **u** described above. For the purpose of defining the normal strain ε_x , one needs to take into account only the x component u_x of the displacement vector and its influence on the spatial translation of these two points in a direction parallel to the x axis as illustrated by Figure 2.4. The other two displacement components u_y and u_z which are both orthogonal to x axis and do not significantly affect the outcome, therefore, do not need to be considered.

In an initial state prior to u_x coming into play, points *A* and *B* are assumed an infinitesimal distance dx [L] apart

$$\overline{AB} = \mathrm{dx} \quad (> 0) \,. \tag{2.12}$$

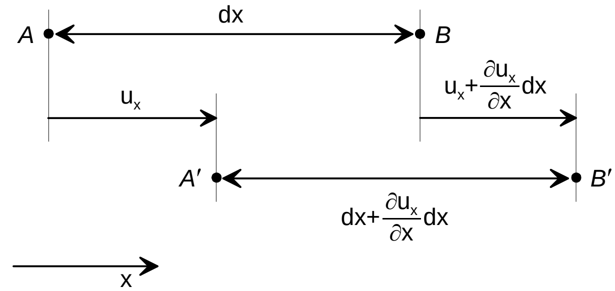

Figure 2.4 Translation of points *A* and *B* due to u_x, the x component of an infinitesimal displacement vector field **u**.

Under the imposed displacement field, translation at both points occurs. Points *A* and *B* move by the quantities u_x and $u_x + \frac{\partial u_x}{\partial x} dx$, respectively, to new locations which are schematically denoted by *A′* and *B′*, Figure 2.4. $\frac{\partial u_x}{\partial x}$ represents the quantity of change of u_x with respect to the change of position in x direction and $\frac{\partial u_x}{\partial x} dx$ represents the resulting total change of u_x over the distance dx. Hence, the new distance between the two points *A* and *B* in a deformed state becomes

$$\overline{A'B'} = dx + \left(u_x + \frac{\partial u_x}{\partial x} dx \right) - u_x = dx + \frac{\partial u_x}{\partial x} dx \quad . \tag{2.13}$$

Note that deformation of the bar material (Figure 2.2) is physically analogous to that of the element $\overline{AB}$ (Figure 2.4) except that the nature of deformation of the element $\overline{AB}$ now is not physically explicit and is completely dependent on the spatial variation of u_x. The displacement u_x under consideration does not necessarily carry both ends of the element toward, or away from, each other at the same time as opposed to that of the deformed bar example. Instead, both end points *A* and *B* may exhibit movement toward the same direction but with different magnitudes as illustrated here, Figure 2.4. In contrast, one can discern whether compression or extension occurs to the bar material (Figure 2.2) based intuitively on the increase or decrease in bar length.

Engineering Mechanics Convention

If u_x increases in the increasing x direction or $\frac{\partial u_x}{\partial x} > 0$, from (2.13) and (2.12),

$$\overline{A'B'} = dx + \frac{\partial u_x}{\partial x} dx > dx = \overline{AB} \quad \text{or} \quad \overline{A'B'} - \overline{AB} > 0$$

which suggests that points *A* and *B* are moving farther apart from each other. The element under consideration is said to be expanding or in tensile strain under the displacement field. Normal strain ε_x is then calculated using (2.12) and (2.13) as

$$\varepsilon_x = \frac{\overline{A'B'} - \overline{AB}}{\overline{AB}} = \frac{\left(dx + \frac{\partial u_x}{\partial x} dx\right) - dx}{dx} = \frac{\partial u_x}{\partial x} \tag{2.14e}$$

if a positive (> 0) quantity is desired for such a tensile strain. And, a normal strain if compressive will otherwise take on a negative (< 0) value. (2.14e) is essentially the definition of the x-component normal strain (Timoshenko and Goodier, 1970, p. 7; Brady and Brown, 1985, p. 33) under the engineering mechanics convention (see Section 1.4.3 for definition of a sign convention).

Geomechanics Convention

If u_x decreases, on the other hand, in the increasing x direction, that is, $\frac{\partial u_x}{\partial x} < 0$, from (2.13) and (2.12),

$$\overline{A'B'} = dx + \frac{\partial u_x}{\partial x} dx < dx = \overline{AB} \quad \text{or} \quad \overline{A'B'} - \overline{AB} < 0$$

which indicates that the distance between points *A* and *B* is getting shorter, albeit infinitesimally shorter. In other words, the element is in a state of compression under the displacement field. Again, if a positive quantity of compressive normal strain is desired, ε_x ought to be calculated using also (2.12) and (2.13), but, as

$$\varepsilon_x = \frac{\overline{AB} - \overline{A'B'}}{\overline{AB}} = \frac{dx - \left(dx + \frac{\partial u_x}{\partial x} dx\right)}{dx} = -\frac{\partial u_x}{\partial x} \tag{2.15g}$$

which guarantees a compressional strain with a positive value. Also, by (2.15g), a tensile strain which is satisfied by the condition $\frac{\partial u_x}{\partial x} > 0$, as addressed before, will then quantitatively turn out to be negative. (2.15g), hence, is the expression of normal strain ε_x (Goodman, 1980, p. 355; Brady and Brown, 1985, p. 42) according to the geomechanics convention. Note that, in contrast to (2.14e), a negative sign "–" appears in (2.15g). The existence of this algebraic sign is thus justified if one adopts the geomechanics sign convention.

There is, of course, one last trivial case to be considered in which $\frac{\partial u_x}{\partial x} = 0$ or u_x simply remains constant with respect to, at least, x. Similar to the above discussions, it follows from (2.13) and (2.12) that when $\frac{\partial u_x}{\partial x} = 0$,

$$\overline{A'B'} = dx + \frac{\partial u_x}{\partial x} dx = dx = \overline{AB} \quad \text{or} \quad \overline{A'B'} - \overline{AB} = 0 \ .$$

In other words, points *A* and *B* are drawn no closer or farther by u_x which, however, may not be zero. In this case, (2.14e) and (2.15g) are both applicable in arriving at a zero normal strain.

Based on the above discussions, a conclusion can be drawn, regardless of which sign convention adopted, that the state of deformation (tensional or compressional) of a material is manifested solely by whether the first-order spatial differential is positive or negative. Taking the normal strain ε_x as an example, a deformed material undergoes a tensional strain if $\frac{\partial u_x}{\partial x} > 0$ or, otherwise, undergoes a compressional strain if $\frac{\partial u_x}{\partial x} < 0$.

2.2.2 Shear Strain

For a two-dimensional square element, with the bottom edge fixed, subjected to a shearing force F as displayed in Figure 2.5, the extent of distortion is characterized by an angle γ measured in radians but taken to be dimensionless in unit. It is customary to consider the geometry of Figure 2.5(a) in order to quantify the shear strain according to the engineering mechanics convention. The rule of thumb is to let the shearing force F be pointed in the positive x direction and the element be distorted in shape accordingly. A classical development in pursuit of the angular quantity γ then proceeds as if, quantitatively, γ is positive (> 0) and, therefore, the angle θ under consideration is less than $\pi/2$ in units of radian (or 90°), Figure 2.5(a). If, however, γ turns out to be negative (< 0), the

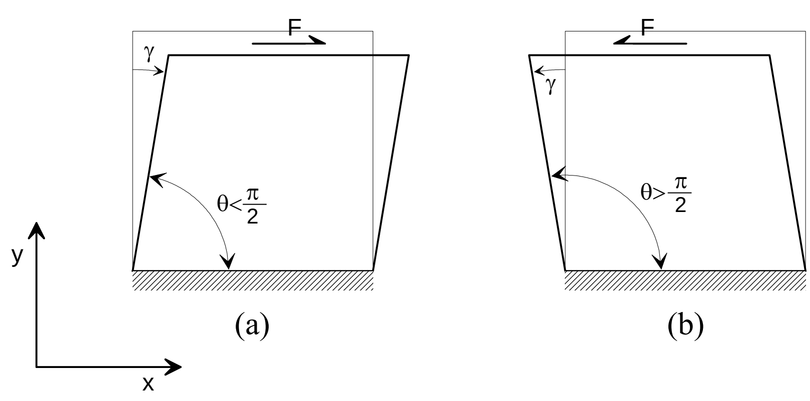

Figure 2.5 Distortion of a square by pure shearing due to a shearing force F acting in the (a) positive or (b) negative x direction.

distortion taking place is actually opposite to what was imagined: the force F involved is acting in an opposite direction and the resulting angle θ is greater than $\pi/2$ as depicted by Figure 2.5(b).

By the same principle and under a different convention, say, the geomechanics convention, the aforementioned shearing force F is then let pointed in an opposite direction, that is, to the negative x direction as indicated in Figure 2.5(b). The angle θ under consideration is now perceivably larger than $\pi/2$.

In order to define the shear distortion γ_{xy}, in terms of the continuous and infinitesimal displacement field **u**, for the cubic element (Figure 2.3) one needs only to find the relative angular distortion among points *A*, *B* and *C* due to the x and y components, u_x and u_y, of the displacement vector **u**.

Assume that prior to deformation, the initial positions of points *C*, *A* and *B* together form a right angle ($\pi/2$ radians or 90°), Figure 2.3 for which an xy-plane profile is portrayed in Figure 2.6. In a deformed state, all three points move presumably by pure translation. Point *A* moves to a new location schematically marked by *A′* of which the net travel path can be represented by the vector $[u_x \;\; u_y \;\; u_z]^T$ indicating changes of u_x, u_y and u_z in x, y and z coordinates, respectively. The relative movement of points *A*, *B* and *C* projected on xy plane is not affected by the z component u_z for it is in a direction orthogonal to the xy plane.

Translation of point *C* takes place as well and is associated with an x-coordinate change by the quantity $u_x + \frac{\partial u_x}{\partial y} dy$ of which $\frac{\partial u_x}{\partial y}$ represents the spatial change of u_x along the dy direction parallel to the y axis and $\frac{\partial u_x}{\partial y} dy$ represents the total change of u_x

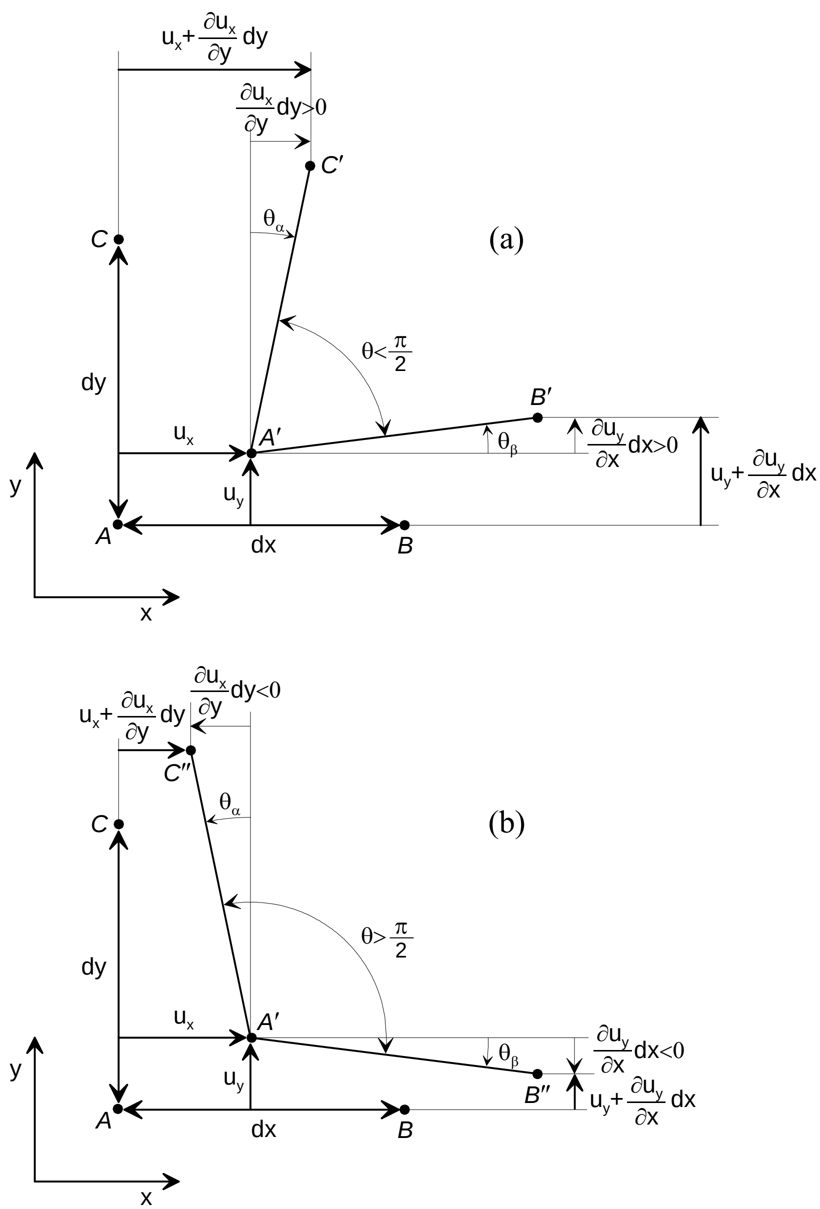

Figure 2.6 Distortion of angles in xy plane for (a) $\theta < \frac{\pi}{2}$ and (b) $\theta > \frac{\pi}{2}$.

(dx > 0 and dy > 0)

over the distance dy [L] between *C* and *A*. Under the imposed displacement, point *C* then occupies a different locale where the net deviation from the reference line perpendicular to the x axis and passing through point *A′*, Figure 2.6, occurs by the quantity

$$\left(u_x + \frac{\partial u_x}{\partial y} dy\right) - u_x = \frac{\partial u_x}{\partial y} dy \qquad (dy > 0)\ . \tag{2.16}$$

The exact new location where point *C* moves to in a deformed state is, therefore, dependent on the quantity $\frac{\partial u_x}{\partial y} dy$ of which the element length dy under consideration is greater than zero. (2.16) also represents the change in x coordinate relative to *A′*, the new residence of *A*.

If $\frac{\partial u_x}{\partial y} > 0$, that is, the x-component displacement u_x increases along the positive y direction, the offset defined by (2.16) is quantitatively positive (> 0), since $\frac{\partial u_x}{\partial y} > 0$ and also dy > 0, and is responsible for depressing the magnitude of the angle θ, Figure 2.6(a). In this case, point *C′* as depicted registers the new location of *C* due to the imposed displacement.

If, on the other hand, $\frac{\partial u_x}{\partial y} < 0$, the offset according to (2.16) becomes negative-valued, and, therefore, the angle θ of interest widens. Thus, point *C″* instead displayed in Figure 2.6(b) marks the new location of *C*.

By the same token, net y-coordinate offset of *B* with respect to *A′* in a deformed state occurs by the quantity

$$\left(u_y + \frac{\partial u_y}{\partial x} dx\right) - u_y = \frac{\partial u_y}{\partial x} dx \qquad (dx > 0)\ . \tag{2.17}$$

And, the new location of *B* is schematically marked by *B′* if $\frac{\partial u_y}{\partial x} > 0$ (Figure 2.6(a)) or by *B″* if $\frac{\partial u_y}{\partial x} > 0$ (Figure 2.6(b)).

Without further declaring which sign convention is adopted, it follows from the above discussions that a positive transverse spatial differential of displacement, namely, $\frac{\partial u_x}{\partial y} > 0$ or $\frac{\partial u_y}{\partial x} > 0$, contributes to the reduction of the right angle initially spanned by *C*, *A* and *B* (Figure 2.6(a)) and a negative transverse differential, $\frac{\partial u_x}{\partial y} < 0$ or $\frac{\partial u_y}{\partial x} < 0$, has an opposite effect and contributes to the widening of this right angle (Figure 2.6(b)), assuming that the angular changes involved are discernable although infinitesimal.

<u>Engineering Mechanics Convention</u>

A shear strain associated with the reduction of the right angle, under the engineering mechanics convention, should be quantitatively considered positive (> 0). And, therefore, the angles of distortion θ_α and θ_β as depicted in Figure 2.6(a), regardless clockwise or counterclockwise, should be perceived as positive (> 0) as well since shear strain itself is the collective sum of these two angles θ_α and θ_β. Hence, the following trigonometric relations

$$\tan\theta_\alpha = \frac{\left(u_x + \frac{\partial u_x}{\partial y} dy\right) - u_x}{dy} = \frac{\partial u_x}{\partial y} \tag{2.18}$$

and

$$\tan\theta_\beta = \frac{\left(u_y + \frac{\partial u_y}{\partial x} dx\right) - u_y}{dx} = \frac{\partial u_y}{\partial x} \tag{2.19}$$

can be established so that if $\frac{\partial u_x}{\partial y} > 0$, then $\tan\theta_\alpha > 0$ and, therefore, $\theta_\alpha > 0$ and, similarly, $\theta_\beta > 0$ provided that $\frac{\partial u_y}{\partial x} > 0$. In other words, positive angular distortions θ_α (> 0) and θ_β (> 0) as defined by (2.18) and (2.19), respectively, are contingent upon the conditions that $\frac{\partial u_x}{\partial y} > 0$ and $\frac{\partial u_y}{\partial x} > 0$ and, accordingly, magnitude of the initial right angle diminishes as indicated earlier.

Since the displacement under consideration is assumed infinitesimally small, the corresponding angles of distortion are conceivably small as well and, thus, can be approximated, from (2.18) and (2.19), respectively, by

$$\theta_\alpha \approx \tan\theta_\alpha = \frac{\partial u_x}{\partial y} \quad \text{(for small } \theta_\alpha\text{)} \tag{2.20}$$

and

$$\theta_\beta \approx \tan\theta_\beta = \frac{\partial u_y}{\partial x} \quad \text{(for small } \theta_\beta\text{)} . \tag{2.21}$$

The resulting total angular distortion is then accounted for as the sum of these two angles, measured in units of radian, or

$$\gamma_{xy} = \theta_\alpha + \theta_\beta \approx \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} . \tag{2.22e}$$

However, the units of a shear strain are taken to be dimensionless by definition. (2.22e) is the classical definition of shear strain under the engineering mechanics convention (Timoshenko and Goodier, 1970, p. 7; Brady and Brown, 1985, p. 34).

Furthermore, following the configuration of Figure 2.6(a) for which $\frac{\partial u_x}{\partial y} > 0$ and $\frac{\partial u_y}{\partial x} > 0$, the magnitude of the angle θ under consideration can be shown, indeed, to be less than that of a right angle, by taking (2.20) and (2.21) into account simultaneously, as

$$\theta = \frac{\pi}{2} - \theta_\alpha - \theta_\beta \approx \frac{\pi}{2} - \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) < \frac{\pi}{2} \ .$$

It is evident, at this point, that magnitude of the angle θ is actually dictated collectively by the two transverse spatial differentials of displacement: $\theta < \frac{\pi}{2}$ if $\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} > 0$ and, otherwise, $\theta > \frac{\pi}{2}$ if $\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} < 0$ (Figure 2.6). Note that this statement is mathematically true as well without further declaration on which sign convention selected. For the latter case with widening angle θ, shear strain so defined by (2.22e) will take on a negative value as required by the engineering mechanics convention.

Geomechanics Convention

If, on the other hand, the geomechanics sign convention is preferred, the angles of distortion θ_α and θ_β depicted in Figure 2.6(b), instead, regardless counterclockwise or not, are considered positive (> 0). Coordinate offsets previously defined by (2.16) and (2.17) using *A′* as the reference point must now be redefined with the reference reversed as, for x-coordinate offset of *A′* with respect to *C″*,

$$u_x - \left(u_x + \frac{\partial u_x}{\partial y} dy \right) = -\frac{\partial u_x}{\partial y} dy \qquad (dy > 0) \tag{2.23}$$

and, for y-coordinate offset of *A′* with respect to *B″*,

$$u_y - \left(u_y + \frac{\partial u_y}{\partial x} dx \right) = -\frac{\partial u_y}{\partial x} dx \qquad (dx > 0) \tag{2.24}$$

so that both are quantitatively positive contingent upon the conditions that the angle θ widens. As mentioned earlier, the negative transverse spatial differentials $\frac{\partial u_x}{\partial y} < 0$ and

$\frac{\partial u_y}{\partial x} < 0$ are both associated with the widening of the angle θ to beyond $\frac{\pi}{2}$. It is the same statement to assert that $-\frac{\partial u_x}{\partial y} dy > 0$ and $-\frac{\partial u_y}{\partial x} dx > 0$, respectively, play the same role in widening of the angle θ. Therefore, for infinitesimal angles, θ_α and θ_β can be approximated by the respective tangent functions expressed using the offsets defined, instead, by (2.23) and (2.24), as

$$\theta_\alpha \approx \tan\theta_\alpha = \frac{u_x - \left(u_x + \frac{\partial u_x}{\partial y} dy\right)}{dy} = -\frac{\partial u_x}{\partial y} \quad \text{(for small } \theta_\alpha\text{)} \tag{2.25}$$

and

$$\theta_\beta \approx \tan\theta_\beta = \frac{u_y - \left(u_y + \frac{\partial u_y}{\partial x} dx\right)}{dx} = -\frac{\partial u_y}{\partial x} \quad \text{(for small } \theta_\beta\text{)} . \tag{2.26}$$

Adding both angles together leads to a slightly different definition of shear strain

$$\gamma_{xy} = \theta_\alpha + \theta_\beta \approx -\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{2.27g}$$

under the geomechanics convention (Goodman, 1980, p. 355; Brady and Brown, 1985, p. 42). Comparing with (2.22e), (2.27g) differs by merely a negative sign "–" due to a different sign convention considered.

Also, by considering the angular distortions defined by (2.25) and (2.26) for which $\frac{\partial u_x}{\partial y} < 0$ and $\frac{\partial u_y}{\partial x} < 0$ as illustrated in Figure 2.6(b), the angle θ is found to be

$$\theta = \frac{\pi}{2} + \left(\theta_\alpha + \theta_\beta\right) \approx \frac{\pi}{2} - \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) > \frac{\pi}{2}$$

which is a contingent requirement of the geomechanics convention. If, however, $\frac{\partial u_x}{\partial y} > 0$

and $\frac{\partial u_y}{\partial x} > 0$, then $\theta < \frac{\pi}{2}$ and shear strain defined by (2.27g) will naturally turn out to be negative (< 0) as required.

Finally, note that by switching the orders of the subscripts x and y and the two differential terms, (2.22e) and (2.27g) become, respectively,

$$\gamma_{yx} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \equiv \gamma_{xy} \quad \text{and} \quad \gamma_{yx} = -\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \equiv \gamma_{xy}$$

which demonstrates, without rigorous proof, that the expressions of γ_{xy} and γ_{yx} in terms of displacements u_x and u_y are the same regardless of which sign convention selected. By the same argument, for the rest of the shear strain components, $\gamma_{yz} \equiv \gamma_{zy}$ and $\gamma_{zx} \equiv \gamma_{xz}$.

2.2.3 Rotation

When subjected to an infinitesimal displacement field **u**, the cubic element displayed in Figure 2.3 undergoes infinitesimal rotation as well which can be characterized by a rotation vector with orientation and magnitude indicating, respectively, orientation of the axis about which rotation occurs and the magnitude of rotation.

Consider the rotation of the element (Figure 2.3) about a line parallel to the z axis and passing through the point A. In the deformed state, the angles of distortion in xy plane θ_α and θ_β as depicted in Figure 2.6(a) and (b) are opposite in direction in both sign conventions and are both positive angular quantities ($\theta_\alpha \geq 0$ and $\theta_\beta \geq 0$ with directions indicated in Figure 2.6(a) and (b)). An angle of rotation is considered a positive angle if the angular direction follows the "right-hand rule" which is the same rule employed in operation of vector cross product. If **i**, **j** and **k** are the unit vectors of the coordinate axes, x, y and z, respectively, then $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. Therefore, by imagining that in Figure 2.6, the third coordinate axis z is pointing upward (away from the page), a counterclockwise angle of rotation is considered a positive angle.

<u>Engineering Mechanics Convention</u>

If one adopts the engineering mechanics convention, the net angle of rotation about *A′* ,which is the new location of point *A* in the deformed state, (Figure 2.6(a)) is $\theta_\beta - \theta_\alpha$ with θ_α and θ_β defined by (2.20) and (2.21), respectively. However, the actual rotation about the center of the element is only half of $\theta_\beta - \theta_\alpha$, that is, if denoting the angle of rotation by ω_z,

$$\omega_z = \frac{1}{2}\left(\theta_\beta - \theta_\alpha\right) . \tag{2.28e}$$

Hence, with θ_α and θ_β expressed in terms of displacement, (2.20) and (2.21), (2.28e) becomes

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{2.29}$$

which is the magnitude of rotation of the cubic element (Figure 2.3) about its center line orthogonal to the xy plane (Timoshenko and Goodier, 1970, p. 233; Brady and Brown, 1985, p. 34) and the units of ω_z are taken to be dimensionless.

<u>Geomechanics Convention</u>

In contrast, under the geomechanics convention, the angles of distortion θ_α and θ_β with directions indicated in Figure 2.6(b) are considered positive angles and are respectively defined by (2.25) and (2.26). The angle of rotation ω_z should then be calculated as

$$\omega_z = \frac{1}{2}\left(\theta_\alpha - \theta_\beta\right) . \tag{2.30g}$$

Substitution of (2.25) and (2.26) into (2.30g) simultaneously leads to

$$\omega_z = \frac{1}{2}\left[\left(-\frac{\partial u_x}{\partial y}\right) - \left(-\frac{\partial u_y}{\partial x}\right)\right] = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$$

of which the final expression is identical to that of (2.29) under the engineering mechanics convention.

Therefore, the angle of rotation ω_z is defined in terms of displacement by the same expression (2.29) for both sign conventions. The rotation ω_z as expressed by (2.29) occurs about an axis parallel to the z coordinate axis and, therefore, is always assigned to the z component of a so-called "rotation vector". Hence, if the rotation vector is denoted by $\boldsymbol{\omega} = [\,\omega_x \;\; \omega_y \;\; \omega_z\,]^T$, then (2.29) defines the z component of the vector regardless of sign convention. And, ω_x and ω_y can be defined similarly.

2.2.4 Summary of Strain and Rotation

Although derivations of only two strain components are demonstrated above, it can be concluded that the two consistent sign conventions, engineering mechanics and geomechanics, considered differ by merely an algebraic sign in strain definition ((2.14e) vs. (2.15g) and (2.22e) vs. (2.27g)). These two sign conventions are essentially the two opposite sides of a coin and the physical phenomena of material movement and deformation under scrutiny remain unchanged. Moreover, both conventions share the same definition of rotation (2.29).

Based on the previous discussions and published references, definitions of infinitesimal strain and rotation for both conventions are compiled and summarized below.

For a rectangular Cartesian coordinate system x y z,

$$\begin{cases} \varepsilon_x = \pm \dfrac{\partial u_x}{\partial x} \ , \quad \gamma_{xy} \equiv \gamma_{yx} = \pm\left(\dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x}\right) \\ \varepsilon_y = \pm \dfrac{\partial u_y}{\partial y} \ , \quad \gamma_{yz} \equiv \gamma_{zy} = \pm\left(\dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y}\right) \\ \varepsilon_z = \pm \dfrac{\partial u_z}{\partial z} \ , \quad \gamma_{zx} \equiv \gamma_{xz} = \pm\left(\dfrac{\partial u_z}{\partial x} + \dfrac{\partial u_x}{\partial z}\right) \end{cases} \tag{2.31}$$

(Timoshenko and Goodier, 1970, p. 7; Brady and Brown, 1985, pp. 33~34; Goodman, 1980, p. 355) and for a cylindrical coordinate system r θ z,

$$\begin{cases} \varepsilon_r = \pm \dfrac{\partial u_r}{\partial r} \ , \quad \gamma_{r\theta} \equiv \gamma_{\theta r} = \pm\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} + \dfrac{\partial u_\theta}{\partial r}\right) \\ \varepsilon_\theta = \pm\left(\dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}\right) \ , \quad \gamma_{\theta z} \equiv \gamma_{z\theta} = \pm\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \varepsilon_z = \pm \dfrac{\partial u_z}{\partial z} \ , \quad \gamma_{zr} \equiv \gamma_{rz} = \pm\left(\dfrac{\partial u_z}{\partial r} + \dfrac{\partial u_r}{\partial z}\right) \end{cases} \tag{2.32}$$

(Timoshenko and Goodier, 1970, p. 342; Goodman, 1980, p. 394). Since displacement **u** = **u**(**x**,t) is defined in (2.5) and (2.6) as a function of space **x** and time t, strains of (2.31) and (2.32) must be also functions of space **x** and time t.

Also, volumetric strain [dimensionless], if denoted by ε_v, is defined as the sum of the three components of normal strain. It follows from (2.31) and (2.32) that in Cartesian and cylindrical coordinates, respectively,

$$\varepsilon_v = \pm\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = \pm\nabla\cdot\mathbf{u} \tag{2.33}$$

and

$$\varepsilon_v = \pm\left[\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right] = \pm\nabla\cdot\mathbf{u} \tag{2.34}$$

(Gere, 2004, p. 507; Brady and Brown, 1985, pp. 34 & 40).

The three components of the rotation vector $\boldsymbol{\omega}$ in Cartesian coordinates ($\boldsymbol{\omega} = [\,\omega_x \;\; \omega_y \;\; \omega_z\,]^T$) and cylindrical coordinates ($\boldsymbol{\omega} = [\,\omega_r \;\; \omega_\theta \;\; \omega_z\,]^T$), respectively, are defined as

$$\begin{cases} \omega_x = \dfrac{1}{2}\left(\dfrac{\partial u_z}{\partial y} - \dfrac{\partial u_y}{\partial z}\right) \\ \omega_y = \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial z} - \dfrac{\partial u_z}{\partial x}\right) \\ \omega_z = \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y}\right) \end{cases} \tag{2.35}$$

(Timoshenko and Goodier, 1970, p. 233; Brady and Brown, 1985, p. 34) and

$$\begin{cases} \omega_r = \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta} - \dfrac{\partial u_\theta}{\partial z}\right) \\ \omega_\theta = \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} - \dfrac{\partial u_z}{\partial r}\right) \\ \omega_z = \dfrac{1}{2}\left(\dfrac{u_\theta}{r} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}\right) \end{cases} \tag{2.36}$$

(Lliboutry, 1987, p. 477) where $\omega_x = \omega_x(\mathbf{x},t)$, $\omega_y = \omega_y(\mathbf{x},t)$, $\omega_z = \omega_z(\mathbf{x},t)$ ($\mathbf{x} = [x \;\; y \;\; z]^T$), $\omega_r = \omega_r(\mathbf{x},t)$, $\omega_\theta = \omega_\theta(\mathbf{x},t)$ and $\omega_z = \omega_z(\mathbf{x},t)$ ($\mathbf{x} = [r \;\; \theta \;\; z]^T$).

Expressions (2.31)~(2.36) are all applicable to deformable materials as long as displacements u_x, u_y and u_z are infinitesimally small (Lliboutry, 1987, pp. 43~44; Ranalli, 1995, pp. 45~48) and all coordinate variables are functions of time t, that is, x = x(t), y = y(t), z = z(t), r = r(t) and θ = θ(t).

The notation "±" in an expression in (2.31)~(2.34) is meant to represent two alternative expressions as explained in Section 1.4.3. By (2.31), for example, the normal strain ε_z is written as, for engineering mechanics convention,

$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

and, for geomechanics convention,

$$\varepsilon_z = -\frac{\partial u_z}{\partial z} \ .$$

2.3 Stress State at A Point

2.3.1 Stress Components

A complete description of the stress state at a point consists of nine components and is often represented by a stress tensor, say, $\boldsymbol{\sigma}$ which for a cubic element dx dy dz in a Cartesian coordinate system x y z (Figure 2.7) can be expressed in matrix form as

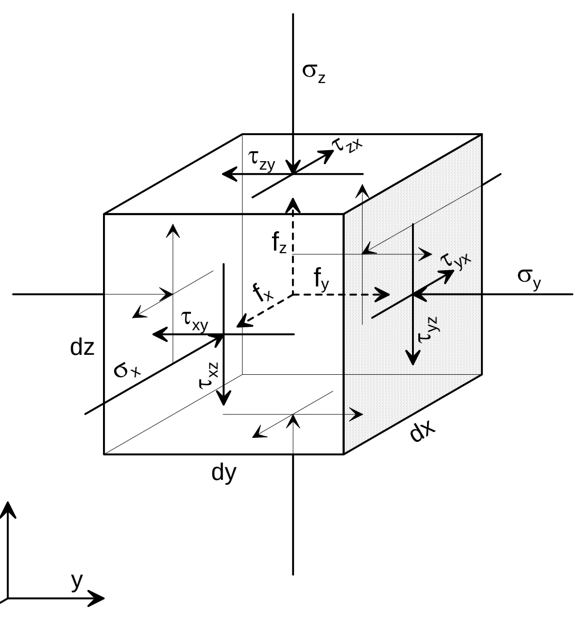

Figure 2.7 Stress components of a cubic element dx dy dz in a Cartesian coordinate system. (Geomechanics convention)

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{2.37}$$

where σ_x, σ_y and σ_z are the components of normal stress [M/Lt^2] in x, y and z directions, respectively, and τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx} and τ_{zy} are the components of shear stress [M/Lt^2] with orientations indicated by the subscripts (similar to the notation of the shear strain components in Section 2.2). The only subscript of a normal stress denotes the surface or face which the normal stress acts upon as well as the direction of the force. The normal stress σ_x, for example, as illustrated in Figure 2.7 points in a direction parallel to the x axis and exerts its force on the front face, or "+x face" (a term to be explained in the following paragraph), of the cube. For a shear stress with dual subscript, however, the first subscript denotes the surface plane on which the shear stress exerts force while the second subscript indicates the direction in which the shear stress lies. Take the pair of shear stresses τ_{xy} and τ_{xz} as examples. Both stresses exert forces on the same +x face (front face) of the cubic element, Figure 2.7, as indicated by the first subscript x. But, τ_{xy} and τ_{xz} point in different directions, parallel to y and z axes, respectively, as indicated by the second subscripts y and z.

With the volumetric element conceptually shrinking to an infinitesimal size, or, dx $\rightarrow$ 0, dy $\rightarrow$ 0 and dz $\rightarrow$ 0 [L], $\boldsymbol{\sigma}$, (2.37) is said to represent the stress state at a point and is a function of space $\mathbf{x}$ (=$[x \;\; y \;\; z]^T$) and time t.

There are six surfaces or faces enclosing a cubic body. A face is considered a positive (+) face if its unit outward normal vector coincides with the positive direction of a coordinate axis. Otherwise, it's considered a negative (−) face. The shaded area in

Figure 2.7 is then a +y (or positive-y) face according to this rule. Also, the front (+x) face and the top (+z) face of the cube are positive faces as well. And, the rest of the hidden faces are negative faces: –x face (rear), –y face (left) and –z face (bottom).

In a typical stress diagram, there are always three stress components exerting forces on one surface plane and their orientations depend on which sign convention is followed.

Consider the orientations of stresses exerting forces on a positive face. If the engineering mechanics convention is adopted, the direction of a normal stress is perceived as though it is in the same direction of the unit outward normal of this positive face and the directions of the two shear stresses are in the positive directions of the respective coordinate axes. Meanwhile on the opposing negative face, stress directions are perceived in exactly the opposite directions. Note that the normal stress so considered is physically a tensile stress under the engineering mechanics convention.

If, however, the geomechanics convention is followed and also the stress orientation on a positive face is considered, a normal stress is directed as if it points in a direction opposite to the unit outward normal of the positive face and the two shear stresses are let point in the negative directions of the corresponding coordinate axes. As to the stresses on the opposing negative face, their directions are exactly opposite to that on the positive face. Clearly, a normal stress so orientated relative to the cubic element under consideration is physically a compressive stress. Thus, the stress diagram illustrated by Figure 2.7 is said to be compatible with the geomechanics convention.

Another force that commonly appears in a stress diagram is the body force (per unit volume), if denoted by **f**,

$$\mathbf{f} = [f_x \;\; f_y \;\; f_z]^T \tag{2.38}$$

where $f_x = f_x(\mathbf{x}, t)$, $f_y = f_y(\mathbf{x}, t)$ and $f_z = f_z(\mathbf{x}, t)$ are the vector components of the body force **f** [M/L^2t^2] in x, y and z directions, respectively, as depicted in Figure 2.7. Body force for a porous medium is the bulk gravitational force per unit volume of the medium and the values of its vector components depend on the choice of coordinate axes. If a set of coordinate axes is chosen so that the z axis is positive upward against the direction of gravitational force, then $f_x = f_y = 0$ and $f_z < 0$. And, body force along with stress and inertia force is often required to satisfy certain conditions.

