CHAPTER 3

A NEW METHODOLGY TO EVALUATE PRODUCTIVE EFFICIENCY IN A DYNAMIC ENVIRONMENT

Dynamic productive efficiency is a measure of a systems ability to convert inputs into outputs at a specific time t, during a transient period, such that either the largest possible outcome is achieved given a fixed set of inputs (maximization principle), or the least possible inputs are employed given a fixed set of outputs (minimization principle). Dynamic productive efficiency is distinguished from productive efficiency by: (i) the element of time is introduced; and (ii) a disturbance is introduced to cause the system to seek a new equilibrium.

Heretofore, the study of productive efficiency has concentrated on system efficiency in a steady state. For example, the DDEA methodology developed by Färe and Grosskopf (1996) is an evaluation of a system in steady state which progress over time. The study of systems in dynamic and complex environments is important because by their very nature, these transient states are the most disruptive during a system's life-cycle.

A organization's ability to master these periods is fundamental to achieving steady state operations more efficiently, thus reducing losses due to poor productivity. For example, figure 3-1 (replicated from figure 1-1) illustrates the productive efficiency problem during the transient phase. At time t_0, a disturbance is introduced into the system causing the system to seek a new steady state. The system performance resulting from a normal transient phase shows the system eventually achieving a steady state, but accumulates significant performance degradation during the transient phase. The more efficient path for achieving a steady state is via the enhanced system transient performance line. When this path is followed, performance losses due to the transient period are minimized. The methodology derived in this chapter prescribes a solution for mastering transient periods.

Before the methodology can be illustrated, several legacy productivity concepts must be expanded. Section 3.1 reviews the theoretical constructs of dynamical systems and expands them to include the exploration of dynamical, causal, and closed systems.

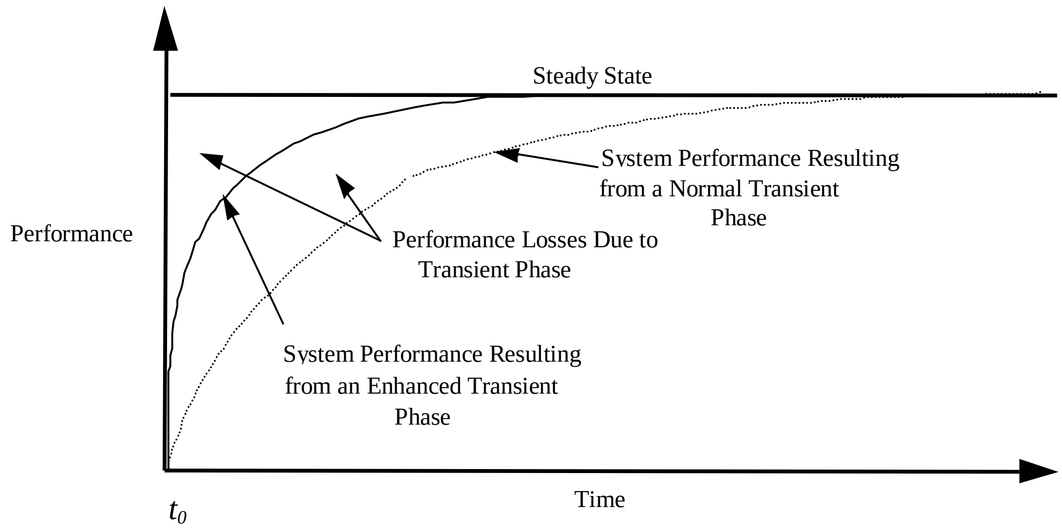

Figure 3-1. The Productive Efficiency Problem during a Transient Period.

Section 3.2 expands the static production axioms into the dynamic realm. Section 3.3 builds the bridge between the production axioms and systems dynamic behavior by relating the two concepts. Section 3.4 expands the theory of production frontiers into the dynamic realm. With the stage being set, section 3.5 develops the SD optimization model used to evaluate productive efficiency. Section 3.6 uses an example developed by Kopp (1981) to illustrate and validate this model.

3.1 Dynamical Systems

This research is concerned with studying productive efficiency in a dynamic environment. Samuelson (1947, p. 314) states a *"system is dynamical if its behavior over time is determined by functional equations in which 'variables at different points of time' are involved in an 'essential' way."*

Dynamical systems can be classified in three distinct ways: (1) dynamic and historical; (2) dynamic and causal (Samuelson, 1947); and (3) dynamic, causal, and closed. Systems that are dynamic and historical exhibit a high degree of correlation between the variables at the initial time t_0 with the variables at the final time t. Neither the passage of time nor the structure of the system are considered, thus variables which become active during the interval between the initial time and final time, (t_0, t), are not considered when determining the final state of the system. Dynamic and historical systems can be expressed (Samuelson, 1947) as:

$$y_{jt} = f\{t_0; t; x_{it_0}\} \tag{3-1}$$

Where:

x_{it_0} is the ith input at the initial time t_0, $i=1,2,3,\ldots n$

y_{jt} is the jth output at time t, $j=1,2,3,\ldots m$.

Dynamic and historical systems correlate the initial conditions of the system to the final conditions, and do not contain information about the structure or behavior of the system. As an example, consider a farmer who plants a crop in the spring and returns to harvest his crop in the autumn. The farmer does not witness the behavior of the crops during the growing season, nor does he care about the structure of the system. His only concern is the initial inputs and outputs after a given period of time.

Dynamic and causal systems consider the initial system inputs, x_{it_0}, along with the passage of time. Dynamic and causal systems can be expressed:

$$y_{jt} = f\{t - t_0; x_{it_0}; x_{it_d}\} \tag{3-2}$$

This type of system allows inputs to be added at some intermediate time t_d, where $t_0 < t_d < t$, and for the behavior of the system to be evaluated at any given time. To continue with the farm example, in a dynamic and causal system the farmer would be able to monitor the growth of the crop from the time the crop was planted until the time it was harvested. Inputs designed to enhance crop production (e.g., fertilizer, water, and pesticides) can be added at intermediate times. However, the inputs added during the interval (t_0, t) are not a result of feedback mechanism from within the system. Thus in this example, the farmer may be adding fertilizer at an intermediate time, regardless if the crop needs it or not.

Heretofore the systems discussed could be classified as open systems. An open system converts inputs into outputs, but the outputs are isolated from the system so that they cannot influence the inputs further. Conversely outputs are not isolated in a closed system, thus are aware of and can influence the system behavior by providing information to modify the inputs via a feedback mechanism.

Dynamic, causal, and closed systems can be expressed as:

$$y_{jt} = f\{t - t_0; x_{it_0}; x_{it_d}; y_{j(t_d - t_o)}\} \tag{3-3}$$

Where $y_{j(t_d - t_0)}$ is the jth output resulting from action during the interval [t_0, t_d]. By adding the output variable to (3), this relationship is defined in an "essential way" such that the system brings results from past actions to influence or control future actions via a feedback mechanism. To further expand the farm example, with a dynamic, causal, and closed system the farmer can now monitor his crop growth (i.e., $y_{j(t_d - t_0)}$), and intervene in the process by applying water, pesticides, and fertilizer as needed.

In the context of evaluating the efficiency performance of dynamic, causal, and closed systems, one must also investigate their equilibrium and stability. Systems can be categorized as being in equilibrium or in disequilibrium, stable or unstable. Equilibrium can be further categorized as either static or dynamic. Static equilibrium is defined as the condition that exists when there is no flow within the system (Sterman, 2000). Two conditions must be satisfied for a system to be in static equilibrium: (1) All first order derivatives x'_{it}, y'_{jt} are zero at the time considered; and (2) all higher order derivatives are also zero. A system in which only condition (1) is satisfied is said to be momentarily at rest (, 1935-36).

A system in dynamic equilibrium is a system where there is a constant flow going through the system. Viewing the system from a macro level, dynamic equilibrium gives the appearance that nothing within the system changes over time. A closer look reveals that there is a constant flow of inputs into the system, and a constant flow of outputs from the system (Sterman, 2000). All derivatives will have non-zero values for dynamic equilibrium.

A system that does not meet the criteria for either static or dynamic equilibrium is said to be in a state of disequilibrium (Frisch, 1935-36). An example of a system in disequilibrium is a manufacturing plant where there is a constant influx of orders, and the number of orders exceeds the plant capacity. In this case the queue of orders will continue to grow, thus creating a state of disequilibrium.

System stability refers to how a system that was previously in equilibrium behaves when a disturbance is introduced. Consider a small disturbance introduced to the system at time t_d. If the system returns to its original (or closely related) state of equilibrium after being disturbed, the system is considered stable (Frisch, 1935-36; Sterman, 2000). If the small disturbance forces the system further away from equilibrium with the passage of time, the system is said to be in unstable equilibrium (Sterman, 2000).

3.2 The Dynamic Production Axioms

The production axioms are a set of properties that explain and govern the activities that occur within a production environment. The production technology can be represented in a variety of ways (i.e. output correspondence, input correspondence, and graph theory) (Färe and Grosskopf, 1996). Without the loss of generality, this paper addresses the production axioms from the output correspondence perspective.

Treatment of the production axioms to date has assumed static conditions. Thus they assume inputs into a production system during one period are transformed into outputs during that same period. These axioms are also well suited to address dynamical systems that can be characterized as dynamic and historical as they do not contain information about intermediate times. One can also argue that these static axioms can also be applied to dynamic and causal systems only if inputs added at intermediate times are assumed to originate at the initial time period since in dynamic and causal systems variables are considered at the beginning and the end of the time horizon.

In realistic production scenarios, outputs from a system are often not realized until after a significant time delay. Furthermore, realistic production scenarios are not performed in a vacuum, thus they are aware, and react to, changes within the environment. Thus there is a need to develop and examine the production axioms from a dynamic, causal and closed system perspective.

From this perspective, production technology uses a set of input vectors x defined in the input space $\Re_+^N$,over a given time horizon $[t_0, t]$ such that:

$$x_{t-t_0} = [x_i]_{t_0}^{t} \in \Re_+^N, \quad i = 1,2,3,\ldots n \qquad (3\text{-}4)$$

From these inputs, a set of output vectors y_t in output space $\Re_+^M$ (final and intermediate) is produced:

$$y_t = (y_j)_t \in \Re_+^M, \quad j = 1,2,3,...m \tag{3-5}$$

The *dynamic* production technology is represented by the output correspondence P as:

$$P : \Re_+^N \rightarrow P(x_{t-t_0}) = \{y_t : x_{t-t_o}, y_{t_d - t_0} \in S(y_t)\} \tag{3-6}$$

where $S(y_t)$ is the input correspondence.

This relationship shows when considering production in a dynamic environment, outputs from the system at time *t*, result from inputs that comprise the initial conditions of the system, inputs at intermediate time(s), intermediate outputs that provide feedback relationships to the inputs within the system, and the passage of time. The transformation process, represented by *P*, occurs during the interval [t_0, *t*]. The dynamic and historical, dynamic and causal, dynamic, causal, and closed production axioms are shown in Table 3-1 (Färe and Primont, 1995; Färe and Grosskopf, 1996; Vaneman and Triantis, 2003). We will present the dynamic, causal, and closed system production axioms in the ensuing discussion.

The first production axiom is separated into two parts. Axiom 1(a) states that it is always possible to produce no outputs. The dynamic axiom is represented by:

$$DA.1(a) \qquad 0_t \in P(x_{t-t_0}; y_{t_d - t_0}), \forall (x_{t-t_o}; y_{t_t - t_0}) \in \Re_+^N \tag{3-7}$$

The spirit behind this axiom is if no inputs are entered into the system during the time horizon [t_0, *t*], the system would remain at rest or in a state of static equilibrium. Axiom 1(b) further amplifies the premise established in DA.1(a). This axiom is known as the "No Free Lunch" axiom (Färe and Primont, 1995, p. 27) because it states that it is not possible to produce outputs at time *t*, in the absence of inputs during the time interval [t_0, *t*]. Axiom 1(b) is represented dynamically as:

$$DA.1(b) \qquad y_t \notin P(x_{t-t_0}; y_{t_d - t_0}) = 0, y_t > 0 \tag{3-8}$$

Table 3-1. Dynamical Systems Representations of the Production Axioms (Output Correspondence Perspective).

