Appendix B Equations for Residual Forces

Although numerous papers pertaining to the FE equations governing pore fluid flow in a deforming porous solid are available, none outline the equations required to calculate the residual forces. In this section, these equations are developed for a nonlinear incremental finite element program that employs a modified Newton-Raphson solution scheme.

The time integration of Equations 2.5 and 2.6 is performed using the following approximation:

$$\int_{t}^{t+\Delta t} \chi \, dt = \alpha \, \Delta t \, {}^{t+\Delta t}\chi + (1 - \alpha) \, \Delta t \, {}^{t}\chi \qquad \text{B.1}$$

for $0 \leq \alpha \leq 1$. From Equation B.1, the following are developed:

$${}^{t+\alpha\Delta t}\dot{\chi} = \frac{{}^{t+\Delta t}\chi - {}^{t}\chi}{\Delta t} \qquad \text{B.2}$$

and

$${}^{t+\alpha\Delta t}\chi = (1-\alpha)\, {}^{t}\chi + \alpha \, {}^{t+\Delta t}\chi \qquad \text{B.3}$$

Table B.1.
Time Integration Parameters

Value α	Difference Scheme	Stability
0	Forward or Euler	Conditionally
½	Crank-Nicolson	Unconditionally
⅔	Galerkin	Unconditionally
1	Backward	Conditionally

Table B.1 gives the most common difference schemes adopted by the selection of a given value of α. Equation 3.5 may be written for a given time $t+\alpha\Delta t$ as:

$$ {}^{t+\alpha\Delta t}[K]\;{}^{t+\alpha\Delta t}\dot{u} \;-\; {}^{t+\alpha\Delta t}[L]\;{}^{t+\alpha\Delta t}\dot{\pi} \;=\; {}^{t+\alpha\Delta t}\dot{R} \quad . \qquad \text{B.4} $$

Equations B.2 and B.3 are introduced into Equation B.4 to produce the following

$$ {}^{t+\Delta t}[K]\left\{\frac{{}^{t+\Delta t}u \;-\; {}^{t}u}{\Delta t}\right\} \;-\; {}^{t+\Delta t}[L]\left\{\frac{{}^{t+\Delta t}\pi \;-\; {}^{t}\pi}{\Delta t}\right\} = \frac{{}^{t+\Delta t}R - {}^{t}R}{\Delta t} \qquad \text{B.5} $$

Multiplying by Δt and collecting terms, one obtains

$$ {}^{t+\Delta t}[K]\left\{{}^{t+\Delta t}u \;-\; {}^{t}u\right\} \;-\; {}^{t+\Delta t}[L]\left\{{}^{t+\Delta t}\pi \;-\; {}^{t}\pi\right\} = {}^{t+\Delta t}R - {}^{t}R \quad . \qquad \text{B.6} $$

In general, Equation B.6 represents nonlinear behavior. The relationship may be linearized with the following expressions:

$$ {}^{t+\Delta t}u^{(i)} \;=\; {}^{t+\Delta t}u^{(i-1)} \;+\; \delta u^{(i)} \qquad \text{B.7} $$

$$ {}^{t+\Delta t}\pi^{(i)} \;=\; {}^{t+\Delta t}\pi^{(i-1)} \;+\; \delta \pi^{(i)} \qquad \text{B.8} $$

where i represents the current iteration, and the initial conditions are ${}^{t+\Delta t}u^{(0)} = {}^{t}u$ and

${}^{t+\Delta t}\pi^{(0)} = {}^{t}\pi$. This linearization can be used as the first step in a Newton-Raphson iteration (Bathe 1982). If Equations B.7 and B.8 are substituted into Equation B.6 and the terms for iterations (*i-1*) and (*0*) are brought to the right hand side of the equation, then one obtains:

$$\begin{aligned} {}^{t+\Delta t}[K]\delta u^{(i)} - {}^{t+\Delta t}[L]\delta\pi^{(i)} &= {}^{t+\Delta t}R - {}^{t}R \\ &- {}^{t+\Delta t}[K]\left\{{}^{t+\Delta t}u^{(i-1)} - {}^{t+\Delta t}u^{(0)}\right\} \\ &+ {}^{t+\Delta t}[L]\left\{{}^{t+\Delta t}\pi^{(i-1)} - {}^{t+\Delta t}\pi^{(0)}\right\} \quad . \end{aligned} \tag{B.9}$$