Similar to (2.37) and (2.38), stress tensor **σ** and body force **f** in cylindrical coordinates are denoted by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_\theta & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_z \end{bmatrix} \tag{2.39}$$

and

$$\mathbf{f} = [f_r \;\; f_\theta \;\; f_z]^T \tag{2.40}$$

where all tensor and vector components are function of space $\mathbf{x} = [r \;\; \theta \;\; z]^T$ and time t.

In order to relate stress to strain, a constitutive law (to be introduced in Chapter 3) must be invoked.

2.3.2 Total Stress and Effective Stress

For a porous medium saturated with fluid, shearing resistance between solid grains is diminished by virtue of reduced grain-to-grain force due to the existence of pore fluid pressure (Terzaghi, 1936, p.54). The stress between solid grains is referred to as the "effective stress" or "intergranular stress" and is reduced in magnitude by the buoyant force in the form of fluid pressure. The stress state of a point within the porous medium can thus be decomposed into an effective stress and a fluid pressure. The term "point" as applied to a porous medium is used to indicate the centroid of a conceptually infinitesimal volume of the medium treated as a continuum; and any kinetic and dynamic properties associated with the point are assumed to be averages taken over the Representative Elementary Volume (REV) containing the point. (Please see Section 1.4.1 for details of the conceptual definition of a point and the Representative Elementary Volume.)

According to Terzaghi (1936, p. 54), the principle of effective stress can be stated by the expression

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' \mp p\mathbf{I} \quad \text{or} \quad \boldsymbol{\sigma}' = \boldsymbol{\sigma} \pm p\mathbf{I} \tag{2.41}$$

where $\boldsymbol{\sigma}' = \boldsymbol{\sigma}'(\mathbf{x}, t)$ is the effective stress, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$ is referred to as the "total stress" (to distinguish from $\boldsymbol{\sigma}'$), p = p($\mathbf{x}$,t) is the fluid pressure [M/Lt^2] and $\mathbf{I}$ is the identity tensor [dimensionless] and, in a three-dimensional space,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.42}$$

Furthermore, in a Cartesian coordinate system x y z,

$$\boldsymbol{\sigma}' = \begin{bmatrix} \sigma'_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma'_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma'_z \end{bmatrix} \tag{2.43}$$

where $\sigma'_x = \sigma'_x(\mathbf{x}, t)$, $\sigma'_y = \sigma'_y(\mathbf{x}, t)$ and $\sigma'_z = \sigma'_z(\mathbf{x}, t)$ are the components of effective normal stress [M/Lt^2] and the shear stress components are defined previously in Section 2.3.1. It follows from (2.37) and (2.41)~(2.43) that

$$\sigma_x = \sigma'_x \mp p \ , \ \ \sigma_y = \sigma'_y \mp p \ \text{ and } \ \sigma_z = \sigma'_z \mp p \tag{2.44}$$

and shear stresses remain unchanged. In a cylindrical coordinate system r θ z, (2.44) is denoted as

$$\sigma_r = \sigma'_r \mp p \ , \ \ \sigma_\theta = \sigma'_\theta \mp p \ \text{ and } \ \sigma_z = \sigma'_z \mp p \tag{2.45}$$

where pressure p, total stresses and effective stresses are all functions of space $\mathbf{x} = [r\ \theta\ z]^T$ and time t.

2.3.3 Stress Invariants and Mean Stress

The stress state at a point $\boldsymbol{\sigma}$ as denoted by (2.37), for Cartesian coordinates, is based on a coordinate system with orientation of the coordinate axes arbitrarily selected at the point. All stress components thus vary and depend on orientation of the coordinate axes. However, no matter what the state of stress and orientation of the coordinate axes are, there are some stress identities that never change, notably, the first I_1, second I_2 and third I_3 stress invariants [M/Lt^2]:

$$\begin{cases} I_1 = \sigma_x + \sigma_y + \sigma_z \\ I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}\tau_{yx} - \tau_{yz}\tau_{zy} - \tau_{zx}\tau_{xz} \\ I_3 = \sigma_x\sigma_y\sigma_z + \tau_{xy}\tau_{yz}\tau_{zx} + \tau_{xz}\tau_{zy}\tau_{yx} - \sigma_x\tau_{yz}\tau_{zy} - \sigma_y\tau_{xz}\tau_{zx} - \sigma_z\tau_{xy}\tau_{yx} \end{cases} \tag{2.46}$$

(Timoshenko and Goodier, 1970, p. 224; Brady and Brown, 1985, pp. 24; Goodman, 1980, pp. 349) based on the principle of force conservation. The first stress invariant I_1 is sometimes referred to as the "trace" and the second and third invariants also appear as

$$I_2 = \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix} \tag{2.47}$$

and

$$I_3 = |\boldsymbol{\sigma}| = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix} \tag{2.48}$$

of which I_3 is identical to the third-order determinant of the second-rank stress tensor (2.37) and I_2 is the sum of the three second-order determinants of the submatrices associated with the respective diagonal entries of the stress tensor (2.37).

It follows from the definition of the first stress invariant I_1 of (2.46) that

$$\sigma_m = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) = \frac{1}{3} I_1 \tag{2.49}$$

where $\sigma_m = \sigma_m(\mathbf{x}, t)$ is referred to as the "mean total stress" [M/Lt^2] in this thesis and evidently is also an invariant independent of the orientation of coordinate axes. If a mean effective stress σ'_m [M/Lt^2] is also defined in terms of the effective stresses σ'_x, σ'_y and σ'_z of (2.43) as

$$\sigma'_m = \frac{1}{3}\left(\sigma'_x + \sigma'_y + \sigma'_z\right) , \tag{2.50}$$

then substituting (2.44) into (2.49) and rearranging terms leads to

$$p = \mp\left(\sigma_m - \sigma'_m\right) . \tag{2.51}$$

Since the mean total stress σ_m has been demonstrated to be a stress invariant in (2.49) and pore fluid pressure p is isotropic by nature, it can be concluded from (2.51) that the mean effective stress σ'_m defined by (2.50) must be also an invariant and does not depend on orientation of the coordinate axes chosen.

In cylindrical coordinates, mean total stress σ_m and mean effective stress σ'_m are denoted as

$$\sigma_m = \frac{1}{3}\left(\sigma_r + \sigma_\theta + \sigma_z\right) \tag{2.52}$$

and

$$\sigma'_m = \frac{1}{3}\left(\sigma'_r + \sigma'_\theta + \sigma'_z\right) \tag{2.53}$$

which are functions of $\mathbf{x} = [r \;\; \theta \;\; z]^T$ and time t.

2.4 Hydraulic Conductivity

Hydraulic conductivity is the ability of a porous medium to transmit fluids through the pore space under some hydraulic gradient which can be interpreted as the driving force per unit weight of fluid. Hydraulic conductivity is an intrinsic property dependent on both fluid and the medium. The range of hydraulic conductivity for water is around 10^{-12}~1 m/sec for unconsolidated earth materials and around 10^{-14}~10^{-12} m/sec for consolidated rocks (Freeze and Cherry, 1979, p. 29).

For a three-dimensional heterogeneous and anisotropic medium in general, hydraulic conductivity is represented by a second-rank tensor **K** [L/t] which is denote as, in Cartesian coordinates x y z,

$$\mathbf{K} = \begin{bmatrix} K_x & K_{yx} & K_{zx} \\ K_{xy} & K_y & K_{zy} \\ K_{xz} & K_{yz} & K_z \end{bmatrix} \tag{2.54}$$

and, in cylindrical coordinates r θ z,

$$\mathbf{K} = \begin{bmatrix} K_r & K_{\theta r} & K_{zr} \\ K_{r\theta} & K_\theta & K_{z\theta} \\ K_{rz} & K_{\theta z} & K_z \end{bmatrix} \tag{2.55}$$

where K_x, K_y and K_z are the components in x, y and z directions, respectively, K_r, K_θ and K_z are the respective components in r, θ and z directions and, likewise, off-diagonal components govern the fluid transmissibility in respective directions denoted by the subscripts.

Hydraulic conductivity is known to be a stress-dependent variable since compaction induced by changes of grain-to-grain effective stress often leads to the closure of available pore space for fluid flow which inevitably results in the reduction of hydraulic conductivity. In this thesis, hydraulic conductivity is considered a stress-dependent variable according to Helm (1976, pp. 378~379) and is assumed to be generally a continuous function of space $\mathbf{x}$ and time t, namely,

$$\mathbf{K} = \mathbf{K}(\mathbf{x}, t) \tag{2.56}$$

where $\mathbf{x} = [x \ \ y \ \ z]^T$ in Cartesian coordinates and $\mathbf{x} = [r \ \ \theta \ \ z]^T$ in cylindrical coordinates.

Chapter 3 The Poroviscosity Constitutive Law

The main purpose of this chapter is to introduce the subject of poroviscosity as a constitutive relation underlying the mathematical model proposed in this thesis. By first presenting some commonly known constitutive laws in Section 3.1, the one-dimensional poroviscosity as it was first published (Helm, 1998) is introduced in Section 3.2 followed by the proposed three-dimensional extension of the theory in Section 3.3. As noted in Section 1.4.2, the discussions in this chapter focus on an infinitesimal material element that is located at or near a stress boundary and is in direct contact with the applied stress.

3.1 Introduction to Constitutive Laws

Displacement and strain as discussed in Sections 2.1 and 2.2 are kinematic quantities and stress (Section 2.3) is a dynamic quantity. The mechanical state of a material is specified by means of the kinematic and dynamic quantities with certain fundamental laws referred to as "constitutive relations" or "constitutive laws".

One of the most commonly known constitutive laws is the linear elasticity or Hooke's law. Deformation, within a certain limit, of an elastic material is perceived as that of an elastic spring and is theoretically reversible in response to loading and unloading. For a one-dimensional elastic material, axial strain ε is linearly proportional to the axial stress σ applied as

$$\sigma = E\varepsilon \quad \text{or} \quad \varepsilon = \frac{\sigma}{E} \tag{3.1}$$

(Gere, 2004, p. 23) where the constant of proportionality E [M/Lt^2] is a material property known as the "modulus of elasticity" or "Young's modulus" and represents the stiffness of a material. For stiff materials, Young's modulus has relatively large values. Also, the value of elastic modulus usually varies depending on testing conditions and its typical range for rocks under compression is from a few 10^7 Pa to some 10^{11} Pa (Gere, 2004, p. 913; Hoek and Brown, 1980, p. 91). The reciprocal of Young's modulus, 1/E, is referred to as the "compliance".

Figure 3.1 displays the strain behavior of an ideal elastic material, previously unstrained, in response to a loading with constant stress. If a series of loading with varying stresses occurs, a stress-strain curve will theoretically follow a straight line (Figure 3.1(a)) and its slope defines the modulus of elasticity E. However, for a constant-stress loading, the resulting deformation is at a constant level of ε (Figure 3.1(b)) at all times.

Note that deformation is always assumed instantaneous in linear elasticity and, at time t = 0, the very beginning of stress loading, there are two strain values ε_p and ε_i if, as in this example, the part of material under consideration happens to be at or in the vicinity of the boundary point where stress is applied. To distinguish the conditions at

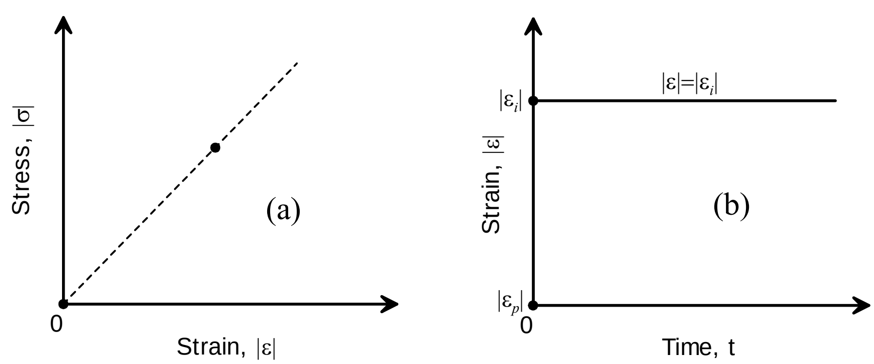

Figure 3.1 Strain in relation to (a) stress and (b) time for an ideal elastic material under a constant-stress loading since t = 0 assuming at preloading time $t = 0^-$, $\varepsilon = \varepsilon_p = 0$.

time t = 0, the notations "0^-" and "0^+" are used in this thesis to indicate the prestressed (or preloading) time ($t = 0^-$) and the initial time ($t = 0^+$) as mentioned in Section 1.4.2. Thus, by using the italic subscripts "*p*" and "*i*" to denote the strains under preloading and initial conditions, respectively, $\varepsilon = \varepsilon_p$ for $t = 0^-$ and $\varepsilon = \varepsilon_i$ for $t = 0^+$. In the example illustrated by Figure 3.1, it is assumed that $\varepsilon_p = 0$ for the prestressed time ($t = 0^-$).

The distinction between preloading and initial conditions is of critical importance in cases that an instantaneous mechanical response may occur at stress boundaries.

For a mechanically isotropic material according to linear elasticity, the three-dimensional relationship between stresses and strains is governed by

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E} \\ \varepsilon_y = -\nu\dfrac{\sigma_x}{E} + \dfrac{\sigma_y}{E} - \nu\dfrac{\sigma_z}{E} \\ \varepsilon_z = -\nu\dfrac{\sigma_x}{E} - \nu\dfrac{\sigma_y}{E} + \dfrac{\sigma_z}{E} \end{cases} \tag{3.2}$$

or the compliance form

$$\begin{cases} \sigma_x = \dfrac{1}{(1+\nu)(1-2\nu)}\left[(1-\nu)E\varepsilon_x + \nu E\varepsilon_y + \nu E\varepsilon_z\right] \\ \sigma_y = \dfrac{1}{(1+\nu)(1-2\nu)}\left[\nu E\varepsilon_x + (1-\nu)E\varepsilon_y + \nu E\varepsilon_z\right] \\ \sigma_z = \dfrac{1}{(1+\nu)(1-2\nu)}\left[\nu E\varepsilon_x + \nu E\varepsilon_y + (1-\nu)E\varepsilon_z\right] \end{cases} \tag{3.3}$$

where ν is the Poisson's ratio [dimensionless] and the shear components of stress and strain are related by a shear modulus G [M/Lt^2] as $\tau_{xy} = G\gamma_{xy}$, $\tau_{yz} = G\gamma_{yz}$ and $\tau_{zx} = G\gamma_{zx}$ (Gere, 2004, p. 507). Poisson's ratio is the ratio of the magnitude of a lateral strain to that of the corresponding axial strain and varies from about 0.15 to 0.30 for rock materials

(Gere, 2004, p. 913; Hoek and Brown, 1980, p. 91). Shear modulus is a measure of shearing resistance of materials and, for typical rocks, ranges between 10^9 Pa and 10^{10} Pa (Hoek and Brown, 1980, p. 91).

Another commonly known constitutive law applies to Newtonian fluids. For an incompressible Newtonian fluid confined in a damping dashpot, axial strain rate $\dot{\varepsilon}$ of the material is related to the applied stress σ by

$$\sigma = 3\eta\dot{\varepsilon} \qquad (\eta\text{: constant}) \tag{3.4}$$

in which η is the dynamic viscosity of the fluid [M/Lt] and $\dot{\varepsilon}$ $(= d\varepsilon/dt)$ is the time rate of change of the axial strain [1/t] (Obert and Duvall, 1967, pp. 164~166). The coefficient "3" in (3.4) arises due to the incompressibility assumption and the deformation in the axial direction is essentially the result of viscous resistance of the fluid confined in the dashpot. Under experimental conditions, viscosity of fluids is dependent on temperature and atmospheric pressure and its typical values for water and air are 1.1×10^{-3} Pa-sec and 1.8×10^{-5} Pa-sec, respectively, at 15 °C and 1 atm (Crowe *et al*., 2000, pp. A-14 & A-12). Also, the inferred viscosity of glacial ice is on the order of 10^{13} Pa-sec (Ranalli, 1995, p. 71). For a Newtonian fluid, viscosity η is considered a constant under a constant temperature and a constant atmospheric pressure. Otherwise, if a fluid exhibits a viscosity dependency on strain, strain rate and/or other factors, it is then considered a non-Newtonian fluid. The main topics presented following this introduction is the poroviscosity constitutive law (Helm, 1998) which can be applied to non-Newtonian fluids or solid materials with the stress-strain behavior resembling that of non-Newtonian fluids.

If Equation (3.4) is applied to a Newtonian fluid subjected to a constant stress σ

since t = 0, then holding the stress σ constant and completing the integral of (3.4) from t = 0 to an arbitrary later time t (≥ 0) leads to

$$\varepsilon = \frac{\sigma}{3\eta} t \tag{3.5}$$

assuming that ε = 0 at t = 0. Figure 3.2 shows that for an incompressible Newtonian fluid confined in a dashpot and initially unstrained, axial strain is a linear function of time according to Equation (3.5) and the slope of the strain-time curve is $\sigma/3\eta$. The less viscous the fluid is, the steeper the slope of the strain curve will be.

For an ideal Newtonian fluid, the mechanical energy applied is gradually dissipated within the material. The resulting deformation in itself is rather instantaneous and is continuous from the preloading state. Thus, the initial strain and the prestressed strain are the same in this example.

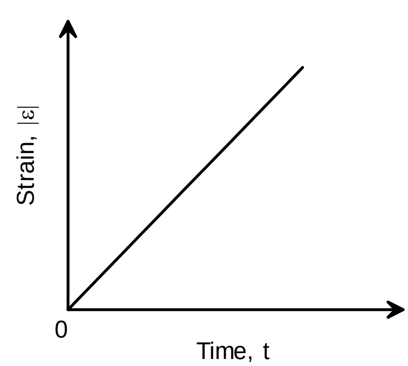

Figure 3.2 Axial deformation ε as a linear function of time t for an ideal Newtonian fluid confined in a dashpot under a constant axial stress σ since t = 0 for which ε = 0 is assumed.

3.2 One-Dimensional Poroviscosity

Helm (1998) introduced a new poroviscosity constitutive relation which governs the axial deformation of a laterally confined material according to

$$\begin{cases} \sigma = \eta\dot{\varepsilon} & (3.6) \\ \sigma = \mp A\dot{\eta} & (3.7) \end{cases}$$

where

σ ($\neq 0$) is the normal stress [M/Lt^2],

ε is the normal strain [dimensionless],

$\dot{\varepsilon} = \dfrac{d\varepsilon}{dt}$ ($\neq 0$) is the time rate of change of strain ε [1/t],

η (> 0) is the dynamic viscosity [M/Lt],

$\dot{\eta} = \dfrac{d\eta}{dt}$ ($\neq 0$) is the time rate of change of viscosity [M/Lt^2], and

A (> 0) is a poroviscous constitutive coefficient [dimensionless].

For the sake of argument, a zero-valued stress σ is not considered, for now. Viscosity η is a material property and, therefore, is limited to be positive-valued. The poroviscous coefficient A is considered to be a constant during any transient loading event and has a value somewhere in the range 10^{-4}~10^{-3} (Helm, 1998) for a clay material (Chicago clay) based on analysis on a set of consolidation test data (Taylor, 1948, pp. 248~249). Also, in order to mathematically suffice, η and ε must be continuous functions, respectively, of space and time t and their total time derivatives $\dot{\varepsilon}$ and $\dot{\eta}$ must exist.

By the definitions of (3.6) and (3.7) and recalling that $A > 0$ and $\eta > 0$, the following conditions are inferred for two different sign conventions (defined in Section 1.4.3). Under the engineering mechanics convention,

$$\begin{cases} \text{If } \sigma > 0 \text{ and } \varepsilon > 0 \text{ (tensile stress and strain), } \dot{\eta} < 0 \text{ and } \dot{\varepsilon} > 0. \\ \text{If } \sigma < 0 \text{ and } \varepsilon < 0 \text{ (compressive stress and strain), } \dot{\eta} > 0 \text{ and } \dot{\varepsilon} < 0. \end{cases} \tag{3.8e}$$

Under the geomechanics convention,

$$\begin{cases} \text{If } \sigma > 0 \text{ and } \varepsilon > 0 \text{ (compressive stress and strain), } \dot{\eta} > 0 \text{ and } \dot{\varepsilon} > 0. \\ \text{If } \sigma < 0 \text{ and } \varepsilon < 0 \text{ (tensile stress and strain), } \dot{\eta} < 0 \text{ and } \dot{\varepsilon} < 0. \end{cases} \tag{3.9g}$$

Also, the initial conditions pertaining to time $t = 0^+$ are assumed known to be

$$\begin{cases} \sigma = \sigma_i \ (\neq 0) \\ \varepsilon = \varepsilon_i \qquad\qquad \text{(for } t = 0^+) \\ \dot{\varepsilon}_i = \left.\dfrac{d\varepsilon}{dt}\right|_{t=0^+} \ (\neq 0) \end{cases} \tag{3.10}$$

and, from (3.6),

$$\eta_i = \frac{\sigma_i}{\dot{\varepsilon}_i} \ (> 0) \quad \text{(for } t = 0^+) . \tag{3.11}$$

The initial stress σ_i and the corresponding viscosity η_i, strain ε_i, strain rate $\dot{\varepsilon}_i$ as appear in (3.10) and (3.11) can be simply viewed as constants of integration in the following discussions.

The set of two simultaneous equations (3.6) and (3.7) with the initial conditions (3.10) and (3.11) has some alternative expressions. By first equating (3.7) to (3.6) and by subsequent operations of separation of variables and integrations, an expression of viscosity in terms of strain for any time $t \geq 0^+$ can be reaches as

$$\eta\frac{d\varepsilon}{dt} = \mp A\frac{d\eta}{dt} \quad \Rightarrow \quad \int_{\eta_i}^{\eta}\frac{1}{\eta}d\eta = \mp\frac{1}{A}\int_{\varepsilon_i}^{\varepsilon}d\varepsilon \quad \Rightarrow \quad \ln\eta\,\Big|_{\eta_i}^{\eta} = \mp\frac{1}{A}\varepsilon\,\Big|_{\varepsilon_i}^{\varepsilon} \quad (\eta > 0)$$

$$\Rightarrow \quad \ln\eta - \ln\eta_i = \ln\left(\frac{\eta}{\eta_i}\right) = \mp\frac{1}{A}(\varepsilon - \varepsilon_i) \quad \Rightarrow \quad \frac{\eta}{\eta_i} = e^{\mp\left(\frac{\varepsilon - \varepsilon_i}{A}\right)}$$

$$\Rightarrow \quad \eta = \eta_i \, e^{\mp\left(\frac{\varepsilon-\varepsilon_i}{A}\right)} \; (> 0) \tag{3.12}$$

where initial viscosity η_i and initial strain ε_i are assumed as constants of integration. Alternatively, with Equation (3.11), expression (3.12) becomes

$$\eta = \frac{\sigma_i}{\dot{\varepsilon}_i} e^{\mp\left(\frac{\varepsilon-\varepsilon_i}{A}\right)} \; . \tag{3.13}$$

Equations (3.12) and (3.13) indicate that, according to the current poroviscosity constitutive relation, viscosity η is allowed to vary and is dependent on strain ε.

To reach alternative expressions of Equations (3.6) and (3.7), the time rate of change of viscosity defined by (3.12) or (3.13) must be evaluated. By holding σ_i, $\dot{\varepsilon}_i$ and ε_i as constants, it follows from (3.13) that

$$\dot{\eta} = \frac{d\eta}{dt} = \frac{\sigma_i}{\dot{\varepsilon}_i} e^{\mp\left(\frac{\varepsilon-\varepsilon_i}{A}\right)} \left(\mp \frac{1}{A}\right) \frac{d\varepsilon}{dt} \quad \Rightarrow \quad \dot{\eta} = \mp \frac{1}{A} \frac{\sigma_i}{\dot{\varepsilon}_i} \dot{\varepsilon} \, e^{\mp\left(\frac{\varepsilon-\varepsilon_i}{A}\right)} \; . \tag{3.14}$$

Substitution of (3.14) into (3.7) and rearrangement of terms leads to the following three alternative expressions

$$\begin{cases} \sigma = \dfrac{\sigma_i}{\dot{\varepsilon}_i} \dot{\varepsilon} \, e^{\mp\left(\frac{\varepsilon-\varepsilon_i}{A}\right)} \\ \varepsilon = \varepsilon_i \mp A\left[\ln\left(\dfrac{\sigma}{\sigma_i}\right) - \ln\left(\dfrac{\dot{\varepsilon}}{\dot{\varepsilon}_i}\right)\right] \\ \dot{\varepsilon} = \dfrac{\dot{\varepsilon}_i}{\sigma_i} \sigma \, e^{\pm\left(\frac{\varepsilon-\varepsilon_i}{A}\right)} \end{cases} \tag{3.15}$$

or, expressed in terms of viscosity η defined by (3.12) with the initial state by (3.6),

$$\begin{cases} \sigma = \eta_i \dot{\varepsilon} \, e^{\mp\left(\frac{\varepsilon-\varepsilon_i}{A}\right)} \\ \varepsilon = \varepsilon_i \mp A\left(\ln \eta - \ln \eta_i\right) \; . \\ \dot{\varepsilon} = \dfrac{1}{\eta_i} \sigma \, e^{\pm\left(\frac{\varepsilon-\varepsilon_i}{A}\right)} \end{cases} \tag{3.16}$$

Expressions of (3.15) and (3.16) are therefore the alternative one-dimensional expressions for the poroviscosity constitutive relation posed by the two simultaneous equations (3.6) and (3.7).

Also, the two equations (3.6) and (3.7) together representing the constitutive relation can be combined to from one single compact equation as

$$\frac{\dot{\sigma}}{\sigma} = \frac{\ddot{\varepsilon}}{\dot{\varepsilon}} \mp \frac{\dot{\varepsilon}}{A} \tag{3.17}$$

(see Appendix A.1 for details on how the differential equation is reached). With the initial condition (3.10) and (3.11), expressions of (3.15) and (3.16) can also be reached from Equation (3.17). Equation (3.17) as written implies that $\dot{\sigma}$ and $\ddot{\varepsilon}$ must also exist.

According to (3.6) and (3.7) (or (3.17)) or the alternative forms of (3.15) and (3.16), there is only one constant coefficient, namely, A, that needs to be determined through laboratory testing. Therefore, for modeling one-dimensional material compression behavior, poroviscosity shares the same principal advantage for invoking linear elasticity as a constitutive relation. In comparison, for one-dimensional elasticity, Young's modulus E, in Equations (3.1), is the coefficient that must be determined.

The following two subsections present two special cases of the one-dimensional poroviscosity constitutive relation, namely, a special case with constant-stress loading (Section 3.2.1) and the other with loading maintained at a constant strain rate (Section 3.2.2).

3.2.1 A Special Case: Constant Stress

For the special case that the applied stress σ is hold constant since t = 0, stress is maintained at the initial level, that is, $\sigma = \sigma_i$ for $t \geq 0^+$. It then follows from (3.7) that

$$\sigma = \mp A \frac{d\eta}{dt} \quad \Rightarrow \quad \int_{\eta_i}^{\eta} d\eta = \mp \frac{\sigma}{A} \int_0^t dt \quad \Rightarrow \quad \eta \Big|_{\eta_i}^{\eta} = \mp \frac{\sigma}{A} t \Big|_0^t \quad \Rightarrow \quad \eta - \eta_i = \mp \frac{\sigma}{A}(t - 0)$$

$$\Rightarrow \quad \eta = \eta_i \mp \frac{\sigma}{A} t \ (> 0) \tag{3.18}$$

or, with (3.11) and recognizing that $\sigma_i = \sigma$,

$$\eta = \sigma\left(\frac{1}{\dot{\varepsilon}_i} \mp \frac{1}{A} t\right) \ (> 0) \ . \tag{3.19}$$

Time t is considered not only for $t \geq 0$ but also with certain upper limit (to be defined later) so that for tensile loading, $\eta \neq 0$ is satisfied as defined in (3.6) and (3.7) if theoretically material failure does not occur.

By taking Equation (3.19) into account and holding the applied stress σ constant, integration of (3.6) from the initial state ($t = 0^+$) to that at an arbitrary later time t ($\geq 0^+$) and rearrangement of terms leads to

$$\sigma = \eta \frac{d\varepsilon}{dt} = \sigma\left(\frac{1}{\dot{\varepsilon}_i} \mp \frac{1}{A} t\right)\frac{d\varepsilon}{dt} \quad \Rightarrow \quad \int_{\varepsilon_i}^{\varepsilon} d\varepsilon = \int_0^t \frac{1}{\frac{1}{\dot{\varepsilon}_i} \mp \frac{1}{A} t} dt = \mp A \int_{u(0)}^{u(t)} \frac{1}{u} du \,\Bigg|_{u = \frac{1}{\dot{\varepsilon}_i} \mp \frac{1}{A} t}$$

$$\Rightarrow \quad \varepsilon \Big|_{\varepsilon_i}^{\varepsilon} = \mp A \ln|u| \Big|_{u(0)}^{u(t)} \,\Bigg|_{u = \frac{1}{\dot{\varepsilon}_i} \mp \frac{1}{A} t} = \mp A \ln\left|\frac{1}{\dot{\varepsilon}_i} \mp \frac{1}{A} t\right| \Bigg|_0^t$$

$$\Rightarrow \quad \varepsilon - \varepsilon_i = \mp A\left(\ln\left|\frac{1}{\dot{\varepsilon}_i} \mp \frac{1}{A} t\right| - \ln\left|\frac{1}{\dot{\varepsilon}_i}\right|\right) = \mp A \ln\left(\frac{\frac{1}{\dot{\varepsilon}_i} \mp \frac{1}{A} t}{\frac{1}{\dot{\varepsilon}_i}}\right)$$

$$\Rightarrow \quad \varepsilon = \varepsilon_i \mp A \ln\left|1 \mp \frac{\dot{\varepsilon}_i}{A} t\right| \ . \tag{3.20}$$

In order to satisfy that the material property $\eta > 0$ as defined by (3.18) or (3.19), domain of time t and the term $1 \mp \frac{\dot{\varepsilon}_i}{A} t$ in Equation (3.20) must satisfy the limits:

$$\begin{cases} \text{If } \sigma \text{ is a constant compressive stress, } t \geq 0 \text{ and } 1 \mp \frac{\dot{\varepsilon}_i}{A} t \geq 1 . \\ \text{If } \sigma \text{ is a constant tensile stress, } 0 \leq t < \pm \frac{A}{\dot{\varepsilon}_i} \text{ and } 0 < 1 \mp \frac{\dot{\varepsilon}_i}{A} t \leq 1 . \end{cases} \tag{3.21}$$

(See Section A.2, Appendix, for detailed discussions.) In both cases, the condition $1 \mp \frac{\dot{\varepsilon}_i}{A} t > 0$ is satisfied and Equation (3.20) with the absolute sign eliminated can be simply written as

$$\varepsilon = \varepsilon_i \mp A \ln\left(1 \mp \frac{\dot{\varepsilon}_i}{A} t\right) . \tag{3.22}$$

Hence, strain rate $\dot{\varepsilon}$ can be evaluated as

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \mp A \frac{1}{1 \mp \frac{\dot{\varepsilon}_i}{A} t}\left(\mp \frac{\dot{\varepsilon}_i}{A}\right) = \frac{\dot{\varepsilon}_i}{1 \mp \frac{\dot{\varepsilon}_i}{A} t}$$

$$\Rightarrow \quad \dot{\varepsilon} = \frac{A}{\frac{A}{\dot{\varepsilon}_i} \mp t} . \tag{3.23}$$

Under a constant stress σ, viscosity varies in time t according to (3.19) or, by further taking the initial condition (3.11) into account and recognizing that $\sigma_i = \sigma$,

$$\eta = \eta_i \left(1 \mp \frac{\dot{\varepsilon}_i}{A} t\right) \; (> 0) . \tag{3.24}$$

Equations (3.22) and (3.24) suggests that under a constant-stress loading, strain and viscosity are logarithmic and linear functions of time t, respectively. Also, the time rate of change of strain is inversely proportional to time according to Equation (3.23)

Degradation of Compressional Strain Rate

If t_c is defined as the time when $\dot{\varepsilon} = \dot{\varepsilon}_i/2$ in a loading with constant compressive stress, it then follows from (3.23) that

$$\frac{1}{2}\dot{\varepsilon}_i = \frac{A}{\dfrac{A}{\dot{\varepsilon}_i} \mp t_c} \quad \Rightarrow \quad \frac{1}{2}A \mp \frac{1}{2}\dot{\varepsilon}_i t_c = A$$

$$\Rightarrow \quad t_c = \mp\frac{A}{\dot{\varepsilon}_i} \qquad (\text{for } \dot{\varepsilon} = \dot{\varepsilon}_i/2) \tag{3.25}$$

where $\mp\dot{\varepsilon}_i > 0$ for compression according to (3.8e) and (3.9g). Also, with $t = t_c$ and the definition of (3.25), Equation (3.22) becomes

$$\varepsilon = \varepsilon_i \mp A\ln\left[1 \mp \frac{\dot{\varepsilon}_i}{A}\left(\mp\frac{A}{\dot{\varepsilon}_i}\right)\right]$$

$$\Rightarrow \quad \varepsilon = \varepsilon_i \mp A\ln 2 \cong \varepsilon_i \mp 0.6931A \qquad (\text{for } t = t_c)\,. \tag{3.26}$$

At time $t = t_c$, it follows from Equations (3.24) and (3.25) that

$$\eta = \eta_i\left[1 \mp \frac{\dot{\varepsilon}_i}{A}\left(\mp\frac{A}{\dot{\varepsilon}_i}\right)\right] \quad \Rightarrow \quad \eta = 2\eta_i \qquad (\text{for } t = t_c) \tag{3.27}$$

which indicates that under a constant compressive stress, viscosity is doubled exactly from the initial viscosity when the strain rate is decreased by half.

Equations (3.22)~(3.24) can be alternatively expressed in terms of t_c as

$$\begin{cases} \dfrac{\varepsilon}{\varepsilon_i} = 1 \mp \dfrac{A}{\varepsilon_i}\ln\left(1 + \dfrac{t}{t_c}\right) & (3.28) \\ \dfrac{\dot{\varepsilon}}{\dot{\varepsilon}_i} = \dfrac{1}{1 + \dfrac{t}{t_c}} & (3.29) \end{cases}$$

and

$$\frac{\eta}{\eta_i} = 1 + \frac{t}{t_c} \ . \tag{3.30}$$

Figure 3.3 displays the dimensionless plots of strain $\varepsilon/\varepsilon_i$, strain rate $\dot{\varepsilon}/\dot{\varepsilon}_i$ and viscosity η/η_i versus dimensionless time t/t_c according to Equations (3.28)~(3.30).

Growth of Tensional Strain Rate

Similarly, if t_t is defined as the time when $\dot{\varepsilon} = 2\dot{\varepsilon}_i$ in a loading with constant tensile stress, if follows from (3.23) that

$$2\dot{\varepsilon}_i = \frac{A}{\dfrac{A}{\dot{\varepsilon}_i} \mp t_t} \quad \Rightarrow \quad 2A \mp 2\dot{\varepsilon}_i t_t = A$$

$$\Rightarrow \quad t_t = \pm\frac{1}{2}\frac{A}{\dot{\varepsilon}_i} \quad (\text{for } \dot{\varepsilon} = 2\dot{\varepsilon}_i) \tag{3.31}$$

where $\pm\dot{\varepsilon}_i > 0$ for tension according to (3.8e) and (3.9g). From (3.22) with t_t defined by (3.31),

$$\varepsilon = \varepsilon_i \mp A \ln\left[1 \mp \frac{\dot{\varepsilon}_i}{A}\left(\pm\frac{1}{2}\frac{A}{\dot{\varepsilon}_i}\right)\right]$$

$$\Rightarrow \quad \varepsilon = \varepsilon_i \pm A\ln 2 \cong \varepsilon_i \pm 0.6931A \quad (\text{for } t = t_t)\ . \tag{3.32}$$

From (3.24) and (3.31),

$$\eta = \eta_i\left[1 \mp \frac{\dot{\varepsilon}_i}{A}\left(\pm\frac{1}{2}\frac{A}{\dot{\varepsilon}_i}\right)\right] \quad \Rightarrow \quad \eta = \frac{1}{2}\eta_i \quad (\text{for } t = t_t) \tag{3.33}$$

which indicates that under a constant tensile stress, viscosity is reduced by half from the initial state when strain rate is doubled from the initial value.