Axiom	Dynamic and Historical Representation	Dynamic and Causal Representation	Dynamic, Causal, and Closed Representation
1(a) Inactivity	$0 \in P(x), \forall x \in \Re_+^N$	$0_t \in P(x_{t-t_0}), \forall x_{t-t_0} \in \Re_+^N$	$0_t \in P(x_{t-t_0}; y_{t_d-t_0}),$ $\forall (x_{t-t_0}; y_{t_d-t_0}) \in \Re_+^N$
1(b) No Free Lunch	$y \notin P(x=0)$, if $y>0$	$y_t \notin P(x_{t-t_0}=0)$, if $y_t>0$	$y_t \notin P(x_{t-t_0}; y_{t_d-t_0})=0,$ if $y_t>0$
2(a) Weak Input Disposability	If $y \in P(x) \wedge \lambda \geq 1$ $\Rightarrow y \in P(\lambda x)$	If $y_t \in P(x_{t-t_0}) \wedge \lambda \geq 1$ $\Rightarrow y_t \in P(\lambda x_{t-t_0})$	If $y_t \in P(x_{t-t_0}; y_{t_d-t_0}) \wedge$ $\lambda \geq 1 \Rightarrow y_t \in P(\lambda x_{t-t_0}; y_{t_d-t_0})$
2(b) Strong Input Disposability	If $y \in P(\breve{x}) \wedge x \geq \breve{x}$ $\Rightarrow y \in P(x)$	If $y_t \in P(\breve{x}_{t-t_0}) \wedge$ $x_{t-t_0} \geq \breve{x}_{t-t_0} \Rightarrow y_t \in P(x_{t-t_0})$	If $y_t \in P(\breve{x}_{t-t_0}; y_{t_d-t_0}) \wedge$ $x_{t-t_0} \geq \breve{x}_{t-t_0}$ $\Rightarrow y_t \in P(x_{t-t_0}; y_{t_d-t_0})$
3(a) Weak Output Disposability	If $y \in P(x) \wedge 0 \leq \varphi \leq 1$ $\Rightarrow \varphi y \in P(x)$	If $y_t \in P(x_{t-t_0}) \wedge 0 \leq \varphi \leq 1$ $\Rightarrow \varphi y_t \in P(x_{t-t_0})$	If $y_t \in P(x_{t-t_0}; y_{t_d-t_0}) \wedge$ $0 \leq \varphi \leq 1$ $\Rightarrow \varphi y_t \in P(x_{t-t_0}; y_{t_d-t_0})$
3(b) Strong Output Disposability	If $y \in P(x) \wedge \breve{y} \leq y$ $\Rightarrow \breve{y} \in P(x)$	If $y_t \in P(x_{t-t_0}) \wedge$ $\breve{y}_t \leq y_t \Rightarrow \breve{y}_t \in P(x_{t-t_0})$	If $y_t \in P(x_{t-t_0}; y_{t_d-t_0}) \wedge$ $\breve{y}_t \leq y_t \Rightarrow \breve{y}_t \in P(x_{t-t_0}; y_{t_d-t_0})$
4 Scarcity	$\forall x \in \Re_+^N, P(x)$ is a bounded set	$\forall x_{t-t_0} \in \Re_+^N, y_t \in P(x_{t-t_0})$ is a bounded set	$\forall x_{t-t_0} \in \Re_+^N, y_t \in P(x_{t-t_0}; y_{t_d-t_0})$ is a bounded set
5 Closedness	$\forall x \in \Re_+^N, P(x)$ is a closed set	$\forall x_{t-t_0} \in \Re_+^N, y_t \in P(x_{t-t_0})$ is a closed set	$\forall x_{t-t_0} \in \Re_+^N, y_t \in P(x_{t-t_0}; y_{t_d-t_0})$ is a closed set
6 Convexity	$\forall x \in S(y) \in \Re_+^N if\ 0 \leq \lambda \leq 1 \Rightarrow$ $\lambda x + (1-\lambda)\breve{x} \in S(y)$	$\forall x_{t-t_0} \in S(y_t) \in \Re_+^N if$ $0 \leq \lambda \leq 1 \Rightarrow$ $\lambda x_{t-t_0} + (1-\lambda)\breve{x}_{t-t_0} \in S(y_t)$	$\forall x_{t-t_0} \in S(y_t) \in \Re_+^N if$ $0 \leq \lambda \leq 1 \Rightarrow$ $\lambda(x_{t-t_0}; y_{t_d-t_0}) +$ $(1-\lambda)(\breve{x}_{t-t_0}; y_{t_d-t_0}) \in S(y_t)$

The weak input disposability axiom states if all inputs are increased proportionally, outputs will not decrease. If inputs are not increased proportionally, outputs may decrease. (Färe and Grosskopf, 1996). The dynamic axiom further expands upon this concept because if inputs are increased proportionally during the time horizon [t_0, t] then output will not decrease at the corresponding time t. However, if inputs are not increased proportionally during [t_0, t], then outputs may decrease at the corresponding t. Dynamic axiom 2(a) is stated as:

$$DA.2(a) \qquad y_t \in P(x_{t-t_0}; y_{t_d-t_0}) \wedge \lambda \geq 1 \Rightarrow y_t \in P(\lambda x_{t-t_0}; y_{t_d-t_0}) \qquad (3\text{-}9)$$

System dynamic models can be characterized as continuous at discrete points in time, thus the model values change smoothly, but are only accessed at specific time steps, as defined by the granularity of the model's time bounds (Pidd, 1998). In this case, the weighting factor λ is unique for all y_t and is assumed to be constant within each infinitesimal time step t.

The strong input disposability axiom states that if any input increases, whether proportional or not, output will not decrease (Färe and Grosskopf, 1996). In a dynamic environment, if the increases to x_{t-t_0} during the time horizon [t_0, t] are proportional or not, than the output y_t at the corresponding time period t will not be reduced. Axiom 2(b) is stated as:

$$DA.2(b) \qquad y_t \in P(\breve{x}_{t-t_0}; y_{t_d-t_0}) \wedge x_{t-t_0} \geq \breve{x}_{t-t_0} \Rightarrow y_t \in P(x_{t-t_0}; y_{t_d-t_0}) \qquad (3\text{-}10)$$

If axiom DA.2(b) is true, then axiom DA.2(a) is also true. However the converse is not true because DA.2(b) states that outputs y_t will not decrease regardless if inputs x_{t-t_0} are increased proportionally or not (Färe and Grosskopf, 1996).

The weak output disposability axiom states that a proportional reduction of outputs is possible (Färe and Primont, 1995). Thus if output y_t is produced by input x_{t-t_0}, an output φy_t can also be produced by input x_{t-t_0}, when $\varphi y_t \leq y_t$.

The dynamic weak output disposability axiom is represented as:

$$DA.3(a) \qquad y_t \in P(x_{t-t_0}; y_{t_d-t_0}) \wedge 0 \leq \varphi \leq 1 \Rightarrow \varphi y_t \in P(x_{t-t_0}; y_{t_d-t_0}) \qquad (3\text{-}11)$$

Similar to the weighting factor λ in DA.2(a), the weighting factor φ is unique for all y_t and is assumed to be constant within each infinitesimal time step *t*. However, φ can vary between time steps.

The utility of this axiom is most commonly found when the system produces both desirable and undesirable outputs (e.g., consequences of pollution regulations) (Färe and Grosskopf, 1996). If DA.3(a) is applicable, then in order reduce the undesirable output(s), the desirable output(s) must also be reduced by the same proportional amount.

Strong or free output disposability states that outputs can be costlessley disposed of. The cause of this condition may be an inefficient production process that generates waste that can be discarded without consequences (e.g., smoke from a production process being emitted to the environment can be considered a costless disposal of an undesirable output in the absence of pollution regulations) (Färe and Grosskopf, 1996). In a dynamic system, production processes during the interval [t_0, *t*] may yield outputs that are disposed of without costs, if and only if at least one output variable is exogenous to the system. Otherwise, they will create feedbacks into the system endogenously that will have performance consequences. DA.3(b) is represented by:

$$DA.3(b) \qquad y_t \in P(x_{t-t_0}; y_{t_d-t_0}) \wedge \breve{y}_t \le y_t \Rightarrow \breve{y}_t \in P(x_{t-t_0}; y_{t_d-t_0}) \qquad (3\text{-}12)$$

If axiom DA.3(b) is true, then axiom DA.3(a) is also true. However the converse is not true (Färe and Grosskopf, 1996). This condition can be easily seen because the output variables may include a combination of endogenous and exogenous variables, while DA.3(b) must include at least one exogenous variable.

DA.4 is known as the scarcity axiom because the output set is bounded. Thus finite amounts of input can only yield finite amounts of output (Färe and Primont, 1995). In a dynamic environment, the inputs are also bounded within the time domain. Thus bounded resources within [t_0, *t*] can only yield finite outputs at corresponding time *t*. DA.4 is represented by:

$$DA.4 \qquad \forall x_{t-t_0} \in \Re_+^N, y_t \in P(x_{t-t_0}; y_{t_d-t_0}) \text{ is a bounded set} \qquad (3\text{-}13)$$

The mathematical condition of closedness is addressed in DA.5. Assume the output *y* is a series of vectors $y_j = (y_1, y_2, y_3, ... y_m)$ such that $\lim_{j\to\infty} y_j = y$. If every sequence of output vectors y_j can be produced from inputs x_i, then x can produce y (Takayama, 1985; Färe and Primont, 1995). Expanding this concept to a dynamic environment, if the inputs x at time t-t_0 can produce every sequence of vectors y_j at time t, then x_{t-t_0} can produce y_t. DA.5 is shown as:

$$DA.5 \qquad \forall x_{t-t_0} \in \Re_+^N, y_t \in P(x_{t-t_0}; y_{t_d-t_0}) \text{ is a closed set} \tag{3-14}$$

DA.4 and DA.5 ensure that the function $P(x)$ is a compact set. A compact set is a finite set in space that contains all possible limit points.

DA.6 is the convexity axiom (Färe and Primont, 1995). A convex set is the resultant of a weighted combination of two extreme points. The weighted combination yields a line segment that joins the two points (Hillier and Lieberman, 1995). Thus if x_i and $\breve{x}_i$ are both a series of inputs that can produce y, then any weighted combination of x_i and $\breve{x}_i$ can produce y. In a dynamic sense, x_i and $\breve{x}_i$ can be inputs during different times within the time interval [t_0, t] as long as their respective outputs share common time period t. DA.6 is described as:

$$DA.6 \;\; \forall x_{t-t_0} \in S(y_t) \in \Re_+^N \, if \, 0 \le \lambda \le 1 \Rightarrow \lambda(x_{t-t_0}; y_{t_d-t_0}) + (1-\lambda)(\breve{x}_{t-t_0}; y_{t_d-t_0}) \in S(y_t) \tag{3-15}$$

3.3 The Dynamic Production Axioms related to the System Dynamic Behaviors

3.3.1 Terminology used in this Section

Before the dynamic production axioms can be related to the system dynamics behaviors, the linkage between production theory and system dynamics must be explored. The dynamic production axioms serve to explain the transformation of inputs x_{t-t_0} into outputs y_t. However, the production axioms do not seek to explore the structure of the transformation process. Our current research is employing the system dynamics modeling approach to explore the structure of production processes and the policies that govern them. System dynamics models are a system of equations that describe the transformation of inputs into outputs, the policies that govern this transformation, and the feedback processes that regulate future inputs.

System dynamic structures recognize that there are intermediary variables that explain system behavior during the transformation while converting inputs x into outputs y during the time horizon [t_0, t] These variables are known as the level variables L_t and represent accumulations (or units stored) within the system. To facilitate the comparison of the system dynamic structures to the dynamic production axioms, it will be assumed that for the level variable L_t that no units are being stored at the end of the time horizon *t* because the dynamic axioms introduced in the previous section do not consider accumulations within the system.

Most production systems have defined performance goals $\hat{L}$. These goals are represented by the quantity of system inputs *RI* that can be converted to system outputs *RO* during a given time horizon, where $RI = x_{t-t_0}$, and $RO = y_t$. These performance goals typically are determined by the requirements $\hat{y}$ that are placed on the system. The desired state of the system is defined for the level of transformation (or production) during a given time period to equal the desired performance goal $(L_t = \hat{L})$. This state of the system is desirable because ideally system output should equal system requirements $(y_t = \hat{y})$ since the requirements $\hat{y}$ determine the performance goals $\hat{L}$.

3.3.2 The Relationships between the Dynamic Production Axioms and the System Dynamic Behaviors[1]

To understand the production scenarios that need to be considered when conducting a performance measurement analysis in a dynamic environment, one must understand how the dynamic production axioms relate to system structures and behaviors. The output variables y_t of the dynamic production axioms are evaluated against the behaviors of the system structures, as defined by the system outputs *RO* at the system limits. This approach is used because system dynamics models seek to portray a system in dynamic equilibrium. Dynamic equilibrium can be defined as the condition that exists when inputs x_{t-t_0} are converted into outputs y_t, such that the net change in the state of the system is equal to zero $(dL_t/dt = 0)$. If the production axioms are not valid for the system at dynamic equilibrium, they are not valid for the system overall, despite potentially being valid early in the system's life-cycle.

Since system dynamics models are systems of equations, each system can have a large number of level and rate variables to model the complexities of the production process. However, regardless of the intricate complexities, the behavior of dynamical system structures can be categorized by eight distinct behavior patterns: static equilibrium, exponential growth, goal seeking, exponential decay, oscillation, S-shaped growth, S-Shaped growth with overshoot, and overshoot and collapse (Sterman, 2000). These system behaviors were discussed in Chapter 2. An overview of the system structures and their related behaviors is provided for ease of reference in Appendix A. Table 3-2 (Vaneman and Triantis, 2003) provides an overview of the relationship between the system dynamics structures and the dynamic production axioms.

The easiest system behavior to relate to the dynamic production axioms is that of static equilibrium. Any system structure can exhibit this behavior provided that there is no activity within the system. If the state of the system L_t is represented by:

$$L_t = L_{t_0} + \int_{t_0}^{t} (R)\, dt \qquad (3-16)$$

Where L_{t_0} is the initial state of the system and R is represents the flow within the system such that $R=RI-RO$. Thus if there are no inputs, there are no outputs assuming that $L_{t_o} = 0$[2].

Both DA.1 (a and b) apply to this behavior. Recall that the spirit behind the inactivity axiom is that no activity is taking place within the system. The "no free lunch" axiom is valid with respect to static

equilibrium iff $L_{t_0} = 0$. If $L_{t_0} \neq 0$, then it would be possible to produce a small amount of output y_t without any inputs x_{t-t_0} due to the some residual activity resident within the system before the start of the time horizon [t_0, t]. For this condition to occur there must be more than one time

horizon for the given problem. For example, if one were to study productive efficiency during a specific time period where no new production was planned, no inputs would be added to the system. However, if work was started in another time period that was not considered in the study, and completed during the study time frame, it would appear to be an output that was created without an input during that study's time frame.

The scarcity and closedness axioms also apply because, static equilibrium infers the null set, and by definition, the null set is bounded and closed. The convexity axiom is not a valid

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[1] Section 3.3.3 presents an example that illustrates the concepts discussed in this section.

[2] L_{t_0} may not equal zero initially due to stored inputs from a previous state of the system. If $L_{t_0} > 0$, then the operational rules that govern RO will force the production of an output y_t.

relationship because x_{t-t_0} and $\breve{x}_{t-t_0}$ are assumed to be positive in the relationship $\lambda x_{t-t_0} + (1-\lambda)\breve{x}_{t-t_0}$ where $0 \le \lambda \le 1$.