Recognizing that

$${}^{t}R = {}^{t+\Delta t}[K]\ {}^{t+\Delta t}u^{(0)} - {}^{t+\Delta t}[L]\ {}^{t+\Delta t}\pi^{(0)} \tag{B.10}$$

Equation B.9 may be written in terms of the incremental or accumulative stresses and pore pressures as

$$\begin{aligned} &{}^{t+\Delta t}[K]\delta u^{(i)} - {}^{t+\Delta t}[L]\delta\pi^{(i)} = \\ &\quad {}^{t+\Delta t}R - {}^{t+\Delta t}F^{(i-1)} + {}^{t+\Delta t}C^{(i-1)} \end{aligned} \tag{B.11}$$

where $\quad {}^{t+\Delta t}F^{(i-1)} = \int_V B^T\ {}^{t+\Delta t}\sigma^{(i-1)}\, dV \quad$ and

$${}^{t+\Delta t}C^{(i-1)} = {}^{t+\Delta t}[L]^{(i-1)}\ {}^{t+\Delta t}\pi^{(i-1)}.$$

Equation B.11 is one of the two equations required to solve for pore fluid flow in a deforming nonlinear porous solid. The second equation is developed from Equation 3.6 and is written at time $t+\alpha\Delta t$ as:

$$[H]\ {}^{t+\alpha\Delta t}\pi - [S]\ {}^{t+\alpha\Delta t}\dot{\pi} - [L]^T\ {}^{t+\alpha\Delta t}\dot{u} = {}^{t+\alpha\Delta t}Q \quad . \tag{B.12}$$

Equations B.2 and B.3 are introduced into Equation B.12 to produce

$$[H]\left\{(1-\alpha)\ {}^{t}\pi + \alpha\ {}^{t+\Delta t}\pi\right\} - [S]\left\{\frac{{}^{t+\Delta t}\pi - {}^{t}\pi}{\Delta t}\right\}$$

$$- [L]^{T}\left\{\frac{{}^{t+\Delta t}u - {}^{t}u}{\Delta t}\right\} \qquad \text{B.13}$$

$$= (1-\alpha)\ {}^{t}Q + \alpha\ {}^{t+\Delta t}Q \quad .$$

Collecting terms and multiplying by Δt one obtains:

$$\left\{-[S] + \alpha\Delta t[H]\right\}\ {}^{t+\Delta t}\pi - [L]^{T}\left\{{}^{t+\Delta t}u - {}^{t}u\right\}$$

$$+ \left\{[S] + (1-\alpha)\Delta t[H]\right\}{}^{t}\pi = P \qquad \text{B.14}$$

where $P = \Delta t\left\{(1-\alpha)\ {}^{t}Q + \alpha\ {}^{t+\Delta t}Q\right\}$. If the following equations

$$\hat{S} = -[S] + \alpha\Delta t[H] \qquad \text{B.15}$$

and

$$\hat{H} = [S] + (1-\alpha)\Delta t[H] = \Delta t[H] - \hat{S} \qquad \text{B.16}$$

are substituted into Equation B.14 for simplification, then one obtains

$$\hat{S}\ {}^{t+\Delta t}\pi - [L]^{T}\ {}^{t+\Delta t}u = P - \hat{H}\ {}^{t}\pi - [L]^{T}\ {}^{t}u \quad . \qquad \text{B.17}$$