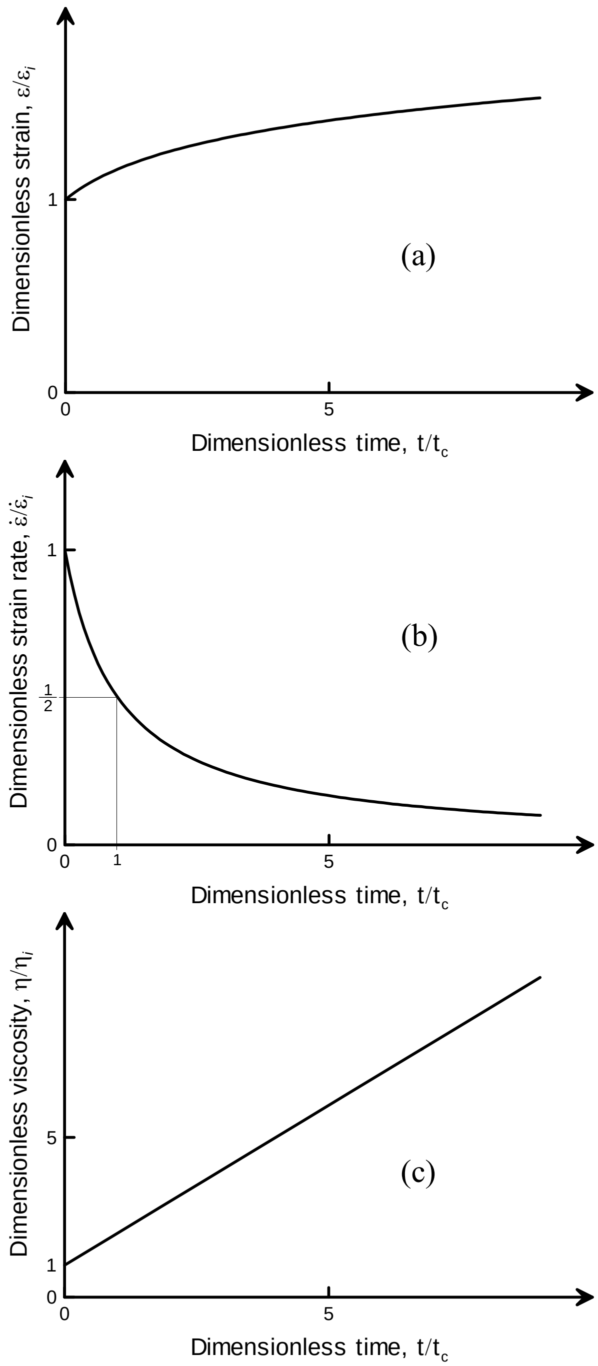

Figure 3.3 Normalized (a) strain $\varepsilon/\varepsilon_i$ (b) strain rate $\dot{\varepsilon}/\dot{\varepsilon}_i$, and (c) viscosity η/η_i of a one-dimensional poroviscous element previously unstrained and subjected to a constant compressive stress σ since t = 0. ($t_c = \mp A/\dot{\varepsilon}_i$)

Equations (3.22)~(3.24) thus can be alternatively expressed in terms of t_t as

$$\begin{cases} \dfrac{\varepsilon}{\varepsilon_i} = 1 \mp \dfrac{A}{\varepsilon_i} \ln\left(1 - \dfrac{1}{2}\dfrac{t}{t_t}\right) & (3.34) \\ \dfrac{\dot{\varepsilon}}{\dot{\varepsilon}_i} = \dfrac{1}{1 - \dfrac{1}{2}\dfrac{t}{t_t}} & (3.35) \end{cases}$$

and

$$\frac{\eta}{\eta_i} = 1 - \frac{1}{2}\frac{t}{t_t} \quad . \tag{3.36}$$

Figure 3.4 displays the dimensionless plots of strain $\varepsilon/\varepsilon_i$, strain rate $\dot{\varepsilon}/\dot{\varepsilon}_i$ and viscosity η/η_i versus dimensionless time t/t_t according to Equations (3.34)~(3.36).

3.2.2 A Special Case: Constant Strain Rate

Another case is a loading condition under which strain rate is maintained at a constant level at all times. If strain rate $\dot{\varepsilon}$ is hold constant for t ≥ 0, that is, $\dot{\varepsilon}_i = \dot{\varepsilon}$, then from definition of total time derivative of strain,

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} \quad \Rightarrow \quad \int_{\varepsilon_i}^{\varepsilon} d\varepsilon = \dot{\varepsilon}\int_0^t dt \quad \Rightarrow \quad \varepsilon \Big|_{\varepsilon_i}^{\varepsilon} = \dot{\varepsilon}\, t \Big|_0^t \quad \Rightarrow \quad \varepsilon - \varepsilon_i = \dot{\varepsilon}(t-0)$$

$$\Rightarrow \quad \varepsilon = \varepsilon_i + \dot{\varepsilon}\, t \tag{3.37}$$

By holding strain rate $\dot{\varepsilon}$ constant, equate (3.6) to (3.7), rearrange terms and integrate to solve for η as

$$\eta\dot{\varepsilon} = \mp A\frac{d\eta}{dt} \quad \Rightarrow \quad \int_{\eta_i}^{\eta} \frac{1}{\eta} d\eta = \mp\frac{\dot{\varepsilon}}{A}\int_0^t dt \quad \Rightarrow \quad \ln\eta \Big|_{\eta_i}^{\eta} = \mp\frac{\dot{\varepsilon}}{A} t \Big|_0^t \quad (\eta > 0)$$

$$\Rightarrow \quad \ln\eta - \ln\eta_i = \mp\frac{\dot{\varepsilon}}{A}(t-0) \quad \Rightarrow \quad \ln\left(\frac{\eta}{\eta_i}\right) = \mp\frac{\dot{\varepsilon}}{A} t \Rightarrow \quad \frac{\eta}{\eta_i} = e^{\mp\left(\frac{\dot{\varepsilon}}{A} t\right)}$$

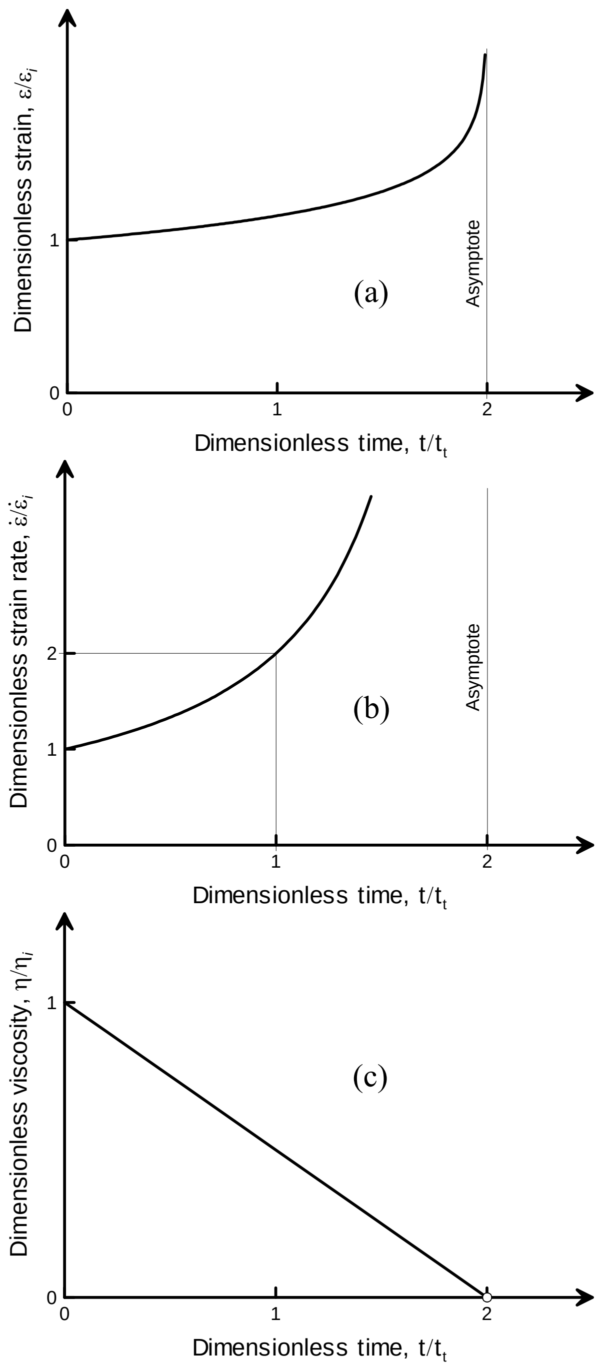

Figure 3.4 Normalized (a) strain $\varepsilon/\varepsilon_i$, (b) strain rate $\dot{\varepsilon}/\dot{\varepsilon}_i$ and (c) viscosity η/η_i of a one-dimensional poroviscous element previously unstrained and subjected to a constant tensile stress σ since $t = 0$. ($t_t = \pm A/2\dot{\varepsilon}_i$)

$$\Rightarrow \quad \eta = \eta_i \, e^{\mp\left(\frac{\dot{\varepsilon}}{A}t\right)} \; (> 0) \tag{3.38}$$

or, with the initial condition (3.11) and recognizing that $\dot{\varepsilon}_i = \dot{\varepsilon}$,

$$\eta = \frac{\sigma_i}{\dot{\varepsilon}} e^{\mp\left(\frac{\dot{\varepsilon}}{A}t\right)} \; (> 0) \; . \tag{3.39}$$

Substitution of (3.39) into (3.6) leads to

$$\sigma = \left[\frac{\sigma_i}{\dot{\varepsilon}} e^{\mp\left(\frac{\dot{\varepsilon}}{A}t\right)} \right] \dot{\varepsilon} \quad \Rightarrow \quad \sigma = \sigma_i \, e^{\mp\left(\frac{\dot{\varepsilon}}{A}t\right)} \; . \tag{3.40}$$

Equation (3.40) indicates that in order to maintain a constant strain rate, the applied loading stress must be regulated in the lab so that it changes exponentially in time. As a result, variation of viscosity also follows an exponential relation with respect to time according to (3.38) or (3.39).

Growth of Compressional Stress

If t'_c is defined as the time when $\sigma = 2\sigma_i$ in a compressive stress loading with constant strain rate, it follows from (3.40) that

$$2\sigma_i = \sigma_i \, e^{\mp\left(\frac{\dot{\varepsilon}}{A}t'_c\right)} \quad \Rightarrow \quad \mp \frac{\dot{\varepsilon}}{A} t'_c = \ln 2$$

$$\Rightarrow \quad t'_c = \mp \frac{A}{\dot{\varepsilon}} \ln 2 \cong \mp 0.6931 \frac{A}{\dot{\varepsilon}} \qquad (\text{for } \sigma = 2\sigma_i) \tag{3.41}$$

where $\mp\dot{\varepsilon} > 0$ for compression according to (3.8e) and (3.9g). From (3.37) with $t = t'_c$ and by taking the definition of t'_c (3.41) into account,

$$\varepsilon = \varepsilon_i + \dot{\varepsilon}\left(\mp \frac{A}{\dot{\varepsilon}} \ln 2 \right)$$

$$\Rightarrow \quad \varepsilon = \varepsilon_i \mp A \ln 2 \cong \varepsilon_i \mp 0.6931 A \qquad (\text{for } t = t'_c) \; . \tag{3.42}$$

At time $t = t'_c$, it follows from (3.38) and (3.41) that

$$\eta = \eta_i \, e^{\mp\left[\frac{\dot{\varepsilon}}{A}\left(\mp\frac{A}{\dot{\varepsilon}}\ln 2\right)\right]} = \eta_i \, e^{\ln 2} \quad \Rightarrow \quad \eta = 2\eta_i \quad (\text{for } t = t'_c) \,. \tag{3.43}$$

Viscosity is doubled when compressive stress reaches twice the initial loading stress. Equations (3.37), (3.38) and (3.40) can be alternatively expressed in terms of t'_c as

$$\begin{cases} \dfrac{\sigma}{\sigma_i} = e^{\frac{t}{t'_c}\ln 2} & (3.44) \\ & (\text{for } t \ge 0) \\ \dfrac{\varepsilon}{\varepsilon_i} = 1 \mp \dfrac{A}{\varepsilon_i}(\ln 2)\dfrac{t}{t'_c} & (3.45) \end{cases}$$

and

$$\frac{\eta}{\eta_i} = e^{\frac{t}{t'_c}\ln 2} \,. \tag{3.46}$$

Figure 3.5 displays the dimensionless plots of stress σ/σ_i, strain $\varepsilon/\varepsilon_i$ and viscosity η/η_i versus dimensionless time t/t'_c according to Equations (3.44)~(3.46).

<u>Degradation of Tensional Stress</u>

Similarly, if t'_t is defined as the time when $\sigma = \sigma_i/2$ in a tensile loading with constant strain rate, from (3.40),

$$\frac{1}{2}\sigma_i = \sigma_i \, e^{\mp\left(\frac{\dot{\varepsilon}}{A}t'_t\right)} \quad \Rightarrow \quad \mp\frac{\dot{\varepsilon}}{A}t'_t = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$\Rightarrow \quad t'_t = \pm\frac{A}{\dot{\varepsilon}}\ln 2 \cong \pm 0.6931\frac{A}{\dot{\varepsilon}} \qquad (\text{for } \sigma = \sigma_i/2) \tag{3.47}$$

where $\pm\dot{\varepsilon} > 0$ for tension according to (3.8e) and (3.9g). From (3.37) with $t = t'_t$, (3.47),

$$\varepsilon = \varepsilon_i + \dot{\varepsilon}\left(\pm\frac{A}{\dot{\varepsilon}}\ln 2\right)$$

$$\Rightarrow \quad \varepsilon = \varepsilon_i \pm A\ln 2 \cong \varepsilon_i \pm 0.6931A \qquad (\text{for } t = t'_t) \,. \tag{3.48}$$

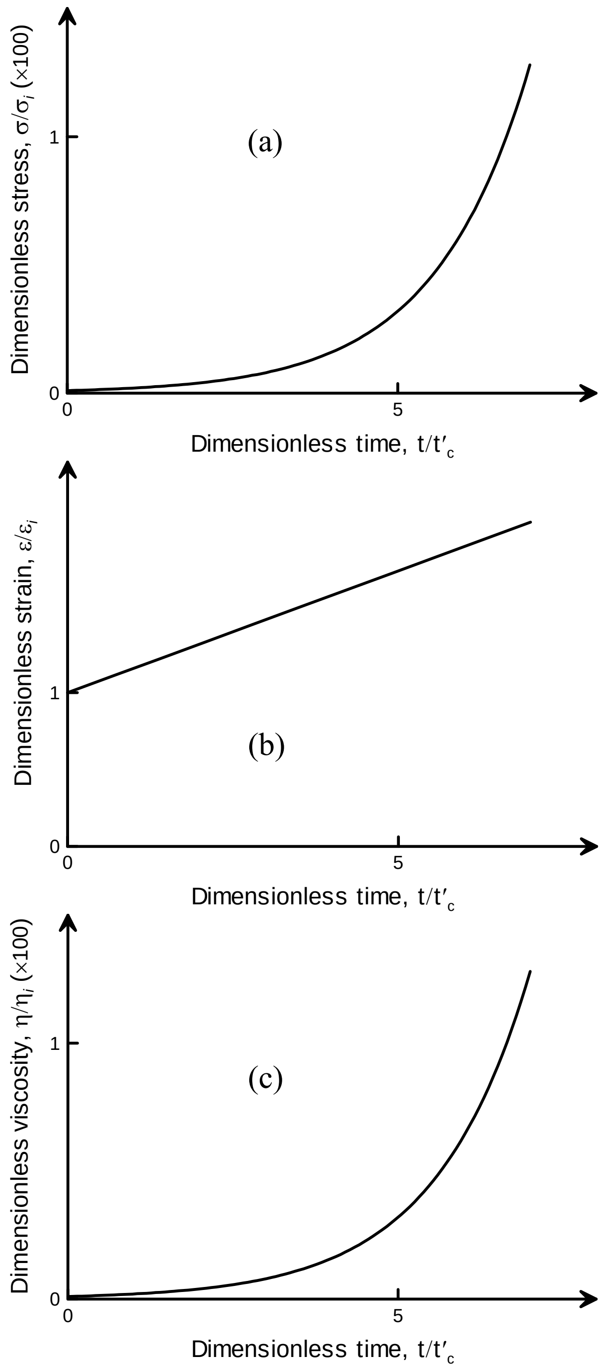

Figure 3.5 Normalized (a) stress σ/σ_i, (b) strain $\varepsilon/\varepsilon_i$ and (c) viscosity η/η_i of a one-dimensional poroviscous element previously unstrained and under compression with a constant strain rate $\dot{\varepsilon}$ since $t = 0$. ($t'_c = \mp(A/\dot{\varepsilon})\ln 2$)

And, from (3.38),

$$\eta = \eta_i\, e^{\mp\left[\frac{\dot{\varepsilon}}{A}\left(\pm\frac{A}{\dot{\varepsilon}}\ln 2\right)\right]} = \eta_i\, e^{\ln\left(\frac{1}{2}\right)} \quad \Rightarrow \quad \eta = \frac{1}{2}\eta_i \quad (\text{for } t = t'_t) \tag{3.49}$$

which indicates that under a tensile loading with constant strain rate, viscosity is reduced by half from the initial value when tensile stress reaches exactly half of its initial value.

Therefore, Equations (3.37), (3.38) and (3.40) can be expressed in terms of t'_t as

$$\begin{cases} \dfrac{\sigma}{\sigma_i} = e^{-\left(\frac{t}{t'_t}\ln 2\right)} & (3.50) \\ \dfrac{\varepsilon}{\varepsilon_i} = 1 \pm \dfrac{A}{\varepsilon_i}(\ln 2)\dfrac{t}{t'_t} & (3.51) \end{cases}$$

and

$$\frac{\eta}{\eta_i} = e^{-\left(\frac{t}{t'_t}\ln 2\right)} . \tag{3.52}$$

Figure 3.6 displays the dimensionless plots of stress σ/σ_i, strain $\varepsilon/\varepsilon_i$ and viscosity η/η_i versus dimensionless time t/t'_t according to (3.50)~(3.52).

The stress σ in the discussions of this section, up to this point, has been limited to $\sigma \neq 0$. However, in a fairly trivial case where $\sigma = 0$, from Equations (3.6) and (3.7) for $A \neq 0$ and $\eta \neq 0$, it can be concluded that $\dot{\varepsilon} = d\varepsilon/dt = 0$ and $\dot{\eta} = d\eta/dt = 0$. In other words, when there is no stress involved, strain ε and viscosity η do not change with time and remain at a constant level pertaining to $t = 0$.

Newton's first law of motion (or law of inertia) mandates that if not exerted by any force, an object initially at rest will stay at rest and an object initially in motion will stay in motion with the same velocity. Stress itself is a form of force (per unit area) and strain rate can be regarded as the spatial gradient of velocity. Therefore, for the trivial case that $\sigma = 0$, strain and viscosity remain at a constant level pertaining to $t = 0$.

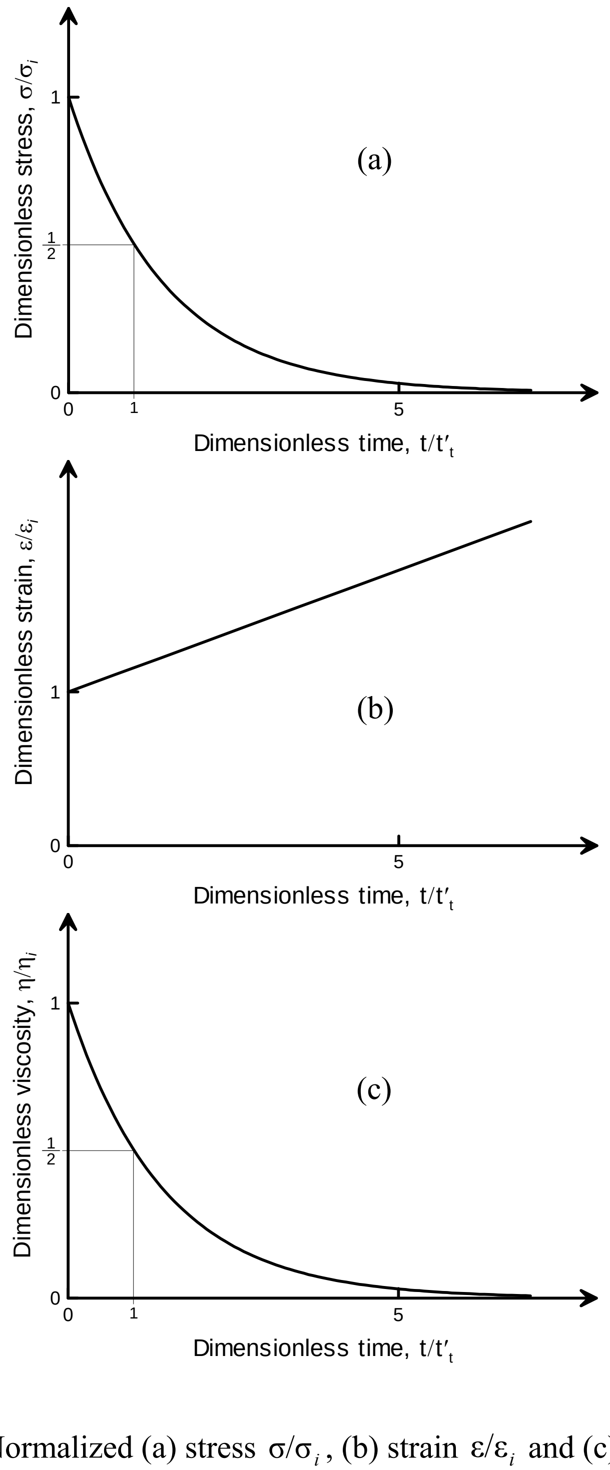

Figure 3.6 Normalized (a) stress σ/σ_i, (b) strain $\varepsilon/\varepsilon_i$ and (c) viscosity η/η_i of a one-dimensional poroviscous element previously unstrained and under tension with a constant strain rate $\dot{\varepsilon}$ since $t = 0$. ($t'_t = \pm(A/\dot{\varepsilon})\ln 2$)

3.3 Three-Dimensional Extension of Poroviscosity

For a three-dimensional anisotropic orthotropic poroviscous material in a Cartesian coordinate system x y z, define the relationship between normal stresses and normal strain rates as

$$\begin{cases} \sigma_x = \dfrac{1}{B_{xx}}\eta_x\dot{\varepsilon}_x + \dfrac{1}{B_{yx}}\eta_y\dot{\varepsilon}_y + \dfrac{1}{B_{zx}}\eta_z\dot{\varepsilon}_z \\ \sigma_y = \dfrac{1}{B_{xy}}\eta_x\dot{\varepsilon}_x + \dfrac{1}{B_{yy}}\eta_y\dot{\varepsilon}_y + \dfrac{1}{B_{zy}}\eta_z\dot{\varepsilon}_z \\ \sigma_z = \dfrac{1}{B_{xz}}\eta_x\dot{\varepsilon}_x + \dfrac{1}{B_{yz}}\eta_y\dot{\varepsilon}_y + \dfrac{1}{B_{zz}}\eta_z\dot{\varepsilon}_z \end{cases} \tag{3.53}$$

where σ_x, σ_y and σ_z are the normal stresses [M/Lt^2], $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$ and $\dot{\varepsilon}_z$ are the normal strain rates [1/t] and η_x, η_y and η_z are the dynamic viscosities [M/Lt]. A variable is associated with the axis direction denoted by the subscript "x", "y" or "z". The coefficients B_{jk} (j,k = x,y,z) are assumed some constants [dimensionless] to be defined later. Both stresses and strain rates are considered continuous functions of space **x** and time t. The term "orthotropic" (or "rhombic") is borrowed from linear elasticity to refer to otherwise an anisotropic elastic material of which the resulting normal strains are independent of the applied shear stresses and shear strains are independent of the applied normal stresses (Ting, 1996, p. 45; Rand and Rovenski, 2005, pp. 56~58).

The three simultaneous equations of (3.53) can be inverted to appear as

$$\begin{cases} \dot{\varepsilon}_x = b_{xx}\dfrac{\sigma_x}{\eta_x} + b_{xy}\dfrac{\sigma_y}{\eta_x} + b_{xz}\dfrac{\sigma_z}{\eta_x} \\ \dot{\varepsilon}_y = b_{yx}\dfrac{\sigma_x}{\eta_y} + b_{yy}\dfrac{\sigma_y}{\eta_y} + b_{yz}\dfrac{\sigma_z}{\eta_y} \\ \dot{\varepsilon}_z = b_{zx}\dfrac{\sigma_x}{\eta_z} + b_{zy}\dfrac{\sigma_y}{\eta_z} + b_{zz}\dfrac{\sigma_z}{\eta_z} \end{cases} \tag{3.54}$$

with constant coefficients b_{jk} (j,k = x,y,z) [dimensionless] having some relationships with B_{jk} (j,k = x,y,z) according to

$$\begin{cases} \dfrac{1}{B_{xx}} = \dfrac{1}{J_b}\begin{vmatrix} b_{yy} & b_{yz} \\ b_{zy} & b_{zz} \end{vmatrix}, & \dfrac{1}{B_{yx}} = -\dfrac{1}{J_b}\begin{vmatrix} b_{xy} & b_{xz} \\ b_{zy} & b_{zz} \end{vmatrix}, & \dfrac{1}{B_{zx}} = \dfrac{1}{J_b}\begin{vmatrix} b_{xy} & b_{xz} \\ b_{yy} & b_{yz} \end{vmatrix} \\ \dfrac{1}{B_{xy}} = -\dfrac{1}{J_b}\begin{vmatrix} b_{yx} & b_{yz} \\ b_{zx} & b_{zz} \end{vmatrix}, & \dfrac{1}{B_{yy}} = \dfrac{1}{J_b}\begin{vmatrix} b_{xx} & b_{xz} \\ b_{zx} & b_{zz} \end{vmatrix}, & \dfrac{1}{B_{zy}} = -\dfrac{1}{J_b}\begin{vmatrix} b_{xx} & b_{xz} \\ b_{yx} & b_{yz} \end{vmatrix} \\ \dfrac{1}{B_{xz}} = \dfrac{1}{J_b}\begin{vmatrix} b_{yx} & b_{yy} \\ b_{zx} & b_{zy} \end{vmatrix}, & \dfrac{1}{B_{yz}} = -\dfrac{1}{J_b}\begin{vmatrix} b_{xx} & b_{xy} \\ b_{zx} & b_{zy} \end{vmatrix}, & \dfrac{1}{B_{zz}} = \dfrac{1}{J_b}\begin{vmatrix} b_{xx} & b_{xy} \\ b_{yx} & b_{yy} \end{vmatrix} \end{cases} \tag{3.55}$$

where

$$J_b = \begin{vmatrix} b_{xx} & b_{xy} & b_{xz} \\ b_{yx} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{vmatrix} \ (\neq 0)\ . \tag{3.56}$$

Due to the physical meanings of the coefficients b_{jk} (j,k = x,y,z) that are to be explained later, a symmetry property exists and can be stated by the three mathematical identities

$$\frac{b_{xy}}{\eta_x} = \frac{b_{yx}}{\eta_y},\ \frac{b_{xz}}{\eta_x} = \frac{b_{zx}}{\eta_z}\ \text{ and }\ \frac{b_{yz}}{\eta_y} = \frac{b_{zy}}{\eta_z}\ . \tag{3.57}$$

from which, along with (3.55) and (3.56), it is verifiable that

$$\frac{\eta_y}{B_{yx}} = \frac{\eta_x}{B_{xy}},\ \frac{\eta_z}{B_{zx}} = \frac{\eta_x}{B_{xz}}\ \text{ and }\ \frac{\eta_z}{B_{zy}} = \frac{\eta_y}{B_{yz}}\ . \tag{3.58}$$

By the symmetry identities of (3.57), equations of (3.54) can be rewritten as

$$\begin{cases} \dot{\varepsilon}_x = b_{xx}\dfrac{\sigma_x}{\eta_x} + b_{yx}\dfrac{\sigma_y}{\eta_y} + b_{zx}\dfrac{\sigma_z}{\eta_z} \\ \dot{\varepsilon}_y = b_{xy}\dfrac{\sigma_x}{\eta_x} + b_{yy}\dfrac{\sigma_y}{\eta_y} + b_{zy}\dfrac{\sigma_z}{\eta_z} \\ \dot{\varepsilon}_z = b_{xz}\dfrac{\sigma_x}{\eta_x} + b_{yz}\dfrac{\sigma_y}{\eta_y} + b_{zz}\dfrac{\sigma_z}{\eta_z} \end{cases}\ . \tag{3.59}$$

Consider a uniaxial unconfined testing in the lab where a sample of poroviscous material is subjected to only an axial stress σ_x (and, therefore, lateral stresses $\sigma_y = \sigma_z = 0$). It then follows from (3.59) that for the normal strain rate in y direction, $\dot{\varepsilon}_y = b_{xy}(\sigma_x/\eta_x)$ or

$$b_{xy} = \frac{\dot{\varepsilon}_y}{\sigma_x/\eta_x} \quad (\text{for } \sigma_y = \sigma_z = 0) . \tag{3.60}$$

The ratio σ_x/η_x in the above expression can be considered, in this example, as the axial strain rate if compared with the one-dimensional relation between stress and strain rate according to (3.6). Therefore, the assumed constant coefficient b_{xy} can be interpreted as the ratio of the lateral strain rate in y direction to the axial strain rate in x direction. This physical interpretation thus should apply to b_{jk} (j,k = x,y,z) in general which can be determined only through laboratory testing. Note that, intuitively, $b_{jk} = 1$ for j = k or

$$b_{xx} = b_{yy} = b_{zz} = 1 \ . \tag{3.61}$$

The three viscosity components η_x, η_y and η_z in (3.53)~(3.54) and (3.59) are defined in terms of normal strains in the respective directions as

$$\begin{cases} \eta_x = \eta_{xi}\, e^{\mp\left(\frac{\varepsilon_x - \varepsilon_{xi}}{A_x}\right)} \\ \eta_y = \eta_{yi}\, e^{\mp\left(\frac{\varepsilon_y - \varepsilon_{yi}}{A_y}\right)} \\ \eta_z = \eta_{zi}\, e^{\mp\left(\frac{\varepsilon_z - \varepsilon_{zi}}{A_z}\right)} \end{cases} \tag{3.62}$$

where A_x, A_y and A_z are the poroviscous constitutive coefficients [dimensionless], η_{xi}, η_{yi} and η_{zi} are the initial viscosities [M/Lt] and ε_{xi}, ε_{yi} and ε_{zi} are the initial strains [dimensionless].

The initial states of all variables, pertaining to time $t = 0^+$,

$$\begin{cases} \eta_x = \eta_{xi}\ ,\ \ \eta_y = \eta_{yi}\ ,\ \ \eta_z = \eta_{zi} \\ \varepsilon_x = \varepsilon_{xi}\ ,\ \ \varepsilon_y = \varepsilon_{yi}\ ,\ \ \varepsilon_z = \varepsilon_{zi} \\ \dot{\varepsilon}_x = \dot{\varepsilon}_{xi}\ ,\ \ \dot{\varepsilon}_y = \dot{\varepsilon}_{yi}\ ,\ \ \dot{\varepsilon}_z = \dot{\varepsilon}_{zi} \end{cases} \qquad (\text{for } t = 0^+) \tag{3.63}$$

and

$$\sigma_x = \sigma_{xi}\ ,\ \ \sigma_y = \sigma_{yi} \ \ \text{and} \ \ \sigma_z = \sigma_{zi} \qquad (\text{for } t = 0^+) \tag{3.64}$$

are considered to be constants of integrations and are assumed functions of space **x**.

For convenience, the following are defined, from (3.54), as the "equivalent stresses"

$$\begin{cases} \sigma_{xe} = \eta_x \dot{\varepsilon}_x = b_{xx}\sigma_x + b_{xy}\sigma_y + b_{xz}\sigma_z \\ \sigma_{ye} = \eta_y \dot{\varepsilon}_y = b_{yx}\sigma_x + b_{yy}\sigma_y + b_{yz}\sigma_z \\ \sigma_{ze} = \eta_z \dot{\varepsilon}_z = b_{zx}\sigma_x + b_{zy}\sigma_y + b_{zz}\sigma_z \end{cases} \tag{3.65}$$

and the initial "equivalent stresses"

$$\begin{cases} \sigma_{xei} = \eta_{xi}\,\dot{\varepsilon}_{xi} = b_{xx}\sigma_{xi} + b_{xy}\sigma_{yi} + b_{xz}\sigma_{zi} \\ \sigma_{yei} = \eta_{yi}\,\dot{\varepsilon}_{yi} = b_{yx}\sigma_{xi} + b_{yy}\sigma_{yi} + b_{yz}\sigma_{zi} \\ \sigma_{zei} = \eta_{zi}\,\dot{\varepsilon}_{zi} = b_{zx}\sigma_{xi} + b_{zy}\sigma_{yi} + b_{zz}\sigma_{zi} \end{cases} \qquad (\text{for } t = 0^+) \tag{3.66}$$

so that the initial viscosities can be written, in short, as

$$\eta_{xi} = \frac{\sigma_{xei}}{\dot{\varepsilon}_{xi}}\ ,\ \ \eta_{yi} = \frac{\sigma_{yei}}{\dot{\varepsilon}_{yi}} \ \ \text{and} \ \ \eta_{zi} = \frac{\sigma_{zei}}{\dot{\varepsilon}_{zi}} \qquad (\text{for } t = 0^+)\ . \tag{3.67}$$

Isotropic Poroviscosity

The above defines the poroviscosity constitutive relation for an anisotropic poro-viscous material. However, for a mechanically isotropic material, viscosity is equal in all three directions and so are poroviscous constants. Also, the number of independent constant coefficients in (3.53) reduces to only two and (3.54) and (3.59) have only two independent coefficients as well. Thus, let

$$\begin{cases} \eta = \eta_x = \eta_y = \eta_z \,, & A = A_x = A_y = A_z \\ B_A = B_{xx} = B_{yy} = B_{zz} \,, & B_B = B_{xy} = B_{yx} = B_{yz} = B_{zy} = B_{zx} = B_{xz} \\ b_a = b_{xx} = b_{yy} = b_{zz} \,, & b_b = b_{xy} = b_{yx} = b_{yz} = b_{zy} = b_{zx} = b_{xz} \end{cases} . \tag{3.68}$$

Also, from (3.55) and (3.56), it can be verified that

$$\begin{cases} \dfrac{1}{B_A} = \dfrac{b_a + b_b}{(b_a - b_b)(b_a + 2b_b)} \\ \dfrac{1}{B_B} = -\dfrac{b_b}{(b_a - b_b)(b_a + 2b_b)} \end{cases} . \tag{3.69}$$

Therefore, for an isotropic poroviscous material, the constitutive equations (3.53) and (3.59) simplify to

$$\begin{cases} \sigma_x = \dfrac{1}{B_A}\eta\dot{\varepsilon}_x + \dfrac{1}{B_B}\eta\dot{\varepsilon}_y + \dfrac{1}{B_B}\eta\dot{\varepsilon}_z \\ \sigma_y = \dfrac{1}{B_B}\eta\dot{\varepsilon}_x + \dfrac{1}{B_A}\eta\dot{\varepsilon}_y + \dfrac{1}{B_B}\eta\dot{\varepsilon}_z \\ \sigma_z = \dfrac{1}{B_B}\eta\dot{\varepsilon}_x + \dfrac{1}{B_B}\eta\dot{\varepsilon}_y + \dfrac{1}{B_A}\eta\dot{\varepsilon}_z \end{cases} \tag{3.70}$$

and

$$\begin{cases} \dot{\varepsilon}_x = b_a \dfrac{\sigma_x}{\eta} + b_b \dfrac{\sigma_y}{\eta} + b_b \dfrac{\sigma_z}{\eta} \\ \dot{\varepsilon}_y = b_b \dfrac{\sigma_x}{\eta} + b_a \dfrac{\sigma_y}{\eta} + b_b \dfrac{\sigma_z}{\eta} \\ \dot{\varepsilon}_z = b_b \dfrac{\sigma_x}{\eta} + b_b \dfrac{\sigma_y}{\eta} + b_a \dfrac{\sigma_z}{\eta} \end{cases} . \tag{3.71}$$

In comparing equations of (3.71) with the elasticity constitutive equations of (3.2), viscosity η can be perceived as a measure of "dynamic rigidity" of a poroviscous material.