The exponential growth behavior is governed by the positive feedback structure. The state of the system represented by the level variable L_t can be described equivalently by:

$$L_t = \int_{t_0}^{t} (R)dt \Leftrightarrow \frac{dL_t}{dt} = R \tag{3-17}$$

For positive feedback structures, the system appears to be linear for small intervals of time. However, this appearance often masks exponential growth because, linear growth requires that there be no feedback from the state of the system to the net increase rate (Sterman, 2000). Since the net inflow rate R is increasing proportionally with respect to the state of the system, it can be defined as $R = cL_t$, where c is the system's constant rate of growth. Thus $dL_t/dt = cL_t$ is obtained through substitution in equation (17). If the variables are separated and both sides of the equation are integrated, the resultant equation is (Drew, 1994; Sterman, 2000):

$$\int_{L_{t_0}}^{L_t} \frac{dL_t}{L_t} = \int_{t_0}^{t} cdt \tag{3-18}$$

This yields $ln\, L_t = ct$. Taking the exponential of both sides yields (Sterman, 2000):

$$L_t = e^{ct} \tag{3-19}$$

Clearly DA.1(a) does not apply because exponential growth implies activity within the system. Nor does DA.1(b) apply because this axiom assumes that there are no inputs into the system. The positive feedback structure requires some initial input even if that input is small. Since zero production and zero input utilization is not part the production possibility set this does not allow for inactivity and no free lunch.

Exponential growth implies $\lim_{t\to\infty} y_t = \infty$. DA.2(a) states in part that if inputs are not increased proportionally, then outputs may decrease. Clearly this production axiom is violated when considering exponential behavior at the limit. DA.2(b) states that $y_t \in P(\breve{x}_{t-t_0})$ and $x \ge \breve{x}$, then $y_t \in P(x_{t-t_0})$. Since DA.2(b) allows output to remain constant, then it is not consistent with the spirit of this behavior.

Table 3-2. Relationships between System Dynamic Behaviors and the Dynamic Production Axioms.

Axiom	Static Equilibrium	Exponential Growth	Exponential Decay	Goal Seeking	Oscillation	S-shaped Growth	S-shaped Growth with Overshoot	Overshoot and Collapse
1(a) Inactivity	yes	no	no	no	no	no	no	no
1(b) No Free Lunch	yes	no	no	no	no	no	no	no
2(a) Weak Input Disposability	no	no	no	yes	yes	yes	yes	no
2(b) Strong Input Disposability	no	no	no	yes	yes	yes	yes	no
3(a) Weak Output Disposability	no	no	yes	no	yes	no	yes	yes
3(b) Strong Output Disposability	no	no	yes	no	yes	no	yes	yes
4 Scarcity	yes	no	yes	yes	yes	yes	yes	yes
5 Closedness	yes	yes	yes	yes	yes	yes	yes	yes
6 Convexity	no	yes	yes	yes	yes	yes	yes	yes

Note: A “yes” within a cell indicates that the dynamic production axiom holds for system dynamic behavior.

A “no’ indicates that the dynamic production axiom does not hold for the system dynamic behavior.

The weak output disposability axiom (DA.3(a)) contains the condition that $\varphi y_t \in P(x_{t-t_0})$ where $0 \le \varphi \le 1$. This means that outputs can be reduced proportionally for the same level of inputs. This situation does not hold for exponential growth whether one considers its limit or this behavior earlier in its life cycle. Likewise, in DA.3(b), outputs may decrease, because the condition $\breve{y}_t \le y_t$ exists. Since exponential growth by nature increases the state of the system, this production axiom does not hold for this behavior.

In theory, exponential growth allows a finite amount of inputs x_{t-t_0} to yield an infinite amount of outputs y_t. This condition implies an unbounded system. Therefore, the scarcity axiom (DA.4) violates exponential growth behavior. The closedness axiom (DA.5) does apply to exponential growth because in an exponential environment, if every sequence of output vectors $(y_j)_t = (y_1, y_2, y_3, ..., y_m)_t$ can be produced from inputs $(x_i)_{t-t_0}$, then x_{t-t_0} can produce y_{jt} where y_{jt} is the limit of the sequence (y_1,…, y_m). The convexity axiom (DA.6) also applies to exponential growth because if x_{t-t_0} and $\breve{x}_{t-t_0}$ are both inputs that can produce y_t, then any weighted combination of x_{t-t_0} and $\breve{x}_{t-t_0}$ can produce y_t provided that $\lambda x_{t-t_0} + (1-\lambda)\breve{x}_{t-t_0}$ exists where $0 \le \lambda \le 1$.

Negative feedback structures lead to both goal seeking and exponential decay behaviors. Goal seeking behavior will be discussed first. The presence of a desired state of the system $\hat{L}$ provides the system with upper limit (or goal) that the system strives to achieve. The amount of change required to bring the current system state in line with the desired system state is determined by the discrepancy Δ, such that $\Delta = \hat{L} - L$. The time constant t_c is the time that it would take to place the system in dynamic equilibrium, assuming that the system's present rate of change is constant (Drew, 1994). The ultimate objective of goal seeking behavior is to achieve a state of the system where dynamic equilibrium exists. Thus every system input x_{t-t_0}, is transformed into a system output y_t, to satisfy a known system requirement $\hat{y}$.

Since dynamic equilibrium is the desired state, the net inflow rate can be defined as $R = dL_t \big/ dt = \hat{L} - L_t \big/ t_c$. If the variables are separated and both sides of the equation are integrated, the resultant equation is:

$$\int_{L_{t0}}^{L_t} \frac{dL_t}{\hat{L} - L_t} = \int_{t_0}^{t} \frac{1}{t_c} dt \tag{3-20}$$

Solving yields:

$$-\ln(\hat{L} - L_t)\Big|_{L_{t0}}^{L_t} = \frac{t}{t_c} \tag{3-21}$$

Applying the limits of integration:

$$-\ln \frac{\hat{L} - L_t}{\hat{L} - L_{t_0}} = \frac{t}{t_c} \Rightarrow \frac{\hat{L} - L_t}{\hat{L} - L_{t_t}} = e^{-t/t_c} \tag{3-22}$$

The final solution to goal seeking structures is (Drew, 1994; Sterman, 2000):

$$L_t = \hat{L} - (\hat{L} - L_{t_0})e^{-t/t_c} \tag{3-23}$$

From this solution, it is clear that $\lim_{t \to \infty} L_t = \hat{L}$. If the system's goal $\hat{L}$ is to fulfill the system requirements $\hat{y}$, then as the system achieves dynamic equilibrium, $\lim_{t \to \infty} y_t = \hat{y}$. Since a dynamic equilibrium condition is assumed, DA.1(a and b) do not hold. Recall that the weak input disposability axiom (DA.2(a)) states that if all inputs are increased proportionally outputs will not decrease. This means that $y_t \ge \hat{y}$ because outputs cannot decrease, they will either increase or remain constant, and at the limit $y_t = \hat{y}$. DA.2(b) is also a valid assumption because it states that outputs will not decrease regardless if inputs are increased proportionally of not.

The weak output disposability axiom (DA.3(a)) states that the output can be reduced proportionally such that φy_t, where 0≤φ≤1. This means that it is possible for $y_t \le \hat{y}$ because outputs will either decrease or remain constant. If $y_t < \hat{y}$ the $\lim_{t \to \infty} y_t = \hat{y}$ is violated, and this axiom does not hold. Dynamic axiom 3(b) states that undesirable outputs can be reduced "freely" without reducing the desirable outputs. Since DA.3(b) can be described by $\breve{y}_t \leq y_t$, this axiom is also violated.

The goal seeking behavior is bounded in two ways, i.e., (1) by a defined upper limit and (2) by a net change of the state of the system equal to zero. Therefore, DA.4 is a valid axiom because it is bounded by definition. DA.5 is also valid with respect to goal seeking behavior because every sequence of output vectors $(y_j)_t = (y_1, y_2, y_3, ... y_m)_t$ can be produced from inputs $(x_i)_{t-t_0}$, then x_{t-t_0} can produce y_t. The convexity axiom also is valid because the relationship $\lambda x_{t-t_0} + (1-\lambda)\breve{x}_{t-t_0}$ implies there may be a different mixture of inputs that produces y_t, but not a decrease in outputs y_t.

The other system behavior that is governed by a first-order (or single) feedback loop is exponential decay. Exponential decay is a special case of the negative feedback structure that assumes that *RI*<*RO*. Thus the system inputs x_{t-t_0} are transformed in outputs y_t but are not meeting the system output requirements $\hat{y}$ therefore the system is deteriorating as $t \to \infty$.

Since *RI*<*RO*, a net outflow R_0 is assumed such that $R_0 = dL_t / dt = -fL_t$, where f is the fractional rate of decay (Sterman, 2000). Separating the variables and integrating both sides of the equation yields:

$$\int_{L_{t_0}}^{L_t} \frac{dL_t}{L_t} = \int_{t_0}^{t} (-f)dt \qquad (3\text{-}24)$$

Which when solved yields:

$$L_t = L_{t_0} e^{-ft} \qquad (3\text{-}25)$$

From equation 3-25 it can be seen that the $\lim_{t\to\infty} y_t = 0$ exists. However, inputs are entering the system, and the system does have activity during the time interval [t_0, t], therefore the inactivity and "no free lunch" axioms do not hold for this behavior. The weak input disposability states that if inputs are increased whether proportional or not, then outputs will not decrease. Since $\lim_{t\to\infty} y_t = 0$, for exponential decay behavior, DA.2(a) is not relevant. DA.2(b) also states that outputs will not decrease regardless if inputs are increased proportionally or not. This axiom violates the exponential decay behavior.

Both output disposability axioms allow for a decrease in outputs y_t while the inputs x_{t-t_0} can remain constant. Therefore both DA.3(a and b) adhere to the condition $\lim_{t\to\infty} y_t = 0$, thus they apply to exponential decay behavior.

The negative feedback loop that governs exponential decay is bounded by an upper limit that is achieved at time t_0, and a lower limit as $t \to \infty$. Thus the scarcity axiom applies because the output y_t is bounded. The closedness axiom also applies because every sequence of output vectors $(y_j)_t = (y_1, y_2, y_3, ... y_m)_t$ can be produced from inputs $(x_i)_{t-t_0}$ then x_{t-t_0} can produce y_t. The convexity axiom also applies, because as $y_t \to 0$, any $\lambda x_{t-t_0} + (1-\lambda)\breve{x}_{t-t_0}$ can provide decreasing and zero output.

Oscillatory behavior in its most basic form is represented by a negative feedback structure with a delay in at least one of its causal linkages. Recall that the structure contains a reporting delay between the state of the system L_t and the discrepancy Δ. The discrepancy at time *t* is calculated as $\Delta = \hat{L} - L_t$. Thus, when the system is adjusting its inflow rate *R*, old information is being used. This causes the system to either overshoot or undershoot the desired state of the system, causing an oscillatory behavior.

Sterman (2000) identifies four types of oscillation, i.e., damped oscillation, expanding oscillation, limit cycle oscillation, and chaotic oscillation. For the purposes of this discussion, the expanding oscillation will be discounted because this type of behavior will drive a system to perform between -∞ and ∞, as $t \to \infty$. This is not acceptable for production systems since output cannot be less than zero. Damped oscillation can be explained by $\lim_{t\to\infty} y_t = \hat{y}$, and limit cycle and chaotic oscillatory behavior are explained by $\lim_{t\to\infty} y_t = \bar{y}$, where $\bar{y}$ is the mean output. Note that under ideal oscillatory conditions, the system output y_t will eventually settle at the system requirement $\hat{y}$ if damped oscillation is present, and oscillate about the mean output $\bar{y}$ (where $\bar{y} = \hat{y}$) in the presence of limit cycle and chaotic oscillation.

Like the goal seeking structure, this structure converts inputs into outputs. Therefore DA.1 (a and b) do not hold. The weak input disposability contains the condition λ≥1. If during a certain time horizon *t* the value of λ fluctuates such that λ≥1, an oscillatory behavior is created. In this case the outputs y_t are evaluated against the outputs at a base time period y_{t_0}. If $y_t \geq y_{t_o}$, then the oscillatory condition does not violate this axiom. This condition is depicted in Figure 3-2 (Vaneman and Triantis, 2003). In the strong input disposability axiom, inputs can be disproportionately reduced but outputs do not decrease. Therefore outputs y_t may vary during the time horizon [t_0, t] but $y_t \geq y_{t_0}$.

DA.3(a) states that outputs can be reduced proportionately, while holding inputs constant. Thus if the condition $y_t \leq y_{t_0}$ exists, this production axiom is valid for oscillatory behavior. When outputs are reduced disproportionately, as in DA.3(b), the condition $y_t \leq y_{t_0}$ also exists. Therefore DA.3(b) is a valid assumption for oscillation behavior. With the elimination of expanding oscillation, all oscillatory behaviors are bounded. Therefore the scarcity axiom applies. The closedness and convexity assumptions also apply because all y_t can be produced from x_{t-t_0}, and y_t can be produced from a combination of x_{t-t_0} and $\breve{x}_{t-t_0}$, such that $\lambda x_{t-t_0} + (1-\lambda)\breve{x}_{t-t_0}$.

S-shaped growth behavior is gleaned from a multiple-loop structure, with one feedback loop being positive, and the other negative. The system behavior corresponds to the behavior of the dominant loop. Initially, the positive feedback structure is dominant, and it appears that the system has unlimited production capacity. Therefore, the system is governed by Equation 3-19 as $t \to 0$. As the system approaches its fixed production capacity the system production becomes more constrained, and the negative feedback structure becomes dominant. Thus as $t \to \infty$, the system is governed by equation 3-23. The system can be described by $\lim_{t \to 0} y_t = \infty$ when the positive feedback loop is dominant, and by $\lim_{t \to \infty} y_t = \hat{y}$ when the negative feedback loop is dominant.