Equation B.17 must now be formulated for general nonlinear behavior using the same linearization process applied to Equation B.6, i.e., Equations B.7 and B.8 must be substituted. This operation produces

$${}^{t+\Delta t}\hat{S}\left\{{}^{t+\Delta t}\pi^{(i-1)} + \delta\pi^{(i)}\right\} - {}^{t+\Delta t}[L]^{T}\left\{{}^{t+\Delta t}u^{(i-1)} + \delta u^{(i)}\right\}$$

$$= {}^{t+\Delta t}P - {}^{t+\Delta t}\hat{H}\ {}^{t+\Delta t}\pi^{(0)} - {}^{t+\Delta t}[L]^{T}\ {}^{t+\Delta t}u^{(0)} \quad . \qquad \text{B.18}$$

Collecting the δu and $\delta \pi$ terms on the left hand side of the equation, one gets

$$
\begin{aligned}
{}^{t+\Delta t}\hat{S}\delta\pi^{(i)} - {}^{t+\Delta t}[L]^{T}\delta u^{(i)} &= {}^{t+\Delta t}P - {}^{t+\Delta t}\hat{H}\,{}^{t+\Delta t}\pi^{(0)} \\
&\quad - {}^{t+\Delta t}\hat{S}\,{}^{t+\Delta t}\pi^{(i-1)} + {}^{t+\Delta t}[L]^{T}\left\{{}^{t+\Delta t}u^{(i-1)} - {}^{t+\Delta t}u^{(0)}\right\} \ .
\end{aligned}
\qquad \text{B.19}
$$

Equations B.15 and B.16 can be used to eliminate the terms $\hat{H}$ and $\hat{S}$ from the right hand side of Equation B.19 to produce

$$
\begin{aligned}
{}^{t+\Delta t}\hat{S}\delta\pi^{(i)} - {}^{t+\Delta t}[L]^{T}\delta u^{(i)} &= {}^{t+\Delta t}P \\
&\quad - \Delta t\ {}^{t+\Delta t}H\,{}^{t+\Delta t}\pi^{(0)} - {}^{t+\Delta t}G^{(i-1)} + {}^{t+\Delta t}M^{(i-1)}
\end{aligned}
\qquad \text{B.20}
$$

where
$$
\begin{aligned}
{}^{t+\Delta t}G^{(i-1)} &= \left\{ - {}^{t+\Delta t}[S]^{(i-1)} + \alpha\Delta t\ {}^{t+\Delta t}[H]^{(i-1)}\right\}{}^{t+\Delta t}\Delta\pi^{(i-1)}, \\
{}^{t+\Delta t}M^{(i-1)} &= {}^{t+\Delta t}[L]^{T(i-1)}\ {}^{t+\Delta t}\Delta u^{(i-1)} \ , \\
{}^{t+\Delta t}\Delta\pi^{(i-1)} &= {}^{t+\Delta t}\pi^{(i-1)} - {}^{t+\Delta t}\pi^{(0)} \ , \\
{}^{t+\Delta t}\Delta u^{(i-1)} &= {}^{t+\Delta t}u^{(i-1)} - {}^{t+\Delta t}u^{(0)} \ , \\
{}^{t+\Delta t}\Delta u^{(0)} &= 0 \ \text{ and } \ {}^{t+\Delta t}\Delta\pi^{(0)} = 0 \ .
\end{aligned}
$$

Equations B.11 and B.20 are written in matrix form for increment $t+\Delta t$ as:

$$
\begin{bmatrix} K & -L \\ -L^{T} & -S + \alpha\Delta t H \end{bmatrix}
\begin{Bmatrix} \delta u^{(i)} \\ \delta\pi^{(i)} \end{Bmatrix}
=
\begin{Bmatrix} R - F^{(i-1)} + C^{(i-1)} \\ P - \Delta t[H]\pi^{(0)} - G^{(i-1)} + M^{(i-1)} \end{Bmatrix}
\qquad \text{B.21}
$$

This is the system of equations that must be solved to calculate displacements and pore fluid pressures in a deforming porous solid.