Chapter 4 The Poroviscosity Model

The purpose of this chapter is to establish a mathematical model using the definitions of variables and the poroviscosity constitutive law described in the previous chapters. The subject of focus in this chapter is a general porous medium consisting of a mixture of water and solids in field conditions. A porous medium is treated as a continuum by taking into account the concept of REV discussed in Section 1.4.1. Although stress equilibrium and moment balance are not invoked in the governing equations, the topics are briefly presented in Section 4.3.3 for completeness.

4.1 Fundamental Variables

4.1.1 Displacement and Velocity Fields and Bulk Volume

The definition of displacement of a moving particle in general is given by (2.1) which can also be used to define the displacement field for both the solid and fluid constituents of a porous medium. If displacements of solid skeletal frame and interstitial water are denoted by $\mathbf{u}_\mathrm{s}$ and $\mathbf{u}_\mathrm{w}$ [L], respectively, and are assumed to be continuous functions of space $\mathbf{x}$ and time t, that is, $\mathbf{u}_\mathrm{s} = \mathbf{u}_\mathrm{s}(\mathbf{x}, \mathrm{t})$ and $\mathbf{u}_\mathrm{w} = \mathbf{u}_\mathrm{w}(\mathbf{x}, \mathrm{t})$, then

$$\begin{cases} \mathbf{u}_\mathrm{s} = \mathbf{x}_\mathrm{s} - \mathbf{x}_{\mathrm{s}p} = \int_0^\mathrm{t} \mathbf{v}_\mathrm{s}\, \mathrm{dt} \\ \mathbf{u}_\mathrm{w} = \mathbf{x}_\mathrm{w} - \mathbf{x}_{\mathrm{w}p} = \int_0^\mathrm{t} \mathbf{v}_w\, \mathrm{dt} \end{cases} \tag{4.1}$$

in which $\mathbf{v}_\mathrm{s} = \mathbf{v}_\mathrm{s}(\mathbf{x}, \mathrm{t})$ and $\mathbf{v}_\mathrm{w} = \mathbf{v}_\mathrm{w}(\mathbf{x}, \mathrm{t})$ are the velocities [L/t], $\mathbf{x}_{\mathrm{s}p} = \mathbf{x}_\mathrm{s}(0)$ and $\mathbf{x}_{\mathrm{w}p} = \mathbf{x}_\mathrm{w}(0)$ are the position vectors [L] at time t = 0^- and $\mathbf{x}_\mathrm{s} = \mathbf{x}_\mathrm{s}(\mathrm{t})$ and $\mathbf{x}_\mathrm{w} = \mathbf{x}_\mathrm{w}(\mathrm{t})$ are the position vectors [L] at an arbitrary later time t ($\geq 0^-$). A variable with a subscript "s" pertains to solid skeletal frame and "w" to water; and an extra subscript "*p*" indicates the state of a variable at time t = 0^- (for interior points remote from any stress and/or hydraulic boundary as explained in Section 1.4.2). It also follows from (4.1) that

$$\begin{cases} \mathbf{v}_s = \dfrac{d\mathbf{u}_s}{dt} \\ \mathbf{v}_w = \dfrac{d\mathbf{u}_w}{dt} \end{cases} ; \tag{4.2}$$

and at time t = 0^-,

$$\begin{cases} \mathbf{u}_{sp} = \mathbf{u}_s(\mathbf{x},0) = \mathbf{0} \\ \mathbf{u}_{wp} = \mathbf{u}_w(\mathbf{x},0) = \mathbf{0} \end{cases} \quad (\text{for t} = 0^-) \tag{4.3}$$

and

$$\begin{cases} \mathbf{v}_{sp} = \left.\dfrac{d\mathbf{u}_s}{dt}\right|_{t=0} \\ \mathbf{v}_{wp} = \left.\dfrac{d\mathbf{u}_w}{dt}\right|_{t=0} \end{cases} \quad (\text{for t} = 0^-) \tag{4.4}$$

where $\mathbf{v}_{sp} = \mathbf{v}_s(\mathbf{x},0)$ and $\mathbf{v}_{wp} = \mathbf{v}_w(\mathbf{x},0)$.

Similar to equations of (4.1)~(4.4), displacement and velocity of any point of interest P in general at any time t (≥ 0) are defined as

$$\mathbf{u}_P = \mathbf{x}_P - \mathbf{x}_{Pp} = \int_0^t \mathbf{v}_P \, dt \tag{4.5}$$

and

$$\mathbf{v}_P = \frac{d\mathbf{u}_P}{dt} \tag{4.6}$$

for which

$$\mathbf{u}_{Pp} = \mathbf{u}_P(\mathbf{x},0) = \mathbf{0} \quad (\text{for t} = 0^-) \tag{4.7}$$

and

$$\mathbf{v}_{Pp} = \mathbf{v}_P(\mathbf{x},0) = \left.\frac{d\mathbf{u}_P}{dt}\right|_{t=0} \qquad (\text{for } t = 0^-) \; . \tag{4.8}$$

In this thesis, the point P is employed to represent the centroid of the control or bulk volume of a porous medium consisting of solids and water saturated in the pore space.

Note that in equations of (4.4) and (4.8), the velocities $\mathbf{v}_{Pp} = \mathbf{v}_{Pp}(\mathbf{x})$, $\mathbf{v}_{sp} = \mathbf{v}_{sp}(\mathbf{x})$ and $\mathbf{v}_{wp} = \mathbf{v}_{wp}(\mathbf{x})$ become functions of only space **x**, that is, independent of time t, and $\mathbf{v}_{Pp} = \mathbf{0}$, $\mathbf{v}_{sp} = \mathbf{0}$ and $\mathbf{v}_{wp} = \mathbf{0}$ and are not necessarily assumed.

Displacement of solid skeleton $\mathbf{u}_s$ is denoted in terms of vector components as

$$\mathbf{u}_s = \begin{cases} [u_x \;\; u_y \;\; u_z]^T & (\text{for Cartesian coordinates}) \\ [u_r \;\; u_\theta \;\; u_z]^T & (\text{for cylindrical coordinates}) \end{cases} \tag{4.9}$$

where $u_x = u_x(\mathbf{x},t)$, $u_y = u_y(\mathbf{x},t)$, $u_r = u_r(\mathbf{x},t)$, $u_\theta = u_\theta(\mathbf{x},t)$ and $u_z = u_z(\mathbf{x},t)$ are all functions of space **x** and time t and $\mathbf{x} = [x \;\; y \;\; z]^T$ for Cartesian coordinates and $\mathbf{x} = [r \;\; \theta \;\; z]^T$ for cylindrical coordinates. Also, radial distance as a component of the position vector in cylindrical coordinates is denoted by r_P for the centroid of the bulk volume P and by, simply, r for the solid skeletal frame. And, according to Equations (4.5) and (4.1),

$$\begin{cases} r_P = r_{Pp} + u_{Pr} \\ r = r_p + u_r \end{cases} \tag{4.10}$$

where r_{Pp} and r_p are the radial distance of P and solid skeleton, respectively, at time t = 0^- and u_{Pr} is the displacement of P in radial direction at any time $t \geq 0^-$.

For an infinitesimal bulk volume V [L^3] (=V(**x**,t), a function of space **x** and time t) of a deforming porous medium containing the same amount of mass with centroid P moving at the velocity $\mathbf{v}_P$, an important relationship

$$\nabla \cdot \mathbf{v}_P = \frac{1}{V}\frac{dV}{dt} \tag{4.11}$$

arises (Kellogg, 1929, p. 36). If $dV/dt > 0$, the volume is said to be expanding; if $dV/dt < 0$, the volume is contracting; and, otherwise, if $dV/dt = 0$, the material is said to be incompressible according to (4.11). It must be noted that the bulk volume under consideration depends on whether the centroid P is moving or not and, if it is, on which physical entity it is following.

Helm (1975, p. 466; 1984, p. 2) points out that soil engineers and geohydrologists had obliviously viewed the physical representation of the bulk volume differently and demonstrates (Helm, 1987, p. 374) with great insight and clarity that a subtle distinction must be made. If the centroid P is fixed in space, namely, $\mathbf{v}_P = \mathbf{0}$, the corresponding volume, if denoted by V_a, should remain constant with respect to time t according to Equation (4.11). If the point P follows the centroid of solids within the bulk volume, that is, $\mathbf{v}_P = \mathbf{v}_s$, then the volume is required to maintain a certain size, if denoted by V_b, so that the mass of solids remains unchanged in time as the bulk volume moves in space. In another case that P follows the centroid of water, then the mass of water within the volume with a certain size, say, V_c, must be constant with respect to time. Note that in all three cases mentioned, V_a, V_b and V_c are three physically distinct volumes and are not at all the same in size.

Helm (1987, p. 374) concludes that bulk volume of a deforming porous medium in question is theoretically governed by the porosity n of the medium [dimensionless] and the density of solids ρ_s [M/L^3] or interstitial water ρ_w [M/L^3] as

$$\nabla\cdot \mathbf{v}_P = \begin{cases} \dfrac{1}{V_a}\dfrac{dV_a}{dt} = 0 & (\text{for } \mathbf{v}_P = \mathbf{0}) \\ \dfrac{1}{V_b}\dfrac{dV_b}{dt} = -\left(\dfrac{1}{\rho_s}\dfrac{d\rho_s}{dt} - \dfrac{1}{1-n}\dfrac{dn}{dt}\right) & (\text{for } \mathbf{v}_P = \mathbf{v}_s) \\ \dfrac{1}{V_c}\dfrac{dV_c}{dt} = -\left(\dfrac{1}{\rho_w}\dfrac{d\rho_w}{dt} - \dfrac{1}{n}\dfrac{dn}{dt}\right) & (\text{for } \mathbf{v}_P = \mathbf{v}_w) \end{cases} \tag{4.12}$$

of which $V_a = V_b = V_c$ is not necessarily true.

4.1.2 Hydraulic Head

As first postulated by Hubbert (1940, p. 801; 1969, p. 41), for a porous medium saturated with water, the mechanical energy Φ [L^2/t^2] per unit mass of water which slowly moves within the pore space is defined as

$$\Phi = gh_H = g\int_{z_{sl}}^{z_H} dz \mp \int_{p_{sl}}^{p_a} \frac{1}{\rho_w} dp_a \tag{4.13}$$

in which g is the gravitational acceleration [L/t^2], z_{sl} is the elevation of sea level [L], z_H is the elevation [L] of a point fixed in space where mechanical energy is evaluated in a coordinate system with vertical z axis pointed upward against the direction of gravity, h_H is the hydraulic head [L] at the point with elevation z_H, p_{sl} and p_a are the respective absolute pressures [M/Lt^2] at sea level z_{sl} and at the elevation z_H and $\rho_w = \rho_w(p_a)$ is the density of water [M/L^3] which, in general, is assumed to be a function of pressure p_a. Hydraulic head h_H so defined by (4.13) essentially represents the potential energy per unit weight of water.

If density of water is assumed to be independent of pressure p_a, completing the integrals of Equation (4.13) and dividing both sides by the gravitational constant g leads to

$$h_H = z_H - z_{sl} \mp \frac{p_a - p_{sl}}{\rho_w g} \quad \text{(for } \rho_w \text{ independent of } p_a\text{)} . \tag{4.14}$$

Furthermore, if the origin of a set of coordinate axes is selected to be at the sea level, that is, $z_{sl}=0$, and the gauge pressure, instead of absolute pressure, is used, then (4.14) becomes

$$h_H = z_H \mp \frac{p_w}{\rho_w g} \quad \text{(for } \rho_w \text{ independent of } p_w\text{)} \tag{4.15}$$

with

$$p_w = p_a - p_{sl} \tag{4.16}$$

where p_w is the gauge pressure [M/Lt^2] at the point with the fixed elevation z_H. (4.15) is the classical definition of hydraulic head for a point fixed in space within a porous medium in question.

However, for a deforming medium, any point under consideration is not always fixed in space at all times and its position changes as the medium deforms, although infinitesimally. Hydraulic head therefore must account for this dynamic change of position. In contrast to (4.15), hydraulic head for a deformable medium is redefined using a different set of notations as

$$h = z_P \mp \frac{p}{\rho_w g} \tag{4.17}$$

where $z_P = z_P(t)$ is the elevation [L] of a point in motion within the medium, $p = p(\mathbf{x},t)$ is the gauge pressure [M/Lt^2] of pore water and $h = h(\mathbf{x},t)$ is the hydraulic head [L]. A coordinate system must be adopted so that the vertical z axis is pointed upward against the direction of gravity and the reference datum of z is at the sea level. An assumption inherent to the newly defined hydraulic head (4.17) is that the density of water ρ_w is independent of pressure p.

It follows from Equation (4.17) that

$$\nabla h = \mathbf{k} \mp \frac{1}{g} \nabla \left(\frac{p}{\rho_w} \right) \tag{4.18}$$

where **k** is the unit vector of the vertical z coordinate axis. Although one of the assumptions inherent to the definition of hydraulic head (4.17) is that the density of water ρ_w is independent of the water pressure p, it is not necessarily assumed that water density ρ_w is a constant. In general, ρ_w may be still a function of space **x** and time t.

4.2 Poroviscosity Constitutive Law

For a porous medium saturated with water, the stress that is responsible for deformation is the effective stress. For an anisotropic poroviscous material, the relationship between the effective stress and strain rate for an arbitrary time t ($\geq 0^-$), according to (3.53), is

$$\begin{cases} \sigma'_x = \dfrac{1}{B_{xx}} \eta_x \dot{\varepsilon}_x + \dfrac{1}{B_{yx}} \eta_y \dot{\varepsilon}_y + \dfrac{1}{B_{zx}} \eta_z \dot{\varepsilon}_z \\ \sigma'_y = \dfrac{1}{B_{xy}} \eta_x \dot{\varepsilon}_x + \dfrac{1}{B_{yy}} \eta_y \dot{\varepsilon}_y + \dfrac{1}{B_{zy}} \eta_z \dot{\varepsilon}_z \\ \sigma'_z = \dfrac{1}{B_{xz}} \eta_x \dot{\varepsilon}_x + \dfrac{1}{B_{yz}} \eta_y \dot{\varepsilon}_y + \dfrac{1}{B_{zz}} \eta_z \dot{\varepsilon}_z \end{cases} \tag{4.19}$$

where η_x, η_y and η_z are the dynamic viscosities [M/Lt] defined by (3.62) (with the initial values to be replaced by prestressed values and redefined accordingly in terms of prestressed effective stresses below), $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$ and $\dot{\varepsilon}_z$ are the normal strain rates [1/t] and σ'_x, σ'_y and σ'_z are the normal effective stresses [M/Lt^2]. Also, B_{jk} (j,k = x,y,z) are some assumed constant coefficients. Similar to the "equivalent stresses" defined by

(3.65), the "equivalent effective stresses" σ'_{xe}, σ'_{ye} and σ'_{ze} [M/Lt^2] for a porous medium saturated with fluid are defined as

$$\begin{cases} \sigma'_{xe} = \eta_x \dot{\varepsilon}_x = b_{xx}\sigma'_x + b_{xy}\sigma'_y + b_{xz}\sigma'_z \\ \sigma'_{ye} = \eta_y \dot{\varepsilon}_y = b_{yx}\sigma'_x + b_{yy}\sigma'_y + b_{yz}\sigma'_z \\ \sigma'_{ze} = \eta_z \dot{\varepsilon}_z = b_{zx}\sigma'_x + b_{zy}\sigma'_y + b_{zz}\sigma'_z \end{cases} \tag{4.20}$$

with the corresponding stresses σ'_{xep}, σ'_{yep} and σ'_{zep} at prestressed time

$$\begin{cases} \sigma'_{xep} = \eta_{xp} \dot{\varepsilon}_{xp} = b_{xx}\sigma'_{xp} + b_{xy}\sigma'_{yp} + b_{xz}\sigma'_{zp} \\ \sigma'_{yep} = \eta_{yp} \dot{\varepsilon}_{yp} = b_{yx}\sigma'_{xp} + b_{yy}\sigma'_{yp} + b_{yz}\sigma'_{zp} \\ \sigma'_{zep} = \eta_{zp} \dot{\varepsilon}_{zp} = b_{zx}\sigma'_{xp} + b_{zy}\sigma'_{yp} + b_{zz}\sigma'_{zp} \end{cases} \quad (\text{for } t = 0^-) \tag{4.21}$$

where η_{xp}, η_{yp} and η_{zp} are the prestressed viscosities, $\dot{\varepsilon}_{xp}$, $\dot{\varepsilon}_{yp}$ and $\dot{\varepsilon}_{zp}$ are the prestressed strain rates and σ'_{xp}, σ'_{yp} and σ'_{zp} are the effective stresses at prestressed time.

It follows immediately from (4.21) that

$$\eta_{xp} = \frac{\sigma'_{xep}}{\dot{\varepsilon}_{xp}}, \quad \eta_{yp} = \frac{\sigma'_{yep}}{\dot{\varepsilon}_{yp}} \quad \text{and} \quad \eta_{zp} = \frac{\sigma'_{zep}}{\dot{\varepsilon}_{zp}} \quad (\text{for } t = 0^-) \,. \tag{4.22}$$

4.3 Fundamental Principles

4.3.1 Darcy-Gersevanov Law and Bulk Flux

Gersevanov's (1937) generalization of Darcy's law expresses Darcy flux $\mathbf{q}$ [L/t] as the velocity of water $\mathbf{v}_w$ relative to that of the solid skeletal frame $\mathbf{v}_s$ factored by the porosity n [dimensionless] of the porous medium as

$$\mathbf{q} = n(\mathbf{v}_w - \mathbf{v}_s) = -\mathbf{K}\nabla h \tag{4.23}$$

(Darcy, 1856; Gersevanov, 1937, pp. 34~35) where h is the hydraulic head [L] and $\mathbf{K} = \mathbf{K}(\mathbf{x}, t)$ is the hydraulic conductivity [L/t] assumed to be a variable in this thesis.

Equations (4.23) can also be mathematically rewritten as

$$\mathbf{q} = n[(\mathbf{v}_w - \mathbf{v}_P) - (\mathbf{v}_s - \mathbf{v}_P)] = -\mathbf{K}\nabla h \qquad (4.24)$$

without altering the identity. By (4.24), the specific flux $\mathbf{q}$ can therefore be viewed conceptually as a flux of water relative to the movement of the control volume $\mathbf{v}_P$. When the movement of solids is far slower than that of water, or $|\mathbf{v}_s| << |\mathbf{v}_w|$, as in the case of most ground-water flow, Equation (4.23) simplifies to

$$\mathbf{q} = -\mathbf{K}\nabla h \cong n\mathbf{v}_w \qquad (\text{for } |\mathbf{v}_w| >> |\mathbf{v}_s|) \qquad (4.25)$$

which is often used to estimate the velocity of interstitial water $\mathbf{v}_w$ ($\cong \mathbf{q}/n$) in contaminant transport problems. In other words, Darcy's law can be considered a special case of Gersevanov's generalization, (4.23).

On the basis of volume fraction, Helm (1987) postulates that a bulk flux $\mathbf{q}_b$ [L/t] of a porous medium saturated with water can be defined to take into account both velocities of the skeletal frame and water weighted by the respective physical spaces occupied as

$$\mathbf{q}_b = n\mathbf{v}_w + (1-n)\mathbf{v}_s \qquad (4.26)$$

(Helm, 1987, p. 371). Equation (4.26) can be rearranged to incorporate (4.23) as

$$\mathbf{q}_b = n(\mathbf{v}_w - \mathbf{v}_s) + \mathbf{v}_s$$

$$\Rightarrow \quad \mathbf{q}_b = \mathbf{v}_s + \mathbf{q} = \mathbf{v}_s - \mathbf{K}\nabla h \qquad (4.27)$$

which unveils that the so-called "bulk flux" $\mathbf{q}_b$ bears some relationship with the Darcy-Gersevanov law of (4.23).

Certain mechanical and hydraulic boundaries are characterized by some specific bulk fluxes. For an impermeable boundary where $\mathbf{q} = \mathbf{0}$, it follows from Equations (4.23) and (4.27), that $\mathbf{v}_w = \mathbf{v}_s$ and $\mathbf{q}_b = \mathbf{v}_s$. For a boundary fixed in space, that is, $\mathbf{v}_s = \mathbf{0}$, then it can be deduced from (4.27) and (4.23) that $\mathbf{q}_b = \mathbf{q}$ and $\mathbf{q} = n\mathbf{v}_w$. For a boundary that is both impermeable and fixed in space, $\mathbf{q} = \mathbf{0}$ and $\mathbf{v}_s = \mathbf{0}$ which leads to $\mathbf{q}_b = \mathbf{0}$. Also, in the case that $\mathbf{v}_s$ and $\mathbf{q}$ are equal in magnitude but opposite in direction, bulk flux $\mathbf{q}_b = \mathbf{0}$ as well. Therefore, it can be summarized that for various boundaries,

$$\mathbf{q}_b = \begin{cases} \mathbf{v}_s = \mathbf{v}_w & (\text{for } \mathbf{q} = \mathbf{0}) \\ \mathbf{q} = n\mathbf{v}_w & (\text{for } \mathbf{v}_s = \mathbf{0}) \\ \mathbf{0} & (\text{for } \mathbf{v}_s = -\mathbf{q} \text{ or } \mathbf{v}_s = \mathbf{v}_w = \mathbf{q} = \mathbf{0}) \end{cases} . \tag{4.28}$$

4.3.2 Mass Balance

For a porous medium consisting of solid and water phases, the principle of mass balance requires that for each phase, the net total of the mass leaving the bulk control volume V [L^3] through the boundary *S* and the mass produced within the volume V be equal to the rate of mass decrease within the control volume. Accordingly, a general mass balance equation can be written for phase α as

$$\int_S \rho^\alpha \left(\mathbf{v}_\alpha - \mathbf{v}_P\right) \cdot d\mathbf{S} + \int_V \dot{m}_\alpha dV = -\frac{d}{dt}\left(\int_V \rho^\alpha dV\right) \tag{4.29}$$

(Helm, 1987, p. 373) where ρ^α and $\mathbf{v}_\alpha$ are the partial density [M/L^3] and velocity [L/t] of the constituent α, $\mathbf{v}_P$ is the velocity of the bulk control volume, $\dot{m}_\alpha$ $(= dm_\alpha/dt)$ is the mass decrease (or production) rate of phase α per unit bulk volume [M/L^3t] and d$\mathbf{S}$ is the unit outward normal vector of an infinitesimal area of the surface *S* enclosing the control volume. For a saturated porous medium consisting of only solid and water with porosity n, partial density of each constituent is defined by

$$\begin{cases} \rho^s = (1-n)\rho_s \\ \rho^w = n\rho_w \end{cases} \tag{4.30}$$

where ρ_s and ρ_w are the densities [M/L^3] of solid grains and water, respectively. From the definitions of (4.30), bulk density of the porous medium ρ_b [M/L^3] is defined as

$$\rho_b = n\rho_w + (1-n)\rho_s \ . \tag{4.31}$$

With (4.30), the mass balance equation (4.29) can be written specifically for both the solid and the fluid phases, respectively, as

$$\begin{cases} \frac{d}{dt}\left[(1-n)\rho_s V\right] + \int_S (1-n)\rho_s(\mathbf{v}_s - \mathbf{v}_P)\cdot d\mathbf{S} = -\int_V \dot{m}_s dV & (4.32) \\ \frac{d}{dt}(n\rho_w V) + \int_S n\rho_w(\mathbf{v}_w - \mathbf{v}_P)\cdot d\mathbf{S} = -\int_V \dot{m}_w dV & (4.33) \end{cases}$$

where $\dot{m}_s$ and $\dot{m}_w$ are the mass decrease rates [M/L^{3}t] of solids and water, respectively. If it is assumed that there is no mass production or loss of each constituent, the integrals on the right-hand side of both Equations (4.32) and (4.33) can be eliminated. Therefore, with the assumption of no source or sink of both constituents, taking both Equations (4.32) and (4.33) into account simultaneously will lead to

$$\nabla\cdot\mathbf{q}_b = -\left\{\frac{1-n}{\rho_s}\left[\frac{\partial\rho_s}{\partial t} + \mathbf{v}_s\cdot\nabla\rho_s\right] + \frac{n}{\rho_w}\left[\frac{\partial\rho_w}{\partial t} + \mathbf{v}_w\cdot\nabla\rho_w\right]\right\} \tag{4.34}$$

(Helm, 1987, p. 374) (See Appendix A.3 for detailed derivation).

In arriving at (4.34), the identity of (4.11) has been invoked. The terms within the first and the second pairs of brackets are equivalent to the material derivatives of solid density ρ_s and water density ρ_w, respectively, as if the movements of two completely different material identities, namely, solid skeletal frame and water are followed as suggested by the presence of $\mathbf{v}_s$ and $\mathbf{v}_w$ in the two convective terms of (4.34). Therefore, if Equation (4.34) is to be written in terms of the total derivatives $d\rho_s/dt$ and $d\rho_w/dt$, the expression should include this information of using simultaneously two different reference frames as

$$\nabla\cdot\mathbf{q}_b = -\left(\frac{1-n}{\rho_s}\frac{d\rho_s}{dt}\bigg|_{\mathbf{v}_P=\mathbf{v}_s} + \frac{n}{\rho_w}\frac{d\rho_w}{dt}\bigg|_{\mathbf{v}_P=\mathbf{v}_w}\right)$$

in which the subscripts $\mathbf{v}_P = \mathbf{v}_s$ and $\mathbf{v}_P = \mathbf{v}_w$ respectively emphasize that the reference frames follow two different moving centroids of solid frame and water.

Equation (4.34) can be considered as another statement of the mass balance principle of a porous medium saturated with water and as an alternative to that of (4.29), (4.32) and (4.33). If solid grains are considered incompressible, that is, ρ_s is constant, the terms inside the first pair of brackets of Equation (4.34) can, therefore, be eliminated. Or, if water is considered incompressible (or ρ_w is constant), then the terms inside the second brackets can be ignored. Thus, for some conditions, the mass balance equation of (4.34) can be simplified as

$$\nabla\cdot\mathbf{q}_b = \begin{cases} -\dfrac{1-n}{\rho_s}\left(\dfrac{\partial\rho_s}{\partial t} + \mathbf{v}_s\cdot\nabla\rho_s\right) & \text{(for constant } \rho_w\text{)} \\ -\dfrac{n}{\rho_w}\left(\dfrac{\partial\rho_w}{\partial t} + \mathbf{v}_w\cdot\nabla\rho_w\right) & \text{(for constant } \rho_s\text{)} \\ 0 & \text{(for constant } \rho_w \text{ and } \rho_s\text{)} \end{cases} \quad . \tag{4.35}$$

For a porous medium, individual solid grains are relatively incompressible and solid skeletal frame is highly compressible. Deformation of the medium therefore is mostly attributed to the rearrangement of solid grains within the medium which is reflected by porosity change. If solid grains and water are both considered incompressible, it follows from Equations (4.35), (4.27) and (4.2) that

$$\nabla\cdot\mathbf{q}_b = \nabla\cdot\left(\frac{d\mathbf{u}_s}{dt}\right) + \nabla\cdot\mathbf{q} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\nabla\cdot\mathbf{u}_s\right) + \nabla\cdot\mathbf{q} = 0$$

which, by further taking the definition of volumetric strain ε_v (2.33) of solid skeletal frame into account, becomes

$$\nabla\cdot\mathbf{q} = \nabla\cdot\left(-\mathbf{K}\nabla h\right) = \mp\frac{d\varepsilon_v}{dt} \qquad \text{(for constant } \rho_w \text{ and } \rho_s\text{)} . \tag{4.36}$$

Equation (4.36) suggests that for a porous medium consisting of incompressible water and solid grains, the net outflow of pore water is simply the decrease of volumetric strain rate of the solid skeletal frame.

4.3.3 Stress Equilibrium and Moment Balance

The stress state of a porous medium treated as a continuum is represented by a total stress tensor $\boldsymbol{\sigma}$, (2.37), which can be decomposed, according to Equation (2.41), into an effective stress $\boldsymbol{\sigma}'$, (2.43), and pore fluid pressure p as discussed Section 2.3. In a coordinate system x y z, the principle of force balance requires that the total stress $\boldsymbol{\sigma}$, body force $\mathbf{f} = [\,f_x \;\; f_y \;\; f_z\,]^T$, (2.38), and the inertia force of a porous medium satisfy the relationship

$$\begin{cases} \dfrac{\partial\sigma_x}{\partial x} + \dfrac{\partial\tau_{yx}}{\partial y} + \dfrac{\partial\tau_{zx}}{\partial z} \pm f_x = \pm\rho_b a_{bx} \\ \dfrac{\partial\tau_{xy}}{\partial x} + \dfrac{\partial\sigma_y}{\partial y} + \dfrac{\partial\tau_{zy}}{\partial z} \pm f_y = \pm\rho_b a_{by} \\ \dfrac{\partial\tau_{xz}}{\partial x} + \dfrac{\partial\tau_{yz}}{\partial y} + \dfrac{\partial\sigma_z}{\partial z} \pm f_z = \pm\rho_b a_{bz} \end{cases} \tag{4.37}$$

(Timoshenko and Goodier, 1970, p. 236; Fung, 1977, pp.72~76; Brady and Brown, 1985, pp. 26 & 42) in which a_{bx}, a_{by} and a_{bz} are the accelerations of the bulk porous medium

[L/t^2] in x, y and z directions, respectively, ρ_b is the bulk density of the medium [M/L^3] defined by (4.31). $\rho_b [a_{bx} \;\; a_{by} \;\; a_{bz}]^T$ represents the inertia force per unit volume [M/L^2t^2] of the bulk medium.

Also, the principle of moment balance requires that

$$\tau_{xy} = \tau_{yx}, \;\; \tau_{xz} = \tau_{zx} \;\; \text{and} \;\; \tau_{yz} = \tau_{zy} \tag{4.38}$$

(Timoshenko and Goodier, 1970, p. 5; Fung, 1977, pp.74~76) so that an infinitesimal bulk element dx dy dz of a porous medium is not subjected to a rotating moment.

If it is assumed that the inertia force is negligibly small (that is, $\rho_b [a_{bx} \;\; a_{by} \;\; a_{bz}]^T \approx \mathbf{0}$) and that the only body force of interest is the gravity force acting in the negative z direction, then the only body force component in equations of (4.37) remained is f_z, that is,

$$\begin{cases} f_x = f_y = 0 \\ f_z = -\rho_b g \end{cases} \tag{4.39}$$

and equations of (4.37) simplifies to

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} \pm f_z = 0 \end{cases} . \tag{4.40}$$

As written, the force balance equations of (4.37) and the stress equilibrium equations of (4.40) have not invoked the principle of moment balance, (4.38).

In cylindrical coordinates, equations of stress equilibrium appear differently and are written (Timoshenko and Goodier, 1970, p.342; Brady and Brown, 1985, p. 39) as

$$
\begin{cases}
\dfrac{\partial \sigma_r}{\partial r} + \dfrac{\sigma_r - \sigma_\theta}{r_P} + \dfrac{1}{r_P}\dfrac{\partial \tau_{\theta r}}{\partial \theta} + \dfrac{\partial \tau_{zr}}{\partial z} = 0 \\
\dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{\tau_{r\theta} + \tau_{\theta r}}{r_P} + \dfrac{1}{r_P}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{\partial \tau_{z\theta}}{\partial z} = 0 \\
\dfrac{\partial \tau_{rz}}{\partial r} + \dfrac{\tau_{rz}}{r_P} + \dfrac{1}{r_P}\dfrac{\partial \tau_{\theta z}}{\partial \theta} + \dfrac{\partial \sigma_z}{\partial z} \pm f_z = 0
\end{cases}
\tag{4.41}
$$

in which r_P is the radial distance of the centroid of the bulk volume under consideration. According to Equation (4.5), $r_P = r_{P_p} + u_r$ and, hence, equations of (4.41) can be written as

$$
\begin{cases}
\dfrac{\partial \sigma_r}{\partial r} + \dfrac{\sigma_r - \sigma_\theta}{r_{P_p} + u_{Pr}} + \dfrac{1}{r_{P_p} + u_{Pr}}\dfrac{\partial \tau_{\theta r}}{\partial \theta} + \dfrac{\partial \tau_{zr}}{\partial z} = 0 \\
\dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{\tau_{r\theta} + \tau_{\theta r}}{r_{P_p} + u_{Pr}} + \dfrac{1}{r_{P_p} + u_{Pr}}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{\partial \tau_{z\theta}}{\partial z} = 0 \\
\dfrac{\partial \tau_{rz}}{\partial r} + \dfrac{\tau_{rz}}{r_{P_p} + u_{Pr}} + \dfrac{1}{r_{P_p} + u_{Pr}}\dfrac{\partial \tau_{\theta z}}{\partial \theta} + \dfrac{\partial \sigma_z}{\partial z} \pm f_z = 0
\end{cases}
\quad .
\tag{4.42}
$$

Also, similar to (4.38), the principle of moment balance in cylindrical coordinates requires that

$$
\tau_{r\theta} = \tau_{\theta r}\,, \quad \tau_{rz} = \tau_{zr} \quad \text{and} \quad \tau_{\theta z} = \tau_{z\theta}\ . \tag{4.43}
$$

Note that the stress components which appear in the equilibrium equations (4.37) and (4.40)~(4.42) are the total stresses.

4.4 Equation of Motion and Governing Equations

From Darcy-Gersevanov's equation (4.27) with the definition of hydraulic gradient (4.18) if, for simplicity, assuming water is incompressible and from the relationship between water pressure and mean stresses (2.51),

$$
\mathbf{q}_b = \mathbf{v}_s - \mathbf{K}\nabla h = \mathbf{v}_s - \mathbf{K}\left(\mathbf{k} \mp \frac{\nabla p}{\rho_w g}\right) = \mathbf{v}_s - \mathbf{K}\left\{\mathbf{k} \mp \frac{1}{\rho_w g}\nabla\left[\mp\left(\sigma_m - \sigma'_m\right)\right]\right\}
$$

$$\Rightarrow \quad \mathbf{q}_b = \mathbf{v}_s - \mathbf{K}\mathbf{k} \pm \frac{\mathbf{K}}{\rho_w g}\left(\mp \nabla\sigma_m \pm \nabla\sigma'_m\right)$$

in which **k** is the unit vector of the vertical coordinate axis (positive upward against the direction of gravity). By further recognizing (4.2), the above equation can be rearranged to yield the equation of motion (Helm, 1987, p. 372)

$$\frac{d\mathbf{u}_s}{dt} + \frac{\mathbf{K}}{\rho_w g}\nabla\sigma'_m = \mathbf{R} \tag{4.44}$$

where

$$\mathbf{R} = \mathbf{q}_b + \mathbf{K}\mathbf{k} + \frac{\mathbf{K}}{\rho_w g}\nabla\sigma_m \ . \tag{4.45}$$

In Cartesian and cylindrical coordinates, **R** [L/t] and bulk flux $\mathbf{q}_b$ are denoted in terms of vector components by

$$\begin{cases} \mathbf{R} = [R_x \;\; R_y \;\; R_z]^T \\ \mathbf{q}_b = [q_{bx} \;\; q_{by} \;\; q_{bz}]^T \end{cases} \tag{4.46}$$

and

$$\begin{cases} \mathbf{R} = [R_r \;\; R_\theta \;\; R_z]^T \\ \mathbf{q}_b = [q_{br} \;\; q_{b\theta} \;\; q_{bz}]^T \end{cases} , \tag{4.47}$$

respectively.

The right-hand side of Equation (4.44) that is defined by (4.45) represents a forcing function and can be conceptually utilized to account for various boundary conditions as outlined by (4.28) and some field conditions including force differentials due to overburden and tectonic force.

For a control or bulk volume element fixed in space, $\mathbf{v}_P = \mathbf{0}$ and, by definition of total derivative,

$$\frac{d\mathbf{u}_s}{dt} = \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{v}_P \cdot \nabla \mathbf{u}_s = \frac{\partial \mathbf{u}_s}{\partial t} \quad .$$

Equation (4.44) becomes

$$\frac{\partial \mathbf{u}_s}{\partial t} + \frac{\mathbf{K}}{\rho_w g} \nabla \sigma'_m = \mathbf{R} \qquad (\text{for } \mathbf{v}_P = \mathbf{0}) \ . \tag{4.48}$$

Similarly, if $\mathbf{v}_P = \mathbf{v}_s$ or the control volume follows the moving centroid of solid skeletal frame, Equation (4.44) becomes

$$\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{u}_s + \frac{\mathbf{K}}{\rho_w g} \nabla \sigma'_m = \mathbf{R} \qquad (\text{for } \mathbf{v}_P = \mathbf{v}_s) \ . \tag{4.49}$$

Equation (4.44) along with the right-hand side defined by (4.45) is the general form of equation of motion whereas Equations (4.48) and (4.49) as written are associated with specific control or bulk volume.