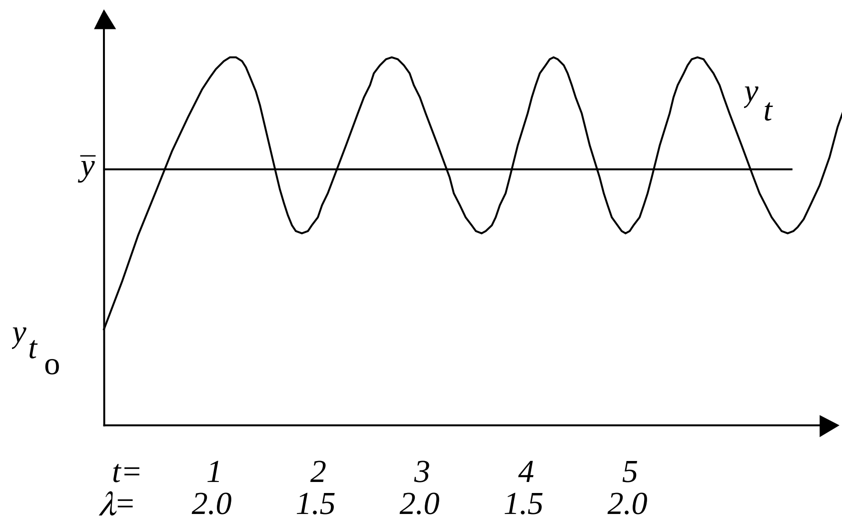

Figure 3-2. Limit Cycle Oscillation with λ≥1.

Since S-shaped growth is ultimately governed by goal seeking behavior, the dynamic production axioms that hold for this behavior are those production axioms that hold for S-shaped growth. Thus the weak and strong input disposability, scarcity, closedness, and convexity dynamic production axioms hold for S-shaped growth behavior.

S-shaped growth with overshoot is another system that contains two feedback structures – a negative and a positive feedback structure. However, unlike S-shaped growth, the negative feedback contains a significant delay in at least one of its causal linkages, which causes an oscillatory behavior as $t \to \infty$. Given these two feedback structures, the S-shaped growth with overshoot behavior can be described by $\lim_{t\to\infty} y_t = \infty$ when the positive feedback loop is dominant, and $\lim_{t\to\infty} y_t = \hat{y}$ (for damped oscillation) or $\lim_{t\to\infty} y_t = \bar{y}$ (for limit cycle and chaotic oscillation) when the negative feedback loop is dominant.

Given that the system eventually is governed by oscillatory behavior, only the dynamic production axioms that hold for oscillation hold for S-shaped growth with overshoot. Therefore all of the disposability axioms (DA.2(a and b) and DA.3(a and b), scarcity, closedness, and convexity axioms hold.

S-shaped growth with overshoot and collapse follows the assumption that as the system achieves its capacity and this capacity begins to erode until the system achieves a state of collapse as defined by $\lim_{t\to\infty} y_t = 0$. An example of this type of behavior would be found in a system that uses a non-renewal resource. This behavior is governed by a system of three feedback structures – a positive and two negative feedback loops. Initially, as the system is dominated by the positive feedback loop, it appears that system production is unlimited $(\lim_{t\to 0} y_t = \infty)$, as the system resources are plentiful. Thus exponential growth characterizes the system. As the system approaches its capacity, the behavior is dominated by a goal-seeking behavior $(\lim_{t\to t_k} y_t = \hat{y}$, where t_k is the time when the system requirement $\hat{y}$ is achieved). As the system achieves its production capacity, the behaviors of the two feedback loops are combined that accelerates the decline of the system capacity. The erosion phase of the system's life-cycle can be defined by $\lim_{t\to\infty} y_t = 0$.

Given that the final phases of the system's life-cycle are characterized by exponential decay, only those dynamic production axioms that are associated with exponential decay can be applied here. Thus the output disposability axioms (DA.3(a and b)), scarcity, closedness and convexity axioms hold. At this juncture it is important to note that the production axioms that apply are dependent upon the system's dynamic behavior. Thus different axioms may apply at different times within the same time horizon.

3.3.3 Example – A Precursor to Chapter 4's Implementation Problem

Our current research studies the productive efficiency of a organization that compiles and disseminates information via an Internet-like media. The organization has recently entered into the business of providing information. Their capability to produce information is increasing after every time period (i.e., monthly) as a function of the learning that takes place in the organization. However, while production capacity is increasing, they are still not producing as much information as their customers require.

In addition to the traditional inputs of labor, capital and overhead, the organization receives a variety of source materials in various formats. Information is extracted from the source materials and is used in the compilation of the information that is ultimately placed upon the organization's servers. The source materials vary in their applicability and usefulness to the end product.

The system outputs are four distinct types of information. As the outputs are placed upon the organization's servers, the production process is considered complete. Customers download the information from the servers at undetermined periodicity. The information placed upon the server is time-sensitive. That is as time passes, information deteriorates at a certain rate and is a function of the data type. When the information deteriorates beyond a certain point, the information is discarded and the compilation process begins again.

Two types of behavior are predominant in this production system. Each of the information types (represented in the system as sub-systems) exhibits S-shaped growth and exponential decay behaviors. One may argue that such a system usually exhibits exponential growth as opposed to S-shaped growth due to the learning that takes place. This is true because during the course of this study we observed the positive feedback loop to be dominant. However, at some point in the system's future the production capacity of the organization and the requirements from the customers equalize, and the system achieves a state of dynamic equilibrium. The exponential decay behavior comes from the data deteriorating as a function of time.

Discovering two dominant dynamic behaviors within the system is significant because each behavior type corresponds to a different set of production axioms. The significance of this finding is that the decision-maker must work with two distinct sets of assumptions when making improvements within the system. Since the production process is governed by the S-shaped growth behavior, and the data resident on the servers are dominated by exponential decay behavior, the behaviors are treated as separate structures when considering production assumptions.

First consider the production process assumptions and S-shaped growth behavior. Since there are inputs into the system and the system has activity, DA.1(a and b) do not apply. The system uses various source materials as inputs. Since some of those source materials are more heavily weighted than others, the source materials can be increased at a disproportionate rate (by including additional source materials that contribute more to the production process) without negatively affecting output. Since this is true, the strong input disposability (DA.3(b)) applies. Since a disproportionate increase in certain inputs is possible, then a proportionate increase in inputs will not negatively affect outputs, thus DA.2(a) applies. The S-shaped growth behavior has an upper limit that represents the maximum production capacity. Thus outputs are finite and bounded, so the scarcity axiom (DA.4) applies. The system also is a closed and convex combinations of inputs and outputs are part of the production possibility set, therefore DA.5 and DA.6 apply.

Next consider the data resident on the organization's servers where the exponential decay behavior is dominant. The data deterioration as a function of time is analogous to a system producing a desirable output (useful information) and an undesirable output (data that deteriorates over time). Since activity takes place within the system, DA.1(b) does not apply. Since the server does not have a feedback mechanism to the production process, it knows only of the inputs are arriving as a result of the production process, and is not concerned with any other aspect of the inputs. Therefore the input disposability axioms (DA.2(a and b)) do not hold. The strong output disposability axioms state that outputs can be costlessly disposed of. If the system's output is an exogenous variable (i.e., no feedback mechanism from the output to the system), then as data deteriorates it is disposed of without cost. As shown in section 3.2, if DA.3(b) applies than DA.3(a) also applies. As in the S-shaped growth behavior, the system is bounded, closed and convex combinations of inputs and outputs are part of the production possibility set, therefore DA.4, DA.5, and DA.6 apply.

Currently the organization allows the information on the server to degrade until it is no longer of any value to their customers. Then they re-evaluate if there is a true demand for the information. If there is not a demand, the information simply expires. If there is a continued demand, then the information is compiled from scratch. One of the policies that is being explored by this research is the concept of creating a data maintenance environment. In this environment data would be recompiled using the existing information base as long as there was a demand for the information. Implementing this policy/process improvement creates a feedback mechanism that triggers the recompilation process from the deteriorating data. This change in the structure eliminates the exponential decay behavior, thus changes the production assumptions. The new system structure combines the once decoupled production process and data deterioration structures into an S-shaped growth with overshoot structure (the oscillatory behavior is caused by the deterioration delay that occurs before information is recompiled). Therefore the dynamic axioms that apply to this structure (DA. 2(a and b), DA.3(a and b), DA.4, DA.4, and DA.6) now apply to the entire system.

3.4 Expanding Production Frontiers into the Dynamic Realm

Dynamic productive efficiency is concerned with optimizing system performance during a transient period [t_0 , t], from the initiation of the disturbance at time t_0 until the time when the system achieves a new level of equilibrium at time t. Dynamic productive efficiency has two important characteristics that distinguish it from the traditional notion of productivity. First, the element of time is introduced. The importance of time is realized during a transient period, as the projected system performance is compared against a dynamic production frontier or another DMU for the same time period. Second, a disturbance (i.e. a change in inputs, outputs, or processes that triggers a change in the others) is introduced into the system that causes it to seek a new equilibrium level.

Recall from Figure 3-1 that a disturbance was introduced into a system at time t_0 , causing the system to seek a new steady state level. Further recall that two options were presented to allow the system to achieve equilibrium over a time horizon [t_0 , t]. Figure 3-3 illustrates the two transient options. By inspection, transient option 2 is more efficient than transient option 1. The algorithms developed later in this chapter will describe how the optimal path to achieve a new level of equilibrium is found. But for the purposes of this discussion assume that transient option 2 was the calculated production frontier.

In static productive efficiency evaluations, data for a single DMU is generally collected over time, with the assumption that the system conditions are the same at each data collection point, thus allowing data collected a times t-1 and t to be treated and compared equivalently. Thus the data point with the best performance score is deemed the most efficient irrespective of time. If data is collected from multiple DMUs, a similar comparison occurs and again the best performance score is deemed the most efficient.

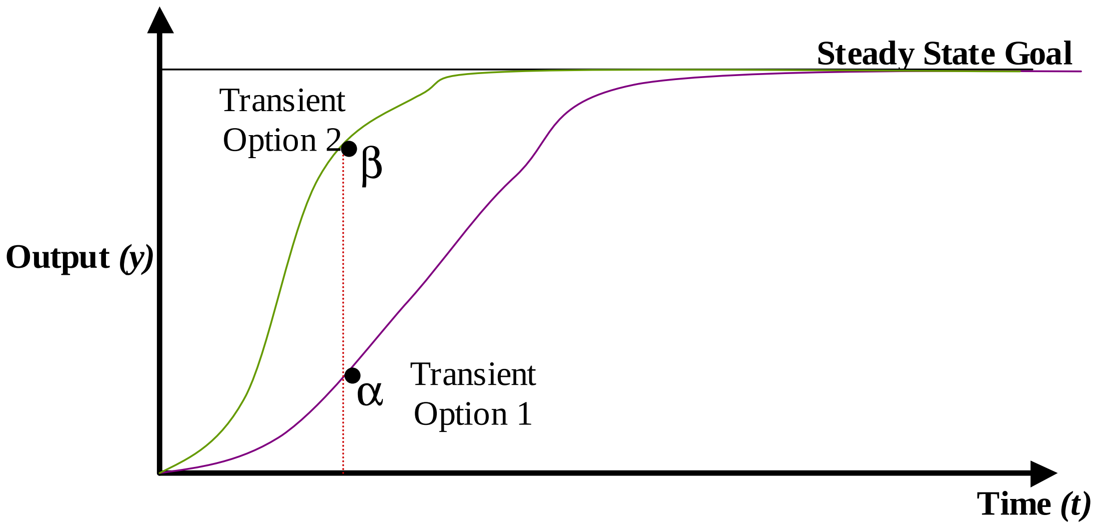

Figure 3-3. The Transient Options.

In dynamic productive efficiency, the assumption that all time periods are equivalent does not hold (thus data collected at time *t-1* cannot be compared with data collected at time *t*). Making such an assumption under normal conditions will always yield a higher productivity value for the data collected at *t*. By doing this, the decision-maker is not provided with how efficiently the system is seeking equilibrium, but simply that the current period was better then the former period(s). Therefore when comparing data points a DMU must be compared either to another DMU data point for the same time period, or against a point on the dynamic production frontier.

Figure 3-3 illustrates this fundamental concept. Points α and β are data points collected at the same time *t*, and therefore can be compared for performance evaluations. Examining points α and β more closely (Figure 3-4 (replicated from Figure 2-22)) the traditional isoquant and isocost lines are revealed for a single time period. Point β lies at the intersection of the isoquant and isocost line, thus is deemed both technically and allocatively efficient with respect to point α. Point α is initially located to the northeast of the isoquant and is deemed technically inefficient. For point α to be technically efficient, either inputs must be reduced by a factor of $0\alpha'/0\alpha$ to produce the same level of output, or outputs must increase by a factor of $0\alpha/0\alpha'$ when holding the inputs constant.

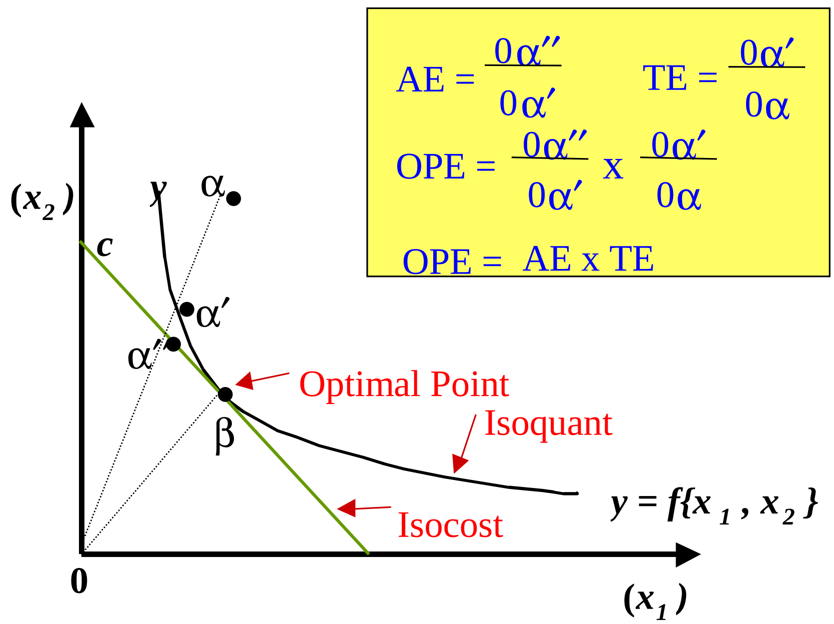

Figure 3-4. A Two-Dimensional, Static View of Efficiency.

