Appendix D The Proportionality Factor

The proportionality factor $d\lambda$ must be determined prior to evaluating any of the plastic strains. Baladi and Akers(1981), Chen and Baladi(1985), and Rohani(1977) outline the methods required to evaluate the proportionality factor. Those methods are included here for completeness.

Using Equations 4.4, 4.5, and 4.6 or 4.7, the total derivative of the loading function f may be expressed as

$$df = \frac{\partial f}{\partial J_1} dJ_1 + \frac{1}{2\sqrt{J_{2D}}} \frac{\partial f}{\partial \sqrt{J_{2D}}} s_{ij} ds_{ij} + \frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial \varepsilon^p_{kk}} d\varepsilon^p_{mm} = 0 \qquad \text{D.1}$$

This expression is known as the "consistent condition" for strain-hardening materials (Chen and Baladi 1985). Using Equations 4.15, 4.16, and 4.20, Equation D.1 may be manipulated to give

$$3K d\varepsilon^e_{kk} \frac{\partial f}{\partial J_1} + \frac{G de^e_{ij}}{\sqrt{J_{2D}}} \frac{\partial f}{\partial \sqrt{J_{2D}}} s_{ij} + 3 d\lambda \frac{\partial f}{\partial J_1} \frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial \varepsilon^p_{kk}} = 0 \qquad \text{D.2}$$

Substitution of Equation 4.10 into Equation D.2 produces

$$\begin{aligned} 3K(d\varepsilon_{kk} - d\varepsilon^p_{kk}) \frac{\partial f}{\partial J_1} + \frac{G}{\sqrt{J_{2D}}}(de_{ij} - de^p_{ij}) \frac{\partial f}{\partial \sqrt{J_{2D}}} s_{ij} = \\ - 3 d\lambda \frac{\partial f}{\partial J_1} \frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial \varepsilon^p_{kk}} \end{aligned} \qquad \text{D.3}$$

By substituting Equations 4.20 and 4.22 into Equation D.3, one obtains

$$3K\frac{\partial f}{\partial J_1}d\varepsilon_{kk} + \frac{G}{\sqrt{J_{2D}}}\frac{\partial f}{\partial\sqrt{J_{2D}}}s_{ij}de_{ij} = d\lambda\left[9K\left(\frac{\partial f}{\partial J_1}\right)^2 + G\left(\frac{\partial f}{\partial\sqrt{J_{2D}}}\right)^2 - 3\frac{\partial f}{\partial J_1}\frac{\partial f}{\partial\kappa}\frac{\partial\kappa}{\partial\varepsilon_{kk}^p}\right] \tag{D.4}$$

Solving for the proportionality factor $d\lambda$ yields

$$d\lambda = \frac{3K\frac{\partial f}{\partial J_1}d\varepsilon_{kk} + \frac{G}{\sqrt{J_{2D}}}\frac{\partial f}{\partial\sqrt{J_{2D}}}s_{ij}de_{ij}}{9K\left(\frac{\partial f}{\partial J_1}\right)^2 + G\left(\frac{\partial f}{\partial\sqrt{J_{2D}}}\right)^2 - 3\frac{\partial f}{\partial J_1}\frac{\partial f}{\partial\kappa}\frac{\partial\kappa}{\partial\varepsilon_{kk}^p}} \tag{D.5}$$

By using Equations 4.17, 4.19, and D.5, the total strain increment tensor may be written as

$$\begin{aligned} d\varepsilon_{ij} &= \frac{dJ_1}{9K}\delta_{ij} + \frac{ds_{ij}}{2G} \\ &+ \left[\frac{3K\frac{\partial f}{\partial J_1}d\varepsilon_{kk} + \frac{G}{\sqrt{J_{2D}}}\frac{\partial f}{\partial\sqrt{J_{2D}}}s_{ij}de_{ij}}{9K\left(\frac{\partial f}{\partial J_1}\right)^2 + G\left(\frac{\partial f}{\partial\sqrt{J_{2D}}}\right)^2 - 3\frac{\partial f}{\partial J_1}\frac{\partial f}{\partial\kappa}\frac{\partial\kappa}{\partial\varepsilon_{kk}^p}}\right] \\ &\times \left(\frac{\partial f}{\partial J_1}\delta_{ij} + \frac{s_{ij}}{2\sqrt{J_{2D}}}\frac{\partial f}{\partial\sqrt{J_{2D}}}\right) \end{aligned} \tag{D.6}$$

The stress increment tensor may be written as

$$d\sigma_{ij} = K\,d\varepsilon_{kk}\,\delta_{ij} + 2\,G\,de_{ij}$$

$$+\left[\frac{3\,K\dfrac{\partial f}{\partial J_1}\,d\varepsilon_{kk} + \dfrac{G}{\sqrt{J_{2D}}}\,\dfrac{\partial f}{\partial\sqrt{J_{2D}}}\,s_{ij}\,de_{ij}}{9K\left(\dfrac{\partial f}{\partial J_1}\right)^2 + G\left(\dfrac{\partial f}{\partial\sqrt{J_{2D}}}\right)^2 - 3\,\dfrac{\partial f}{\partial J_1}\,\dfrac{\partial f}{\partial\kappa}\,\dfrac{\partial\kappa}{\partial\varepsilon_{kk}^p}}\right] \qquad \text{D.7}$$

$$\times\left(3K\frac{\partial f}{\partial J_1}\,\delta_{ij} + \frac{G}{\sqrt{J_{2D}}}\,\frac{\partial f}{\partial\sqrt{J_{2D}}}\,s_{ij}\right)$$

Equations D.6 and D.7 are the general constitutive equations for an elastic work-hardening plastic isotropic material (Chen and Baladi 1985). To use these equations, one must first define the loading function f, the functional forms of the elastic moduli K and G, and the hardening parameter κ for the material of interest.