In order to reach the governing equations applicable to field conditions, the gradient term of the mean effective stress $\nabla \sigma'_m$ in the equation of motion (4.44) needs to be evaluated by taking the poroviscosity constitutive relation, introduced in Section 4.2, into account.

For convenience, define B_x, B_y and B_z as some constants such that

$$\begin{cases} \dfrac{1}{B_x} = \dfrac{1}{B_{xx}} + \dfrac{1}{B_{xy}} + \dfrac{1}{B_{xz}} \\ \dfrac{1}{B_y} = \dfrac{1}{B_{yx}} + \dfrac{1}{B_{yy}} + \dfrac{1}{B_{yz}} \\ \dfrac{1}{B_z} = \dfrac{1}{B_{zx}} + \dfrac{1}{B_{zy}} + \dfrac{1}{B_{zz}} \end{cases} \tag{4.50}$$

where B_{jk} (j,k = x,y,z) are constant coefficients of the constitutive equations of (4.19). It then follows from Equations (2.50) and (4.19) that

$$\sigma'_m = \frac{1}{3}\left(\sigma'_x + \sigma'_y + \sigma'_z\right) = \frac{1}{3}\left(\frac{1}{B_x}\eta_x\dot{\varepsilon}_x + \frac{1}{B_y}\eta_y\dot{\varepsilon}_y + \frac{1}{B_z}\eta_z\dot{\varepsilon}_z\right) \tag{4.51}$$

where σ'_m is the mean effective stress and a stress invariant as well according to (2.50). Furthermore, from Equation (4.51),

$$\nabla\sigma'_m = \frac{1}{3}\left[\frac{1}{B_x}\nabla\left(\eta_x\dot{\varepsilon}_x\right) + \frac{1}{B_y}\nabla\left(\eta_y\dot{\varepsilon}_y\right) + \frac{1}{B_z}\nabla\left(\eta_z\dot{\varepsilon}_z\right)\right]$$

$$\Rightarrow \quad \nabla\sigma'_m = \frac{1}{3}\left[\frac{1}{B_x}\left(\eta_x\nabla\dot{\varepsilon}_x + \dot{\varepsilon}_x\nabla\eta_x\right) + \frac{1}{B_y}\left(\eta_y\nabla\dot{\varepsilon}_y + \dot{\varepsilon}_y\nabla\eta_y\right) + \frac{1}{B_z}\left(\eta_z\nabla\dot{\varepsilon}_z + \dot{\varepsilon}_z\nabla\eta_z\right)\right] \tag{4.52}$$

in which gradients of viscosities and strain rates need to be evaluated separately.

Viscosity components of a mechanically anisotropic porous medium are defined by (3.62) with the prestressed states given by (4.22) with (4.21). For x component of the nonlinear viscosities of (3.62),

$$\nabla\eta_x = \nabla\left[\eta_{xp} e^{\mp\left(\frac{\varepsilon_x - \varepsilon_{xp}}{A_x}\right)}\right] = e^{\mp\left(\frac{\varepsilon_x - \varepsilon_{xp}}{A_x}\right)}\nabla\eta_{xp} + \eta_{xp}\left\{e^{\mp\left(\frac{\varepsilon_x - \varepsilon_{xp}}{A_x}\right)}\left[\mp\frac{1}{A_x}\left(\nabla\varepsilon_x - \nabla\varepsilon_{xp}\right)\right]\right\}$$

$$= e^{\mp\left(\frac{\varepsilon_x - \varepsilon_{xp}}{A_x}\right)}\nabla\eta_{xp} \mp \frac{1}{A_x}\left[\eta_{xp} e^{\mp\left(\frac{\varepsilon_x - \varepsilon_{xp}}{A_x}\right)}\right]\left(\nabla\varepsilon_x - \nabla\varepsilon_{xp}\right)$$

in which $\varepsilon_{xp} = 0$ when $t = 0^-$ and, therefore, $\nabla\varepsilon_{xp} = \mathbf{0}$. By further recognizing the definition of viscosity (3.62) again, the above expression becomes

$$\nabla\eta_x = \frac{\eta_x}{\eta_{xp}}\nabla\eta_{xp} \mp \frac{1}{A_x}\eta_x\nabla\varepsilon_x \ .$$

Therefore, for all three components of viscosity in Cartesian coordinates,

$$
\begin{cases}
\nabla\eta_x = \dfrac{1}{A_x}\eta_x\left(\dfrac{A_x}{\eta_{xp}}\nabla\eta_{xp} \mp \nabla\varepsilon_x\right) \\
\nabla\eta_y = \dfrac{1}{A_y}\eta_y\left(\dfrac{A_y}{\eta_{yp}}\nabla\eta_{yp} \mp \nabla\varepsilon_y\right) \\
\nabla\eta_z = \dfrac{1}{A_z}\eta_z\left(\dfrac{A_z}{\eta_{zp}}\nabla\eta_{zp} \mp \nabla\varepsilon_z\right)
\end{cases}
\tag{4.53}
$$

where η_{xp}, η_{yp} and η_{zp} are defined by (4.22) with (4.21). Substituting all three expressions of (4.53) simultaneously into Equation (4.52) leads to

$$
\nabla\sigma'_m = \frac{1}{3}\left\{\frac{1}{B_x}\left[\eta_x\nabla\dot{\varepsilon}_x + \frac{1}{A_x}\eta_x\dot{\varepsilon}_x\left(\frac{A_x}{\eta_{xp}}\nabla\eta_{xp} \mp \nabla\varepsilon_x\right)\right] + \frac{1}{B_y}\left[\eta_y\nabla\dot{\varepsilon}_y \right.\right.
$$
$$
\left.\left. + \frac{1}{A_y}\eta_y\dot{\varepsilon}_y\left(\frac{A_y}{\eta_{yp}}\nabla\eta_{yp} \mp \nabla\varepsilon_y\right)\right] + \frac{1}{B_z}\left[\eta_z\nabla\dot{\varepsilon}_z + \frac{1}{A_z}\eta_z\dot{\varepsilon}_z\left(\frac{A_z}{\eta_{zp}}\nabla\eta_{zp} \mp \nabla\varepsilon_z\right)\right]\right\}
$$

$$
\Rightarrow \quad \nabla\sigma'_m = \frac{1}{3}\left[\frac{1}{A_xB_x}\eta_x\dot{\varepsilon}_x\left(\frac{A_x}{\eta_{xp}}\nabla\eta_{xp} + \frac{A_x}{\dot{\varepsilon}_x}\nabla\dot{\varepsilon}_x \mp \nabla\varepsilon_x\right) + \frac{1}{A_yB_y}\eta_y\dot{\varepsilon}_y\left(\frac{A_y}{\eta_{yp}}\nabla\eta_{yp} \right.\right.
$$
$$
\left.\left. + \frac{A_y}{\dot{\varepsilon}_y}\nabla\dot{\varepsilon}_y \mp \nabla\varepsilon_y\right) + \frac{1}{A_zB_z}\eta_z\dot{\varepsilon}_z\left(\frac{A_z}{\eta_{zp}}\nabla\eta_{zp} + \frac{A_z}{\dot{\varepsilon}_z}\nabla\dot{\varepsilon}_z \mp \nabla\varepsilon_z\right)\right]. \tag{4.54}
$$

Furthermore, substitution of (4.54) into equation of motion (4.44) yields

$$
\frac{d\mathbf{u}_s}{dt} + \frac{\mathbf{K}}{3\rho_w g}\left[\frac{1}{A_xB_x}\eta_x\dot{\varepsilon}_x\left(\frac{A_x}{\eta_{xp}}\nabla\eta_{xp} + \frac{A_x}{\dot{\varepsilon}_x}\nabla\dot{\varepsilon}_x \mp \nabla\varepsilon_x\right) + \frac{1}{A_yB_y}\eta_y\dot{\varepsilon}_y\left(\frac{A_y}{\eta_{yp}}\nabla\eta_{yp} \right.\right.
$$
$$
\left.\left. + \frac{A_y}{\dot{\varepsilon}_y}\nabla\dot{\varepsilon}_y \mp \nabla\varepsilon_y\right) + \frac{1}{A_zB_z}\eta_z\dot{\varepsilon}_z\left(\frac{A_z}{\eta_{zp}}\nabla\eta_{zp} + \frac{A_z}{\dot{\varepsilon}_z}\nabla\dot{\varepsilon}_z \mp \nabla\varepsilon_z\right)\right] = \mathbf{R} \tag{4.55}
$$

where **R** on the right-hand side is defined by (4.45).

The equation of motion of (4.55) in cylindrical coordinates takes the form

$$\frac{d\mathbf{u}_s}{dt}+\frac{\mathbf{K}}{3\rho_w g}\left[\frac{1}{A_r B_r}\eta_r\dot{\varepsilon}_r\left(\frac{A_r}{\eta_{rp}}\nabla\eta_{rp}+\frac{A_r}{\dot{\varepsilon}_r}\nabla\dot{\varepsilon}_r\mp\nabla\varepsilon_r\right)+\frac{1}{A_\theta B_\theta}\eta_\theta\dot{\varepsilon}_\theta\left(\frac{A_\theta}{\eta_{\theta p}}\nabla\eta_{\theta p}\right.\right.$$
$$\left.\left.+\frac{A_\theta}{\dot{\varepsilon}_\theta}\nabla\dot{\varepsilon}_\theta\mp\nabla\varepsilon_\theta\right)+\frac{1}{A_z B_z}\eta_z\dot{\varepsilon}_z\left(\frac{A_z}{\eta_{zp}}\nabla\eta_{zp}+\frac{A_z}{\dot{\varepsilon}_z}\nabla\dot{\varepsilon}_z\mp\nabla\varepsilon_z\right)\right]=\mathbf{R} \quad . \quad (4.56)$$

where the gradient operator is defined differently from that in Cartesian coordinates.

4.5 Simplification of Governing Equations

In order to simplify the equation of motions (4.55), further assumptions need to be made. One of the major assumptions is

$$\begin{cases} \left|\frac{A_x}{\eta_{xp}}\nabla\eta_{xp}+\frac{A_x}{\dot{\varepsilon}_x}\nabla\dot{\varepsilon}_x\right| = A_x\left|\nabla\left(\ln\left|\eta_{xp}\right|\right)+\nabla\left(\ln\left|\dot{\varepsilon}_x\right|\right)\right| << \left|\nabla\varepsilon_x\right| \\ \left|\frac{A_y}{\eta_{yp}}\nabla\eta_{yp}+\frac{A_y}{\dot{\varepsilon}_y}\nabla\dot{\varepsilon}_y\right| = A_y\left|\nabla\left(\ln\left|\eta_{yp}\right|\right)+\nabla\left(\ln\left|\dot{\varepsilon}_y\right|\right)\right| << \left|\nabla\varepsilon_y\right| \\ \left|\frac{A_z}{\eta_{zp}}\nabla\eta_{zp}+\frac{A_z}{\dot{\varepsilon}_z}\nabla\dot{\varepsilon}_z\right| = A_z\left|\nabla\left(\ln\left|\eta_{zp}\right|\right)+\nabla\left(\ln\left|\dot{\varepsilon}_z\right|\right)\right| << \left|\nabla\varepsilon_z\right| \end{cases} . \quad (4.57)$$

The implications of (4.57) can be physically interpreted as, in any coordinate direction j (= x,y,z),

(1) The spatial change of the prestressed viscosity η_{jp} (at t = 0^-) is negligibly small relative to that of strain ε_j at any later time (t ≥ 0^-); and

(2) At any time (t ≥ 0^-) after loading starts or the stress condition of an aquifer system changes, the spatial variation of aquifer deformation ε_j is far greater than that of the time rate of change of strain $\dot{\varepsilon}_j$.

Note that for a clay sediment, the poroviscous coefficient A_j was estimated to be around 10^{-4}~10^{-3} (Helm, 1998) according to a one-dimensional analysis. Due to the possibly low values of such coefficients A_j, the condition of (4.57) is even more easier to be satisfied.

If the gradient terms of the prestressed viscosities $A_j \nabla\left(\ln\left|\eta_{jp}\right|\right)$ (j = x,y,z)) are considered not only negligibly small but also zero-valued as assumed by Helm (1998), it is equivalent to assume that the prestressed viscosities are constants with respect to space **x** and time t, that is, $\nabla\eta_{jp} = \mathbf{0}$ and $\left.\frac{\partial \eta_{jp}}{\partial t}\right|_{t=0} = 0$ so that the material derivative $\left.\frac{d\eta_{jp}}{dt}\right|_{t=0} = 0$.

With the assumption of (4.57) and the definition of equivalent stresses (4.20), Equation (4.55) simplifies to

$$\frac{d\mathbf{u}_s}{dt} \mp \frac{\mathbf{K}}{3\rho_w g}\left(\frac{1}{A_x B_x}\sigma'_{xe}\nabla\varepsilon_x + \frac{1}{A_y B_y}\sigma'_{ye}\nabla\varepsilon_y + \frac{1}{A_z B_z}\sigma'_{ze}\nabla\varepsilon_z\right) = \mathbf{R} \ . \tag{4.58}$$

Furthermore, with the definitions of strain (2.31) for a Cartesian coordinate system, Equation (4.58) becomes

$$\frac{d\mathbf{u}_s}{dt} \mp \frac{\mathbf{K}}{3\rho_w g}\left[\frac{1}{A_x B_x}\sigma'_{xe}\nabla\left(\pm\frac{\partial u_x}{\partial x}\right) + \frac{1}{A_y B_y}\sigma'_{ye}\nabla\left(\pm\frac{\partial u_y}{\partial y}\right) + \frac{1}{A_z B_z}\sigma'_{ze}\nabla\left(\pm\frac{\partial u_z}{\partial z}\right)\right] = \mathbf{R}$$

$$\Rightarrow \quad \frac{d\mathbf{u}_s}{dt} - \frac{\mathbf{K}}{3\rho_w g}\left[\frac{1}{A_x B_x}\sigma'_{xe}\nabla\left(\frac{\partial u_x}{\partial x}\right) + \frac{1}{A_y B_y}\sigma'_{ye}\nabla\left(\frac{\partial u_y}{\partial y}\right) + \frac{1}{A_z B_z}\sigma'_{ze}\nabla\left(\frac{\partial u_z}{\partial z}\right)\right] = \mathbf{R} \ . \tag{4.59}$$

Another major assumption is that the principal hydraulic directions are also aligned along that of the coordinate axes such that all off-diagonal entries of the hydraulic conductivity tensors in (2.54) vanish. As a result, hydraulic conductivity in Cartesian coordinates becomes

$$\mathbf{K} = \begin{bmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{bmatrix}. \tag{4.60}$$

If the principal hydraulic directions coincide with the directions of coordinate axes, (4.59) can be written as

$$\begin{cases} \dfrac{du_x}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{xx}}{A_x B_x}\dfrac{\partial^2 u_x}{\partial x^2} + \dfrac{c_{xy}}{A_y B_y}\dfrac{\partial^2 u_y}{\partial x \partial y} + \dfrac{c_{xz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial x \partial z}\right) = R_x \\ \dfrac{du_y}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{yx}}{A_x B_x}\dfrac{\partial^2 u_x}{\partial y \partial x} + \dfrac{c_{yy}}{A_y B_y}\dfrac{\partial^2 u_y}{\partial y^2} + \dfrac{c_{yz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial y \partial z}\right) = R_y \\ \dfrac{du_z}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{zx}}{A_x B_x}\dfrac{\partial^2 u_x}{\partial z \partial x} + \dfrac{c_{zy}}{A_y B_y}\dfrac{\partial^2 u_y}{\partial z \partial y} + \dfrac{c_{zz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial z^2}\right) = R_z \end{cases} \tag{4.61}$$

in which

$$\begin{cases} R_x = q_{bx} + \dfrac{K_x}{\rho_w g}\dfrac{\partial \sigma_m}{\partial x} \\ R_y = q_{by} + \dfrac{K_y}{\rho_w g}\dfrac{\partial \sigma_m}{\partial y} \\ R_z = q_{bz} + K_z + \dfrac{K_z}{\rho_w g}\dfrac{\partial \sigma_m}{\partial z} \end{cases} \tag{4.62}$$

and

$$\begin{cases} c_{xx} = K_x \sigma'_{xe}, & c_{xy} = K_x \sigma'_{ye}, & c_{xz} = K_x \sigma'_{ze} \\ c_{yx} = K_y \sigma'_{xe}, & c_{yy} = K_y \sigma'_{ye}, & c_{yz} = K_y \sigma'_{ze} \\ c_{zx} = K_z \sigma'_{xe}, & c_{zy} = K_z \sigma'_{ye}, & c_{zz} = K_z \sigma'_{ze} \end{cases} \tag{4.63}$$

are assumed constants with respect to time (Helm, 1976, pp. 378~379).

Equation (4.58) for cylindrical coordinates is written as

$$\frac{d\mathbf{u}_s}{dt} \mp \frac{\mathbf{K}}{3\rho_w g}\left(\frac{1}{A_r B_r}\sigma'_{re}\nabla\varepsilon_r + \frac{1}{A_\theta B_\theta}\sigma'_{\theta e}\nabla\varepsilon_\theta + \frac{1}{A_z B_z}\sigma'_{ze}\nabla\varepsilon_z\right) = \mathbf{R} \tag{4.64}$$

which can be derived also from Equation (4.56) with the assumption

$$\begin{cases} \left|\frac{A_r}{\eta_{rp}}\nabla\eta_{rp}+\frac{A_r}{\dot{\varepsilon}_r}\nabla\dot{\varepsilon}_r\right| = A_r\left|\nabla\left(\ln\left|\eta_{rp}\right|\right)+\nabla\left(\ln\left|\dot{\varepsilon}_r\right|\right)\right| << \left|\nabla\varepsilon_r\right| \\ \left|\frac{A_\theta}{\eta_{\theta p}}\nabla\eta_{\theta p}+\frac{A_\theta}{\dot{\varepsilon}_\theta}\nabla\dot{\varepsilon}_\theta\right| = A_\theta\left|\nabla\left(\ln\left|\eta_{\theta p}\right|\right)+\nabla\left(\ln\left|\dot{\varepsilon}_\theta\right|\right)\right| << \left|\nabla\varepsilon_\theta\right| \\ \left|\frac{A_z}{\eta_{zp}}\nabla\eta_{zp}+\frac{A_z}{\dot{\varepsilon}_z}\nabla\dot{\varepsilon}_z\right| = A_z\left|\nabla\left(\ln\left|\eta_{zp}\right|\right)+\nabla\left(\ln\left|\dot{\varepsilon}_z\right|\right)\right| << \left|\nabla\varepsilon_z\right| \end{cases} . \tag{4.65}$$

The physical implications of (4.65) is similar to that of (4.57) for Cartesian coordinates.

For a porous medium in cylindrical coordinates with coordinate axes r, θ and z coinciding with the principal hydraulic directions, conductivity tensor (2.55) simplifies to

$$\mathbf{K} = \begin{bmatrix} K_r & 0 & 0 \\ 0 & K_\theta & 0 \\ 0 & 0 & K_z \end{bmatrix} \tag{4.66}$$

and the equation of motion (4.64) in cylindrical coordinates, if taking strain definition of (2.32) into account, represents three simultaneous equations given by

$$\begin{cases} \frac{du_r}{dt} - \frac{1}{3\rho_w g}\left[\frac{c_{rr}}{A_r B_r}\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial r}\right) \right. \\ \qquad \left. + \frac{c_{r\theta}}{A_\theta B_\theta}\frac{\partial}{\partial r}\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + \frac{c_{rz}}{A_z B_z}\frac{\partial}{\partial r}\left(\frac{\partial u_z}{\partial z}\right)\right] = R_r \\ \frac{du_\theta}{dt} - \frac{1}{3\rho_w g}\left[\frac{c_{\theta r}}{A_r B_r}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_r}{\partial r}\right) \right. \\ \qquad \left. + \frac{c_{\theta\theta}}{A_\theta B_\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + \frac{c_{\theta z}}{A_z B_z}\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_z}{\partial z}\right)\right] = R_\theta \\ \frac{du_z}{dt} - \frac{1}{3\rho_w g}\left[\frac{c_{zr}}{A_r B_r}\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial r}\right) \right. \\ \qquad \left. + \frac{c_{z\theta}}{A_\theta B_\theta}\frac{\partial}{\partial z}\left(\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\right) + \frac{c_{zz}}{A_z B_z}\frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial z}\right)\right] = R_z \end{cases} \tag{4.67}$$

in which

$$\begin{cases} R_r = q_{br} + \dfrac{K_r}{\rho_w g}\dfrac{\partial\sigma_m}{\partial r} \\ R_\theta = q_{b\theta} + \dfrac{K_\theta}{\rho_w g}\dfrac{1}{r}\dfrac{\partial\sigma_m}{\partial\theta} \\ R_z = q_{bz} + K_z + \dfrac{K_z}{\rho_w g}\dfrac{\partial\sigma_m}{\partial z} \end{cases} \tag{4.68}$$

and

$$\begin{cases} c_{rr} = K_r\sigma'_{re}, \quad c_{r\theta} = K_r\sigma'_{\theta e}, \quad c_{rz} = K_r\sigma'_{ze} \\ c_{\theta r} = K_\theta\sigma'_{re}, \quad c_{\theta\theta} = K_\theta\sigma'_{\theta e}, \quad c_{\theta z} = K_\theta\sigma'_{ze} \\ c_{zr} = K_z\sigma'_{re}, \quad c_{z\theta} = K_z\sigma'_{\theta e}, \quad c_{zz} = K_z\sigma'_{ze} \end{cases} . \tag{4.69}$$

Completing the partial derivatives of (4.67) and recognizing that $r = r_p + u_r$, (4.10), leads to

$$\begin{cases} \dfrac{du_r}{dt} - \dfrac{1}{3\rho_w g}\left[\dfrac{c_{rr}}{A_rB_r}\dfrac{\partial^2 u_r}{\partial r^2} + \dfrac{c_{r\theta}}{A_\theta B_\theta}\dfrac{1}{r_p+u_r}\left(-\dfrac{u_r}{r_p+u_r} + \dfrac{\partial u_r}{\partial r} - \dfrac{1}{r_p+u_r}\dfrac{\partial u_\theta}{\partial\theta} + \dfrac{\partial^2 u_\theta}{\partial r\partial\theta}\right) + \dfrac{c_{rz}}{A_zB_z}\dfrac{\partial^2 u_z}{\partial r\partial z}\right] = R_r \\ \dfrac{du_\theta}{dt} - \dfrac{1}{3\rho_w g}\left[\dfrac{c_{\theta r}}{A_rB_r}\dfrac{1}{r_p+u_r}\dfrac{\partial^2 u_r}{\partial\theta\partial r} + \dfrac{c_{\theta\theta}}{A_\theta B_\theta}\dfrac{1}{r_p+u_r}\left(\dfrac{1}{r_p+u_r}\dfrac{\partial u_r}{\partial\theta} + \dfrac{1}{r_p+u_r}\dfrac{\partial^2 u_\theta}{\partial\theta^2}\right) + \dfrac{c_{\theta z}}{A_zB_z}\dfrac{1}{r_p+u_r}\dfrac{\partial^2 u_z}{\partial\theta\partial z}\right] = R_\theta \\ \dfrac{du_z}{dt} - \dfrac{1}{3\rho_w g}\left[\dfrac{c_{zr}}{A_rB_r}\dfrac{\partial^2 u_r}{\partial z\partial r} + \dfrac{c_{z\theta}}{A_\theta B_\theta}\dfrac{1}{r_p+u_r}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial^2 u_\theta}{\partial z\partial\theta}\right) + \dfrac{c_{zz}}{A_zB_z}\dfrac{\partial^2 u_z}{\partial z^2}\right] = R_z \end{cases} \tag{4.70}$$

with R_r and R_z given by (4.68) and R_θ with an alternative expression of

$$R_\theta = q_{b\theta} + \frac{K_\theta}{\rho_w g}\frac{1}{r_p+u_r}\frac{\partial\sigma_m}{\partial\theta} . \tag{4.71}$$

4.5.1 Assumption of Mechanical Isotropy

As explained in Section 3.3, for a mechanically isotropic poroviscous material, the poroviscous constitutive coefficients A_x, A_y and A_z are all equal (= A), (3.68). Also, it follows from Equations (3.68), (3.69) and (4.50) that $B_x = B_y = B_z$. Therefore, for convenience, let

$$\frac{1}{B} = \frac{1}{B_x} = \frac{1}{B_y} = \frac{1}{B_z} = \frac{1}{B_A} + \frac{2}{B_B} = \frac{1-2b_b}{(b_a - b_b)(b_a + 2b_b)} \tag{4.72}$$

and B then must be some non-zero constant. With (4.72), the three simultaneous equations of (4.61) in Cartesian coordinates become

$$\begin{cases} \dfrac{du_x}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{xx}\dfrac{\partial^2 u_x}{\partial x^2} + c_{xy}\dfrac{\partial^2 u_y}{\partial x \partial y} + c_{xz}\dfrac{\partial^2 u_z}{\partial x \partial z} \right) = R_x \\ \dfrac{du_y}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{yx}\dfrac{\partial^2 u_x}{\partial y \partial x} + c_{yy}\dfrac{\partial^2 u_y}{\partial y^2} + c_{yz}\dfrac{\partial^2 u_z}{\partial y \partial z} \right) = R_y \\ \dfrac{du_z}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{zx}\dfrac{\partial^2 u_x}{\partial z \partial x} + c_{zy}\dfrac{\partial^2 u_y}{\partial z \partial y} + c_{zz}\dfrac{\partial^2 u_z}{\partial z^2} \right) = R_z \end{cases} \tag{4.73}$$

where R_x, R_y and R_z are defined by (4.62) and c_{jk} (j,k = x,y,z) by (4.63).

Similarly, in cylindrical coordinates, equations of (4.70) simplify to

$$\begin{cases} \dfrac{du_r}{dt} - \dfrac{1}{3AB\rho_w g}\Bigg[c_{rr}\dfrac{\partial^2 u_r}{\partial r^2} + c_{r\theta}\dfrac{1}{r_p + u_r}\left(-\dfrac{u_r}{r_p + u_r} + \dfrac{\partial u_r}{\partial r} \right. \\ \qquad\qquad \left. - \dfrac{1}{r_p + u_r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{\partial^2 u_\theta}{\partial r \partial \theta} \right) + c_{rz}\dfrac{\partial^2 u_z}{\partial r \partial z} \Bigg] = R_r \\ \dfrac{du_\theta}{dt} - \dfrac{1}{3AB\rho_w g}\Bigg[c_{\theta r}\dfrac{1}{r_p + u_r}\dfrac{\partial^2 u_r}{\partial \theta \partial r} + c_{\theta\theta}\dfrac{1}{r_p + u_r}\left(\dfrac{1}{r_p + u_r}\dfrac{\partial u_r}{\partial \theta} \right. \\ \qquad\qquad \left. + \dfrac{1}{r_p + u_r}\dfrac{\partial^2 u_\theta}{\partial \theta^2} \right) + c_{\theta z}\dfrac{1}{r_p + u_r}\dfrac{\partial^2 u_z}{\partial \theta \partial z} \Bigg] = R_\theta \\ \dfrac{du_z}{dt} - \dfrac{1}{3AB\rho_w g}\left[c_{zr}\dfrac{\partial^2 u_r}{\partial z \partial r} + c_{z\theta}\dfrac{1}{r_p + u_r}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial^2 u_\theta}{\partial z \partial \theta} \right) + c_{zz}\dfrac{\partial^2 u_z}{\partial z^2} \right] = R_z \end{cases} \tag{4.74}$$

where R_r, R_θ and R_z are defined by (4.68) and c_{jk} (j,k = r,θ,z) by (4.69).

4.5.2 Assumption of Irrotational Deformation

If deformation of solid skeletal frame is assumed irrotational and from Equation (2.35), for which $\omega_x = \omega_y = \omega_z = 0$, then

$$\frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x}, \quad \frac{\partial u_y}{\partial z} = \frac{\partial u_z}{\partial y} \quad \text{and} \quad \frac{\partial u_z}{\partial x} = \frac{\partial u_x}{\partial z} \tag{4.75}$$

and equations of (4.61) become

$$\begin{cases} \dfrac{du_x}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{xx}}{A_x B_x}\dfrac{\partial^2 u_x}{\partial x^2} + \dfrac{c_{xy}}{A_y B_y}\dfrac{\partial^2 u_x}{\partial y^2} + \dfrac{c_{xz}}{A_z B_z}\dfrac{\partial^2 u_x}{\partial z^2}\right) = R_x \\ \dfrac{du_y}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{yx}}{A_x B_x}\dfrac{\partial^2 u_y}{\partial x^2} + \dfrac{c_{yy}}{A_y B_y}\dfrac{\partial^2 u_y}{\partial y^2} + \dfrac{c_{yz}}{A_z B_z}\dfrac{\partial^2 u_y}{\partial z^2}\right) = R_y \\ \dfrac{du_z}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{zx}}{A_x B_x}\dfrac{\partial^2 u_z}{\partial x^2} + \dfrac{c_{zy}}{A_y B_y}\dfrac{\partial^2 u_z}{\partial y^2} + \dfrac{c_{zz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial z^2}\right) = R_z \end{cases} \tag{4.76}$$

where the coefficients c_{jk} (j,k = x,y,z) are defined by (4.63).

If mechanical isotropy is further assumed, it follows from (4.76) or from (4.73) that

$$\begin{cases} \dfrac{du_x}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{xx}\dfrac{\partial^2 u_x}{\partial x^2} + c_{xy}\dfrac{\partial^2 u_x}{\partial y^2} + c_{xz}\dfrac{\partial^2 u_x}{\partial z^2}\right) = R_x \\ \dfrac{du_y}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{yx}\dfrac{\partial^2 u_y}{\partial x^2} + c_{yy}\dfrac{\partial^2 u_y}{\partial y^2} + c_{yz}\dfrac{\partial^2 u_y}{\partial z^2}\right) = R_y \\ \dfrac{du_z}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{zx}\dfrac{\partial^2 u_z}{\partial x^2} + c_{zy}\dfrac{\partial^2 u_z}{\partial y^2} + c_{zz}\dfrac{\partial^2 u_z}{\partial z^2}\right) = R_z \end{cases} . \tag{4.77}$$

For irrotational deformation in cylindrical coordinates, from Equation (2.36) for which $\omega_r = \omega_\theta = \omega_z = 0$,

$$\frac{1}{r}\frac{\partial u_z}{\partial \theta} = \frac{\partial u_\theta}{\partial z}, \quad \frac{\partial u_r}{\partial z} = \frac{\partial u_z}{\partial r} \quad \text{and} \quad \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} \tag{4.78}$$

where $r = r_p + u_r$ according to (4.10). With (4.78), equations of (4.70) become

$$
\left\{
\begin{aligned}
&\frac{du_r}{dt} - \frac{1}{3\rho_w g}\left[-C_{r\theta}\frac{u_r}{(r_p+u_r)^2} + C_{r\theta}\frac{1}{r_p+u_r}\frac{\partial u_r}{\partial r} + C_{rr}\frac{\partial^2 u_r}{\partial r^2} \right. \\
&\qquad\qquad \left. + C_{r\theta}\frac{1}{(r_p+u_r)^2}\frac{\partial^2 u_r}{\partial \theta^2} - 2C_{r\theta}\frac{1}{(r_p+u_r)^2}\frac{\partial u_\theta}{\partial \theta} + C_{rz}\frac{\partial^2 u_r}{\partial z^2} \right] = R_r \\
&\frac{du_\theta}{dt} - \frac{1}{3\rho_w g}\left[C_{\theta\theta}\frac{u_\theta}{(r_p+u_r)^2} + (2C_{\theta r} + C_{\theta\theta})\frac{1}{r_p+u_r}\frac{\partial u_\theta}{\partial r} \right. \\
&\qquad\qquad \left. + C_{\theta r}\frac{\partial^2 u_\theta}{\partial r^2} + C_{\theta\theta}\frac{1}{(r_p+u_r)^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + C_{\theta z}\frac{\partial^2 u_\theta}{\partial z^2} \right] = R_\theta \\
&\frac{du_z}{dt} - \frac{1}{3\rho_w g}\left[C_{z\theta}\frac{1}{r_p+u_r}\frac{\partial u_z}{\partial r} + C_{zr}\frac{\partial^2 u_z}{\partial r^2} \right. \\
&\qquad\qquad \left. + C_{z\theta}\frac{1}{(r_p+u_r)^2}\frac{\partial^2 u_z}{\partial \theta^2} + C_{zz}\frac{\partial^2 u_z}{\partial z^2} \right] = R_z
\end{aligned}
\right. \tag{4.79}
$$

where

$$
\left\{
\begin{aligned}
&C_{rr} = \frac{c_{rr}}{A_r B_r}, \quad C_{r\theta} = \frac{c_{r\theta}}{A_\theta B_\theta}, \quad C_{rz} = \frac{c_{rz}}{A_z B_z} \\
&C_{\theta r} = \frac{c_{\theta r}}{A_r B_r}, \quad C_{\theta\theta} = \frac{c_{\theta\theta}}{A_\theta B_\theta}, \quad C_{\theta z} = \frac{c_{\theta z}}{A_z B_z} \\
&C_{zr} = \frac{c_{zr}}{A_r B_r}, \quad C_{z\theta} = \frac{c_{z\theta}}{A_\theta B_\theta}, \quad C_{zz} = \frac{c_{zz}}{A_z B_z}
\end{aligned}
\right. \tag{4.80}
$$

and c_{jk} (j,k = r,θ,z) are assumed constants defined by (4.69) and, hence, C_{jk} (j,k = r,θ,z) must be some constants as well.

If mechanical isotropy is further assumed, the three simultaneous equations of (4.79) become

$$
\begin{cases}
\dfrac{du_r}{dt} - \dfrac{1}{3AB\rho_w g}\left[-c_{r\theta}\dfrac{u_r}{\left(r_p + u_r\right)^2} + c_{r\theta}\dfrac{1}{r_p + u_r}\dfrac{\partial u_r}{\partial r} + c_{rr}\dfrac{\partial^2 u_r}{\partial r^2} \right. \\
\qquad \left. + c_{r\theta}\dfrac{1}{\left(r_p + u_r\right)^2}\dfrac{\partial^2 u_r}{\partial \theta^2} - 2c_{r\theta}\dfrac{1}{\left(r_p + u_r\right)^2}\dfrac{\partial u_\theta}{\partial \theta} + c_{rz}\dfrac{\partial^2 u_r}{\partial z^2} \right] = R_r \\
\dfrac{du_\theta}{dt} - \dfrac{1}{3AB\rho_w g}\left[c_{\theta\theta}\dfrac{u_\theta}{\left(r_p + u_r\right)^2} + \left(2c_{\theta r} + c_{\theta\theta}\right)\dfrac{1}{r_p + u_r}\dfrac{\partial u_\theta}{\partial r} \right. \\
\qquad \left. + c_{\theta r}\dfrac{\partial^2 u_\theta}{\partial r^2} + c_{\theta\theta}\dfrac{1}{\left(r_p + u_r\right)^2}\dfrac{\partial^2 u_\theta}{\partial \theta^2} + c_{\theta z}\dfrac{\partial^2 u_\theta}{\partial z^2} \right] = R_\theta \\
\dfrac{du_z}{dt} - \dfrac{1}{3AB\rho_w g}\left[c_{z\theta}\dfrac{1}{r_p + u_r}\dfrac{\partial u_z}{\partial r} + c_{zr}\dfrac{\partial^2 u_z}{\partial r^2} \right. \\
\qquad \left. + c_{z\theta}\dfrac{1}{\left(r_p + u_r\right)^2}\dfrac{\partial^2 u_z}{\partial \theta^2} + c_{zz}\dfrac{\partial^2 u_z}{\partial z^2} \right] = R_z
\end{cases}
\tag{4.81}
$$

in which c_{jk} $(j,k = r,\theta,z)$ are defined by (4.69).

4.5.3 Assumption of Mechanical Axisymmetry

In cylindrical coordinates, if it is assumed that there is no displacement or strain in tangential direction, then $u_\theta = 0$ and two equations of (4.70) reduce to

$$
\begin{cases}
\dfrac{du_r}{dt} - \dfrac{1}{3\rho_w g}\left[\dfrac{c_{rr}}{A_r B_r}\dfrac{\partial^2 u_r}{\partial r^2} \right. \\
\qquad \left. + \dfrac{c_{r\theta}}{A_\theta B_\theta}\dfrac{1}{r_p + u_r}\left(-\dfrac{u_r}{r_p + u_r} + \dfrac{\partial u_r}{\partial r} \right) + \dfrac{c_{rz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial r \partial z} \right] = R_r \\
\dfrac{du_z}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{zr}}{A_r B_r}\dfrac{\partial^2 u_r}{\partial z \partial r} + \dfrac{c_{z\theta}}{A_\theta B_\theta}\dfrac{1}{r_p + u_r}\dfrac{\partial u_r}{\partial z} + \dfrac{c_{zz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial z^2} \right) = R_z
\end{cases}
\tag{4.82}
$$

where R_r and R_z are defined by (4.68) and c_{jk} $(j,k = r,\theta,z)$ by (4.69).