To become allocatively efficient the cost of point α must be reduced by $0\alpha''/0\alpha'$. If point α is improved by a factor of $0\alpha''/0\alpha$, it will attain the same overall productive efficiency score as that obtained when at point β (Farrell, 1957).

Figure 3-4 is representative of a single time period. As subsequent time steps are encountered, the isoquants progress in a northeasterly direction (Figure 3-5). This procession indicates that output *y* is increasing from different input combinations. Along with the changing isoquant, the isocost lines also change. While not explicitly labeled, the pattern of outputs in Figure 3-5 could represent data collected at discrete points in time.

Adding a third input dimension x_3, and eliminating the isocost lines from Figure 3-5, yields the graph depicted in Figure 3-6. While the isoquants in Figure 3-5 represent discrete points, the surface in Figure 3-6 represents a surface, as would be found in a continuous time model, with the isoquants being represented at discrete points in time. As in Figure 3-5, the isoquants are progressing towards the northeast, as more output is being produced (i.e. $y_1<y_2<y_3<y_4$). Each isoquant represents a production frontier where all values on that frontier are efficient transformations of inputs into outputs within their

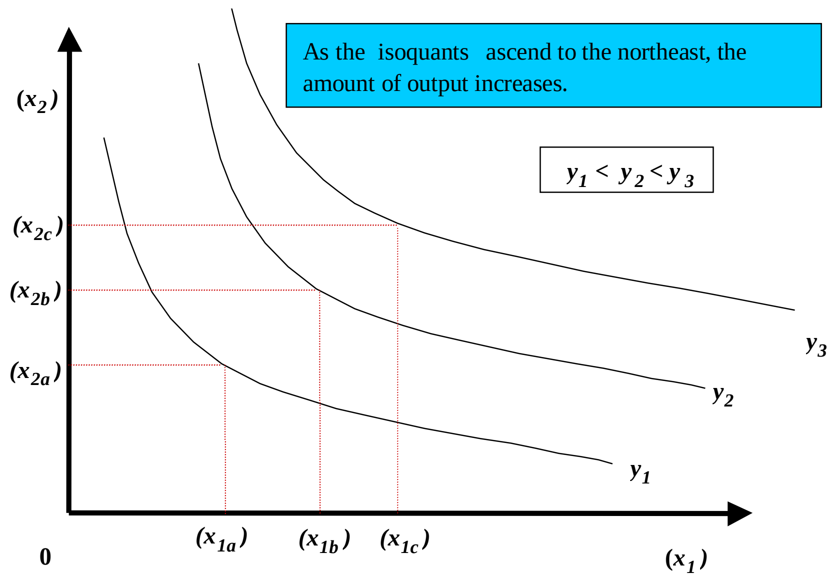

Figure 3-5. Isoquants Increase to the Northeast as Output Increases.

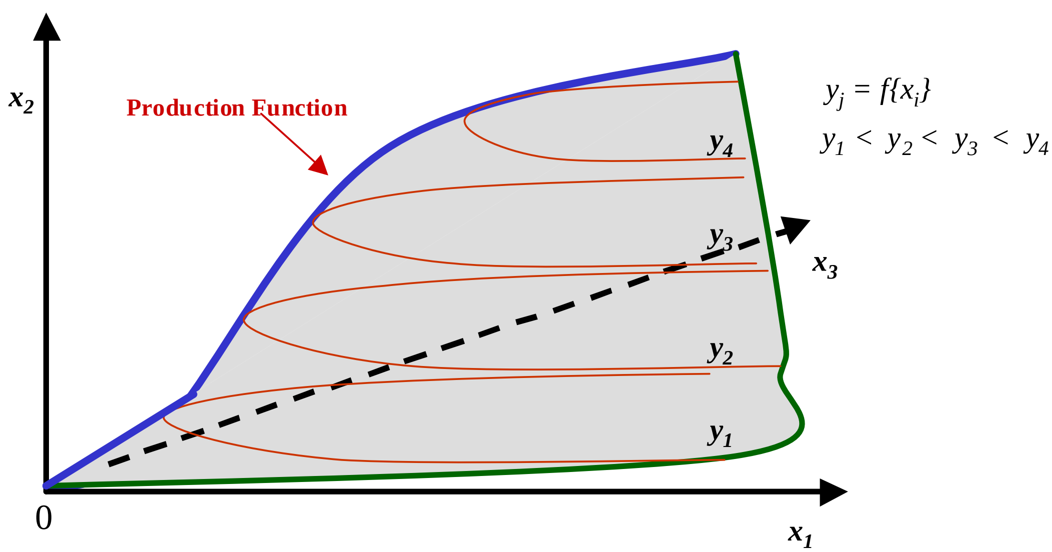

Figure 3-6. Three-Dimensional View of Productivity with the Production Function.

own time period t. The production function $y_j=f\{x_i\}$ is a sample efficient path that a system may take to efficiently achieve its new steady state level.

In hill-climbing search algorithms, the isoquants will make large increases at first, and then will gradually decline as they approach the new system equilibrium. Isoquant y_4 in Figure 3-6 represents the new equilibrium level.

The dynamic efficiency plane (Figure 3-7) expands the discussion further. This graph portrays two input axis x_1 and x_2, and a time axis t. Like the previous two graphs, the isoquants are increasing to the northeast. In a continuous time problem the level of output y is represented by an isoquantic plane. Likewise, the isocost lines are also represented in continuous time, and are depicted as the isocost plane.

In a two-dimensional (or static) representation, overall productive efficiency is achieved at the point of tangency between the isoquants and isocost lines (as depicted by point β in figure 3-4). In a continuous time environment overall productive efficiency is achieved along a line where the isoquant and isocost planes are tangential. This line is known as the *dynamic expansion line*[3]. The dynamic expansion line represents the most efficient (overall productive efficiency) path to traverse during a transient period $[t_0 , t]$. Section 3.5 presents an SD methodology that is designed to find the dynamic expansion line.

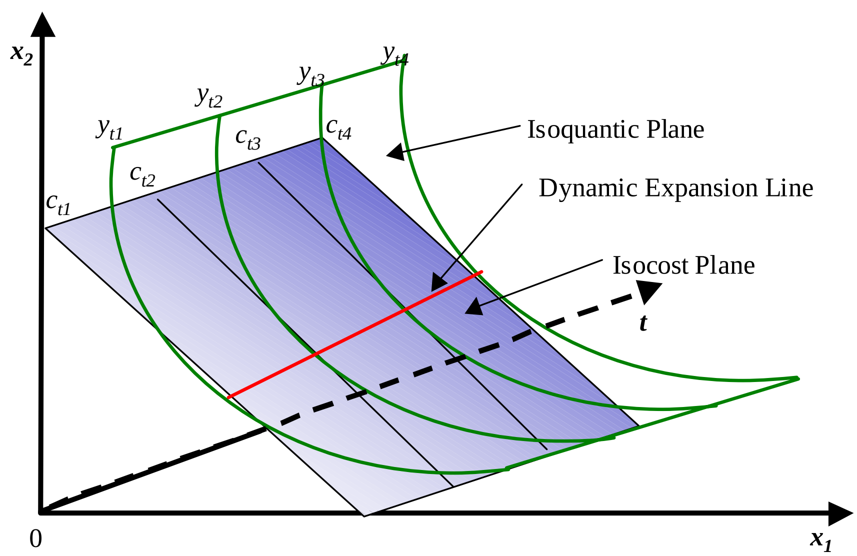

Figure 3-7. The Dynamic Efficiency Plane.

[3] For illustrative ease, the dynamic expansion line is represented as a straight line in Figure 3-7. In reality the dynamic expansion line will most likely not be a straight line, by vary with time.

3.5 The Dynamic Productive Efficiency Model

Wagner (1969, p. 515) states, "Most non-linearities encompassed in a programming model fall into two categories: (i) Empirically observed relationships, such as non-proportional variations in costs, process yields, and quality characteristics; (ii) Structurally derived relationships, which encompass postulated physical phenomena, mathematical deduced behavior, and managerially determined rules."

Examining dynamic productive efficiency against the backdrop of Wagner's (1969) definition, it is easy to see why an SD optimization model is needed to evaluate productivity during a transient period. First, transient behavior describes the changes in a system's character overtime. For example, a system that exhibits growth (i.e. market development, new plant construction, new production system implementation) experiences a change in character (Forrester, 1961). This change usually is a unique occurrence, as each new system initiative is different. Transient behavior usually exhibits non-proportional variations in, costs, use of inputs, and outputs produced. For example, Wright (1936) discovered an exponential relationship existed in the form of items produced per unit time in the aircraft industry. This illustration of transient behavior is commonly known as the learning curve phenomena.

Second, *production functions*[4] are not linear during a transient period. Takayama (1993, p. 49) defines a production function a "a function which describes the technological relation between various outputs and various inputs." The production function must satisfy three conditions to be compatible with the dynamic productive efficiency model. First, the production must be a well-behaved function (i.e. strict monotonicity, continuity, and quasi concavity). Second, the production function should be bounded such that observations are finite. Third, the production function should attribute all variation in output due to technical inefficiencies in the model (Wagner, 1969 and Kopp, 1981). When considering production functions, managerially determined rules and policies must be considered. Production functions are generally not known in

[4] I argue that production functions are generally non-linear for steady state operations, as they are a mathematical replication of real world production environments. However, I do concede that in some instances production functions can be approximated with linear solutions when the loss of the non-linear relationships is acceptable.

practice, and must be either mathematically deduced is gleaned from an understanding of the structural relationships.

The dynamic productive efficiency model takes various forms to correspond with the given problem. However, one of two basic forms (input-decreasing or output-increasing) serves as the foundation for all dynamic productive efficiency models. The input-decreasing model holds the level of output required constant and minimizes the input variables. The output-increasing model holds the input variables constant and maximizes the output variables.

3.5.1 The Input-decreasing Model

The generic input-decreasing model is shown in Figure 3-8. In its most basic form, the input-decreasing model is governed by three feedback loops – the system effect loop, the input x_i adjustment loop, and the input x_i search loop.

The purpose of the system effect loop is to represent the effects of the objective function and system constraints on the optimization structure. The input variables to the model are represents by the level variable x_i. The objective function for the model is the system's production function $y=f(x_i)$. Production goals y^* behave as in a typical goal seeking structure, however represent a form of a constraint. Production goals feed into a negative feedback loop and are designed to ensure that the optimal value of the inputs x_i lie within a finite interval such that $0 < x_i < \infty$. Production goals keep the system in balance through a variable that calculates the discrepancy (i.e. discrepancy comparison variable) between the current state of the objective function and the production goals.

The production goal is compared against the production function through the discrepancy comparison variable. When comparing the production function to the production goal, the following relationship is used:

$$\tilde{y} = \frac{y^*}{y_p} \qquad (3-26)$$

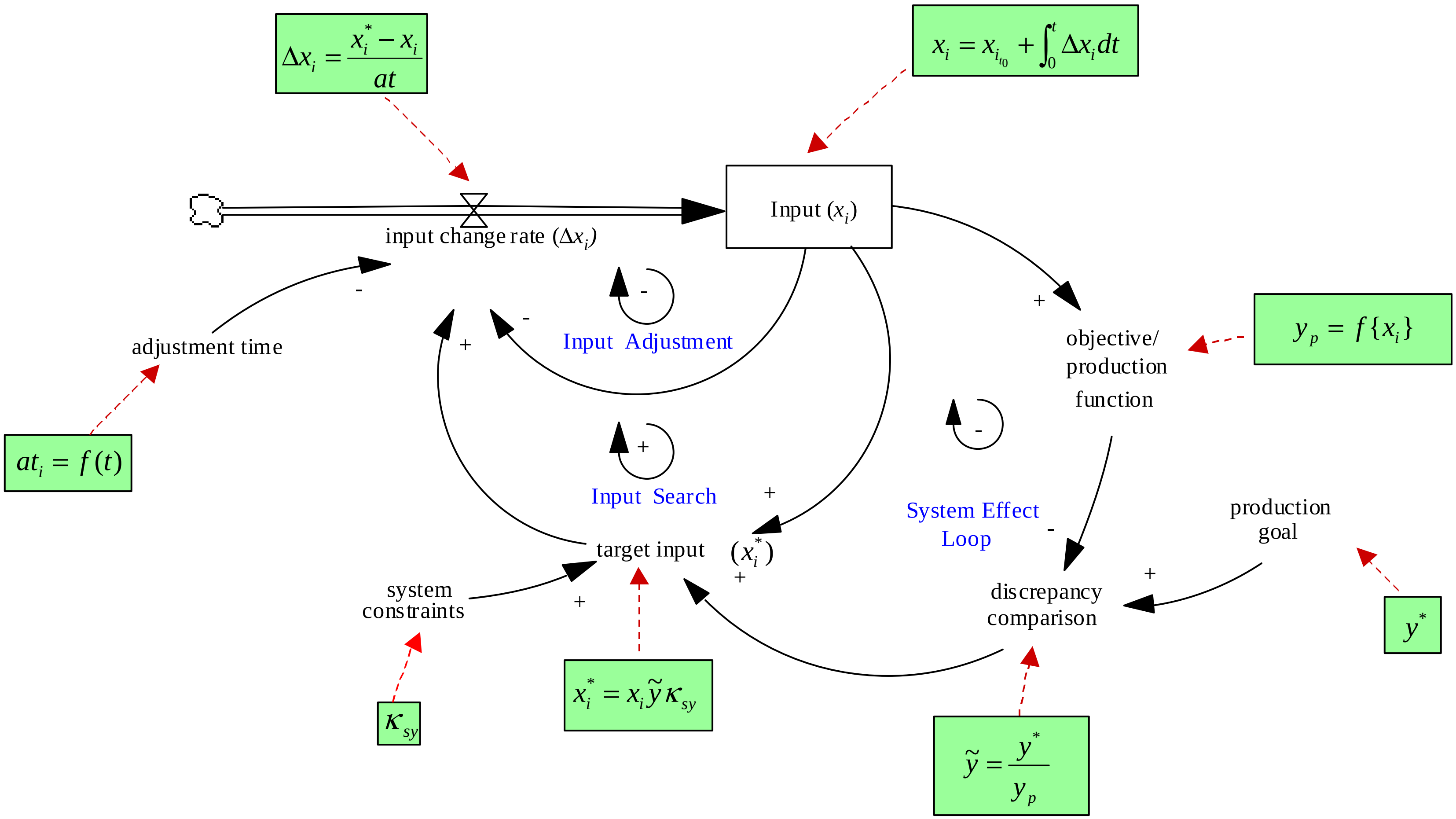

Figure 3-8. The Generic Dynamic Productive Efficiency (Input-Decreasing) Model.