Note that for equations of (4.82), the condition of no stress or hydraulic flow in tangential direction is not necessarily assumed.

If mechanical isotropy is further assumed, (4.82) simplifies to

$$\begin{cases} \dfrac{du_r}{dt} - \dfrac{1}{3AB\rho_w g}\left[c_{rr}\dfrac{\partial^2 u_r}{\partial r^2} + c_{r\theta}\dfrac{1}{r_p + u_r}\left(-\dfrac{u_r}{r_p + u_r} + \dfrac{\partial u_r}{\partial r} \right) + c_{rz}\dfrac{\partial^2 u_z}{\partial r \partial z} \right] = R_r \\ \dfrac{du_z}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{zr}\dfrac{\partial^2 u_r}{\partial z \partial r} + c_{z\theta}\dfrac{1}{r_p + u_r}\dfrac{\partial u_r}{\partial z} + c_{zz}\dfrac{\partial^2 u_z}{\partial z^2} \right) = R_z \end{cases} . \quad (4.83)$$

If both mechanical isotropy and irrotational deformation are assumed, (4.82) becomes

$$\begin{cases} \dfrac{du_r}{dt} - \dfrac{1}{3AB\rho_w g}\left[-c_{r\theta}\dfrac{u_r}{(r_p + u_r)^2} + c_{r\theta}\dfrac{1}{r_p + u_r}\dfrac{\partial u_r}{\partial r} \right. \\ \qquad \left. + c_{rr}\dfrac{\partial^2 u_r}{\partial r^2} + c_{r\theta}\dfrac{1}{(r_p + u_r)^2}\dfrac{\partial^2 u_r}{\partial \theta^2} + c_{rz}\dfrac{\partial^2 u_r}{\partial z^2} \right] = R_r \\ \dfrac{du_z}{dt} - \dfrac{1}{3AB\rho_w g}\left[c_{z\theta}\dfrac{1}{r_p + u_r}\dfrac{\partial u_z}{\partial r} + c_{zr}\dfrac{\partial^2 u_z}{\partial r^2} \right. \\ \qquad \left. + c_{z\theta}\dfrac{1}{(r_p + u_r)^2}\dfrac{\partial^2 u_z}{\partial \theta^2} + c_{zz}\dfrac{\partial^2 u_z}{\partial z^2} \right] = R_z \end{cases} . \quad (4.84)$$

Chapter 5 Incremental Formulation of Governing Equations

In this chapter, the incremental forms of the governing equations from Chapter 4 are formulated. All variables are expressed in terms of the respective increments from the prestressed states of the variables pertaining to time $t = 0^{-}$.

5.1 Basic Variables

Displacement of a point of interest P in general that is moving in space is given by Equation (4.5). Also, by definition, $\mathbf{u}_{\mathrm{P}} = \mathbf{0}$ for $t = 0$, (4.7). If displacement of P at prestressed time $t = 0^{-}$ is denoted in particular by $\mathbf{u}_{\mathrm{P}p}$ (= $\mathbf{0}$) and the increment from the prestressed value is denoted by $\mathit{\Delta}\mathbf{u}_{\mathrm{P}}$ [L] for any time t ($\geq 0^{-}$), then $\mathbf{u}_{\mathrm{P}} = \mathbf{u}_{\mathrm{P}p} + \mathit{\Delta}\mathbf{u}_{\mathrm{P}}$ or simply

$$\mathit{\Delta}\mathbf{u}_{\mathrm{P}} = \mathbf{u}_{\mathrm{P}} \ . \tag{5.1}$$

By the same token, incremental solid displacement is

$$\mathit{\Delta}\mathbf{u}_{\mathrm{s}} = \mathbf{u}_{\mathrm{s}} \ . \tag{5.2}$$

Since $\mathbf{u}_{\mathrm{P}}$ and $\mathbf{u}_{\mathrm{s}}$ are assumed functions of space $\mathbf{x}$ and time t as defined by (4.5) and (4.1), $\mathit{\Delta}\mathbf{u}_{\mathrm{P}}$ and $\mathit{\Delta}\mathbf{u}_{\mathrm{s}}$ [L] must be functions of $\mathbf{x}$ and t as well.

In both Cartesian and cylindrical coordinates, the components of normal strain of solid skeletal frame at prestressed time $t = 0^{-}$ are

$$\begin{cases} \varepsilon_{xp} = \varepsilon_{yp} = \varepsilon_{zp} = 0 \\ \varepsilon_{rp} = \varepsilon_{\theta p} = \varepsilon_{zp} = 0 \end{cases} \quad (t = 0^{-}) \tag{5.3}$$

for solid displacement, by definition of (4.3), is zero-valued at prestressed time and is no longer a function of space.

Write normal strain ε_{x} of (2.31) in incremental form as

$$\varepsilon_x = \varepsilon_{xp} + \Delta\varepsilon_x = \pm\frac{\partial}{\partial x}\left(u_{xp} + \Delta u_x\right)$$

which, by further taking (5.3) and (4.3) into account, leads to

$$\Delta\varepsilon_x = \varepsilon_x = \pm\frac{\partial\left(\Delta u_x\right)}{\partial x} \; .$$

In cylindrical coordinates, the incremental form of normal strain ε_θ (2.32) is

$$\varepsilon_\theta = \varepsilon_{\theta p} + \Delta\varepsilon_\theta = \pm\left[\frac{u_{rp} + \Delta u_r}{r_p + \Delta r} + \frac{1}{r_p + \Delta r}\frac{\partial}{\partial\theta}\left(u_{\theta p} + \Delta u_\theta\right)\right] \tag{5.4}$$

where r_p is the radial distance of solids at prestressed time t = 0^- and Δr is the increment of radial distance [L] at any time t (≥ 0^-). The increment of radial distance in fact is also the radial displacement, that is, $\Delta r = u_r$ and from (5.2), $u_r = \Delta u_r$. Hence, for solids,

$$\Delta r = \Delta u_r = u_r \; . \tag{5.5}$$

By taking (5.3), (4.3), (5.2) and (5.5) into account, (5.4) becomes

$$\Delta\varepsilon_\theta = \varepsilon_\theta = \pm\left[\frac{\Delta u_r}{r_p + \Delta u_r} + \frac{1}{r_p + \Delta u_r}\frac{\partial\left(\Delta u_\theta\right)}{\partial\theta}\right] \; .$$

Based on the above two examples of defining incremental normal strain in terms of incremental displacements, it can be summarized that for Cartesian coordinates,

$$\begin{cases} \Delta\varepsilon_x = \varepsilon_x = \pm\dfrac{\partial\left(\Delta u_x\right)}{\partial x} \\ \Delta\varepsilon_y = \varepsilon_y = \pm\dfrac{\partial\left(\Delta u_y\right)}{\partial y} \\ \Delta\varepsilon_z = \varepsilon_z = \pm\dfrac{\partial\left(\Delta u_z\right)}{\partial z} \end{cases} \tag{5.6}$$

and for cylindrical coordinates,

$$\begin{cases} \Delta\varepsilon_r = \varepsilon_r = \pm\dfrac{\partial(\Delta u_r)}{\partial r} \\ \Delta\varepsilon_\theta = \varepsilon_\theta = \pm\left[\dfrac{\Delta u_r}{r_p + \Delta u_r} + \dfrac{1}{r_p + \Delta u_r}\dfrac{\partial(\Delta u_\theta)}{\partial\theta}\right] \\ \Delta\varepsilon_z = \varepsilon_z = \pm\dfrac{\partial(\Delta u_z)}{\partial z} \end{cases} . \tag{5.7}$$

All incremental strains must be functions of space **x** and time t.

In a similar manner, total stresses σ_x, σ_y and σ_z and effective stresses σ'_x, σ'_y and σ'_z in Cartesian coordinates can be expressed in incremental form as

$$\begin{cases} \sigma_x = \sigma_{xp} + \Delta\sigma_x \\ \sigma_y = \sigma_{yp} + \Delta\sigma_y \\ \sigma_z = \sigma_{zp} + \Delta\sigma_z \end{cases} \tag{5.8}$$

and

$$\begin{cases} \sigma'_x = \sigma'_{xp} + \Delta\sigma'_x \\ \sigma'_y = \sigma'_{yp} + \Delta\sigma'_y \\ \sigma'_z = \sigma'_{zp} + \Delta\sigma'_z \end{cases} \tag{5.9}$$

where $\Delta\sigma_x$, $\Delta\sigma_y$ and $\Delta\sigma_y$ are the increments [M/Lt^2] of total stresses from the prestressed values σ_{xp}, σ_{yp} and σ_{zp}, respectively, and, likewise, $\Delta\sigma'_x$, $\Delta\sigma'_y$ and $\Delta\sigma'_y$ are the incremental effective stresses [M/Lt^2] and σ'_{xp}, σ'_{yp} and σ'_{zp} are the prestressed effective stresses. Note that, at t = 0$^-$, σ_{xp}, σ_{yp}, σ_{zp}, σ'_{xp}, σ'_{yp} and σ'_{zp} become only functions of space **x** and are independent of time t.

Recalling that in Cartesian coordinates, mean total and effective stresses are respectively defined as $\sigma_m = (\sigma_x + \sigma_y + \sigma_z)/3$ and $\sigma'_m = (\sigma'_x + \sigma'_y + \sigma'_z)/3$ ((2.49) and (2.50)), it then follows from (5.8) and (5.9) that

$$\begin{cases} \sigma_m = \sigma_{mp} + \Delta\sigma_m \\ \sigma'_m = \sigma'_{mp} + \Delta\sigma'_m \end{cases} \tag{5.10}$$

with

$$\begin{cases} \Delta\sigma_m = \frac{1}{3}\left(\Delta\sigma_x + \Delta\sigma_y + \Delta\sigma_z\right) \\ \Delta\sigma'_m = \frac{1}{3}\left(\Delta\sigma'_x + \Delta\sigma'_y + \Delta\sigma'_z\right) \end{cases} \tag{5.11}$$

and

$$\begin{cases} \sigma_{mp} = \frac{1}{3}\left(\sigma_{xp} + \sigma_{yp} + \sigma_{zp}\right) \\ \sigma'_{mp} = \frac{1}{3}\left(\sigma'_{xp} + \sigma'_{yp} + \sigma'_{zp}\right) \end{cases} \quad (\text{for } t = 0^-) \tag{5.12}$$

where $\Delta\sigma_m$ and $\Delta\sigma'_m$ are the incremental mean total and effective stresses [M/Lt^2], respectively, and σ_{mp} and σ'_{mp} are the corresponding prestressed values. Also, by definition, spatial gradients of mean stresses at prestressed time are denoted as

$$\begin{cases} \nabla\sigma_{mp} = \nabla\sigma_m \big|_{t=0} \\ \nabla\sigma'_{mp} = \nabla\sigma'_m \big|_{t=0} \end{cases} \quad (\text{for } t = 0^-) \,. \tag{5.13}$$

in which the notation " $\big|_{t=0}$ " accompanying a gradient emphasizes that the corresponding gradient must be evaluated at time $t = 0^-$.

5.2 Bulk Flux

Definition of bulk flux is given by (4.26) which for prestressed time $t = 0^-$, can be expressed as

$$\mathbf{q}_{bp} = n_p \mathbf{v}_{wp} + \left(1 - n_p\right)\mathbf{v}_{sp} \quad (\text{for } t = 0^-) \,. \tag{5.14}$$

For any time t ($\geq 0^-$), write (4.26) in incremental form as

$$\mathbf{q}_{bp} + \Delta\mathbf{q}_b = \left(n_p + \Delta n\right)\left(\mathbf{v}_{wp} + \Delta\mathbf{v}_w\right) + \left[1 - \left(n_p + \Delta n\right)\right]\left(\mathbf{v}_{sp} + \Delta\mathbf{v}_s\right) \,. \tag{5.15}$$

Subtracting Equation (5.14) from Equation (5.15) and rearranging terms yields

$$\Delta \mathbf{q}_b = \left(n_p + \Delta n\right)\Delta \mathbf{v}_w + \left[1 - \left(n_p + \Delta n\right)\right]\Delta \mathbf{v}_s - (\Delta n)\left(\mathbf{v}_{sp} - \mathbf{v}_{wp}\right) \tag{5.16}$$

or

$$\Delta \mathbf{q}_b = n(\Delta \mathbf{v}_w) + \left(1 - n\right)\Delta \mathbf{v}_s - (\Delta n)\left(\mathbf{v}_{sp} - \mathbf{v}_{wp}\right) \tag{5.17}$$

where n $(= n_p + \Delta n)$ is non-incremental porosity. "Δ" is a symbol employed to indicate the increment of a variable and is rather a derivative operator. This can be demonstrated below.

By recognizing the incremental expressions of water and solid velocities $\mathbf{v}_w = \mathbf{v}_{wp} + \Delta \mathbf{v}_w$ and $\mathbf{v}_s = \mathbf{v}_{sp} + \Delta \mathbf{v}_s$, Equation (5.16) can be rearranged to appear as

$$\Delta \mathbf{q}_b = \mathbf{v}_w(\Delta n) + n(\Delta \mathbf{v}_w) - \mathbf{v}_s(\Delta n) + \left(1 - n\right)\Delta \mathbf{v}_s + (\Delta n)\left(\Delta \mathbf{v}_s + \Delta \mathbf{v}_w\right) . \tag{5.18}$$

The derivative of bulk flux, as defined by (4.26), is

$$d\mathbf{q}_b = \mathbf{v}_w dn + n\, d\mathbf{v}_w - \mathbf{v}_s dn + \left(1 - n\right) d\mathbf{v}_s \tag{5.19}$$

where "d" is a derivative operator. If all increments ("Δ") of variables in (5.18) are perceived as the derivatives or the infinitesimal changes of the variables, the very last term would appear as $dn\left(d\mathbf{v}_s + d\mathbf{v}_w\right)$ which represents the product of an infinitesimal change of porosity dn and the sum of two infinitesimal vector changes $\left(d\mathbf{v}_s + d\mathbf{v}_w\right)$. As a result, $dn\left(d\mathbf{v}_s + d\mathbf{v}_w\right)$ is negligibly small. Therefore, Equations (5.18) and (5.19) are identical if and only if all increments of variables are infinitesimally small. However, an increment is meant to be a discrete variable change which may not necessarily be infinitesimal.

5.3 Stress Equilibrium and Moment Balance Equations

The stress equilibrium equations of (4.40) can be rewritten for any time t ($\geq 0^-$) in incremental form, for Cartesian coordinates, as

$$\begin{cases} \frac{\partial}{\partial x}\left(\sigma_{xp}+\Delta\sigma_x\right)\Big|_t+\frac{\partial}{\partial y}\left(\tau_{yxp}+\Delta\tau_{yx}\right)\Big|_t+\frac{\partial}{\partial z}\left(\tau_{zxp}+\Delta\tau_{zx}\right)\Big|_t=0 \\ \frac{\partial}{\partial x}\left(\tau_{xyp}+\Delta\tau_{xy}\right)\Big|_t+\frac{\partial}{\partial y}\left(\sigma_{yp}+\Delta\sigma_y\right)\Big|_t+\frac{\partial}{\partial z}\left(\tau_{zyp}+\Delta\tau_{zy}\right)\Big|_t=0 \\ \frac{\partial}{\partial x}\left(\tau_{xzp}+\Delta\tau_{xz}\right)\Big|_t+\frac{\partial}{\partial y}\left(\tau_{yzp}+\Delta\tau_{yz}\right)\Big|_t+\frac{\partial}{\partial z}\left(\sigma_{zp}+\Delta\sigma_z\right)\Big|_t\pm\left(f_{zp}+\Delta f_z\right)=0 \end{cases} \qquad (5.20)$$

in which $\Delta\sigma_j$ and $\Delta\tau_{jk}$ (j,k = x,y,z) are the increments of total stresses [M/Lt^2] at any time t ($\geq 0^-$) from the corresponding values σ_{jp} and τ_{jkp} [M/Lt^2] (j,k = x,y,z) at prestressed time ($t = 0^-$) and f_{zp} and Δf_z are the prestressed body force and its increment [M/L^2t^2], respectively. The notation "$\big|_t$" in (5.20) indicates that the associated partial derivative is evaluated at time t ($\geq 0^-$). The stress variables σ_{jp} and τ_{jkp} (j,k = x,y,z) associates with a specific time ($t = 0^-$) are functions of space **x** only.

Similarly, equations of (4.40) can be written particularly for the prestressed time as

$$\begin{cases} \frac{\partial\sigma_x}{\partial x}\Big|_{t=0}+\frac{\partial\tau_{yx}}{\partial y}\Big|_{t=0}+\frac{\partial\tau_{zx}}{\partial z}\Big|_{t=0}=0 \\ \frac{\partial\tau_{xy}}{\partial x}\Big|_{t=0}+\frac{\partial\sigma_y}{\partial y}\Big|_{t=0}+\frac{\partial\tau_{zy}}{\partial z}\Big|_{t=0}=0 \\ \frac{\partial\tau_{xz}}{\partial x}\Big|_{t=0}+\frac{\partial\tau_{yz}}{\partial y}\Big|_{t=0}+\frac{\partial\sigma_z}{\partial z}\Big|_{t=0}\pm f_{zp}=0 \end{cases} \qquad (\text{for } t=0^-) \qquad (5.21)$$

in which all partial derivatives must be evaluated at $t = 0^-$ as indicated by "$\big|_{t=0}$".

(5.20) and (5.21) both consist of three simultaneous equations which represent stress equilibrium in x, y and z directions, respectively. For z direction, subtraction of the third equation of (5.21) from that of (5.20) leads to

$$\left(\left.\frac{\partial\tau_{xzp}}{\partial x}\right|_t-\left.\frac{\partial\tau_{xz}}{\partial x}\right|_{t=0}\right)+\left.\frac{\partial(\Delta\tau_{xz})}{\partial x}\right|_t+\left(\left.\frac{\partial\tau_{yzp}}{\partial y}\right|_t-\left.\frac{\partial\tau_{yz}}{\partial y}\right|_{t=0}\right)$$
$$+\left.\frac{\partial(\Delta\tau_{yz})}{\partial y}\right|_t+\left(\left.\frac{\partial\sigma_{zp}}{\partial z}\right|_t-\left.\frac{\partial\sigma_z}{\partial z}\right|_{t=0}\right)+\left.\frac{\partial(\Delta\sigma_z)}{\partial z}\right|_t\pm\Delta f_z=0 \quad . \quad (5.22)$$

By further recognizing that

$$\frac{\partial\tau_{xzp}}{\partial x}=\left.\frac{\partial\tau_{xz}}{\partial x}\right|_{t=0}, \quad \frac{\partial\tau_{yzp}}{\partial y}=\left.\frac{\partial\tau_{yz}}{\partial y}\right|_{t=0} \quad \text{and} \quad \frac{\partial\sigma_{zp}}{\partial z}=\left.\frac{\partial\sigma_z}{\partial z}\right|_{t=0},$$

Equation (5.22) can be simplified to have only four remaining terms. The incremental forms of the x and y components of equilibrium equations of (4.40) can be formulated in a similar manner. In summary,

$$\begin{cases}\dfrac{\partial(\Delta\sigma_x)}{\partial x}+\dfrac{\partial(\Delta\tau_{yx})}{\partial y}+\dfrac{\partial(\Delta\tau_{zx})}{\partial z}=0\\ \dfrac{\partial(\Delta\tau_{xy})}{\partial x}+\dfrac{\partial(\Delta\sigma_y)}{\partial y}+\dfrac{\partial(\Delta\tau_{zy})}{\partial z}=0\\ \dfrac{\partial(\Delta\tau_{xz})}{\partial x}+\dfrac{\partial(\Delta\tau_{yz})}{\partial y}+\dfrac{\partial(\Delta\sigma_z)}{\partial z}\pm\Delta f_z=0\end{cases} \quad (5.23)$$

where Δf_z is the incremental body force and is usually assumed negligible.

The moment balance equations of (4.38) in incremental form can be formulated using the same methodology and are summarized as

$$\Delta\tau_{xy}=\Delta\tau_{yx}, \quad \Delta\tau_{xz}=\Delta\tau_{zx} \quad \text{and} \quad \Delta\tau_{yz}=\Delta\tau_{zy} \ . \quad (5.24)$$

Note that the principle of moment balance as represented by (5.24) has not been invoked in the stress equilibrium equations of (5.23).

In cylindrical coordinates, the incremental stress equilibrium equations should appear as

$$\begin{cases} \dfrac{\partial(\Delta\sigma_r)}{\partial r}+\dfrac{\Delta\sigma_r-\Delta\sigma_\theta}{r_{Pp}+\Delta u_{Pr}}+\dfrac{1}{r_{Pp}+\Delta u_{Pr}}\dfrac{\partial(\Delta\tau_{\theta r})}{\partial\theta} \\ \qquad +\dfrac{\partial(\Delta\tau_{zr})}{\partial z}-\dfrac{\Delta u_{Pr}}{r_{Pp}\left(r_{Pp}+\Delta u_{Pr}\right)}\left(\sigma_{rp}-\sigma_{\theta p}+\dfrac{\partial\tau_{\theta rp}}{\partial\theta}\right)=0 \\ \dfrac{\partial(\Delta\tau_{r\theta})}{\partial r}+\dfrac{\Delta\tau_{r\theta}+\Delta\tau_{\theta r}}{r_{Pp}+\Delta u_{Pr}}+\dfrac{1}{r_{Pp}+\Delta u_{Pr}}\dfrac{\partial(\Delta\sigma_\theta)}{\partial\theta} \\ \qquad +\dfrac{\partial(\Delta\tau_{z\theta})}{\partial z}-\dfrac{\Delta u_{Pr}}{r_{Pp}\left(r_{Pp}+\Delta u_{Pr}\right)}\left(\tau_{r\theta p}+\tau_{\theta rp}+\dfrac{\partial\sigma_{\theta p}}{\partial\theta}\right)=0 \\ \dfrac{\partial(\Delta\tau_{rz})}{\partial r}+\dfrac{\Delta\tau_{rz}}{r_{Pp}+\Delta u_{Pr}}+\dfrac{1}{r_{Pp}+\Delta u_{Pr}}\dfrac{\partial(\Delta\tau_{\theta z})}{\partial\theta} \\ \qquad +\dfrac{\partial(\Delta\sigma_z)}{\partial z}\pm\Delta f_z-\dfrac{\Delta u_{Pr}}{r_{Pp}\left(r_{Pp}+\Delta u_{Pr}\right)}\left(\tau_{rzp}+\dfrac{\partial\tau_{\theta zp}}{\partial\theta}\right)=0 \end{cases} \tag{5.25}$$

and the moment balance equations as

$$\Delta\tau_{r\theta}=\Delta\tau_{\theta r},\quad \Delta\tau_{rz}=\Delta\tau_{zr}\quad \text{and}\quad \Delta\tau_{\theta z}=\Delta\tau_{z\theta}\ . \tag{5.26}$$

In (5.25), r_{Pp} and Δu_{Pr} are respectively the radial distance at prestressed time and the incremental radial displacement of the control volume centroid.

5.4 Poroviscosity Constitutive Law

Written for prestressed time, the poroviscosity constitutive equations of (4.19) are denoted in terms of prestressed variables in Cartesian coordinates as

$$\begin{cases} \sigma'_{xp}=\dfrac{1}{B_{xx}}\eta_{xp}\dot{\varepsilon}_{xp}+\dfrac{1}{B_{xy}}\eta_{yp}\dot{\varepsilon}_{yp}+\dfrac{1}{B_{xz}}\eta_{zp}\dot{\varepsilon}_{zp} \\ \sigma'_{yp}=\dfrac{1}{B_{yx}}\eta_{xp}\dot{\varepsilon}_{xp}+\dfrac{1}{B_{yy}}\eta_{yp}\dot{\varepsilon}_{yp}+\dfrac{1}{B_{yz}}\eta_{zp}\dot{\varepsilon}_{zp} \\ \sigma'_{zp}=\dfrac{1}{B_{zx}}\eta_{xp}\dot{\varepsilon}_{xp}+\dfrac{1}{B_{zy}}\eta_{yp}\dot{\varepsilon}_{yp}+\dfrac{1}{B_{zz}}\eta_{zp}\dot{\varepsilon}_{zp} \end{cases} \quad (\text{for } t=0^-) \tag{5.27}$$

where B_{jk} (j,k = x,y,z) are assumed to be constant coefficients. For any time t ($\geq 0^-$), the three simultaneous equations of (4.19) can be expressed in incremental form. Consider, for now, the x component of (4.19) and rewrite σ'_x as

$$\begin{aligned}\sigma'_x = \sigma'_{xp} + \Delta\sigma'_x = \frac{1}{B_{xx}}\left(\eta_{xp}+\Delta\eta_x\right)\left(\dot{\varepsilon}_{xp}+\Delta\dot{\varepsilon}_x\right) \\ + \frac{1}{B_{xy}}\left(\eta_{yp}+\Delta\eta_y\right)\left(\dot{\varepsilon}_{yp}+\Delta\dot{\varepsilon}_y\right) + \frac{1}{B_{xz}}\left(\eta_{zp}+\Delta\eta_z\right)\left(\dot{\varepsilon}_{zp}+\Delta\dot{\varepsilon}_z\right)\end{aligned} \quad . \quad (5.28)$$

The expression of $\Delta\sigma'_x$ can be found by subtracting the first equation of (5.27) from Equation (5.28) and rearranging terms as

$$\begin{aligned}\Delta\sigma'_x = \sigma'_x - \sigma'_{xp} = \frac{1}{B_{xx}}\left[(\Delta\eta_x)\dot{\varepsilon}_{xp} + \left(\eta_{xp}+\Delta\eta_x\right)\Delta\dot{\varepsilon}_x\right] + \frac{1}{B_{xy}}\left[(\Delta\eta_y)\dot{\varepsilon}_{yp}\right. \\ \left. + \left(\eta_{yp}+\Delta\eta_y\right)\Delta\dot{\varepsilon}_y\right] + \frac{1}{B_{xz}}\left[(\Delta\eta_z)\dot{\varepsilon}_{zp} + \left(\eta_{zp}+\Delta\eta_z\right)\Delta\dot{\varepsilon}_z\right]\end{aligned}$$

$$\Rightarrow \begin{aligned}\Delta\sigma'_x = \frac{1}{B_{xx}}\left(\eta_{xp}+\Delta\eta_x\right)\Delta\dot{\varepsilon}_x + \frac{1}{B_{xy}}\left(\eta_{yp}+\Delta\eta_y\right)\Delta\dot{\varepsilon}_y + \frac{1}{B_{xz}}\left(\eta_{zp}+\Delta\eta_z\right)\Delta\dot{\varepsilon}_z \\ + \left[\frac{1}{B_{xx}}(\Delta\eta_x)\dot{\varepsilon}_{xp} + \frac{1}{B_{xy}}(\Delta\eta_y)\dot{\varepsilon}_{yp} + \frac{1}{B_{xz}}(\Delta\eta_z)\dot{\varepsilon}_{zp}\right]\end{aligned} \quad . \quad (5.29)$$

Similar expressions arise as well for $\Delta\sigma'_y$ and $\Delta\sigma'_z$, namely,

$$\begin{cases}\Delta\sigma'_y = \frac{1}{B_{yx}}\left(\eta_{xp}+\Delta\eta_x\right)\Delta\dot{\varepsilon}_x + \frac{1}{B_{yy}}\left(\eta_{yp}+\Delta\eta_y\right)\Delta\dot{\varepsilon}_y + \frac{1}{B_{yz}}\left(\eta_{zp}+\Delta\eta_z\right)\Delta\dot{\varepsilon}_z \\ \qquad + \left[\frac{1}{B_{yx}}(\Delta\eta_x)\dot{\varepsilon}_{xp} + \frac{1}{B_{yy}}(\Delta\eta_y)\dot{\varepsilon}_{yp} + \frac{1}{B_{yz}}(\Delta\eta_z)\dot{\varepsilon}_{zp}\right] \\ \Delta\sigma'_z = \frac{1}{B_{zx}}\left(\eta_{xp}+\Delta\eta_x\right)\Delta\dot{\varepsilon}_x + \frac{1}{B_{zy}}\left(\eta_{yp}+\Delta\eta_y\right)\Delta\dot{\varepsilon}_y + \frac{1}{B_{zz}}\left(\eta_{zp}+\Delta\eta_z\right)\Delta\dot{\varepsilon}_z \\ \qquad + \left[\frac{1}{B_{zx}}(\Delta\eta_x)\dot{\varepsilon}_{xp} + \frac{1}{B_{zy}}(\Delta\eta_y)\dot{\varepsilon}_{yp} + \frac{1}{B_{zz}}(\Delta\eta_z)\dot{\varepsilon}_{zp}\right]\end{cases} \quad . \quad (5.30)$$

Incremental mean effective stress $\Delta\sigma'_m$ is the average of incremental effective stresses $\Delta\sigma'_x$, $\Delta\sigma'_y$ and $\Delta\sigma'_z$, (5.11). Adding the three expressions of (5.29) and (5.30) together and taking the average yields

$$\Delta\sigma'_m = \frac{1}{3}\left(\Delta\sigma'_x + \Delta\sigma'_y + \Delta\sigma'_z\right) = \frac{1}{3}\left\{\frac{1}{B_x}\left[\left(\eta_{xp} + \Delta\eta_x\right)\Delta\dot{\varepsilon}_x + (\Delta\eta_x)\dot{\varepsilon}_{xp}\right]\right.$$
$$\left. + \frac{1}{B_y}\left[\left(\eta_{yp} + \Delta\eta_y\right)\Delta\dot{\varepsilon}_y + (\Delta\eta_y)\dot{\varepsilon}_{yp}\right] + \frac{1}{B_z}\left[\left(\eta_{zp} + \Delta\eta_z\right)\Delta\dot{\varepsilon}_z + (\Delta\eta_z)\dot{\varepsilon}_{zp}\right]\right\} \tag{5.31}$$

where B_x, B_y and B_z are constant coefficients defined by (4.50). From Equation (5.31), gradient of the incremental mean effective stress is

$$\nabla(\Delta\sigma'_m) = \frac{1}{3}\left(\frac{1}{B_x}\left\{\nabla\left[\left(\eta_{xp} + \Delta\eta_x\right)\Delta\dot{\varepsilon}_x\right] + \nabla\left[(\Delta\eta_x)\dot{\varepsilon}_{xp}\right]\right\} + \frac{1}{B_y}\left\{\nabla\left[\left(\eta_{yp} + \Delta\eta_y\right)\Delta\dot{\varepsilon}_y\right]\right.\right.$$
$$\left.\left. + \nabla\left[(\Delta\eta_y)\dot{\varepsilon}_{yp}\right]\right\} + \frac{1}{B_z}\left\{\nabla\left[\left(\eta_{zp} + \Delta\eta_z\right)\Delta\dot{\varepsilon}_z\right] + \nabla\left[(\Delta\eta_z)\dot{\varepsilon}_{zp}\right]\right\}\right) \tag{5.32}$$

of which all gradient terms on the right-hand side are to be evaluated separately.