The discrepancy comparison variable $\tilde{y}$ is one of the key drivers in determining if the level variable should be increased or decreased. Since a system may have many production goals the discrepancy comparison variables must be constructed such that the resulting units are dimensionless. When multiple constraints are present, their dimensionless nature allows the effects to be combined. For example, when the effect of the production goals are multiplicative, the total effect of discrepancy comparison $\tilde{y}$ of the system can be represented by:

$$\tilde{y} = \prod_{g=0}^{g=m} \tilde{y}_g \tag{3-27}$$

where $\tilde{y}_g = \tilde{y}_1, \tilde{y}_2, ... \tilde{y}_m$

While the multiplicative effects are the most common and realistic, additive effects are also possible and are mentioned here for completeness. These effects may be used occasionally because linear models are solved easily and facilitate parameter estimation by linear regression (Sterman, 2000). When using the additive effects the formulation is:

$$\tilde{y} = a_0 + a_1(\tilde{y}_1) + a_2(\tilde{y}_2) + ... + a_m(\tilde{y}_m) \tag{3-28}$$

where a_m is a weighting factor.

The hill-climbing optimization structure encompasses two feedback loops – input x_i adjustment and – input x_i search. Three primary variables form the basic composition of these feedback loops. As with any SD model, the level variable serves as the centerpiece of the system and it represents the state or condition of the system, and is the initiation point for any simulations (Forrester, 1968, Richardson and Pugh, 1981). In the input-decreasing model, the level variables are the inputs that are being minimized. The level variables in the input-decreasing model are defined as (Sterman, 2000):

$$x_i = x_{i_0} + \int_0^t \Delta x_i dt \tag{3-29}$$

As seen in 3-29, the level variable is atypical of level variables commonly found in SD models because it only contains an inflow rate. The reason for this is the hill-climbing optimization structure is designed to find the optimal values, thus is an artifact (i.e. algorithm that is used for a specific computational purpose, but would not be found in the causal mapping of an organizational structure) of the model. The level variable is adjusted by the structure of the hill-climbing algorithm.

The level variable is the hill-climbing algorithm link with the feedback loops that represent the physical system's structure. As previously discussed, the input x_i feeds directly into the production function, which is used to calculate the system's current level of performance. The level variable is adjusted by a rate variable (known as the input change rate (Δx_i)) which determines the change in the system's state due to optimization. The input change rate adjusts the level by considering the previous input level x_i, the target input Δx_i^*, and the adjustment time *at*. The input change rate is defined as (Sterman, 2000):

$$\Delta x_i = \frac{x_i^* - x_i}{at_i} \qquad (3-30)$$

The adjustment time at_i is the time required to place the input variable x_i into equilibrium (i.e. to achieve a new steady state) at its present rate of change. Adjustment time should not be confused with a time delay. A variable being affected by an adjustment time factor is a variable that is being used in a reduced capacity. For example, when a firm hires a new employee, the new employee does not contribute fully until after an adjustment period commonly known as the learning curve effect. A delay states that there is a time gap before the variable is available for use. Thus in our new employee example, the time delay exists from the time that the job vacancy is announced until the new employee is hired and starts working. In this case, the adjustment time would commence at the end of the time delay. The adjustment time is defined as:

$$at_i = f(t) \qquad (3-31)$$

The target input variable x_i^* is an artifact that is introduced into the hill climbing optimization structure to aid in searching for the optimal value of x_i. The variable considers the current state of the system (i.e. the value of the input variable x_i) and the effects of the system constraints κ_{sy} on the system, and the influence from the discrepancy comparison variable $\tilde{y}$. The target variable is defined as:

$$x*_i = x_i \tilde{y}_g \kappa_{sy} \qquad (3-32)$$

When working in tandem, the three feedback loops form a unimodal function[5] where there is a unique optimal value for x_i over the interval $0 < x_i < \infty$. The purpose of the input adjustment loop is to govern the rate at the gap between x_i and x_i^* is closed. The system effects loop applies the external pressure to cause x_i to seek its optimal value. If $\tilde{y}_g > 1$, the system effect loop will cause $x_i^* > x_i$ and hence put a pressure on the input search loop to increase the input variable x_i. If $\tilde{y}_g < 1$, the system effect loop will cause $x_i^* < x_i$ and will put pressure on the input search loop to decrease the input variable x_i. When $\tilde{y}_g = 1$, the system effect loop will apply no further pressure to the input search loop, and will cause $x_i^* = x_i$ (Wagner, 1969; Sterman, 2000). When this occurs the optimal conditions have been achieved.

There are a few caveats to this approach. First, the production function must be a continuous function and the input variables must be bounded. Second, a sufficient number of simulation iterations must be made to ensure the model achieves it global maximum value. Since it is possible for the system to have many local maximums, the only way to ensure that the global maximum is found is by repeated simulation runs and varying the time steps used in the simulations. Third, as in the real world, there is no guarantee that the model will converge. If it does converge, the achieved state of the system may or may not be the optimal value even after a sufficient number of simulation iterations (Wagner, 1969; Sterman, 2000).

[5] A unimodal function has one mode (usually a maximum, but could mean a minimum, depending on the context). If *f* is defined on the interval [*a*, *b*], let x^* be its mode. Then, *f* strictly increases from *a* to x^* and strictly decreases from x^* to *b* (Wagner, 1969).

Until now, the discussion has centered on the input-decreasing model with multiple inputs and a single output. This may be a realistic model for a firm that has one production line and produces only one output. However, a more flexible model is needed when the firm produces more than one product, or has more than one production line with dissimilar production activities (e.g. two production plants producing the same output with different processes). Figure 3-9 shows the generic dynamic productive efficiency input-decreasing model with two dissimilar production activities.

Since the system has two dissimilar production activities, it contains two different production functions. The goal to minimize the common inputs x_i between the two processes. Each input's contribution factor *cf* for each production function is known and is $cf_i \leq 1$, such that $\sum cf_i = 1$. Total production for the systems is the cumulative total of all the production functions such that:

$$y = \sum_{j=1}^{j-m} y_j \qquad (3-33)$$

Similar system constraints are also combined and compared in an appropriate way. In Figure 3-9, the production goals represent the system constraints. This they are combined by:

$$y^* = \sum_{j=1}^{j=m} y*_j \qquad (3-34)$$

The production goal is compare the total production through the discrepancy comparison variable $\tilde{y}$ as shown in equation 3-26. The cumulative effects of the system are computed as shown in equation 3-27. This allows the dynamic productive efficiency model to be reduced to its basic form as shown in Figure 3-8. Once is this basic form, the model is solved as before.

<u>3.5.2 The Output-increasing Model</u>

The purpose of traditional output-increasing models is to determine the amount that the output can increase while holding the inputs constant. Like the input-decreasing model, the output-increasing model is governed by the three feedback loops, as shown in Figure 3-10. The production function serves as the objective function, but unlike the input-decreasing model, the production function is exogenous the three loop dynamic productive efficiency structure. Since the inputs are exogenous to the structure, they are treated as auxiliary variables, and the outputs are the level variables, and thus optimized. The output variables y_j are defined as:

$$y_j = y_{j_{t_o}} + \int_0^t \Delta y_j dt \tag{3-35}$$

Where $y_{j_{t_0}}$ is the initial output variable, and Δy_j is the change factor output change rate as defined by:

$$\Delta y_j = \frac{y_j^* - y_j}{at_j} \tag{3-36}$$

The production function in the output-increasing model behaves like the system goal in the input-decreasing model. The production function is designed to provide the maximum theoretical value of a production process. Since this equation also serves as the production goal, the dynamic productive efficiency structure will always seek its theoretical value as its optimal value.

This point can be easily demonstrated because during the initial iteration of the model, the relative output will be greater than or equal to one:

$$\tilde{y} = \frac{y_p}{y_{i_0}} \geq 1 \tag{3-37}$$

because the output's initial value $y_{j_{t_0}} \leq y_p$. Thus if no other system constraints are present, the target output variable y_j^* and the output change rate Δy_j will cause the system to adjust to the production function value. If $at_j = 1$, then the output y_j will assume the production function value on the second iteration.

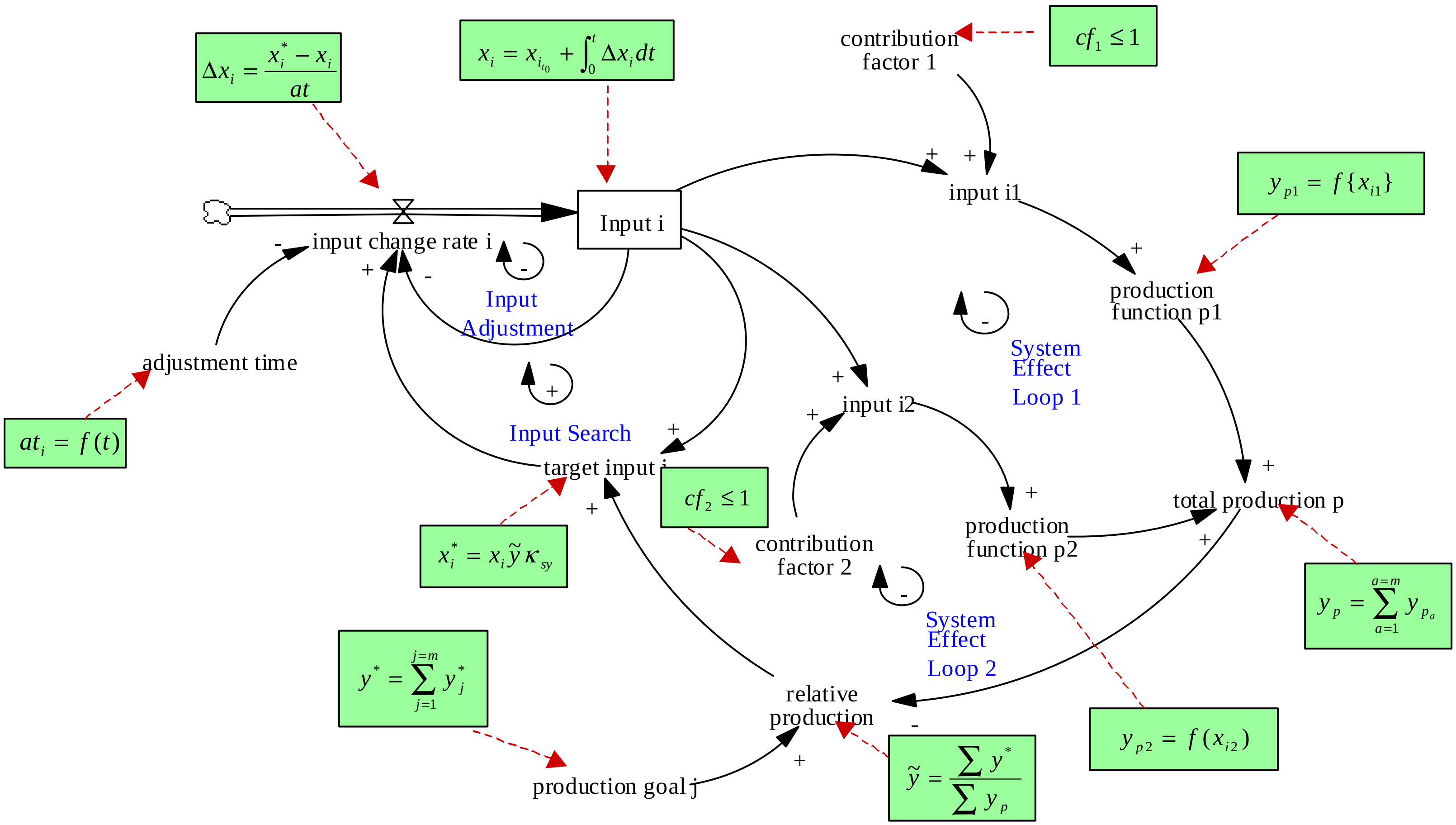

Figure 3-9. Multiple Inputs and Multiple Outputs (Input-Decreasing) Model.

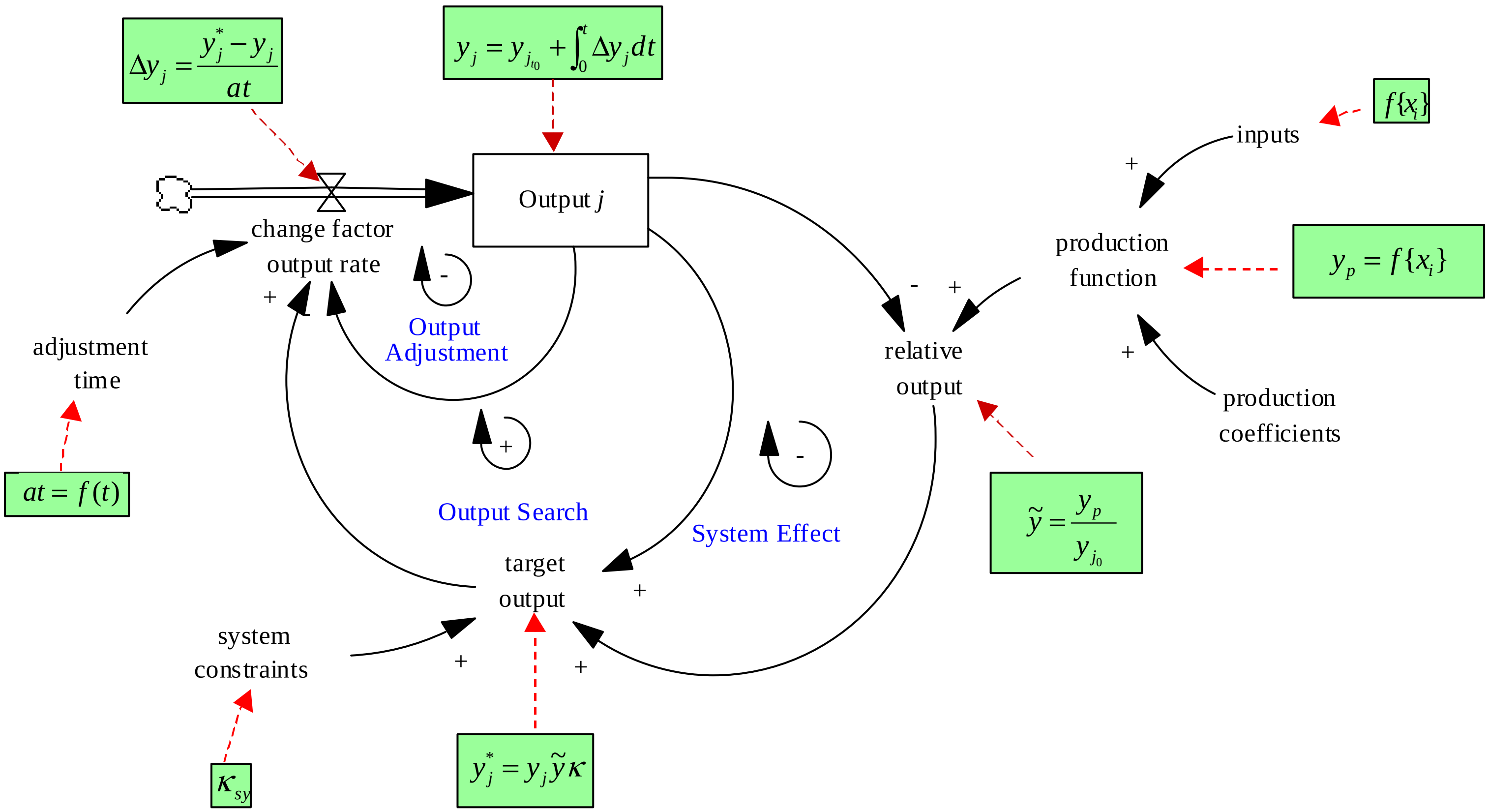

Figure 3-10. The Generic Dynamic Productive Efficiency (Output-Increasing) Model.

The value of this model is realized when system constraints κ_{sy} are present, and $at_j > 1$. When these conditions exist, the system output y_j may or may not achieve the value of the production function y_p, and the model will provide the maximum achievable value for y_j. The value of y_j is important because it allows the decision-maker to: (i) realize the impact of constraints on the system: and (ii) know when the system has achieved its optimal value.

Another strength of this model is that the inputs do not necessarily need to be held constant. The variable nature of these inputs may be a result of seasonal fluctuations. Treating the inputs x_i as variables, coupled with an adjustment time where at_j >1, will show how fast the system reacts to seasonal variation. This is important because it allows the decision-maker the ability to readjust inputs to achieve the maximum output as the market demands.

3.6 An Illustration of the Dynamic Productive Efficiency Model

To illustrate the use of the dynamic productive efficiency model, an widely used (albeit rudimentary) example (Schmidt and Lovell, 1979;Kopp, 1981) was selected. The example examines an oil-fired, steam-generating plant that transforms capital *K*, fuel oil *F*, and labor *L* into electric power. The mathematical form of this transformation takes the form of a Cobb-Douglas (1928) production function[6], and is defined as:

$$Q = 0.049K^{.25}L^{.1}F^{.7} \qquad (3-38)$$

Since the exponents in the production function do not sum to one, the systems exhibits non-constant returns to scale[7].

The initial plant operating conditions and related characteristics are shown in Table 3-3 (Kopp, 1981).

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[6] The Cobb-Douglas (Cobb and Douglas, 1928) production function is defined as: $Q = A\prod_{i-1}^{i=n} x_i^{\alpha i}$.

[7] A system governed by a Cobb-Douglas production function exhibits if the sum of the exponents is equal to one. If the sum of the exponents is less than one, the system experiences decreasing returns to scale, and if greater than one, experiences increasing returns to scale (Cobb and Douglas, 1928, Sterman, 2000).

Table 3-3. Initial Plant Conditions and Operating Characteristics.

Variable	Inputs Consumed	Input Prices
K	500 megawatts	$12,000/ megawatt
L	50 employees	$20,000/employee
F	162,754 x 10^8 BTUs	$65/$10^8$ BTUs
Q	1,300 x 10^6 kilowatt hrs.	Total Cost = $17,579,010

The goal of this analysis is to determine the most efficient technical solution (technical efficiency), the most efficient cost solution (allocative efficiency) and the overall productive efficiency.

Figure 3-11(a-c) illustrates the dynamic productive efficiency structure for this problem. Figure 3-11(a) represents the structure for capital and all of the model's common variables (e.g. the production function and the production goal). Figure 3-11(b) represents the structure for the fuel variable, and Figure 3-11(c) represents the structure for the labor variable.

The system will be solved for technical efficiency first. The initial values for the technical efficiency problem are $K_{t_0} = 500$ megawatts, F_{t_0} = 162,754 x 10^8 BTUs, L_{t_0} = 50 employees, and is producing 1,300 x 10^6 kilowatt hrs of electricity. Applying the initial inputs to equation 3-38, it can be shown that at t=0, the initial system output should be Q = 1,524 x 10^6 kilowatt hrs. Therefore since 1,524 x 10^6 kilowatt hrs > 1,300 x 10^6 kilowatt hrs, the system is working inefficiently, and the inputs can be reduced to achieve the same level of output.

Simulating the model in Figure 3-11 (a-c), with the procedure described in Section 3.5.1, yields the optimized values for the input parameters (K = 430 megawatts, F = 139,910 x 10^8 BTUs, L = 43 employees), when Q = 1,300 x 10^6 kilowatt hrs. Since this application of the model is multi-factored, technical efficiency is calculated by comparing the total cost gleaned from the optimal values to the total initial cost[8]. The technical efficiency and costs module (Figure 3-11(d)) performs this calculation. The total cost for the optimized values is $15,114, 150. Comparing the total cost for the optimized values with the initial total costs yields a technical efficiency equal to .86.

[8] Single-factor productivity would be derived through a similar structure. However, since like quantities are being compared, cost data is not necessary to determine technical productivity.

Subtracting .86 from 1 and multiplying by 100, shows that by using the inputs more efficiently, the electric plant can save 14% in their operating costs.

Table 3-4 illustrates the technical efficiency comparison between Kopp's (1981) solution and the single-adjustment period of the dynamic productive efficiency model. Differences between the solutions can be attributed to rounding errors.

Kopp (1981) solves this problem through simultaneous equations. The problem with this approach is that it assumes that there will be an immediate adjustment time for each input variable (i.e. $at_i = 1$). While this condition allowed Kopp to determine the potential new system steady state, it failed on two accounts. First, since it was assumed that the input variables would contribute their full potential, losses associated with the reorganizing the employees into a smaller work-force, or the reconfiguration of the equipment was not considered. Second, Kopp's formulation did not provide decision-makers with a benchmark for comparison against actual progress during the transient period. The dynamic productive efficiency model achieved a steady state in six iterations. This is significant because even with the relaxed adjustment assumptions presented in this problem, the system did not adjust immediately.

To test these claims against the dynamic productive efficiency model, the adjustment times for the capital and labor variables are changed to *Kat* = 10 months, and *Lat* = 4 months. When these adjustment times are simulated, the new steady state

Table 3-4. Technical Efficiency Comparison.

	Kopp's Solution	**Dynamic Productive efficiency Model Solution**
Capital (*K*) (megawatts)	430	430
Fuel (*F*) (x 10^8 BTUs)	139,965	139,910
Labor (*L*) (employees)	43	43
Electricity Produced (*Q*) (x 10^6 kilowatt hours)	1,300	1,300
Total Cost (*TCTE*) ($)	15,117,725[9]	15,114,150
Technical Efficiency (*TE*)	.86	.86

[9] Kopp (1981) actually records the total cost as $15,020,365, however calculating the total cost based upon the optimal inputs found and the fixed prices given, $15,117,725 results.

conditions are: K = 490 megawatts , F = 131,603 x 10^8 BTUs, L = 48 employees. The total costs to produce 1,300 x 10^6 kilowatt hrs is $15,394,195 at its new steady state. The technical efficiency with the is .88, thus only a 12 percent gain can be made when adjustment times are included. The period-by-period inputs and cost projections are provided for an 11-month period in Table 3-5. The incremental changes are small dues to the simplicity of this model. However, while the changes in total cost are small between periods, the changes between the more detailed costs in this simulation compared to the total cost of the simulation where all adjustment times equal unity is significant (approximately one quarter of a million dollars over an 11 month period!).

The structure in Figure 3-11 (a-c) is used again to solve for both technical and allocative efficiency. To solve this problem the objective function (equation 3-38) is expanded to include the cost, and the Cobb-Douglas production function takes the form (Schmidt and Lovell, 1979; Triantis, 1995):

$$Z = p_1 x_1 + p_2 x_2 + p_3 x_3 + b + \xi\left[Q - A x_1^{\alpha 1} x_2^{\alpha 2} x_3^{\alpha 3}\right] \tag{3-39}$$

Table 3-5. Period-by-Period Input and Cost Projections.

Month	**Capital (K)** (megawatts)	**Fuel (F)** (x 10^8 BTUs)	**Labor** (employees)	**Total Cost** ($)
0	500	162,754	50	17,579,010
1	487.76	138,856	48	15,838,760
2	484.75	133,718	48	15,468,670
3	484.04	132,542	48	15,383,710
4	483.87	132,269	48	15,363,925
5	483.84	132,205	48	15,359,405
6	483.83	132,190	48	15,358,310
7	483.82	132,187	48	15,357,995
8	483.82	132,186	48	15,357,930
9	483.82	132, 186	48	15,357,930
10	483.82	132,186	48	15,357,930
			Total	**168,508,425**

Where ξ ensures the rate of change is in the proper direction with respect to cost. Taking the partial derivative of Z with respect to x_1 , x_2 , x_3 , and ξ yields:

$$\frac{\partial Z}{\partial x_1} = p_1 - \xi A\alpha 1 x_1^{\alpha 1-1} x_2^{\alpha 2} x_3^{\alpha 3} = p_1 - \xi \frac{\alpha 1 Q}{x_1} \Rightarrow p_1 x_1 = \xi \alpha 1 Q \tag{3-40}$$

$$\frac{\partial Z}{\partial x 2} = p_2 - \xi A\alpha 2 x_1^{\alpha 1} x_2^{\alpha 2-1} x_3^{\alpha 3} = p_2 - \xi \frac{\alpha 2 Q}{x_2} \Rightarrow p_2 x_2 = \xi \alpha 2 Q \tag{3-41}$$

$$\frac{\partial Z}{\partial x 3} = p_3 - \xi A\alpha 3 x_1^{\alpha 1} x_2^{\alpha 2} x_3^{\alpha 3-1} = p_3 - \xi \frac{\alpha 3 Q}{x_3} \Rightarrow p_3 x_3 = \xi \alpha 3 Q \tag{3-42}$$

$$\frac{\partial Z}{\partial \xi} = 0 \Rightarrow Q = A x_1^{\alpha 1} x_2^{\alpha 2} x_3^{\alpha 3} \tag{3-43}$$

To solve equation 3-43, make all input terms equivalent by using the partial derivatives found in equations 3-40, 3-41, and 3-42. Thus placing x_2 and x_3 in terms of x_1 is achieved by dividing 3-40 by 3-41 to yield:

$$x_2 = \left(\frac{p_1 \alpha 2}{p_2 \alpha 1} \right) x_1 \tag{3-44}$$

And dividing 3-40 by 3-42, yields (Triantis, 1995):

$$x_3 = \left(\frac{p_1 \alpha 3}{p_3 \alpha 1} \right) x_1 \tag{3-45}$$

Thus from equations 3-44 and 3-45, the input variables must be stated in the terms of a single input variable, and they can be related through the cost data.

The objective for the oil-fired, steam generating plant's technical and allocative efficiency is gleaned by substituting the system's variables into equation 3-40, yielding:

$$Q = (pK)(K) + (pF)(F) + (pL)(L) + \left[Q - .049K^{.25}L^{.1}F^{.7}\right] \qquad (3\text{-}46)$$

Equation 3-46 can be reduced to a standard Cobb-Douglas production function (equation 3-38), by restating the labor and fuel variables as capital variables. This is achieved by substituting the variables related to *F* into equation 3-44:

$$F = \left[\frac{(pK)(.7)}{(pF)(.25)}\right]K \qquad (3\text{-}47)$$

And substituting the variables related to *L* into equation 3-45:

$$L = \left[\frac{(pK)(.1)}{(pL)(.25)}\right]K \qquad (3\text{-}48)$$

The computations for 3-47 and 3-48 are included in the dynamic productive efficiency model (Figure 3-11(e)).