With Equation (5.6), the anisotropic viscosities of (3.62) can be expressed as

$$\begin{cases} \eta_x = \eta_{xp} + \Delta\eta_x = \eta_{xp} e^{\mp\left(\frac{\Delta\varepsilon_x}{A_x}\right)} \\ \eta_y = \eta_{yp} + \Delta\eta_y = \eta_{yp} e^{\mp\left(\frac{\Delta\varepsilon_y}{A_y}\right)} \\ \eta_z = \eta_{zp} + \Delta\eta_z = \eta_{zp} e^{\mp\left(\frac{\Delta\varepsilon_z}{A_z}\right)} \end{cases} . \tag{5.33}$$

It follows immediately that

$$\begin{cases} \Delta\eta_x = \eta_{xp} e^{\mp\left(\frac{\Delta\varepsilon_x}{A_x}\right)} - \eta_{xp} \\ \Delta\eta_y = \eta_{yp} e^{\mp\left(\frac{\Delta\varepsilon_y}{A_y}\right)} - \eta_{yp} \\ \Delta\eta_z = \eta_{zp} e^{\mp\left(\frac{\Delta\varepsilon_z}{A_z}\right)} - \eta_{zp} \end{cases} \tag{5.34}$$

or

$$e^{\mp\left(\frac{\Delta\varepsilon_x}{A_x}\right)} = \frac{\eta_x}{\eta_{xp}} , \quad e^{\mp\left(\frac{\Delta\varepsilon_y}{A_y}\right)} = \frac{\eta_y}{\eta_{yp}} \quad \text{and} \quad e^{\mp\left(\frac{\Delta\varepsilon_z}{A_z}\right)} = \frac{\eta_z}{\eta_{zp}} . \tag{5.35}$$

Using the expression of $\eta_{xp}+\varDelta\eta_x$ in (5.33),

$$\nabla\left[\left(\eta_{xp}+\varDelta\eta_x\right)\varDelta\dot{\varepsilon}_x\right]=\nabla\left\{\left[\eta_{xp}e^{\mp\left(\frac{\varDelta\varepsilon_x}{A_x}\right)}\right]\varDelta\dot{\varepsilon}_x\right\}$$

$$=(\varDelta\dot{\varepsilon}_x)e^{\mp\left(\frac{\varDelta\varepsilon_x}{A_x}\right)}\nabla\eta_{xp}+\eta_{xp}e^{\mp\left(\frac{\varDelta\varepsilon_x}{A_x}\right)}\nabla(\varDelta\dot{\varepsilon}_x)+\eta_{xp}(\varDelta\dot{\varepsilon}_x)e^{\mp\left(\frac{\varDelta\varepsilon_x}{A_x}\right)}\left[\mp\frac{1}{A_x}\nabla(\varDelta\varepsilon_x)\right].$$

$$=(\varDelta\dot{\varepsilon}_x)\frac{\eta_{xp}+\varDelta\eta_x}{\eta_{xp}}\nabla\eta_{xp}+\left(\eta_{xp}+\varDelta\eta_x\right)\nabla(\varDelta\dot{\varepsilon}_x)\mp\frac{1}{A_x}\left(\eta_{xp}+\varDelta\eta_x\right)(\varDelta\dot{\varepsilon}_x)\nabla(\varDelta\varepsilon_x)$$

Also, using the expression of $\varDelta\eta_x$ in (5.34),

$$\nabla\left[(\varDelta\eta_x)\dot{\varepsilon}_{xp}\right]=\nabla\left\{\left[\eta_{xp}e^{\mp\left(\frac{\varDelta\varepsilon_x}{A_x}\right)}-\eta_{xp}\right]\dot{\varepsilon}_{xp}\right\}=\nabla\left[\eta_{xp}\dot{\varepsilon}_{xp}e^{\mp\left(\frac{\varDelta\varepsilon_x}{A_x}\right)}-\eta_{xp}\dot{\varepsilon}_{xp}\right]$$

$$=\left\{\dot{\varepsilon}_{xp}e^{\mp\left(\frac{\varDelta\varepsilon_x}{A_x}\right)}\nabla\eta_{xp}+\eta_{xp}e^{\mp\left(\frac{\varDelta\varepsilon_x}{A_x}\right)}\nabla\dot{\varepsilon}_{xp}+\eta_{xp}\dot{\varepsilon}_{xp}e^{\mp\left(\frac{\varDelta\varepsilon_x}{A_x}\right)}\left[\mp\frac{1}{A_x}\nabla(\varDelta\varepsilon_x)\right]\right\}$$
$$-\left(\dot{\varepsilon}_{xp}\nabla\eta_{xp}+\eta_{xp}\nabla\dot{\varepsilon}_{xp}\right).$$

$$=\dot{\varepsilon}_{xp}\frac{\eta_{xp}+\varDelta\eta_x}{\eta_{xp}}\nabla\eta_{xp}+\left(\eta_{xp}+\varDelta\eta_x\right)\nabla\dot{\varepsilon}_{xp}\mp\frac{1}{A_x}\left(\eta_{xp}+\varDelta\eta_x\right)\dot{\varepsilon}_{xp}\nabla(\varDelta\varepsilon_x)$$
$$-\dot{\varepsilon}_{xp}\nabla\eta_{xp}-\eta_{xp}\nabla\dot{\varepsilon}_{xp}$$

$$=\dot{\varepsilon}_{xp}\left(\frac{\eta_{xp}+\varDelta\eta_x}{\eta_{xp}}-1\right)\nabla\eta_{xp}+(\varDelta\eta_x)\nabla\dot{\varepsilon}_{xp}\mp\frac{1}{A_x}\left(\eta_{xp}+\varDelta\eta_x\right)\dot{\varepsilon}_{xp}\nabla(\varDelta\varepsilon_x)$$

Similar gradient terms can be evaluated in the same manner. In summary,

$$\begin{cases}\nabla\left[\left(\eta_{xp}+\varDelta\eta_x\right)\varDelta\dot{\varepsilon}_x\right]=\frac{1}{A_x}\left(\eta_{xp}+\varDelta\eta_x\right)(\varDelta\dot{\varepsilon}_x)\left[\frac{A_x}{\eta_{xp}}\nabla\eta_{xp}+\frac{A_x}{\varDelta\dot{\varepsilon}_x}\nabla(\varDelta\dot{\varepsilon}_x)\mp\nabla(\varDelta\varepsilon_x)\right]\\ \nabla\left[\left(\eta_{yp}+\varDelta\eta_y\right)\varDelta\dot{\varepsilon}_y\right]=\frac{1}{A_y}\left(\eta_{yp}+\varDelta\eta_y\right)(\varDelta\dot{\varepsilon}_y)\left[\frac{A_y}{\eta_{yp}}\nabla\eta_{yp}+\frac{A_y}{\varDelta\dot{\varepsilon}_y}\nabla(\varDelta\dot{\varepsilon}_y)\mp\nabla(\varDelta\varepsilon_y)\right]\\ \nabla\left[\left(\eta_{zp}+\varDelta\eta_z\right)\varDelta\dot{\varepsilon}_z\right]=\frac{1}{A_z}\left(\eta_{zp}+\varDelta\eta_z\right)(\varDelta\dot{\varepsilon}_z)\left[\frac{A_z}{\eta_{zp}}\nabla\eta_{xp}+\frac{A_z}{\varDelta\dot{\varepsilon}_z}\nabla(\varDelta\dot{\varepsilon}_z)\mp\nabla(\varDelta\varepsilon_z)\right]\end{cases}\quad(5.36)$$

and

$$
\begin{cases}
\nabla\left[(\Delta\eta_{x})\dot{\varepsilon}_{xp}\right]=\dfrac{1}{A_{x}}\left(\eta_{xp}+\Delta\eta_{x}\right)\dot{\varepsilon}_{xp}\left[A_{x}\left(\dfrac{1}{\eta_{xp}}-\dfrac{1}{\eta_{xp}+\Delta\eta_{x}}\right)\nabla\eta_{xp}\right. \\
\qquad\qquad\left.+\dfrac{\Delta\eta_{x}}{\eta_{xp}+\Delta\eta_{x}}\dfrac{A_{x}}{\dot{\varepsilon}_{xp}}\nabla\dot{\varepsilon}_{xp}\mp\nabla(\Delta\varepsilon_{x})\right] \\
\nabla\left[(\Delta\eta_{y})\dot{\varepsilon}_{yp}\right]=\dfrac{1}{A_{y}}\left(\eta_{yp}+\Delta\eta_{y}\right)\dot{\varepsilon}_{yp}\left[A_{y}\left(\dfrac{1}{\eta_{yp}}-\dfrac{1}{\eta_{yp}+\Delta\eta_{y}}\right)\nabla\eta_{yp}\right. \\
\qquad\qquad\left.+\dfrac{\Delta\eta_{y}}{\eta_{yp}+\Delta\eta_{y}}\dfrac{A_{y}}{\dot{\varepsilon}_{yp}}\nabla\dot{\varepsilon}_{yp}\mp\nabla(\Delta\varepsilon_{y})\right] \\
\nabla\left[(\Delta\eta_{z})\dot{\varepsilon}_{zp}\right]=\dfrac{1}{A_{z}}\left(\eta_{zp}+\Delta\eta_{z}\right)\dot{\varepsilon}_{zp}\left[A_{z}\left(\dfrac{1}{\eta_{zp}}-\dfrac{1}{\eta_{zp}+\Delta\eta_{z}}\right)\nabla\eta_{zp}\right. \\
\qquad\qquad\left.+\dfrac{\Delta\eta_{z}}{\eta_{zp}+\Delta\eta_{z}}\dfrac{A_{z}}{\dot{\varepsilon}_{zp}}\nabla\dot{\varepsilon}_{zp}\mp\nabla(\Delta\varepsilon_{z})\right]
\end{cases}
. \quad (5.37)
$$

Substitution of all gradient expressions of (5.36) and (5.37) simultaneously into Equation (5.32) yields

$$
\begin{aligned}
\nabla(\Delta\sigma'_{m})=\frac{1}{3}\Bigg(&\frac{1}{A_{x}B_{x}}\left(\eta_{xp}+\Delta\eta_{x}\right)\Bigg\{(\Delta\dot{\varepsilon}_{x})\left[\frac{A_{x}}{\eta_{xp}}\nabla\eta_{xp}+\frac{A_{x}}{\Delta\dot{\varepsilon}_{x}}\nabla(\Delta\dot{\varepsilon}_{x})\mp\nabla(\Delta\varepsilon_{x})\right] \\
&+\dot{\varepsilon}_{xp}\left[A_{x}\left(\frac{1}{\eta_{xp}}-\frac{1}{\eta_{xp}+\Delta\eta_{x}}\right)\nabla\eta_{xp}+\frac{\Delta\eta_{x}}{\eta_{xp}+\Delta\eta_{x}}\frac{A_{x}}{\dot{\varepsilon}_{xp}}\nabla\dot{\varepsilon}_{xp}\mp\nabla(\Delta\varepsilon_{x})\right]\Bigg\} \\
&+\frac{1}{A_{y}B_{y}}\left(\eta_{yp}+\Delta\eta_{y}\right)\Bigg\{(\Delta\dot{\varepsilon}_{y})\left[\frac{A_{y}}{\eta_{yp}}\nabla\eta_{yp}+\frac{A_{y}}{\Delta\dot{\varepsilon}_{y}}\nabla(\Delta\dot{\varepsilon}_{y})\mp\nabla(\Delta\varepsilon_{y})\right] \\
&+\dot{\varepsilon}_{yp}\left[A_{y}\left(\frac{1}{\eta_{yp}}-\frac{1}{\eta_{yp}+\Delta\eta_{y}}\right)\nabla\eta_{yp}+\frac{\Delta\eta_{y}}{\eta_{yp}+\Delta\eta_{y}}\frac{A_{y}}{\dot{\varepsilon}_{yp}}\nabla\dot{\varepsilon}_{yp}\mp\nabla(\Delta\varepsilon_{y})\right]\Bigg\} \\
&+\frac{1}{A_{z}B_{z}}\left(\eta_{zp}+\Delta\eta_{z}\right)\Bigg\{(\Delta\dot{\varepsilon}_{z})\left[\frac{A_{z}}{\eta_{zp}}\nabla\eta_{xp}+\frac{A_{z}}{\Delta\dot{\varepsilon}_{z}}\nabla(\Delta\dot{\varepsilon}_{z})\mp\nabla(\Delta\varepsilon_{z})\right] \\
&+\dot{\varepsilon}_{zp}\left[A_{z}\left(\frac{1}{\eta_{zp}}-\frac{1}{\eta_{zp}+\Delta\eta_{z}}\right)\nabla\eta_{zp}+\frac{\Delta\eta_{z}}{\eta_{zp}+\Delta\eta_{z}}\frac{A_{z}}{\dot{\varepsilon}_{zp}}\nabla\dot{\varepsilon}_{zp}\mp\nabla(\Delta\varepsilon_{z})\right]\Bigg\}\Bigg)
\end{aligned}
\quad (5.38)
$$

which is the gradient of incremental mean effective stress in Cartesian coordinates.

In cylindrical coordinates r θ z, the same expression as (5.38) will arise with the exception that the subscripts “x” and “y” are replaced by “r” and “θ”, respectively.

5.5 Equation of Motion and Governing Equations

If all variables of the equation of motion, (4.44) with (4.45), are expressed in terms of the incremental quantities from that of the prestressed state when $t = 0^-$, the equation of motion at any time t ($\geq 0^-$) can be written as

$$\frac{d}{dt}\left(\mathbf{u}_{sp} + \varDelta\mathbf{u}_s\right)\bigg|_t + \frac{\mathbf{K}_p + \varDelta\mathbf{K}}{\rho_w g}\nabla\left(\sigma'_{mp} + \varDelta\sigma'_m\right)\bigg|_t = \left(\mathbf{q}_{bp} + \varDelta\mathbf{q}_b\right) + \left(\mathbf{K}_p + \varDelta\mathbf{K}\right)\mathbf{k} + \frac{\mathbf{K}_p + \varDelta\mathbf{K}}{\rho_w g}\nabla\left(\sigma_{mp} + \varDelta\sigma_m\right)\bigg|_t \tag{5.39}$$

in which all derivatives must be evaluated at time t ($\geq 0^-$). For time $t = 0^-$, the equation of motion (4.44) is written as

$$\frac{d\mathbf{u}_s}{dt}\bigg|_{t=0} + \frac{\mathbf{K}_p}{\rho_w g}\nabla\sigma'_m\bigg|_{t=0} = \mathbf{q}_{bp} + \mathbf{K}_p\mathbf{k} + \frac{\mathbf{K}_p}{\rho_w g}\nabla\sigma_m\bigg|_{t=0} \qquad (\text{for } t = 0^-)\ . \tag{5.40}$$

Subtracting Equation (5.40) from Equation (5.39) yields

$$\left(\frac{d\mathbf{u}_{sp}}{dt}\bigg|_t - \frac{d\mathbf{u}_s}{dt}\bigg|_{t=0}\right) + \frac{d(\varDelta\mathbf{u}_s)}{dt}\bigg|_t + \frac{\mathbf{K}_p}{\rho_w g}\left(\nabla\sigma'_{mp}\Big|_t - \nabla\sigma'_m\Big|_{t=0}\right) + \frac{\varDelta\mathbf{K}}{\rho_w g}\nabla\sigma'_{mp}\Big|_t + \frac{\mathbf{K}_p + \varDelta\mathbf{K}}{\rho_w g}\nabla(\varDelta\sigma'_m)\Big|_t = \varDelta\mathbf{q}_b + (\varDelta\mathbf{K})\mathbf{k} + \frac{\mathbf{K}_p}{\rho_w g}\left(\nabla\sigma_{mp}\Big|_t - \nabla\sigma_m\Big|_{t=0}\right) + \frac{\varDelta\mathbf{K}}{\rho_w g}\nabla\sigma_{mp}\Big|_t + \frac{\mathbf{K}_p + \varDelta\mathbf{K}}{\rho_w g}\nabla(\varDelta\sigma_m)\Big|_t\ . \tag{5.41}$$

The solid displacement $\mathbf{u}_{sp}$ at prestressed time is no longer dependent on time t, according to (4.3), and is only a function of space $\mathbf{x}$. Hence, the total derivative of $\mathbf{u}_{sp}$ is, by definition, denoted as

$$\frac{d\mathbf{u}_{sp}}{dt} = \frac{d\mathbf{u}_s}{dt}\bigg|_{t=0} \qquad (\text{for } t = 0^-) \ . \tag{5.42}$$

With (5.13) and (5.42), Equation (5.43) simplifies to

$$\frac{d(\Delta\mathbf{u}_s)}{dt}\bigg|_t + \frac{\mathbf{K}_p+\Delta\mathbf{K}}{\rho_w g}\nabla(\Delta\sigma'_m)\Big|_t - \Delta\mathbf{q}_b - \frac{\mathbf{K}_p+\Delta\mathbf{K}}{\rho_w g}\nabla(\Delta\sigma_m)\Big|_t$$

$$= (\Delta\mathbf{K})\left[\mathbf{k} + \frac{1}{\rho_w g}\nabla\left(\sigma_{mp} - \sigma'_{mp}\right)\right] = (\Delta\mathbf{K})\left[\nabla z_P + \frac{1}{\rho_w g}\nabla\left(\mp p_p\right)\right]$$

$$= (\Delta\mathbf{K})\nabla\left(z_{Pp} \mp \frac{p_p}{\rho_w g}\right)$$

$$\Rightarrow \quad \frac{d(\Delta\mathbf{u}_s)}{dt}\bigg|_t + \frac{\mathbf{K}_p+\Delta\mathbf{K}}{\rho_w g}\nabla(\Delta\sigma'_m)\Big|_t = \Delta\mathbf{q}_b + \frac{\mathbf{K}_p+\Delta\mathbf{K}}{\rho_w g}\nabla(\Delta\sigma_m)\Big|_t + (\Delta\mathbf{K})\nabla h_p \tag{5.43}$$

where

$$\nabla h_p = \nabla\left(z_{Pp} \mp \frac{p_p}{\rho_w g}\right) = \nabla h\Big|_{t=0} \qquad (\text{for } t = 0^-) \tag{5.44}$$

according to the definition of hydraulic gradient (4.18), with the assumption that $\nabla\rho_w = \mathbf{0}$.

Equation (5.43) is the equation of motion in incremental form and, with the expression of (5.38), can be regarded as the incremental governing equation.

5.6 Simplification of The Incremental Governing Equation

In order to simplify the incremental equation of motion (5.43), either $\Delta\mathbf{K} = \mathbf{0}$ or $\nabla h_p = \mathbf{0}$ can be assumed. The implication is either that hydraulic conductivity remains unchanged at all times or that the hydraulic condition is hydrostatic at prestressed time t = 0^-.

If the hydrostatic condition is assumed for $t = 0^-$, that is,

$$\nabla h_p = \nabla h\Big|_{t=0} = \mathbf{0} \ , \tag{5.45}$$

then Equation (5.43) simplifies to

$$\frac{d(\mathit{\Delta}\mathbf{u}_s)}{dt}+\frac{\mathbf{K}_p+\mathit{\Delta}\mathbf{K}}{\rho_w g}\nabla(\mathit{\Delta}\sigma'_m)=\mathit{\Delta}\mathbf{q}_b+\frac{\mathbf{K}_p+\mathit{\Delta}\mathbf{K}}{\rho_w g}\nabla(\mathit{\Delta}\sigma_m) \tag{5.46}$$

or

$$\frac{d(\mathit{\Delta}\mathbf{u}_s)}{dt}+\frac{\mathbf{K}}{\rho_w g}\nabla(\mathit{\Delta}\sigma'_m)=\mathit{\Delta}\mathbf{q}_b+\frac{\mathbf{K}}{\rho_w g}\nabla(\mathit{\Delta}\sigma_m) \tag{5.47}$$

which is a simplified form of incremental equation of motion if the hydrostatic condition is assumed for prestressed time. Note that in Equation (5.47), hydraulic conductivity tensor $\mathbf{K}$ $(=\mathbf{K}_p+\mathit{\Delta}\mathbf{K})$ is the only non-incremental variable. Besides that variables are incremental, Equations (5.46) and (5.47) differ from Equation (4.44) by the absence of a $\mathbf{Kk}$ term.

Recall from (4.28) that under hydrostatic condition ($\mathbf{q}=\mathbf{0}$ or $\nabla h=\mathbf{0}$), bulk flux $\mathbf{q}_b$, velocity of solid $\mathbf{v}_s$ and velocity of water $\mathbf{v}_w$ are equal. Therefore, for the hydrostatic condition to occur at prestressed time, the relationship

$$\mathbf{q}_{bp}=\mathbf{v}_{sp}=\mathbf{v}_{wp} \qquad (t=0^-) \tag{5.48}$$

must be satisfied.

With the requirement of (5.48), expression of the incremental bulk flux (5.16) derived previously in Section 5.2 simplifies to

$$\mathit{\Delta}\mathbf{q}_b=\left(n_p+\mathit{\Delta}n\right)\mathit{\Delta}\mathbf{v}_w+\left[1-\left(n_p+\mathit{\Delta}n\right)\right]\mathit{\Delta}\mathbf{v}_s \tag{5.49}$$

or

$$\mathit{\Delta}\mathbf{q}_b=n\left(\mathit{\Delta}\mathbf{v}_w\right)+\left(1-n\right)\mathit{\Delta}\mathbf{v}_s \tag{5.50}$$

where n $(=n_p+\mathit{\Delta}n)$ is the non-incremental porosity.

In order to simplify the gradient of incremental stress (5.38) in Cartesian coordinates, an assumption similar to (4.57) can be made, that is,

$$\begin{cases} \left|\dfrac{A_x}{\eta_{xp}}\nabla\eta_{xp}+\dfrac{A_x}{\Delta\dot{\varepsilon}_x}\nabla(\Delta\dot{\varepsilon}_x)\right| = A_x\left|\nabla\left(\ln\left|\eta_{xp}\right|\right)+\nabla\left(\ln\left|\Delta\dot{\varepsilon}_x\right|\right)\right| << \left|\nabla(\Delta\varepsilon_x)\right| \\ \left|\dfrac{A_y}{\eta_{yp}}\nabla\eta_{yp}+\dfrac{A_y}{\Delta\dot{\varepsilon}_y}\nabla(\Delta\dot{\varepsilon}_y)\right| = A_y\left|\nabla\left(\ln\left|\eta_{yp}\right|\right)+\nabla\left(\ln\left|\Delta\dot{\varepsilon}_y\right|\right)\right| << \left|\nabla(\Delta\varepsilon_y)\right| \\ \left|\dfrac{A_z}{\eta_{zp}}\nabla\eta_{zp}+\dfrac{A_z}{\Delta\dot{\varepsilon}_z}\nabla(\Delta\dot{\varepsilon}_z)\right| = A_z\left|\nabla\left(\ln\left|\eta_{zp}\right|\right)+\nabla\left(\ln\left|\Delta\dot{\varepsilon}_z\right|\right)\right| << \left|\nabla(\Delta\varepsilon_z)\right| \end{cases} . \tag{5.51}$$

It can be inferred mathematically that

$$\begin{cases} \left|A_x\left(\dfrac{1}{\eta_{xp}}-\dfrac{1}{\eta_{xp}+\Delta\eta_x}\right)\nabla\eta_{xp}+\dfrac{\Delta\eta_x}{\eta_{xp}+\Delta\eta_x}\dfrac{A_x}{\dot{\varepsilon}_{xp}}\nabla\dot{\varepsilon}_{xp}\right| \le \left|\dfrac{A_x}{\eta_{xp}}\nabla\eta_{xp}+\dfrac{A_x}{\Delta\dot{\varepsilon}_x}\nabla(\Delta\dot{\varepsilon}_x)\right| \\ \left|A_y\left(\dfrac{1}{\eta_{yp}}-\dfrac{1}{\eta_{yp}+\Delta\eta_y}\right)\nabla\eta_{yp}+\dfrac{\Delta\eta_y}{\eta_{yp}+\Delta\eta_y}\dfrac{A_y}{\dot{\varepsilon}_{yp}}\nabla\dot{\varepsilon}_{yp}\right| \le \left|\dfrac{A_y}{\eta_{yp}}\nabla\eta_{yp}+\dfrac{A_y}{\Delta\dot{\varepsilon}_y}\nabla(\Delta\dot{\varepsilon}_y)\right| \\ \left|A_z\left(\dfrac{1}{\eta_{zp}}-\dfrac{1}{\eta_{zp}+\Delta\eta_z}\right)\nabla\eta_{zp}+\dfrac{\Delta\eta_z}{\eta_{zp}+\Delta\eta_z}\dfrac{A_z}{\dot{\varepsilon}_{zp}}\nabla\dot{\varepsilon}_{zp}\right| \le \left|\dfrac{A_z}{\eta_{zp}}\nabla\eta_{zp}+\dfrac{A_z}{\Delta\dot{\varepsilon}_z}\nabla(\Delta\dot{\varepsilon}_z)\right| \end{cases} . \tag{5.52}$$

Hence, the condition that both (5.51) and (5.52) are satisfied simultaneously is guaranteed by

$$\begin{cases} \left|A_x\left(\dfrac{1}{\eta_{xp}}-\dfrac{1}{\eta_{xp}+\Delta\eta_x}\right)\nabla\eta_{xp}+\dfrac{\Delta\eta_x}{\eta_{xp}+\Delta\eta_x}\dfrac{A_x}{\dot{\varepsilon}_{xp}}\nabla\dot{\varepsilon}_{xp}\right| << \left|\nabla(\Delta\varepsilon_x)\right| \\ \left|A_y\left(\dfrac{1}{\eta_{yp}}-\dfrac{1}{\eta_{yp}+\Delta\eta_y}\right)\nabla\eta_{yp}+\dfrac{\Delta\eta_y}{\eta_{yp}+\Delta\eta_y}\dfrac{A_y}{\dot{\varepsilon}_{yp}}\nabla\dot{\varepsilon}_{yp}\right| << \left|\nabla(\Delta\varepsilon_y)\right| \\ \left|A_z\left(\dfrac{1}{\eta_{zp}}-\dfrac{1}{\eta_{zp}+\Delta\eta_z}\right)\nabla\eta_{zp}+\dfrac{\Delta\eta_z}{\eta_{zp}+\Delta\eta_z}\dfrac{A_z}{\dot{\varepsilon}_{zp}}\nabla\dot{\varepsilon}_{zp}\right| << \left|\nabla(\Delta\varepsilon_z)\right| \end{cases} . \tag{5.53}$$

Therefore, if the condition of (5.53) is assumed, Equation (5.38) greatly simplifies to

$$\nabla(\mathit{\Delta}\sigma'_{\mathrm{m}}) = \mp\frac{1}{3}\left[\frac{1}{\mathrm{A_x B_x}}\left(\eta_{xp}+\mathit{\Delta}\eta_{\mathrm{x}}\right)\left(\dot{\varepsilon}_{xp}+\mathit{\Delta}\dot{\varepsilon}_{\mathrm{x}}\right)\nabla(\mathit{\Delta}\varepsilon_{\mathrm{x}})\right.$$
$$\left. + \frac{1}{\mathrm{A_y B_y}}\left(\eta_{yp}+\mathit{\Delta}\eta_{\mathrm{y}}\right)\left(\dot{\varepsilon}_{yp}+\mathit{\Delta}\dot{\varepsilon}_{\mathrm{y}}\right)\nabla(\mathit{\Delta}\varepsilon_{\mathrm{y}}) + \frac{1}{\mathrm{A_z B_z}}\left(\eta_{zp}+\mathit{\Delta}\eta_{\mathrm{z}}\right)\left(\dot{\varepsilon}_{zp}+\mathit{\Delta}\dot{\varepsilon}_{\mathrm{z}}\right)\nabla(\mathit{\Delta}\varepsilon_{\mathrm{z}})\right] \quad (5.54)$$

or

$$\nabla(\mathit{\Delta}\sigma'_{\mathrm{m}}) = \mp\frac{1}{3}\left[\frac{1}{\mathrm{A_x B_x}}\eta_{\mathrm{x}}\dot{\varepsilon}_{\mathrm{x}}\nabla(\mathit{\Delta}\varepsilon_{\mathrm{x}}) + \frac{1}{\mathrm{A_y B_y}}\eta_{\mathrm{y}}\dot{\varepsilon}_{\mathrm{y}}\nabla(\mathit{\Delta}\varepsilon_{\mathrm{y}}) + \frac{1}{\mathrm{A_z B_z}}\eta_{\mathrm{z}}\dot{\varepsilon}_{\mathrm{z}}\nabla(\mathit{\Delta}\varepsilon_{\mathrm{z}})\right] \quad (5.55)$$

where $\eta_{\mathrm{j}}\ (=\eta_{jp}+\ \mathit{\Delta}\eta_{\mathrm{j}})$ and $\dot{\varepsilon}_{\mathrm{j}}\ (=\dot{\varepsilon}_{jp}+\ \mathit{\Delta}\dot{\varepsilon}_{\mathrm{j}})$ (j = x,y,z) are non-incremental variables.

Substituting (5.55) into the incremental equation of motion (5.47) and taking into account the definition of incremental strain (5.6) yields

$$\frac{\mathrm{d}(\mathit{\Delta}\mathbf{u}_{\mathrm{s}})}{\mathrm{dt}} - \frac{\mathbf{K}}{3\rho_{\mathrm{w}}\mathrm{g}}\left\{\frac{1}{\mathrm{A_x B_x}}\sigma'_{\mathrm{xe}}\nabla\left[\frac{\partial(\mathit{\Delta}\mathrm{u_x})}{\partial \mathrm{x}}\right]\right.$$
$$\left. + \frac{1}{\mathrm{A_y B_y}}\sigma'_{\mathrm{ye}}\nabla\left[\frac{\partial(\mathit{\Delta}\mathrm{u_y})}{\partial \mathrm{y}}\right] + \frac{1}{\mathrm{A_z B_z}}\sigma'_{\mathrm{ze}}\nabla\left[\frac{\partial(\mathit{\Delta}\mathrm{u_z})}{\partial \mathrm{z}}\right]\right\} = \mathbf{R}_{\mathit{\Delta}} \quad (5.56)$$

where

$$\mathbf{R}_{\mathit{\Delta}} = \mathit{\Delta}\mathbf{q}_{\mathrm{b}} + \frac{\mathbf{K}}{\rho_{\mathrm{w}}\mathrm{g}}\nabla(\mathit{\Delta}\sigma_{\mathrm{m}})\ . \quad (5.57)$$

Equation (5.56) is the incremental form of the governing equation (4.59) although $\mathbf{K}$ $(=\mathbf{K}_p+\ \mathit{\Delta}\mathbf{K})$, σ'_{xe}, σ'_{ye} and σ'_{ze} are not incremental variables. Note that $\mathbf{R}_{\mathit{\Delta}}$ is not exactly the increment of $\mathbf{R}$ given by (4.45) due to the hydrostatic assumption (5.45) for prestressed time ($t = 0^{-}$) and, hence, a different notation is used.

In cylindrical coordinates, however, incremental normal strains are defined by (5.7), instead, and the resulting incremental governing equation appears as

$$
\frac{d(\Delta \mathbf{u}_s)}{dt} - \frac{\mathbf{K}}{3\rho_w g}\left\{\frac{1}{A_r B_r}\sigma'_{re}\nabla\left[\frac{\partial(\Delta u_r)}{\partial r}\right] + \frac{1}{A_\theta B_\theta}\sigma'_{\theta e}\nabla\left[\frac{\Delta u_r}{r_p + \Delta u_r} + \frac{1}{r_p + \Delta u_r}\frac{\partial(\Delta u_\theta)}{\partial \theta}\right] + \frac{1}{A_z B_z}\sigma'_{ze}\nabla\left[\frac{\partial(\Delta u_z)}{\partial z}\right]\right\} = \mathbf{R}_\Delta \tag{5.58}
$$

where the radial distance of solid skeletal frame $r = r_p + \Delta r = r_p + \Delta u_r$ according to (5.5). In order for the simplified form of (5.58) to be reached, the following assumption has been made.

$$
\begin{cases}
\left| A_r\left(\frac{1}{\eta_{rp}} - \frac{1}{\eta_{rp}+\Delta\eta_r}\right)\nabla\eta_{rp} + \frac{\Delta\eta_r}{\eta_{rp}+\Delta\eta_r}\frac{A_r}{\dot{\varepsilon}_{rp}}\nabla\dot{\varepsilon}_{rp}\right| << \left|\nabla(\Delta\varepsilon_r)\right| \\
\left| A_\theta\left(\frac{1}{\eta_{\theta p}} - \frac{1}{\eta_{\theta p}+\Delta\eta_\theta}\right)\nabla\eta_{\theta p} + \frac{\Delta\eta_\theta}{\eta_{\theta p}+\Delta\eta_\theta}\frac{A_\theta}{\dot{\varepsilon}_{\theta p}}\nabla\dot{\varepsilon}_{\theta p}\right| << \left|\nabla(\Delta\varepsilon_\theta)\right| \\
\left| A_z\left(\frac{1}{\eta_{zp}} - \frac{1}{\eta_{zp}+\Delta\eta_z}\right)\nabla\eta_{zp} + \frac{\Delta\eta_z}{\eta_{zp}+\Delta\eta_z}\frac{A_z}{\dot{\varepsilon}_{zp}}\nabla\dot{\varepsilon}_{zp}\right| << \left|\nabla(\Delta\varepsilon_z)\right|
\end{cases} \tag{5.59}
$$

which is similar to the assumption (5.53) for Cartesian coordinates.

In Cartesian coordinates, the series of simplified governing equations of (4.61), (4.73) and (4.76)~(4.77) with some simplifying assumptions are reduced from the general equation (4.59). In cylindrical coordinates, equations of (4.70), (4.74), (4.79), (4.81)~(4.84) are simplified from the general equation (4.64) with the infinitesimal strains defined by (2.32). In much the same way, the general incremental governing equations of (5.56) and (5.58) can also be reduced to some simpler forms.

For example, besides the inherent assumptions of (5.53) and (5.45), the general equation (5.56) will simplify, if the principal hydraulic directions are further assumed to coincide with the directions of Cartesian coordinate axes, to

$$\begin{cases} \frac{d(\Delta u_x)}{dt} - \frac{1}{3\rho_w g}\left(\frac{c_{xx}}{A_x B_x}\frac{\partial^2(\Delta u_x)}{\partial x^2} + \frac{c_{xy}}{A_y B_y}\frac{\partial^2(\Delta u_y)}{\partial x \partial y} + \frac{c_{xz}}{A_z B_z}\frac{\partial^2(\Delta u_z)}{\partial x \partial z} \right) = R_{\Delta x} \\ \frac{d(\Delta u_y)}{dt} - \frac{1}{3\rho_w g}\left(\frac{c_{yx}}{A_x B_x}\frac{\partial^2(\Delta u_x)}{\partial y \partial x} + \frac{c_{yy}}{A_y B_y}\frac{\partial^2(\Delta u_y)}{\partial y^2} + \frac{c_{yz}}{A_z B_z}\frac{\partial^2(\Delta u_z)}{\partial y \partial z} \right) = R_{\Delta y} \\ \frac{d(\Delta u_z)}{dt} - \frac{1}{3\rho_w g}\left(\frac{c_{zx}}{A_x B_x}\frac{\partial^2(\Delta u_x)}{\partial z \partial x} + \frac{c_{zy}}{A_y B_y}\frac{\partial^2(\Delta u_y)}{\partial z \partial y} + \frac{c_{zz}}{A_z B_z}\frac{\partial^2(\Delta u_z)}{\partial z^2} \right) = R_{\Delta z} \end{cases} \quad (5.60)$$

with

$$\begin{cases} R_{\Delta x} = \Delta q_{bx} + \frac{K_x}{\rho_w g}\frac{\partial(\Delta\sigma_m)}{\partial x} \\ R_{\Delta y} = \Delta q_{by} + \frac{K_y}{\rho_w g}\frac{\partial(\Delta\sigma_m)}{\partial y} \\ R_{\Delta z} = \Delta q_{bz} + \frac{K_z}{\rho_w g}\frac{\partial(\Delta\sigma_m)}{\partial z} \end{cases} \quad (5.61)$$

where Δu_x, Δu_y and Δu_z are the vector components of incremental solid displacement $\Delta \mathbf{u}_s$, $[R_{\Delta x} \;\; R_{\Delta y} \;\; R_{\Delta z}]^T = \mathbf{R}_\Delta$, $[\Delta q_{bx} \;\; \Delta q_{by} \;\; \Delta q_{bz}]^T = \Delta \mathbf{q}_b$ is the incremental bulk flux and $\Delta\sigma_m$ is the incremental mean total stress. The three simultaneous equations of (5.60) are the incremental forms of (4.61) with the same material coefficients A_j, B_j, c_{jk} (j,k = x,y,z) and non-incremental hydraulic conductivities K_x, K_y and K_z. Note that the coefficients c_{jk} (j,k = x,y,z) are still defined by (4.63) which consists of products of non-incremental equivalent stresses (4.20) and non-incremental hydraulic conductivities.

By comparing (5.60) with (4.61), the obvious differences are that all non-incremental displacement variables u_x, u_y and u_z are replaced by the corresponding incremental displacements Δu_x, Δu_y and Δu_z and that the right-hand sides $R_{\Delta x}$, $R_{\Delta y}$ and $R_{\Delta z}$ are defined, (5.61), differently from R_x, R_y and R_z, (4.62). However, implicit in equations of (5.60) as written is the assumption of hydrostatic condition (5.45) for the prestressed time.

If mechanical isotropy ((3.68) and (4.72)) and irrotational deformation ((4.75) with (5.2)) are further assumed, the three equations of (5.60) simplify to

$$\begin{cases} \dfrac{d(\varDelta u_x)}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{xx}\dfrac{\partial^2(\varDelta u_x)}{\partial x^2} + c_{xy}\dfrac{\partial^2(\varDelta u_x)}{\partial y^2} + c_{xz}\dfrac{\partial^2(\varDelta u_x)}{\partial z^2} \right) = R_{\varDelta x} \\ \dfrac{d(\varDelta u_y)}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{yx}\dfrac{\partial^2(\varDelta u_y)}{\partial x^2} + c_{yy}\dfrac{\partial^2(\varDelta u_y)}{\partial y^2} + c_{yz}\dfrac{\partial^2(\varDelta u_y)}{\partial z^2} \right) = R_{\varDelta y} \\ \dfrac{d(\varDelta u_z)}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{zx}\dfrac{\partial^2(\varDelta u_z)}{\partial x^2} + c_{zy}\dfrac{\partial^2(\varDelta u_z)}{\partial y^2} + c_{zz}\dfrac{\partial^2(\varDelta u_z)}{\partial z^2} \right) = R_{\varDelta z} \end{cases} \quad (5.62)$$

which are the incremental forms of equations of (4.77).

By the same token, for Cartesian coordinates, the incremental forms of the simplified governing equations of (4.73) and (4.76) can be reached from the general equation (5.56) or directly from the simplified equations of (5.60); and, for cylindrical coordinates, Equation (5.58) with the condition (5.59) satisfied will lead to the incremental forms of the equations of (4.70), (4.74), (4.79) and (4.81)~(4.84) with some appropriate assumptions. For brevity, these simplified governing equations in incremental form are not listed.

In summary, the incremental form of the equation of motion (5.46) or (5.47) is valid only when the prestressed hydraulic condition is hydrostatic, (5.45), ($\nabla h_p = \mathbf{0}$ and, therefore, $\mathbf{q}_p = \mathbf{0}$). One of the implications of this hydrostatic condition is that bulk flux and solid and water velocities are all equal at prestressed time ($t = 0^-$), that is, $\mathbf{q}_{bp} = \mathbf{v}_{sp} = \mathbf{v}_{wp}$ according to (5.48). Note that $\mathbf{q}_{bp} = \mathbf{v}_{sp} = \mathbf{v}_{wp} = \mathbf{0}$ is not necessarily implied nor assumed. The incremental forms of the governing equations reached in Section 4.5 of the previous chapter can be formulated with ease from the general incremental equations (5.56) and (5.58) in a fairly similar manner in arriving at equations of (5.51) and (5.62).