From equations 3-47 and 3-48, the initial conditions for the technical and allocative efficiency model are K_{t_0} = 500 megawatts, F_{t_0} = 258, 461 x 10^8 BTUs, and $L_{t_0} = 120$ employees. Applying the initial condition to the model (Figure 3-11(a-c)) yields the new optimized steady state conditions for technical and allocative efficiency as *K* = 291 megawatts , *F* = 150,174 x 10^8 BTUs, *L* = 70 employees.

Figure 3-11(f) depicts the structure used for calculating the total cost of the allocative efficiency option (*TCAE*), the allocative efficiency (*AE*), and the overall productive efficiency (*OPE*). Allocative efficiency is calculated by the ratio:

$$AE = \frac{TCAE}{TCTE} \tag{3-49}$$

And, overall productive efficiency is calculated by:

$$OPE = \frac{TCAE}{TCw/o} \tag{3-50}$$

From the cost and allocative efficiency structure (Figure 3-11(f)), *TCAE* = \$14,653,310, *AE* = .97, and *OPE* = .83. Thus of the steam generation plant implemented the technical and allocative efficiency solution, the production cost would be reduced by 17 percent (i.e. (1-.*OPE*)(100%)). This is a 3 percent reduction in cost with respect to the reductions that can be made with technical efficiency alone (i.e. (1-*AE*)(100%)).

Table 3-6 shows a comparison between the problem solved with Kopp's (1981) approach, and the solution gleaned from the dynamic productive efficiency model. Difference in the results can be attributed to rounding errors in the original solution.

Table 3-6. Allocative Efficiency Comparison.

	Kopp's Solution	**Dynamic Productive efficiency Model Solution**
Capital (*K*) (megawatts)	286	291
Fuel (*F*) (x 10^8 BTUs)	147,912	150,174
Labor (*L*) (employees)	69	70
Electricity Produced (*Q*) (x 10^6 kilowatt hours)	1,300	1,300
Total Cost (*TCAE*)(\$)	14,426,280	14,653,310
Allocative Efficiency (*AE*)	.95	.97
Overall Productive Efficiency (*OPE*)	.82	.83

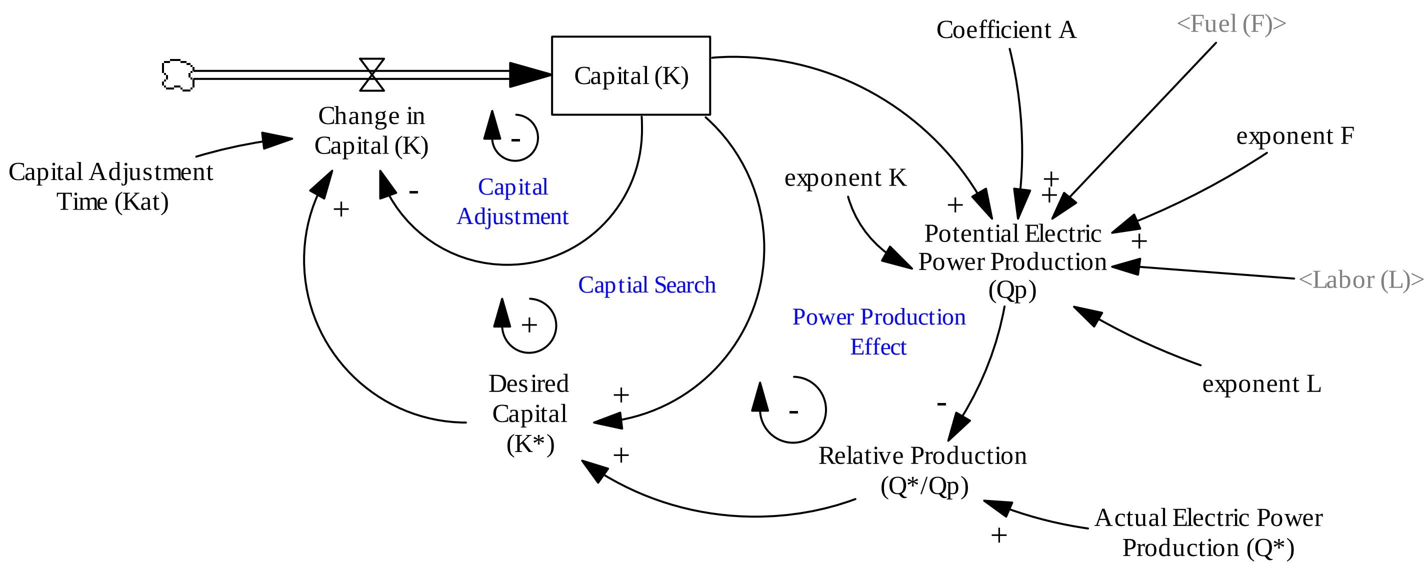

Figure 3-11(a). Capital Optimization Structure.[10]

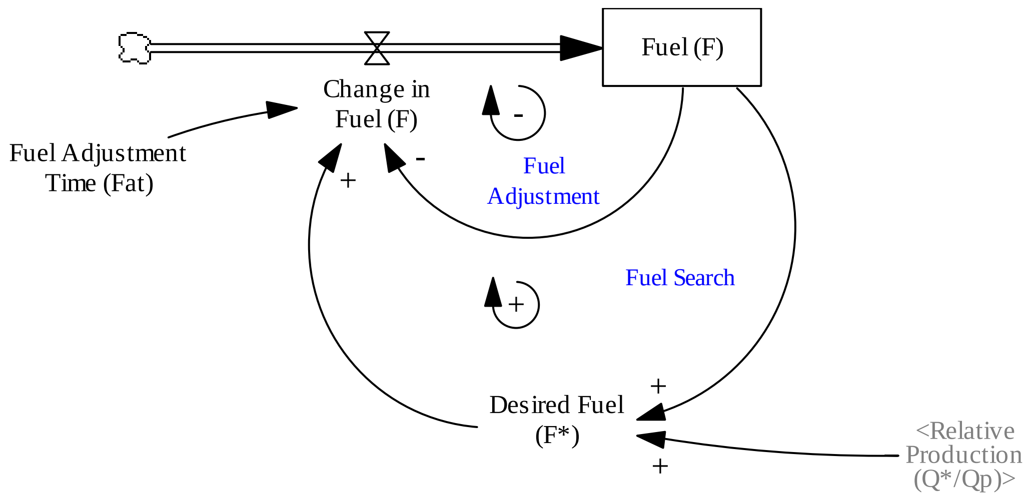

Figure 3-11(b). Fuel Optimization Structure.

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10 The text of the electricity generation – technical efficiency model is located in Appendix B, and the allocative model is located in Appendix C.

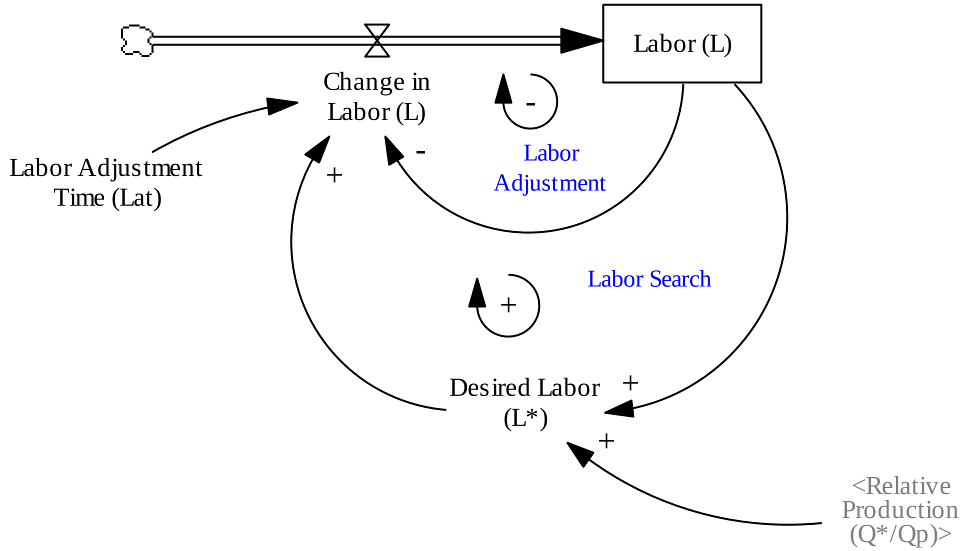

Figure 3-11(c). Labor Optimization Structure.

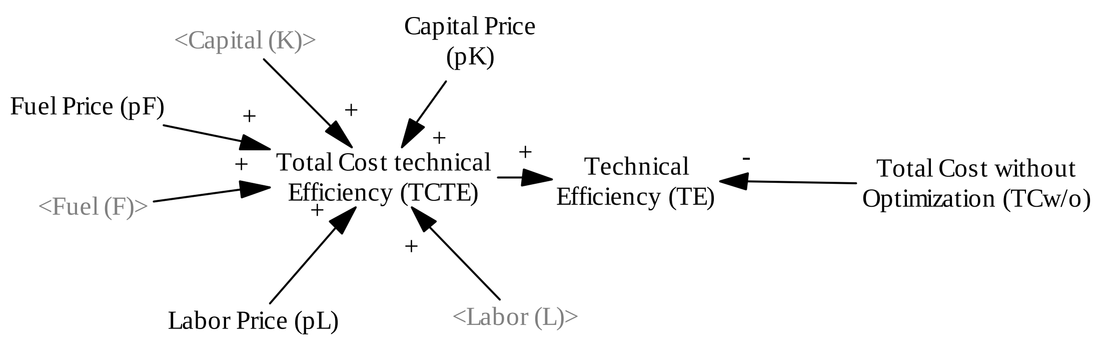

Figure 3-11(d). Technical Efficiency and Cost Calculation Structure.

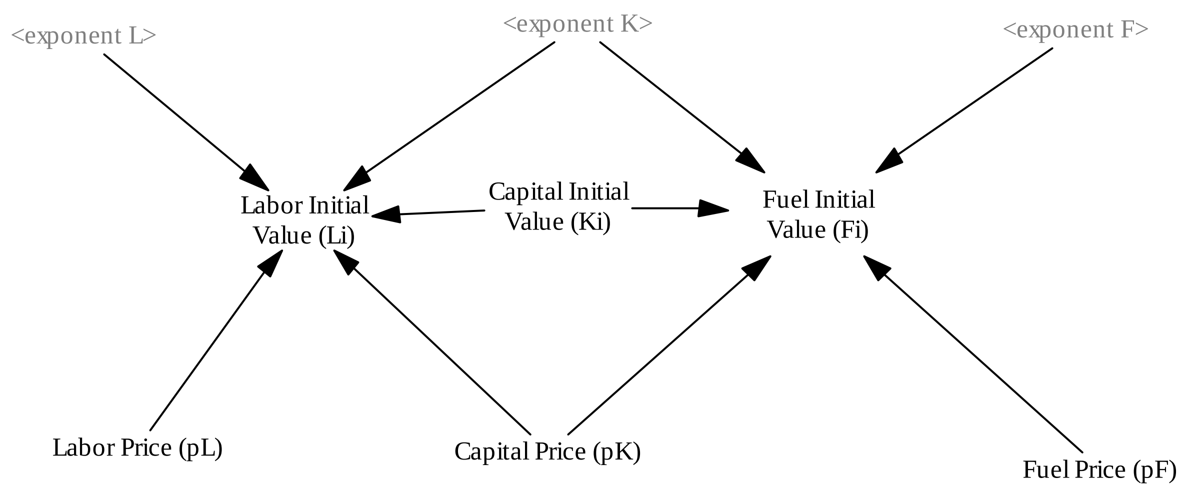

Figure 3-11(e). Allocative Efficiency Initial Conditions Calculation Structure.

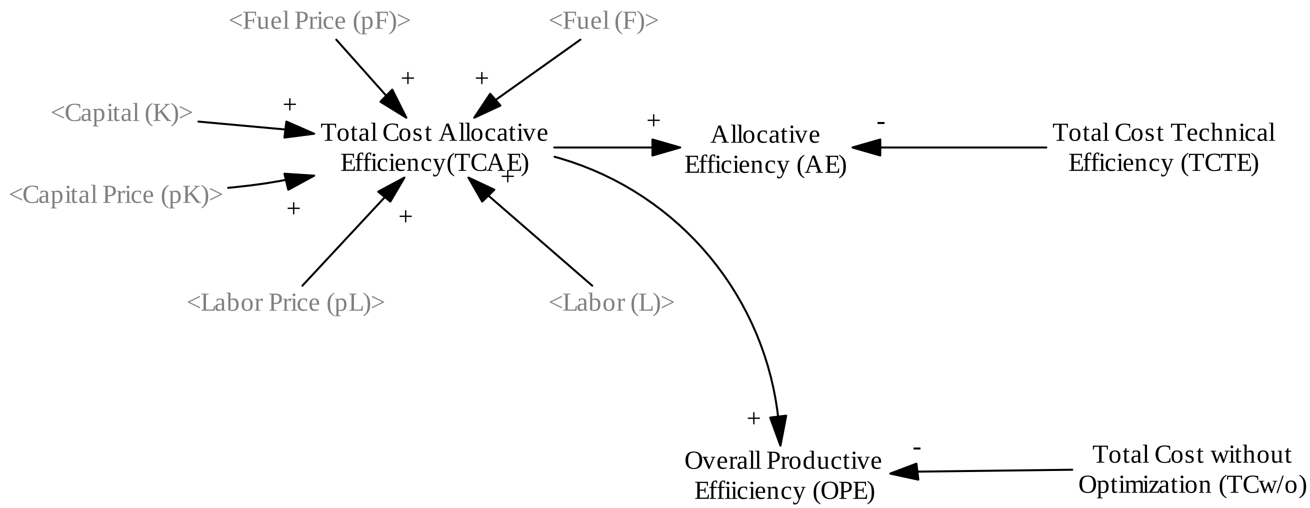

Figure 3-11(f). Allocative and Overall Productive Efficiency and Cost Calculation Structure.

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