Chapter 6 Summary and Conclusions

Historically, mathematical and numerical models aimed at characterizing the behavior of compressible aquifer systems have been formulated using poroelasticity as a constitutive relation. However, laboratory and field data alike indicate that three separate types of compression occur: an instantaneous strain, followed by a primary strain and a secondary or creep strain. The overall behavior of the stress-strain curve tends to be inelastic, nonlinear and strongly dependent on time. Hence, poroelastic models do not adequately predict the behavior of compressible porous media, particularly those consist of significant portion of fine-grained sediments like clays. A new constitutive relation called "poroviscosity" (Helm, 1998) was developed and found to more accurately reproduce laboratory strain behavior in one dimension. This thesis represents the formulation of the three-dimensional governing equations invoking the poroviscous constitutive relation. The governing equations of the poroviscous model proposed in this thesis are the simultaneous equations of (4.61) and (4.70), respectively, for a Cartesian coordinate system x y z and a cylindrical coordinate system r θ z.

$$\begin{cases} \dfrac{du_x}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{xx}}{A_x B_x}\dfrac{\partial^2 u_x}{\partial x^2} + \dfrac{c_{xy}}{A_y B_y}\dfrac{\partial^2 u_y}{\partial x \partial y} + \dfrac{c_{xz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial x \partial z} \right) = R_x \\ \dfrac{du_y}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{yx}}{A_x B_x}\dfrac{\partial^2 u_x}{\partial y \partial x} + \dfrac{c_{yy}}{A_y B_y}\dfrac{\partial^2 u_y}{\partial y^2} + \dfrac{c_{yz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial y \partial z} \right) = R_y \\ \dfrac{du_z}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{zx}}{A_x B_x}\dfrac{\partial^2 u_x}{\partial z \partial x} + \dfrac{c_{zy}}{A_y B_y}\dfrac{\partial^2 u_y}{\partial z \partial y} + \dfrac{c_{zz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial z^2} \right) = R_z \end{cases} \qquad (4.61)$$

in which c_{jk} (j,k = x,y,z) are assumed constant coefficients defined by (4.63) and

$$\begin{cases} R_x = q_{bx} + \dfrac{K_x}{\rho_w g}\dfrac{\partial \sigma_m}{\partial x} \\ R_y = q_{by} + \dfrac{K_y}{\rho_w g}\dfrac{\partial \sigma_m}{\partial y} \\ R_z = q_{bz} + K_z + \dfrac{K_z}{\rho_w g}\dfrac{\partial \sigma_m}{\partial z} \end{cases} ; \tag{4.62}$$

and

$$\begin{cases} \dfrac{du_r}{dt} - \dfrac{1}{3\rho_w g}\left[\dfrac{c_{rr}}{A_r B_r}\dfrac{\partial^2 u_r}{\partial r^2} + \dfrac{c_{r\theta}}{A_\theta B_\theta}\dfrac{1}{r_p + u_r}\left(-\dfrac{u_r}{r_p + u_r} + \dfrac{\partial u_r}{\partial r} \right.\right. \\ \qquad \left.\left. - \dfrac{1}{r_p + u_r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{\partial^2 u_\theta}{\partial r \partial \theta}\right) + \dfrac{c_{rz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial r \partial z}\right] = R_r \\ \dfrac{du_\theta}{dt} - \dfrac{1}{3\rho_w g}\left[\dfrac{c_{\theta r}}{A_r B_r}\dfrac{1}{r_p + u_r}\dfrac{\partial^2 u_r}{\partial \theta \partial r} + \dfrac{c_{\theta\theta}}{A_\theta B_\theta}\dfrac{1}{r_p + u_r}\left(\dfrac{1}{r_p + u_r}\dfrac{\partial u_r}{\partial \theta} \right.\right. \\ \qquad \left.\left. + \dfrac{1}{r_p + u_r}\dfrac{\partial^2 u_\theta}{\partial \theta^2}\right) + \dfrac{c_{\theta z}}{A_z B_z}\dfrac{1}{r_p + u_r}\dfrac{\partial^2 u_z}{\partial \theta \partial z}\right] = R_\theta \\ \dfrac{du_z}{dt} - \dfrac{1}{3\rho_w g}\left[\dfrac{c_{zr}}{A_r B_r}\dfrac{\partial^2 u_r}{\partial z \partial r} + \dfrac{c_{z\theta}}{A_\theta B_\theta}\dfrac{1}{r_p + u_r}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial^2 u_\theta}{\partial z \partial \theta}\right) + \dfrac{c_{zz}}{A_z B_z}\dfrac{\partial^2 u_z}{\partial z^2}\right] = R_z \end{cases} \tag{4.70}$$

with the assumed constant coefficients c_{jk} (j,k = r,θ,z) defined by (4.69) and the right-hand sides defined as

$$\begin{cases} R_r = q_{br} + \dfrac{K_r}{\rho_w g}\dfrac{\partial \sigma_m}{\partial r} \\ R_\theta = q_{b\theta} + \dfrac{K_\theta}{\rho_w g}\dfrac{1}{r_p + u_r}\dfrac{\partial \sigma_m}{\partial \theta} \\ R_z = q_{bz} + K_z + \dfrac{K_z}{\rho_w g}\dfrac{\partial \sigma_m}{\partial z} \end{cases} . \tag{4.68}$$

Both sets of governing equations (4.61) and (4.70) assume sea level as the datum of the z coordinate axis which points upward against the direction of gravity. The common assumptions inherent to (4.61) and (4.70) are

(1) Density of water ρ_w is independent of both water pressure p and spatial position **x** (or $\partial\rho_w/\partial p = 0$ and $\nabla\rho_w = \mathbf{0}$),

(2) Deformation of a porous medium in general is infinitesimal and

(3) Principal hydraulic directions coincide with the directions of coordinate axes.

Also, for Cartesian and cylindrical coordinates, the respective conditions of (4.57) and (4.65) are assumed, that is,

$$\left|\frac{A_j}{\eta_{jp}}\nabla\eta_{jp} + \frac{A_j}{\dot{\varepsilon}_j}\nabla\dot{\varepsilon}_j\right| = A_j\left|\nabla\left(\ln\left|\eta_{jp}\right|\right) + \nabla\left(\ln\left|\dot{\varepsilon}_j\right|\right)\right| << \left|\nabla\varepsilon_j\right| \qquad (4.57) \text{ or } (4.65)$$

where j = x,y,z for (4.57) and j = r,θ,z for (4.65).

The much more simplified governing equations of (4.61) and (4.70) are the equations of (4.77) and (4.81), respectively,

$$\begin{cases} \dfrac{du_x}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{xx}\dfrac{\partial^2 u_x}{\partial x^2} + c_{xy}\dfrac{\partial^2 u_x}{\partial y^2} + c_{xz}\dfrac{\partial^2 u_x}{\partial z^2}\right) = R_x \\ \dfrac{du_y}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{yx}\dfrac{\partial^2 u_y}{\partial x^2} + c_{yy}\dfrac{\partial^2 u_y}{\partial y^2} + c_{yz}\dfrac{\partial^2 u_y}{\partial z^2}\right) = R_y \\ \dfrac{du_z}{dt} - \dfrac{1}{3AB\rho_w g}\left(c_{zx}\dfrac{\partial^2 u_z}{\partial x^2} + c_{zy}\dfrac{\partial^2 u_z}{\partial y^2} + c_{zz}\dfrac{\partial^2 u_z}{\partial z^2}\right) = R_z \end{cases} \qquad (4.77)$$

and

$$\begin{cases} \dfrac{du_r}{dt} - \dfrac{1}{3AB\rho_w g}\left[-c_{r\theta}\dfrac{u_r}{\left(r_p + u_r\right)^2} + c_{r\theta}\dfrac{1}{r_p + u_r}\dfrac{\partial u_r}{\partial r} + c_{rr}\dfrac{\partial^2 u_r}{\partial r^2}\right. \\ \qquad\qquad \left. + c_{r\theta}\dfrac{1}{\left(r_p + u_r\right)^2}\dfrac{\partial^2 u_r}{\partial \theta^2} - 2c_{r\theta}\dfrac{1}{\left(r_p + u_r\right)^2}\dfrac{\partial u_\theta}{\partial \theta} + c_{rz}\dfrac{\partial^2 u_r}{\partial z^2}\right] = R_r \\ \dfrac{du_\theta}{dt} - \dfrac{1}{3AB\rho_w g}\left[c_{\theta\theta}\dfrac{u_\theta}{\left(r_p + u_r\right)^2} + \left(2c_{\theta r} + c_{\theta\theta}\right)\dfrac{1}{r_p + u_r}\dfrac{\partial u_\theta}{\partial r}\right. \\ \qquad\qquad \left. + c_{\theta r}\dfrac{\partial^2 u_\theta}{\partial r^2} + c_{\theta\theta}\dfrac{1}{\left(r_p + u_r\right)^2}\dfrac{\partial^2 u_\theta}{\partial \theta^2} + c_{\theta z}\dfrac{\partial^2 u_\theta}{\partial z^2}\right] = R_\theta \end{cases} \qquad (4.81)$$

$$\left| \begin{array}{l} \frac{du_z}{dt} - \frac{1}{3AB\rho_w g}\left[c_{z\theta} \frac{1}{r_p + u_r} \frac{\partial u_z}{\partial r} + c_{zr} \frac{\partial^2 u_z}{\partial r^2} \right. \\ \qquad\qquad\qquad\qquad \left. + c_{z\theta} \frac{1}{(r_p + u_r)^2} \frac{\partial^2 u_z}{\partial \theta^2} + c_{zz} \frac{\partial^2 u_z}{\partial z^2} \right] = R_z \end{array} \right.$$

with additional assumptions of mechanical isotropy and irrotational deformation.

A one-dimensional governing equation can be acquired from equations of (4.77) or (4.81). Consider only the deformation in the vertical z direction. By ignoring the lateral displacements u_x, u_y, u_r and u_θ and assuming u_z is independent of the coordinates of the other two axes (x and y; or, r and θ), both (4.77) and (4.81) reduce to the one-dimensional consolidation equation of Helm (1998, p. 400):

$$\frac{du_z}{dt} - c\frac{\partial^2 u_z}{\partial z^2} = R_z \tag{6.1}$$

in which R_z is defined by (4.62) or (4.68) and

$$c = \frac{c_{zz}}{3AB\rho_w g} \tag{6.2}$$

is a consolidation parameter assumed to be constant.

The incremental equation of motion, (5.46) (or (5.47))

$$\frac{d(\Delta \mathbf{u}_s)}{dt} + \frac{\mathbf{K}_p + \Delta\mathbf{K}}{\rho_w g}\nabla(\Delta\sigma'_m) = \Delta\mathbf{q}_b + \frac{\mathbf{K}_p + \Delta\mathbf{K}}{\rho_w g}\nabla(\Delta\sigma_m) \tag{5.46}$$

in which $\Delta\mathbf{u}_s$, $\Delta\sigma'_m$, $\Delta\sigma_m$ and $\Delta\mathbf{q}_b$ are incremental variables, is found to differ from Helm's (1987, p. 372)

$$\frac{d(\Delta \mathbf{u}_s)}{dt} + \frac{\mathbf{K}_p + \Delta\mathbf{K}}{\rho_w g}\nabla(\Delta\sigma'_m) = \Delta\mathbf{q}_b + (\mathbf{K}_p + \Delta\mathbf{K})\mathbf{k} + \frac{\mathbf{K}_p + \Delta\mathbf{K}}{\rho_w g}\nabla(\Delta\sigma_m) \tag{6.3}$$

(modified from Equations (4.44) and (4.45)) by (1) the absence of a $(\mathbf{K}_p + \Delta\mathbf{K})\mathbf{k}$ term and (2) the necessary hydrostatic assumption for the prestressed time ($t = 0^-$): $\nabla h_p = \mathbf{0}$, (5.45).

The only set of incremental governing equations formulated is

$$\begin{cases} \dfrac{d(\Delta u_x)}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{xx}}{A_x B_x}\dfrac{\partial^2(\Delta u_x)}{\partial x^2} + \dfrac{c_{xy}}{A_y B_y}\dfrac{\partial^2(\Delta u_y)}{\partial x \partial y} + \dfrac{c_{xz}}{A_z B_z}\dfrac{\partial^2(\Delta u_z)}{\partial x \partial z} \right) = R_{\Delta x} \\ \dfrac{d(\Delta u_y)}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{yx}}{A_x B_x}\dfrac{\partial^2(\Delta u_x)}{\partial y \partial x} + \dfrac{c_{yy}}{A_y B_y}\dfrac{\partial^2(\Delta u_y)}{\partial y^2} + \dfrac{c_{yz}}{A_z B_z}\dfrac{\partial^2(\Delta u_z)}{\partial y \partial z} \right) = R_{\Delta y} \\ \dfrac{d(\Delta u_z)}{dt} - \dfrac{1}{3\rho_w g}\left(\dfrac{c_{zx}}{A_x B_x}\dfrac{\partial^2(\Delta u_x)}{\partial z \partial x} + \dfrac{c_{zy}}{A_y B_y}\dfrac{\partial^2(\Delta u_y)}{\partial z \partial y} + \dfrac{c_{zz}}{A_z B_z}\dfrac{\partial^2(\Delta u_z)}{\partial z^2} \right) = R_{\Delta z} \end{cases} \quad (5.60)$$

which assumes that the coordinate axes coincide with the principal hydraulic directions. Equations of (5.60) are applicable to mechanically anisotropic aquifers in general as long as the additional assumption

$$\left| A_j \left(\frac{1}{\eta_{jp}} - \frac{1}{\eta_{jp} + \Delta\eta_j} \right) \nabla\eta_{jp} + \frac{\Delta\eta_j}{\eta_{jp} + \Delta\eta_j} \frac{A_j}{\dot{\varepsilon}_{jp}} \nabla\dot{\varepsilon}_{jp} \right| << \left| \nabla(\Delta\varepsilon_j) \right| \qquad (5.53)$$

(j = x,y,z) is satisfied. Note that (4.57) is the assumption for the non-incremental equations of (4.61) and is not the necessary condition for the incremental equations of (5.60) to be valid.

Although more simple forms of the incremental governing equations are not explicitly derived, they can be formulated without extensive labor, provided that the inherent assumption (5.53) is recognized, as demonstrated in the derivation of equations of (5.62) in the end of Section 5.6.

The incremental form of Helm's (1998, p. 400) one-dimensional consolidation equation (6.1) can be reduced from the general incremental equations of (5.60) to appear as

$$\frac{d(\Delta u_z)}{dt} - c\frac{\partial^2(\Delta u_z)}{\partial z^2} = R_{\Delta z} \tag{6.4}$$

where $R_{\Delta z}$ is defined by (5.61) and the coefficient c is till given by (6.2).

In conclusion, the proposed model employing the newly extended poroviscosity constitutive relation (Helm, 1998) possesses the ability to quantify the short-term (instantaneous and primary) and long-term (secondary) behaviors of aquifer deformation in one single unified model. Based on the perfect performance of the poroviscosity theory in laboratory prediction of one-dimensional sediment compaction (Helm, 1998), the proposed three-dimensional model is expected to quantitatively describe aquifer deformation more accurately than the theory of poroelasticity (Terzaghi, 1925a, 1925b; Biot, 1941, 1973). The resulting mathematical model can be employed to develop numerical models for complex field applications. Although the principle of stress equilibrium is not invoked in the model, the stress equilibrium equations ((4.40), (4.42), (5.23) and (5.25)) can potentially be used as criteria for checking the validity of numerical solutions.

6.1 Suggested Future Studies

To facilitate the use of the proposed model as a modeling and prediction tool, the next stage of development would be to work out a numerical solution to a practical and somewhat simple field problem in order to test the newly developed mathematical model.

In addition, laboratory experiments should be conducted to acquire the valid ranges of model parameters so that a set of reasonable values can be used for modeling purposes. The parameters and coefficients that need to be determined in the lab include, in general, A_j, B_j and c_{jk} (j,k = x,y,z for (4.61) and (5.60) and j,k = r,θ,z for (4.70) and its incremental form) (or $c_{jk}/(3\rho_w g A_k B_k)$ as a group). For a mechanically isotropic porous material, the six parameters A_j and B_j (j,k = x,y,z or j,k = r,θ,z) reduce to only two, namely, A and B in (4.77), (4.81) (listed above), (4.73)~(4.74), (4.83)~(4.84) and (5.62) (in Chapters 4 and 5). Also, A and B can be combined with $c_{jk}/(3\rho_w g)$ into a single parameter $c_{jk}/(3\rho_w gAB)$. The collective term $c_{jk}/(3\rho_w gA_k B_k)$ or $c_{jk}/(3\rho_w gAB)$ can be viewed as a new three-dimensional consolidation parameter. However, the technical details on laboratory design for obtaining these new model parameters depend very much on the expertise of soil and rock mechanics technicians and are beyond the scope of this thesis.

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Appendix: Supplemental Derivations

A.1 An Alternative Constitutive Equation of Poroviscosity

Equating (3.6) to (3.7) and rearranging terms leads to

$$\frac{\dot{\eta}}{\eta} = \mp \frac{\dot{\varepsilon}}{A} \qquad (A, \eta \neq 0) \ . \tag{A.1}$$

Making time derivative on both sides of Equation (3.6) and applying the chain rule yields

$$\dot{\sigma} = \dot{\eta}\dot{\varepsilon} + \eta\ddot{\varepsilon} \tag{A.2}$$

if furthermore $\dot{\sigma}$ and $\ddot{\varepsilon}$ both exist. Dividing (A.2) by (3.6) leads to

$$\frac{\dot{\sigma}}{\sigma} = \frac{\eta\ddot{\varepsilon} + \dot{\eta}\dot{\varepsilon}}{\eta\dot{\varepsilon}} = \frac{\ddot{\varepsilon}}{\dot{\varepsilon}} + \frac{\dot{\eta}}{\eta} \qquad (\sigma, \eta, \dot{\varepsilon} \neq 0)$$

which, with Equation (A.1), becomes

$$\frac{\dot{\sigma}}{\sigma} = \frac{\ddot{\varepsilon}}{\dot{\varepsilon}} \mp \frac{\dot{\varepsilon}}{A} \ . \tag{3.17}$$

In order to find some forms of alternative expressions using the assumed initial condition (3.10), Equation (3.17) is first written as

$$\frac{1}{\sigma}\frac{d\sigma}{dt} = \frac{1}{\dot{\varepsilon}}\frac{d\dot{\varepsilon}}{dt} \mp \frac{1}{A}\frac{d\varepsilon}{dt} \quad \text{or} \quad \frac{1}{\sigma}d\sigma = \frac{1}{\dot{\varepsilon}}d\dot{\varepsilon} \mp \frac{1}{A}d\varepsilon \ . \tag{A.3}$$

Integration of (A.3), a different form of (3.17), from the initial time t = 0^+ to a somewhat arbitrarily later time t ($\geq 0^+$), yields

$$\int_{\sigma_i}^{\sigma} \frac{1}{\sigma} d\sigma = \int_{\dot{\varepsilon}_i}^{\dot{\varepsilon}} \frac{1}{\dot{\varepsilon}} d\dot{\varepsilon} \mp \frac{1}{A}\int_{\varepsilon_i}^{\varepsilon} d\varepsilon \quad \Rightarrow \quad \ln|\sigma| \Big|_{\sigma_i}^{\sigma} = \ln|\dot{\varepsilon}| \Big|_{\dot{\varepsilon}_i}^{\dot{\varepsilon}} \mp \frac{1}{A}\varepsilon \Big|_{\varepsilon_i}^{\varepsilon}$$

$$\Rightarrow \quad \ln|\sigma| - \ln|\sigma_i| = \ln|\dot{\varepsilon}| - \ln|\dot{\varepsilon}_i| \mp \frac{1}{A}(\varepsilon - \varepsilon_i) \quad \Rightarrow \quad \ln\left|\frac{\sigma}{\sigma_i}\right| = \ln\left|\frac{\dot{\varepsilon}}{\dot{\varepsilon}_i}\right| \mp \frac{1}{A}(\varepsilon - \varepsilon_i)$$

$$\Rightarrow \quad \ln\left(\frac{\sigma}{\sigma_i}\right) - \ln\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_i}\right) = \mp\frac{1}{A}(\varepsilon - \varepsilon_i) \quad (\sigma_i\sigma > 0,\ \dot{\varepsilon}_i\dot{\varepsilon} > 0)\ .$$

Rearrangement and simplification of terms in the above equation will lead to (3.15).

A.2 Time Limits of Poroviscosity Constitutive Law

Equation (3.19) as written represents two alternative expressions of viscosity, that is, for engineering mechanics convention,

$$\eta = \sigma\left(\frac{1}{\dot{\varepsilon}_i} - \frac{1}{A}t\right) \tag{A.4e}$$

and for geomechanics convention,

$$\eta = \sigma\left(\frac{1}{\dot{\varepsilon}_i} + \frac{1}{A}t\right) \tag{A.4g}$$

of which the material property $\eta > 0$ must be satisfied. Limits of time t and bounds of $1 \mp \frac{\dot{\varepsilon}_i}{A}t$ can be found from the following discussions.

For engineering mechanics convention,

(i) If $\sigma > 0$ (tensile stress), then $\dot{\varepsilon} > 0$ for $t \geq 0$ according to (3.8e). For $\eta > 0$, it follows from Equation (A.4e) that

$$\eta = \sigma\left(\frac{1}{\dot{\varepsilon}_i} - \frac{1}{A}t\right) > 0 \quad (\sigma > 0) \quad \Rightarrow \quad \frac{1}{\dot{\varepsilon}_i} - \frac{1}{A}t > 0 \quad (A > 0,\ \dot{\varepsilon}_i > 0)$$

$$\Rightarrow \quad t < \frac{A}{\dot{\varepsilon}_i} \ (> 0)\ .$$

Since time is considered only for $t \geq 0$, t is therefore required to be bound by

$$0 \leq t < \frac{A}{\dot{\varepsilon}_i}\ . \tag{A.5}$$

Furthermore, since $A > 0$ and $\dot{\varepsilon}_i > 0$,

$$0 \geq -\frac{\dot{\varepsilon}_i}{A}t > -1 \quad \Rightarrow \quad 0 < 1 - \frac{\dot{\varepsilon}_i}{A}t \leq 1 \; . \tag{A.6}$$

(ii) If $\sigma < 0$ (compressive stress), then $\dot{\varepsilon} < 0$ for $t \geq 0$, (3.8e), and, from (A.4e),

$$\eta = \sigma\left(\frac{1}{\dot{\varepsilon}_i} - \frac{1}{A}t\right) > 0 \quad (\sigma < 0) \quad \Rightarrow \quad \frac{1}{\dot{\varepsilon}_i} - \frac{1}{A}t < 0 \quad (A > 0,\ \dot{\varepsilon}_i < 0)$$

$$\Rightarrow \quad t > \frac{A}{\dot{\varepsilon}_i} \quad (< 0) \; .$$

At the same time, $t \geq 0$ must be satisfied. Therefore,

$$t \geq 0 \tag{A.7}$$

which, in turn, recalling that $A > 0$ and $\dot{\varepsilon}_i < 0$, leads to

$$-\frac{\dot{\varepsilon}_i}{A}t \geq 0 \quad \Rightarrow \quad 1 \leq 1 - \frac{\dot{\varepsilon}_i}{A}t \; . \tag{A.8}$$

For geomechanics convention,

(iii) If $\sigma > 0$ (compressive stress), then $\dot{\varepsilon} > 0$ for $t \geq 0$, (3.9g), and, from (A.4g),

$$\eta = \sigma\left(\frac{1}{\dot{\varepsilon}_i} + \frac{1}{A}t\right) > 0 \quad (\sigma > 0) \quad \Rightarrow \quad \frac{1}{\dot{\varepsilon}_i} + \frac{1}{A}t > 0 \quad (A > 0,\ \dot{\varepsilon}_i > 0)$$

$$\Rightarrow \quad t > -\frac{A}{\dot{\varepsilon}_i} \quad (< 0) \; .$$

Hence,

$$t \geq 0 \; . \tag{A.9}$$

Also, by recognizing that $A > 0$ and $\dot{\varepsilon}_i > 0$, it then follows from (A.9) that

$$\frac{\dot{\varepsilon}_i}{A}t \geq 0 \quad \Rightarrow \quad 1 \leq 1 + \frac{\dot{\varepsilon}_i}{A}t \; . \tag{A.10}$$

(iv) If $\sigma < 0$ (tensile stress), then $\dot{\varepsilon} < 0$ for $t \geq 0$, (3.9g), and, from (A.4g),

$$\eta = \sigma\left(\frac{1}{\dot{\varepsilon}_i} + \frac{1}{A}t\right) > 0 \quad (\sigma < 0) \quad \Rightarrow \quad \frac{1}{\dot{\varepsilon}_i} + \frac{1}{A}t < 0 \quad (A > 0,\ \dot{\varepsilon}_i < 0)$$

$$\Rightarrow \quad t < -\frac{A}{\dot{\varepsilon}_i} \ (>0)\ .$$

Thus, time t is required to be bound according to

$$0 \le t < -\frac{A}{\dot{\varepsilon}_i} \tag{A.11}$$

which leads to

$$0 \ge \frac{\dot{\varepsilon}_i}{A} t > -1 \quad \Rightarrow \quad 0 < 1 + \frac{\dot{\varepsilon}_i}{A} t \le 1 \tag{A.12}$$

since A > 0 and $\dot{\varepsilon}_i < 0$.

From (A.5)~(A.8) and (A.9)~(A.12), it can be concluded that

$$\begin{cases} \text{If } \sigma \text{ is a constant compressive stress, } t \ge 0 \text{ and } 1 \mp \frac{\dot{\varepsilon}_i}{A} t \ge 1\ . \\ \text{If } \sigma \text{ is a constant tensile stress, } 0 \le t < \pm\frac{A}{\dot{\varepsilon}_i} \text{ and } 0 < 1 \mp \frac{\dot{\varepsilon}_i}{A} t \le 1\ . \end{cases} \tag{3.21}$$

In either cases, $1 \mp \frac{\dot{\varepsilon}_i}{A} t > 0$ is always satisfied.

A.3 Mass Balance Equation

For a porous medium consisting of a mixture of solids and water, the principle of mass balance requires that Equations (4.32) and (4.33) be satisfied as discussed in Section 4.3.2. If it is assumed that there is no mass production or loss of any constituent, that is, $\dot{m}_s = 0$ and $\dot{m}_w = 0$, then dividing (4.32) and (4.33) by $\rho_s V$ and $\rho_w V$, respectively, completing the derivatives and rearranging terms leads to

$$\begin{cases} \frac{1}{\rho_s V}\left\{\left[\rho_s V \frac{d}{dt}(1-n) + (1-n)V\frac{d\rho_s}{dt} + (1-n)\rho_s \frac{dV}{dt}\right] + \int_S (1-n)\rho_s(\mathbf{v}_s - \mathbf{v}_P)\cdot d\mathbf{S}\right\} = 0 \\ \frac{1}{\rho_w V}\left[\left(\rho_w V \frac{dn}{dt} + nV\frac{d\rho_w}{dt} + n\rho_w \frac{dV}{dt}\right) + \int_S n\rho_w(\mathbf{v}_w - \mathbf{v}_P)\cdot d\mathbf{S}\right] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} -\dfrac{dn}{dt}+\dfrac{1-n}{\rho_s}\dfrac{d\rho_s}{dt}+\dfrac{1-n}{V}\dfrac{dV}{dt}+\dfrac{1}{\rho_s}\left[\dfrac{1}{V}\displaystyle\int_S (1-n)\rho_s(\mathbf{v}_s-\mathbf{v}_P)\cdot d\mathbf{S}\right]=0 \\ \dfrac{dn}{dt}+\dfrac{n}{\rho_w}\dfrac{d\rho_w}{dt}+\dfrac{n}{V}\dfrac{dV}{dt}+\dfrac{1}{\rho_w}\left[\dfrac{1}{V}\displaystyle\int_S n\rho_w(\mathbf{v}_w-\mathbf{v}_P)\cdot d\mathbf{S}\right]=0 \end{cases}$$

which, after applying the Gauss divergence theorem (Kreyszig, 1999, pp. 506 & 510~511) to transform the surface integrals to volume integrals, become

$$\begin{cases} -\dfrac{dn}{dt}+\dfrac{1-n}{\rho_s}\dfrac{d\rho_s}{dt}+\dfrac{1-n}{V}\dfrac{dV}{dt}+\dfrac{1}{\rho_s}\nabla\cdot[(1-n)\rho_s(\mathbf{v}_s-\mathbf{v}_P)]=0 \\ \dfrac{dn}{dt}+\dfrac{n}{\rho_w}\dfrac{d\rho_w}{dt}+\dfrac{n}{V}\dfrac{dV}{dt}+\dfrac{1}{\rho_w}\nabla\cdot[n\rho_w(\mathbf{v}_w-\mathbf{v}_P)]=0 \end{cases}. \qquad (A.13)$$

For a scalar F and a vector **v** in general, divergence of their product F**v** can be expanded as $\nabla\cdot(F\mathbf{v})=F\nabla\cdot\mathbf{v}+\mathbf{v}\cdot\nabla F$ (Kreyszig, 1999, p. 463). Therefore, the two equations of (A.13) can be expanded for the divergence terms and rearranged as

$$\begin{cases} -\dfrac{dn}{dt}+\dfrac{1-n}{\rho_s}\dfrac{d\rho_s}{dt}+\dfrac{1-n}{V}\dfrac{dV}{dt} \\ \qquad +\dfrac{1}{\rho_s}\left\{(1-n)\rho_s\nabla\cdot(\mathbf{v}_s-\mathbf{v}_P)+(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla[(1-n)\rho_s]\right\}=0 \\ \dfrac{dn}{dt}+\dfrac{n}{\rho_w}\dfrac{d\rho_w}{dt}+\dfrac{n}{V}\dfrac{dV}{dt}+\dfrac{1}{\rho_w}[n\rho_w\nabla\cdot(\mathbf{v}_w-\mathbf{v}_P)+(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla(n\rho_w)]=0 \end{cases}$$

$$\Rightarrow \begin{cases} -\dfrac{dn}{dt}+\dfrac{1-n}{\rho_s}\dfrac{d\rho_s}{dt}+\dfrac{1-n}{V}\dfrac{dV}{dt} \\ \qquad +(1-n)\nabla\cdot(\mathbf{v}_s-\mathbf{v}_P)+\dfrac{1}{\rho_s}(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla[(1-n)\rho_s]=0 \\ \dfrac{dn}{dt}+\dfrac{n}{\rho_w}\dfrac{d\rho_w}{dt}+\dfrac{n}{V}\dfrac{dV}{dt}+n\nabla\cdot(\mathbf{v}_w-\mathbf{v}_P)+\dfrac{1}{\rho_w}(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla(n\rho_w)=0 \end{cases}$$

$$
\Rightarrow \begin{cases} -\dfrac{dn}{dt}+\dfrac{1-n}{\rho_s}\dfrac{d\rho_s}{dt}+\dfrac{1-n}{V}\dfrac{dV}{dt}+\nabla\cdot(\mathbf{v}_s-\mathbf{v}_P)-n\nabla\cdot(\mathbf{v}_s-\mathbf{v}_P) \\ \qquad\qquad +\dfrac{1}{\rho_s}(\mathbf{v}_s-\mathbf{v}_P)\cdot\left[-\rho_s\nabla n+(1-n)\nabla\rho_s\right]=0 \\ \dfrac{dn}{dt}+\dfrac{n}{\rho_w}\dfrac{d\rho_w}{dt}+\dfrac{n}{V}\dfrac{dV}{dt}+n\nabla\cdot(\mathbf{v}_w-\mathbf{v}_P) \\ \qquad\qquad +\dfrac{1}{\rho_w}(\mathbf{v}_w-\mathbf{v}_P)\cdot(\rho_w\nabla n+n\nabla\rho_w)=0 \end{cases}
$$

$$
\Rightarrow \begin{cases} -\dfrac{dn}{dt}+\dfrac{1-n}{\rho_s}\dfrac{d\rho_s}{dt}+\dfrac{1}{V}\dfrac{dV}{dt}-\dfrac{n}{V}\dfrac{dV}{dt}+\nabla\cdot(\mathbf{v}_s-\mathbf{v}_P)-n\nabla\cdot(\mathbf{v}_s-\mathbf{v}_P) \\ \qquad\qquad -(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla n+\dfrac{1-n}{\rho_s}(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla\rho_s=0 \\ \dfrac{dn}{dt}+\dfrac{n}{\rho_w}\dfrac{d\rho_w}{dt}+\dfrac{n}{V}\dfrac{dV}{dt}+n\nabla\cdot(\mathbf{v}_w-\mathbf{v}_P) \\ \qquad\qquad +(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla n+\dfrac{n}{\rho_w}(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla\rho_w=0 \end{cases}.
$$

Furthermore, add the above two equations together and rearrange terms as

$$
\begin{aligned} &\frac{1-n}{\rho_s}\frac{d\rho_s}{dt}+\frac{1-n}{\rho_s}(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla\rho_s+\frac{n}{\rho_w}\frac{d\rho_w}{dt}+\frac{n}{\rho_w}(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla\rho_w \\ &\quad+\left[n\nabla\cdot(\mathbf{v}_w-\mathbf{v}_P)-n\nabla\cdot(\mathbf{v}_s-\mathbf{v}_P)\right]+\left[(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla n-(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla n\right] \\ &\quad+\frac{1}{V}\frac{dV}{dt}+\nabla\cdot(\mathbf{v}_s-\mathbf{v}_P)=0 \end{aligned}
$$

$$
\Rightarrow \begin{aligned} &\frac{1-n}{\rho_s}\left[\frac{d\rho_s}{dt}+(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla\rho_s\right]+\frac{n}{\rho_w}\left[\frac{d\rho_w}{dt}+(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla\rho_w\right] \\ &\quad+n\nabla\cdot(\mathbf{v}_w-\mathbf{v}_P-\mathbf{v}_s+\mathbf{v}_P)+(\mathbf{v}_w-\mathbf{v}_P-\mathbf{v}_s+\mathbf{v}_P)\cdot\nabla n \\ &\quad+\frac{1}{V}\frac{dV}{dt}+\nabla\cdot\mathbf{v}_s-\nabla\cdot\mathbf{v}_P=0 \end{aligned}
$$

$$
\Rightarrow \begin{aligned} &\frac{1-n}{\rho_s}\left[\frac{d\rho_s}{dt}+(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla\rho_s\right]+\frac{n}{\rho_w}\left[\frac{d\rho_w}{dt}+(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla\rho_w\right] \\ &\quad+\left[n\nabla\cdot(\mathbf{v}_w-\mathbf{v}_s)+(\mathbf{v}_w-\mathbf{v}_s)\cdot\nabla n\right]+\nabla\cdot\mathbf{v}_s-\left(\nabla\cdot\mathbf{v}_P-\frac{1}{V}\frac{dV}{dt}\right)=0 \end{aligned}
$$

$$\Rightarrow \quad \frac{1-n}{\rho_s}\left[\frac{d\rho_s}{dt}+(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla\rho_s\right]+\frac{n}{\rho_w}\left[\frac{d\rho_w}{dt}+(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla\rho_w\right] + \nabla\cdot\left[n(\mathbf{v}_w-\mathbf{v}_s)+\mathbf{v}_s\right]-\left(\nabla\cdot\mathbf{v}_P-\frac{1}{V}\frac{dV}{dt}\right)=0$$

$$\Rightarrow \quad \frac{1-n}{\rho_s}\left[\frac{d\rho_s}{dt}+(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla\rho_s\right]+\frac{n}{\rho_w}\left[\frac{d\rho_w}{dt}+(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla\rho_w\right] + \nabla\cdot\left[n\mathbf{v}_w+(1-n)\mathbf{v}_s\right]-\left(\nabla\cdot\mathbf{v}_P-\frac{1}{V}\frac{dV}{dt}\right)=0 \quad . \text{(A.14)}$$

By recalling that the definition of bulk flux $\mathbf{q}_b = n\mathbf{v}_w + (1-n)\mathbf{v}_s$, (4.28), and the relationship between velocity $\mathbf{v}_P$ and the time rate of change of volume V, (4.11), Equation (A.14) becomes

$$\frac{1-n}{\rho_s}\left[\frac{d\rho_s}{dt}+(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla\rho_s\right]+\frac{n}{\rho_w}\left[\frac{d\rho_w}{dt}+(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla\rho_w\right]+\nabla\cdot\mathbf{q}_b=0$$

$$\Rightarrow \quad \nabla\cdot\mathbf{q}_b=-\left\{\frac{1-n}{\rho_s}\left[\frac{d\rho_s}{dt}+(\mathbf{v}_s-\mathbf{v}_P)\cdot\nabla\rho_s\right]+\frac{n}{\rho_w}\left[\frac{d\rho_w}{dt}+(\mathbf{v}_w-\mathbf{v}_P)\cdot\nabla\rho_w\right]\right\}. \quad \text{(A.15)}$$

By expanding the total derivatives inside the brackets and rearranging terms, Equation (A.15) can be further simplified as

$$\nabla\cdot\mathbf{q}_b=-\left\{\frac{1-n}{\rho_s}\left[\left(\frac{\partial\rho_s}{\partial t}+\mathbf{v}_P\cdot\nabla\rho_s\right)+\mathbf{v}_s\cdot\nabla\rho_s-\mathbf{v}_P\cdot\nabla\rho_s\right] + \frac{n}{\rho_w}\left[\left(\frac{\partial\rho_w}{\partial t}+\mathbf{v}_P\cdot\nabla\rho_w\right)+\mathbf{v}_w\cdot\nabla\rho_w-\mathbf{v}_P\cdot\nabla\rho_w\right]\right\}$$

$$\Rightarrow \quad \nabla\cdot\mathbf{q}_b=-\left[\frac{1-n}{\rho_s}\left(\frac{\partial\rho_s}{\partial t}+\mathbf{v}_s\cdot\nabla\rho_s\right)+\frac{n}{\rho_w}\left(\frac{\partial\rho_w}{\partial t}+\mathbf{v}_w\cdot\nabla\rho_w\right)\right] \qquad (4.34)$$

which is the mass balance equation of a porous medium consisting of only solids and water provided that there is no source or sink of both solids and water within the control volume under consideration.